Problem 10.4.3.

Evaluate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ counterclockwise around the boundary C of the region R by Green's theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla\times \mathbf{F} \cdot \mathbf{k}\ dA$$

where

$$\mathbf{F} = \left[x^2 e^y, y^2 e^x\right], \qquad R \text{ the rectangle with vertices } (0,\,0),\, (2,\,0),\, (2,\,3),\, (0,\,3)$$

Solution.

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \nabla\times \mathbf{F} \cdot \mathbf{k}\ dA \\
&= \int_0^3 \int_0^2 \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\ dx\ dy \\
&= \int_0^3 \int_0^2 \left(y^2 e^x - x^2 e^y\right)\ dx\ dy \\
&= \int_0^3 \left[y^2 e^x - \frac{1}{3}x^3 e^y\right]_{x=0}^{x=2}\ dy \\
&= \int_0^3 \left[\left(y^2 e^2 - \frac{1}{3}2^3 e^y\right) - \left(y^2 e^0 - \frac{1}{3}0^3 e^y\right)\right]\ dy \\
&= \int_0^3 \left[\left(y^2 e^2 - \frac{8}{3}e^y\right) - \left(y^2\right)\right]\ dy \\
&= \int_0^3 \left(y^2(e^2 - 1) - \frac{8}{3}e^y\right)\ dy \\
&= \frac{1}{3}y^3(e^2-1) - \frac{8}{3}e^y\bigg|_0^3 \\
&= \left(\frac{1}{3}3^3(e^2-1) - \frac{8}{3}e^3\right) - \left(\frac{1}{3}0^3(e^2-1) - \frac{8}{3}e^0\right) \\
&= \left(9(e^2-1) - \frac{8}{3}e^3\right) - \left(-\frac{8}{3}\right) \\
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \boxed{9(e^2-1) - \frac{8}{3}(e^3-1)}
\end{aligned}$$

Problem 10.4.7.

Evaluate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ counterclockwise around the boundary C of the region R by Green's theorem, where

$$\mathbf{F} = \nabla\left(x^3\cos^2(xy)\right), \qquad R: 1 \leqq y \leqq 2 - x^2$$

Solution.

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \nabla\times \mathbf{F} \cdot \mathbf{k}\ dA \\
&= \iint_R \cancelto{0}{\nabla\times(\nabla f)} \cdot \mathbf{k}\ dA \\
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \boxed{0}
\end{aligned}$$

Problem 10.4.15.

Using the following equation,

$$\iint_R \Delta\, w\ dA = \oint_C \frac{\partial w}{\partial n}\ ds$$

Find the value of $\int_C \frac{\partial w}{\partial n}\ ds$ taken counterclockwise over the boundary C of the region R.

$$w = e^x \cos y + xy^3, \qquad R : 1 \leqq y \leqq 10 - x^2, \quad x \geqq 0$$

Solution.

$$\begin{aligned}
\Delta\, w &= w_{xx} + w_{yy} \\
&= (e^x \cos y) + (-e^x \cos y + 6xy) \\
&= 6xy
\end{aligned}$$

$$\begin{aligned}
\oint_C \frac{\partial w}{\partial n}\ ds &= \iint_R \Delta\, w\ dA \\
&= \int_0^3 \int_1^{10-x^2} 6xy\ dy\ dx \\
&= \int_0^3 \left[3xy^2\right]_{y=1}^{y=10-x^2}\ dx \\
&= \int_0^3 \left(3x(10-x^2)^2 - 3x\right)\ dx \\
&= \int_0^3 \left(3x(100 - 20x^2 + x^4) - 3x\right)\ dx \\
&= \int_0^3 \left(297x - 60x^3 + 3x^5\right)\ dx \\
&= \frac{297}{2}x^2 - 15x^4 + \frac{1}{2}x^6\Big|_0^3 \\
&= 1336.5 - 1215 + 364.5 \\
&= \boxed{486}
\end{aligned}$$

Problem 10.4.16.

Using the following equation,

$$\iint_R \Delta\, w\ dx\ dy = \oint_C \frac{\partial w}{\partial n}\ ds$$

find the value of $\int_C \frac{\partial w}{\partial n}\ ds$ taken counterclockwise over the boundary C of the region R.

$$w = x^2 + y^2, \qquad C : x^2 + y^2 = 4$$

Solution.

$$\begin{aligned}\Delta w &= w_{xx} + w_{yy} \\ &= (2) + (2) \\ &= 4\end{aligned}$$

$$\begin{aligned}\oint_C \frac{\partial w}{\partial n}\, ds &= \iint_R \Delta w\, dA \\ &= 4 \iint_R dA \\ &= 4 \cdot \text{Area}(R) \\ &= 4 \cdot \pi \cdot 2^2 \\ &= \boxed{16\pi}\end{aligned}$$

Confirm the answer by direct integration.

$$C \implies \mathbf{r}(t) = [2\cos t, 2\sin t], \quad t \in [0, 2\pi]$$

The arc length of C is,

$$\begin{aligned}ds &= |\mathbf{r}'(t)|\, dt \\ &= \sqrt{(-2\sin t)^2 + (2\cos t)^2}\, dt \\ &= 2\, dt\end{aligned}$$

The normal vector is,

$$\begin{aligned}\mathbf{n} &= \frac{\nabla x^2 + y^2}{|\nabla x^2 + y^2|} \\ &= \frac{[2x, 2y]}{\sqrt{(2x)^2 + (2y)^2}} \\ &= \frac{[x, y]}{\sqrt{x^2 + y^2}}\end{aligned}$$

On C, $r = 2$,

$$\mathbf{n} = \frac{[x, y]}{2}$$

The integral of the normal derivative is then,

$$\begin{aligned}\oint_C \frac{\partial w}{\partial n}\, ds &= \oint_C \nabla w \cdot \mathbf{n}\, ds \\ &= \frac{1}{2} \oint_C [2x, 2y] \cdot [x, y]\, ds \\ &= \frac{1}{2} \oint_C 2x^2 + 2y^2\, ds \\ &= \oint_C x^2 + y^2\, ds\end{aligned}$$

On C, $x^2 + y^2 = 4$, and $ds = 2\, dt$ where $t \in [0, 2\pi]$,

$$\begin{aligned}\oint_C x^2 + y^2 \ ds &= 8 \int_0^{2\pi} \ dt \\ &= 8 \cdot 2\pi \\ &= \boxed{16\pi}\end{aligned}$$

Problem 10.4.19.

Show that $w = e^x \sin y$ satisfies Laplace's equation $\Delta w = 0$ and, using the following equation,

$$\iint_R \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] \ dx \ dy = \oint_C w \left(\frac{\partial w}{\partial n} \right) \ ds$$

integrate $w\left(\frac{\partial w}{\partial n}\right)$ counter-clockwise around the boundary curve C of the rectangle $0 \leqq x \leqq 2$, $0 \leqq y \leqq 5$.

Solution.

$$\nabla w = [e^x \sin y, e^x \cos y]$$

$$\begin{aligned}\Delta w &= (e^x \sin y) + (-e^x \sin y) \\ &= \boxed{0}\end{aligned}$$

$$\begin{aligned}\oint_C w \left(\frac{\partial w}{\partial n} \right) \ ds &= \iint_R \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] \ dx \ dy \\ &= \iint_R (e^x \sin y)^2 + (e^x \cos y)^2 \ dx \ dy \\ &= \iint_R \left(e^2 x[\sin^2 y + \cos^2 y \right) \ dx \ dy \\ &= \iint_R \left(e^2 x \right) \ dx \ dy \\ &= \int_0^5 \int_0^2 \left(e^2 x \right) \ dx \ dy \\ &= \frac{1}{2} \int_0^5 \left[e^2 x \right]_0^2 \ dy \\ &= \frac{1}{2}(e^4 - 1) \int_0^5 \ dy \\ &= \frac{1}{2}(e^4 - 1) \cdot 5 \\ &= \boxed{\frac{5}{2}(e^4 - 1)}\end{aligned}$$

Problem 10.5.2.

Derive a parametric representation, $z = f(x, y)$ or $g(x, y, z) = 0$, by finding the **parameter curves** (curves $u =$ const and $v =$ const) of the surface and a normal vector $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$ of the surface.

$$xy\text{-plane in polar coordinates } \mathbf{r}(u, v) = [u \cos v, u \sin v], \quad (\text{thus } u = r, v = \theta)$$

Solution.

$$\boxed{z = 0}$$

The parameter curves are,

- $u = \text{const} \rightarrow$ concentric circles
- $v = \text{const} \rightarrow$ radial straight lines

$$\begin{aligned}
\mathbf{r}_u &= [\cos v, \sin v] \\
\mathbf{r}_v &= [-u \sin v, u \cos v]
\end{aligned}$$

$$\begin{aligned}
\mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 0 \end{vmatrix} \\
&= (\cos v \cdot u \cos v - \sin v \cdot (-u \sin v))\, \mathbf{k} \\
&= (u \cos^2 v + u \sin^2 v)\mathbf{k} \\
&= u\mathbf{k} \\
\mathbf{N} &= \boxed{[0, 0, u]}
\end{aligned}$$

Problem 10.5.3.

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

$$\text{Cone } \mathbf{r}(u, v) = [u \cos v, u \sin v, cu]$$

Solution.

$$\begin{aligned}
x &= u \cos v \\
y &= u \sin v \\
z &= cu
\end{aligned}$$

$$\begin{aligned}
x^2 + y^2 + z^2 &= u^2(\cos^2 v + \sin^2 v + c^2) \\
x^2 + y^2 + z^2 &= u^2(1 + c^2)
\end{aligned}$$

But $u = \frac{z}{c}$,

$$\begin{aligned}
x^2 + y^2 + z^2 &= \frac{z^2}{c^2}(1 + c^2) \\
x^2 + y^2 + \cancel{z^2} &= \frac{z^2}{c^2} + \cancel{z^2}
\end{aligned}$$

$$\boxed{z = c\sqrt{x^2 + y^2}}$$

The parameter curves are,

- $u = \text{const} \rightarrow$ concentric circles
- $v = \text{const} \rightarrow$ straight lines through the origin

$$\begin{aligned}
\mathbf{r}_u &= [\cos v, \sin v, c] \\
\mathbf{r}_v &= [-u \sin v, u \cos v, 0]
\end{aligned}$$

$$\begin{aligned}
\mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & c \\ -u \sin v & u \cos v & 0 \end{vmatrix} \\
&= \left[(0 - cu \cos v), -(0 + cu \sin v), (u \cos^2 v + u \sin^2 v)\right] \\
&= [-cu \cos v, -cu \sin v, u] \\
&= \boxed{u[-c \cos v, -c \sin v, 1]}
\end{aligned}$$

Problem 10.5.5.

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

$$\text{Paraboloid of revolution } \mathbf{r}(u, v) = [u \cos v, u \sin v, u^2]$$

Solution.

$$\begin{aligned}
x &= u \cos v \\
y &= u \sin v \\
z &= u^2
\end{aligned}$$

$$\begin{aligned}
x^2 + y^2 + z &= u^2 \cos^2 v + u^2 \sin^2 v + u^2 \\
&= u^2(\cos^2 v + \sin^2 v + 1) \\
&= u^2(1 + 1) \\
&= 2u^2
\end{aligned}$$

But $u^2 = z$,

$$x^2 + y^2 + z = 2z$$

$$\boxed{z = x^2 + y^2}$$

The parameter curves are,

- $u = \text{const} \rightarrow$ concentric circles at a given height
- $v = \text{const} \rightarrow$ parabolas with vertex at the origin

$$\begin{aligned}
\mathbf{r}_u &= [\cos v, \sin v, 2u] \\
\mathbf{r}_v &= [-u \sin v, u \cos v, 0]
\end{aligned}$$

$$\begin{aligned}
\mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} \\
&= \left[(0 - 2u^2\cos v), -(0 + 2u^2\sin v), (u\cos^2 v + u\sin^2 v)\right] \\
&= \boxed{\left[-2u^2\cos v, -2u^2\sin v, u\right]}
\end{aligned}$$

Problem 10.5.7.

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

$$\text{Ellipsoid } \mathbf{r}(u, v) = [a\cos v\cos u, b\cos v\sin u, c\sin v]$$

Solution.

$$\begin{aligned}
x &= a\cos v\cos u \\
y &= b\cos v\sin u \\
z &= c\sin v
\end{aligned}$$

$$\begin{aligned}
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= \cos^2 v\cos^2 u + \cos^2 v\sin^2 u + \sin^2 v \\
&= (\cos^2 u + \sin^2 u)\cos^2 v + \sin^2 v \\
&= \cos^2 v + \sin^2 v \\
&= 1
\end{aligned}$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$$

The parameter curves are,

- $u = \text{const} \rightarrow \text{elipses}$
- $v = \text{const} \rightarrow \text{elipses}$

$$\begin{aligned}
\mathbf{r}_u &= [-a\cos v\sin u, b\cos v\cos u, 0] \\
\mathbf{r}_v &= [-a\sin v\cos u, -b\sin v\sin u, c\cos v]
\end{aligned}$$

$$\begin{aligned}
\mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\cos v\sin u & b\cos v\cos u & 0 \\ -a\sin v\cos u & -b\sin v\sin u & c\cos v \end{vmatrix} \\
&= \left[(bc\cos^2 v\cos u - 0), -(-ac\cos^2 v\sin u - 0), (ab\cos v\sin v\sin^2 u + ab\cos v\sin v\cos^2 u)\right] \\
&= \left[(bc\cos^2 v\cos u), (ac\cos^2 v\sin u), (ab\cos v\sin v(\sin^2 u + \cos^2 u))\right] \\
&= \boxed{\left[(bc\cos^2 v\cos u), (ac\cos^2 v\sin u), (ab\cos v\sin v)\right]}
\end{aligned}$$

Problem 10.5.14.

Find a normal vector. Sketch the surface and parameter curves.

$$\text{Plane } 4x + 3y + 2z = 12$$

Solution.

Problem 10.5.15.

Find a normal vector. Sketch the surface and parameter curves.

$$\text{Cylinder of revolution } (x-2)^2 + (y+1)^2 = 25$$

Solution.

Problem 10.5.18.

Find a normal vector. Sketch the surface and parameter curves.

$$\text{Elliptic cone } z = \sqrt{x^2 + 4y^2}$$

Solution.

Problem 10.6.3.

Evaluate the flux integral, $\int_S \mathbf{F} \cdot \mathbf{n}\, dA$, for the given data. Describe the kind of surface.

$$\mathbf{F} = [0, x, 0], \qquad S : x^2 + y^2 + z^2 = 1, \quad x, y, z \geqq 0$$

Solution.

Problem 10.6.5.

Evaluate the flux integral for the given data. Describe the kind of surface.

$$\mathbf{F} = [x, y, z], \qquad S : \mathbf{r} = [u \cos v, u \sin v, u^2], \quad 0 \leqq u \leqq 4, \quad -\pi \leqq v \leqq \pi$$

Solution.

Problem 10.6.7.

Evaluate the flux integral for the given data. Describe the kind of surface.

$$\mathbf{F} = [0, \sin y, \cos z], \qquad S \text{ the cylinder } x = y^2, \text{ where } 0 \leqq y \leqq \frac{\pi}{4}, \quad 0 \leqq z \leqq y$$

Solution.

Problem 10.6.13.

Evaluate the surface integral, $\iint_S G(\mathbf{r})\ dA$, for the following data. Indicate the kind of surface.

$$G = x + y + z, \qquad z = x + 2y, \quad 0 \leqq x \leqq \pi, \quad 0 \leqq y \leqq x$$

Solution.

Problem 10.6.15.

Evaluate the surface integral for the following data. Indicate the kind of surface.

$$G = (1 + 9xz)^{3/2}, \qquad S : \mathbf{r} = [u, v, u^3], 0 \leqq u \leqq 1, \quad -2 \leqq v \leqq 2$$

Solution.