NTNU Norwegian University of Science and Technology

Exercise No. 3

Course: Numerical Linear Algebra

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1 Conjugate Gradient on Poisson 2D

For this task Conjugate Gradient method was implemented by working directly on the grid of size $(N+1) \times (N+1)$. Figure 1 shows convergence of our implemented CG method with relative residual on the y-axis. It can be seen that, 2-norm of relative residual was minimized to a tolerance of 10^{-12}. Figure 2 shows 2-norm of residual at each iteration i.e convergence graph, for three different grids i.e $N = [10, 20, 30]$. Figure 3 shows the solution of Conjugate Gradient method applied on 2D Poisson problem.

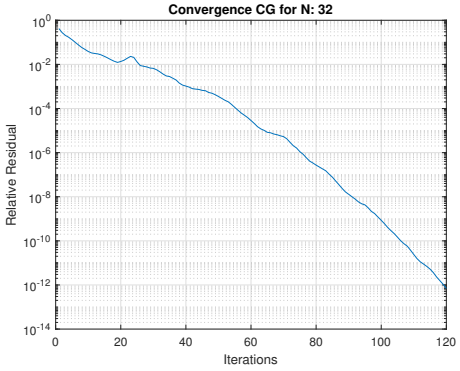

Figure 1: Convergence with Relative residual for $N = 2^5$, $tol : 10^{-12}$

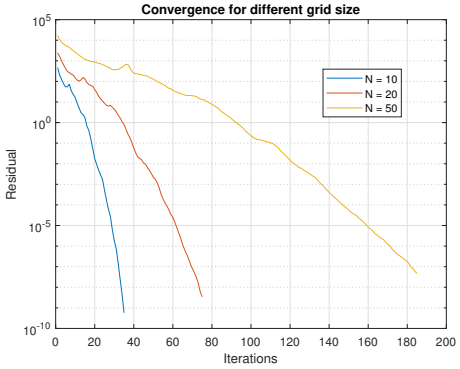

Figure 2: Convergence with residual for N = [10, 20, 50]

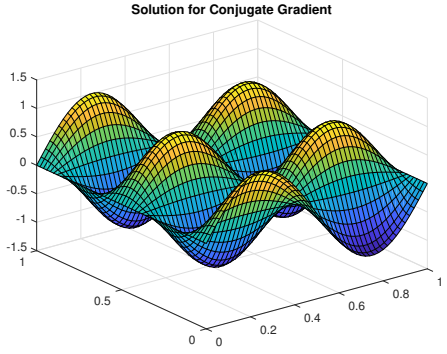

Figure 3: CG Solution $N = 2^5$

2 Multigrid V-cycle for Poisson 2D

In this task Poisson problem was solved using multigrid V-cycle. 3 pre-smoothings were done using weighted Jacboi, then residual was restricted to the coarsest level using restriction function where problem was solved using conjugate gradient.After solving problem on the coarsest problem, error was linearly interpolated. The corrected solution was post-smoothed using 3 iterations of weighted jacobi method. Figure 5 shows the solution for first five multigrid V-cycles for grid size $N = 2^5$ and Figure 4 shows convergence for Multigrid V-cycle.

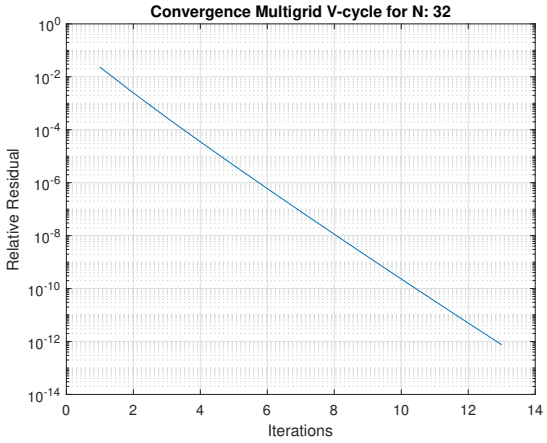

Figure 4: Convergence Multigrid V-cycle

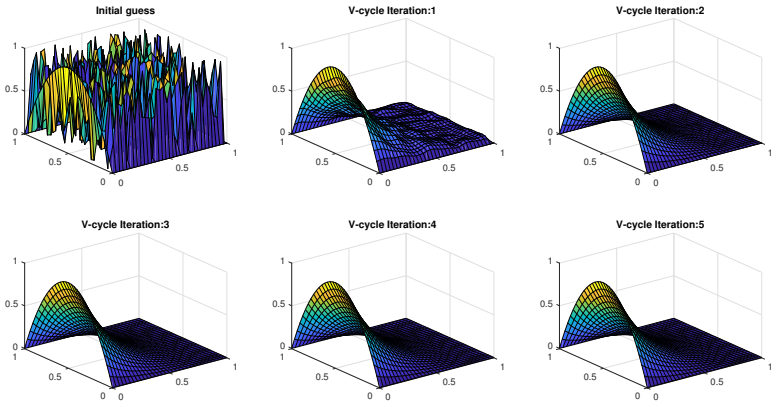

Figure 5: Solutions for Multigrid V-cycle Parameters: N = 2^5, maximum level: 4 with 3 post and pre-smoothings each

Figure 6 and Table 1 shows how number of pre and post-smoothing affect the convergence of 2D Poisson solved using Multigrid V-cycle. Results are as expected, more the pre/post-smoothing less the number of iterations to converge to the set tolerance. Relative residual is used as a stopping criteria, however the plots used for verification of convergence are made with residual vs iterations. It was also observed that increasing the size of problem did not have big effect on the convergence speed of problem.

Pre-Smoothing	Post-Smoothing	Iteration count
1	1	32
1	2	22
1	3	17
2	1	22
3	1	18
2	2	17
3	3	13
4	4	12
10	10	8
1	10	10
10	1	11

Table 1: Dependence of No. of Pre/Post-Smoothing on Convergence for $N = 2^5$

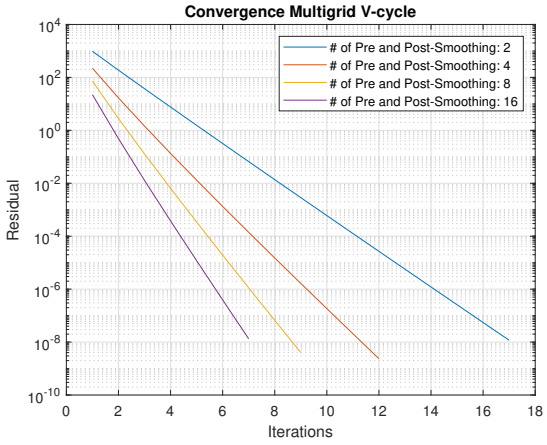

Figure 6: Convergence for different Pre/Post Smoothing Parameters: N: 2^5, maximum level: 4

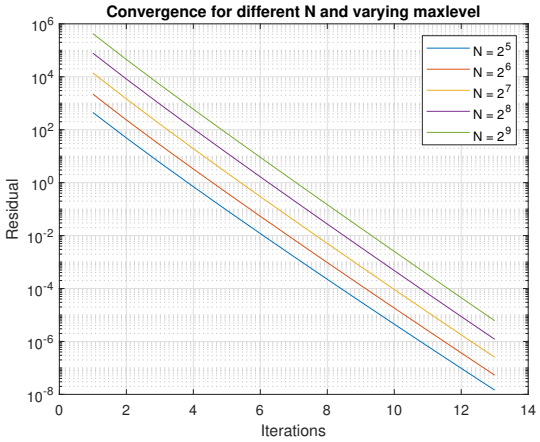

Figure 7: Convergence for varying grid size, Multigrid V-cycle, Parameters: N: 2^L, Max-level: L

3 Preconditioned CG with Multigrid V-cycle as Preconditioner

In this task, Conjugate Gradient as implemented in task 1 was modified to Preconditioned CG such that result of one multigrid V-cycle was used as a preconditioner to solve the problem. Figure 10 shows solution of initial guess and first 5 iterations of preconditioned conjugate gradient method. Figure 8 show the convergence of the method. Parameters used to solve the problem are exactly the same as used for the previous tasks.

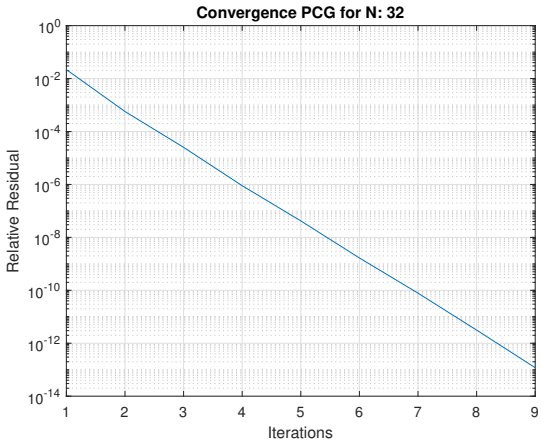

Figure 8: Convergence Preconditioned CG with Multigrid V-cycle as Preconditioner

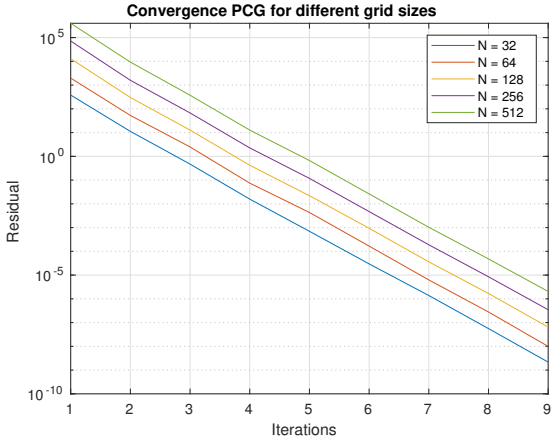

Figure 9: Convergence PCG for different grid sizes, maximum level: 4 with 3 post and pre-smoothings each

When solving the problem with Multigrid as a preconditioner care should be taken that it satisfies the conditions of a good preconditioner i.e the condition number of the of the linear system should reduce after applying the preconditioner. Since, we are solving the problem with CG, it is good practice to check positive definiteness and symmetry of the preconditioner.

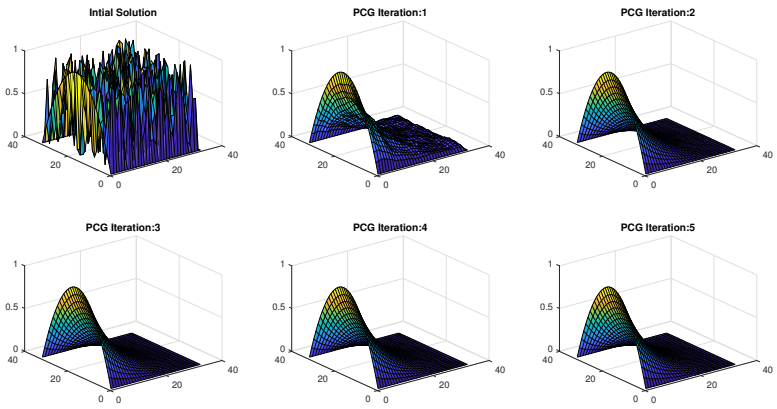

Figure 10: Solutions for PCG, Parameters: N = 2^5, maximum level: 4 with 3 post and pre-smoothings each

4 Conclusion

It was observed, number of iterations required to solve Poisson equation with usual conjugate gradient method were greatly reduced from 120 to 9 which in reality is a good achievement. We can therefore say that, multigrid V-cycle acted as a good preconditioner for Poisson equation solved with Conjugate gradient. It would be interesting to work on how different methods of smoothing, like Gauss-Seidal, SOR etc affect the convergence of this particular problem.