Problem 10.7.11

Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} \; dA$ by the divergence theorem:

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} \; dA = \iiint_T \nabla \cdot \mathbf{F} \; dV}$$

$$\mathbf{F} = [e^x, e^y, e^z], \quad S \text{ the surface of the cube } |x| \leqq 1, \; |y| \leqq 1, \; |z| \leqq 1$$

Solution

$$\nabla \cdot \mathbf{F} = e^x + e^y + e^z$$

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n} \; dA &= \iiint_T \nabla \cdot \mathbf{F} \; dV \\
&= \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} (e^x + e^y + e^z) \; dx \; dy \; dz \\
&= \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} e^x \; dx \; dy \; dz + \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} e^y \; dx \; dy \; dz + \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} e^z \; dx \; dy \; dz \\
&= 3 \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} e^x \; dx \; dy \; dz \\
&= 12 \int_{-1}^{1} e^x \; dx \\
&= 12 e^x \Big|_{-1}^{1} \\
&= \boxed{12(e - e^{-1})}
\end{aligned}$$

Problem 10.7.13

Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} \; dA$ by the divergence theorem.

$$\mathbf{F} = [\sin y, \cos x, \cos z], \quad S \text{ the surface of } x^2 + y^2 \leqq 4, \; |z| \leqq 2 \text{ (a cylinder and two disks!)}$$

Solution

Problem 10.7.17

Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} \; dA$ by the divergence theorem.

$$\mathbf{F} = \left[x^2, y^2, z^2\right], \quad S \text{ the surface of the cone } x^2 + y^2 \leqq z^2, \; 0 \leqq z \leqq h$$

Solution

Problem 10.7.22

Given a mass of density 1 in a region T of space, find the moment of intertia about the x-axis,

$$I_x = \iiint_T \left(y^2 + z^2\right)\ dV$$

$$T : \text{The paraboloid } y^2 + z^2 \leqq x,\ 0 \leqq x \leqq h$$

Solution

Problem 10.8.1. Harmonic functions.

Verify Theorem 1 for $f = 2z^2 - x^2 - y^2$ and S the surface of the box $0 \leqq x \leqq a,\ 0 \leqq y \leqq b,\ 0 \leqq z \leqq c$.

Theorem 1. A Basic Property of Harmonic Functions

Let $f(x, y, z)$ be a harmonic function in some domain D is space. Let S be any piecewise smooth closed orientable surface in D whose entire region it encloses belongs to D. Then the integral of the normal derivative of f taken over S is zero.

Solution

Problem 10.8.3. Green's First Identity

$$\boxed{\iiint_T \left(f\,\Delta\, g + \nabla f \cdot \nabla g\right)\ dV = \iint_S f \frac{\partial g}{\partial n}\ dA}$$

Verify for $f = 4y^2$, $g = x^2$, S the surface of the "unit cube" $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, $0 \leqq z \leqq 1$. What are the assumptions of f and g? Must f and g be harmonic?

Solution

Problem 10.8.5. Green's Second Identity

$$\boxed{\iiint_T \left(f\,\Delta\, g - g\,\Delta\, f\right)\ dV = \iint_S \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n}\right)\ dA}$$

Verify for $f = 6y^2$, $g = 2x^2$, S the surface of the "unit cube" $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, $0 \leqq z \leqq 1$.

Solution

Problem 10.8.7

Use the divergence theorem, assuming that the assumptions on T and S are satisfied.

Show that a region T with boundary surface S has the volume

$$V = \iint_S x\ dy\ dz = \iint_S y\ dz\ dx = \iint_S z\ dx\ dy = \frac{1}{3} \iint_S \left(x\ dy\ dz + y\ dz\ dx + z\ dx\ dy\right)$$

Solution

Problem 10.9.3

Evaluate the surface integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dA$ for the given $\mathbf{F}$ and S.

$$\mathbf{F} = [e^{-z}, e^{-z}\cos y, e^{-z}\sin y], \quad S : z = \frac{y^2}{2},\ -1 \leqq x \leqq 1,\ 0 \leqq y \leqq 1$$

Solution

Problem 10.9.5

Evaluate the surface integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dA$ for the given $\mathbf{F}$ and S.

$$\mathbf{F} = [z^2, \frac{3}{2}x, 0], \quad S : 0 \leqq x \leqq a,\ 0 \leqq y \leqq a,\ z = 1$$

Solution

Problem 10.9.13

Calculate $\oint_C \mathbf{F} \cdot \mathbf{r}'(s)\ ds$ by Stokes's theorem,

$$\boxed{\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dA = \oint_C \mathbf{F} \cdot \mathbf{r}'(s)\ ds}$$

for the given $\mathbf{F}$ and C. Assume the Cartesian coordinates to be right-handed and the z-component of the surface normal to be nonnegative.

$$\mathbf{F} = [-5y, 4x, z], \quad C \text{ the circle } x^2 + y^2 = 16,\ z = 4$$

Solution

Problem 10.9.15

Calculate $\oint_C \mathbf{F} \cdot \mathbf{r}'(s)\ ds$ by Stokes's theorem for the given $\mathbf{F}$ and C. Assume the Cartesian coordinates to be right-handed and the z-component of the surface normal to be nonnegative.

$$\mathbf{F} = [z^3, x^3, y^3], \text{ around the triangle with vertices } (0,0,0),\ (1,0,0),\ (1,1,0)$$

Solution

Problem 10.9.19

Calculate $\oint_C \mathbf{F} \cdot \mathbf{r}'(s)\ ds$ by Stokes's theorem for the given $\mathbf{F}$ and C. Assume the Cartesian coordinates to be right-handed and the z-component of the surface normal to be nonnegative.

$$\mathbf{F} = [z, e^z, 0], \quad C \text{ the boundary curve of the portion of the cone } z = \sqrt{x^2 + y^2},\ x \geqq 0,\ y \geqq 0, 0 \leqq z \leqq 1$$

Solution