

Problem 10.7.11

Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} dA$ by the divergence theorem:

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} dA = \iiint_T \nabla \cdot \mathbf{F} dVR}$$

$$\mathbf{F} = [e^x, e^y, e^z], \quad S \text{ the surface of the cube } |x| \leq 1, |y| \leq 1, |z| \leq 1$$

Solution**Problem 10.7.13**

Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} dA$ by the divergence theorem.

$$\mathbf{F} = [\sin y, \cos x, \cos z], \quad S \text{ the surface of } x^2 + y^2 \leq 4, |z| \leq 2 \text{ (a cylinder and two disks!)}$$

Solution**Problem 10.7.17**

Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} dA$ by the divergence theorem.

$$\mathbf{F} = [x^2, y^2, z^2], \quad S \text{ the surface of the cone } x^2 + y^2 \leq z^2, 0 \leq z \leq h$$

Solution**Problem 10.7.22**

Given a mass of density 1 in a region T of space, find the moment of inertia about the x-axis,

$$I_x = \iiint_T (y^2 + z^2) dV$$

$$T : \text{The paraboloid } y^2 + z^2 \leq x, 0 \leq x \leq h$$

Solution**Problem 10.8.1. Harmonic functions.**

Verify Theorem 1 for $f = 2z^2 - x^2 - y^2$ and S the surface of the box $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$.

Theorem 1. A Basic Property of Harmonic Functions

Let $f(x, y, z)$ be a harmonic function in some domain D in space. Let S be any piecewise smooth closed orientable surface in D whose entire region it encloses belongs to D . Then the integral of the normal derivative of f taken over S is zero.

Solution**Problem 10.8.3. Green's First Identity**

$$\iiint_T (f \Delta g + \nabla f \cdot \nabla g) dV = \iint_S f \frac{\partial g}{\partial n} dA$$

Verify for $f = 4y^2$, $g = x^2$, S the surface of the “unit cube” $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$. What are the assumptions of f and g ? Must f and g be harmonic?

Solution**Problem 10.8.5. Green's Second Identity**

$$\iiint_T (f \Delta g - g \Delta f) dV = \iint_S \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) dA$$

Verify for $f = 6y^2$, $g = 2x^2$, S the surface of the “unit cube” $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Solution**Problem 10.8.7**

Use the divergence theorem, assuming that the assumptions on T and S are satisfied.

Show that a region T with boundary surface S has the volume

$$V = \iint_S x dy dz = \iint_S y dz dx = \iint_S z dx dy = \frac{1}{3} \iint_S (x dy dz + y dz dx + z dx dy)$$

Solution**Problem 10.9.3**

Evaluate the surface integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA$ for the given \mathbf{F} and S .

$$\mathbf{F} = [e^{-z}, e^{-z} \cos y, e^{-z} \sin y], \quad S : z = \frac{y^2}{2}, \quad -1 \leq x \leq 1, \quad 0 \leq y \leq 1$$

Solution**Problem 10.9.5**

Evaluate the surface integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA$ for the given \mathbf{F} and S .

$$\mathbf{F} = [z^2, \frac{3}{2}x, 0], \quad S : 0 \leq x \leq a, \quad 0 \leq y \leq a, \quad z = 1$$

Solution**Problem 10.9.13**

Calculate $\oint_C \mathbf{F} \cdot \mathbf{r}'(s) ds$ by Stokes's theorem,

$$\boxed{\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA = \oint_C \mathbf{F} \cdot \mathbf{r}'(s) ds}$$

for the given \mathbf{F} and C . Assume the Cartesian coordinates to be right-handed and the z -component of the surface normal to be nonnegative.

$$\mathbf{F} = [-5y, 4x, z], \quad C \text{ the circle } x^2 + y^2 = 16, z = 4$$

Solution**Problem 10.9.15**

Calculate $\oint_C \mathbf{F} \cdot \mathbf{r}'(s) ds$ by Stokes's theorem for the given \mathbf{F} and C . Assume the Cartesian coordinates to be right-handed and the z -component of the surface normal to be nonnegative.

$$\mathbf{F} = [z^3, x^3, y^3], \text{ around the triangle with vertices } (0, 0, 0), (1, 0, 0), (1, 1, 0)$$

Solution**Problem 10.9.19**

Calculate $\oint_C \mathbf{F} \cdot \mathbf{r}'(s) ds$ by Stokes's theorem for the given \mathbf{F} and C . Assume the Cartesian coordinates to be right-handed and the z -component of the surface normal to be nonnegative.

$$\mathbf{F} = [z, e^z, 0], \quad C \text{ the boundary curve of the portion of the cone } z = \sqrt{x^2 + y^2}, x \geq 0, y \geq 0, 0 \leq z \leq 1$$

Solution