MEMO-F524 - Master Thesis

Inherited vertex colouring of graphs - research on Krenn's conjecture

Merlin Hannon

Supervisors : Gwenaël JORET, Yelena YUDITSKY

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Introduction

Quantum computing has hugely developed in the last few years, with the rise of new technologies and the development of new algorithms. Advancements made in quantum information and computation promise theoretical ways to solve problems that are currently intractable with classical computers. It is already used a lot in the field of cryptography [KG21], and its potential uses extend to many other fields such as artificial intelligence [Ott+17], financial planning and medicine [Has+20]. However, quantum computers are still in their infancy and are not yet able to solve a lot of practical problems. One of the reasons for this is the difficulty to physically create some specific quantum states.

In his research, the physicist Mario Krenn has been working on the generation of high-dimensional multipartite quantum states [KGZ17]. In this same paper, he discovered a new surprising link between the generation of these quantum states and a specific problem in graph theory. The problem was summarized by Mario Krenn himself on his website [Kre]. Furthermore, to promote research on the subject, the physicist promised a reward of 3000€ to the first person who will solve his conjecture, and an additional reward of 1000€ was offered during the last 2 years to the best paper on the subject [Kre]. This problem, that we will refer to as the *Krenn's conjecture*, is about proving the existence of a bound on a quantity called the *weighted matching index* of a graph. It introduces a unique form of vertex colourings in graphs that incorporates perfect matchings, and that was never investigated before. It will be defined in details in Chapter 1 of this thesis.

The main points of this master thesis are the following: first, the Krenn's conjecture will be rigorously defined. This will be done through a progressive approach, starting from reminders of basic graph theory concepts (section 1.1), then looking at a simplified version of the conjecture (section 1.2), and at last introducing the Krenn's conjecture (section 1.3). This definition will be directly followed by a detailed explanation of its physical interpretations (section 1.4).

Then, a state of the art in the domain will be established in Chapter 2. In dressing the state of the art, we will focus on the special cases of the problem that were already solved by other researchers. Also, we will rewrite some proofs in our own terms, to get a better understanding of the reasoning.

Lastly, I will present my own new contributions to the problem, which can be summarized in 2 main points. Firstly, I proved by using a theoretical approach a new special case of the Krenn's conjecture in Chapter 3. This proof will be presented by taking a simple known case of the problem, and then relaxing some of its constraints to generalize it. And secondly, I'm introducing a new tool I developed, called EGPI, to experimentally test the conjecture on a large number of graphs in Chapter 4. The use cases and technical implementation of this tool will be discussed in the sections 4.3 and 4.4 of this chapter. At last, and to finish this master thesis, I present some experimental results I got using EGPI, and my interpretation of them in section 4.5.

1 Description of the problem

1.1 Prerequisites and used notations

In order to understand the following chapters, it is needed to (re)introduce some useful notations that are widely used in the context of graph theory [Die17].

1.1.1 The different definitions of graphs

A reader who is not familiar with the field of graph theory needs first to understand the basic concepts of graphs. The following definitions constitute the foundation of the domain, and I personally learned them in a course based on the book [Die17].

In the context of discrete mathematics, a *graph* $G = (V, E)$ is a set V of objects, called *vertices*, and a set E of pairs of elements from V, called *edges*. The notations $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. The following definitions cover the graphs that will be used in the context of this master thesis.

Definition 1.1.1 (Undirected graph)**.** A graph G is said to be *undirected* if $E(G)$ is a set of unordered pairs of elements of $V(G)$. It is opposed to the concept of *directed graphs*, in which the edges are ordered tuples.

Definition 1.1.2 (Simple graph)**.** A graph G is said to be *simple* if it has at most one edge between each pair of its vertices. In the context of this master thesis, simple graphs are considered to be undirected and have no edges that begin and end at the same vertex (called *loops*).

Definition 1.1.3 (Multigraph)**.** A graph G is said to be a *multigraph* if it is allowed to have multiple edges between the same pair of vertices. In the context of this master thesis, multigraphs are considered to be undirected and have no loops.

The difference between simple graphs and multigraphs is illustrated in figure 1.1.

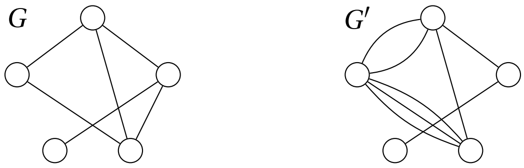

Figure 1.1: Example of simple graph G and multigraph G'. In the simple graph, there is at most one edge between each pair of vertices. In the multigraph, there can be multiple edges between the same pair of vertices.

Definition 1.1.4 (Edge-coloured graph)**.** An *edge-colouring* f of a graph G is a function that maps each edge $e = \{v_1, v_2\} \in E(G)$ to two colours $f(e, v_1)$ and $f(e, v_2)$ (often represented as unsigned integers).

- The edge colouring is said to be *pure*, and is denoted by the letter η, if $\forall e = \{v_1, v_2\} \in E(G), \eta(e, v_1) = \eta(e, v_2)$. A *pure edge-coloured graph* G_η is a graph G equipped with a pure edge-colouring η.

- The edge colouring is said to be *mixed*, and is denoted by the letter μ, if it is not necessarily pure. A *mixed edge-coloured graph* G_μ is a graph G equipped with a mixed edge-colouring μ.

Given an edge set S and a (pure or mixed) edge-colouring f, the image of f on S, denoted $f(S)$, is

$$f(S) = \bigcup_{e \in S} \{f(e, v_1), f(e, v_2)\}$$

Definition 1.1.5 (Vertex-coloured graph)**.** A *vertex-colouring* κ of a graph G is a function that maps each vertex $v \in V(G)$ to a colour $\kappa(v)$ (often represented as unsigned integers). A *vertex-coloured graph* G_κ is a graph G equipped with a vertex-colouring κ.

Definition 1.1.6 (Weighted graph)**.** A *weighting* of a graph G is a function $w : E(G) \to \mathbb{C}$, that maps each edge $e \in E(G)$ to a weight $w(e)$. In the context of this master thesis, this weight is a complex number. A *weighted graph* G^w is a graph G equipped with a weighting w.

This definition of a weighting might surprise some readers. Indeed, the weight of an edge is often assumed to be a real number in the literature and not a complex one. However, this definition is more general and will be convenient to study the Krenn's conjecture in section 1.3.

Definition 1.1.7 (Bipartite graph)**.** A graph G is said to be *bipartite* if its vertex set $V(G)$ can be partitioned into two disjoint sets V_1 and V_2 such that every edge in $E(G)$ has one endpoint in V_1 and the other in V_2.

An example of a bipartite graph is shown in figure 1.2.

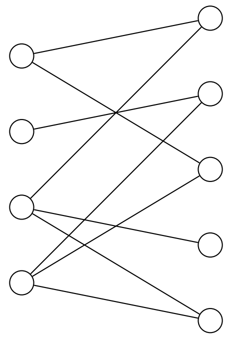

Figure 1.2: Example of bipartite graph. In this example, all the edges have one endpoint on the left and the other endpoint on the right.

1.1.2 Paths and connectivity concepts

The concepts of paths and connectivity are widely used in the mathematical field of graph theory. The definitions of these concepts are recalled here to ensure a better understanding of the following chapters [BM+76].

Definition 1.1.8 (Path)**.** Given a graph G, a *path* $P = (e_1, e_2, \dots, e_l)$ is an ordered sequence of different edges from $E(G)$ respecting the following property.

- $\forall i \in \{1, 2, \cdots, l-1\}$, the edges e_i and e_{i+1} share one and only one common vertex.
- $\forall$ vertex $v \in V(G)$, there are at most two edges $e \in P$ that contain v. In other words, P does not pass through the same vertex twice.
- If $l \geq 2$, then e_1 and e_l do not share a vertex. In other words, P begins and ends at different vertices.

Definition 1.1.9 (Cycle)**.** Given a graph G, a *cycle* $C = (e_1, e_2, \dots, e_l)$ is an ordered sequence of different edges from $E(G)$ respecting the following property.

- $\forall i \in \{1, 2, \cdots, l-1\}$, the edges e_i and e_{i+1} share one common vertex.
- $\forall$ vertex $v \in V(G)$, there are at most two edges $e \in P$ that contain v. In other words, P does not pass through the same vertex twice.
- e_1 and e_l share a vertex. In other words, C begins and ends at the same vertex.

Definition 1.1.10 (Hamiltonian cycle)**.** A cycle H of a graph G is said to be *Hamiltonian* if and only if all the vertices in $V(G)$ appear in H.

Definition 1.1.11 (Arc of Hamiltonian cycle)**.** Let G be a graph that admits a Hamiltonian cycle $H = (\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{n-1}, v_n\})$. In the context of this master thesis, what is called the *arc* from v_i to v_j on H, denoted by $H_{i,j}$, is defined as follows.

$$H_{i,j} = \begin{cases} \{\{v_i, v_{i+1}\}, \{v_{i+1}, v_{i+2}\}, \ldots, \{v_{j-1}, v_j\}\} & \text{if } i < j \\ \{\{v_i, v_{i+1}\}, \{v_{i+1}, v_{i+2}\}, \ldots, \{v_n - 1, v_n\}, \{v_n, v_1\}, \{v_1, v_2\}, \ldots, \{v_{j-1}, v_j\}\} & \text{if } j < i \\ \{\} & \text{if } i = j \end{cases}$$

In other words, $H_{i,j}$ is the set of edges that builds the path from v_i to v_j along H when being allowed to go only in the positive direction. An example is shown in figure 1.3.

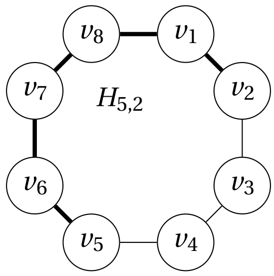

Figure 1.3: Visualization of the arc $H_{5,2}$ on a Hamiltonian path H of some graph G from v_5 to v_2. The arc is marked using thicker edges.

Definition 1.1.12 (Connected graph)**.** A graph G is said to be *connected* if and only if, $\forall v_1, v_2 \in V(G), \exists$ a path P between v_1 and v_2 in G. If this property is not satisfied, then G is *disconnected.*

Definition 1.1.13 (Edge connectivity)**.** The *edge connectivity* of a graph G is the minimum number of edges we have to remove from $E(G)$ to make it disconnected.

Definition 1.1.14 (Maximum and minimum degree)**.** The *degree* of a vertex v in a graph G, denoted $d(v)$, is the number of edges adjacent to v in $E(G)$. The *minimum degree* of G, denoted $\delta(G)$, and the *maximum degree* of G, denoted $\Delta(G)$, are defined as follows.

$$\begin{cases} \delta(G) = \min\limits_{v \in V(G)} d(v) \\ \Delta(G) = \max\limits_{v \in V(G)} d(v) \end{cases}$$

1.1.3 Matching definitions and notations

The studied problem in this master thesis is centered about the (non-)existence of matchings in specific graphs. For this reason, the matching-related concepts are (re-)defined here [Die17].

Definition 1.1.15 (Matching)**.** Given a graph G, two edges in $E(G)$ are said to be *independent* if their intersection is empty. A *matching* M of G is a set $M \subseteq E(G)$ of independent edges.

Definition 1.1.16 (Perfect matching)**.** A matching M of a graph G is said to be *perfect* if $\bigcup_{e\in M} e = V(G)$, i.e. if every vertex $v \in V(G)$ appears in one and at most one edge of M.

An example of a perfect matching is shown in figure 1.4.

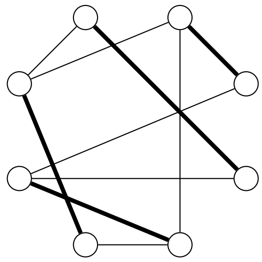

Figure 1.4: Example of a perfect matching in a graph G. The edges in bold form a perfect matching.

1.2 Simplified version of the conjecture (solved)

The following statements and definitions directly come from Mario Krenn's formulation of the problem in [Kre]. He first defines the notion of monochromatic graphs, followed by the notion of its matching index.

Definition 1.2.1 (Monochromatic graph)**.** In the context of this master thesis, a pure edge-coloured graph G_η is said to be *monochromatic* if all its perfect matchings are monochromatic, i.e., for all perfect matchings M, the edges of M are all the same colour.

An example of monochromatic graph is shown in figure 1.5. This concept is different from the one of a monochromatic edge set.

Definition 1.2.2. An edge set S is said to be *monochromatic* if all of its edges have the same colour.

Definition 1.2.3 (Matching index)**.** Let G be a simple graph. For any pure edge colouring η such that G_η is monochromatic, let $c(G,\eta)$ be defined as the number of colours such that $\exists$ a monochromatic perfect matching of G_η of that colour. The *matching index* of G, denoted by $c(G)$, is the maximum value that $c(G,\eta)$ can take.

$$c(G) = \max_{\eta\in\mathscr{E}(G)} (c(G,\eta))$$

Where $\mathscr{E}(G)$ describes here the set of all possible pure edge colourings η of G such that G_η is monochromatic.

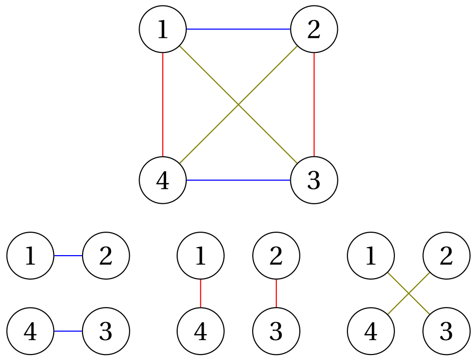

Figure 1.5: Example of monochromatic graph: a pure edge coloured version of K_4. It has at most three monochromatic perfect matchings of different colours. Therefore, $c(K_4) = 3$.

Equipped with these concepts, the reader has now all the knowledge to understand the simplified version of the conjecture.

Theorem 1.2.4 (Simplified version of Krenn's conjecture)**.** For all simple graphs G, if G is isomorphic to K_4, then $c(G) = 3$. Otherwise, $c(G) \leq 2$.

The proof of Theorem 1.2.4 was first proposed by Ilya Bogdanov in a post on a forum [Bog]. I rewrote it in my own terms to make it clearer, according to our own formulation of the problem.

Proof of Theorem 1.2.4. Let G be a monochromatic graph with a matching index $c(G) \geq 2$, and let η be a pure edge colouring of G such that $c(G,\eta) = c(G)$. Let M_1, M_2 be two monochromatic perfect matchings of G_η of different colours. Then, they are disjoint.

Claim 1.2.5. The union of M_1 and M_2 form a disjoint union of cycles of even length.

Proof of Claim 1.2.5. For each vertex $v \in V(G_\eta)$, v is touched by exactly one edge from M_1 and one edge of M_2 (by definition 1.1.16 of a perfect matching). So all the vertices are of degree 2 in $M_1 \bigcup M_2$. This already proves the first statement. Then, on each cycle in $M_1 \bigcup M_2$, the edges must be alternating between edges from M_1 and edges from M_2 (otherwise we would have two edges from the same perfect matching that touch the same vertex, which is forbidden by definition 1.1.15). This condition is satisfied only if the cycles are of even length. $\square$

Let say there are $\mathscr{C}$ different cycles formed by $M_1 \bigcup M_2$. If $\mathscr{C} \geq 2$, then a new non-monochromatic perfect matching N can be found as follows.

$$N = \begin{cases} M_1 & \text{on the } \mathscr{C} - 1 \text{ first cycles} \\ M_2 & \text{on the last cycle} \end{cases}$$

As N contains edges from M_1 and from M_2, it is not monochromatic, and can't exist. Therefore, $\mathscr{C} = 1$ and the union of any 2 perfect matchings of different colours in a monochromatic graph forms a Hamiltonian cycle. We will denote $H = (v_1, v_2, \ldots, v_n)$ the Hamiltonian cycle formed by M_1 and M_2. We notice that n is an even number, since $|H| = 2 \cdot |M_1| = 2 \cdot |M_2|$.

Now let's consider the case where $c(G) = c(G, \eta) \geq 3$. Let M_3 be a third monochromatic perfect matching such that the colour of M_3 differs from the colours of M_1 and M_2. Let $e = \{v_i, v_j\} \in M_3$, where the indices i and j are denoting positions in H. From now, it will be considered without loss of generality that the colour of $\{v_i, v_{i+1}\}$ is 1. Indeed, if it is not the case, the colours 1 and 2 can be exchanged in the following reasoning.

- **If $j - i$ is odd:** using the notation introduced in definition 1.1.11, e splits H in two arcs $H_{i+1,j-1}$ and $H_{j+1,i-1}$ of odd lengths. Then we build a new non-monochromatic perfect matching N as follows.

$$\begin{aligned} N \;=\; & \{e\} \\ & \cup M_1 \cap H_{i+1,j-1} \\ & \cup M_2 \cap H_{j+1,i-1} \end{aligned}$$

The construction of N is shown in figure 1.6. Because N is not monochromatic, it can't exist in G. Therefore, this case is impossible.

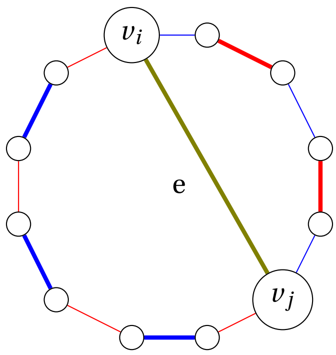

Figure 1.6: Construction of N (in bold) when $j - i$ is odd.

- **If $j - i$ is even:** then let's assume without loss of generality that e cuts the smallest arc possible $H_{i,j}$ in H. Indeed, if it is not the case, it is always possible to choose another edge from M_3. Let $e' = \{v_{i+1}, v_k\} \in M_3$. The existence of e' is certain by definition 1.1.16 of a perfect matching. By the previous case, we know that $k-(i+1)$ can not be odd, so it is even. The vertex v_k does not appear in $H_{i,j}$ since $H_{i,j}$ was the smallest arc possible delimited by e. Let C be defined as the following cycle.

$$C = \{\{v_i, v_{i+1}\}, \{v_{i+1}, v_k\}, \{v_k, v_{k-1}\}, \ldots, \{v_{j+1}, v_j\}, \{v_j, v_i\}\}$$

Note that the cycle C has an even length, $H_{i+1,j}$ and $H_{k,i}$ have also an even length. This observation can be easily understood when looking at figure 1.7. Knowing all these parities, it is possible to find a new non-monochromatic perfect matching N.

$$\begin{aligned} N \quad = \quad & \{e \cup e'\} \\ & \cup M_1 \cap H_{i+1,j} \\ & \cup M_1 \cap H_{k,i} \\ & \cup M_2 \cap H_{j,k} \end{aligned}$$

This process is shown in figure 1.7. If G has more than 4 vertices, N is a non-monochromatic perfect matching and can not exist.

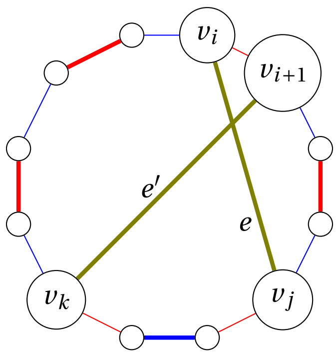

Figure 1.7: Construction of N (in bold) when $j - i$ is even.

In conclusion, K_4 is the only graph that has a matching index of 3. For all graphs G that are different from K_4, $c(G) \leq 2$. □

1.3 Krenn's conjecture (unsolved)

All the graphs that were studied in the previous section were simple pure edge coloured graphs. (Referring to the definitions 1.1.2, and 1.1.4). However, in the context of Mario Krenn's research, these graphs constitute a particular case of all the graphs that need to be studied [KGZ17; Kre]. The exact reasons to extend the problem to a whole new level are directly motivated by quantum optics experiments, and will be described into details in section 1.4.

Definition 1.3.1 (Experiment graph)**.** In the context of this master thesis, what is called an *experiment graph* G^w_μ is a mixed edge coloured weighted multigraph, referring to the definitions 1.1.4, 1.1.6 and 1.1.3.

The denomination of experiment graph was used by Gajjala and Chandran in [CG23]. The reasons behind this name will become clearer in section 1.4. The newly added weights in these experiment graphs introduce much more freedom compared to the previous case, and allow the definitions of new generalized concepts about perfect matchings. Again, these definitions come directly from [Kre].

Definition 1.3.2 (Weight of a matching)**.** The *weight* of a matching M, denoted $w(M)$, is the product of all the weights of its edges.

$$w(M) = \prod_{e \in M} w(e)$$

Definition 1.3.3 (Induced vertex colouring)**.** Given an experiment graph G_μ^w, it is said that a vertex colouring κ is *induced* by a perfect matching M of G if and only if, for each vertex $v \in V(G)$, $\exists e \in M$ such that $\kappa(v) = \mu(e, v)$. (Using the notation introduced in definition 1.1.4). The induced vertex colouring by a perfect matching M is denoted $\kappa(M)$ in the context of this master thesis. The other way around, $\mathscr{M}_\kappa$ denotes the set of all perfect matchings that induce a vertex colouring κ.

Definition 1.3.4 (Feasible vertex colouring)**.** Let G_μ^w be an experiment graph. It is said of a vertex colouring κ that it is *feasible* for G_μ^w if and only if there is at least one perfect matching of G_μ^w that induces κ.

Definition 1.3.5 (Weight of a vertex colouring)**.** The *weight* of a feasible vertex colouring κ of an experiment graph G_μ^w, denoted $w(\kappa)$, is the sum of all the weights of the perfect matchings that induce this vertex colouring.

$$w(\kappa) = \sum_{M \in \mathscr{M}_\kappa} w(M)$$

This definition uses a notation introduced in definition 1.3.3.

Definition 1.3.6 (Perfectly monochromatic graph)**.** An experiment graph G_μ^w is said to be *perfectly monochromatic* if the weights of all its feasible monochromatic vertex colourings are equal to 1, and the weights of all its feasible non-monochromatic vertex colourings are equal to 0.

An example of a perfectly monochromatic graph is presented in the figure 1.8. Notice that in a perfectly monochromatic graph, non-monochromatic perfect matchings are allowed as long as the weight of their induced vertex colouring is 0. This was not the case in simple monochromatic graphs. Just like the matching index of a monochromatic graph was defined in the previous section (definition 1.2.3), we generalize this notion by introducing a weighted matching index in perfectly monochromatic graphs.

Definition 1.3.7 (Weighted matching index)**.** Given a perfectly monochromatic graph G_μ^w, let $\tilde{c}(G,\mu,w)$ be the number of different feasible monochromatic vertex colourings in G_μ^w. We define the *weighted matching index* of G, denoted $\tilde{c}(G)$, as the maximum number that $\tilde{c}(G,\mu,w)$ can take for a mixed-edge colouring μ and a weighting w.

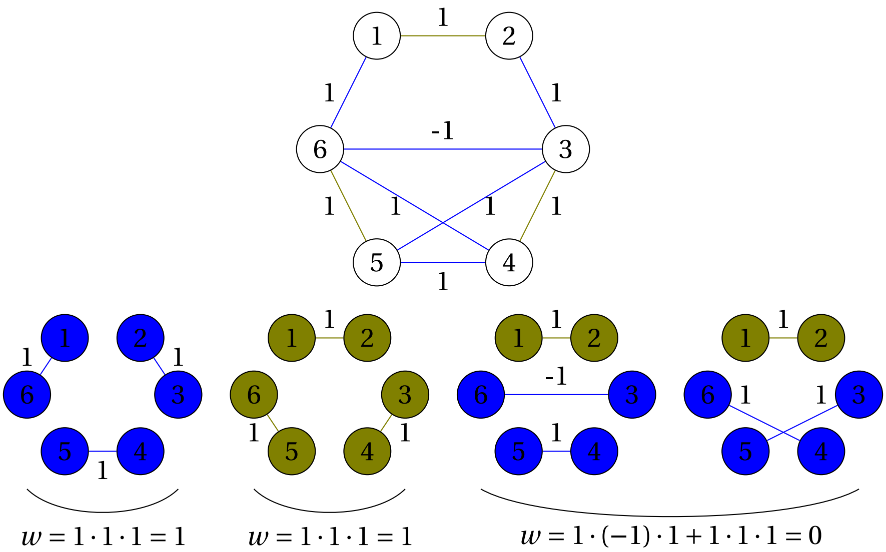

Figure 1.8: Example of a perfectly monochromatic graph G_μ^w. With this mixed edge colouring μ and weighting w, we have $\tilde{c}(G,\mu,w) = 2$ because there are 2 feasible monochromatic vertex colourings in G_μ^w. It can be shown that it is not possible to find a mixed edge colouring μ' and a weighting w' such that $\tilde{c}(G,\mu',w') > \tilde{c}(G,\mu,w)$. Therefore, $\tilde{c}(G) = 2$.

Lemma 1.3.8 (Link between c and $\tilde{c}$)**.** For all graphs G, $\tilde{c}(G) \geq c(G)$.

Proof of lemma 1.3.8. The structure of this proof follows the one proposed by Chandran and Gajjala in [CG22]. Let $G_\eta = (V,E)$ be a monochromatic graph such that $c(G,\eta) = c(G)$. Let $n_M(b)$ be the number of different monochromatic perfect matchings in G of the colour b. Now let's define the following weighting w. For each edge $e \in E(G)$:

$$w(e) = \begin{cases} \left(\frac{1}{n_M(\eta(e))}\right)^{\frac{2}{|V|}} & \text{if there are PMs of the colour } \eta(e) \\ 1 & \text{if there are none} \end{cases}$$

Because G_η is monochromatic, we know that all of its perfect matchings are monochromatic. For each of these monochromatic perfect matching M, we have

$$w(M) = \left(\left(\frac{1}{n_M(\eta(M))}\right)^{\frac{2}{|V|}}\right)^{\frac{|V|}{2}} = \frac{1}{n_M(\eta(M))}$$

Where $\eta(M)$ is the colour of the edges in the monochromatic perfect matching M, using a notation introduced in definition 1.1.4. It follows that for each monochromatic vertex colouring κ of colour b,

$$w(\kappa) = n_M(b)\frac{1}{n_M(b)} = 1$$

Hence, G^w_η is perfectly monochromatic and $c(G,\eta) = c(G) = \tilde{c}(G,\eta,w)$. By definition 1.3.7, we have

$$\tilde{c}(G) = \max_{\mu,w'}(\tilde{c}(G,\mu,w')) \geq \tilde{c}(G,\eta,w).$$

We conclude that $\tilde{c}(G) \geq c(G)$, which ends our proof. $\square$

The reader now possesses all the needed tools to understand the Krenn's conjecture, which is the heart of this master thesis research.

Conjecture 1.3.9 (Krenn's conjecture)**.** Let G be a multigraph, as defined in definition 1.1.3. If G is isomorphic to K_4, then $\tilde{c}(G) = 3$. Otherwise, $\tilde{c}(G) \leq 2$.

This last statement is, as its name says, a conjecture: it was not yet proven and no constant upper bound is currently known for the weighted matching index of an experiment graph. However, some special cases were already studied. We analyse them in the next chapter. But before diving into the study of special cases, we first present in the next section the motivations behind the Krenn's conjecture. Indeed, the conjecture originated in quantum optics, and the answer to it has a direct impact on this field.

1.4 Motivations

While the Krenn's conjecture as defined in conjecture 1.3.9 may seem very theoretical at first sight, it is actually directly motivated by important questions in the field of quantum physics. To understand why the answer to this problem has a real impact, the reader must acquire some very basic knowledge of quantum physics. Please note, however, that a reader only interested in the graph theory question can pass this section, as it is not a prerequisite to understand the rest of the master thesis.

In 1935, A. Einstein, B. Podolsky and N. Rosen discovered that quantum physics theory implied a strange phenomenon considered as impossible. They observed for the first time that the theory predicts the ability of two particles' quantum states to depend on each other even if the particles are separated by an arbitrary distance [EPR35]. Such two particles are said to be in an *entangled* state. Furthermore, in 1964, J.S. Bell stated three inequalities that must be respected by any quantum theory in the hypothesis of a deterministic local theory using hidden variables [Bel64]. The phenomenon of quantum entanglement was breaking these inequalities, and many scientists were therefore very sceptical about it. It was specifically redefining the ancient vision of the locality principle, defended by Albert Einstein. However, quantum entanglement still doesn't allow information exchange at a speed greater than the speed of light [Sie20].

Since then, quantum entanglement has been verified experimentally [J12], proving the Bell's inequalities to be wrong and opening new exciting usages of the phenomenon, mainly in the field of quantum computation. In 1989, Greenberger, Horne and Zeilinger theorized for the first time special entangled states involving three particles [GHZ89]. These states are known today as GHZ-states, and were experimentally observed in 1999 [Bou+99]. Using the Dirac's notation [Dir39], a GHZ-state is formally defined as follows.

$$GHZ = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

A reader who's not familiar with the Dirac's formalism can interpret such a state as a system composed of 3 qubits that can't be described separately from each other. The whole system has eight classical states $\{\,|\,0\rangle, |\,1\rangle\}^3$, and is in a perfect mixed state between the classical states $|000\rangle$ and $|111\rangle$. In this base case, the involved particles have only two classical states, $|0\rangle$ and $|1\rangle$. Therefore, the system is said to be of dimension $d = 2$ [KGZ17]. Generalized GHZ-states accept other dimensions, and can involve more than $n = 3$ qubits.

$$GHZ_{n,d} = \frac{1}{\sqrt{d}}(|0\rangle^n + |1\rangle^n + \cdots + |d-1\rangle^n)$$

The use of photonic technologies allowed physicists to experimentally create generalized GHZ-states [Wan+16]. One method to create such states is called *path identity*, it consists in a class of experiments designed to control the paths of photons pairs [Kre+17]. However, it is still unknown if every generalized GHZ-state can be created using path identity. In particular it seems difficult to create higher dimensional states. In 2017, Mario Krenn studied this question and discovered an unexpected link between the ability to experimentally create high-dimensional GHZ-states by path identity and graph theory. Indeed, he could show that path identity experiments could be represented as perfectly monochromatic graphs [KGZ17]. Furthermore, given an experiment that allows to create GHZ-states of dimension d involving n qubits and its corresponding perfectly monochromatic graph G:

$$\begin{cases} n &= |V(G)| \\ d &= \tilde{c}(G) \end{cases}$$

This lead to the formulation of the **Krenn's conjecture** 1.3.9 in graph theory, which remains unsolved [Kre]. Finding any constant bound on the weighted matching index of a perfectly monochromatic graph would allow physicists to know the limits of this class of experiments. On the other hand, finding counter-examples to the conjecture would lead to the ability to create higher dimensional GHZ-states.

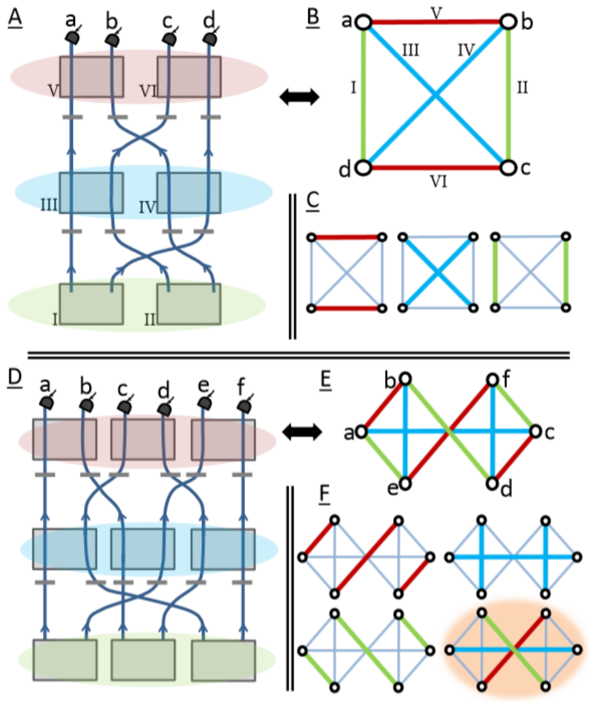

Figure 1.9: This figure was taken from Mario Krenn's paper of 2017 [KGZ17]. It represents two examples of experiments to create entangled states using photonic technologies. On one hand, the experiment described in A corresponds to the graph B that is monochromatic as shown in C. Therefore, it allows to create a GHZ-state of dimension 3 involving four particles $|\psi\rangle = GHZ_{4,3} = \frac{1}{\sqrt{3}}(|0000\rangle + |1111\rangle + |2222\rangle)$. On the other hand, the experiment described in D is represented by the graph in E, which is non-monochromatic as shown in F. Therefore, the state it creates is not a GHZ-state. $|\psi'\rangle = \frac{1}{2}(|000000\rangle + |111111\rangle + |222222\rangle + |121200\rangle)$.

At last, please note that the study of GHZ-states is considered as a fundamental question in quantum physics. For instance, Alain Aspect, John F. Clauser and Anton Zeilinger received in 2022 a Nobel Prize for their experiments on quantum entanglement, and their work was closely related to the topic [Sci22]. Finding new ways to experimentally create such states would open new intriguing gates in the fields of quantum computing [GK20] and cryptography [PHM18].

2 State of the art

During the last few years, the problem was studied by a few researchers around the world [KGZ17; Bog; CG22; CG23]. While none of them could find any constant upper bound on the weighted matching index of perfectly monochromatic graphs (defined in definitions 1.3.7 and 1.3.6), some special cases of the conjecture were proven to be true. Furthermore, non-constant bounds could also be found in more general cases of experiment graphs. This section aims to present these results. But before getting started, it could be interesting to introduce some useful definitions and tools that will be used a lot in the different proofs. A variant of these tools was used by Chandran and Gajjala in [CG22] to prove some of the results presented in this section.

Definition 2.0.1 (Redundant edge)**.** An edge e from a graph G is said to be *redundant* if it does not belong to any perfect matching of G.

Definition 2.0.2 (Redundant colour)**.** Let G_μ^w be an experiment graph. Let $r \in \mu(E(G))$, using a notation introduced in definition 1.1.4. r is said to be *redundant* if there is no monochromatic perfect matching of colour r in G_μ.

Definition 2.0.3 (Redundant mixed-colour)**.** Let G_μ^w be an experiment graph. Let $r, g \in \left(\mu(E(G))\right)^2$, using a notation introduced in definition 1.1.4.

The mixed-colour $\{r, g\}$ is said to be *redundant* if at least one of the two following conditions is true.

$$\begin{cases} r \text{ is a redundant colour} \\ g \text{ is a redundant colour} \end{cases}$$

Definition 2.0.4 ((Non-)redundant experiment graph)**.** In the context of this master thesis, an experiment graph G_μ^w as defined in definition 1.3.1 is said to be *non-redundant* if it satisfies all the following properties.

- $\forall e \in E(G), w(e) \neq 0$

- G_μ^w does not have any redundant edge.
- $\mu(E(G))$ does not have any redundant colour.
- $\forall e \in E(G)$, $\mu(e)$ is not a redundant mixed-colour.
- $\forall (v_1, v_2) \in (V(G))^2, v_1 \neq v_2$, and $\forall$ colour pair (r, g), $\exists$ at most one unique edge e between v_1 and v_2 such that $\mu(e, v_1) = r$ and $\mu(e, v_2) = g$, using notations introduced in definition 1.1.4.

Otherwise, it is said to be *redundant*.

To build the *non-redundant induced subgraph* of a redundant experiment graph G_μ^w, apply successively the following operations.

- $\forall e \in E(G)$ such that $w(e) = 0$, delete e.
- $\forall e \in E(G)$, if e is redundant, delete e.
- $\forall e \in E(G)$, if $\mu(e)$ is a redundant (mixed) colour, delete e.
- $\forall (v_1, v_2) \in (V(G))^2, v_1 \neq v_2$, and $\forall$ colour pair (r, g), if there are multiple edges $e_1, e_2, \ldots, e_m$ such that

  $$\forall i \in \{1, \ldots, m\},\ \mu(e_i, v_1) = r \text{ and } \mu(e_i, v_2) = g$$

  Then, replace them by one single edge $e = (v_1, v_2)$ that has the following properties.

  $$\mu(e, v_1) = r \text{ and } \mu(e, v_2) = g$$
  $$w(e) = \sum_{i=1}^{m} w(e_i)$$

An example of such transformation is shown in figure 2.1.

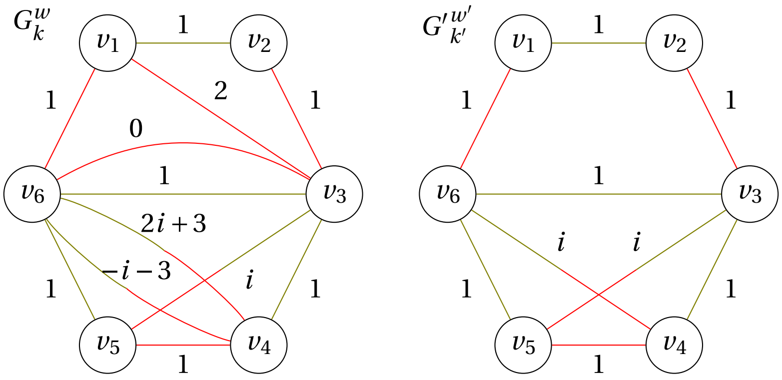

Figure 2.1: In this figure, G_μ^w is an experiment graph and $G'^{w'}_{\mu'}$ is its non-redundant induced graph. In this particular case, it consisted in doing the following operations. First of all, the 0-weighted edge between v_3 and v_6 is removed. Secondly, the edge $\{v_1, v_3\}$ is also deleted because it does not belong to any perfect matching (indeed, including it in a perfect matching M would prevent M to cover v_2). Lastly, the two edges between v_4 and v_6 are combined in one single edge, since they have the same colour at each endpoint. The reader can verify that the resulting graph $G'^{w'}_{\mu'}$ has a weighted matching index $\tilde{c}(G', \mu', w') = 2$. Therefore, $\tilde{c}(G, \mu, w) = 2$ by observation 2.0.5.

This definition and denomination of redundancy in a graph is entirely motivated by the following observation.

Observation 2.0.5 (Non-redundancy is enough)**.** Let G_μ^w be a redundant experiment graph, and let $G'^{w'}_{\mu'}$ be its non-redundant induced subgraph. Then $\tilde{c}(G, \mu, w) = \tilde{c}(G', \mu', w')$.

Proof of observation 2.0.5. To prove this last observation, the procedure will be to show that each of the transformations that was applied on G_μ^w did not change its weighted matching index.

1. **Transformation 1:** $\forall e \in E(G)$ such that $w(e) = 0$, delete e.

   Let $e = \{v_1, v_2\}$ be a zero-weighted edge of G_μ^w. At most, e can contribute to the weights of all the feasible vertex colourings κ such that $\kappa(v_1) = \mu(e, v_1)$ and $\kappa(v_2) = \mu(e, v_2)$. But since $w(e) = 0$, then all the perfect matchings M such that $e \in M$ have a weight $w(M) = 0$. Therefore, the contribution of e to κ is null, and removing it doesn't change anything to the feasibility of κ.

2. **Transformation 2:** $\forall e \in E(G)$, if e is redundant, delete e.

   Since e does not belong to any perfect matching M by definition 2.0.1, its weight can not contribute to any feasible vertex colouring κ. Removing it has therefore no impact.

3. **Transformation 3:** $\forall e \in E(G)$, if $\mu(e)$ is a redundant (mixed) colour, delete e.

   Indeed, let $e = \{v_1, v_2\}$ be an edge of G_μ^w such that $\mu(e) = \{r, g\}$ is a redundant (mixed) colour. Then, r or g is a redundant colour. Let say without loss of generality that r is a redundant colour. Then, there is no monochromatic perfect matching of colour r in G_μ^w. Therefore, e has no impact on the weight of any monochromatic feasible vertex colouring.

   However, e can still be part of a non-monochromatic perfect matching N that induces the non-monochromatic feasible vertex colouring κ_N. But κ_N has at least one r-coloured vertex, and no r-coloured edges remain in G_μ^w after the transformation. Therefore, κ_N is not feasible anymore, so we do not have to worry about its weight.

4. **Transformation 4:** $\forall (v_1, v_2) \in (V(G))^2, v_1 \neq v_2$ and $\forall$ colour pair (r, g), if there are multiple edges $e_1, e_2, \ldots, e_m$ such that

   $$\forall i \in \{1, \ldots, m\}, \mu(e_i, v_1) = r \text{ and } \mu(e_i, v_2) = g$$

   Then, replace them by one single edge $e = (v_1, v_2)$ that has the following properties.

   $$\mu(e, v_1) = r \text{ and } \mu(e, v_2) = g$$
   $$w(e) = \sum_{i=1}^{m} w(e_i)$$

   Indeed, let κ be a feasible vertex colouring such that $\kappa(v_1) = r$ and $\kappa(v_2) = g$. We need to introduce some new notations for this specific part of the proof, which are inspired by the notations we previously defined in definition 1.3.3 of an induced vertex colouring.

   - $\mathcal{M}_\kappa$ is the set of all perfect matching M of G that induce a vertex colouring κ on G.
   - $\mathcal{M}_\kappa^{e_i}$ is the set of perfect matchings M of G that induce a vertex colouring κ on G such that $e_i \in M$.
   - $\mathcal{M}_\kappa^*$ is the set of perfect matchings M of G that have no edges from $\{e_1, \ldots, e_m\}$.
   - $\mathcal{M'}_\kappa$ is the set of all perfect matching of G' that induce a vertex colouring κ on G'.
   - $\mathcal{M'}_\kappa^e$ is the set of perfect matchings of G' that induce a vertex colouring κ on G' such that $e_i \in M$.
   - $\mathcal{M'}_\kappa^* = \mathcal{M}_\kappa^*$ is the set of perfect matchings M of G' that have do not contain e.

Using these new notations, and by the definition 1.3.5 of the weight of a vertex colouring,

$$
\begin{aligned}
w(\kappa) &= \sum_{M \in \mathcal{M}_\kappa} w(M) \\
&= \sum_{i=1}^{m} \sum_{M \in \mathcal{M}_\kappa^{e_i}} w(M) + \sum_{M \in \mathcal{M}_\kappa^{*}} w(M) \\
&= \sum_{i=1}^{m} w(e^i) \sum_{M \in \mathcal{M}_\kappa^{e_i}} w(M \backslash e^i) + \sum_{M \in \mathcal{M}_\kappa^{*}} w(M) \\
&= w(e) \sum_{M \in \mathcal{M}'^{e}_\kappa} w(M \backslash e) + \sum_{M \in \mathcal{M}'^{*}_\kappa} w(M) \\
&= \sum_{M \in \mathcal{M}'_\kappa} w(M)
\end{aligned}
$$

This result proves the transformation 3 did not have any influence on the weight of all induced vertex colourings.

□

This last observation is convenient since it allows researchers to focus on non-redundant experiment graphs to prove bounds on their matching index, and these bounds are still valid in redundant experiment graphs without any loss of generality.

2.1 Special cases that were already proven

2.1.1 Restrictions on the weights

Lemma 2.1.1 (Real, positive weights)**.** Let G_μ^w be a perfectly monochromatic graph which has only positive, real weights. If G is isomorphic to K_4, then $\tilde{c}(G,\mu,w) \leq 3$. Otherwise, $\tilde{c}(G,\mu,w) \leq 2$.

Proof of Lemma 2.1.1. This is a direct result from Bogdanov's proof, described in Theorem 1.2.4 [Bog]. Because all the weights of G_μ^w are real, positive numbers, it means that the weight of any perfect matching is positive. Therefore, the weight of any feasible vertex colouring is positive by definition 1.3.5. But all the non-monochromatic feasible vertex colourings in a perfectly monochromatic graph should have a weight of 0 by definition 1.2.1. It follows that G_μ^w has no non-monochromatic perfect matching. Then $\tilde{c}(G,\mu,w) = c(G,\mu)$. □

2.1.2 Restrictions on the matching index

In 2022, Chandran and Gajjala analysed in [CG22] the Krenn's conjecture by separating the graphs in different subclasses according to their matching index (see definition 1.2.3). Here are their results.

Lemma 2.1.2 (Graphs with a matching index of 0)**.** If G is a simple graph with a matching index of 0, then the Krenn's conjecture is true for G and $\tilde{c}(G) = c(G) = 0$.

Proof of Lemma 2.1.2. Let G be a graph with a matching index of 0. Then, G has no perfect matchings (otherwise it would be feasible to colour all the edges of G in the same colour and find a monochromatic perfect matching, which contradicts the fact that $c(G) = 0$ by definition 1.2.3 of a matching index). Because it has no perfect matchings, there is no mixed-edge colouring μ and weighting w such that G_μ^w has a feasible monochromatic vertex colouring. It follows that $\tilde{c}(G) = 0$. □

Lemma 2.1.3 (Graphs with a matching index of 2)**.** If G is a simple graph with a matching index of 2, then the Krenn's conjecture is true for G and $\tilde{c}(G) = c(G) = 2$.

Lemma 2.1.3 is proved by Chandran and Gajjala in [CG22]. The proof is not trivial and uses some very interesting observations about the structure of a perfectly monochromatic graph with $\tilde{c}(G) = 2$. The reader is highly encouraged to have a look at it in the original paper in order to believe this claim. Nevertheless, the proof will not be discussed here. Thanks to their proof, Chandran and Gajjala could pose the following theorem, which is a summary of the two last Lemmas 2.1.2 and 2.1.3.

Theorem 2.1.4 (Chandran - Gajjala's Theorem)**.** Let G be a simple graph. If $c(G) \neq 1$, then the Krenn's conjecture is true and $\tilde{c}(G) = c(G)$.

Having Theorem 2.1.4, it would be a natural question to ask ourselves if, for any graph G, $c(G) = \tilde{c}(G)$. Unfortunately, it is not the case, since a counter-example was found by Chandran and Gajjala [CG22].

Observation 2.1.5. There exists a graph G that satisfies $c(G) = 1$ and $\tilde{c}(G) = 2$. Such a graph is shown in figure 2.2.

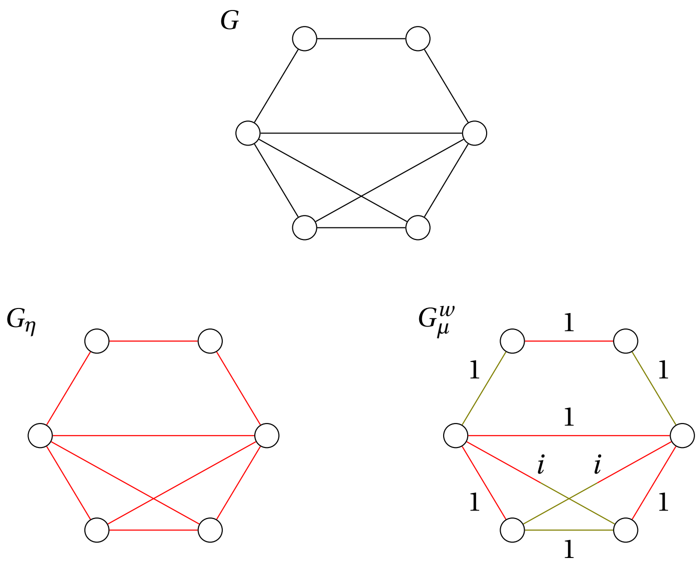

Figure 2.2: Example of graph G that has a matching index of 1 and a weighted matching index of 2. On this figure, the pure edge colouring η is an example of pure edge colouring that gives $c(G,\eta) = 1$. Furthermore, the mixed-edge colouring μ and the weighting w are examples of mixed-edge colourings and weightings that result in $\tilde{c}(G,\mu,w) = 2$. Another example of mixed-edge colouring and weighting that gives the same weighted matching index on G is available in figure 1.8. It has still to be shown that it is impossible to find other edge-colourings / weightings that would lead to bigger (weighted) matching indexes. This proof is not presented here, but is available in Chandran and Gajjala's original paper [CG22].

2.2 Known non-constant bounds on the weighted matching index

Despite the failure to discover any constant bound on the weighted matching index of experiment graphs up to now, some interesting non-constant bounds were nevertheless found. Chandran and Gajjala discovered in [CG22] two interesting bounds in terms of minimum degree and edge connectivity that I considered to be relevant to rewrite here. The two following theorems are due to them.

Lemma 2.2.1 (Upper bound in terms of minimum degree)**.** Let G be a multigraph, and $\delta(G)$ be its minimum degree as defined in definition 1.1.14. Then $\tilde{c}(G) \le \delta(G)$.

Proof of Lemma 2.2.1. Let G_μ^w be a perfectly monochromatic graph such that $\tilde{c}(G,\mu,w) = \tilde{c}(G)$. It is known by definition 1.3.7 of the weighted matching index that G_μ^w has at least one monochromatic perfect matching for $\tilde{c}(G)$ different colour classes. In the context of this proof, these perfect matchings are denoted $M_1, M_2, \ldots, M_{\tilde{c}(G)}$. Because these perfect matchings are of different colours, they can not share any edge.

Let $v \in V(G)$ be a vertex of minimum degree $d(v) = \delta(G)$. This vertex must be covered by $M_1, M_2, \ldots, M_{\tilde{c}(G)}$. M_1 covers it through an edge e_1, M_2 through an edge e_2, ..., $M_{\tilde{c}(G)}$ through an edge $e_{\tilde{c}(G)}$ (as shown in figure 2.3). Then v has at least $\tilde{c}(G)$ incident edges, and $d(v) = \delta(G) \geq \tilde{c}(G)$. □

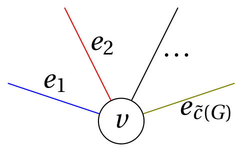

Figure 2.3: Visualization of the proof that $\tilde{c}(G) \leq \delta(G)$ for every perfectly monochromatic graph G.

Theorem 2.2.2 (Upper bound in terms of edge connectivity)**.** Let G be a multigraph, and $\lambda(G)$ be its edge connectivity as defined in definition 1.1.13. Then $\tilde{c}(G) \leq \lambda(G)$.

The proof of Theorem 2.2.2 is not trivial. It was first described by Chandran and Gajjala in [CG22], and we are using the same concepts as them here.

Proof of Theorem 2.2.2. Let G' be a multigraph, and let μ' be a mixed-edge colouring and w' be a weighting such that $\tilde{c}(G', \mu', w') = \tilde{c}(G')$. Let's say by contradiction that $\tilde{c}(G', \mu', w') = \tilde{c}(G') \geq \lambda(G') + 1$. Let G^w_μ be the non-redundant induced subgraph of $G'^{w'}_{\mu'}$. By observation 2.0.5, we know that

$$\tilde{c}(G, \mu, w) = \tilde{c}(G) = \tilde{c}(G', \mu', w') = \tilde{c}(G').$$

We also notice that the edge-connectivity of a graph can not increase when we remove edges. Therefore

$$\begin{aligned} \tilde{c}(G) &\geq \lambda(G') + 1 \\ &\geq \lambda(G) + 1 \end{aligned}$$

By definition 1.1.13 of the edge-connectivity of a graph, G can be cut in two disconnected parts S and S' by removing $\lambda(G)$ edges. Note that, because G has perfect matchings, $|V(G)|$ is an even number and therefore $|S|$ and $|S'|$ are of the same parity.

1. **If** $|S|$ **and** $|S'|$ **are odd** then, for every monochromatic perfect matching M, M contains at least one crossing edge, i.e. an edge with one endpoint in S and the other endpoint in S' (see figure 2.4).

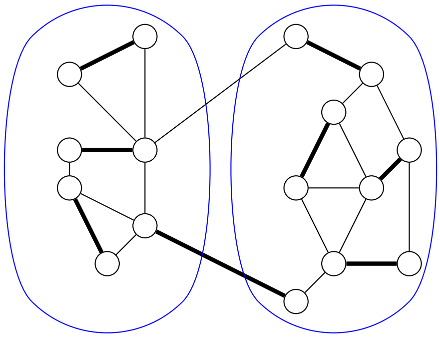

Figure 2.4: Existence of a crossing edge between S and S' in a perfect matching M of G if $|S|$ and $|S'|$ are odd.

Let $E(S, S')$ be the set of crossing edges from S to S'. The maximum number of different monochromatic edges that can be found on $E(S, S')$ is at most $\lambda(G)$ (otherwise more edges would be needed). The conclusion is that the monochromatic perfect matchings of G^w_μ can be of a most $\lambda(G)$ different colours. Hence,

$$\tilde{c}(G') = \tilde{c}(G) \leq \lambda(G) \leq \lambda(G')$$

This forms a contradiction with the statement that $\tilde{c}(G') \geq \lambda(G') + 1$.

2. **If** $|S|$ **and** $|S'|$ **are even** then let's separate all the colours of $\mu(E(G))$ in

$$\begin{cases} [r] & = & \{1, 2, \ldots, r\} & = & \text{all the colours such that none of the monochromatic} \\ & & & & \text{perfect matchings of these colours intersects } E(S, S') \\ [r+1, r+r'] & = & \{r+1, r+2, \ldots, r+r'\} & = & \text{all the colours such that there exists a monochromatic} \\ & & & & \text{perfect matching of this colour intersecting } E(S, S') \end{cases}$$

Clearly, $\tilde{c}(G') = \tilde{c}(G) = r + r'$. For all colours $i \in [r+1, r+r']$, there exists a monochromatic perfect matching M which intersects $E(S, S')$. And since $|S|$ and $|S'|$ are even, M contains at least two edges from $E(S, S')$, as shown in figure 2.5.

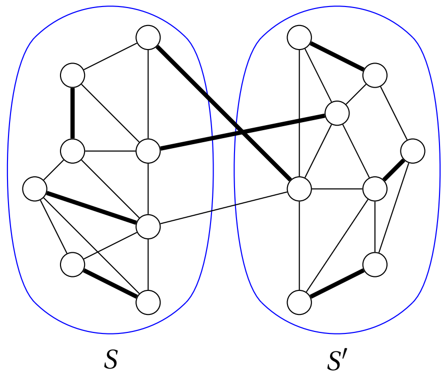

Figure 2.5: Existence of 2 crossing edges in a crossing perfect matching M of G if $|S|$ and $|S'|$ are even numbers.

Then there exist at least 2 edges of colour i in $E(S,S')$ for all $i \in [r+1, r+r']$. It follows that $\lambda(G') \geq \lambda(G) \geq 2 \cdot r'$.

2.1 **If** $r \leq 1$ **and** $r' \geq 1$, then

$$\begin{aligned} \tilde{c}(G') &= r + r' \\ &\leq 2 \cdot r' \\ &\leq \lambda(G') \end{aligned}$$

2.2 **If** $r \leq 1$ **and** $r' = 0$, then it is trivial because G is connected.

$$\begin{aligned} \lambda(G') &\geq 1 \\ &\geq r + r' \\ &= \tilde{c}(G') \end{aligned}$$

2.3 **Otherwise,** $r \geq 2$. Then one can pick two colours from $[r]$ (let say 1 and 2).

Claim 2.2.3. There should be at least two mixed edges of colour $(1,2)$ in $E(S,S')$.

Proof of Claim 2.2.3. By contradiction, suppose not. Let's consider 2 monochromatic perfect matchings M_1 and M_2 of colour 1 and 2 that induce the monochromatic vertex colourings $1_{V(G)}$ and $2_{V(G)}$ respectively. M_1 and M_2 do not have any edge in $E(S,S')$. In general, in the context of this proof, i_A will denote the monochromatic vertex colouring of colour i of the vertices in A. Because G^w_μ is perfectly monochromatic, the weights of $1_{V(G)}$ and $2_{V(G)}$ must be 1. Since $1,2 \in [r]$, there are no normal edges of colour 1 or 2 in $E(S,S')$. Then we have the following relations, where $w(i_A)$ denotes the weight of the monochromatic vertex colouring of colour i on the subgraph A.

$$\begin{aligned} w(1_{V(G)}) &= w(1_S) \cdot w(1_{S'}) &= 1 \\ w(2_{V(G)}) &= w(2_S) \cdot w(2_{S'}) &= 1 \end{aligned}$$

Therefore, $w(1_S)$, $w(1_{S'})$, $w(2_S)$, $w(2_{S'})$ must be non-zeros.

Now let's consider a non-monochromatic vertex colouring κ in which S is coloured 1 and S' is coloured 2. κ is feasible because we can build a perfect matching by taking all the edges from M_1 on S and all the edges from M_2 on S'. Because G_μ^w is perfectly monochromatic, $w(\kappa)$ is 0 by definition 1.3.6. Also, since $|S|$ and $|S'|$ are even, every perfect matching contains an even number of edges from $E(S,S')$. But by assumption, there is at most one crossing edge of colour $\{1,2\}$ and no crossing edges of colour 1 or 2. In conclusion, no perfect matching that induces κ can contain an edge from $E(S,S')$. Then,

$$w(\kappa) = w(1_S) \cdot w(2_{S'}) = 0$$

But $w(1_S)$ and $w(2_{S'})$ are non-zeros, so this is impossible and creates a contradiction. This proves the claim. □

We proved our claim. Therefore, for a pair of colours $i, j \in [r]$, there should be at least two mixed edges of colour $\{i, j\}$ in $\{S, S'\}$.

$$\text{Minimum number of edges in } E(S,S') = 2\binom{r}{2} + 2 \cdot r' \leq \lambda(G')$$

Finally, it means that

$$\begin{aligned} \tilde{c}(G') &= r + r' \\ &\leq 2\binom{r}{2} + 2 \cdot r' \\ &\leq \lambda(G') \end{aligned}$$

And this ends the proof.

□

This proof was interesting to rewrite here since it uses some very interesting reasoning about the structure of perfectly monochromatic graphs and the definitions of their weights. Some similar reasoning is sometimes reused in this master thesis in order to prove my own new results.

Last but not least, Chandran and Gajjala showed in another paper an upper bound on the weighted matching index of graphs in terms of their number of vertices [CG23].

Theorem 2.2.4 (Upper bound in terms of number of vertices)**.** Let G_μ^w be an experiment graph that has n vertices. Then,

$$\tilde{c}(G, \mu, w) \leq \frac{n}{\sqrt{2}}$$

The proof of Theorem 2.2.4 is detailed in Chandran and Gajjala's publication [CG23], and will not be discussed here.

3

New theoretical contributions

There are two intuitive ways to tackle the problem of the Krenn's conjecture. The first way, done in this chapter, is to use a theoretical approach to find interesting properties about perfectly monochromatic graphs. In this approach, I prove the Krenn's conjecture in a very restrained subcase in subsection 3.2.1. Then, I show how to relax the constraints of this analysed subcase in the subsection 3.2.2. Also, I present two different ways of thinking about the problems by showing an equivalency between two cases in the subsection 3.1.

The second approach is a computational approach, and is presented in chapter 4. The goal of this second approach will be to generate a big number of random perfectly monochromatic experiment graphs, and extract conclusions from the data. To do so, I will present a new tool I developed, called EGPI, that aims to perform such experiments.

3.1 Problem reduction

Let $\beta \in \mathbb{N}_0$ be a strictly positive integer. Having β, we can formulate the two following conjectures.

Conjecture 3.1.1 (Weighted matching index bounded by β if signum weighting)**.** Let G_μ^w be a perfectly monochromatic graph such that $w : E(G) \to \{-1, 1\}$. Then, $\tilde{c}(G, \mu, w) \leq \beta$.

Conjecture 3.1.2 (Weighted matching index bounded by β if integer weights)**.** Let G_μ^w be a perfectly monochromatic graphs such that $w : E(G) \to \mathbb{Z}$. Then, $\tilde{c}(G, \mu, w) \leq \beta$.

Since the conjecture 3.1.1 is a particular case of the conjecture 3.1.2, the conjecture 3.1.1 seems to be easier to prove. However, the following lemma holds.

Lemma 3.1.3 (Conjectures 3.1.1 and 3.1.2 are equivalent)**.** Let $\beta \in \mathbb{N}_0$ be a strictly positive integer. The conjecture 3.1.1 is true for β if and only if the conjecture 3.1.2 is true for β. In other words, the two conjectures are equivalent.

Proof of Lemma 3.1.3. Let G_μ^w be a perfectly monochromatic graph such that $w : E(G) \to \mathbb{Z}$. We will show that we can build a perfectly monochromatic graph that has only weights included in $\{-1, 1\}$ and that has the same weighted matching index as G_μ^w.

First, we choose an edge in G_μ^w that has a weight different from 1 or -1 (if such an edge doe not exist, we are done). We replace this edge by $|w(e)|$ parallel edges of the same colour, and that have all a weight of $\frac{w(e)}{|w(e)|}$ (1 if $w(e) > 0$, -1 if $w(e) < 0$). We say that each of these new edges is *derived* from e. Remember this can be done because experiment graphs allow multi-edges by definition 1.3.1. This creates a new graph ${G'_{\mu'}}^{w'}$. That process is illustrated in figure 3.1.

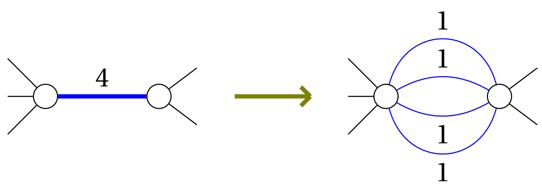

Figure 3.1: Illustration of the transformation from an experiment graph with integer weights to an experiment graph with weights included in $\{-1, 1\}$.

Let κ be a feasible vertex colouring of G_μ^w. To compute the weight of κ in G_μ^w, we will denote by

$$\begin{cases} M_\kappa & \text{the set of perfect matchings of } G_\mu^w \text{ that induce the vertex colouring } \kappa \\ M_\kappa^e & \text{the set of perfect matchings of } G_\mu^w \text{ that induce the vertex colouring } \kappa \text{ and contain } e \\ M_\kappa^{\neg e} & \text{the set of perfect matchings of } G_\mu^w \text{ that induce the vertex colouring } \kappa \text{ and do not contain } e \end{cases}$$

The weight of κ in G_μ^w is

$$\begin{aligned} w(\kappa \text{ in } G_\mu^w) &= \sum_{M \in M_\kappa} w(M) \\ &= \sum_{M \in M_\kappa^{\neg e}} w(M) + \sum_{M \in M_\kappa^e} w(M) \end{aligned}$$

The next step, which is the heart of the proof, consists of computing the weight of the vertex colouring κ in ${G'_{\mu'}}^{w'}$. We will denote by

$$\begin{cases} M'_\kappa & \text{the set of perfect matchings of } {G'_{\mu'}}^{w'} \text{ that induce the vertex colouring } \kappa \\ {M'_\kappa}^e & \text{the set of perfect matchings of } {G'_{\mu'}}^{w'} \text{ that induce the vertex colouring } \kappa \\ & \text{and contain an edge that was derived from } e \\ {M'_\kappa}^{\neg e} & \text{the set of perfect matchings of } {G'_{\mu'}}^{w'} \text{ that induce the vertex colouring } \kappa \\ & \text{and does not contain an edge that was derived from } e \end{cases}$$

In addition to these concepts, for every perfect matching $M \in M_\kappa^e$, let $\mathcal{M}'(M)$ be the set of corresponding perfect matchings M' in ${M'_\kappa}^e$. In other words, $\mathcal{M}'(M)$ denotes the set of perfect matchings that are the same as M on every edge except e, and that contain one of the edges that were added when e was removed. It follows that, for every $M \in M_\kappa^e$, $|\mathcal{M}'(M)| = w(e)$. Also, given $M \in M_\kappa^e$, for each perfect matching $M' \in \mathcal{M}'(M)$, $w(M') = \frac{w(M)}{|w(e)|}$. Finally, we notice the following relations between the different sets we defined :

$$\begin{cases} {M'_\kappa}^{\neg e} & = & M_\kappa^{\neg e} \\ {M'_\kappa}^{e} & = & \bigcup\limits_{M \in M_\kappa^e} \mathcal{M}'(M) \end{cases}$$

Having all these observations in mind, we can now compute the weight of κ in ${G'_{\mu'}}^{w'}$.

$$\begin{aligned}
w(\kappa \text{ in } {G'_{\mu'}}^{w'}) &= \sum_{M' \in M'_\kappa} w(M') \\
&= \sum_{M' \in {M'_\kappa}^{\neg e}} w(M') + \sum_{M' \in {M'_\kappa}^{e}} w(M') \\
&= \sum_{M \in {M_\kappa}^{\neg e}} w(M) + \sum_{M \in M_\kappa^e} \left(\sum_{M' \in \mathcal{M}'(M)} w(M') \right) \\
&= \sum_{M \in {M_\kappa}^{\neg e}} w(M) + \sum_{M \in M_\kappa^e} \left(\sum_{M' \in \mathcal{M}'(M)} \frac{w(M)}{|w(e)|} \right) \\
&= \sum_{M \in {M_\kappa}^{\neg e}} w(M) + \sum_{M \in M_\kappa^e} \left(|w(e)| \frac{w(M)}{|w(e)|} \right) \\
&= \sum_{M \in {M_\kappa}^{\neg e}} w(M) + \sum_{M \in M_\kappa^e} w(M) \\
&= w(\kappa \text{ in } G_\mu^w)
\end{aligned}$$

So, since the weight of each feasible vertex colouring in G_μ^w remains unchanged in ${G'_{\mu'}}^{w'}$, the monochromatic feasible vertex colourings have still a weight of 1, and the non-monochromatic feasible vertex colourings have still a weight of 0. So, ${G'_{\mu'}}^{w'}$ is still perfectly monochromatic and $\tilde{c}(G', \mu', w') = \tilde{c}(G, \mu, w)$. Now, we can rename G' as G and repeat the whole procedure while G_μ^w has still edges with a weight different from $\{-1, 1\}$. The resulting graph has only edges that have signed unitary weights and has the same weighted chromatic index as the initial graph. So if any upper bound can be found on the weighted matching index of the final graph with signed unitary weights, it will still be valid for the weighted matching index of the initial one, with integer weights. □

The implication of Lemma 3.1.3 is that there are actually two ways to reason about the Krenn's conjecture when we are interested in integer weights. The first way is to consider only non-redundant graphs (as defined in definition 2.0.4) and to try to find a bound on their weighted matching index. And the second way is to consider redundant graphs in which each edge has a weight included in $\{-1, 1\}$. Every result discovered in the second approach can be translated to the first approach, and vice versa.

3.2 Constraints' relaxation

As it was explained in the introduction, a simplified version of the conjecture was already proven thanks to Bogdanov.[Bog] This version, presented in Lemma 2.1.1, is only valid when all the weights of a perfectly monochromatic graph G_μ^w are positive. In this section, our main goal will be to relax these constraints.

3.2.1 Allowing one negative edge

Since the conjecture is proven to be true when all the weights are positive, it is natural to ask ourselves how the proof would be affected if this constraint is relaxed. The most simple case is the one where one edge is allowed to have a negative weight. In this section, I show that the Krenn's conjecture is true for simple graphs in the absence of bicoloured edges and when maximum one edge has a negative weight. Let's begin by proving the two following claims.

Claim 3.2.1 (Existence of a Hamiltonian cycle)**.** Let G_η^w be a perfectly monochromatic graph that respects the following properties.

- G is a simple graph, referring to definition 1.1.2.
- η is a pure edge colouring, referring to definition 1.1.4.
- G_η^w has a weighted matching index $\tilde{c}(G, \eta, w) \geq 3$.
- $\exists$ two colours $r, g \in \left(\eta(E(G))\right)^2, r \neq g$, such that all the edges coloured r or g in G_η^w have a real, positive weight.

Let M_r and M_g be 2 monochromatic perfect matchings of G_η^w coloured r and g respectively. Then, the union of M_r and M_g forms a Hamiltonian cycle of even length.

Proof of Claim 3.2.1. Since M_r and M_g are disjoint, they form a disjoint union $\mathscr{C}$ of cycles of even length (this was formally showed in Claim 1.2.5). If $|\mathscr{C}| \geq 2$, we denote by C_i the i^{th} cycle. Then, we can build the following non-monochromatic perfect matching :

$$N = (C_1 \cap M_r) \cup \left(\bigcup_{i=2}^{|\mathscr{C}|} C_i \cap M_g\right)$$

The construction of N is highlighted in figure 3.2.

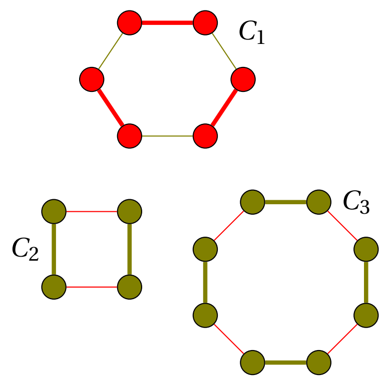

Figure 3.2: In this example, the non-monochromatic perfect matching N (represented by thick edges) is constructed from a red perfect matching and a blue one. The induced vertex colouring is also visible.

Since M_r and M_g include no negatively weighted edge, $w(N) = \prod_{e \in N} w(e) > 0$. But, by definition 1.3.6 of a perfectly monochromatic graph, and using the notations introduced in definitions 1.3.2 and 1.3.4,

$$w(\kappa(N)) = \sum_{N_i \in \mathscr{M}_{\kappa(N)}} N_i = 0$$

Therefore, we know that $\exists N' \in \mathscr{M}_{\kappa(N)}$ such that $w(N') < 0$. This is impossible, because the only way for N' to have a negative weight is to include at least one negative edge. And negative edges don't exist in colours r or g. We can conclude that $|\mathscr{C}| = 1$, which means that the union of M_r and M_g forms a Hamiltonian cycle. $\square$

Claim 3.2.2 (Parity of crossing edges)**.** Let G_η^w be a perfectly monochromatic graph that respects the following properties.

- G is a simple graph, referring to definition 1.1.2.
- η is a pure edge colouring, referring to definition 1.1.4.
- G_η^w has a weighted matching index $\tilde{c}(G, \eta, w) \geq 3$.
- $\exists$ two colours $r, g \in \big(\eta(E(G))\big)^2, r \neq g$, such that all the edges coloured r or g in G_η^w have a real, positive weight.

Let M_r, M_g be 2 monochromatic perfect matchings of different colours r and g respectively. Let $H = (v_1, \ldots, v_n)$ be the Hamiltonian cycle of G_η^w formed by M_r and M_g. Let $e = (v_i, v_j) \in E(G)$ be an edge whose colour is $b \notin \{r, g\}$. Then, $j - i$ is even.

Proof of Claim 3.2.2. Let's assume by contradiction that $j - i$ is odd. Without loss of generality, we can assume that the colour of (v_i, v_{i+1}) is r (otherwise, inverse colours r and g in the following arguments). Then the following non-monochromatic perfect matching can be built.

$$N = e \cup (M_g \cap H_{i+1,j-1}) \cup (M_r \cap H_{j+1,i-1})$$

The construction of the non-monochromatic perfect matching N is shown in figure 3.3.

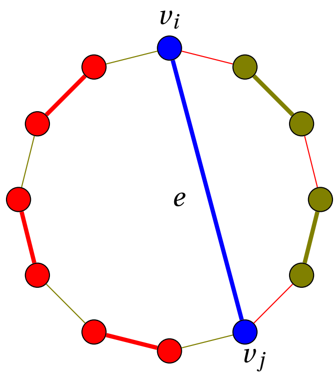

Figure 3.3: Construction of N from M_r, M_g and e if $j-i$ is odd. On this figure, N is represented by thick edges. The induced vertex colouring $\kappa(N)$ is also visible.

- If $w(e) > 0$, then, since e was the only edge that had a colour different from r or g in N,

  $$w(N) = \prod_{e' \in N} w(e') > 0$$

  by definition 1.3.2. But $w(\kappa(N)) = \sum_{N_i \in \mathcal{M}_{\kappa(N)}} N_i = 0$ by definitions 1.3.7 and 1.3.6 (using a notation introduced in definition 1.3.3). Therefore, $\exists$ another non-monochromatic perfect matching $N' \in \mathcal{M}_\kappa(N)$ (using notations introduced in 1.3.4) such that $w(N') < 0$. To satisfy this constraint, $e \in N'$. But e is the only edge that has a colour different from r or g in N', which means that the sign of $w(e)$ determines the sign of $w(N')$ by definition 1.3.2. Therefore, $w(N')$ can not be negative. This is a contradiction.

- If $w(e) < 0$, then, since e was the only edge that had a colour different from r or g in N,

  $$w(N) = \prod_{e' \in N} w(e') = w(e) \cdot \prod_{e' \in N \setminus \{e\}} w(e') < 0$$

  by definition 1.3.2. But $w(\kappa(N)) = \sum_{N_i \in \mathcal{M}_{\kappa(N)}} N_i = 0$ by definitions 1.3.5 and 1.3.6 (using a notation introduced in definition 1.3.3). Therefore, $\exists$ another non-monochromatic perfect matching $N' \in \mathcal{M}_\kappa(N)$ such that $w(N') > 0$. To satisfy this constraint, $e \in N'$. But e is the only edge that has a colour different from r or g in N', which means that the sign of $w(e)$ determines the sign of $w(N')$ by definition 1.3.2. Therefore, $w(N')$ can not be positive. This is a contradiction.

$\square$

These observations may seem obscure at first, but they are necessary to prove the following lemma.

Lemma 3.2.3 (One negative edge allowed)**.** Let G_η^w be a perfectly monochromatic graph that respects the following properties.

- G is a simple graph, referring to definition 1.1.2.
- η is a pure edge colouring, referring to definition 1.1.4.
- $\forall e \in E(G_\eta^w)$, $w(e) \in \mathbb{R}$. Also, $E(G_\eta^w)$ has at most one edge that has a negative weight.

Then, if G_η^w is not isomorphic to K_4, $\tilde{c}(G,\eta,w) \leq 2$.

The sketch of the proof of Lemma 3.2.3 goes as follows. Using observations 3.2.1 and 3.2.2, we find a Hamiltonian cycle formed of 2 distinct monochromatic perfect matchings. Then, we build non-monochromatic perfect matchings in it using edges from a third monochromatic perfect matching. At last, we show that they create a disbalance in the weight of their feasible vertex colouring that can not be counterbalanced with another perfect matching. This is done by analysing every possible location of the only negatively weighted edge allowed. Let's dive into it.

Proof of Lemma 3.2.3. Let's assume by contradiction that the weighted matching index of G_η^w is $\tilde{c}(G,\eta,w) \geq 3$.

1. If G_η^w has only positive weights, then we're done — this case is already solved by Bogdanov in [Bog] and was presented in the Lemma 2.1.1 of this master thesis.

2. If G_η^w has exactly one negative weight: let M_r, M_g and M_b be three distinct monochromatic perfect matchings of G_η^w that have colours r, g and b respectively. They exist by definition 1.3.7 of the weighted matching index. Let e^- be the only negatively weighted edge of G_η^w. Without loss of generality, I will say that the colour of e^- is b. From Claim 3.2.1, M_r and M_g form a Hamiltonian cycle $H = (v_1, v_2, \ldots, v_n)$ of even length. Let $e = \{v_i, v_j\} \in M_b$ be a minimal cutting edge of H, which means it respects the following property.

$$\left|H_{i,j}\right| = \min_{\{v_k, v_l\} \in M_3} \left|H_{k,l}\right|$$

We know from Claim 3.2.2 that $j - i$ is even. Without loss of generality, I can assume that the color of (v_i, v_{i+1}) is r (otherwise we exchange colours r and g in the following reasoning). Let $e' = (v_{i+1}, v_k) \in M_b$ (we are certain of the existence of e' because v_{i+1} must be covered by M_b). We observe that v_k must appear in a vertex of $H_{j+1,i-1}$, because otherwise we would have $|H_{i,j}| > |H_{j+1,k}|$, which contradicts the minimality of e.

Also, $j - i$ and $k - (i+1)$ are even numbers, by the Claim 3.2.2. We can now build a non-monochromatic perfect matching N as follows.

$$\begin{aligned} N &= \{e\} \\ &\quad \cup \{e'\} \\ &\quad \cup (H_{j+1,k-1} \cap M_g) \\ &\quad \cup (H_{i+2,j-1} \cap M_r) \\ &\quad \cup (H_{k+1,i-1} \cap M_r) \end{aligned}$$

The construction of N is illustrated in figure 3.4.

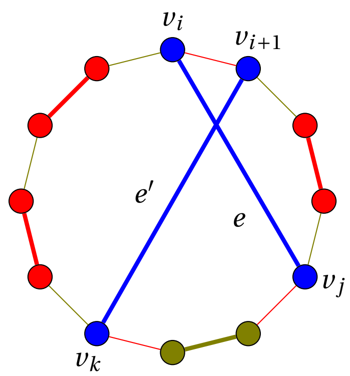

Figure 3.4: Illustration of the construction of N using the fact that e is a minimal cutting edge. This construction works because of the parity argument proved in Claim 3.2.2. The induced vertex colouring $\kappa(N)$ is also visible.

The weight of N is computed as follows.

$$\begin{aligned} w(N) &= \prod_{e_i \in N} w(e_i) \\ &= w(e) \cdot w(e') \cdot \prod_{e_i \in N \setminus \{e, e'\}} w(e_i) \end{aligned}$$

Since e^- is a b-coloured edge, three situations can occur.

2.1 If $e = e^-$, then $w(N) < 0$. But $w(\kappa(N)) = \sum_{N_i \in \mathcal{M}_{\kappa(N)}} N_i = 0$ by definitions 1.3.7 and 1.3.6 (using a notation introduced in definition 1.3.3). Therefore, $\exists N' \in \mathcal{M}_\kappa(N)$ (defined in 1.3.4) such that $w(N') > 0$. To satisfy this constraint, $e^- = e \notin N'$.

The only way to match v_i with a b-coloured edge different from e^- is that $\exists$ a b-coloured edge $e'' = (v_i, v_k) \in N'$. Indeed, e'' can't be between v_i and v_{i+1} because there's already an r-coloured edge there, and G is a simple graph. But this is impossible, because $k - i$ is an odd number, which is forbidden by Claim 3.2.2. This is a contradiction.

2.2 If $e' = e^-$, then $w(N) < 0$. But $w(\kappa(N)) = \sum_{N_i \in \mathcal{M}_{\kappa(N)}} N_i = 0$ by definitions 1.3.7 and 1.3.6 (using a notation introduced in definition 1.3.3). Therefore, $\exists N' \in \mathcal{M}_\kappa(N)$ (defined in 1.3.4) such that $w(N') > 0$. To satisfy this constraint, $e^- = e' \notin N'$.

The only way to match v_{i+1} with a b-coloured edge different from e^- is that $\exists$ a b-coloured edge $e'' = (v_{i+1}, v_j) \in N'$. Indeed, e'' can't be between v_i and v_{i+1} because there's already an r-coloured edge there, and G is a simple graph. But this is impossible, because $j - (i+1)$ is an odd number, which is forbidden by Claim 3.2.2. This is a contradiction.

2.3 If $e^- \notin \{e, e'\}$, then $w(N) > 0$. But $w(\kappa(N)) = \sum_{N_i \in \mathcal{M}_{\kappa(N)}} N_i = 0$ by definitions 1.3.7 and 1.3.6 (using a notation introduced in definition 1.3.3). Therefore, $\exists N' \in \mathcal{M}_\kappa(N)$ (using notations introduced in 1.3.4) such that $w(N') < 0$. To satisfy this constraint, $e^- \in N'$.

Because e^- is b-coloured, it connects 2 vertices $\in \{v_i, v_{i+1}, v_j, v_k\}$

- $e^- \neq \{v_i, v_{i+1}\}$ because there's already an r-coloured edge there, and G is a simple graph.
- Also, $e^- \neq \{v_j, v_k\}$, since it would imply again the existence of a b-coloured edge between v_i and v_{i+1}.
- At last, $e^- \neq \{\{v_i, v_j\}, \{v_{i+1}, v_k\}\}$, because it is different from e and e'.

The last possibilities are that $e^- = \{v_i, v_k\}$ or $e^- = \{v_{i+1}, v_j\}$. But this is impossible, because $k - i$ and $j - (i+1)$ are odd numbers, which is forbidden by Claim 3.2.2. This is a contradiction, and it ends our proof.

□

3.2.2 Allowing all the classes to have arbitrary weights except 2

The main argument of the proof of the previous analysed case was that there were two colours containing only positively weighted edges. This suggests that the structure of the proof might work as well if more than one single negatively weighted edge was present in the combination of the other colour classes. Such a result would be even more powerful since it would prove the conjecture to be true whenever 2 colours have only positive weighted edges, no matter the weights of the other edges. In this section, we will reuse the arguments from section 3.2.1 to verify them in the situation where multiple negative edge-weights are allowed.

Lemma 3.2.4 (2 positive colours)**.** Let G^w_η be a perfectly monochromatic graph that respects the following properties.

- G is a simple graph, referring to definition 1.1.2.
- η is a pure edge colouring, referring to definition 1.1.4.
- $\exists r, g \in \left(\eta(E(G))\right)^2, r \neq g$ such that all the edges coloured r or g in G^w_η have a real, positive weight.

Then, if G^w_η is not isomorphic to K_4, $\tilde{c}(G,\eta,w) \leq 2$.

Proof of Lemma 3.2.4. By contradiction, let's assume that $\tilde{c}(G,\eta,w) \geq 3$. Let M_r, M_g and M_b be 3 distinct monochromatic perfect matchings of G^w_η that have colours r, g and b respectively, and let's assume that the colours r and g have only positive weighted edges. From Claim 3.2.1, M_r and M_g form a Hamiltonian cycle $H = (v_1, v_2, \ldots, v_n)$ of even length. Let $e = \{v_i, v_j\} \in M_b$ be a minimal cutting edge of H, which means it respects the following property.

$$\left|H_{i,j}\right| = \min_{\{v_k, v_l\} \in M_3} \left|H_{k,l}\right|$$

Let $e' = (v_{i+1}, v_k) \in M_b$ (we are certain of the existence of e' because v_{i+1} must be covered by M_b). We know from Claim 3.2.2 that $j - i$ and $k - (i+1)$ are even numbers. We observe that v_k must appear in a vertex of $H_{j+1,i-1}$, because otherwise we would have $\left|H_{i,j}\right| > \left|H_{j+1,k}\right|$, which contradicts the minimality of e.

Without loss of generality, let's assume that the color of $\{v_i, v_{i+1}\}$ is r (otherwise we exchange colours r and g in the following arguments). It is then possible to find a non-monochromatic perfect matching as follows.

$$\begin{aligned} N &= \{e\} \\ &\cup \{e'\} \\ &\cup M_r \cap H_{i+2,j-1} \\ &\cup M_r \cap H_{k+1,i-1} \\ &\cup M_g \cap H_{j+1,k-1} \end{aligned}$$

The construction of N is visualized in figure 3.5.

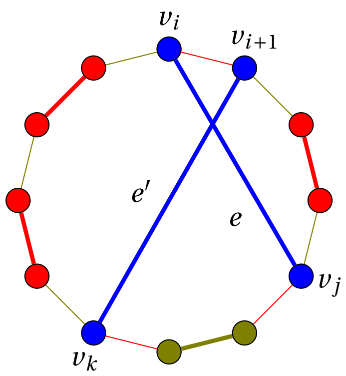

Figure 3.5: Construction of N from M_r, M_g, e and e'. N is represented by the thick edges. $\kappa(N)$ is also represented.

By definition 1.3.2, the weight of N computed as follows.

$$\begin{aligned} w(N) &= \prod_{e_i \in N} w(e_i) \\ &= w(e) \cdot w(e') \cdot \prod_{e_i \in N \setminus \{e,e'\}} w(e_i) \end{aligned}$$

We notice that $N \setminus \{e, e'\}$ has only r and g-coloured edges, which have positive weights. Therefore, the sign of $w(N)$ is determined by the signs of $w(e)$ and $w(e')$.

1. if $w(e)$ and $w(e')$ have the same sign: then, $w(N) > 0$.

   But $w(\kappa(N)) = \sum_{N_i \in \mathcal{M}_{\kappa(N)}} N_i = 0$ by definitions 1.3.5 and 1.3.6. This means that $\exists N' \in \mathcal{M}_{\kappa(N)}$ such that $w(N') < 0$.

   The only b-coloured vertices in $\kappa(N) = \kappa(N')$ are v_i, v_{i+1}, v_j and v_k. For that reason, we can not have $\{e, e'\} \in N'$ (otherwise the signs of $w(N)$ and $w(N')$ would be the same). This last condition is satisfied only if $\exists e'' = (v_i, v_k)$ and $e''' = (v_{i+1}, v_j)$ of colour b. But this is forbidden by Claim 3.2.2 because $(k - i)$ and $\big(j - (i+1)\big)$ are odd numbers.

2. if $w(e)$ and $w(e')$ have different signs: then, $w(N) < 0$.

   But $w(\kappa(N)) = \sum_{N_i \in \mathcal{M}_{\kappa(N)}} N_i = 0$ by definitions 1.3.5 and 1.3.6. This means that $\exists N' \in \mathcal{M}_{\kappa(N)}$ such that $w(N') > 0$.

   The only b-coloured vertices in $\kappa(N) = \kappa(N')$ are v_i, v_{i+1}, v_j and v_k. For that reason, we can not have $\{e, e'\} \in N'$ (otherwise the signs of $w(N)$ and $w(N')$ would be the same). This last condition is satisfied only if $\exists e'' = (v_i, v_k)$ and $e''' = (v_{i+1}, v_j)$ of colour b. But this is forbidden by Claim 3.2.2 because $(k - i)$ and $\big(j - (i+1)\big)$ are odd numbers.

This ends the proof of Lemma 3.2.4.

□

3.3 Other explored cases

Before finding these interesting results, I explored other subcases of the Krenn's conjecture that were less successful. Nevertheless, I could make some interesting observations in one of them: the case of bipartite graphs. This lead me to assemble my observations into a lemma, presented in the next subsection.

3.3.1 Focus on bipartite graphs

Many problems in graph theory about matchings are easier to solve when restricted to bipartite graphs. For this reason, I spent some time trying to find a proof of the Krenn's conjecture that would be valid only for bipartite graphs. This was unsuccessful, but I will present the main observations I made during this exploration by proving the following lemma.

Lemma 3.3.1 (Bipartite graphs)**.** Let G_η^w be a perfectly monochromatic graph of size n that respects the following properties.

- G is a simple graph, referring to definition 1.1.2.
- η is a pure edge colouring, referring to definition 1.1.4.
- G is bipartite.
- $\forall e \in E(G_\eta^w),\ w(e) \in \{-1, 1\}$
- $\tilde{c}(G, \eta, w) \geq 3$
- $\forall$ pair of monochromatic perfect matching M_r and M_g of G_η^w of different colours, M_r and M_g form a Hamiltonian cycle.

Then, G_η^w has at least $n + 7$ distinct perfect matchings.

Proof of Lemma 3.3.1. Let M_r, M_g and M_b be three monochromatic perfect matchings of G_η^w that have the different colours r, g and b respectively. Their existence is guaranteed by definition 1.3.7 of the weighted matching index. Let $H = (v_1, v_2, \ldots, v_n)$ be the Hamiltonian cycle formed by M_r and M_g. Let $e = (v_i, v_j) \in M_b$. Since G_η^w is bipartite,

$j-i$ is odd. We can assume without loss of generality that the colour of (v_i, v_{i+1}) is r (otherwise, we exchange colours r and g in the following reasoning). Then, we can build a non-monochromatic perfect matching as follows.

$$\begin{aligned} N &= \{e\} \\ &\cup (H_{i+1,j-1} \cap M_g) \\ &\cup (H_{j+1,i-1} \cap M_r) \end{aligned}$$

Since N is non-monochromatic, the weight of its induced vertex colouring $w(\kappa(N)) = \sum_{N_i \in \mathcal{M}_{\kappa(N)}} N_i = 0$ by definition 1.3.6. Also, $w(N) \in \{-1, 1\}$ because all its edges have a weight in $\{-1, 1\}$. Therefore, $\exists N'$ such that $\kappa(N') = \kappa(N)$ and $w(N') = -w(N)$.

This reasoning can be applied to all the edges of M_b. Let

- N_e, N'_e be the 2 distinct non-monochromatic perfect matchings that can be built from $e \in M_b$.
- $N_{e'}, N'_{e'}$ be the 2 distinct non-monochromatic perfect matchings that can be built from $e' \in M_b$ ($e' \neq e$).

We observe that $\kappa(N_e) = \kappa(N'_e) \neq \kappa(N_{e'}) = N'_{e'}$. Indeed, the only b-edge in N_e and N'_e is e, and the only b-edge in $N_{e'}$ and $N'_{e'}$ is e'. This means that N_e, N'_e, $N_{e'}$ and $N'_{e'}$ are all distinct.

This reasoning can be applied to all pair of edges of M_b. We conclude that G^w_η has at least $2 \cdot \frac{n}{2} = n$ distinct non-monochromatic perfect matchings.

Actually, the reasoning can still go a bit further. Having $e \in M_b$, we notice that both N_e and N'_e contain e. This means that the only way for N'_e to differ from N_e is that they differ in the edges that have colour r or g. Let's say without loss of generality that they differ (at least) in their red parts. Let $N'_e[r]$ be the set of red edges of N'_e. Then, we can find a new monochromatic red perfect matching as follows.

$$\begin{aligned} M'_r &= N'_e[r] \\ &\cup (H_{j+1,i-1} \cap M_r) \end{aligned}$$

We will denote by κ_r the red monochromatic vertex colouring. Let's compute the current weight of κ_r.

$$w(\kappa_r) = w(M_r) + w(M'_r) \in \{-2, 0, 2\}$$

Currently, this weight can not be 1. This means that $\exists M_r''$, different from M_r' and M_r, such that $\kappa(M_r'') = \kappa_r$.

This leaves us with (at least) 3 monochromatic perfect matchings that induce the vertex colouring κ_r. The same reasoning is possible by forming an initial hamiltonian cycle with M_g and M_b. Therefore, κ_g or κ_b is also induced by (at least) 3 monochromatic perfect matchings. The other is induced by (at least) 1 perfect matching because $\tilde{c}(G,\eta,w) \geq 3$. By summing everything up, we find that G_η^w has at least $n+7$ distinct perfect matchings.

□

4 Computational approach

In order to find a counter-example to the Krenn's conjecture in the best case scenario, or just to find interesting new properties in perfectly monochromatic graphs, having a more experimental approach has many interests. In this section, I am introducing a new tool I developed, called EGPI, that computes the weighted matching index of experiment graphs. Then, some of its potential uses are presented.

4.1 Motivation

Finding interesting examples of experimental graphs to study can be challenging due to the high degree of freedom experiment graphs can have, by definition 1.3.1. Also, the weighted matching index of experiment graphs as defined in definition 1.3.7 is hard to find by hand since it requires finding all perfect matchings of a graph. To my knowledge, no public access tool exists at the time of writing to easily encode experimental graphs and compute their weighted matching index. My hope is that such a tool could help me and other researchers to quickly verify some properties on instances of graphs they want to test, without losing more time on it. For this reason, I developed a new program that does exactly that. The program is called **EGPI**, which stands for **Experiment Graphs Properties Identifier**. I believe this name resumes the whole meaning of its utilization. The program is available on my Github repository [Han24].

4.2 Limitations of the experimental approach

Assuming that we successfully find a way to compute in a quick time the weighted matching index of an experiment graph, we still have to face some limitations. It may be tempting to use such a tool to prove the conjecture in some very restrained cases by generating and testing every possible experiment graph that respects some properties. While this idea is theoretically possible, the user must be aware that he is extremely limited by the computational time of such an algorithm. Indeed, the number of possible graphs grows exponentially with their degrees of freedom.

Let's consider a quick and simple example by trying to prove experimentally the following conjecture.

Conjecture 4.2.1. Let G_μ^w be a non-redundant experiment graph with $n = 6$ vertices, and let say that, for all edge $e \in E(G)$,

$$\begin{cases} Re(w(e)) & \in \{-1, 0, 1\} \\ Im(w(e)) & \in \{-1, 0, 1\} \\ w(e) & \neq 0 \end{cases}$$

Then, the weighted matching index $\tilde{c}(G, \mu, w) \leq 2$.

One way to prove the conjecture 4.2.1 is to imagine a brute force algorithm A that generates every possible graph respecting the pre-conditions of the conjecture 4.2.1, with $\mu(e) \in \{r, g, b\}^2$, and check their weighted matching index. If none of their weighted matching index is 3, then the conjecture 4.2.1 is true. Such an algorithm is much simpler to implement if the generated graphs are *labelled*.

Definition 4.2.2 (Labelled graph)**.** A graph G is said to be *labelled* if each of its vertices has a unique label.

In other words, two different isomorphic labelled graphs are considered different in algorithm A, while they would be considered the same if they were not labelled. Working with labelled graphs is simpler since isomorphism is not something to take into account. However, the number of possible labelled graphs is bigger than the number of possible unlabelled graphs respecting these pre-conditions. Therefore, A will check multiple times the same unlabelled graph, which is not efficient.

The question is then if it is still interesting to implement A. Unfortunately, it is not, even if the graphs it is about seem tiny, because of the following observation.

Observation 4.2.3. Let G_μ^w be an experiment graph that has n vertices, and let say that, for all edge $e = \{u, v\} \in E(G)$, the edge can have a value different from 0 chosen among W different values. Furthermore, $\mu(e, u)$ and $\mu(e, v)$ are chosen among K different colours. At last, $\forall e' = (u', v')$, we have $\mu(e', u') \neq \mu(e, u)$ or $\mu(e', v') \neq \mu(e, v)$. Then, the number of different possible labelled graphs that G_μ^w can be, denoted N_{graphs}, is

$$N_{graphs} = (1 + W)^{\left(\frac{K^2 \cdot n \cdot (n-1)}{2}\right)}$$

Proof of observation 4.2.3. Indeed, each edge e is characterized by the following properties.

- It has a position $\{u, v\}$, that can take $\frac{n\cdot(n-1)}{2}$ different values.
- It has 2 colours, one at each of its endpoint, that can take K different values. Then, in total, there are K^2 different ways to bicolour it.
- It has a complex weight that can have W different value.

A first observation is that the maximum number of edges between 2 different vertices of $V(G)$ is K^2. Indeed, more edges would imply that 2 edges at least would have the same positions and colours, which is forbidden.

Then, we observe that there are W^x possibilities to assign weights to an edge set of size x. Using these 2 last observations, it is possible to compute the number of different weighted bicoloured edges sets between 2 defined vertices u and v. Let's denote this number N_{edges}.

$$\begin{aligned} N_{edges} &= \binom{K^2}{0}W^0 + \binom{K^2}{1}W^1 + \binom{K^2}{2}W^2 + \cdots + \binom{K^2}{K^2}W^{K^2} \\ &= \sum_{x=0}^{K^2} \binom{K^2}{x} W^x \end{aligned}$$

It is possible to simplify this expression by using the binomial theorem, a well known mathematical formula [Coo49].

$$\begin{aligned} N_{edges} &= \sum_{x=0}^{K^2} \binom{K^2}{x} W^x \\ &= \sum_{x=0}^{K^2} \binom{K^2}{x} 1^{(K^2-x)} W^x \\ &= (1+W)^{K^2} \end{aligned}$$

Having this number, it is easy to find the number N_{graphs} of possible labelled experiment graphs. Indeed, we just have to choose one of the possible combinations of weighted bicoloured edges between 2 defined vertices for each possible pair of vertices. The number of possible pairs of vertices is $\frac{n\cdot(n-1)}{2}$. Therefore:

$$\begin{aligned} N_{graphs} &= N_{edges}^{\frac{n\cdot(n-1)}{2}} \\ &= \left((1+W)^{K^2}\right)^{\frac{n\cdot(n-1)}{2}} \\ &= (1+W)^{\left(K^2\cdot\frac{n\cdot(n-1)}{2}\right)} \end{aligned}$$

□

Returning to conjecture 4.2.1, let's compute as an example the number of graphs needed to verify to prove it by using algorithm A. The graphs described in conjecture 4.2.1

have $n = 6$ vertices, $K = 3$ different possible colours per edge's endpoint, and $W = 8$ possible weights different from 0. Therefore, in this case,

$$\begin{aligned} N_{graphs} &= (1+W)^{\left(\frac{K^2 \cdot n \cdot (n-1)}{2}\right)} \\ &= (1+8)^{\left(\frac{3^2 \cdot 6 \cdot (6-1)}{2}\right)} \\ &= 9^{270} \end{aligned}$$

This number is absurdly big, and it is totally impossible to imagine being able to generate all of those graphs on a classical computer and compute their matching index, no matter of how efficient this operation is. Therefore, conjecture 4.2.1 cannot be solved by using the algorithm A, even if the graphs it is about are tiny and restricted compared to the space of all experiment graphs possible.

This observation leads us to consider that EGPI, in its implementation, should not try to generate all possible graphs. Instead, it should focus on generating as wisely as possible random graphs that have a high probability to have a big weighted matching index. This is the approach I took in the development of EGPI, and it is detailed in the next subsections.

4.3 Functionalities of EGPI

At the current state of its development, EGPI implements functions that allow the user to do the following:

1. **Discover all the perfect matchings of an encoded experiment graph**. The perfect matchings are encoded as edge sets. Their encoding includes access to their weights and to the feasible vertex colouring they induce.

2. **Discover all the feasible vertex colourings of an encoded experiment graph**. The set of all feasible vertex colourings of a graph is encoded as a Python dictionary, where the vertex colourings (Python tuples) are the keys and their weights (complex numbers) are the values.

3. **Find out if an encoded experiment graph is perfectly monochromatic or not.**

4. **Compute the weighted matching index of an encoded experiment graph.**

5. **Find out if an encoded experiment graph is bipartite.**

6. **Save an encoded experiment graph and its properties in a JSON file.** The JSON file contains all the properties of the graph, including its perfect matchings, its

feasible vertex colourings, its weighted matching index, and the fact it is bipartite or not.

7. **Draw an encoded experiment graph in a pdf file.** This is done through the creation of a tex file containing the LaTeX representation of the graph. This LaTeX file is still available after the program ends, so that its code can be re-used if wanted so.

8. **Randomly create candidate experiment graph**, a process that can be defined as follows.

   Definition 4.3.1 (Random creation of a candidate experiment graph)**.** Given an even number of vertices $n \in \mathbb{N}$, a list of colours $L_{colours}$, a list of complex numbers $L_{weights}$, and a complexity bound $b \in \mathbb{N}$, the *random creation of a candidate experiment graph* respecting n, $L_{colours}$, $L_{weights}$ and b is the process described in algorithm 1.

Algorithm 1 Random creation of a candidate experiment graph

Require: n is a positive even integer
Require: $L_{colours}$ is a list of colours
Require: $L_{weights}$ is a list of complex numbers different from 0
Require: $b \in \mathbb{N}$
  $G_\mu^w \leftarrow$ an empty graph with n vertices and no edge
  for $r \in L_{colours}$ **do**
    $S \leftarrow \frac{n}{2}$ independent edges at random positions of G_μ^w
    Colour each edge $\in S$ with colour r
    Assign random weights $\in L_{weights}$ to each edge $\in S$
    Add S to $E(G_\mu^w)$ $\qquad\triangleright$ These edges form a monochromatic perfect matching
  end for
  for $i \in \{1, 2, \ldots, b\}$ **do**
    $S \leftarrow \frac{n}{2}$ independent edges at random positions of G_μ^w
    for each pair of vertices $\{u, v\} \in S$ **do**
      if there is no edge between u and v in G_μ^w **then**
        Add an edge e between u and v
        Colour e with a random bicolour $\in L_{colours}^2$
        Assign a random weight $\in L_{weights}$ to e
      else
        $add\_edge \leftarrow True$ with probability $\frac{1}{(\text{\#edges between u and v})+1}$
        if add_edge **then**
          Add an edge e between u and v
          Colour e with a random bicolour $\in L_{colours}^2$ different from the bicolour of the other edges between u and v
          Assign a random weight $\in L_{weights}$ to e
        end if
      end if
    end for
  end for

The denomination *candidate* experiment graph comes from the fact that graphs built that way have higher chances to have big weighted matching index than completely random graphs. Also, this construction ensures that each of the edges included in a candidate graph is part of a perfect matching. Furthermore, they do not have multiple edges of the same bicolour between 2 vertices. Assuming $0 \notin L_{weights}$, this makes them non-redundant, according to the definition 2.0.4. All of these observations make them indeed good *candidates* to study.

9. **Perform a random experiment graphs research process**, defined as follows.

   Definition 4.3.2 (Random Experiment Graphs Research Process)**.** Given a number of vertices $n \in \mathbb{N}$, a list of colours $L_{colours}$, a complexity bound $b \in \mathbb{N}$, a list of complex numbers $L_{weights}$, and a number of trials $m \in \mathbb{N}$, a *Random Experiment Graphs Research Process* is the process of randomly creating m candidate experiment graphs respecting n, $L_{colours}$, $L_{weights}$ and b, and analyse them. More specifically, each of these graphs G_μ^w gets verified to check if it is perfectly monochromatic and if it has at least one non-monochromatic feasible vertex colouring. If it is, G_μ^w is saved as a JSON file containing all its properties (including its perfect matchings, its feasible vertex colourings, its weighted matching index, and the fact it is bipartite or not). Then, G_μ^w is drawn in another pdf file.

   In short, this experiment allows the user to generate a big number of experiment graphs and analyse properties on them. The condition that the graphs must have at least one non-monochromatic feasible vertex colouring allows the user to focus on non-trivial graphs. Indeed, if it is not satisfied, then the graph is monochromatic, according to definition 1.2.1, and this case of the Krenn's conjecture is already proven thanks to Bogdanov [Bog].

4.4 Details of implementation

This section explains all the technical aspects of the implementation of EGPI. Readers only interested in its use can pass this section if they want to.

EGPI was implemented in Python, this choice was motivated by two reasons. Firstly, Python is a high-level programming language that offers many functionalities and data types by default [YH21]. This made the implementation a lot easier and compacter than if it had to be done in other programming languages. Secondly, Python is easy to read and to learn for people who are not familiar with programming. This may allow future researchers interested in re-using EGPI to perform modifications to it without too many difficulties if they want to.

The main purpose of EGPI is to encode and apply operations on experiment graphs, rigorously defined in definition 1.3.1. It is therefore important to build a good data structure of experiment graphs. The architecture of the program to do so is detailed in a state diagram available in figure 4.1.

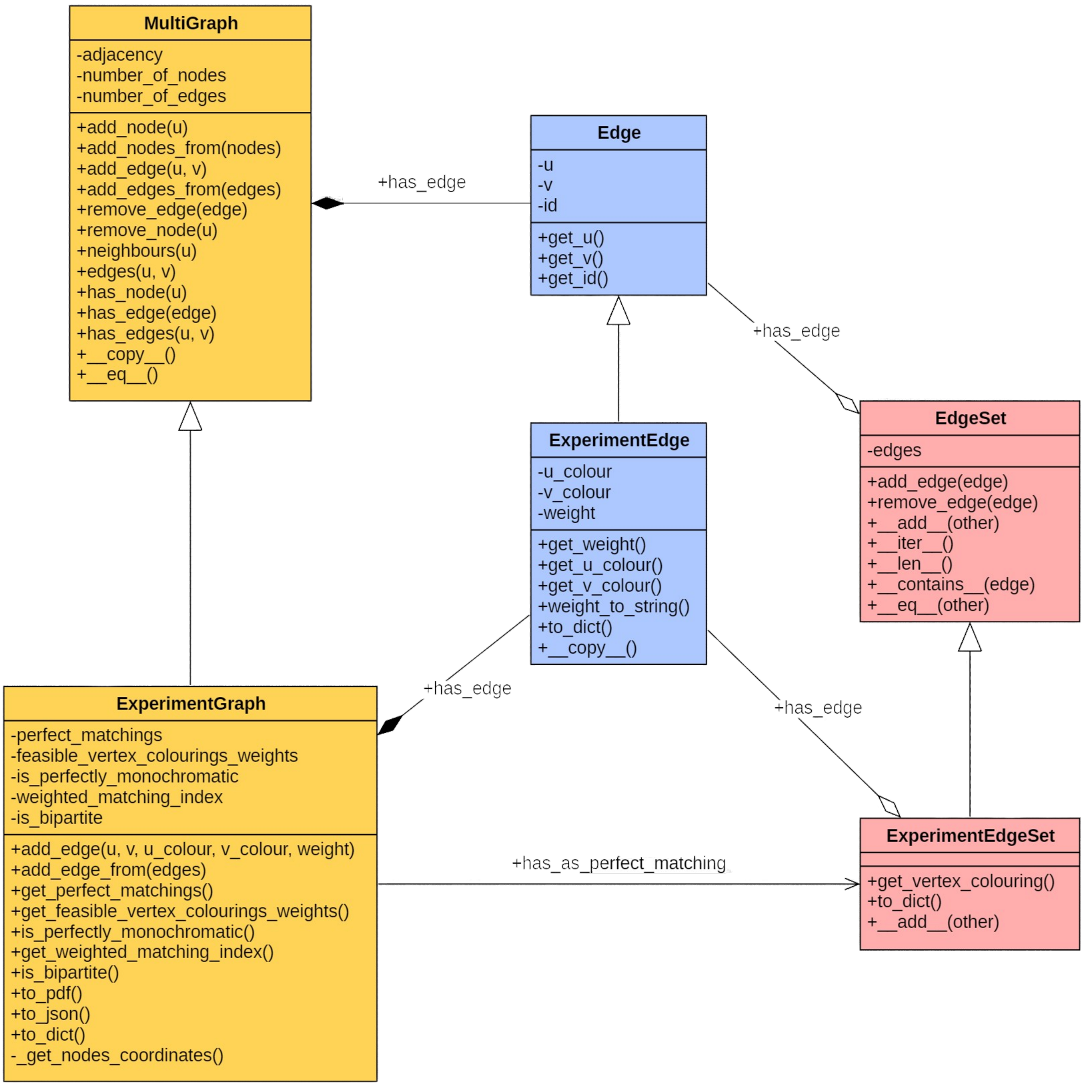

Figure 4.1: Structure diagram of EGPI showing the important relations between the different implemented data structures. Most of the implemented methods of the program are shown in the diagram.

Here is an exhaustive list of all non-trivial experiment graphs' properties we are interested to compute, and the algorithms we use in EGPI to do so.

1. **Their perfect matchings:** to find all the perfect matchings of a graph, EGPI uses an algorithm written in pseudocode in algorithm 2.

Algorithm 2 Find all perfect matchings of an experiment graph G

Require: G is an experiment graph
  if G has no vertex **then**
    The only perfect matching is $\varnothing$
  else if G has vertices **then**
    $PMs \leftarrow$ empty list
    choose an arbitrary vertex $u \in V(G)$
    for all $v \in$ neighbours of u **do**
      $subPMs \leftarrow$ all perfect matchings of $G - \{u, v\}$
      for all $subPM \in subPMs$ **do**
        for all $e \in$ edges between u and v **do**
          add $subPM \cup e$ to PMs
        end for
      end for
    end for
  end if
  return PMs

Complexity of algorithm 2: The algorithm is a backtracking algorithm. In the worst case scenario, G_μ^w is a complete graph with n vertices. In other words, each vertex has $n-1$ neighbours in G_μ^w. Also, at each recursive step of the algorithm, we remove 2 vertices from the graph. This means that the depth of the search tree is $\frac{n}{2}$. The running time of this search algorithm is then $O\left(n^{\frac{n}{2}}\right)$.

2. **The weights of their feasible vertex colourings:** according to their definition 1.3.4, finding the feasible vertex colourings of an experiment graph requires to find their perfect matchings. The next steps to find them and their weights are simpler and are described in algorithm 3.

Algorithm 3 Find all feasible vertex colourings of an experiment graph G_μ^w

---

Require: G_μ^w is an experiment graph
  $PMs \leftarrow$ all perfect matchings of G_μ^w
  $FVCs \leftarrow$ empty Python dictionary
  for all $PM \in PMs$ **do**
    $FVC \leftarrow$ feasible vertex colouring induced by PM
    $w \leftarrow$ weight of PM
    $FVCs[FVC] \leftarrow w$
  end for
  return $FVCs$

The hardest step of this algorithm is to find all the perfect matchings of G_μ^w. Therefore, the complexity of this algorithm is the same as the one of algorithm 2, which is $O\left(n^{\frac{n}{2}}\right)$ in the worst case.

3. **If the graph is perfectly monochromatic:** by definition 1.3.6, finding if an experiment graph is perfectly monochromatic or not just requires looking at all its feasible vertex colourings. This is done in algorithm 4.

Algorithm 4 Check if an experiment graph G^w_μ is perfectly monochromatic

---

Require: G^w_μ is an experiment graph
  $FVCs \leftarrow$ all feasible vertex colourings of G
  $isPM \leftarrow$ True
  for all $FVC \in FVCs$ **do**
    if FVC is monochromatic and $w(FVC) \neq 1$ **then**
      $isPM \leftarrow$ False
    else if FVC is not monochromatic and $w(FVC) \neq 0$ **then**
      $isPM \leftarrow$ False
    end if
  end for
  return $isPM$

The only hard step of this algorithm is to find all the feasible vertex colourings of G^w_μ. Therefore, the complexity of this algorithm is the same as the one of algorithm 3, which is $O\left(n^{\frac{n}{2}}\right)$ in the worst case.

4. **The weighted matching index of the graph:** defined in definition 1.3.7, the weighted matching index of a perfectly monochromatic experiment graph is the number of monochromatic feasible vertex colourings in this graph. If the graph is not perfectly monochromatic, the weighted matching index is 0 by definition. EGPI uses the algorithm 5 to compute the weighted matching index of a graph.

Algorithm 5 Compute the weighted matching index of an experiment graph G^w_μ

Require: G^w_μ is an experiment graph
  $FVCs \leftarrow$ all feasible vertex colourings of G^w_μ
  $isPM \leftarrow$ is G^w_μ perfectly monochromatic ?
  if $isPM$ **then**
    $c \leftarrow 0$
    for all $FVC \in FVCs$ **do**
      if $w(FVC) = 1$ **then**
        $c \leftarrow c + 1$
      end if
    end for
    return c
  else
    return 0
  end if

Again, the complexity of this algorithm is determined by the one of algorithm 3, which is $O\left(n^{\frac{n}{2}}\right)$ in the worst case.

5. **If the graph is bipartite:** finding if a graph is bipartite or not is a well-known problem in graph theory. EGPI uses the following greedy algorithm to find if a graph is bipartite or not. The algorithm is written in algorithm 6.

Algorithm 6 Check if an experiment graph G_μ^w is bipartite

Require: G_μ^w is an experiment graph
  $Q \leftarrow$ empty queue
  add $v_0 \in V(G_\mu^w)$ to Q
  colour of $v_0 \leftarrow$ red
  $isBipartite \leftarrow$ True
  while Q is not empty **do**
    $u \leftarrow$ pop Q
    for all $v \in$ neighbours of u **do**
      if colour of v is not defined **then**
        colour of $v \leftarrow$ opposite colour of u
        add v to Q
      else if colour of v is the same as colour of u **then**
        $isBipartite \leftarrow$ False
      end if
    end for
  end while
  return $isBipartite$

This algorithm visits all the vertices of G_μ^w only once. Therefore, its complexity is $\mathcal{O}(n)$, where n is the number of vertices of G_μ^w.

4.5 Realized experiments with EGPI

In this section, I present some of the experiments I realized with EGPI. The goal of these experiments was to find interesting properties of experiment graphs, and to try to find a counter-example to the Krenn's conjecture.

4.5.1 Research for counter-examples

The first experiment I realized with EGPI was to try to find a counter-example to the Krenn's conjecture. To do so, I performed different random experiment graphs research processes, defined in definition 4.3.2. Each of these processes used the following parameters.

- The number of trials was $m = 10^6$.
- The possible colours of the edges were $L_{colours} = \{red, green\}$.
- The possible complex weights of the edges were $L_{weights} = \{-1, 1, -i, i\}$.

I performed 21 different random experiment graphs research processes, with $n \in 6, 8, 10$ and $b \in \{1, 2, 3, 4, 5, 6\}$.

On all the 21 millions generated candidate graphs, none of them was perfectly monochromatic. This result is a new argument in favour of the Krenn's conjecture, at least for graphs that have less than 10 vertices. Nevertheless, it does not constitute a proof of it in any case, and I encourage future researchers to continue looking for counterexamples to it alongside their researches (by using EGPI or any other tool).

4.5.2 Research for perfectly monochromatic graphs with a weighted matching index of 2

The second experiment I realized with EGPI was to try to find perfectly monochromatic graphs that have a weighted matching index of 2. These graphs are authorized by the Krenn's conjecture, and their properties are interesting to study. Indeed, understanding better the structure of these graphs might help researchers to find new ideas in their seek for a proof to the conjecture.

But this is not the main and only interest of the experiment. Exploring this class of experiment graphs is a great way to test the different functionalities of EGPI, and to experimentally check the impact of the different parameters of the program. Using the following parameters,

- The number of trials was $m = 10^6$.
- The possible colours of the edges were $L_{colours} = \{red, green\}$.
- The possible complex weights of the edges were $L_{weights} = \{-1, 1, -i, i\}$.

I performed 15 different random experiment graphs research processes, with $n \in 6, 8, 10$ and $b \in \{1, 2, 3, 4, 5\}$. I got results summarized in figure 4.2.

Results of the experiment						
n	$b = 1$	$b = 2$	$b = 3$	$b = 4$	$b = 5$	total
6	103	18	4	0	0	125
8	39	1	0	0	0	40
10	3	0	0	0	0	3

Figure 4.2: Results of 3 different random experiment graphs research process with $n = 6$, $n = 8$ and $n = 10$ respectively.

An example of discovered perfectly monochromatic graph by the program is shown in figure 4.3.

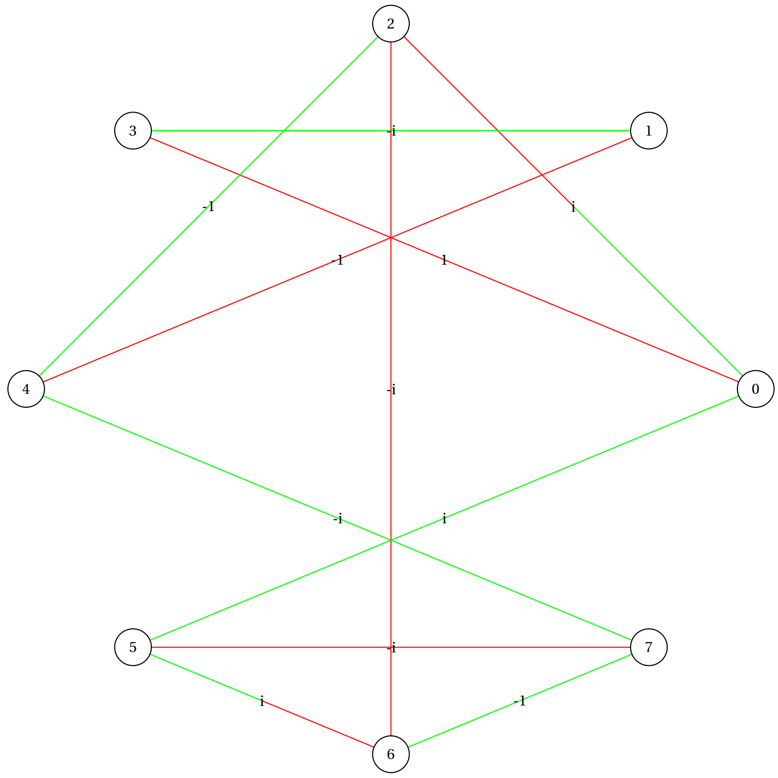

Figure 4.3: A perfectly monochromatic graph G_{μ}^{w} with 8 vertices that has a weighted matching index $\tilde{c}(G, \mu, w) = 2$, discovered by EGPI during the experiment.

From the results in figure 4.2, we can do the following observations. Firstly, it is a lot harder to find perfectly monochromatic graphs in these conditions when their size is bigger. Indeed, the number of possibilities grows exponentially with the number of vertices, and the probability to find a perfectly monochromatic graph decreases. Secondly, the complexity bound has a big impact on the number of found graphs: a higher complexity bound results in a smaller number of found graphs. This was expected, since the graphs generated with a higher complexity bound have higher chances to have more edges and perfect matchings. The probability that all these randomly generated edges and perfect matchings together satisfy the definition 1.3.6 are small.

The last thing I am interested in is to count bipartite graphs among all the perfectly monochromatic graphs found. The result is the following: **none** of the perfectly monochromatic graphs found were bipartite. This is a very interesting new observation: indeed, it is consistent with the arguments we used in the proofs of Lemmas 3.2.3 and 3.2.4. One of the observations these proofs used is that, in these particular cases, the perfectly monochromatic graphs found were not bipartite. This allows me to formulate the following conjecture.

Conjecture 4.5.1. Let G_{μ}^{w} be a non-redundant perfectly monochromatic graph that respects the following properties.

- $\tilde{c}(G, \mu, w) \geq 2$
- $\forall e \in E(G_{\mu}^{w}), w(e) \in \{-1, 1, -i, i\}$
- G_{μ}^{w} has at least one non-monochromatic feasible vertex colouring.

Then, G_{μ}^{w} is not bipartite.

This conjecture might constitute a new interesting subcase of the Krenn's conjecture to investigate in the future.

4.6 Possible improvements for EGPI

EGPI is a tool that can still be improved in many ways.

Firstly, currently it does not offer the functionality of generating only non-isomorphic graphs. This functionality could be interesting if the user wants to count the number of different graphs generated by the tool. This could be coded from scratch, or by using an external program. For instance, in Python, the package *NetworkX* offers a function to check if two graphs are isomorphic or not [HSS08]. A more common program used a lot in the field of graph theory is Nauty, a program originally designed by Brendan McKay [MP14]. Using it would offer a lot of advantages.

A second improvement that could be brought to EGPI is to improve the efficiency of the algorithms used to find the perfect matchings of a graph. Indeed, this step bounds the efficiency of most of the tasks performed by EGPI. Improving this algorithm would allow the program to search more graphs in a given time, and to find more results in general. However, doing so is a hard task, as the problem of finding all perfect matchings of a graph is known to be in the complexity class $\#\mathscr{P}$ citevaliant1979complexity, where $\#\mathscr{P}$ is the class of counting problems associated with $\mathscr{N}\mathscr{P}$-time computations. Indeed, Valiant showed in 1979 that counting all the perfect matchings of a graph is equivalent to computing the permanent of a $(0,1)$-matrix, which he proved to be a complete problem for the class $\#\mathscr{P}$ citevaliant1979complexity.

At last, one more thing that could be improved in EGPI is its ability to check some properties in perfectly monochromatic graphs. For now, the program only checks if a graph is bipartite or not, and it already lead us to the intriguing observation that no discovered graphs were bipartite. It would be interesting to check other properties, such as the existence of Hamiltonian cycles, the chromatic index of the graphs for instance. This might lead to new interesting observations and conjectures to investigate in the future.

Conclusion

In this master thesis, we started by understanding and summarizing the concept of the Krenn's conjecture: a conjecture about perfect matchings and their intersections in edge-weighted edge-bi-coloured multigraphs.[Kre] We dived deeper in the connection between this conjecture and quantum physics, investigated for the first time by Mario Krenn in [KGZ17]. Mario Krenn indeed discovered that the (non-)existence of certain specific graphs could lead to the (non-)feasibility to create specific quantum states, called GHZ-states, through the use of certain experiments. Being able to create such states is a key point in quantum computing [GK20] and cryptography [PHM18]. This discovery was the starting point of his conjecture in graph theory, which remains unsolved.

We then focused on dressing a state of the art of the research on the Krenn's conjecture. In it, we could see that the conjecture was already solved for real, positive weights by Bogdanov in [Bog]. We also spoke about Chandran and Gajjala, who proved the conjecture in the special case of graphs that have a matching index (defined in 1.2.3) different from 1 [CG22]. In that same paper, they could find some upper bounds on the weighted matching index in terms of minimum degree and of edge connectivity. Their researches were extremely valuable for us, as they introduced a lot of notations and concepts that we used in our own work.

After that, we adopted a theoretical approach to find new interesting results related to the problem. We first tried to perform problem reductions to find out what restrained cases were interesting to study. By doing so, we discovered that the study of non-redundant experiment graphs (defined in 2.0.4) with integer weights could be reduced to the study of redundant experiment graphs that have weights included in $\{-1, 1\}$. We then focused on proving some new special cases of the Krenn's conjecture. Our main results are the following: we could extend Bogdanov's results[Bog] by allowing one weight to be negative in the absence of bicoloured edges. This attempt was a success, and encouraged us to extend the used arguments to a more general case. This is how we came by with what I consider to be the main result of this master thesis, formulated in Lemma 3.2.4.

Lemma 3.2.4 Let G_μ^w be a simple perfectly monochromatic graph that has only real weights, and that has at least 2 colour classes that have only positive weighted edges. Then, if G is not isomorphic to K_4, $\tilde{c}(G, \mu, w) \leq 2$.

At last, we adopted a more experimental approach to the problem by looking for counter-examples to the Krenn's conjecture. In practice, this was done through the implementation of a new program called EGPI (Experiment Graphs Properties Identifier) — a program that generates random experiment graphs and checks properties related to the Krenn's conjecture on them. In my opinion, such a tool was currently missing in the researchers' community. Indeed, finding interesting graphs to analyse or showcase by hand is hard, and the EGPI program could help in this task. Also, even though I strongly believe the conjecture to be true at this point, it is not excluded that the EGPI program (or any similar tool) finds a counter-examples to it by chance one day. Using EGPI, I looked for such counter-examples and could not find any, which reinforces my belief in the conjecture's truth. EGPI also allowed me to analyse a big number of random perfectly monochromatic graphs with a weighted matching index of 2, and to discover that none of them were bipartite. Therefore, I finished my work by stating conjecture 4.5.1 about the non-existence of bipartite perfectly-monochromatic graphs in certain conditions.

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List of theorems and definitions

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