Problem 10.4.3.

Evaluate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ counterclockwise around the boundary C of the region R by Green's theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla\times \mathbf{F} \cdot \mathbf{k}\ dA$$

where

$$\mathbf{F} = \left[x^2 e^y, y^2 e^x\right], \qquad R \text{ the rectangle with vertices } (0,\,0),\,(2,\,0),\,(2,\,3),\,(0,\,3)$$

Solution.

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \nabla\times \mathbf{F} \cdot \mathbf{k}\ dA \\
&= \int_0^3 \int_0^2 \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)\ dx\ dy \\
&= \int_0^3 \int_0^2 \left(y^2 e^x - x^2 e^y \right)\ dx\ dy \\
&= \int_0^3 \left[y^2 e^x - \frac{1}{3} x^3 e^y \right]_{x=0}^{x=2}\ dy \\
&= \int_0^3 \left[\left(y^2 e^2 - \frac{1}{3} 2^3 e^y \right) - \left(y^2 e^0 - \frac{1}{3} 0^3 e^y \right) \right]\ dy \\
&= \int_0^3 \left[\left(y^2 e^2 - \frac{8}{3} e^y \right) - \left(y^2 \right) \right]\ dy \\
&= \int_0^3 \left(y^2 (e^2 - 1) - \frac{8}{3} e^y \right)\ dy \\
&= \frac{1}{3} y^3 (e^2 - 1) - \frac{8}{3} e^y \Big|_0^3 \\
&= \left(\frac{1}{3} 3^3 (e^2 - 1) - \frac{8}{3} e^3 \right) - \left(\frac{1}{3} 0^3 (e^2 - 1) - \frac{8}{3} e^0 \right) \\
&= \left(9(e^2 - 1) - \frac{8}{3} e^3 \right) - \left(-\frac{8}{3} \right) \\
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \boxed{9(e^2 - 1) - \frac{8}{3}(e^3 - 1)}
\end{aligned}$$

Problem 10.4.7.

Evaluate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ counterclockwise around the boundary C of the region R by Green's theorem, where

$$\mathbf{F} = \nabla \left(x^3 \cos^2(xy) \right), \qquad R : 1 \leqq y \leqq 2 - x^2$$

Solution.

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \nabla\times \mathbf{F} \cdot \mathbf{k}\ dA \\
&= \iint_R \cancelto{0}{\nabla\times (\nabla f) \cdot} \mathbf{k}\ dA \\
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \boxed{0}
\end{aligned}$$

Problem 10.4.15.

Using the following equation,

$$\iint_R \Delta w \; dx \; dy = \oint_C \frac{\partial w}{\partial n} ds$$

Find the value of $\int_C \frac{\partial w}{\partial n} \; ds$ taken counterclockwise over the boundary C of the region R.

$$w = e^x \cos y + xy^3, \qquad R: 1 \leqq y \leqq 10 - x^2, \quad x \geqq 0$$

Solution.

Problem 10.4.16.

Using the following equation,

$$\iint_R \Delta w \; dx \; dy = \oint_C \frac{\partial w}{\partial n} \; ds$$

find the value of $\int_C \frac{\partial w}{\partial n} \; ds$ taken counterclockwise over the boundary C of the region R.

$$W = x^2 + y^2, \qquad C: x^2 + y^2 = 4$$

Confirm the answer by direct integration.

Solution.

Problem 10.4.19.

Show that $w = e^x \sin y$ satisfies Laplace's equation $\Delta w = 0$ and, using the following equation,

$$\iint_R \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] \; dx \; dy = \oint_C w \left(\frac{\partial w}{\partial n} \right) \; ds$$

integrate $w \left(\frac{\partial w}{\partial n} \right)$ counter-clockwise around the boundary curve C of the rectangle $0 \leqq x \leqq 2$, $0 \leqq y \leqq 5$.

Solution.

Problem 10.5.2.

Derive a parametric representation, $z = f(x, y)$ or $g(x, y, z) = 0$, by finding the **parameter curves** (curves $u = \text{const}$ and $v = \text{const}$) of the surface and a normal vector $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$ of the surface.

$$xy\text{-plane in polar coordinates } \mathbf{r}(u, v) = [u \cos v, u \sin v], \quad (\text{thus } u = r, v = \theta)$$

Solution.

Problem 10.5.3.

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

$$\text{Cone } \mathbf{r}(u,v) = [u\cos v, u\sin v, cu]$$

Solution.

Problem 10.5.5.

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

$$\text{Paraboloid of revolution } \mathbf{r}(u,v) = [u\cos v, u\sin v, u^2]$$

Solution.

Problem 10.5.7.

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

$$\text{Ellipsoid } \mathbf{r}(u,v) = [a\cos v\cos u, b\cos v\sin u, c\sin v]$$

Solution.

Problem 10.5.14.

Find a normal vector. Sketch the surface and parameter curves.

$$\text{Plane } 4x + 3y + 2z = 12$$

Solution.

Problem 10.5.15.

Find a normal vector. Sketch the surface and parameter curves.

$$\text{Cylinder of revolution } (x-2)^2 + (y+1)^2 = 25$$

Solution.

Problem 10.5.18.

Find a normal vector. Sketch the surface and parameter curves.

$$\text{Elliptic cone } z = \sqrt{x^2 + 4y^2}$$

Solution.

Problem 10.6.3.

Evaluate the flux integral, $\int_S \mathbf{F} \cdot \mathbf{n}\, dA$, for the given data. Describe the kind of surface.

$$\mathbf{F} = [0, x, 0], \qquad S : x^2 + y^2 + z^2 = 1, \quad x, y, z \geqq 0$$

Solution.

Problem 10.6.5.

Evaluate the flux integral for the given data. Describe the kind of surface.

$$\mathbf{F} = [x, y, z], \qquad S : \mathbf{r} = [u \cos v, u \sin v, u^2], \quad 0 \leqq u \leqq 4, \quad -\pi \leqq v \leqq \pi$$

Solution.

Problem 10.6.7.

Evaluate the flux integral for the given data. Describe the kind of surface.

$$\mathbf{F} = [0, \sin y, \cos z], \qquad S \text{ the cylinder } x = y^2, \text{ where } 0 \leqq y \leqq \frac{\pi}{4}, \quad 0 \leqq z \leqq y$$

Solution.

Problem 10.6.13.

Evaluate the surface integral, $\iint_S G(\mathbf{r})\, dA$, for the following data. Indicate the kind of surface.

$$G = x + y + z, \qquad z = x + 2y, \quad 0 \leqq x \leqq \pi, \quad 0 \leqq y \leqq x$$

Solution.

Problem 10.6.15.

Evaluate the surface integral for the following data. Indicate the kind of surface.

$$G = (1 + 9xz)^{3/2}, \qquad S : \mathbf{r} = [u, v, u^3], 0 \leqq u \leqq 1, \quad -2 \leqq v \leqq 2$$

Solution.