

**Problem 10.4.3.**

Evaluate  $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$  counterclockwise around the boundary  $C$  of the region  $R$  by Green's theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA$$

where

$$\mathbf{F} = [x^2 e^y, y^2 e^x], \quad R \text{ the rectangle with vertices } (0, 0), (2, 0), (2, 3), (0, 3)$$

**Solution.**

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA \\ &= \int_0^3 \int_0^2 \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dx \, dy \\ &= \int_0^3 \int_0^2 (y^2 e^x - x^2 e^y) \, dx \, dy \\ &= \int_0^3 \left[ y^2 e^x - \frac{1}{3} x^3 e^y \right]_{x=0}^{x=2} \, dy \\ &= \int_0^3 \left[ \left( y^2 e^2 - \frac{1}{3} 2^3 e^y \right) - \left( y^2 e^0 - \frac{1}{3} 0^3 e^y \right) \right] \, dy \\ &= \int_0^3 \left[ \left( y^2 e^2 - \frac{8}{3} e^y \right) - (y^2) \right] \, dy \\ &= \int_0^3 \left( y^2 (e^2 - 1) - \frac{8}{3} e^y \right) \, dy \\ &= \frac{1}{3} y^3 (e^2 - 1) - \frac{8}{3} e^y \Big|_0^3 \\ &= \left( \frac{1}{3} 3^3 (e^2 - 1) - \frac{8}{3} e^3 \right) - \left( \frac{1}{3} 0^3 (e^2 - 1) - \frac{8}{3} e^0 \right) \\ &= \left( 9(e^2 - 1) - \frac{8}{3} e^3 \right) - \left( -\frac{8}{3} \right) \\ \oint_C \mathbf{F} \cdot d\mathbf{r} &= \boxed{9(e^2 - 1) - \frac{8}{3}(e^3 - 1)} \end{aligned}$$

**Problem 10.4.7.**

Evaluate  $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$  counterclockwise around the boundary  $C$  of the region  $R$  by Green's theorem, where

$$\mathbf{F} = \nabla (x^3 \cos^2(xy)), \quad R : 1 \leq y \leq 2 - x^2$$

**Solution.**

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA \\ &= \iint_R \nabla \times (\nabla f) \cdot \mathbf{k} \, dA \\ \oint_C \mathbf{F} \cdot d\mathbf{r} &= \boxed{0} \end{aligned}$$

**Problem 10.4.15.**

Using the following equation,

$$\iint_R \Delta w \, dA = \oint_C \frac{\partial w}{\partial n} \, ds$$

Find the value of  $\int_C \frac{\partial w}{\partial n} \, ds$  taken counterclockwise over the boundary  $C$  of the region  $R$ .

$$w = e^x \cos y + xy^3, \quad R : 1 \leq y \leq 10 - x^2, \quad x \geq 0$$

**Solution.**

$$\begin{aligned} \Delta w &= \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \\ &= (e^x \cos y) + (-e^x \cos y + 6xy) \\ &= 6xy \end{aligned}$$

$$\begin{aligned} \oint_C \frac{\partial w}{\partial n} \, ds &= \iint_R \Delta w \, dA \\ &= \int_0^3 \int_1^{10-x^2} 6xy \, dy \, dx \\ &= \int_0^3 [3xy^2]_{y=1}^{y=10-x^2} \, dx \\ &= \int_0^3 (3x(10-x^2)^2 - 3x) \, dx \\ &= \int_0^3 (3x(100-20x^2+x^4) - 3x) \, dx \\ &= \int_0^3 (297x - 60x^3 + 3x^5) \, dx \\ &= \left. \frac{297}{2}x^2 - 15x^4 + \frac{1}{2}x^6 \right|_0^3 \\ &= 1336.5 - 1215 + 364.5 \\ &= \boxed{486} \end{aligned}$$

**Problem 10.4.16.**

Using the following equation,

$$\iint_R \Delta w \, dx \, dy = \oint_C \frac{\partial w}{\partial n} \, ds$$

find the value of  $\int_C \frac{\partial w}{\partial n} \, ds$  taken counterclockwise over the boundary  $C$  of the region  $R$ .

$$w = x^2 + y^2, \quad C : x^2 + y^2 = 4$$

Confirm the answer by direct integration.

**Solution.**

**Problem 10.4.19.**

Show that  $w = e^x \sin y$  satisfies Laplace's equation  $\Delta w = 0$  and, using the following equation,

$$\iint_R \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] dx dy = \oint_C w \left( \frac{\partial w}{\partial n} \right) ds$$

integrate  $w \left( \frac{\partial w}{\partial n} \right)$  counter-clockwise around the boundary curve  $C$  of the rectangle  $0 \leq x \leq 2$ ,  $0 \leq y \leq 5$ .

**Solution.**

**Problem 10.5.2.**

Derive a parametric representation,  $z = f(x, y)$  or  $g(x, y, z) = 0$ , by finding the **parameter curves** (curves  $u = \text{const}$  and  $v = \text{const}$ ) of the surface and a normal vector  $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$  of the surface.

$xy$ -plane in polar coordinates  $\mathbf{r}(u, v) = [u \cos v, u \sin v]$ , (thus  $u = r, v = \theta$ )

**Solution.**

**Problem 10.5.3.**

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

Cone  $\mathbf{r}(u, v) = [u \cos v, u \sin v, cu]$

**Solution.**

**Problem 10.5.5.**

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

Paraboloid of revolution  $\mathbf{r}(u, v) = [u \cos v, u \sin v, u^2]$

**Solution.**

**Problem 10.5.7.**

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

Ellipsoid  $\mathbf{r}(u, v) = [a \cos v \cos u, b \cos v \sin u, c \sin v]$

**Solution.**

**Problem 10.5.14.**

Find a normal vector. Sketch the surface and parameter curves.

$$\text{Plane } 4x + 3y + 2z = 12$$

**Solution.**

**Problem 10.5.15.**

Find a normal vector. Sketch the surface and parameter curves.

$$\text{Cylinder of revolution } (x - 2)^2 + (y + 1)^2 = 25$$

**Solution.**

**Problem 10.5.18.**

Find a normal vector. Sketch the surface and parameter curves.

$$\text{Elliptic cone } z = \sqrt{x^2 + 4y^2}$$

**Solution.**

**Problem 10.6.3.**

Evaluate the flux integral,  $\int_S \mathbf{F} \cdot \mathbf{n} dA$ , for the given data. Describe the kind of surface.

$$\mathbf{F} = [0, x, 0], \quad S : x^2 + y^2 + z^2 = 1, \quad x, y, z \geq 0$$

**Solution.**

**Problem 10.6.5.**

Evaluate the flux integral for the given data. Describe the kind of surface.

$$\mathbf{F} = [x, y, z], \quad S : \mathbf{r} = [u \cos v, u \sin v, u^2], \quad 0 \leq u \leq 4, \quad -\pi \leq v \leq \pi$$

**Solution.**

**Problem 10.6.7.**

Evaluate the flux integral for the given data. Describe the kind of surface.

$$\mathbf{F} = [0, \sin y, \cos z], \quad S \text{ the cylinder } x = y^2, \text{ where } 0 \leq y \leq \frac{\pi}{4}, \quad 0 \leq z \leq y$$

**Solution.****Problem 10.6.13.**

Evaluate the surface integral,  $\iint_S G(\mathbf{r}) \, dA$ , for the following data. Indicate the kind of surface.

$$G = x + y + z, \quad z = x + 2y, \quad 0 \leq x \leq \pi, \quad 0 \leq y \leq x$$

**Solution.****Problem 10.6.15.**

Evaluate the surface integral for the following data. Indicate the kind of surface.

$$G = (1 + 9xz)^{3/2}, \quad S : \mathbf{r} = [u, v, u^3], \quad 0 \leq u \leq 1, \quad -2 \leq v \leq 2$$

**Solution.**