

Problem 10.4.3.

Evaluate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ counterclockwise around the boundary C of the region R by Green's theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA$$

where

$$\mathbf{F} = [x^2 e^y, y^2 e^x], \quad R \text{ the rectangle with vertices } (0, 0), (2, 0), (2, 3), (0, 3)$$

Solution.

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA \\ &= \int_0^3 \int_0^2 \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dx \, dy \\ &= \int_0^3 \int_0^2 (y^2 e^x - x^2 e^y) \, dx \, dy \\ &= \int_0^3 \left[y^2 e^x - \frac{1}{3} x^3 e^y \right]_{x=0}^{x=2} \, dy \\ &= \int_0^3 \left[\left(y^2 e^2 - \frac{1}{3} 2^3 e^y \right) - \left(y^2 e^0 - \frac{1}{3} 0^3 e^y \right) \right] \, dy \\ &= \int_0^3 \left[\left(y^2 e^2 - \frac{8}{3} e^y \right) - (y^2) \right] \, dy \\ &= \int_0^3 \left(y^2 (e^2 - 1) - \frac{8}{3} e^y \right) \, dy \\ &= \frac{1}{3} y^3 (e^2 - 1) - \frac{8}{3} e^y \Big|_0^3 \\ &= \left(\frac{1}{3} 3^3 (e^2 - 1) - \frac{8}{3} e^3 \right) - \left(\frac{1}{3} 0^3 (e^2 - 1) - \frac{8}{3} e^0 \right) \\ &= \left(9(e^2 - 1) - \frac{8}{3} e^3 \right) - \left(-\frac{8}{3} \right) \\ \oint_C \mathbf{F} \cdot d\mathbf{r} &= \boxed{9(e^2 - 1) - \frac{8}{3}(e^3 - 1)} \end{aligned}$$

Problem 10.4.7.

Evaluate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ counterclockwise around the boundary C of the region R by Green's theorem, where

$$\mathbf{F} = \nabla (x^3 \cos^2(xy)), \quad R : 1 \leq y \leq 2 - x^2$$

Solution.

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA \\ &= \iint_R \nabla \times (\nabla f) \cdot \mathbf{k} \, dA \\ \oint_C \mathbf{F} \cdot d\mathbf{r} &= \boxed{0} \end{aligned}$$

Problem 10.4.15.

Using the following equation,

$$\iint_R \Delta w \, dA = \oint_C \frac{\partial w}{\partial n} \, ds$$

Find the value of $\int_C \frac{\partial w}{\partial n} \, ds$ taken counterclockwise over the boundary C of the region R .

$$w = e^x \cos y + xy^3, \quad R : 1 \leq y \leq 10 - x^2, \quad x \geq 0$$

Solution.

$$\begin{aligned} \Delta w &= w_{xx} + w_{yy} \\ &= (e^x \cos y) + (-e^x \cos y + 6xy) \\ &= 6xy \end{aligned}$$

$$\begin{aligned} \oint_C \frac{\partial w}{\partial n} \, ds &= \iint_R \Delta w \, dA \\ &= \int_0^3 \int_1^{10-x^2} 6xy \, dy \, dx \\ &= \int_0^3 [3xy^2]_{y=1}^{y=10-x^2} \, dx \\ &= \int_0^3 (3x(10-x^2)^2 - 3x) \, dx \\ &= \int_0^3 (3x(100-20x^2+x^4) - 3x) \, dx \\ &= \int_0^3 (297x - 60x^3 + 3x^5) \, dx \\ &= \left. \frac{297}{2}x^2 - 15x^4 + \frac{1}{2}x^6 \right|_0^3 \\ &= 1336.5 - 1215 + 364.5 \\ &= \boxed{486} \end{aligned}$$

Problem 10.4.16.

Using the following equation,

$$\iint_R \Delta w \, dx \, dy = \oint_C \frac{\partial w}{\partial n} \, ds$$

find the value of $\int_C \frac{\partial w}{\partial n} \, ds$ taken counterclockwise over the boundary C of the region R .

$$w = x^2 + y^2, \quad C : x^2 + y^2 = 4$$

Solution.

$$\begin{aligned}\Delta w &= w_{xx} + w_{yy} \\ &= (2) + (2) \\ &= 4\end{aligned}$$

$$\begin{aligned}\oint_C \frac{\partial w}{\partial n} ds &= \iint_R \Delta w dA \\ &= 4 \iint_R dA \\ &= 4 \cdot \text{Area}(R) \\ &= 4 \cdot \pi \cdot 2^2 \\ &= \boxed{16\pi}\end{aligned}$$

Confirm the answer by direct integration.

$$C \implies \mathbf{r}(t) = [2 \cos t, 2 \sin t], \quad t \in [0, 2\pi]$$

The arc length element of C is,

$$\begin{aligned}ds &= |\mathbf{r}'(t)| dt \\ &= \sqrt{(-2 \sin t)^2 + (2 \cos t)^2} dt \\ &= 2 dt\end{aligned}$$

The normal vector is,

$$\begin{aligned}\mathbf{n} &= \frac{\nabla x^2 + y^2}{|\nabla x^2 + y^2|} \\ &= \frac{[2x, 2y]}{\sqrt{(2x)^2 + (2y)^2}} \\ &= \frac{[x, y]}{\sqrt{x^2 + y^2}}\end{aligned}$$

On C , $r = 2$,

$$\mathbf{n} = \frac{[x, y]}{2}$$

The integral of the normal derivative is then,

$$\begin{aligned}\oint_C \frac{\partial w}{\partial n} ds &= \oint_C \nabla w \cdot \mathbf{n} ds \\ &= \frac{1}{2} \oint_C [2x, 2y] \cdot [x, y] ds \\ &= \frac{1}{2} \oint_C 2x^2 + 2y^2 ds \\ &= \oint_C x^2 + y^2 ds\end{aligned}$$

On C , $x^2 + y^2 = 4$, and $ds = 2 dt$ where $t \in [0, 2\pi]$,

$$\begin{aligned}\oint_C x^2 + y^2 \, ds &= 8 \int_0^{2\pi} dt \\ &= 8 \cdot 2\pi \\ &= [16\pi]\end{aligned}$$

Problem 10.4.19.

Show that $w = e^x \sin y$ satisfies Laplace's equation $\Delta w = 0$ and, using the following equation,

$$\iint_R \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] dx dy = \oint_C w \left(\frac{\partial w}{\partial n} \right) ds$$

integrate $w \left(\frac{\partial w}{\partial n} \right)$ counter-clockwise around the boundary curve C of the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 5$.

Solution.

$$\nabla w = [e^x \sin y, e^x \cos y]$$

$$\begin{aligned}\Delta w &= (e^x \sin y) + (-e^x \sin y) \\ &= [0]\end{aligned}$$

$$\begin{aligned}\oint_C w \left(\frac{\partial w}{\partial n} \right) ds &= \iint_R \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] dx dy \\ &= \iint_R (e^x \sin y)^2 + (e^x \cos y)^2 dx dy \\ &= \iint_R (e^2 x [\sin^2 y + \cos^2 y]) dx dy \\ &= \iint_R (e^2 x) dx dy \\ &= \int_0^5 \int_0^2 (e^2 x) dx dy \\ &= \frac{1}{2} \int_0^5 [e^2 x]_0^2 dy \\ &= \frac{1}{2} (e^4 - 1) \int_0^5 dy \\ &= \frac{1}{2} (e^4 - 1) \cdot 5 \\ &= \boxed{\frac{5}{2} (e^4 - 1)}\end{aligned}$$

Problem 10.5.2.

Derive a parametric representation, $z = f(x, y)$ or $g(x, y, z) = 0$, by finding the **parameter curves** (curves $u = \text{const}$ and $v = \text{const}$) of the surface and a normal vector $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$ of the surface.

xy-plane in polar coordinates $\mathbf{r}(u, v) = [u \cos v, u \sin v]$, (thus $u = r, v = \theta$)

Solution.

$$\boxed{z = 0}$$

The parameter curves are,

- $u = \text{const} \rightarrow$ concentric circles
- $v = \text{const} \rightarrow$ radial straight lines

$$\begin{aligned}\mathbf{r}_u &= [\cos v, \sin v] \\ \mathbf{r}_v &= [-u \sin v, u \cos v]\end{aligned}$$

$$\begin{aligned}\mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 0 \end{vmatrix} \\ &= (\cos v \cdot u \cos v - \sin v \cdot (-u \sin v)) \mathbf{k} \\ &= (u \cos^2 v + u \sin^2 v) \mathbf{k} \\ &= u \mathbf{k} \\ \mathbf{N} &= \boxed{[0, 0, u]}\end{aligned}$$

Problem 10.5.3.

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

$$\text{Cone } \mathbf{r}(u, v) = [u \cos v, u \sin v, cu]$$

Solution.

$$\begin{aligned}x &= u \cos v \\ y &= u \sin v \\ z &= cu\end{aligned}$$

$$\begin{aligned}x^2 + y^2 + z^2 &= u^2(\cos^2 v + \sin^2 v + c^2) \\ x^2 + y^2 + z^2 &= u^2(1 + c^2)\end{aligned}$$

But $u = \frac{z}{c}$,

$$\begin{aligned}x^2 + y^2 + z^2 &= \frac{z^2}{c^2}(1 + c^2) \\ x^2 + y^2 + z^2 &= \frac{z^2}{c^2} + z^2\end{aligned}$$

$$\boxed{z = c\sqrt{x^2 + y^2}}$$

The parameter curves are,

- $u = \text{const} \rightarrow$ concentric circles
- $v = \text{const} \rightarrow$ straight lines through the origin

$$\mathbf{r}_u = [\cos v, \sin v, c]$$

$$\mathbf{r}_v = [-u \sin v, u \cos v, 0]$$

$$\begin{aligned}\mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & c \\ -u \sin v & u \cos v & 0 \end{vmatrix} \\ &= [(0 - cu \cos v), -(0 + cu \sin v), (u \cos^2 v + u \sin^2 v)] \\ &= [-cu \cos v, -cu \sin v, u] \\ &= [u[-c \cos v, -c \sin v, 1]]\end{aligned}$$

Problem 10.5.5.

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

Paraboloid of revolution $\mathbf{r}(u, v) = [u \cos v, u \sin v, u^2]$

Solution.

$$x = u \cos v$$

$$y = u \sin v$$

$$z = u^2$$

$$\begin{aligned}x^2 + y^2 + z &= u^2 \cos^2 v + u^2 \sin^2 v + u^2 \\ &= u^2(\cos^2 v + \sin^2 v + 1) \\ &= u^2(1 + 1) \\ &= 2u^2\end{aligned}$$

But $u^2 = z$,

$$x^2 + y^2 + z = 2z$$

$$z = x^2 + y^2$$

The parameter curves are,

- $u = \text{const} \rightarrow$ concentric circles at a given height
- $v = \text{const} \rightarrow$ parabolas with vertex at the origin

$$\mathbf{r}_u = [\cos v, \sin v, 2u]$$

$$\mathbf{r}_v = [-u \sin v, u \cos v, 0]$$

$$\begin{aligned}
 \mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix} \\
 &= [(0 - 2u^2 \cos v), -(0 + 2u^2 \sin v), (u \cos^2 v + u \sin^2 v)] \\
 &= [-2u^2 \cos v, -2u^2 \sin v, u]
 \end{aligned}$$

Problem 10.5.7.

Derive a parametric representation by finding the parameter curves of the surface and a normal vector of the surface.

$$\text{Ellipsoid } \mathbf{r}(u, v) = [a \cos v \cos u, b \cos v \sin u, c \sin v]$$

Solution.

$$\begin{aligned}
 x &= a \cos v \cos u \\
 y &= b \cos v \sin u \\
 z &= c \sin v
 \end{aligned}$$

$$\begin{aligned}
 \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= \cos^2 v \cos^2 u + \cos^2 v \sin^2 u + \sin^2 v \\
 &= (\cos^2 u + \sin^2 u) \cos^2 v + \sin^2 v \\
 &= \cos^2 v + \sin^2 v \\
 &= 1
 \end{aligned}$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$$

The parameter curves are,

- $u = \text{const} \rightarrow \text{elipses}$
- $v = \text{const} \rightarrow \text{elipses}$

$$\begin{aligned}
 \mathbf{r}_u &= [-a \cos v \sin u, b \cos v \cos u, 0] \\
 \mathbf{r}_v &= [-a \sin v \cos u, -b \sin v \sin u, c \cos v]
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \cos v \sin u & b \cos v \cos u & 0 \\ -a \sin v \cos u & -b \sin v \sin u & c \cos v \end{vmatrix} \\
 &= [(bc \cos^2 v \cos u - 0), -(-ac \cos^2 v \sin u - 0), (ab \cos v \sin v \sin^2 u + ab \cos v \sin v \cos^2 u)] \\
 &= [(bc \cos^2 v \cos u), (ac \cos^2 v \sin u), (ab \cos v \sin v (\sin^2 u + \cos^2 u))] \\
 &= \boxed{[(bc \cos^2 v \cos u), (ac \cos^2 v \sin u), (ab \cos v \sin v)]}
 \end{aligned}$$

Problem 10.5.14.

Find a normal vector. Sketch the surface and parameter curves.

$$\text{Plane } 4x + 3y + 2z = 12$$

Solution.

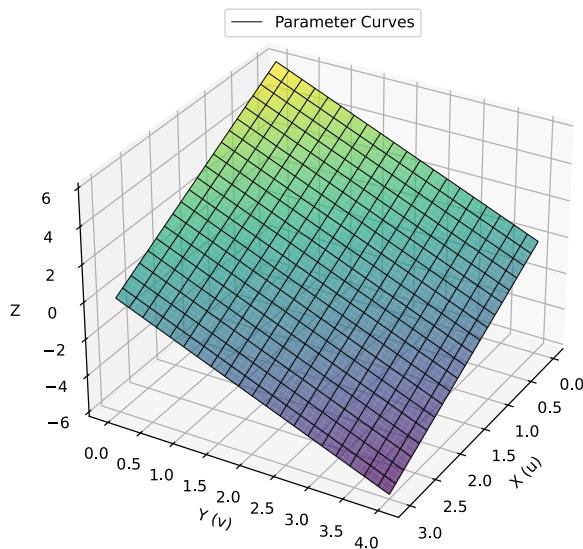
$$4x + 3y + 2z = 12 \implies z = \frac{12 - 4x - 3y}{2}$$

$$\mathbf{r}(u, v) = [u, v, 6 - 2u - 1.5v]$$

$$\mathbf{r}_u = [1, 0, -2]$$

$$\mathbf{r}_v = [0, 1, -1.5]$$

$$\begin{aligned} \mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & -1.5 \end{vmatrix} \\ &= [(0+2), -(-1.5-0), (1-0)] \\ &= [2, 1.5, 1] \end{aligned}$$

**Problem 10.5.15.**

Find a normal vector. Sketch the surface and parameter curves.

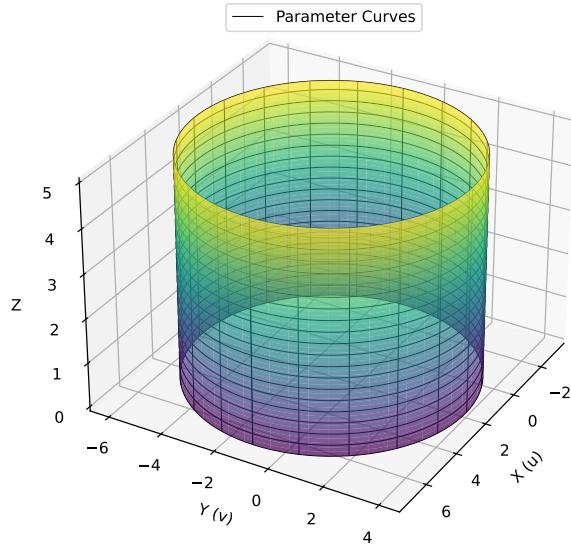
$$\text{Cylinder of revolution } (x - 2)^2 + (y + 1)^2 = 25$$

Solution.

$$\mathbf{r}(u, v) = [2 + 5 \cos u, -1 + 5 \sin u, v]$$

$$\begin{aligned}\mathbf{r}_u &= [-5 \sin u, 5 \cos u, 0] \\ \mathbf{r}_v &= [0, 0, 1]\end{aligned}$$

$$\begin{aligned}\mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 \sin u & 5 \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= [(5 \cos u - 0), -(-5 \sin u - 0), (0 - 0)] \\ &= \boxed{5 [\cos u, \sin u, 0]}\end{aligned}$$



Problem 10.5.18.

Find a normal vector. Sketch the surface and parameter curves.

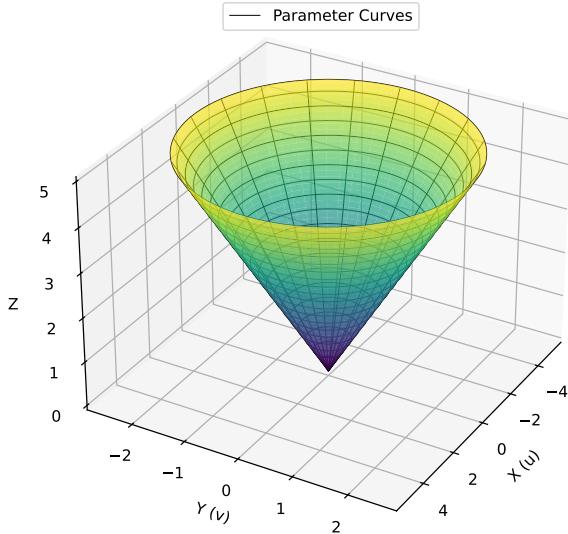
$$\text{Elliptic cone } z = \sqrt{x^2 + 4y^2}$$

Solution.

$$\mathbf{r}(u, v) = u[\cos v, 0.5 \sin v, 1]$$

$$\begin{aligned}\mathbf{r}_u &= [\cos v, 0.5 \sin v, 1] \\ \mathbf{r}_v &= u[-\sin v, 0.5 \cos v, 0]\end{aligned}$$

$$\begin{aligned}
 \mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & 0.5 \sin v & 1 \\ -u \sin v & 0.5u \cos v & 0 \end{vmatrix} \\
 &= [(0 - 0.5u \cos v), -(0 + u \sin v), (0.5u \cos^2 v + 0.5u \sin^2 v)] \\
 &= [u [-0.5 \cos v, -\sin v, 0.5]]
 \end{aligned}$$



Problem 10.6.3.

Evaluate the flux integral, $\int_S \mathbf{F} \cdot \mathbf{n} dA$, for the given data. Describe the kind of surface.

$$\mathbf{F} = [0, x, 0], \quad S : x^2 + y^2 + z^2 = 1, \quad x, y, z \geq 0$$

Solution.

S is $\frac{1}{8}$ of a sphere with radius 1,

$$\mathbf{r}(u, v) = [\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi] \quad u \in [0, \pi/2], \quad v \in [0, \pi/2]$$

$$\begin{aligned}
 \mathbf{N} &= \mathbf{r}_\phi \times \mathbf{r}_\theta \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\ -\sin \phi \sin \theta & \sin \phi \cos \theta & 0 \end{vmatrix} \\
 &= [(0 + \sin^2 \phi \cos \theta), -(0 - \sin^2 \phi \sin \theta), (\cos \phi \sin \phi \cos^2 \theta + \cos \phi \sin \phi \sin^2 \theta)] \\
 &= [(\sin^2 \phi \cos \theta), (\sin^2 \phi \sin \theta), (\cos \phi \sin \phi)]
 \end{aligned}$$

$$\iint_S \mathbf{F} \cdot \mathbf{n} dA = \iint_R \mathbf{F} \cdot \mathbf{N} d\phi d\theta$$

$$\begin{aligned}
\mathbf{F} \cdot \mathbf{N} &= [0, x, 0] \cdot [(\sin^2 \phi \cos \theta), (\sin^2 \phi \sin \theta), (\cos \phi \sin \phi)] \\
&= 0 + x \cdot (\sin^2 \phi \sin \theta) + 0 \\
&= -(\sin \phi \cos \theta)(\sin^2 \phi \sin \theta) \\
&= -\sin^3 \phi \cos \theta \sin \theta
\end{aligned}$$

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n} \, dA &= \int_0^{\pi/2} \int_0^{\pi/2} [-\sin^3 \phi \cos \theta \sin \theta] \, d\phi \, d\theta \\
&= \left(- \int_0^{\pi/2} \sin^3 \phi \, d\phi \right) \left(\int_0^{\pi/2} \cos \theta \sin \theta \, d\theta \right)
\end{aligned}$$

First integral substitute $u = \cos \phi$, $du = -\sin \phi \, d\phi$:

$$\begin{aligned}
-\int_0^{\pi/2} \sin^3 \phi \, d\phi &= -\int_0^{\pi/2} (1 - \cos^2 \phi) \sin \phi \, d\phi \\
&= \int_1^0 (1 - u^2) \, du \\
&= u - \frac{u^3}{3} \Big|_1^0 \\
&= \frac{2}{3}
\end{aligned}$$

Second integral substitute $u = \sin \theta$, $du = \cos \theta \, d\theta$:

$$\begin{aligned}
\int_0^{\pi/2} \cos \theta \sin \theta \, d\theta &= \int_0^1 u \, du \\
&= \frac{u^2}{2} \Big|_1^0 \\
&= \frac{1}{2}
\end{aligned}$$

Therefore,

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \frac{1}{3}}$$

Problem 10.6.5.

Evaluate the flux integral for the given data. Describe the kind of surface.

$$\mathbf{F} = [x, y, z], \quad S : \mathbf{r} = [u \cos v, u \sin v, u^2], \quad 0 \leq u \leq 4, \quad -\pi \leq v \leq \pi$$

Solution.

S is a paraboloid of revolution bounded by $z = 16$.

$$\begin{aligned}
\mathbf{N} &= \mathbf{r}_u \times \mathbf{r}_v \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix} \\
&= [(0 - 2u^2 \cos v), -(0 + 2u^2 \sin v), (u \cos^2 v + u \sin^2 v)] \\
&= [(-2u^2 \cos v), (-2u^2 \sin v), (u)]
\end{aligned}$$

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \iint_R \mathbf{F} \cdot \mathbf{N} \, du \, dv$$

$$\begin{aligned}
\mathbf{F} \cdot \mathbf{N} &= [x, y, z] \cdot [-2u^2 \cos v, -2u^2 \sin v, u] \\
&= [u \cos v, u \sin v, u^2] \cdot [-2u^2 \cos v, -2u^2 \sin v, u] \\
&= -2u^3 \cos^2 v - 2u^3 \sin^2 v + u^3 \\
&= -u^3
\end{aligned}$$

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n} \, dA &= \int_{-\pi}^{\pi} \int_0^4 -u^3 \, du \, dv \\
&= \left(\int_{-\pi}^{\pi} dv \right) \left(\int_0^4 -u^3 \, du \right) \\
&= 2\pi \left[-\frac{u^4}{4} \right]_0^4 \\
&= -128\pi
\end{aligned}$$

Problem 10.6.7.

Evaluate the flux integral for the given data. Describe the kind of surface.

$$\mathbf{F} = [0, \sin y, \cos z], \quad S \text{ the cylinder } x = y^2, \text{ where } 0 \leq y \leq \frac{\pi}{4}, \quad 0 \leq z \leq y$$

Solution.

Problem 10.6.13.

Evaluate the surface integral, $\iint_S G(\mathbf{r}) \, dA$, for the following data. Indicate the kind of surface.

$$G = x + y + z, \quad z = x + 2y, \quad 0 \leq x \leq \pi, \quad 0 \leq y \leq x$$

Solution.

Problem 10.6.15.

Evaluate the surface integral for the following data. Indicate the kind of surface.

$$G = (1 + 9xz)^{3/2}, \quad S : \mathbf{r} = [u, v, u^3], 0 \leq u \leq 1, \quad -2 \leq v \leq 2$$

Solution.