

Problem 10.2.3.

Show that the form under the integral sign is exact in the plane and evaluate the integral. Show the details of your work.

$$\int_{(\pi/2,\pi)}^{(\pi,0)} \left(\frac{1}{2} \cos \frac{1}{2}x \cos 2y \, dx - 2 \sin \frac{1}{2}x \sin 2y \, dy \right)$$

Solution.

The form is exact iff $\nabla \times \mathbf{F} = \mathbf{0}$, where $\mathbf{F} = [\frac{1}{2} \cos \frac{1}{2}x \cos 2y, -2 \sin \frac{1}{2}x \sin 2y]$. For 2D \mathbf{F} ,

$$\begin{aligned} \nabla \times \mathbf{F} &= (F_2)_x - (F_1)_y \\ &= \left(-\cos \frac{1}{2}x \sin 2y \right) - \left(-\cos \frac{1}{2}x \sin 2y \right) \\ &= 0 \end{aligned}$$

Therefore the form is **exact**. The integral takes the form,

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B df = f(B) - f(A)$$

Finding f , such that $\nabla f = \mathbf{F}$,

$$f = \int F_1 dx = \int \frac{1}{2} \cos \frac{1}{2}x \cos 2y \, dx = \sin \frac{1}{2}x \cos 2y + g(x)$$

$$f_y = -2 \sin \frac{1}{2}x \sin 2y + g_y = F_2 = -2 \sin \frac{1}{2}x \sin 2y \implies g_y = 0 \implies g = 0$$

$$f = \sin \frac{1}{2}x \cos 2y$$

The integral is then,

$$\begin{aligned} \int_{(\pi/2,\pi)}^{(\pi,0)} df &= f(\pi, 0) - f(\pi/2, \pi) \\ &= \boxed{1 - \frac{\sqrt{2}}{2}} \end{aligned}$$

Problem 10.2.5.

Show that the form under the integral sign is exact in space and evaluate the integral. Show the details of your work.

$$\int_{(0,0,\pi)}^{(2,1/2,\pi/2)} e^{xy} (y \sin z \, dx + x \sin z \, dy + \cos z \, dz)$$

Solution.

The form is exact iff $\nabla \times \mathbf{F} = \mathbf{0}$, where $\mathbf{F} = e^{xy} [y \sin z, x \sin z, \cos z]$.

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \begin{bmatrix} (F_3)_y - (F_2)_z \\ (F_1)_z - (F_3)_x \\ (F_2)_x - (F_1)_y \end{bmatrix} = \begin{bmatrix} (xe^{xy} \cos z) - (xe^{xy} \cos z) \\ (ye^{xy} \cos z) - (ye^{xy} \cos z) \\ (xye^{xy} \sin z + e^{xy} \sin z) - (xye^{xy} \sin z + e^{xy} \sin z) \end{bmatrix} = \mathbf{0}\end{aligned}$$

Therefore the form is **exact**. The integral takes the form,

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B df = f(B) - f(A)$$

Finding f , such that $\nabla f = \mathbf{F}$,

$$f = \int F_1 dx = \int e^{xy} y \sin z \, dx = e^{xy} \sin z + g(y, z)$$

$$f_y = xe^{xy} \sin z + g_y = F_2 = xe^{xy} \sin z \implies g_y = 0 \implies g = h(z)$$

$$f_z = e^{xy} \cos z + h'(z) = F_3 = e^{xy} \cos z \implies h'(z) = 0 \implies h(z) = 0$$

$$f = e^{xy} \sin z$$

The integral is then,

$$\begin{aligned}\int_{(0,0,\pi)}^{(2,1/2,\pi/2)} df &= f(2, 1/2, \pi/2) - f(0, 0, \pi) \\ &= e \sin(\pi/2) - e^0 \sin(\pi) \\ &= \boxed{e}\end{aligned}$$

Problem 10.2.13.

Check, and if independent, integrate from $(0, 0, 0)$ to (a, b, c) .

$$2e^{x^2} (x \cos 2y \, dx - \sin 2y \, dy)$$

Solution.

The integral is independent iff $\nabla \times \mathbf{F} = \mathbf{0}$, where $\mathbf{F} = 2e^{x^2} [x \cos 2y, -\sin 2y]$. For 2D \mathbf{F} ,

$$\begin{aligned}\nabla \times \mathbf{F} &= (F_2)_x - (F_1)_y \\ &= (-4xe^{x^2} \sin 2y) - (-4xe^{x^2} \sin 2y) \\ &= 0\end{aligned}$$

Therefore the integral is **independent of path**. The integral takes the form,

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B df = f(B) - f(A)$$

Finding f , such that $\nabla f = \mathbf{F}$,

$$f = \int F_1 dx = \int 2e^{x^2} x \cos 2y \, dx = e^{x^2} \cos 2y + g(y)$$

$$f_y = -2e^{x^2} \sin 2y + g_y = F_2 = -2e^{x^2} \sin 2y \implies g_y = 0 \implies g(y) = 0$$

$$f = e^{x^2} \cos 2y$$

The integral is then,

$$\begin{aligned} \int_{(0,0,0)}^{(a,b,c)} df &= f(a, b, c) - f(0, 0, 0) \\ &= e^{a^2} \cos 2b - e^0 \cos 0 \\ &= \boxed{e^{a^2} \cos 2b - 1} \end{aligned}$$

Problem 10.2.16.

Check, and if independent, integrate from $(0, 0, 0)$ to (a, b, c) .

$$e^y \, dx + (xe^y - e^z) \, dy - ye^z \, dz$$

Solution.

Problem 10.3.5.

Describe the region of integration and evaluate.

$$\int_0^1 \int_{x^2}^x (1 - 2xy) \, dy \, dx$$

Solution.

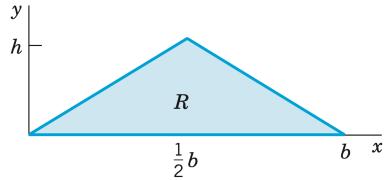
Problem 10.3.10.

Find the volume of the first octant region bounded by the coordinate planes and the surfaces $y = 1 - x^2$ and $z = 1 - x^2$. Sketch it.

Solution.

Problem 10.3.12.

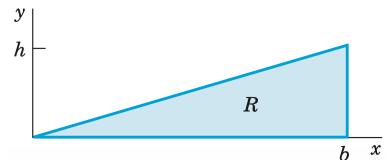
Find the center of gravity (\bar{x}, \bar{y}) of a mass of density $f(x, y) = 1$ in the given region R .



Solution.

Problem 10.3.17.

Find I_x , I_y , I_0 of a mass of density $f(x, y) = 1$ in the region R .



Solution.