

**Problem 10.4.3.**

Evaluate  $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$  counterclockwise around the boundary  $C$  of the region  $R$  by Green's theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA$$

where

$$\mathbf{F} = [x^2 e^y, y^2 e^x], \quad R \text{ the rectangle with vertices } (0, 0), (2, 0), (2, 3), (0, 3)$$

**Solution.**

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA \\ &= \int_0^3 \int_0^2 \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dx \, dy \\ &= \int_0^3 \int_0^2 (y^2 e^x - x^2 e^y) \, dx \, dy \\ &= \int_0^3 \left[ y^2 e^x - \frac{1}{3} x^3 e^y \right]_{x=0}^{x=2} \, dy \\ &= \int_0^3 \left[ \left( y^2 e^2 - \frac{1}{3} 2^3 e^y \right) - \left( y^2 e^0 - \frac{1}{3} 0^3 e^y \right) \right] \, dy \\ &= \int_0^3 \left[ \left( y^2 e^2 - \frac{8}{3} e^y \right) - (y^2) \right] \, dy \\ &= \int_0^3 \left( y^2 (e^2 - 1) - \frac{8}{3} e^y \right) \, dy \\ &= \frac{1}{3} y^3 (e^2 - 1) - \frac{8}{3} e^y \Big|_0^3 \\ &= \left( \frac{1}{3} 3^3 (e^2 - 1) - \frac{8}{3} e^3 \right) - \left( \frac{1}{3} 0^3 (e^2 - 1) - \frac{8}{3} e^0 \right) \\ &= \left( 9(e^2 - 1) - \frac{8}{3} e^3 \right) - \left( -\frac{8}{3} \right) \\ \oint_C \mathbf{F} \cdot d\mathbf{r} &= \boxed{9(e^2 - 1) - \frac{8}{3}(e^3 - 1)} \end{aligned}$$

**Problem 10.4.7.**

Evaluate  $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$  counterclockwise around the boundary  $C$  of the region  $R$  by Green's theorem, where

$$\mathbf{F} = \nabla (x^3 \cos^2(xy)), \quad R : 1 \leq y \leq 2 - x^2$$

**Solution.**

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA \\ &= \iint_R \nabla \times (\nabla f) \cdot \mathbf{k} \, dA \\ \oint_C \mathbf{F} \cdot d\mathbf{r} &= \boxed{0} \end{aligned}$$

**Problem 10.4.15.**

Using the following equation,

$$\iint_R \Delta w \, dA = \oint_C \frac{\partial w}{\partial n} \, ds$$

Find the value of  $\int_C \frac{\partial w}{\partial n} \, ds$  taken counterclockwise over the boundary  $C$  of the region  $R$ .

$$w = e^x \cos y + xy^3, \quad R : 1 \leq y \leq 10 - x^2, \quad x \geq 0$$

**Solution.**

$$\begin{aligned} \Delta w &= w_{xx} + w_{yy} \\ &= (e^x \cos y) + (-e^x \cos y + 6xy) \\ &= 6xy \end{aligned}$$

$$\begin{aligned} \oint_C \frac{\partial w}{\partial n} \, ds &= \iint_R \Delta w \, dA \\ &= \int_0^3 \int_1^{10-x^2} 6xy \, dy \, dx \\ &= \int_0^3 [3xy^2]_{y=1}^{y=10-x^2} \, dx \\ &= \int_0^3 (3x(10-x^2)^2 - 3x) \, dx \\ &= \int_0^3 (3x(100-20x^2+x^4) - 3x) \, dx \\ &= \int_0^3 (297x - 60x^3 + 3x^5) \, dx \\ &= \left. \frac{297}{2}x^2 - 15x^4 + \frac{1}{2}x^6 \right|_0^3 \\ &= 1336.5 - 1215 + 364.5 \\ &= \boxed{486} \end{aligned}$$

**Problem 10.4.16.**

Using the following equation,

$$\iint_R \Delta w \, dx \, dy = \oint_C \frac{\partial w}{\partial n} \, ds$$

find the value of  $\int_C \frac{\partial w}{\partial n} \, ds$  taken counterclockwise over the boundary  $C$  of the region  $R$ .

$$w = x^2 + y^2, \quad C : x^2 + y^2 = 4$$

**Solution.**

$$\begin{aligned}\Delta w &= w_{xx} + w_{yy} \\ &= (2) + (2) \\ &= 4\end{aligned}$$

$$\begin{aligned}\oint_C \frac{\partial w}{\partial n} ds &= \iint_R \Delta w dA \\ &= 4 \iint_R dA \\ &= 4 \cdot \text{Area}(R) \\ &= 4 \cdot \pi \cdot 2^2 \\ &= \boxed{16\pi}\end{aligned}$$

Confirm the answer by direct integration.

$$C \implies \mathbf{r}(t) = [2 \cos t, 2 \sin t], \quad t \in [0, 2\pi]$$

The arc length of  $C$  is,

$$\begin{aligned}ds &= |\mathbf{r}'(t)| dt \\ &= \sqrt{(-2 \sin t)^2 + (2 \cos t)^2} dt \\ &= 2 dt\end{aligned}$$

The normal vector is,

$$\begin{aligned}\mathbf{n} &= \frac{\nabla x^2 + y^2}{|\nabla x^2 + y^2|} \\ &= \frac{[2x, 2y]}{\sqrt{(2x)^2 + (2y)^2}} \\ &= \frac{[x, y]}{\sqrt{x^2 + y^2}}\end{aligned}$$

On  $C$ ,  $r = 2$ ,

$$\mathbf{n} = \frac{[x, y]}{2}$$

The integral of the normal derivative is then,

$$\begin{aligned}\oint_C \frac{\partial w}{\partial n} ds &= \oint_C \nabla w \cdot \mathbf{n} ds \\ &= \frac{1}{2} \oint_C [2x, 2y] \cdot [x, y] ds \\ &= \frac{1}{2} \oint_C 2x^2 + 2y^2 ds \\ &= \oint_C x^2 + y^2 ds\end{aligned}$$

On  $C$ ,  $x^2 + y^2 = 4$ , and  $ds = 2 dt$  where  $t \in [0, 2\pi]$ ,

$$\begin{aligned}\oint_C x^2 + y^2 \, ds &= 8 \int_0^{2\pi} dt \\ &= 8 \cdot 2\pi \\ &= [16\pi]\end{aligned}$$

**Problem 10.4.19.**

Show that  $w = e^x \sin y$  satisfies Laplace's equation  $\Delta w = 0$  and, using the following equation,

$$\iint_R \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] dx dy = \oint_C w \left( \frac{\partial w}{\partial n} \right) ds$$

integrate  $w \left( \frac{\partial w}{\partial n} \right)$  counter-clockwise around the boundary curve  $C$  of the rectangle  $0 \leq x \leq 2$ ,  $0 \leq y \leq 5$ .

**Solution.**

$$\nabla w = [e^x \sin y, e^x \cos y]$$

$$\begin{aligned}\Delta w &= (e^x \sin y) + (-e^x \sin y) \\ &= [0]\end{aligned}$$

$$\begin{aligned}\oint_C w \left( \frac{\partial w}{\partial n} \right) ds &= \iint_R \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] dx dy \\ &= \iint_R (e^x \sin y)^2 + (e^x \cos y)^2 dx dy \\ &= \iint_R (e^2 x [\sin^2 y + \cos^2 y]) dx dy \\ &= \iint_R (e^2 x) dx dy \\ &= \int_0^5 \int_0^2 (e^2 x) dx dy \\ &= \frac{1}{2} \int_0^5 [e^2 x]_0^2 dy \\ &= \frac{1}{2} (e^4 - 1) \int_0^5 dy \\ &= \frac{1}{2} (e^4 - 1) \cdot 5 \\ &= \boxed{\frac{5}{2} (e^4 - 1)}\end{aligned}$$

**Problem 10.5.2.**

Derive a parametric representation,  $z = f(x, y)$  or  $g(x, y, z) = 0$ , by finding the **parameter curves** (curves  $u = \text{const}$  and  $v = \text{const}$ ) of the surface and a normal vector  $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$  of the surface.

*xy*-plane in polar coordinates  $\mathbf{r}(u, v) = [u \cos v, u \sin v]$ , (thus  $u = r, v = \theta$ )

**Solution.**

$$z = f(x, y),$$

$$\begin{aligned}x &= u \cos v \\y &= u \sin v\end{aligned}$$

$$\begin{aligned}x^2 + y^2 &= u^2(\cos^2 v + \sin^2 v) \\x^2 + y^2 &= u^2 \\x^2 + y^2 &= z^2\end{aligned}$$

$$z = \sqrt{x^2 + y^2}$$

The parameter curves are,

- $u = \text{const} \rightarrow$  concentric circles
- $v = \text{const} \rightarrow$  radial straight lines

$$\begin{aligned}\mathbf{r}_u &= [\cos v, \sin v] \\ \mathbf{r}_v &= [-u \sin v, u \cos v]\end{aligned}$$

$$\begin{aligned}\mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 0 \end{vmatrix} \\&= (\cos v \cdot u \cos v - \sin v \cdot (-u \sin v)) \mathbf{k} \\&= (u \cos^2 v + u \sin^2 v) \mathbf{k} \\&= u \mathbf{k} \\ \mathbf{N} &= [0, 0, u]\end{aligned}$$

**Problem 10.5.3.**

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

$$\text{Cone } \mathbf{r}(u, v) = [u \cos v, u \sin v, cu]$$

**Solution.**

$$\begin{aligned}x &= u \cos v \\y &= u \sin v \\z &= cu\end{aligned}$$

$$\begin{aligned}x^2 + y^2 + z^2 &= u^2(\cos^2 v + \sin^2 v + c^2) \\x^2 + y^2 + z^2 &= u^2(1 + c^2)\end{aligned}$$

But  $u = \frac{z}{c}$ ,

$$\begin{aligned}x^2 + y^2 + z^2 &= \frac{z^2}{c^2}(1 + c^2) \\x^2 + y^2 + z^2 &= \frac{z^2}{c^2} + z^2\end{aligned}$$

$$z = c\sqrt{x^2 + y^2}$$

The parameter curves are,

- $u = \text{const} \rightarrow$  concentric circles at a given height
- $v = \text{const} \rightarrow$  lines that extend straight along the edges of the cone and pass through the origin

$$\begin{aligned}\mathbf{r}_u &= [\cos v, \sin v, c] \\ \mathbf{r}_v &= [-u \sin v, u \cos v, 0]\end{aligned}$$

$$\begin{aligned}\mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & c \\ -u \sin v & u \cos v & 0 \end{vmatrix} \\ &= [(0 - cu \cos v), -(0 + cu \sin v), (u \cos^2 v + u \sin^2 v)] \\ &= [-cu \cos v, -cu \sin v, u] \\ &= \boxed{u[-c \cos v, -c \sin v, 1]}\end{aligned}$$

### Problem 10.5.5.

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

$$\text{Paraboloid of revolution } \mathbf{r}(u, v) = [u \cos v, u \sin v, u^2]$$

**Solution.**

### Problem 10.5.7.

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

$$\text{Ellipsoid } \mathbf{r}(u, v) = [a \cos v \cos u, b \cos v \sin u, c \sin v]$$

**Solution.**

### Problem 10.5.14.

Find a normal vector. Sketch the surface and parameter curves.

$$\text{Plane } 4x + 3y + 2z = 12$$

**Solution.**

**Problem 10.5.15.**

Find a normal vector. Sketch the surface and parameter curves.

$$\text{Cylinder of revolution } (x - 2)^2 + (y + 1)^2 = 25$$

**Solution.**

**Problem 10.5.18.**

Find a normal vector. Sketch the surface and parameter curves.

$$\text{Elliptic cone } z = \sqrt{x^2 + 4y^2}$$

**Solution.**

**Problem 10.6.3.**

Evaluate the flux integral,  $\int_S \mathbf{F} \cdot \mathbf{n} dA$ , for the given data. Describe the kind of surface.

$$\mathbf{F} = [0, x, 0], \quad S : x^2 + y^2 + z^2 = 1, \quad x, y, z \geq 0$$

**Solution.**

**Problem 10.6.5.**

Evaluate the flux integral for the given data. Describe the kind of surface.

$$\mathbf{F} = [x, y, z], \quad S : \mathbf{r} = [u \cos v, u \sin v, u^2], \quad 0 \leq u \leq 4, \quad -\pi \leq v \leq \pi$$

**Solution.**

**Problem 10.6.7.**

Evaluate the flux integral for the given data. Describe the kind of surface.

$$\mathbf{F} = [0, \sin y, \cos z], \quad S \text{ the cylinder } x = y^2, \text{ where } 0 \leq y \leq \frac{\pi}{4}, \quad 0 \leq z \leq y$$

**Solution.**

**Problem 10.6.13.**

Evaluate the surface integral,  $\iint_S G(\mathbf{r}) dA$ , for the following data. Indicate the kind of surface.

$$G = x + y + z, \quad z = x + 2y, \quad 0 \leq x \leq \pi, \quad 0 \leq y \leq x$$

**Solution.****Problem 10.6.15.**

Evaluate the surface integral for the following data. Indicate the kind of surface.

$$G = (1 + 9xz)^{3/2}, \quad S : \mathbf{r} = [u, v, u^3], 0 \leq u \leq 1, \quad -2 \leq v \leq 2$$

**Solution.**