Problem 10.2.3.

Show that the form under the integral sign is exact in the plane and evaluate the integral. Show the details of your work.

$$\int_{(\pi/2,\pi)}^{(\pi,0)} \left(\frac{1}{2} \cos \frac{1}{2} x \cos 2y \ dx - 2 \sin \frac{1}{2} x \sin 2y \ dy \right)$$

Solution.

The form is exact iff $\nabla \times \mathbf{F} = \mathbf{0}$, where $\mathbf{F} = \left[\frac{1}{2} \cos \frac{1}{2} x \cos 2y, -2 \sin \frac{1}{2} x \sin 2y \right]$. For 2D $\mathbf{F}$,

$$\nabla \times \mathbf{F} = (F_2)_x - (F_1)_y$$
$$= \left(-\cos \frac{1}{2} x \sin 2y \right) - \left(-\cos \frac{1}{2} x \sin 2y \right)$$
$$= 0$$

Therefore the form is **exact**. The integral takes the form,

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B df = f(B) - f(A)$$

Finding f, such that $\nabla f = \mathbf{F}$,

$$f = \int F_1 dx = \int \frac{1}{2} \cos \frac{1}{2} x \cos 2y \ dx = \sin \frac{1}{2} x \cos 2y + g(x)$$

$$f_y = -2 \sin \frac{1}{2} x \sin 2y + g_y = F_2 = -2 \sin \frac{1}{2} x \sin 2y \implies g_y = 0 \implies g = 0$$

$$f = \sin \frac{1}{2} x \cos 2y$$

The integral is then,

$$\int_{(\pi/2,\pi)}^{(\pi,0)} df = f(\pi,0) - f(\pi/2,\pi)$$

$$= \boxed{1 - \frac{\sqrt{2}}{2}}$$

Problem 10.2.5.

Show that the form under the integral sign is exact in space and evaluate the integral. Show the details of your work.

$$\int_{(0,0,\pi)}^{(2,1/2,\pi/2)} e^{xy} (y \sin z \ dx + x \sin z \ dy + \cos z \ dz)$$

Solution.

The form is exact iff $\nabla \times \mathbf{F} = \mathbf{0}$, where $\mathbf{F} = e^{xy}\,[y\sin z, x\sin z, \cos z]$.

$$
\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \begin{bmatrix} (F_3)_y - (F_2)_z \\ (F_1)_z - (F_3)_x \\ (F_2)_x - (F_1)_y \end{bmatrix} = \begin{bmatrix} (xe^{xy}\cos z) - (xe^{xy}\cos z) \\ (ye^{xy}\cos z) - (ye^{xy}\cos z) \\ (xye^{xy}\sin z + e^{xy}\sin z) - (xye^{xy}\sin z + e^{xy}\sin z) \end{bmatrix} = \mathbf{0}
\end{aligned}
$$

Therefore the form is **exact**. The integral takes the form,

$$
\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B df = f(B) - f(A)
$$

Finding f, such that $\nabla f = \mathbf{F}$,

$$
f = \int F_1 dx = \int e^{xy} y \sin z \; dx = e^{xy} \sin z + g(y,z)
$$

$$
f_y = xe^{xy} \sin z + g_y = F_2 = xe^{xy} \sin z \implies g_y = 0 \implies g = h(z)
$$

$$
f_z = e^{xy} \cos z + h'(z) = F_3 = e^{xy} \cos z \implies h'(z) = 0 \implies h(z) = 0
$$

$$
f = e^{xy} \sin z
$$

The integral is then,

$$
\begin{aligned}
\int_{(0,0,\pi)}^{(2,1/2,\pi/2)} df &= f(2,1/2,\pi/2) - f(0,0,\pi) \\
&= e\sin(\pi/2) - e^0 \sin(\pi) \\
&= \boxed{e}
\end{aligned}
$$

Problem 10.2.13.

Check, and if independent, integrate from $(0,0,0)$ to (a,b,c).

$$
2e^{x^2}\left(x\cos 2y \; dx - \sin 2y \; dy\right)
$$

Solution.

The integral is independent iff $\nabla \times \mathbf{F} = \mathbf{0}$, where $\mathbf{F} = 2e^{x^2}\,[x\cos 2y, -\sin 2y]$. For 2D $\mathbf{F}$,

$$
\begin{aligned}
\nabla \times \mathbf{F} &= (F_2)_x - (F_1)_y \\
&= \left(-4xe^{x^2}\sin 2y\right) - \left(-4xe^{x^2}\sin 2y\right) \\
&= 0
\end{aligned}
$$

Therefore the integral is **independent of path**. The integral takes the form,

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B df = f(B) - f(A)$$

Finding f, such that $\nabla f = \mathbf{F}$,

$$f = \int F_1 dx = \int 2e^{x^2} x \cos 2y \ dx = e^{x^2} \cos 2y + g(y)$$

$$f_y = -2e^{x^2} \sin 2y + g_y = F_2 = -2e^{x^2} \sin 2y \implies g_y = 0 \implies g(y) = 0$$

$$f = e^{x^2} \cos 2y$$

The integral is then,

$$\int_{(0,0,0)}^{(a,b,c)} df = f(a,b,c) - f(0,0,0)$$

$$= e^{a^2} \cos 2b - e^0 \cos 0$$

$$= \boxed{e^{a^2} \cos 2b - 1}$$

Problem 10.2.16.

Check, and if independent, integrate from $(0,0,0)$ to (a,b,c).

$$e^y \ dx + (xe^y - e^z) \ dy - ye^z \ dz$$

Solution.

The integral is independent iff $\nabla \times \mathbf{F} = \mathbf{0}$, where $\mathbf{F} = [e^y, (xe^y - e^z), -ye^z]$.

$$
\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & (xe^y - e^z) & -ye^z \end{vmatrix}
$$

$$= \left[\left(\frac{\partial}{\partial y}(-ye^z) - \frac{\partial}{\partial z}(xe^y - e^z) \right), \left(\frac{\partial}{\partial z}(e^y) - \frac{\partial}{\partial x}(-ye^z) \right), \left(\frac{\partial}{\partial x}(xe^y - e^z) - \frac{\partial}{\partial y}(e^y) \right) \right]$$

$$= [((-e^z) - (-e^y)), (0 - 0), (e^y - e^y)]$$

$$= \mathbf{0}$$

Therefore the integral is **independent of path**. The integral takes the form,

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B df = f(B) - f(A)$$

Finding f, such that $\nabla f = \mathbf{F}$,

$$f = \int F_2 dy = \int (xe^y - e^z)dy = xe^y - ye^z + g(x,z)$$

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$$f_x = e^y + g_x = F_1 = e^y \implies g_x = 0 \implies g(x,z) = h(z)$$

$$f_z = -ye^z + h_z = F_3 = -ye^z \implies h_z = 0 \implies h(z) = 0$$

$$f = xe^y - ye^z$$

The integral is then,

$$\int_{(0,0,0)}^{(a,b,c)} df = f(a,b,c) - f(0,0,0)$$
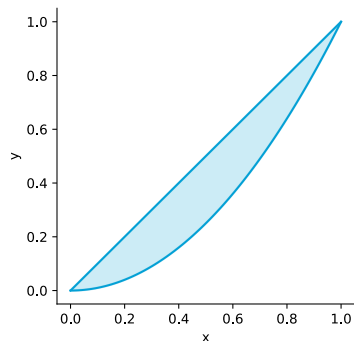$$= (ae^b - be^c) - (0 - 0)$$
$$= \boxed{ae^b - be^c}$$

Problem 10.3.5.

Describe the region of integration and evaluate.

$$\int_0^1 \int_{x^2}^{x} (1 - 2xy) \; dy \; dx$$

Solution.

The region of integration is the volume under the surface $z = 1 - 2xy$ for $y \in [x^2, x]$ and $x \in [0,1]$. The region on the xy-plane looks like a slim leaf-like shape.
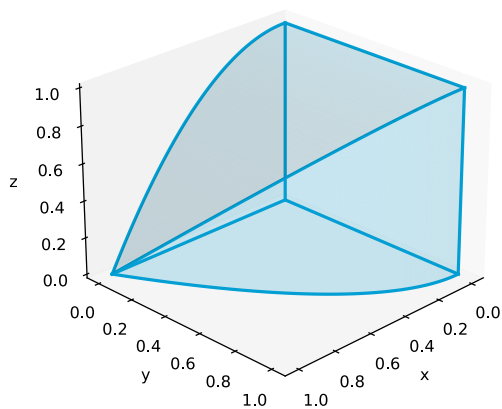

$$\int_0^1 \int_{x^2}^{x} (1 - 2xy) \; dy \; dx = \int_0^1 \left[y - xy^2 \right]_{x^2}^{x} \; dx$$
$$= \int_0^1 \left[x - x^2 - x^3 + x^5 \right] \; dx$$
$$= \left[\frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^6}{6} \right]_0^1$$
$$= \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{6}$$
$$= \boxed{\frac{1}{12}}$$

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Problem 10.3.10.

Find the volume of the first octant region bounded by the coordinate planes and the surfaces $y = 1 - x^2$ and $z = 1 - x^2$. Sketch it.
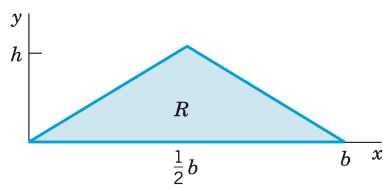
Solution.

$$
\begin{aligned}
V &= \int_0^1 \int_0^{1-x^2} \int_0^{1-x^2} dz \; dy \; dx \\
&= \int_0^1 \int_0^{1-x^2} [z]_{z=0}^{z=1-x^2} \; dy \; dx \\
&= \int_0^1 \int_0^{1-x^2} (1 - x^2) \; dy \; dx \\
&= \int_0^1 [y(1 - x^2)]_{y=0}^{y=1-x^2} \; dx \\
&= \int_0^1 (1 - x^2)(1 - x^2) \; dx \\
&= \int_0^1 (1 - 2x^2 + x^4) \; dx \\
&= x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \Big|_0^1 \\
&= 1 - \frac{2}{3} + \frac{1}{5} \\
&= \boxed{\frac{8}{15}}
\end{aligned}
$$

Problem 10.3.12.

Find the center of gravity $(\bar{x}, \bar{y})$ of a mass of density $f(x, y) = 1$ in the given region R.

Solution.

The center of gravity is found at the coordinates $(\bar{x}, \bar{y})$ where,

$$\bar{x} = \frac{1}{M} \iint_R x f(x,y) \ dx \ dy \qquad \bar{y} = \frac{1}{M} \iint_R y f(x,y) \ dx \ dy \qquad M = \iint_R f(x,y) \ dx \ dy$$

$$M = \iint_R (1) \ dx \ dy$$
$$= \frac{hb}{2}$$

$$\bar{x} = \frac{2}{hb} \iint_R x \ dx \ dy$$

$$= \frac{2}{hb} \left[\int_0^{b/2} \int_0^{(2h/b)x} x \ dy \ dx + \int_{b/2}^b \int_0^{2h-(2h/b)x} x \ dy \ dx \right]$$

$$= \frac{2}{hb} \left[\int_0^{b/2} [xy]_{y=0}^{y=(2h/b)x} \ dx + \int_{b/2}^b [xy]_{y=0}^{y=2h-(2h/b)x} \ dx \right]$$

$$= \frac{2}{hb} \left[\int_0^{b/2} \left(\frac{2h}{b} x^2 \right) \ dx + \int_{b/2}^b \left(2hx - \frac{2h}{b} x^2 \right) \ dx \right]$$

$$= \frac{2}{hb} \left[\left[\frac{2h}{3b} x^3 \right]_0^{b/2} + \left[hx^2 - \frac{2h}{3b} x^3 \right]_{b/2}^b \right]$$

$$= \frac{2}{hb} \left[\left[\frac{2h}{3b} \left(\frac{b}{2} \right)^3 \right] + \left[hb^2 - \frac{2h}{3b}(b)^3 - h \left(\frac{b}{2} \right)^2 + \frac{2h}{3b} \left(\frac{b}{2} \right)^3 \right] \right]$$

$$= \frac{2}{hb} \left[\frac{hb^2}{6} + hb^2 - \frac{2hb^2}{3} - \frac{hb^2}{4} \right]$$

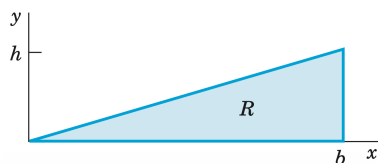$$= b \left[\frac{1}{3} + 2 - \frac{4}{3} - \frac{1}{2} \right]$$

$$\bar{x} = \boxed{\frac{1}{2} b}$$

$$\bar{y} = \frac{2}{hb} \iint_R y \ dx \ dy$$

$$= \frac{2}{hb} \left[\int_0^{b/2} \int_0^{(2h/b)x} y \ dy \ dx + \int_{b/2}^b \int_0^{2h-(2h/b)x} y \ dy \ dx \right]$$

$$= \frac{2}{hb} \cdot \frac{1}{2} \left[\int_0^{b/2} [y^2]_{y=0}^{y=(2h/b)x} \ dx + \int_{b/2}^b [y^2]_{y=0}^{y=2h-(2h/b)x} \ dx \right]$$

$$= \frac{1}{hb} \left[\int_0^{b/2} \left(\frac{2h}{b}x \right)^2 \ dx + \int_{b/2}^b \left(2h - \frac{2h}{b}x \right)^2 \ dx \right]$$

$$= \frac{1}{hb} \left[\int_0^{b/2} \left(\frac{4h^2}{b^2}x^2 \right) \ dx + \int_{b/2}^b \left(4h^2 - \frac{8h^2}{b}x + \frac{4h^2}{b^2}x^2 \right) \ dx \right]$$

$$= \frac{4h}{b} \left[\int_0^{b/2} \left(\frac{1}{b^2}x^2 \right) \ dx + \int_{b/2}^b \left(1 - \frac{2}{b}x + \frac{1}{b^2}x^2 \right) \ dx \right]$$

$$= \frac{4h}{b} \left[\left[\frac{1}{3b^2}x^3 \right]_0^{b/2} + \left[x - \frac{1}{b}x^2 + \frac{1}{3b^2}x^3 \right]_{b/2}^b \right]$$

$$= \frac{4h}{b} \left[\left[\frac{1}{3b^2}\left(\frac{b}{2}\right)^3 \right] + \left[(b) - \frac{1}{b}(b)^2 + \frac{1}{3b^2}(b)^3 - \left(\frac{b}{2}\right) + \frac{1}{b}\left(\frac{b}{2}\right)^2 - \frac{1}{3b^2}\left(\frac{b}{2}\right)^3 \right] \right]$$

$$= \frac{4h}{b} \left[\frac{b}{3} - \frac{b}{2} + \frac{b}{4} \right]$$

$$\bar{y} = \boxed{\frac{h}{3}}$$

Summary.

$$(\bar{x}, \bar{y}) = \left(\frac{b}{2}, \frac{h}{3} \right)$$

Problem 10.3.17.

Find I_x, I_y, I_0 of a mass of density $f(x,y) = 1$ in the region R.

Solution.