

Problem 9.7.3.

Find ∇f . Graph some level curves $f = \text{const.}$ Indicate ∇f by arrows at some points of these curves.

$$f = \frac{y}{x}$$

Solution.

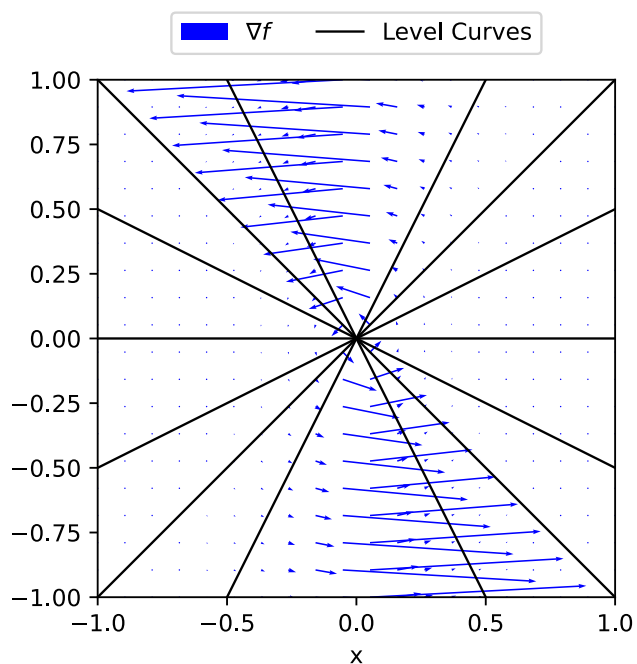
$$\nabla f = [f_x, f_y]$$

$$\nabla f = \left[-\frac{y}{x^2}, \frac{1}{x} \right]$$

Level Surfaces:

$$f(x, y) = \frac{y}{x} = c$$

$$y = cx$$



Problem 9.7.8.

Prove and illustrate by an example

$$\nabla(fg) = f \nabla g + g \nabla f$$

Proof.

$$\nabla(fg) = \left[\frac{\partial}{\partial x_1}(fg), \dots, \frac{\partial}{\partial x_n}(fg) \right]$$

The i -th component can be expanded by chain rule,

$$\frac{\partial}{\partial x_i}(fg) = f \frac{\partial g}{\partial x_i} + g \frac{\partial f}{\partial x_i}$$

Therefore in general,

$$\begin{aligned} \nabla(fg) &= \left[\frac{\partial}{\partial x_1}(fg), \dots, \frac{\partial}{\partial x_n}(fg) \right] \\ &= f \left[\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n} \right] + g \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right] \\ &= \boxed{f \nabla g + g \nabla f} \end{aligned}$$

Example.

$$\nabla(xy) = x \nabla y + y \nabla x$$

$$\nabla(xy) = x [0, 1] + y [1, 0] = [y, x]$$

Problem 9.7.21.

Given the velocity potential f of a flow, find the velocity $\mathbf{v} = \nabla f$ of the field and its value $\mathbf{v}(P)$ at P . Sketch $\mathbf{v}(P)$ and the curve $f = \text{const}$ passing through P .

$$f = e^x \cos y, \quad P : (1, 1)$$

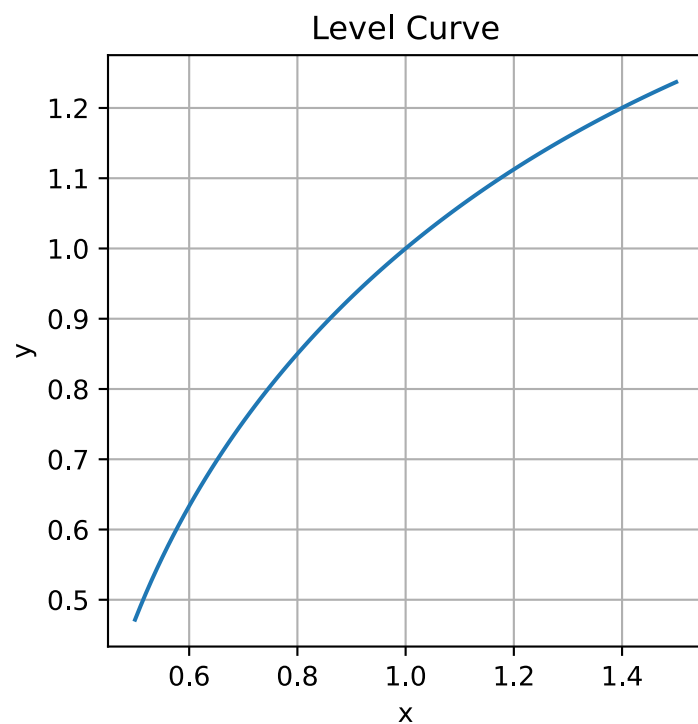
Solution.

$$\begin{aligned} \mathbf{v} &= \nabla(e^x \cos y) \\ &= \boxed{[e^x \cos y, -e^x \sin y]} \end{aligned}$$

$$\boxed{\mathbf{v}(P) = [e \cos 1, -e \sin 1]}$$

At P , $f = (e \cos 1)$. The level curve is

$$y = \cos^{-1}(e^{1-x} \cos 1)$$

**Problem 9.7.26.**

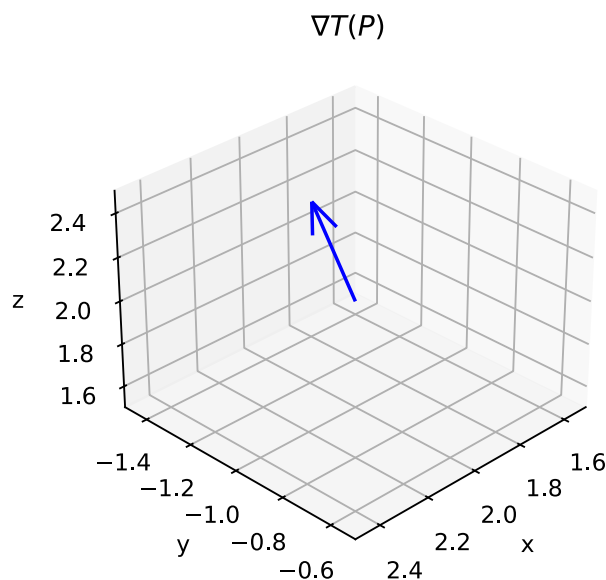
Experiments show that in a temperature field, heat flows in the direction of maximum decrease of temperature T . Find this direction in general and at the given point P . Sketch that direction at P as an arrow.

$$T = x^2 + y^2 + 4z^2, \quad P : (2, -1, 2)$$

Solution.

$$\nabla T = [2x, 2y, 8z]$$

$$\nabla T(P) = [4, -2, 16]$$



Problem 9.8.3.

Find $\nabla \cdot \mathbf{v}$ and its value at P :

$$\mathbf{v} = (x^2 + y^2)^{-1}[x, y]$$

Solution.

$$\begin{aligned} \mathbf{v} &= \left[\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right] \\ \nabla \cdot \mathbf{v} &= \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) \\ &= \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} \\ &= \boxed{0} \end{aligned}$$

Problem 9.8.5.

Find $\nabla \cdot \mathbf{v}$ and its value at P :

$$\mathbf{v} = x^2 y^2 z^2 [x, y, z], \quad P : (3, -1, 4)$$

Solution.

$$\mathbf{v} = [x^3 y^2 z^2, x^2 y^3 z^2, x^2 y^2 z^3]$$

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \frac{\partial}{\partial x}(x^3 y^2 z^2) + \frac{\partial}{\partial y}(x^2 y^3 z^2) + \frac{\partial}{\partial z}(x^2 y^2 z^3) \\ &= \boxed{9x^2 y^2 z^2}\end{aligned}$$

$$\boxed{\nabla \cdot \mathbf{v}(\mathbf{P}) = 1296}$$

Problem 9.8.9. PROJECT. Useful Formulas for the Divergence.

Part a.

Prove $\nabla \cdot (k\mathbf{v}) = k \nabla \cdot \mathbf{v}$, (k constant)

$$k\mathbf{v} = [kv_1, \dots, kv_n]$$

$$\begin{aligned}\nabla \cdot (k\mathbf{v}) &= \frac{\partial}{\partial x_1}(kv_1) + \dots + \frac{\partial}{\partial x_n}(kv_n) \\ &= k \left(\frac{\partial v_1}{\partial x_1} + \dots + \frac{\partial v_n}{\partial x_n} \right) \\ &= \boxed{k \nabla \cdot \mathbf{v}}\end{aligned}$$

Part b.

Prove $\nabla \cdot (f\mathbf{v}) = f(\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla f$

$$f\mathbf{v} = [fv_1, \dots, fv_n]$$

$$\nabla \cdot (f\mathbf{v}) = \frac{\partial}{\partial x_1}(fv_1) + \dots + \frac{\partial}{\partial x_n}(fv_n)$$

The i -th component can be expanded by chain rule,

$$\frac{\partial}{\partial x_i}(fv_i) = f \frac{\partial v_i}{\partial x_i} + \frac{\partial f}{\partial x_i} v_i$$

Therefore in general,

$$\begin{aligned}\nabla \cdot (f\mathbf{v}) &= \frac{\partial}{\partial x_1}(fv_1) + \dots + \frac{\partial}{\partial x_n}(fv_n) \\ &= f \left(\frac{\partial v_1}{\partial x_1} + \dots + \frac{\partial v_n}{\partial x_n} \right) + \left(v_1 \frac{\partial f}{\partial x_1} + \dots + v_n \frac{\partial f}{\partial x_n} \right) \\ &= f \left(\frac{\partial v_1}{\partial x_1} + \dots + \frac{\partial v_n}{\partial x_n} \right) + \mathbf{v} \cdot \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right] \\ &= \boxed{f \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla f}\end{aligned}$$

Verify for $f = e^{xyz}$ and $\mathbf{v} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$.

$$\begin{aligned}\nabla \cdot (f\mathbf{v}) &= \nabla \cdot [axe^{xyz}, bye^{xyz}, cze^{xyz}] \\ &= (ae^{xyz} + axye^{xyz}) + (bye^{xyz} + bxyze^{xyz}) + (cze^{xyz} + cxyze^{xyz}) \\ &= \boxed{(a+b+c)e^{xyz} + (a+b+c)xyze^{xyz}}\end{aligned}$$

$$\begin{aligned}
\nabla \cdot (f \mathbf{v}) &= e^{xyz} \nabla \cdot [ax, by, cz] + [ax, by, cz] \cdot \nabla e^{xyz} \\
&= (a + b + c)e^{xyz} + [ax, by, cz] \cdot [yze^{xyz}, xze^{xyz}, xye^{xyz}] \\
&= \boxed{(a + b + c)e^{xyz} + (a + b + c)xyz e^{xyz}}
\end{aligned}$$

Obtain the answer to Prob. 6.

Find $\nabla \cdot (x^2 + y^2 + z^2)^{-3/2} [x, y, z]$.

$$\begin{aligned}
\mathbf{v} &= [x, y, z], & \nabla \cdot \mathbf{v} &= \nabla \cdot [x, y, z] \\
& & &= 3 \\
f &= (x^2 + y^2 + z^2)^{-3/2}, & \nabla f &= \nabla (x^2 + y^2 + z^2)^{-3/2} \\
& & &= -3(x^2 + y^2 + z^2)^{-5/2} [x, y, z]
\end{aligned}$$

$$\begin{aligned}
\nabla \cdot (f \mathbf{v}) &= f \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla f \\
&= (x^2 + y^2 + z^2)^{-3/2} \cdot 3 + [x, y, z] \cdot -3(x^2 + y^2 + z^2)^{-5/2} [x, y, z] \\
&= \boxed{0}
\end{aligned}$$

Part c.

Prove $\nabla \cdot (f \nabla g) = f \Delta g + \nabla f \cdot \nabla g$

$$\begin{aligned}
f \nabla g &= \left[f \frac{\partial g}{\partial x_1}, \dots, f \frac{\partial g}{\partial x_n} \right] \\
\nabla \cdot (f \nabla g) &= \frac{\partial}{\partial x_1} \left(f \frac{\partial g}{\partial x_1} \right) + \dots + \frac{\partial}{\partial x_n} \left(f \frac{\partial g}{\partial x_n} \right)
\end{aligned}$$

The i -th component can be expanded by chain rule,

$$\frac{\partial}{\partial x_i} \left(f \frac{\partial g}{\partial x_i} \right) = f \frac{\partial^2 g}{\partial x_i^2} + \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i}$$

Therefore in general,

$$\begin{aligned}
\nabla \cdot (f \nabla g) &= \frac{\partial}{\partial x_1} \left(f \frac{\partial g}{\partial x_1} \right) + \dots + \frac{\partial}{\partial x_n} \left(f \frac{\partial g}{\partial x_n} \right) \\
&= f \left(\frac{\partial^2 g}{\partial x_1^2} + \dots + \frac{\partial^2 g}{\partial x_n^2} \right) + \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right] \cdot \left[\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n} \right] \\
&= \boxed{f \Delta g + \nabla f \cdot \nabla g}
\end{aligned}$$

Verify for $f = x^2 - y^2$ **and** $g = e^{x+y}$.

	∇	Δ
f	$[2x, -2y]$	$[e^{x+y}, e^{x+y}]$
g	$[e^{x+y}, e^{x+y}]$	$[e^{x+y}, e^{x+y}]$

$$\begin{aligned}
\nabla \cdot (f \nabla g) &= \nabla \cdot ((x^2 - y^2) [e^{x+y}, e^{x+y}]) \\
&= \nabla \cdot [x^2 e^{x+y} - y^2 e^{x+y}, x^2 e^{x+y} - y^2 e^{x+y}] \\
&= (2x + x^2 - y^2) e^{x+y} + (x^2 - 2y - y^2) e^{x+y} \\
&= \boxed{2(x + x^2 - y - y^2) e^{x+y}}
\end{aligned}$$

$$\begin{aligned}
\nabla \cdot (f \nabla g) &= f \Delta g + \nabla f \cdot \nabla g \\
&= (x^2 - y^2)[e^{x+y}, e^{x+y}] + [2x, -2y] \cdot [e^{x+y}, e^{x+y}] \\
&= \boxed{2(x + x^2 - y - y^2)e^{x+y}}
\end{aligned}$$

Part d.

Prove $\nabla \cdot (f \nabla g) - \nabla \cdot (g \nabla f) = f \Delta g - g \Delta f$

From **Part c**,

$$\nabla \cdot (f \nabla g) - \nabla \cdot (g \nabla f) = (f \Delta g + \nabla f \cdot \nabla g) - (g \Delta f + \nabla g \cdot \nabla f)$$

Therefore,

$$\nabla \cdot (f \nabla g) - \nabla \cdot (g \nabla f) = f \Delta g - g \Delta f$$

Problem 9.8.17.

Calculate Δf by Eq. (3). Check by direct differentiation. Indicate when (3) is simpler. Show the details of your work.

Eq. (3):

$$\nabla \cdot (\nabla f) = \Delta f$$

$$f = \ln(x^2 + y^2)$$

Solution.

$$\begin{aligned}
\Delta f &= \nabla \cdot (\nabla f) \\
&= \nabla \cdot \left[\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right] \\
&= \frac{2(x^2 + y^2) - 4x^2}{(x^2 + y^2)^2} + \frac{2(x^2 + y^2) - 4y^2}{(x^2 + y^2)^2} \\
&= \frac{4(x^2 + y^2) - 4(x^2 + y^2)}{(x^2 + y^2)^2} \\
&= \boxed{0}
\end{aligned}$$

$$\begin{aligned}
\Delta f &= f_{xx} + f_{yy} \\
&= \frac{2(x^2 + y^2) - 4x^2}{(x^2 + y^2)^2} + \frac{2(x^2 + y^2) - 4y^2}{(x^2 + y^2)^2} \\
&= \frac{4(x^2 + y^2) - 4(x^2 + y^2)}{(x^2 + y^2)^2} \\
&= \boxed{0}
\end{aligned}$$

Problem 9.9.4.

Find $\nabla \times \mathbf{v}$ for \mathbf{v} given with respect to right-handed Cartesian coordinates. Show the details of your work.

$$\mathbf{v} = [2y^2, 5x, 0]$$

Solution.

$$\begin{aligned}
 \nabla \times \mathbf{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^2 & 5x & 0 \end{vmatrix} \\
 &= \hat{i} \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(5x) \right) - \hat{j} \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(2y^2) \right) + \hat{k} \left(\frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(2y^2) \right) \\
 &= \hat{k}(5 - 4y) \\
 &= \boxed{[0, 0, 5 - 4y]}
 \end{aligned}$$

Problem 9.9.9.

Let \mathbf{v} be the velocity vector of a steady fluid flow.

$$\mathbf{v} = [0, 3z^2, 0]$$

Is the flow irrotational?

$$\begin{aligned}
 \nabla \times \mathbf{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 3z^2 & 0 \end{vmatrix} \\
 &= \hat{i} \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(3z^2) \right) - \hat{j} \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(0) \right) + \hat{k} \left(\frac{\partial}{\partial x}(3z^2) - \frac{\partial}{\partial y}(0) \right) \\
 &= \hat{i}(-6z) \\
 &= \boxed{[-6z, 0, 0]}
 \end{aligned}$$

$\nabla \times \mathbf{v} \neq 0$ therefore the flow is **rotational**.

Is the flow incompressible?

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(3z^2) + \frac{\partial}{\partial z}(0) = \boxed{0}$$

$\nabla \cdot \mathbf{v} = 0$ therefore the flow is **incompressible**.

Find the streamlines (the paths of the particles).

$$\mathbf{v} = \mathbf{r}' = [x', y', z']$$

$$\begin{cases} x' = 0 \\ y' = 3z^2 \\ z' = 0 \end{cases}$$

Integrating, we get:

$$\begin{cases} x = x_0 \\ y = y_0 + 3z_0^2 t \\ z = z_0 \end{cases}$$

Streamlines are in the form $\boxed{\mathbf{r}(t) = [x_0, y_0 + 3z_0^2 t, z_0]}$.

Problem 9.9.14. PROJECT. Useful Formulas for the Curl.

Assuming sufficient differentiability, show that

Part a.

$$\nabla \times (\mathbf{u} + \mathbf{v}) = \nabla \times \mathbf{u} + \nabla \times \mathbf{v}$$

Solution.

For 3D vectors \mathbf{u} and \mathbf{v} ,

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= [u_x + v_x, u_y + v_y, u_z + v_z] \\ \nabla \times \mathbf{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x + v_x & u_y + v_y & u_z + v_z \end{vmatrix} \\ &= \hat{i} \left(\frac{\partial}{\partial y}(u_z + v_z) - \frac{\partial}{\partial z}(u_y + v_y) \right) - \hat{j} \left(\frac{\partial}{\partial x}(u_z + v_z) - \frac{\partial}{\partial z}(u_x + v_x) \right) + \hat{k} \left(\frac{\partial}{\partial x}(u_y + v_y) - \frac{\partial}{\partial y}(u_x + v_x) \right) \\ &= \left[\left(\frac{\partial}{\partial y}(u_z + v_z) - \frac{\partial}{\partial z}(u_y + v_y) \right), \left(\frac{\partial}{\partial z}(u_x + v_x) - \frac{\partial}{\partial x}(u_z + v_z) \right), \left(\frac{\partial}{\partial x}(u_y + v_y) - \frac{\partial}{\partial y}(u_x + v_x) \right) \right] \\ &= \left[\left(\frac{\partial u_z}{\partial y} + \frac{\partial v_z}{\partial y} - \frac{\partial u_y}{\partial z} - \frac{\partial v_y}{\partial z} \right), \left(\frac{\partial u_x}{\partial z} + \frac{\partial v_x}{\partial z} - \frac{\partial u_z}{\partial x} - \frac{\partial v_z}{\partial x} \right), \left(\frac{\partial u_y}{\partial x} + \frac{\partial v_y}{\partial x} - \frac{\partial u_x}{\partial y} - \frac{\partial v_x}{\partial y} \right) \right] \\ &= \left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right), \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right), \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \right] + \left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right), \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right), \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \right] \\ &= \boxed{\nabla \times \mathbf{u} + \nabla \times \mathbf{v}} \end{aligned}$$

Part b

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0$$

Solution.

For 3D vector \mathbf{v} ,

$$\begin{aligned} \nabla \times \mathbf{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\ &= \hat{i} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) - \hat{j} \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right) + \hat{k} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \\ &= \left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right), \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right), \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \right] \\ \nabla \cdot (\nabla \times \mathbf{v}) &= \nabla \cdot \left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right), \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right), \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \right] \\ &= \frac{\partial}{\partial x} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \\ &= \frac{\partial^2 v_z}{\partial x \partial y} - \frac{\partial^2 v_y}{\partial x \partial z} + \frac{\partial^2 v_x}{\partial y \partial z} - \frac{\partial^2 v_z}{\partial x \partial y} + \frac{\partial^2 v_y}{\partial x \partial z} - \frac{\partial^2 v_x}{\partial y \partial z} \\ &= \boxed{0} \end{aligned}$$

Part c

$$\nabla \times (f \mathbf{v}) = (\nabla f) \times \mathbf{v} + f \nabla \times \mathbf{v}$$

Solution.

For 3D vector \mathbf{v} ,

$$\begin{aligned} \nabla \times \mathbf{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f v_x & f v_y & f v_z \end{vmatrix} \\ &= \hat{i} \left(\frac{\partial}{\partial y} (f v_z) - \frac{\partial}{\partial z} (f v_y) \right) - \hat{j} \left(\frac{\partial}{\partial z} (f v_x) - \frac{\partial}{\partial x} (f v_z) \right) + \hat{k} \left(\frac{\partial}{\partial x} (f v_y) - \frac{\partial}{\partial y} (f v_x) \right) \\ &= \left[\left(\frac{\partial}{\partial y} (f v_z) - \frac{\partial}{\partial z} (f v_y) \right), \left(\frac{\partial}{\partial z} (f v_x) - \frac{\partial}{\partial x} (f v_z) \right), \left(\frac{\partial}{\partial x} (f v_y) - \frac{\partial}{\partial y} (f v_x) \right) \right] \end{aligned}$$

Expanding with chain rule,

$$\begin{aligned} &= \left[\left(f \frac{\partial v_z}{\partial y} + v_z \frac{\partial f}{\partial y} - f \frac{\partial v_y}{\partial z} - v_y \frac{\partial f}{\partial z} \right), \left(f \frac{\partial v_z}{\partial x} - v_z \frac{\partial f}{\partial x} - f \frac{\partial v_x}{\partial z} + v_x \frac{\partial f}{\partial z} \right), \left(f \frac{\partial v_y}{\partial x} + v_y \frac{\partial f}{\partial x} - f \frac{\partial v_x}{\partial y} - v_x \frac{\partial f}{\partial y} \right) \right] \\ &= \left[\left(v_z \frac{\partial f}{\partial y} - v_y \frac{\partial f}{\partial z} \right), \left(v_x \frac{\partial f}{\partial z} - v_z \frac{\partial f}{\partial x} \right), \left(v_y \frac{\partial f}{\partial x} - v_x \frac{\partial f}{\partial y} \right) \right] + \left[f \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right), f \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right), f \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \right] \\ &= \boxed{(\nabla f) \times \mathbf{v} + f \nabla \times \mathbf{v}} \end{aligned}$$

Part d

$$\nabla \times (\nabla f) = \mathbf{0}$$

Solution.

For $f(x, y, z)$,

$$\begin{aligned} \nabla \times (\nabla f) &= \nabla \times \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] \\ &= \left[\left(\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial z} \right) - \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial y} \right) \right), \left(\frac{\partial}{\partial z} \left(\frac{\partial f}{\partial x} \right) - \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial z} \right) \right), \left(\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \right) \right] \\ &= [0, 0, 0] \\ &= \boxed{\mathbf{0}} \end{aligned}$$

Part e

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \nabla \times \mathbf{u} - \mathbf{u} \cdot \nabla \times \mathbf{v}$$

Solution.

For 3D vectors \mathbf{u} and \mathbf{v} ,

$$(\mathbf{u} \times \mathbf{v}) = [(u_y v_z - u_z v_y), (u_z v_x - u_x v_z), (u_x v_y - u_y v_x)]$$

$$\begin{aligned}
\nabla \cdot (\mathbf{u} \times \mathbf{v}) &= \nabla \cdot [(u_y v_z - u_z v_y), (u_z v_x - u_x v_z), (u_x v_y - u_y v_x)] \\
&= \frac{\partial}{\partial x} (u_y v_z - u_z v_y) + \frac{\partial}{\partial y} (u_z v_x - u_x v_z) + \frac{\partial}{\partial z} (u_x v_y - u_y v_x) \\
&= \left(v_z \frac{\partial u_y}{\partial x} - v_y \frac{\partial u_z}{\partial x} \right) + \left(v_x \frac{\partial u_z}{\partial y} - v_z \frac{\partial u_x}{\partial y} \right) + \left(v_y \frac{\partial u_x}{\partial z} - v_x \frac{\partial u_y}{\partial z} \right) + \\
&\quad \left(u_y \frac{\partial v_z}{\partial x} - u_z \frac{\partial v_y}{\partial x} \right) + \left(u_z \frac{\partial v_x}{\partial y} - u_x \frac{\partial v_z}{\partial y} \right) + \left(u_x \frac{\partial v_y}{\partial z} - u_y \frac{\partial v_x}{\partial z} \right) \\
&= v_x \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right) + v_y \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right) + v_z \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) + \\
&\quad u_x \left(\frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y} \right) + u_y \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right) + u_z \left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right) \\
&= [v_x, v_y, v_z] \cdot \left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right), \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right), \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \right] + \\
&\quad [u_x, u_y, u_z] \cdot \left[\left(\frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y} \right), \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right), \left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right) \right] \\
&= \boxed{\mathbf{v} \cdot \nabla \times \mathbf{u} - \mathbf{u} \cdot \nabla \times \mathbf{v}}
\end{aligned}$$

Problem 10.1.2.

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If \mathbf{F} is a force, this gives the work done by the force in the displacement along C . Show the details.

$$\mathbf{F} = [y^2, -x^2], \quad C : y = 4x^2 \text{ from } (0, 0) \text{ to } (1, 4)$$

Solution.

$$\mathbf{r} = [t, 4t^2], \quad t \in [0, 1]$$

$$\begin{aligned}
\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\
&= \int_0^1 [16t^4, -t^2] \cdot [1, 8t] dt \\
&= \int_0^1 (16t^4 - 8t^3) dt \\
&= \left[\frac{16}{5} t^5 - 2t^4 \right]_0^1 \\
&= \frac{16}{5} - 2 \\
&= \boxed{\frac{6}{5}}
\end{aligned}$$

Problem 10.1.3.

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If \mathbf{F} is a force, this gives the work done by the force in the displacement along C . Show the details.

$$\mathbf{F} = [y^2, -x^2], \quad C \text{ from } (0, 0) \text{ straight to } (1, 4)$$

Solution.

$$\mathbf{r} = [t, 4t], \quad t \in [0, 1]$$

$$\begin{aligned} \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \\ &= \int_0^1 [16t^2, -t^2] \cdot [1, 4] \, dt \\ &= \int_0^1 (16t^2 - 4t^2) \, dt \\ &= \int_0^1 12t^2 \, dt \\ &= [4t^3]_0^1 \\ &= \boxed{4} \end{aligned}$$

Problem 10.1.5.

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If \mathbf{F} is a force, this gives the work done by the force in the displacement along C . Show the details.

$$\mathbf{F} = [xy, x^2y^2], \quad C \text{ the quarter-circle from } (2, 0) \text{ to } (0, 2) \text{ with center } (0, 0)$$

Solution.

$$\mathbf{r} = 2[\cos t, \sin t], \quad t \in \left[0, \frac{\pi}{2}\right]$$

$$\begin{aligned} \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \\ &= \int_0^1 [4 \cos t \sin t, 16 \cos^2 t \sin^2 t] \cdot 2[-\sin t, \cos t] \, dt \\ &= \int_0^1 (-8 \cos t \sin^2 t + 32 \cos^3 t \sin^2 t) \, dt \end{aligned}$$

Substituting $u = \sin t$, $du = \cos t \, dt$ and $\cos^2 t = 1 - \sin^2 t$:

$$\begin{aligned} &= \int_0^1 (-8u^2 + 32u^2(1 - u^2)) \, du \\ &= \int_0^1 (-8u^2 + 32u^2 - 32u^4) \, du \\ &= \int_0^1 (24u^2 - 32u^4) \, du \\ &= \left[8u^3 - \frac{32}{5}u^5 \right]_0^1 \\ &= \boxed{\frac{8}{5}} \end{aligned}$$

Problem 10.1.6.

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If \mathbf{F} is a force, this gives the work done by the force in the displacement along C . Show the details.

$$\mathbf{F} = [x - y, y - z, z - x], \quad C : \mathbf{r} = [2 \cos t, t, 2 \sin t] \text{ from } (2, 0, 0) \text{ to } (2, 2\pi, 0)$$

Solution.

$$\begin{aligned} \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^{2\pi} [2 \cos t - t, t - 2 \sin t, 2 \sin t - 2 \cos t] \cdot [-2 \sin t, 1, 2 \cos t] dt \\ &= \int_0^{2\pi} (-4 \cos t \sin t + 2t \sin t + t - 2 \sin t + 4 \cos t \sin t - 4 \cos^2 t) dt \\ &= \int_0^{2\pi} (2t \sin t + t - 2 \sin t - 4 \cos^2 t) dt \end{aligned}$$

Integrate Term 1 by parts ($\int u dv = uv - \int v du$):

$$\begin{aligned} u &= t & du &= dt \\ v &= -\cos t & dv &= \sin t dt \end{aligned}$$

$$\int_0^{2\pi} 2t \sin t dt = 2 \left[-t \cos t - \int_0^{2\pi} \cos t dt \right] = 2 [-t \cos t - \sin t]_0^{2\pi} = -4\pi$$

Term 2:

$$\int_0^{2\pi} t dt = \frac{t^2}{2} \Big|_0^{2\pi} = 2\pi^2$$

Term 3:

$$\int_0^{2\pi} (-2 \sin t) dt = 2 \cos t \Big|_0^{2\pi} = 0$$

In Term 4, substitute $\cos^2 t = \frac{1 + \cos 2t}{2}$:

$$\int_0^{2\pi} (-4 \cos^2 t) dt = -2 \int_0^{2\pi} (1 + \cos 2t) dt = -2 \left[t + \frac{1}{2} \sin 2t \right]_0^{2\pi} = -4\pi$$

Adding all terms together:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = -4\pi + 2\pi^2 + 0 - 4\pi = \boxed{2\pi^2 - 8\pi}$$

Problem 10.1.10.

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If \mathbf{F} is a force, this gives the work done by the force in the displacement along C . Show the details.

$$\mathbf{F} = [x, -z, 2y] \text{ from } (0, 0, 0) \text{ straight to } (1, 1, 0), \text{ then to } (1, 1, 1), \text{ back to } (0, 0, 0)$$

Solution.

$$\begin{array}{lll} (0, 0, 0) \text{ to } (1, 1, 0) & \Rightarrow & \mathbf{r}_1(t) = [t, t, 0] \quad t \in [0, 1] \\ (1, 1, 0) \text{ to } (1, 1, 1) & \Rightarrow & \mathbf{r}_2(t) = [1, 1, t] \quad t \in [0, 1] \\ (1, 1, 1) \text{ to } (0, 0, 0) & \Rightarrow & \mathbf{r}_3(t) = [1-t, 1-t, 1-t] \quad t \in [0, 1] \end{array}$$

$$\begin{aligned} \mathbf{F}(\mathbf{r}_1(t)) \cdot \mathbf{r}'_1(t) &= [t, 0, 2t] \cdot [1, 1, 0] \\ &= t \end{aligned}$$

$$\begin{aligned} \mathbf{F}(\mathbf{r}_2(t)) \cdot \mathbf{r}'_2(t) &= [1, -t, 2] \cdot [0, 0, 1] \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{F}(\mathbf{r}_3(t)) \cdot \mathbf{r}'_3(t) &= [1-t, t-1, 2(1-t)] \cdot [-1, -1, -1] \\ &= -1 + t - t + 1 - 2 + 2t \\ &= 2t - 2 \end{aligned}$$

$$\begin{aligned} \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \\ &= \int_0^1 (t) + (2) + (2t - 2) \, dt \\ &= \int_0^1 3t \, dt \\ &= \left[\frac{3}{2} t^2 \right]_0^1 \\ &= \boxed{\frac{3}{2}} \end{aligned}$$