TMA4212 Finite Difference Methods
Exercise Set 2

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Problem 2(a)

Implement the Backward Euler and the Crank-Nicolson (trapezoidal) scheme for the heat equation,

$$u_t = u_{xx} \quad \text{on} \quad 0 \le x \le 1, 0 \le t \le .5$$

where boundary conditions are,

$$u(0,t) = u(1,t) = 0, \; u(x,0) = f(x),$$

Solution:

- **Convergence in Time**

  Figure 1 shows the convergence plot when the N is varying and M is kept constant.

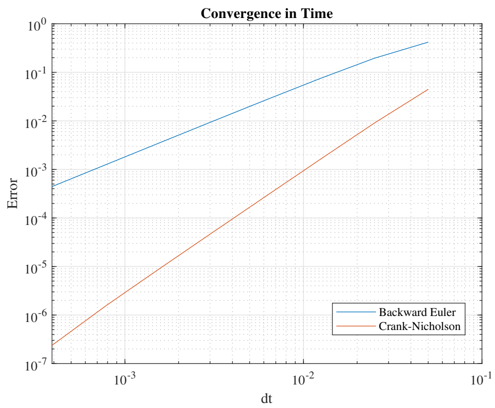

Figure 1: Variables: $M = 2000$, $N = 2^k$, $k = 1:8$.

Choice of M and N plays important role when it comes to convergence. For example for a small value of M will get a result as shown in figure 2. The initial condition function used for the heat equation is $f(x) = sin(\pi x)$ and the exact solution is $U(x,t) = sin(\pi x)e^{-\pi^2 t}$.

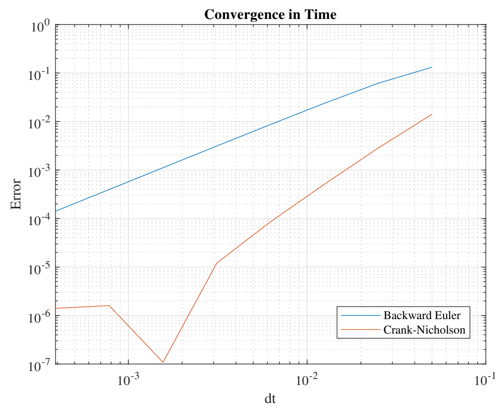

Figure 2: Variables: $M = 200$, $N = 2^k$, $k = 1:8$.

- **Convergence in Space**

  Figure 3 shows the convergence plot when N is constant and M is is varying. Plots look identical at the start but the difference becomes clear as one moves to a smaller step-size.

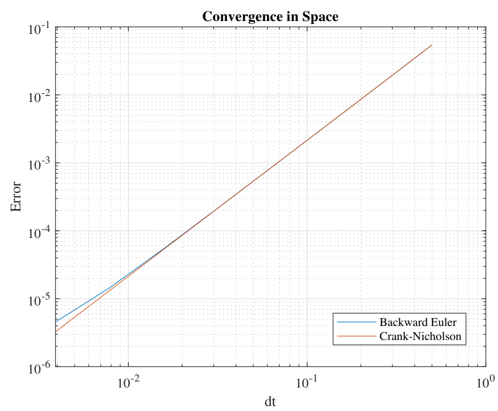

Figure 3: Variables: $N = 100000$, $M = 2^k$, $k = 1:8$.

Just like in convergence in time, step-size plays very important role in convergence. A smaller value of N will result Backward Euler to not give exact straight line in contrast to Crank Nicholson which handled small value of N better. This can be seen in figure 4.

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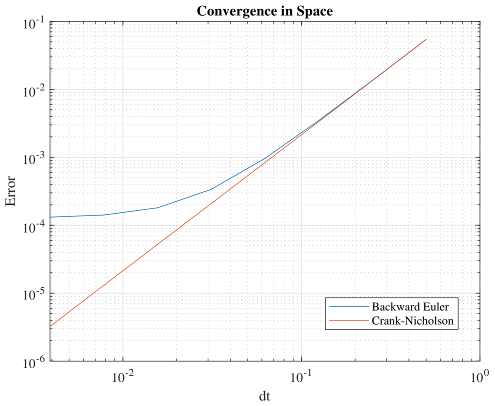

Figure 4: Variables: $N = 1000$, $M = 2^k$, $k = 1 : 8$.

For both space and time convergence we observe the error at time $t = 0.1$ so the u is not close to zero.

Problem 2(b)

Consider heat equation with periodic boundary conditions.

$$\begin{aligned} u_t &= u_{xx}, \\ u(x,0) &= f(x), \\ f(x+1) &= f(x), \\ f(x) &= sin(\pi x), \\ u(x+1,t) &= u(x,t). \end{aligned}$$

Use Crank-Nicholson to solve the problem.

Solution:
When using periodic boundary conditions it is straightforward to say that, the value at the end i.e u_M will be equal to start value u_0. Therefore we tackle the problem by slightly modifying the matrix A and our linear system. The modifications are as following,

- Instead of working on inner grid i.e only points inside the grid, we solve the linear system on complete grid. i.e matrix A is of size $\mathbb{R}^{M+1 \times M+1}$ when previously it was $\mathbb{R}^{M-1 \times M-1}$, keeping in mind we're using MATLAB for programming and it does not count zero.

- Matrices A_1 and A_2, we used for Crank-Nicholson were modified by adding entries in the first and last row.
  For example Matrix A_1 would look like,

$$\begin{bmatrix} 1+2r & -r & & & & -r \\ -r & 1+2r & -r & & & \\ & -r & 1+2r & -r & & \\ & & & & & \\ & & & & & \\ -r & & & & -r & 1+2r \end{bmatrix},$$

  Similarly we can modify A_2 accordingly.

- After above-mentioned modifications we solve the system as following,

$$U = A_1^{-1}\left(A_2 U\right)$$

Now when we've discussed how we implemented Crank-Nicholson with periodic boundary conditions, we'll discuss how we calculate the error. We first solve the problem on a very fine grid, say $M = 1000, N = M^2$. After calculating the solution at time $t = 0.1$ we find the compare results at coarser grid and finer grid to calculate the error.

- **Convergence in Time**

  Figure 5 shows convergence in time with Crank-Nicholson used. Finding error vector for this varying time-steps is not difficult however difficulty is encountered when finding error with space variable changing. This will be explained when we talk about space convergence. For Time convergence, we find error at the $t = 0.1$. Solution on finer grid and coarser grid, in terms of $x - axis$ distribution is same. So we find error using 2-norm and scale it with square root of time-step.

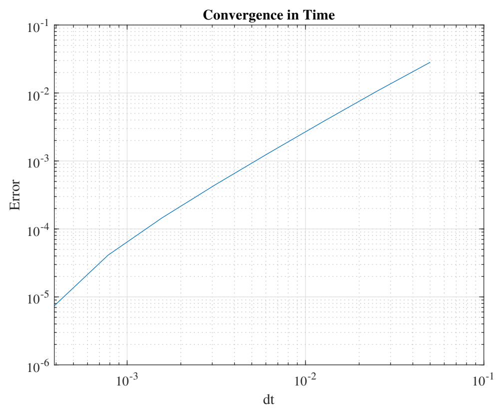

Figure 5: Variables: $M = 200000$, $N = 2^k$, $k = 1 : 8$.

- **Convergence in Space** Figure 6 shows convergence when space variables are varying. Comparing error in space convergence was not an easy task due to their different grid sizes. We compare the error by finding the same positions of x on coarser grid and finer grid, then compare error at those positions. The plot in figure 6 was calculated with a maximum norm of vector error, while other plots have been calculated using 2-norm with scaling of square root of step-size.

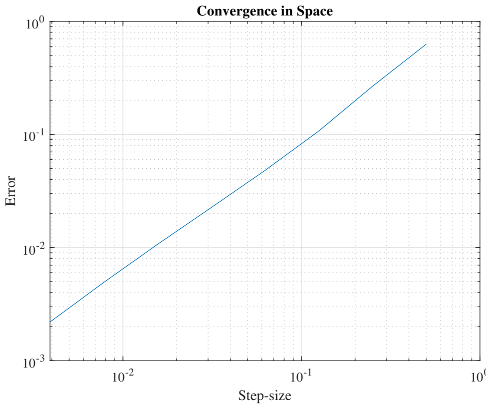

Figure 6: Variables: $N = 100$, $M = 2^k$, $k = 1 : 8$.

We also observe the space convergence when using 2-norm with scaling. It can be seen in figure 7.

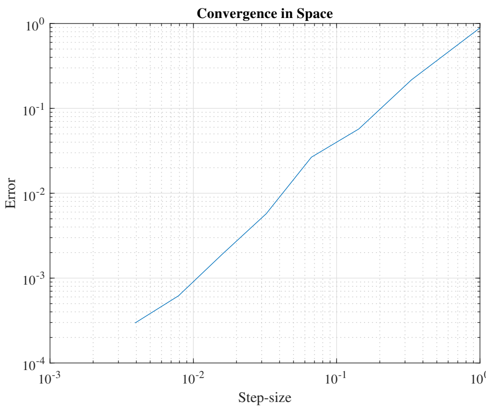

Figure 7: Variables: $N = 100$, $M = 2^k$, $k = 1 : 8$.

Problem 3

Solution of Fisher's equation with diffusion.

$$u_t = 0.01u_{xx} + u(1 - u),$$

Solution:

- **Semi-Discretization**
  The semi-discretization in space for the above problem is as following,

$$\dot{u} = \frac{d}{h^2}\left(u_{m-1} - 2u_m + u_{m+1}\right) + u(1 - u),$$

- **Boundary Conditions handling**
  We dealt with the boundary conditions using fictitious nodes as following,

$$\frac{u_1 - u_1}{2h} = 0 \implies u_{-1} = u_1,$$
$$u_t = \frac{d}{h^2}\left(u_{-1} - 2u_0 + u_1\right) + u_0(1 - u_0),$$
$$u_t = \frac{d}{h^2}\left(-2u_0 + 2u_1\right) + u_0(1 - u_0),$$

  Similarly, we use fictitious nodes at the end node to solve the problem.

- **Stiff ODE solvers**: Since, we are using MATLAB, therefore we'll use stiff solvers like *ode*15*s* or *ode*23*s*. Using *ode*23*s* and integrating from 0 to 10, we get the results as shown in figure 8 and figure 9 with two different views.

- **Results**

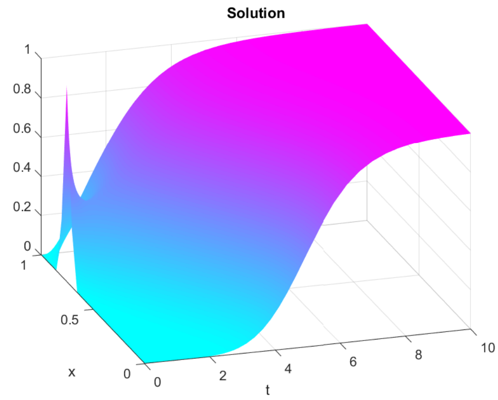

Figure 8: Variables: $M = 20$, $N = 40000$. (View 1)

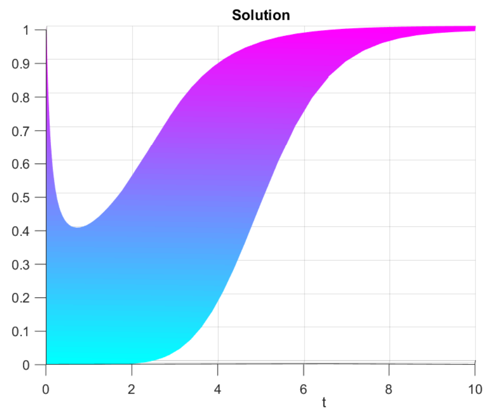

Figure 9: Variables: $M = 20$, $N = 40000$. (View 2)

- **Physical Explanation** The logistic equation alone, without the diffusion term, will yield a sigmoid solution, which has approximately the shape of the lower silhouette/outline of figure (9). The inclusion of the diffusion term makes the numerical solution curve flatten out in the x-direction as t increases. A physical interpretation of this curve may be the local population of deer, u, spread out across a one-dimensional region $0 \leq x \leq 1$. At $t = 0$, the population at each point along the x-axis is given by initial condition $f(x, 0) = sin(\pi(x - .25))^{100}$. This distribution places a lot of deer in the neighbourhood of $x = .75$, while the population is close to zero near boundaries $x = 0, x = 1$. As time t increases, there are two general tendencies that change u in any point x: firstly, deer move from highly populated areas to sparsely populated areas (in order to get more food, have more space to frolic, etc.). Secondly, the local deer population increases due to it being below the carrying capacity K that the local ecological system can sustain. At boundary $x = 0$, u grows almost nothing at first, since it is further away from the main cluster of deer than the other boundary, so it takes longer for it to experience the spillover effect of the diffusion from $x = .75$. As more deer migrate to the more remote boundary $x = 0$, population u grows ever faster until it reaches the value $u = .5$. From here, the growth $\frac{\partial u}{\partial t}$ peters out as u converges to $K = 1$. This is because from $u = .5$ and beyond, the population experiences new challenges to growth (overpopulation means less food, diseases can spread faster, etc.).

Hence, near the boundaries, population u will only increase, while at $x = .75$ the population starkly declines due to local overpopulation until about $t = .8$, as seen from the upper silhouette of figure (9). From here, u picks up again and starts increasing towards the carrying capacity.