

**Problem 10.4.3.**

Evaluate  $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$  counterclockwise around the boundary  $C$  of the region  $R$  by Green's theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA$$

where

$$\mathbf{F} = [x^2 e^y, y^2 e^x], \quad R \text{ the rectangle with vertices } (0, 0), (2, 0), (2, 3), (0, 3)$$

**Solution.**

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA \\ &= \int_0^3 \int_0^2 \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx \, dy \\ &= \int_0^3 \int_0^2 (y^2 e^x - x^2 e^y) dx \, dy \\ &= \int_0^3 \left[ y^2 e^x - \frac{1}{3} x^3 e^y \right]_{x=0}^{x=2} dy \\ &= \int_0^3 \left[ \left( y^2 e^2 - \frac{1}{3} 2^3 e^y \right) - \left( y^2 e^0 - \frac{1}{3} 0^3 e^y \right) \right] dy \\ &= \int_0^3 \left[ \left( y^2 e^2 - \frac{8}{3} e^y \right) - (y^2) \right] dy \\ &= \int_0^3 \left( y^2 (e^2 - 1) - \frac{8}{3} e^y \right) dy \\ &= \frac{1}{3} y^3 (e^2 - 1) - \frac{8}{3} e^y \Big|_0^3 \\ &= \left( \frac{1}{3} 3^3 (e^2 - 1) - \frac{8}{3} e^3 \right) - \left( \frac{1}{3} 0^3 (e^2 - 1) - \frac{8}{3} e^0 \right) \\ &= \left( 9(e^2 - 1) - \frac{8}{3} e^3 \right) - \left( -\frac{8}{3} \right) \\ \oint_C \mathbf{F} \cdot d\mathbf{r} &= \boxed{9(e^2 - 1) - \frac{8}{3}(e^3 - 1)} \end{aligned}$$

**Problem 10.4.7.**

Evaluate  $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$  counterclockwise around the boundary  $C$  of the region  $R$  by Green's theorem, where

$$\mathbf{F} = \nabla (x^3 \cos^2(xy)), \quad R : 1 \leq y \leq 2 - x^2$$

**Solution.**

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA \\ &= \iint_R \nabla \times (\nabla f) \cdot \mathbf{k} \, dA \\ \oint_C \mathbf{F} \cdot d\mathbf{r} &= \boxed{0} \end{aligned}$$

**Problem 10.4.15.**

Using the following equation,

$$\iint_R \Delta w \, dA = \oint_C \frac{\partial w}{\partial n} \, ds$$

Find the value of  $\oint_C \frac{\partial w}{\partial n} \, ds$  taken counterclockwise over the boundary  $C$  of the region  $R$ .

$$w = e^x \cos y + xy^3, \quad R : 1 \leq y \leq 10 - x^2, \quad x \geq 0$$

**Solution.**

$$\begin{aligned} \Delta w &= w_{xx} + w_{yy} \\ &= (e^x \cos y) + (-e^x \cos y + 6xy) \\ &= 6xy \end{aligned}$$

$$\begin{aligned} \oint_C \frac{\partial w}{\partial n} \, ds &= \iint_R \Delta w \, dA \\ &= \int_0^3 \int_1^{10-x^2} 6xy \, dy \, dx \\ &= \int_0^3 [3xy^2]_{y=1}^{y=10-x^2} \, dx \\ &= \int_0^3 (3x(10-x^2)^2 - 3x) \, dx \\ &= \int_0^3 (3x(100 - 20x^2 + x^4) - 3x) \, dx \\ &= \int_0^3 (297x - 60x^3 + 3x^5) \, dx \\ &= \left. \frac{297}{2}x^2 - 15x^4 + \frac{1}{2}x^6 \right|_0^3 \\ &= 1336.5 - 1215 + 364.5 \\ &= \boxed{486} \end{aligned}$$

**Problem 10.4.16.**

Using the following equation,

$$\iint_R \Delta w \, dx \, dy = \oint_C \frac{\partial w}{\partial n} \, ds$$

find the value of  $\oint_C \frac{\partial w}{\partial n} \, ds$  taken counterclockwise over the boundary  $C$  of the region  $R$ .

$$w = x^2 + y^2, \quad C : x^2 + y^2 = 4$$

**Solution.**

$$\begin{aligned}\Delta w &= w_{xx} + w_{yy} \\ &= (2) + (2) \\ &= 4\end{aligned}$$

$$\begin{aligned}\oint_C \frac{\partial w}{\partial n} ds &= \iint_R \Delta w dA \\ &= 4 \iint_R dA \\ &= 4 \cdot \text{Area}(R) \\ &= 4 \cdot \pi \cdot 2^2 \\ &= \boxed{16\pi}\end{aligned}$$

**Confirm the answer by direct integration.**

$$C \implies \mathbf{r}(t) = [2 \cos t, 2 \sin t], \quad t \in [0, 2\pi]$$

The arc length element of  $C$  is,

$$\begin{aligned}ds &= |\mathbf{r}'(t)| dt \\ &= \sqrt{(-2 \sin t)^2 + (2 \cos t)^2} dt \\ &= 2 dt\end{aligned}$$

The normal vector is,

$$\begin{aligned}\mathbf{n} &= \frac{\nabla x^2 + y^2}{|\nabla x^2 + y^2|} \\ &= \frac{[2x, 2y]}{\sqrt{(2x)^2 + (2y)^2}} \\ &= \frac{[x, y]}{\sqrt{x^2 + y^2}}\end{aligned}$$

On  $C$ ,  $r = 2$ ,

$$\mathbf{n} = \frac{[x, y]}{2}$$

The integral of the normal derivative is then,

$$\begin{aligned}\oint_C \frac{\partial w}{\partial n} ds &= \oint_C \nabla w \cdot \mathbf{n} ds \\ &= \frac{1}{2} \oint_C [2x, 2y] \cdot [x, y] ds \\ &= \frac{1}{2} \oint_C 2x^2 + 2y^2 ds \\ &= \oint_C x^2 + y^2 ds\end{aligned}$$

On  $C$ ,  $x^2 + y^2 = 4$ , and  $ds = 2 dt$  where  $t \in [0, 2\pi]$ ,

$$\begin{aligned}
 \oint_C x^2 + y^2 \, ds &= 8 \int_0^{2\pi} dt \\
 &= 8 \cdot 2\pi \\
 &= \boxed{16\pi}
 \end{aligned}$$

### Problem 10.4.19.

Show that  $w = e^x \sin y$  satisfies Laplace's equation  $\Delta w = 0$  and, using the following equation,

$$\iint_R \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] dx \, dy = \oint_C w \left( \frac{\partial w}{\partial n} \right) ds$$

integrate  $w \left( \frac{\partial w}{\partial n} \right)$  counter-clockwise around the boundary curve  $C$  of the rectangle  $0 \leq x \leq 2$ ,  $0 \leq y \leq 5$ .

### Solution.

$$\nabla w = [e^x \sin y, e^x \cos y]$$

$$\begin{aligned}
 \Delta w &= (e^x \sin y) + (-e^x \sin y) \\
 &= \boxed{0}
 \end{aligned}$$

$$\begin{aligned}
 \oint_C w \left( \frac{\partial w}{\partial n} \right) ds &= \iint_R \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] dx \, dy \\
 &= \iint_R (e^x \sin y)^2 + (e^x \cos y)^2 \, dx \, dy \\
 &= \iint_R (e^2 x [\sin^2 y + \cos^2 y]) \, dx \, dy \\
 &= \iint_R (e^2 x) \, dx \, dy \\
 &= \int_0^5 \int_0^2 (e^2 x) \, dx \, dy \\
 &= \frac{1}{2} \int_0^5 [e^2 x]_0^2 \, dy \\
 &= \frac{1}{2} (e^4 - 1) \int_0^5 dy \\
 &= \frac{1}{2} (e^4 - 1) \cdot 5 \\
 &= \boxed{\frac{5}{2} (e^4 - 1)}
 \end{aligned}$$

### Problem 10.5.2.

Derive a parametric representation,  $z = f(x, y)$  or  $g(x, y, z) = 0$ , by finding the **parameter curves** (curves  $u = \text{const}$  and  $v = \text{const}$ ) of the surface and a normal vector  $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$  of the surface.

$xy$ -plane in polar coordinates  $\mathbf{r}(u, v) = [u \cos v, u \sin v]$ , (thus  $u = r, v = \theta$ )

**Solution.**

$$\boxed{z = 0}$$

The parameter curves are,

- $u = \text{const} \rightarrow$  concentric circles
- $v = \text{const} \rightarrow$  radial straight lines

$$\begin{aligned}\mathbf{r}_u &= [\cos v, \sin v] \\ \mathbf{r}_v &= [-u \sin v, u \cos v]\end{aligned}$$

$$\begin{aligned}\mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 0 \end{vmatrix} \\ &= (\cos v \cdot u \cos v - \sin v \cdot (-u \sin v)) \mathbf{k} \\ &= (u \cos^2 v + u \sin^2 v) \mathbf{k} \\ &= u \mathbf{k} \\ \mathbf{N} &= \boxed{[0, 0, u]}\end{aligned}$$

### Problem 10.5.3.

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

$$\text{Cone } \mathbf{r}(u, v) = [u \cos v, u \sin v, cu]$$

**Solution.**

$$\begin{aligned}x &= u \cos v \\ y &= u \sin v \\ z &= cu\end{aligned}$$

$$\begin{aligned}x^2 + y^2 + z^2 &= u^2(\cos^2 v + \sin^2 v + c^2) \\ x^2 + y^2 + z^2 &= u^2(1 + c^2)\end{aligned}$$

But  $u = \frac{z}{c}$ ,

$$\begin{aligned}x^2 + y^2 + z^2 &= \frac{z^2}{c^2}(1 + c^2) \\ x^2 + y^2 + \cancel{z^2} &= \frac{z^2}{c^2} + \cancel{z^2}\end{aligned}$$

$$\boxed{z = c\sqrt{x^2 + y^2}}$$

The parameter curves are,

- $u = \text{const} \rightarrow$  concentric circles
- $v = \text{const} \rightarrow$  straight lines through the origin

$$\begin{aligned}\mathbf{r}_u &= [\cos v, \sin v, c] \\ \mathbf{r}_v &= [-u \sin v, u \cos v, 0]\end{aligned}$$

$$\begin{aligned}\mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & c \\ -u \sin v & u \cos v & 0 \end{vmatrix} \\ &= [(0 - cu \cos v), -(0 + cu \sin v), (u \cos^2 v + u \sin^2 v)] \\ &= [-cu \cos v, -cu \sin v, u] \\ &= \boxed{u[-c \cos v, -c \sin v, 1]}\end{aligned}$$

### Problem 10.5.5.

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

$$\text{Paraboloid of revolution } \mathbf{r}(u, v) = [u \cos v, u \sin v, u^2]$$

**Solution.**

$$\begin{aligned}x &= u \cos v \\ y &= u \sin v \\ z &= u^2\end{aligned}$$

$$\begin{aligned}x^2 + y^2 + z &= u^2 \cos^2 v + u^2 \sin^2 v + u^2 \\ &= u^2(\cos^2 v + \sin^2 v + 1) \\ &= u^2(1 + 1) \\ &= 2u^2\end{aligned}$$

But  $u^2 = z$ ,

$$x^2 + y^2 + z = 2z$$

$$\boxed{z = x^2 + y^2}$$

The parameter curves are,

- $u = \text{const} \rightarrow$  concentric circles at a given height
- $v = \text{const} \rightarrow$  parabolas with vertex at the origin

$$\begin{aligned}\mathbf{r}_u &= [\cos v, \sin v, 2u] \\ \mathbf{r}_v &= [-u \sin v, u \cos v, 0]\end{aligned}$$

$$\begin{aligned}
\mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix} \\
&= [(0 - 2u^2 \cos v), -(0 + 2u^2 \sin v), (u \cos^2 v + u \sin^2 v)] \\
&= \boxed{[-2u^2 \cos v, -2u^2 \sin v, u]}
\end{aligned}$$

### Problem 10.5.7.

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

$$\text{Ellipsoid } \mathbf{r}(u, v) = [a \cos v \cos u, b \cos v \sin u, c \sin v]$$

**Solution.**

$$\begin{aligned}
x &= a \cos v \cos u \\
y &= b \cos v \sin u \\
z &= c \sin v
\end{aligned}$$

$$\begin{aligned}
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= \cos^2 v \cos^2 u + \cos^2 v \sin^2 u + \sin^2 v \\
&= (\cos^2 u + \sin^2 u) \cos^2 v + \sin^2 v \\
&= \cos^2 v + \sin^2 v \\
&= 1
\end{aligned}$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$$

The parameter curves are,

- $u = \text{const} \rightarrow$  ellipses
- $v = \text{const} \rightarrow$  ellipses

$$\begin{aligned}
\mathbf{r}_u &= [-a \cos v \sin u, b \cos v \cos u, 0] \\
\mathbf{r}_v &= [-a \sin v \cos u, -b \sin v \sin u, c \cos v]
\end{aligned}$$

$$\begin{aligned}
\mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \cos v \sin u & b \cos v \cos u & 0 \\ -a \sin v \cos u & -b \sin v \sin u & c \cos v \end{vmatrix} \\
&= [(bc \cos^2 v \cos u - 0), -(-ac \cos^2 v \sin u - 0), (ab \cos v \sin v \sin^2 u + ab \cos v \sin v \cos^2 u)] \\
&= [(bc \cos^2 v \cos u), (ac \cos^2 v \sin u), (ab \cos v \sin v (\sin^2 u + \cos^2 u))] \\
&= \boxed{[(bc \cos^2 v \cos u), (ac \cos^2 v \sin u), (ab \cos v \sin v)]}
\end{aligned}$$

**Problem 10.5.14.**

Find a normal vector. Sketch the surface and parameter curves.

$$\text{Plane } 4x + 3y + 2z = 12$$

**Solution.**

$$4x + 3y + 2z = 12 \implies z = \frac{12 - 4x - 3y}{2}$$

$$\mathbf{r}(u, v) = [u, v, 6 - 2u - 1.5v]$$

$$\mathbf{r}_u = [1, 0, -2]$$

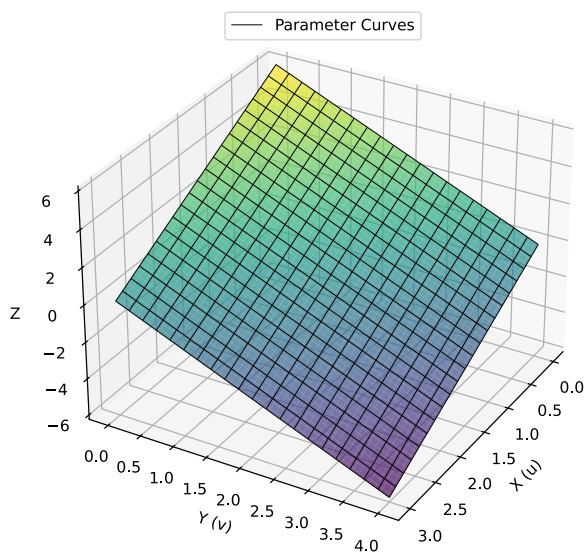
$$\mathbf{r}_v = [0, 1, -1.5]$$

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & -1.5 \end{vmatrix}$$

$$= [(0 + 2), -(-1.5 - 0), (1 - 0)]$$

$$= \boxed{[2, 1.5, 1]}$$

**Problem 10.5.15.**

Find a normal vector. Sketch the surface and parameter curves.

$$\text{Cylinder of revolution } (x - 2)^2 + (y + 1)^2 = 25$$

**Solution.**

$$\mathbf{r}(u, v) = [2 + 5 \cos u, -1 + 5 \sin u, v]$$

$$\mathbf{r}_u = [-5 \sin u, 5 \cos u, 0]$$

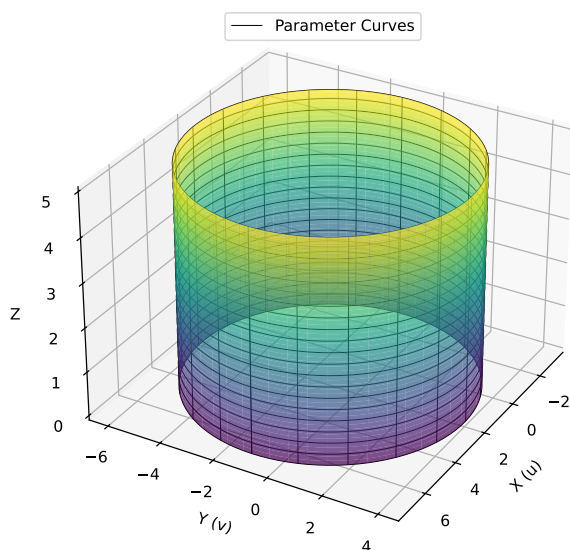
$$\mathbf{r}_v = [0, 0, 1]$$

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 \sin u & 5 \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= [(5 \cos u - 0), -(-5 \sin u - 0), (0 - 0)]$$

$$= \boxed{5 [\cos u, \sin u, 0]}$$



### Problem 10.5.18.

Find a normal vector. Sketch the surface and parameter curves.

$$\text{Elliptic cone } z = \sqrt{x^2 + 4y^2}$$

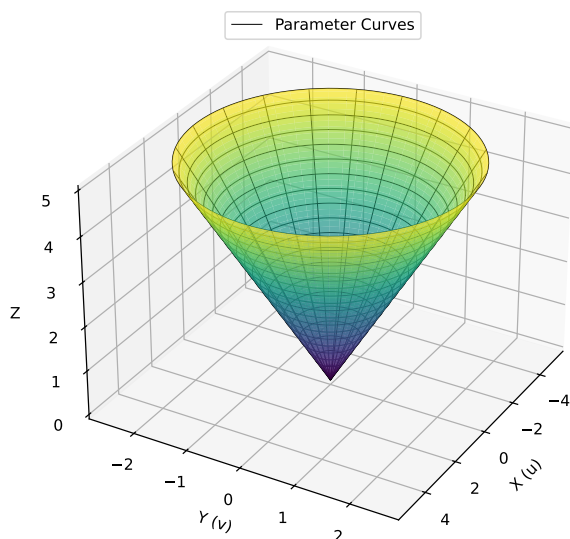
**Solution.**

$$\mathbf{r}(u, v) = u[\cos v, 0.5 \sin v, 1]$$

$$\mathbf{r}_u = [\cos v, 0.5 \sin v, 1]$$

$$\mathbf{r}_v = u[-\sin v, 0.5 \cos v, 0]$$

$$\begin{aligned}
\mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & 0.5 \sin v & 1 \\ -u \sin v & 0.5u \cos v & 0 \end{vmatrix} \\
&= [(0 - 0.5u \cos v), -(0 + u \sin v), (0.5u \cos^2 v + 0.5u \sin^2 v)] \\
&= u[-0.5 \cos v, -\sin v, 0.5]
\end{aligned}$$



### Problem 10.6.3.

Evaluate the flux integral,  $\int_S \mathbf{F} \cdot \mathbf{n} \, dA$ , for the given data. Describe the kind of surface.

$$\mathbf{F} = [0, x, 0], \quad S : x^2 + y^2 + z^2 = 1, \quad x, y, z \geq 0$$

### Solution.

$S$  is  $\frac{1}{8}$  of a sphere with radius 1,

$$\mathbf{r}(u, v) = [\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi] \quad u \in [0, \pi/2], \quad v \in [0, \pi/2]$$

$$\begin{aligned}
\mathbf{N} &= \mathbf{r}_\phi \times \mathbf{r}_\theta \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\ -\sin \phi \sin \theta & \sin \phi \cos \theta & 0 \end{vmatrix} \\
&= [(0 + \sin^2 \phi \cos \theta), -(0 - \sin^2 \phi \sin \theta), (\cos \phi \sin \phi \cos^2 \theta + \cos \phi \sin \phi \sin^2 \theta)] \\
&= [(\sin^2 \phi \cos \theta), (\sin^2 \phi \sin \theta), (\cos \phi \sin \phi)]
\end{aligned}$$

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \iint_R \mathbf{F} \cdot \mathbf{N} \, d\phi \, d\theta$$

$$\begin{aligned}
\mathbf{F} \cdot \mathbf{N} &= [0, x, 0] \cdot [(\sin^2 \phi \cos \theta), (\sin^2 \phi \sin \theta), (\cos \phi \sin \phi)] \\
&= 0 + x \cdot (\sin^2 \phi \sin \theta) + 0 \\
&= -(\sin \phi \cos \theta)(\sin^2 \phi \sin \theta) \\
&= -\sin^3 \phi \cos \theta \sin \theta
\end{aligned}$$

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n} \, dA &= \int_0^{\pi/2} \int_0^{\pi/2} [-\sin^3 \phi \cos \theta \sin \theta] \, d\phi \, d\theta \\
&= \left( -\int_0^{\pi/2} \sin^3 \phi \, d\phi \right) \left( \int_0^{\pi/2} \cos \theta \sin \theta \, d\theta \right)
\end{aligned}$$

First integral substitute  $u = \cos \phi$ ,  $du = -\sin \phi \, d\phi$ :

$$\begin{aligned}
-\int_0^{\pi/2} \sin^3 \phi \, d\phi &= -\int_0^{\pi/2} (1 - \cos^2 \phi) \sin \phi \, d\phi \\
&= \int_1^0 (1 - u^2) \, du \\
&= u - \frac{u^3}{3} \Big|_1^0 \\
&= \frac{2}{3}
\end{aligned}$$

Second integral substitute  $u = \sin \theta$ ,  $du = \cos \theta \, d\theta$ :

$$\begin{aligned}
\int_0^{\pi/2} \cos \theta \sin \theta \, d\theta &= \int_0^1 u \, du \\
&= \frac{u^2}{2} \Big|_1^0 \\
&= \frac{1}{2}
\end{aligned}$$

Therefore,

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \frac{1}{3}}$$

### Problem 10.6.5.

Evaluate the flux integral for the given data. Describe the kind of surface.

$$\mathbf{F} = [x, y, z], \quad S : \mathbf{r} = [u \cos v, u \sin v, u^2], \quad 0 \leq u \leq 4, \quad -\pi \leq v \leq \pi$$

**Solution.**

$S$  is a paraboloid of revolution bounded by  $z = 16$ .

$$\begin{aligned}
\mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix} \\
&= [(0 - 2u^2 \cos v), -(0 + 2u^2 \sin v), (u \cos^2 v + u \sin^2 v)] \\
&= [(-2u^2 \cos v), (-2u^2 \sin v), (u)]
\end{aligned}$$

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \iint_R \mathbf{F} \cdot \mathbf{N} \, du \, dv$$

$$\begin{aligned}
\mathbf{F} \cdot \mathbf{N} &= [x, y, z] \cdot [-2u^2 \cos v, -2u^2 \sin v, u] \\
&= [u \cos v, u \sin v, u^2] \cdot [-2u^2 \cos v, -2u^2 \sin v, u] \\
&= -2u^3 \cos^2 v - 2u^3 \sin^2 v + u^3 \\
&= -u^3
\end{aligned}$$

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n} \, dA &= \int_{-\pi}^{\pi} \int_0^4 -u^3 \, du \, dv \\
&= \left( \int_{-\pi}^{\pi} dv \right) \left( \int_0^4 -u^3 \, du \right) \\
&= 2\pi \left[ -\frac{u^4}{4} \right]_0^4 \\
&= \boxed{-128\pi}
\end{aligned}$$

### Problem 10.6.7.

Evaluate the flux integral for the given data. Describe the kind of surface.

$$\mathbf{F} = [0, \sin y, \cos z], \quad S \text{ the cylinder } x = y^2, \text{ where } 0 \leq y \leq \frac{\pi}{4}, \quad 0 \leq z \leq y$$

### Solution.

$S$  is a parabolic cylinder that is sliced into a triangular shape,

$$\mathbf{r}(u, v) = [u^2, u, v] \quad u \in [0, \pi/4], \quad v \in [0, u]$$

$$\begin{aligned}
\mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
&= [1, -2u, 0]
\end{aligned}$$

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \iint_R \mathbf{F} \cdot \mathbf{N} \, du \, dv$$

$$\begin{aligned}
\mathbf{F} \cdot \mathbf{N} &= [0, \sin y, \cos z] \cdot [1, -2u, 0] \\
&= [0, \sin u, \cos v] \cdot [1, -2u, 0] = -2u \sin u
\end{aligned}$$

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n} \, dA &= \int_0^{\pi/4} \int_0^u -2u \sin u \, dv \, du \\
&= \int_0^{\pi/4} -2u \sin u [v]_0^u \, du \\
&= \int_0^{\pi/4} -2u^2 \sin u \, du
\end{aligned}$$

Integrating by parts ( $\int x dy = xy - \int y dx$ ),

$$\begin{aligned}
x &= u^2 & dx &= 2u \, du \\
dy &= \sin u \, du & y &= -\cos u
\end{aligned}$$

$$\begin{aligned}
\int_0^{\pi/4} 2u^2 \sin u \, du &= -u^2 \cos u \Big|_0^{\pi/4} + \int_0^{\pi/4} 2u \cos u \, du \\
&= -\frac{\pi^2}{16\sqrt{2}} + 2 \int_0^{\pi/4} u \cos u \, du
\end{aligned}$$

And again,

$$\begin{aligned}
x &= u & dx &= du \\
dy &= \cos u \, du & y &= \sin u
\end{aligned}$$

$$\begin{aligned}
\int_0^{\pi/4} u \cos u \, du &= u \sin u \Big|_0^{\pi/4} - \int_0^{\pi/4} \sin u \, du \\
&= \frac{\pi}{4\sqrt{2}} + \cos u \Big|_0^{\pi/4} \\
&= \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - 1
\end{aligned}$$

In all,

$$\int_0^{\pi/4} 2u^2 \sin u \, du = -\frac{\pi^2}{16\sqrt{2}} + \frac{\pi}{2\sqrt{2}} + \frac{2}{\sqrt{2}} - 2$$

Therefore,

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \frac{\pi^2}{16\sqrt{2}} - \frac{\pi}{2\sqrt{2}} - \frac{2}{\sqrt{2}} + 2}$$

### Problem 10.6.13.

Evaluate the surface integral,  $\iint_S G(\mathbf{r}) \, dA$ , for the following data. Indicate the kind of surface.

$$G = x + y + z, \quad z = x + 2y, \quad 0 \leq x \leq \pi, \quad 0 \leq y \leq x$$

**Solution.**

The surface is a plane,

$$\mathbf{r}(u, v) = [u, v, u + 2v] \quad u \in [0, \pi], \quad v \in [0, u]$$

$$\begin{aligned} \mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} \\ &= [-1, -2, 1] \end{aligned}$$

$$\begin{aligned} |\mathbf{N}| &= \sqrt{(-1)^2 + (-2)^2 + 1^2} \\ &= \sqrt{6} \end{aligned}$$

$$\begin{aligned} \iint_S G(\mathbf{r}) \, dA &= \iint_R G|\mathbf{N}| \, du \, dv \\ &= \sqrt{6} \int_0^\pi \int_0^u (u + v + u + 2v) \, dv \, du \\ &= \sqrt{6} \int_0^\pi \int_0^u (2u + 3v) \, dv \, du \\ &= \sqrt{6} \int_0^\pi \left[ 2uv + \frac{3}{2}v^2 \right]_0^u \, du \\ &= \sqrt{6} \int_0^\pi \left( 2u^2 + \frac{3}{2}u^2 \right) \, du \\ &= \frac{7\sqrt{6}}{2} \int_0^\pi u^2 \, du \\ &= \frac{7\sqrt{6}}{6} u^3 \Big|_0^\pi \\ &= \boxed{\frac{7\sqrt{6}\pi^3}{6}} \end{aligned}$$

**Problem 10.6.15.**

Evaluate the surface integral for the following data. Indicate the kind of surface.

$$G = (1 + 9xz)^{3/2}, \quad S : \mathbf{r} = [u, v, u^3], 0 \leq u \leq 1, \quad -2 \leq v \leq 2$$

**Solution.**

The surface is a cubic cylinder,

$$\begin{aligned} \mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3u^2 \\ 0 & 1 & 0 \end{vmatrix} \\ &= [-3u^2, 0, 1] \end{aligned}$$

$$\begin{aligned} |\mathbf{N}| &= \sqrt{(-3u^2)^2 + 0^2 + 1^2} \\ &= \sqrt{9u^4 + 1} \end{aligned}$$

$$\begin{aligned} \iint_S G(\mathbf{r}) \, dA &= \iint_R G|\mathbf{N}| \, du \, dv \\ &= \int_{-2}^2 \int_0^1 (1 + 9u^4)^{3/2} \sqrt{9u^4 + 1} \, du \, dv \\ &= \int_{-2}^2 dv \cdot \int_0^1 (1 + 9u^4)^2 \, du \\ &= \int_{-2}^2 dv \cdot \int_0^1 (81u^8 + 18u^4 + 1) \, du \\ &= [v]_{-2}^2 \cdot \left[ 9u^9 + \frac{18u^5}{5} + u \right]_0^1 \\ &= (4) \left( 9 + \frac{18}{5} + 1 \right) \\ &= \boxed{54.4} \end{aligned}$$