

**Problem 10.7.11**

Evaluate the surface integral  $\iint_S \mathbf{F} \cdot \mathbf{n} dA$  by the divergence theorem:

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} dA = \iiint_T \nabla \cdot \mathbf{F} dV}$$

$$\mathbf{F} = [e^x, e^y, e^z], \quad S \text{ the surface of the cube } |x| \leq 1, |y| \leq 1, |z| \leq 1$$

**Solution**

$$\nabla \cdot \mathbf{F} = e^x + e^y + e^z$$

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} dA &= \iiint_T \nabla \cdot \mathbf{F} dV \\ &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (e^x + e^y + e^z) dx dy dz \\ &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 e^x dx dy dz + \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 e^y dx dy dz + \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 e^z dx dy dz \\ &= 3 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 e^x dx dy dz \\ &= 12 \int_{-1}^1 e^x dx \\ &= 12e^x \Big|_{-1}^1 \\ &= \boxed{12(e - e^{-1})} \end{aligned}$$

**Problem 10.7.13**

Evaluate the surface integral  $\iint_S \mathbf{F} \cdot \mathbf{n} dA$  by the divergence theorem.

$$\mathbf{F} = [\sin y, \cos x, \cos z], \quad S \text{ the surface of } x^2 + y^2 \leq 4, |z| \leq 2 \text{ (a cylinder and two disks!)}$$

**Solution**

$$\begin{aligned} \nabla \cdot \mathbf{F} &= 0 + 0 - \sin z \\ &= -\sin z \end{aligned}$$

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} dA &= \iiint_T \nabla \cdot \mathbf{F} dV \\ &= \int_{-2}^2 \int_0^{2\pi} \int_0^2 (-\sin z) r dr d\theta dz \\ &= \int_0^2 r dr \cdot \int_0^{2\pi} d\theta \cdot \int_{-2}^2 (-\sin z) dz \\ &= \left[ \frac{r^2}{2} \right]_0^2 \cdot (2\pi) \cdot [\cos z]_{-2}^2 \\ &= 4\pi(\cos(2) - \cos(-2)) \\ &= \boxed{0} \end{aligned}$$

**Problem 10.7.17**

Evaluate the surface integral  $\iint_S \mathbf{F} \cdot \mathbf{n} dA$  by the divergence theorem.

$$\mathbf{F} = [x^2, y^2, z^2], \quad S \text{ the surface of the cone } x^2 + y^2 \leq z^2, \quad 0 \leq z \leq h$$

**Solution**

$$\nabla \cdot \mathbf{F} = 2x + 2y + 2z$$

In cylindrical coordinates,

$$\nabla \cdot \mathbf{F} = 2(r \cos \theta + r \sin \theta + z)$$

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} dA &= \iiint_T \nabla \cdot \mathbf{F} dV \\ &= 2 \int_0^h \int_0^{2\pi} \int_0^z (r \cos \theta + r \sin \theta + z) r dr d\theta dz \\ &= 2 \int_0^h \int_0^{2\pi} \int_0^z (r^2 \cos \theta + r^2 \sin \theta + rz) dr d\theta dz \\ &= 2 \int_0^h \int_0^{2\pi} \left[ \frac{r^3}{3} \cos \theta + \frac{r^3}{3} \sin \theta + \frac{r^2}{2} z \right]_{r=0}^{r=z} d\theta dz \\ &= 2 \int_0^h \int_0^{2\pi} z^3 \left( \frac{1}{3} \cos \theta + \frac{1}{3} \sin \theta + \frac{1}{2} z \right) d\theta dz \\ &= 2 \int_0^h z^3 dz \cdot \int_0^{2\pi} \left( \frac{1}{3} \cos \theta + \frac{1}{3} \sin \theta + \frac{1}{2} z \right) d\theta \\ &= \frac{1}{2} [z^4]_0^h \cdot \left[ \frac{1}{3} \sin \theta - \frac{1}{3} \cos \theta + \frac{1}{2} \theta \right]_0^{2\pi} \\ &= \boxed{\frac{\pi h^4}{2}} \end{aligned}$$

**Problem 10.7.22**

Given a mass of density 1 in a region  $T$  of space, find the moment of inertia about the x-axis,

$$I_x = \iiint_T (y^2 + z^2) dV$$

$$T : \text{The paraboloid } y^2 + z^2 \leq x, \quad 0 \leq x \leq h$$

**Solution**

In cylindrical coordinates along the  $x$ -axis, the integrand becomes  $r^2$  and

$$T : \text{The paraboloid } r^2 \leq x, \quad 0 \leq x \leq h$$

$$\begin{aligned}
I_x &= \iiint_T (y^2 + z^2) \, dV \\
&= \iiint_T (r^2) r \, dr \, d\theta \, dx \\
&= \int_0^h \int_0^{2\pi} \int_0^{\sqrt{x}} (r^2) r \, dr \, d\theta \, dx \\
&= \int_0^h \int_0^{2\pi} \int_0^{\sqrt{x}} (r^3) \, dr \, d\theta \, dx \\
&= \frac{1}{4} \int_0^h \int_0^{2\pi} (x^2) \, d\theta \, dx \\
&= \frac{1}{4} \int_0^h (x^2) \, dx \cdot \int_0^{2\pi} \, d\theta \\
&= \frac{\pi}{6} x^3 \Big|_0^h \\
&= \boxed{\frac{\pi h^3}{6}}
\end{aligned}$$

### Problem 10.8.1. Harmonic functions.

#### Theorem 1. A Basic Property of Harmonic Functions

Let  $f(x, y, z)$  be a harmonic function in some domain  $D$  in space. Let  $S$  be any piecewise smooth closed orientable surface in  $D$  whose entire region it encloses belongs to  $D$ . Then the integral of the normal derivative of  $f$  taken over  $S$  is zero.

Verify Theorem 1 for  $f = 2z^2 - x^2 - y^2$  and  $S$  the surface of the box  $0 \leq x \leq a$ ,  $0 \leq y \leq b$ ,  $0 \leq z \leq c$ .

#### Solution

$$\begin{aligned}
\Delta f &= f_{xx} + f_{yy} + f_{zz} \\
&= (-2) + (-2) + (4) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\iint_S \frac{\partial f}{\partial n} \, dA &= \iiint_T \Delta f \, dV \\
&= \boxed{0}
\end{aligned}$$

### Problem 10.8.3. Green's First Identity

$$\boxed{\iiint_T (f \Delta g + \nabla f \cdot \nabla g) \, dV = \iint_S f \frac{\partial g}{\partial n} \, dA}$$

Verify for  $f = 4y^2$ ,  $g = x^2$ ,  $S$  the surface of the “unit cube”  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ ,  $0 \leq z \leq 1$ . What are the assumptions of  $f$  and  $g$ ? Must  $f$  and  $g$  be harmonic?

**Solution**

	$f$	$g$
$\nabla$	$4y^2$	$x^2$
$\Delta$	$[0, 8y]$	$[2x, 0]$

$$\begin{aligned} \iiint_T (f \Delta g + \nabla f \cdot \nabla g) \, dV &= \int_0^1 \int_0^1 \int_0^1 (8y^2) \, dx \, dy \, dz \\ &= \frac{8y^3}{3} \Big|_0^1 \\ &= \boxed{\frac{8}{3}} \end{aligned}$$

$$\iint_S f \frac{\partial g}{\partial n} \, dA = \iint_S f (\nabla g \cdot \mathbf{n}) \, dA$$

Integrating over the cube surface, only the  $x = 1$  face is non-zero,

$$\begin{aligned} \iint_S f (\nabla g \cdot \mathbf{n}) \, dA &= \int_0^1 f g_x(x=1) \, dy \\ &= \int_0^1 (4y^2)(2) \, dy \\ &= 8 \int_0^1 y^2 \, dy \\ &= 8 \left[ \frac{y^3}{3} \right]_0^1 \\ &= \boxed{\frac{8}{3}} \end{aligned}$$

The functions do not need to be harmonic, but the divergence theorem assumptions must hold. The functions need to be twice differentiable and  $S$  must be piecewise smooth.

**Problem 10.8.5. Green's Second Identity**

$$\boxed{\iiint_T (f \Delta g - g \Delta f) \, dV = \iint_S \left( f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) \, dA}$$

Verify for  $f = 6y^2$ ,  $g = 2x^2$ ,  $S$  the surface of the “unit cube”  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ ,  $0 \leq z \leq 1$ .

**Solution**

	$f$	$g$
$\nabla$	$6y^2$	$2x^2$
$\Delta$	$[0, 12y]$	$[4x, 0]$

$$\begin{aligned}
\iiint_T (f \Delta g - g \Delta f) \, dV &= 24 \int_0^1 \int_0^1 \int_0^1 (y^2 - x^2) \, dx \, dy \, dz \\
&= 24 \left( \int_0^1 y^2 \, dy - \int_0^1 x^2 \, dx \right) \cdot \int_0^1 \, dz \\
&= 8 \left( y^3 \Big|_0^1 - x^3 \Big|_0^1 \right) \\
&= \boxed{0}
\end{aligned}$$

$$\iint_S \left( f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) \, dA = \iint_S f (\nabla g \cdot \mathbf{n}) - g (\nabla f \cdot \mathbf{n}) \, dA$$

Integrating over the cube surface, only the  $x = 1$  face (first term) and  $y = 1$  face (second term) are non-zero,

$$\begin{aligned}
\iint_S f (\nabla g \cdot \mathbf{n}) \, dA &= \int_0^1 f g_x(x=1) \, dy - \int_0^1 g f_y(y=1) \, dx \\
&= \int_0^1 (6y^2)(4) \, dy - \int_0^1 (2x^2)(12) \, dx \\
&= 24 \int_0^1 y^2 \, dy - 24 \int_0^1 x^2 \, dx \\
&= 8 \left( y^3 \Big|_0^1 - x^3 \Big|_0^1 \right) \\
&= \boxed{0}
\end{aligned}$$

## Problem 10.8.7

Use the divergence theorem, assuming that the assumptions on  $T$  and  $S$  are satisfied.

Show that a region  $T$  with boundary surface  $S$  has the volume

$$V = \iint_S x \, dy \, dz = \iint_S y \, dz \, dx = \iint_S z \, dx \, dy = \frac{1}{3} \iint_S (x \, dy \, dz + y \, dz \, dx + z \, dx \, dy)$$

### Solution

Take  $\mathbf{F} = [x, 0, 0]$ , therefore  $\nabla \cdot \mathbf{F} = 1$  and

$$\begin{aligned}
V &= \iiint_T \nabla \cdot \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \mathbf{n} \, dA \\
V &= \iint_S x \, dy \, dz
\end{aligned}$$

Take  $\mathbf{F} = [0, y, 0]$ , therefore  $\nabla \cdot \mathbf{F} = 1$  and

$$\begin{aligned}
V &= \iiint_T \nabla \cdot \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \mathbf{n} \, dA \\
V &= \iint_S y \, dx \, dz
\end{aligned}$$

Take  $\mathbf{F} = [0, 0, z]$ ,

$$V = \iiint_T \nabla \cdot \mathbf{F} dV = \iint_S \mathbf{F} \cdot \mathbf{n} dA$$

$$V = \iint_S z dx dy$$

Summing the above,

$$3V = \iint_S (x dy dz + y dx dz + z dx dy)$$

$$V = \frac{1}{3} \iint_S (x dy dz + y dz dx + z dx dy)$$

### Problem 10.9.3

Evaluate the surface integral  $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA$  directly for the given  $\mathbf{F}$  and  $S$ .

$$\mathbf{F} = [e^{-z}, e^{-z} \cos y, e^{-z} \sin y], \quad S : z = \frac{y^2}{2}, \quad -1 \leq x \leq 1, \quad 0 \leq y \leq 1$$

#### Solution

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{-z} & e^{-z} \cos y & e^{-z} \sin y \end{vmatrix} \\ &= [(e^{-z} \cos y + e^{-z} \cos y), -(0 + e^{-z}), (0 - 0)] \\ &= [2e^{-z} \cos y, -e^{-z}, 0] \end{aligned}$$

$$S : \mathbf{r}(x, y) = [x, y, \frac{y^2}{2}]$$

$$\begin{aligned} \mathbf{n} dA &= \mathbf{r}_x \times \mathbf{r}_y dx dy \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & y \end{vmatrix} dx dy \\ &= [0, -y, 1] dx dy \end{aligned}$$

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA &= \iint_S [2e^{-z} \cos y, -e^{-z}, 0] \cdot [0, -y, 1] dx dy \\ &= \iint_S (ye^{-z} + 0) dx dy \\ &= \int_{-1}^1 dx \cdot \int_0^1 ye^{-\frac{y^2}{2}} dy \\ &= 2 \int_0^1 ye^{-\frac{y^2}{2}} dy \\ &= -2e^{-\frac{y^2}{2}} \Big|_0^1 \\ &= \boxed{2(1 - e^{-\frac{1}{2}})} \end{aligned}$$

**Problem 10.9.5**

Evaluate the surface integral  $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA$  directly for the given  $\mathbf{F}$  and  $S$ .

$$\mathbf{F} = [z^2, \frac{3}{2}x, 0], \quad S : 0 \leq x \leq a, \quad 0 \leq y \leq a, \quad z = 1$$

**Solution**

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & \frac{3}{2}x & 0 \end{vmatrix} \\ &= \left[ (0 - 0), -(0 - 2z), \left( \frac{3}{2} - 0 \right) \right] \\ &= \left[ 0, 2z, \frac{3}{2} \right]\end{aligned}$$

The surface is a flat plane and therefore  $\mathbf{n} dA = [0, 0, 1] dx dy$ .

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA &= \iint_S \left[ 0, 2z, \frac{3}{2} \right] \cdot [0, 0, 1] dx dy \\ &= \iint_S \frac{3}{2} dx dy \\ &= \frac{3}{2} \int_0^a dx \cdot \int_0^a dy \\ &= \boxed{\frac{3a^2}{2}}\end{aligned}$$

**Problem 10.9.13**

Calculate  $\oint_C \mathbf{F} \cdot \mathbf{r}'(s) ds$  by Stokes's theorem,

$$\boxed{\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA = \oint_C \mathbf{F} \cdot \mathbf{r}'(s) ds}$$

for the given  $\mathbf{F}$  and  $C$ . Assume the Cartesian coordinates to be right-handed and the  $z$ -component of the surface normal to be nonnegative.

$$\mathbf{F} = [-5y, 4x, z], \quad C \text{ the circle } x^2 + y^2 = 16, \quad z = 4$$

**Solution****Problem 10.9.15**

Calculate  $\oint_C \mathbf{F} \cdot \mathbf{r}'(s) ds$  by Stokes's theorem for the given  $\mathbf{F}$  and  $C$ . Assume the Cartesian coordinates to be right-handed and the  $z$ -component of the surface normal to be nonnegative.

$$\mathbf{F} = [z^3, x^3, y^3], \quad \text{around the triangle with vertices } (0, 0, 0), (1, 0, 0), (1, 1, 0)$$

**Solution****Problem 10.9.19**

Calculate  $\oint_C \mathbf{F} \cdot \mathbf{r}'(s) ds$  by Stokes's theorem for the given  $\mathbf{F}$  and  $C$ . Assume the Cartesian coordinates to be right-handed and the  $z$ -component of the surface normal to be nonnegative.

$$\mathbf{F} = [z, e^z, 0], \quad C \text{ the boundary curve of the portion of the cone } z = \sqrt{x^2 + y^2}, \ x \geq 0, \ y \geq 0, \ 0 \leq z \leq 1$$

**Solution**