Problem 10.7.11

Evaluate the surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}\;dA$ by the divergence theorem:

$$\boxed{\iint_S \mathbf{F}\cdot\mathbf{n}\;dA = \iiint_T \nabla\cdot\mathbf{F}\;dV}$$

$$\mathbf{F} = [e^x, e^y, e^z]\,,\quad S \text{ the surface of the cube } |x| \leqq 1,\ |y| \leqq 1,\ |z| \leqq 1$$

Solution

$$\nabla\cdot\mathbf{F} = e^x + e^y + e^z$$

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\;dA &= \iiint_T \nabla\cdot\mathbf{F}\;dV \\
&= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (e^x + e^y + e^z)\;dx\;dy\;dz \\
&= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} e^x\;dx\;dy\;dz + \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} e^y\;dx\;dy\;dz + \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} e^z\;dx\;dy\;dz \\
&= 3\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} e^x\;dx\;dy\;dz \\
&= 12\int_{-1}^{1} e^x\;dx \\
&= 12e^x\Big|_{-1}^{1} \\
&= \boxed{12(e - e^{-1})}
\end{aligned}$$

Problem 10.7.13

Evaluate the surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}\;dA$ by the divergence theorem.

$$\mathbf{F} = [\sin y, \cos x, \cos z]\,,\quad S \text{ the surface of } x^2 + y^2 \leqq 4,\ |z| \leqq 2 \text{ (a cylinder and two disks!)}$$

Solution

$$\begin{aligned}
\nabla\cdot\mathbf{F} &= 0 + 0 - \sin z \\
&= -\sin z
\end{aligned}$$

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\;dA &= \iiint_T \nabla\cdot\mathbf{F}\;dV \\
&= \int_{-2}^{2}\int_{0}^{2\pi}\int_{0}^{2} (-\sin z) r\;dr\;d\theta\;dz \\
&= \int_0^2 r\;dr \cdot \int_0^{2\pi} d\theta \cdot \int_{-2}^{2} (-\sin z)\;dz \\
&= \left[\frac{r^2}{2}\right]_0^2 \cdot (2\pi)\cdot [\cos z]_{-2}^{2} \\
&= 4\pi(\cos(2) - \cos(-2)) \\
&= \boxed{0}
\end{aligned}$$

Problem 10.7.17

Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dA$ by the divergence theorem.

$$\mathbf{F} = \left[x^2, y^2, z^2\right], \quad S \text{ the surface of the cone } x^2 + y^2 \leqq z^2,\ 0 \leqq z \leqq h$$

Solution

$$\nabla \cdot \mathbf{F} = 2x + 2y + 2z$$

In cylindrical coordinates,

$$\nabla \cdot \mathbf{F} = 2(r\cos\theta + r\sin\theta + z)$$

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, dA &= \iiint_T \nabla \cdot \mathbf{F}\, dV \\
&= 2\int_0^h \int_0^{2\pi} \int_0^z (r\cos\theta + r\sin\theta + z) r\ dr\ d\theta\ dz \\
&= 2\int_0^h \int_0^{2\pi} \int_0^z (r^2\cos\theta + r^2\sin\theta + rz\ dr\ d\theta\ dz \\
&= 2\int_0^h \int_0^{2\pi} \left[\frac{r^3}{3}\cos\theta + \frac{r^3}{3}\sin\theta + \frac{r^2}{2}z\right]_{r=0}^{r=z} d\theta\ dz \\
&= 2\int_0^h \int_0^{2\pi} z^3 \left(\frac{1}{3}\cos\theta + \frac{1}{3}\sin\theta + \frac{1}{2}\right) d\theta\ dz \\
&= 2\int_0^h z^3\ dz \cdot \int_0^{2\pi} \left(\frac{1}{3}\cos\theta + \frac{1}{3}\sin\theta + \frac{1}{2}\right) d\theta \\
&= \frac{1}{2}\left[z^4\right]_0^h \cdot \left[\frac{1}{3}\sin\theta - \frac{1}{3}\cos\theta + \frac{1}{2}\theta\right]_0^{2\pi} \\
&= \boxed{\frac{\pi h^4}{2}}
\end{aligned}$$

Problem 10.7.22

Given a mass of density 1 in a region T of space, find the moment of intertia about the x-axis,

$$I_x = \iiint_T \left(y^2 + z^2\right)\ dV$$

$$T: \text{The paraboloid } y^2 + z^2 \leqq x,\ 0 \leqq x \leqq h$$

Solution

In cylindrical coordinates along the x-axis, the integrand becomes r^2 and

$$T: \text{The paraboloid } r^2 \leqq x,\ 0 \leqq x \leqq h$$

$$
\begin{aligned}
I_x &= \iiint_T \left(y^2 + z^2\right)\ dV \\
&= \iiint_T \left(r^2\right) r\ dr\ d\theta\ dx \\
&= \int_0^h \int_0^{2\pi} \int_0^{\sqrt{x}} \left(r^2\right) r\ dr\ d\theta\ dx \\
&= \int_0^h \int_0^{2\pi} \int_0^{\sqrt{x}} \left(r^3\right)\ dr\ d\theta\ dx \\
&= \frac{1}{4} \int_0^h \int_0^{2\pi} \left(x^2\right)\ d\theta\ dx \\
&= \frac{1}{4} \int_0^h \left(x^2\right)\ dx \cdot \int_0^{2\pi}\ d\theta \\
&= \frac{\pi}{6} x^3 \Big|_0^h \\
&= \boxed{\frac{\pi h^3}{6}}
\end{aligned}
$$

Problem 10.8.1. Harmonic functions.

Theorem 1. A Basic Property of Harmonic Functions

Let $f(x, y, z)$ be a harmonic function in some domain D is space. Let S be any piecewise smooth closed orientable surface in D whose entire region it encloses belongs to D. Then the integral of the normal derivative of f taken over S is zero.

Verify Theorem 1 for $f = 2z^2 - x^2 - y^2$ and S the surface of the box $0 \leqq x \leqq a,\ 0 \leqq y \leqq b,\ 0 \leqq z \leqq c$.

Solution

$$
\begin{aligned}
\Delta f &= f_{xx} + f_{yy} + f_{zz} \\
&= (-2) + (-2) + (4) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\iint_S \frac{\partial f}{\partial n}\ dA &= \iiint_T \cancelto{0}{\Delta f}\, dV \\
&= \boxed{0}
\end{aligned}
$$

Problem 10.8.3. Green's First Identity

$$
\boxed{\iiint_T \left(f\, \Delta\, g + \nabla f \cdot \nabla g\right)\ dV = \iint_S f \frac{\partial g}{\partial n}\ dA}
$$

Verify for $f = 4y^2$, $g = x^2$, S the surface of the "unit cube" $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, $0 \leqq z \leqq 1$. What are the assumptions of f and g? Must f and g be harmonic?

Solution

	f	g
	$4y^2$	x^2
∇	$[0, 8y]$	$[2x, 0]$
Δ	8	2

$$\begin{aligned}\iiint_T (f\,\Delta\, g + \nabla\, f \cdot \nabla\, g)\ dV &= \int_0^1 \int_0^1 \int_0^1 \left(8y^2\right)\ dx\ dy\ dz \\ &= \frac{8y^3}{3}\bigg|_0^1 \\ &= \boxed{\frac{8}{3}}\end{aligned}$$

$$\iint_S f\frac{\partial g}{\partial n}\ dA = \iint_S f\,(\nabla\, g \cdot \mathbf{n})\ dA$$

Integrating over the cube surface, only the $x = 1$ face is non-zero,

$$\begin{aligned}\iint_S f\,(\nabla\, g \cdot \mathbf{n})\ dA &= \int_0^1 f g_x(x = 1)\ dy \\ &= \int_0^1 (4y^2)(2)\ dy \\ &= 8\int_0^1 y^2\ dy \\ &= 8\left[\frac{y^3}{3}\right]_0^1 \\ &= \boxed{\frac{8}{3}}\end{aligned}$$

The functions do not need to be harmonic, but the divergence theorem assumptions must hold. The functions need to be twice differentiable and S must be piecewise smooth.

Problem 10.8.5. Green's Second Identity

$$\boxed{\iiint_T (f\,\Delta\, g - g\,\Delta\, f)\ dV = \iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)\ dA}$$

Verify for $f = 6y^2$, $g = 2x^2$, S the surface of the "unit cube" $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, $0 \leqq z \leqq 1$.

Solution

	f	g
	$6y^2$	$2x^2$
∇	$[0, 12y]$	$[4x, 0]$
Δ	12	4

$$\begin{aligned}\iiint_T (f\,\Delta\, g - g\,\Delta\, f)\ dV &= 24\int_0^1\int_0^1\int_0^1 \left(y^2 - x^2\right)\ dx\ dy\ dz \\ &= 24\left(\int_0^1 y^2\ dy - \int_0^1 x^2\ dx\right)\cdot\int_0^1\ dz \\ &= 8\left(y^3\Big|_0^1 - x^3\Big|_0^1\right) \\ &= \boxed{0}\end{aligned}$$

$$\iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)\ dA = \iint_S f\,(\nabla\, g\cdot\mathbf{n}) - g\,(\nabla\, f\cdot\mathbf{n})\ dA$$

Integrating over the cube surface, only the $x = 1$ face (first term) and $y = 1$ face (second term) are non-zero,

$$\begin{aligned}\iint_S f\,(\nabla\, g\cdot\mathbf{n})\ dA &= \int_0^1 fg_x(x=1)\ dy - \int_0^1 gf_y(y=1)\ dx \\ &= \int_0^1 (6y^2)(4)\ dy - \int_0^1 (2x^2)(12)\ dx \\ &= 24\int_0^1 y^2\ dy - 24\int_0^1 x^2\ dx \\ &= 8\left(y^3\Big|_0^1 - x^3\Big|_0^1\right) \\ &= \boxed{0}\end{aligned}$$

Problem 10.8.7

Use the divergence theorem, assuming that the assumptions on T and S are satisfied.

Show that a region T with boundary surface S has the volume

$$V = \iint_S x\ dy\ dz = \iint_S y\ dz\ dx = \iint_S z\ dx\ dy = \frac{1}{3}\iint_S (x\ dy\ dz + y\ dz\ dx + z\ dx\ dy)$$

Solution

Take $\mathbf{F} = [x, 0, 0]$, therefore $\nabla\cdot\mathbf{F} = 1$ and

$$\begin{aligned}V = \iiint_T \nabla\cdot\mathbf{F}\ dV &= \iint_S \mathbf{F}\cdot\mathbf{n}\ dA \\ V &= \iint_S x\ dy\ dz\end{aligned}$$

Take $\mathbf{F} = [0, y, 0]$, therefore $\nabla\cdot\mathbf{F} = 1$ and

$$\begin{aligned}V = \iiint_T \nabla\cdot\mathbf{F}\ dV &= \iint_S \mathbf{F}\cdot\mathbf{n}\ dA \\ V &= \iint_S y\ dx\ dz\end{aligned}$$

Take $\mathbf{F} = [0, 0, z]$,

$$V = \iiint_T \nabla\cdot \mathbf{F}\ dV = \iint_S \mathbf{F}\cdot\mathbf{n}\ dA$$
$$V = \iint_S z\ dx\ dy$$

Summing the above,

$$3V = \iint_S (x\ dy\ dz + y\ dx\ dz + z\ dx\ dy)$$
$$V = \frac{1}{3}\iint_S (x\ dy\ dz + y\ dz\ dx + z\ dx\ dy)$$

Problem 10.9.3

Evaluate the surface integral $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dA$ directly for the given $\mathbf{F}$ and S.

$$\mathbf{F} = [e^{-z}, e^{-z}\cos y, e^{-z}\sin y], \quad S: z = \frac{y^2}{2},\ -1 \leqq x \leqq 1,\ 0 \leqq y \leqq 1$$

Solution

$$\begin{aligned}\nabla\times\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{-z} & e^{-z}\cos y & e^{-z}\sin y \end{vmatrix} \\ &= \left[\left(e^{-z}\cos y + e^{-z}\cos y\right), -\left(0 + e^{-z}\right), (0-0)\right] \\ &= \left[2e^{-z}\cos y, -e^{-z}, 0\right]\end{aligned}$$

$$S: \mathbf{r}(x,y) = [x, y, \frac{y^2}{2}]$$

$$\begin{aligned}\mathbf{n} &= \mathbf{r}_x \times \mathbf{r}_y \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & y \end{vmatrix} \\ &= [0, -y, 1]\end{aligned}$$

$$\begin{aligned}\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dA &= \iint_S \left[2e^{-z}\cos y, -e^{-z}, 0\right]\cdot[0,-y,1]\ dx\ dy \\ &= \iint_S \left(ye^{-z} + 0\right)\ dx\ dy \\ &= \int_{-1}^{1} dx \cdot \int_0^1 ye^{-\frac{y^2}{2}}\ dy \\ &= 2\int_0^1 ye^{-\frac{y^2}{2}}\ dy \\ &= -2e^{-\frac{y^2}{2}}\Big|_0^1 \\ &= \boxed{2(1-e^{-\frac{1}{2}})}\end{aligned}$$

Problem 10.9.5

Evaluate the surface integral $\iint_S (\nabla\times \mathbf{F})\cdot \mathbf{n}\ dA$ directly for the given $\mathbf{F}$ and S.

$$\mathbf{F} = [z^2, \frac{3}{2}x, 0], \quad S : 0 \leqq x \leqq a,\ 0 \leqq y \leqq a,\ z = 1$$

Solution

$$\begin{aligned}
\nabla\times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & \frac{3}{2}x & 0 \end{vmatrix} \\
&= \left[(0-0), -(0-2z), \left(\frac{3}{2}-0\right)\right] \\
&= \left[0, 2z, \frac{3}{2}\right]
\end{aligned}$$

The surface is a flat plane and therefore $\mathbf{n} = [0, 0, 1]$.

$$\begin{aligned}
\iint_S (\nabla\times \mathbf{F})\cdot \mathbf{n}\ dA &= \iint_S \left[0, 2z, \frac{3}{2}\right]\cdot [0,0,1]\ dx\ dy \\
&= \iint_S \frac{3}{2}\ dx\ dy \\
&= \frac{3}{2}\int_0^a\ dx \cdot \int_0^a\ dy \\
&= \boxed{\frac{3a^2}{2}}
\end{aligned}$$

Problem 10.9.13

Calculate $\oint_C \mathbf{F}\cdot \mathbf{r}'(s)\ ds$ by Stokes's theorem,

$$\boxed{\iint_S (\nabla\times \mathbf{F})\cdot \mathbf{n}\ dA = \oint_C \mathbf{F}\cdot \mathbf{r}'(s)\ ds}$$

for the given $\mathbf{F}$ and C. Assume the Cartesian coordinates to be right-handed and the z-component of the surface normal to be nonnegative.

$$\mathbf{F} = [-5y, 4x, z], \quad C \text{ the circle } x^2 + y^2 = 16,\ z = 4$$

Solution

$$\begin{aligned}
\nabla\times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -5y & 4x & z \end{vmatrix} \\
&= [(0-0), -(0-0), (4+5),] \quad = [0,0,9]
\end{aligned}$$

C is the boundary of a flat disc and therefore $\mathbf{n} = [0, 0, 1]$.

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{r}'(s)\ ds &= \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dA \\
&= 9\iint_S dA \\
&= 9\cdot\text{Area}(S) \\
&= 9\cdot 16\pi \\
&= \boxed{144\pi}
\end{aligned}$$

Problem 10.9.15

Calculate $\oint_C \mathbf{F}\cdot\mathbf{r}'(s)\ ds$ by Stokes's theorem for the given $\mathbf{F}$ and C. Assume the Cartesian coordinates to be right-handed and the z-component of the surface normal to be nonnegative.

$$\mathbf{F} = [z^3, x^3, y^3],\ \text{around the triangle with vertices } (0,0,0),\ (1,0,0),\ (1,1,0)$$

Solution

$$\begin{aligned}
\nabla\times\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^3 & x^3 & y^3 \end{vmatrix} \\
&= \left[\left(3y^2-0\right), -\left(0-3z^2\right), \left(3x^2-0\right),\right] \\
&= 3[y^2, z^2, x^2]
\end{aligned}$$

C is the boundary of a flat triangle and therefore $\mathbf{n} = [0,0,1]$.

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{r}'(s)\ ds &= \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dA \\
&= 3\int_0^1\int_0^x [y^2, z^2, x^2]\cdot[0,0,1]\ dy\ dx \\
&= 3\int_0^1\int_0^x x^2\ dy\ dx \\
&= 3\int_0^1 x^3\ dx \\
&= \boxed{\frac{3}{4}}
\end{aligned}$$

Problem 10.9.19

Calculate $\oint_C \mathbf{F}\cdot\mathbf{r}'(s)\ ds$ by Stokes's theorem for the given $\mathbf{F}$ and C. Assume the Cartesian coordinates to be right-handed and the z-component of the surface normal to be nonnegative.

$$\mathbf{F} = [z, e^z, 0],\quad C \text{ the boundary curve of the portion of the cone } z = \sqrt{x^2+y^2},\ x \geqq 0,\ y \geqq 0, 0 \leqq z \leqq 1$$

Solution

$$\begin{aligned}\nabla\times\mathbf{F} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ z & e^z & 0\end{vmatrix}\\ &= \left[(0-e^z)\,,-(0-1)\,,(0-0)\,,\right]\\ &= [-e^z,1,0]\end{aligned}$$

$$S:\mathbf{r}(r,\theta)=r[\cos\theta,\sin\theta,1]\quad r\in[0,1],\ \theta\in\left[0,\frac{\pi}{2}\right]$$

$$\begin{aligned}\mathbf{n} &= \mathbf{r}_r\times\mathbf{r}_\theta\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\theta & \sin\theta & 1\\ -r\sin\theta & r\cos\theta & 0\end{vmatrix}\\ &= \left[(0-r\cos\theta)\,,-(0+r\sin\theta)\,,\left(r\cos^2\theta+r\sin^2\theta\right)\right]\\ &= r\left[-\cos\theta,-\sin\theta,1\right]\end{aligned}$$

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathbf{r}'(s)\ ds &= \iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}\ dA\\ &= \int_0^{\frac{\pi}{2}}\int_0^1[-e^r,1,0]\cdot r[-\cos\theta,-\sin\theta,1]\ dr\ d\theta\\ &= \int_0^{\frac{\pi}{2}}\int_0^1 r\left(e^r\cos\theta-\sin\theta\right)\ dr\ d\theta\end{aligned}$$

$\int_0^1 re^r\ dr = re^r - e^r\big|_0^1 = 1$

$\int_0^1 r\ dr = \frac{r}{2}\big|_0^1 = \frac{1}{2}$

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathbf{r}'(s)\ ds &= \int_0^{\frac{\pi}{2}}\cos\theta-\frac{1}{2}\sin\theta\ d\theta\\ &= \sin\theta+\frac{1}{2}\cos\theta\Big|_0^{\frac{\pi}{2}}\\ &= \boxed{\frac{1}{2}}\end{aligned}$$