Problem 9.7.3.

Find ∇f. Graph some level curves $f = \text{const}$. Indicate ∇f by arrows at some points of these curves.

$$f = \frac{y}{x}$$

Solution.

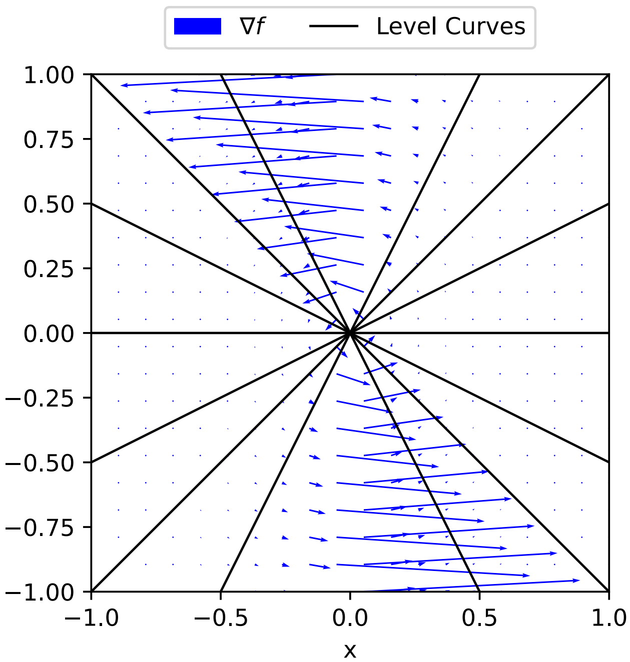
$$\nabla f = [f_x, f_y]$$

$$\boxed{\nabla f = \left[-\frac{y}{x^2}, \frac{1}{x}\right]}$$

Level Surfaces:

$$f(x, y) = \frac{y}{x} = c$$

$$\boxed{y = cx}$$



Problem 9.7.8.

Prove and illustrate by an example

$$\nabla(fg) = f\,\nabla g + g\,\nabla f$$

Proof.

$$\nabla(fg) = \left[\frac{\partial}{\partial x_1}(fg), \ldots, \frac{\partial}{\partial x_n}(fg)\right]$$

The i-th component can be expanded by chain rule,

$$\frac{\partial}{\partial x_i}(fg) = f\frac{\partial g}{\partial x_i} + g\frac{\partial f}{\partial x_i}$$

Therefore in general,

$$\begin{aligned}\nabla(fg) &= \left[\frac{\partial}{\partial x_1}(fg), \dots, \frac{\partial}{\partial x_n}(fg)\right] \\ &= f\left[\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n}\right] + g\left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right] \\ &= \boxed{f\,\nabla g + g\,\nabla f}\end{aligned}$$

Example.

$$\nabla(xy) = x\,\nabla y + y\,\nabla x$$

$$\nabla(xy) = x\,[0,1] + y\,[1,0] = [y,x]$$

Problem 9.7.12.

The force in an electrostatic field given by $f(x,y,z)$ has the direction of the gradient. Find ∇f and its value at P.

$$f = \frac{x}{x^2+y^2}, \quad P:(1,1)$$

Solution.

$$\begin{aligned}\nabla f &= \left[\frac{\partial}{\partial x}(f), \frac{\partial}{\partial y}(f)\right] \\ &= \boxed{\left[\frac{y^2-x^2}{(x^2+y^2)^2}, \frac{-2xy}{(x^2+y^2)^2}\right]}\end{aligned}$$

$$\nabla f(P) = \boxed{\left[0, -\frac{1}{2}\right]}$$

Problem 9.7.15.

The force in an electrostatic field given by $f(x,y,z)$ has the direction of the gradient. Find ∇f and its value at P.

Problem 9.7.21.

Given the velocity potential f of a flow, find the velocity $\mathbf{v} = \nabla f$ of the field and its value $\mathbf{v}(P)$ at P. Sketch $\mathbf{v}(P)$ and the curve $f = \text{const}$ passing through P

$$f = e^x \cos y, \quad P:(1,1)$$

Solution.

$$\begin{aligned}\mathbf{v} &= \nabla(e^x \cos y) \\ &= \boxed{[e^x \cos y, -e^x \sin y]}\end{aligned}$$

$$\boxed{\mathbf{v}(P) = [e \cos 1, -e \sin 1]}$$

Problem 9.7.26.

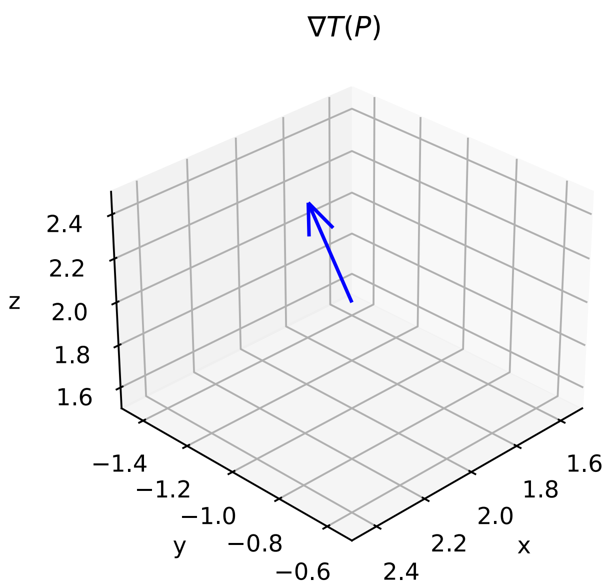
Experiments show that in a temperature field, heat flows in the direction of maximum decrease of temperature T. Find this direction in general and at the given point P. Sketch that direction at P as an arrow.

$$T = x^2 + y^2 + 4z^2, \quad P : (2, -1, 2)$$

Solution.

$$\boxed{\nabla T = [2x, 2y, 8z]}$$

$$\boxed{\nabla T(P) = [4, -2, 16]}$$



Problem 9.8.3.

Find $\nabla \cdot \mathbf{v}$ and its value at P:

$$\mathbf{v} = (x^2 + y^2)^{-1}[x, y]$$

Solution.

$$\mathbf{v} = \left[\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right]$$

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\\ &= \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\\ &= \boxed{0}\end{aligned}$$

Problem 9.8.5.

Find $\nabla\cdot\mathbf{v}$ and its value at P:

$$\mathbf{v} = x^2y^2z^2[x,y,z], \quad P:(3,-1,4)$$

Solution.

$$\mathbf{v} = \left[x^3y^2z^2, x^2y^3z^2, x^2y^2z^3\right]$$

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{\partial}{\partial x}(x^3y^2z^2) + \frac{\partial}{\partial y}(x^2y^3z^2) + \frac{\partial}{\partial z}(x^2y^2z^3)\\ &= \boxed{9x^2y^2z^2}\end{aligned}$$

$$\boxed{\nabla\cdot\mathbf{v}(\mathbf{P}) = 1296}$$

Problem 9.8.9. PROJECT. Useful Formulas for the Divergence.

Part a.

Prove $\nabla\cdot(k\mathbf{v}) = k\,\nabla\cdot\mathbf{v}$, ($k$ **constant**)

$$k\mathbf{v} = [kv_1, \ldots, kv_n]$$

$$\begin{aligned}\nabla\cdot(k\mathbf{v}) &= \frac{\partial}{\partial x_1}(kv_1) + \cdots + \frac{\partial}{\partial x_n}(kv_n)\\ &= k\left(\frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n}\right)\\ &= \boxed{k\,\nabla\cdot\mathbf{v}}\end{aligned}$$

Part b.

Prove $\nabla\cdot(f\mathbf{v}) = f(\nabla\cdot\mathbf{v}) + \mathbf{v}\cdot\nabla f$

$$f\mathbf{v} = [fv_1, \ldots, fv_n]$$

$$\nabla\cdot(f\mathbf{v}) = \frac{\partial}{\partial x_1}(fv_1) + \cdots + \frac{\partial}{\partial x_n}(fv_n)$$

The i-th component can be expanded by chain rule,

$$\frac{\partial}{\partial x_i}(fv_i) = f\frac{\partial v_i}{\partial x_i} + \frac{\partial f}{\partial x_i}v_i$$

Therefore in general,

$$\begin{aligned}\nabla\cdot(f\mathbf{v}) &= \frac{\partial}{\partial x_1}(fv_1) + \cdots + \frac{\partial}{\partial x_n}(fv_n)\\ &= f\left(\frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n}\right) + \left(v_1\frac{\partial f}{\partial x_1} + \cdots + v_n\frac{\partial f}{\partial x_n}\right)\\ &= f\left(\frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n}\right) + \mathbf{v}\cdot\left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right]\\ &= \boxed{f\,\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla f}\end{aligned}$$

Verify for $f = e^{xyz}$ and $\mathbf{v} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$.

$$\begin{aligned}\nabla\cdot(f\mathbf{v}) &= \nabla\cdot[axe^{xyz}, bye^{xyz}, cze^{xyz}]\\ &= (ae^{xyz} + axyze^{xyz}) + (be^{xyz} + bxyze^{xyz}) + (ce^{xyz} + cxyze^{xyz})\\ &= \boxed{(a+b+c)e^{xyz} + (a+b+c)xyze^{xyz}}\end{aligned}$$

$$\begin{aligned}\nabla\cdot(f\mathbf{v}) &= e^{xyz}\,\nabla\cdot[ax, by, cz] + [ax, by, cz]\cdot\nabla e^{xyz}\\ &= (a+b+c)e^{xyz} + [ax, by, cz]\cdot[yze^{xyz}, xze^{xyz}, xye^{xyz}]\\ &= \boxed{(a+b+c)e^{xyz} + (a+b+c)xyze^{xyz}}\end{aligned}$$

Obtain the answer to Prob. 6.

Find $\nabla\cdot(x^2+y^2+z^2)^{-3/2}[x, y, z]$.

$$\begin{aligned}\mathbf{v} &= [x, y, z], & \nabla\cdot\mathbf{v} &= \nabla\cdot[x, y, z]\\ & & &= 3\\ f &= (x^2+y^2+z^2)^{-3/2}, & \nabla f &= \nabla(x^2+y^2+z^2)^{-3/2}\\ & & &= -3(x^2+y^2+z^2)^{-5/2}[x, y, z]\end{aligned}$$

$$\begin{aligned}\nabla\cdot(f\mathbf{v}) &= f\,\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla f\\ &= (x^2+y^2+z^2)^{-3/2}\cdot 3 + [x, y, z]\cdot -3(x^2+y^2+z^2)^{-5/2}[x, y, z]\\ &= \boxed{0}\end{aligned}$$

Part c.

Prove $\nabla\cdot(f\,\nabla g) = f\,\Delta g + \nabla f\cdot\nabla g$

$$f\,\nabla g = \left[f\frac{\partial g}{\partial x_1}, \ldots, f\frac{\partial g}{\partial x_n}\right]$$

$$\nabla\cdot(f\,\nabla g) = \frac{\partial}{\partial x_1}\left(f\frac{\partial g}{\partial x_1}\right) + \cdots + \frac{\partial}{\partial x_n}\left(f\frac{\partial g}{\partial x_n}\right)$$

The i-th component can be expanded by chain rule,

$$\frac{\partial}{\partial x_i}\left(f\frac{\partial g}{\partial x_i}\right) = f\frac{\partial^2 g}{\partial x_i^2} + \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i}$$

Therefore in general,

$$\begin{aligned}\nabla\cdot(f\,\nabla g) &= \frac{\partial}{\partial x_1}\left(f\frac{\partial g}{\partial x_1}\right)+\cdots+\frac{\partial}{\partial x_n}\left(f\frac{\partial g}{\partial x_n}\right)\\ &= f\left(\frac{\partial^2 g}{\partial x_1^2}+\cdots+\frac{\partial^2 g}{\partial x_n^2}\right)+\left[\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right]\cdot\left[\frac{\partial g}{\partial x_1},\ldots,\frac{\partial g}{\partial x_n}\right]\\ &= \boxed{f\,\Delta g+\nabla f\cdot\nabla g}\end{aligned}$$

Verify for $f = x^2 - y^2$ and $g = e^{x+y}$.

	∇	Δ
f	$[2x, -2y]$	
g	$[e^{x+y}, e^{x+y}]$	$[e^{x+y}, e^{x+y}]$

$$\begin{aligned}\nabla\cdot(f\,\nabla g) &= \nabla\cdot\left((x^2-y^2)\left[e^{x+y}, e^{x+y}\right]\right)\\ &= \nabla\cdot\left[x^2e^{x+y}-y^2e^{x+y}, x^2e^{x+y}-y^2e^{x+y}\right]\\ &= (2x+x^2-y^2)e^{x+y}+(x^2-2y-y^2)e^{x+y}\\ &= \boxed{2(x+x^2-y-y^2)e^{x+y}}\end{aligned}$$

$$\begin{aligned}\nabla\cdot(f\,\nabla g) &= f\,\Delta g+\nabla f\cdot\nabla g\\ &= (x^2-y^2)[e^{x+y}, e^{x+y}]+[2x,-2y]\cdot[e^{x+y}, e^{x+y}]\\ &= \boxed{2(x+x^2-y-y^2)e^{x+y}}\end{aligned}$$

Part d.

Prove $\nabla\cdot(f\,\nabla g) - \nabla\cdot(g\,\nabla f) = f\,\Delta g - g\,\Delta f$

From **Part c**,

$$\nabla\cdot(f\,\nabla g) - \nabla\cdot(g\,\nabla f) = (f\,\Delta g + \nabla f\cdot\nabla g) - (g\,\Delta f + \nabla g\cdot\nabla f)$$

Therefore,

$$\nabla\cdot(f\,\nabla g) - \nabla\cdot(g\,\nabla f) = f\,\Delta g - g\,\Delta f$$

Problem 9.8.17.

Calculate Δf by Eq. (3). Check by direct differentiation. Indicate when (3) is simpler. Show the details of your work.

Eq. (3):

$$\nabla\cdot(\nabla f) = \Delta f$$

$$f = \ln(x^2+y^2)$$

Solution.

$$\begin{aligned}
\Delta f &= \nabla\cdot(\nabla f) \\
&= \nabla\cdot\left[\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right] \\
&= \frac{2(x^2+y^2)-4x^2}{(x^2+y^2)^2} + \frac{2(x^2+y^2)-4y^2}{(x^2+y^2)^2} \\
&= \frac{4(x^2+y^2)-4(x^2+y^2)}{(x^2+y^2)^2} \\
&= \boxed{0}
\end{aligned}$$

$$\begin{aligned}
\Delta f &= f_{xx} + f_{yy} \\
&= \frac{2(x^2+y^2)-4x^2}{(x^2+y^2)^2} + \frac{2(x^2+y^2)-4y^2}{(x^2+y^2)^2} \\
&= \frac{4(x^2+y^2)-4(x^2+y^2)}{(x^2+y^2)^2} \\
&= \boxed{0}
\end{aligned}$$

Problem 9.9.4.

Find $\nabla\times\mathbf{v}$ for $\mathbf{v}$ given with respect to right-handed Cartesian coordinates. Show the details of your work.

$$\mathbf{v} = [2y^2, 5x, 0]$$

Solution.

$$\begin{aligned}
\nabla\times\mathbf{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^2 & 5x & 0 \end{vmatrix} \\
&= \hat{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(5x)\right) - \hat{j}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(2y^2)\right) + \hat{k}\left(\frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(2y^2)\right) \\
&= \hat{k}(5-4y) \\
&= \boxed{[0, 0, 5-4y]}
\end{aligned}$$

Problem 9.9.9.

Let $\mathbf{v}$ be the velocity vector of a steady fluid flow.

$$\mathbf{v} = [0, 3z^2, 0]$$

Is the flow irrotational?

$$
\begin{aligned}
\nabla\times \mathbf{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 3z^2 & 0 \end{vmatrix} \\
&= \hat{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(3z^2)\right) - \hat{j}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(0)\right) + \hat{k}\left(\frac{\partial}{\partial x}(3z^2) - \frac{\partial}{\partial y}(0)\right) \\
&= \hat{i}(-6z) \\
&= \boxed{[-6z, 0, 0]}
\end{aligned}
$$

$\nabla\times \mathbf{v} \neq 0$ therefore the flow is **rotational**.

Is the flow incompressible?

$$\nabla\cdot \mathbf{v} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(3z^2) + \frac{\partial}{\partial z}(0) = \boxed{0}$$

$\nabla\cdot \mathbf{v} = 0$ therefore the flow is **incompressible**.

Find the streamlines (the paths of the particles).

$$\mathbf{v} = \mathbf{r}' = [x', y', z']$$

$$\begin{cases} x' = 0 \\ y' = 3z^2 \\ z' = 0 \end{cases}$$

Integrating, we get:

$$\begin{cases} x = x_0 \\ y = y_0 + 3z_0^2 t \\ z = z_0 \end{cases}$$

Streamlines are in the form $\boxed{\mathbf{r}(t) = [x_0, y_0 + 3z_0^2 t, z_0]}$.

Problem 9.9.14. PROJECT. Useful Formulas for the Curl.

Assuming sufficient differentiability, show that

Part a.

$\nabla\times(\mathbf{u} + \mathbf{v}) = \nabla\times \mathbf{u} + \nabla\times \mathbf{v}$

Solution.

For 3D vectors $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{u} + \mathbf{v} = [u_x + v_x, u_y + v_y, u_z + v_z]$$

$$
\begin{aligned}
\nabla\times \mathbf{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x + v_x & u_y + v_y & u_z + v_z \end{vmatrix} \\
&= \hat{i}\left(\frac{\partial}{\partial y}(u_z + v_z) - \frac{\partial}{\partial z}(u_y + v_y)\right) - \hat{j}\left(\frac{\partial}{\partial x}(u_z + v_z) - \frac{\partial}{\partial z}(u_x + v_x)\right) + \hat{k}\left(\frac{\partial}{\partial x}(u_y + v_y) - \frac{\partial}{\partial y}(u_x + v_x)\right) \\
&= \left[\left(\frac{\partial}{\partial y}(u_z + v_z) - \frac{\partial}{\partial z}(u_y + v_y)\right), \left(\frac{\partial}{\partial z}(u_x + v_x) - \frac{\partial}{\partial x}(u_z + v_z)\right), \left(\frac{\partial}{\partial x}(u_y + v_y) - \frac{\partial}{\partial y}(u_x + v_x)\right)\right] \\
&= \left[\left(\frac{\partial u_z}{\partial y} + \frac{\partial v_z}{\partial y} - \frac{\partial u_y}{\partial z} - \frac{\partial v_y}{\partial z}\right), \left(\frac{\partial u_x}{\partial z} + \frac{\partial v_x}{\partial z} - \frac{\partial u_z}{\partial x} - \frac{\partial v_z}{\partial x}\right), \left(\frac{\partial u_y}{\partial x} + \frac{\partial v_y}{\partial x} - \frac{\partial u_x}{\partial y} - \frac{\partial v_x}{\partial y}\right)\right] \\
&= \left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right), \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right), \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\right] + \left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right), \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right), \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right] \\
&= \boxed{\nabla\times \mathbf{u} + \nabla\times \mathbf{v}}
\end{aligned}
$$

Part b

$\nabla\cdot(\nabla\times \mathbf{v}) = 0$

Solution.

For 3D vector $\mathbf{v}$,

$$
\begin{aligned}
\nabla\times \mathbf{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\
&= \hat{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \hat{j}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \hat{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \\
&= \left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right), \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right), \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot(\nabla\times \mathbf{v}) &= \nabla\cdot\left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right), \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right), \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right] \\
&= \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \\
&= \frac{\partial^2 v_z}{\partial x\partial y} - \frac{\partial^2 v_y}{\partial x\partial z} + \frac{\partial^2 v_x}{\partial y\partial z} - \frac{\partial^2 v_z}{\partial x\partial y} + \frac{\partial^2 v_y}{\partial x\partial z} - \frac{\partial^2 v_x}{\partial y\partial z} \\
&= \boxed{0}
\end{aligned}
$$

Part c

$\nabla\times(f\mathbf{v}) = (\nabla f) \times \mathbf{v} + f\,\nabla\times \mathbf{v}$

Solution.

For 3D vector $\mathbf{v}$,

$$
\begin{aligned}
\nabla\times\mathbf{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fv_x & fv_y & fv_z \end{vmatrix} \\
&= \hat{i}\left(\frac{\partial}{\partial y}(fv_z) - \frac{\partial}{\partial z}(fv_y)\right) - \hat{j}\left(\frac{\partial}{\partial z}(fv_x) - \frac{\partial}{\partial x}(fv_z)\right) + \hat{k}\left(\frac{\partial}{\partial y}(fv_x) - \frac{\partial}{\partial x}(fv_y)\right) \\
&= \left[\left(\frac{\partial}{\partial y}(fv_z) - \frac{\partial}{\partial z}(fv_y)\right), \left(\frac{\partial}{\partial x}(fv_z) - \frac{\partial}{\partial z}(fv_x)\right), \left(\frac{\partial}{\partial y}(fv_x) - \frac{\partial}{\partial x}(fv_y)\right)\right]
\end{aligned}
$$

Expanding with chain rule,

$$
\begin{aligned}
&= \left[\left(f\frac{\partial v_z}{\partial y} + v_z\frac{\partial f}{\partial y} - f\frac{\partial v_y}{\partial z} - v_y\frac{\partial f}{\partial z}\right), \left(f\frac{\partial v_z}{\partial x} - v_z\frac{\partial f}{\partial x} - f\frac{\partial v_x}{\partial z} + v_x\frac{\partial f}{\partial z}\right), \left(f\frac{\partial v_y}{\partial x} + v_y\frac{\partial f}{\partial x} - f\frac{\partial v_x}{\partial y} - v_x\frac{\partial f}{\partial y}\right)\right] \\
&= \left[\left(v_z\frac{\partial f}{\partial y} - v_y\frac{\partial f}{\partial z}\right), \left(v_x\frac{\partial f}{\partial z} - v_z\frac{\partial f}{\partial x}\right), \left(v_y\frac{\partial f}{\partial x} - v_x\frac{\partial f}{\partial y}\right)\right] + \left[f\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right), f\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right), f\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right] \\
&= \boxed{(\nabla f)\times\mathbf{v} + f\,\nabla\times\mathbf{v}}
\end{aligned}
$$

Part d

$\nabla\times(\nabla f) = \mathbf{0}$

Solution.

For $f(x, y, z)$,

$$
\begin{aligned}
\nabla\times(\nabla f) &= \nabla\times\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \\
&= \left[\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right), \left(\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)\right), \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)\right] \\
&= [0, 0, 0] \\
&= \boxed{\mathbf{0}}
\end{aligned}
$$

Part e

$\nabla\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\nabla\times\mathbf{u} - \mathbf{u}\cdot\nabla\times\mathbf{v}$

Solution.

For 3D vectors $\mathbf{u}$ and $\mathbf{v}$,

$$
(\mathbf{u}\times\mathbf{v}) = \left[(u_y v_z - u_z v_y), (u_z v_x - u_x v_z), (u_x v_y - u_y v_x)\right]
$$

$$
\begin{aligned}
\nabla\cdot(\mathbf{u}\times\mathbf{v}) &= \nabla\cdot\left[\left(u_y v_z - u_z v_y\right),\left(u_z v_x - u_x v_z\right),\left(u_x v_y - u_y v_x\right)\right] \\
&= \frac{\partial}{\partial x}\left(u_y v_z - u_z v_y\right) + \frac{\partial}{\partial y}\left(u_z v_x - u_x v_z\right) + \frac{\partial}{\partial z}\left(u_x v_y - u_y v_x\right) \\
&= \left(v_z\frac{\partial u_y}{\partial x} - v_y\frac{\partial u_z}{\partial x}\right) + \left(v_x\frac{\partial u_z}{\partial y} - v_z\frac{\partial u_x}{\partial y}\right) + \left(v_y\frac{\partial u_x}{\partial z} - v_x\frac{\partial u_y}{\partial z}\right) + \\
&\quad \left(u_y\frac{\partial v_z}{\partial x} - u_z\frac{\partial v_y}{\partial x}\right) + \left(u_z\frac{\partial v_x}{\partial y} - u_x\frac{\partial v_z}{\partial y}\right) + \left(u_x\frac{\partial v_y}{\partial z} - u_y\frac{\partial v_x}{\partial z}\right) \\
&= v_x\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) + v_y\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) + v_z\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) + \\
&\quad u_x\left(\frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y}\right) + u_y\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + u_z\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right) \\
&= [v_x, v_y, v_z]\cdot\left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right),\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right),\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\right] + \\
&\quad [u_x, u_y, u_z]\cdot\left[\left(\frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y}\right),\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right),\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right)\right] \\
&= \boxed{\mathbf{v}\cdot\nabla\times\mathbf{u} - \mathbf{u}\cdot\nabla\times\mathbf{v}}
\end{aligned}
$$

Problem 10.1.2.

Calculate $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$ for the given data. If $\mathbf{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

$$\mathbf{F} = [y^2, -x^2], \quad C : y = 4x^2 \text{ from } (0,\,0) \text{ to } (1,\,4)$$

Solution.

$$\mathbf{r} = [t, 4t^2], \quad t \in [0,1]$$

$$
\begin{aligned}
\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\ dt \\
&= \int_0^1 [16t^4, -t^2]\cdot[1, 8t]\ dt \\
&= \int_0^1 (16t^4 - 8t^3)\ dt \\
&= \left[\frac{16}{5}t^5 - 2t^4\right]_0^1 \\
&= \frac{16}{5} - 2 \\
&= \boxed{\frac{6}{5}}
\end{aligned}
$$

Problem 10.1.3.

Calculate $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$ for the given data. If $\mathbf{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

$$\mathbf{F} = [y^2, -x^2], \quad C \text{ from } (0,\,0) \text{ straight to } (1,\,4)$$

Solution.

$$\mathbf{r} = [t, 4t], \quad t \in [0, 1]$$

$$\begin{aligned}
\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\
&= \int_0^1 [16t^2, -t^2] \cdot [1, 4]\ dt \\
&= \int_0^1 (16t^2 - 4t^2)\ dt \\
&= \int_0^1 12t^2\ dt \\
&= \left[4t^3\right]_0^1 \\
&= \boxed{4}
\end{aligned}$$

Problem 10.1.5.

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If $\mathbf{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

$$\mathbf{F} = [xy, x^2y^2], \quad C \text{ the quarter-circle from (2, 0) to (0, 2) with center (0, 0)}$$

Solution.

$$\mathbf{r} = 2[\cos t, \sin t], \quad t \in \left[0, \frac{\pi}{2}\right]$$

$$\begin{aligned}
\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\
&= \int_0^1 [4\cos t \sin t, 16\cos^2 t \sin^2 t] \cdot 2[-\sin t, \cos t]\ dt \\
&= \int_0^1 \left(-8\cos t \sin^2 t + 32\cos^3 t \sin^2 t\right)\ dt
\end{aligned}$$

Substituting $u = \sin t$, $du = \cos t\ dt$ and $\cos^2 t = 1 - \sin^2 t$:

$$\begin{aligned}
&= \int_0^1 \left(-8u^2 + 32u^2(1 - u^2)\right)\ du \\
&= \int_0^1 \left(-8u^2 + 32u^2 - 32u^4\right)\ du \\
&= \int_0^1 \left(24u^2 - 32u^4\right)\ du \\
&= \left[8u^3 - \frac{32}{5}u^5\right]_0^1 \\
&= \boxed{\frac{8}{5}}
\end{aligned}$$

Problem 10.1.6.

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If $\mathbf{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

$$\mathbf{F} = [x - y, y - z, z - x], \quad C : \mathbf{r} = [2\cos t, t, 2\sin t] \text{ from } (2,\ 0,\ 0) \text{ to } (2,\ 2\pi,\ 0)$$

Solution.

$$\begin{aligned}
\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\
&= \int_0^{2\pi} [2\cos t - t, t - 2\sin t, 2\sin t - 2\cos t] \cdot [-2\sin t, 1, 2\cos t]\ dt \\
&= \int_0^{2\pi} \left(-4\cos t \sin t + 2t\sin t + t - 2\sin t + 4\cos t \sin t - 4\cos^2 t\right)\ dt \\
&= \int_0^{2\pi} \left(2t\sin t + t - 2\sin t - 4\cos^2 t\right)\ dt
\end{aligned}$$

Integrate Term 1 by parts ($\int u dv = uv - \int v du$):

$$\begin{aligned}
u &= t & du &= dt \\
v &= -\cos t & dv &= \sin t dt
\end{aligned}$$

$$\int_0^{2\pi} 2t\sin t\ dt = 2\left[-t\cos t - \int_0^{2\pi} \cos t\ dt\right] \quad = 2\left[-t\cos t - \sin t\right]_0^{2\pi} = -4\pi$$

Term 2:

$$\int_0^{2\pi} t\ dt = \left.\frac{t^2}{2}\right|_0^{2\pi} = 2\pi^2$$

Term 3:

$$\int_0^{2\pi} (-2\sin t)\ dt = 2\cos t\Big|_0^{2\pi} = 0$$

In Term 4, substitute $\cos^2 t = \frac{1+\cos 2t}{2}$:

$$\int_0^{2\pi} \left(-4\cos^2 t\right)\ dt = -2\int_0^{2\pi} (1 + \cos 2t)\ dt = -2\left[t + \frac{1}{2}\sin 2t\right]_0^{2\pi}\ dt = -4\pi$$

Adding all terms together:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = -4\pi + 2\pi^2 + 0 - 4\pi = \boxed{2\pi^2 - 8\pi}$$

Problem 10.1.10.

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If $\mathbf{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

$$\mathbf{F} = [x, -z, 2y] \text{ from } (0,\ 0,\ 0) \text{ straight to } (1,\ 1,\ 0), \text{ then to } (1,\ 1,\ 1), \text{ back to } (0,\ 0,\ 0)$$

Solution.

$$
\begin{array}{lccr}
(0,\,0,\,0) \text{ to } (1,\,1,\,0) & \Rightarrow & \mathbf{r}_1(t) = [t, t, 0] & t \in [0,1] \\
(1,\,1,\,0) \text{ to } (1,\,1,\,1) & \Rightarrow & \mathbf{r}_2(t) = [1, 1, t] & t \in [0,1] \\
(1,\,1,\,1) \text{ to } (0,\,0,\,0) & \Rightarrow & \mathbf{r}_3(t) = [1-t, 1-t, 1-t] & t \in [0,1]
\end{array}
$$

$$
\begin{aligned}
\mathbf{F}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t) &= [t, 0, 2t] \cdot [1, 1, 0] \\
&= t
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}(\mathbf{r}_2(t)) \cdot \mathbf{r}_2'(t) &= [1, -t, 2] \cdot [0, 0, 1] \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}(\mathbf{r}_3(t)) \cdot \mathbf{r}_3'(t) &= [1-t, t-1, 2(1-t)] \cdot [-1, -1, -1] \\
&= -1 + t - t + 1 - 2 + 2t \\
&= 2t - 2
\end{aligned}
$$

$$
\begin{aligned}
\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\
&= \int_0^1 (t) + (2) + (2t-2)\ dt \\
&= \int_0^1 3t\ dt \\
&= \left[\frac{3}{2}t^2\right]_0^1 \\
&= \boxed{\frac{3}{2}}
\end{aligned}
$$