# Problem 10.7.11

Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\ dA$ by the divergence theorem:

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n}\ dA = \iiint_T \nabla \cdot \mathbf{F}\ dV}$$

$$\mathbf{F} = [e^x, e^y, e^z], \quad S \text{ the surface of the cube } |x| \leqq 1,\ |y| \leqq 1,\ |z| \leqq 1$$

## Solution

$$\nabla \cdot \mathbf{F} = e^x + e^y + e^z$$

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\ dA &= \iiint_T \nabla \cdot \mathbf{F}\ dV \\
&= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (e^x + e^y + e^z)\ dx\ dy\ dz \\
&= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} e^x\ dx\ dy\ dz + \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} e^y\ dx\ dy\ dz + \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} e^z\ dx\ dy\ dz \\
&= 3\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} e^x\ dx\ dy\ dz \\
&= 12\int_{-1}^{1} e^x\ dx \\
&= 12e^x\Big|_{-1}^{1} \\
&= \boxed{12(e - e^{-1})}
\end{aligned}$$

# Problem 10.7.13

Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\ dA$ by the divergence theorem.

$$\mathbf{F} = [\sin y, \cos x, \cos z], \quad S \text{ the surface of } x^2 + y^2 \leqq 4,\ |z| \leqq 2 \text{ (a cylinder and two disks!)}$$

## Solution

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= 0 + 0 - \sin z \\
&= -\sin z
\end{aligned}$$

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\ dA &= \iiint_T \nabla \cdot \mathbf{F}\ dV \\
&= \int_{-2}^{2}\int_{0}^{2\pi}\int_{0}^{2} (-\sin z) r\ dr\ d\theta\ dz \\
&= \int_{0}^{2} r\ dr \cdot \int_{0}^{2\pi} d\theta \cdot \int_{-2}^{2} (-\sin z)\ dz \\
&= \left[\frac{r^2}{2}\right]_0^2 \cdot (2\pi) \cdot [\cos z]_{-2}^{2} \\
&= 4\pi(\cos(2) - \cos(-2)) \\
&= \boxed{0}
\end{aligned}$$

# Problem 10.7.17

Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\ dA$ by the divergence theorem.

$$\mathbf{F} = \left[x^2, y^2, z^2\right], \quad S \text{ the surface of the cone } x^2 + y^2 \leqq z^2,\ 0 \leqq z \leqq h$$

## Solution

$$\nabla\cdot \mathbf{F} = 2x + 2y + 2z$$

In polar coordinates,

$$\nabla\cdot \mathbf{F} = 2(r\cos\theta + r\sin\theta + z)$$

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\ dA &= \iiint_T \nabla\cdot \mathbf{F}\ dV \\
&= 2\int_0^h \int_0^{2\pi} \int_0^z (r\cos\theta + r\sin\theta + z)r\ dr\ d\theta\ dz \\
&= 2\int_0^h \int_0^{2\pi} \int_0^z (r^2\cos\theta + r^2\sin\theta + rz\ dr\ d\theta\ dz \\
&= 2\int_0^h \int_0^{2\pi} \left[\frac{r^3}{3}\cos\theta + \frac{r^3}{3}\sin\theta + \frac{r^2}{2}z\right]_{r=0}^{r=z} d\theta\ dz \\
&= 2\int_0^h \int_0^{2\pi} z^3\left(\frac{1}{3}\cos\theta + \frac{1}{3}\sin\theta + \frac{1}{2}\right) d\theta\ dz \\
&= 2\int_0^h z^3\ dz \cdot \int_0^{2\pi} \left(\frac{1}{3}\cos\theta + \frac{1}{3}\sin\theta + \frac{1}{2}\right) d\theta \\
&= \frac{1}{2}\left[z^4\right]_0^h \cdot \left[\frac{1}{3}\sin\theta - \frac{1}{3}\cos\theta + \frac{1}{2}\theta\right]_0^{2\pi} \\
&= \boxed{\frac{\pi h^4}{2}}
\end{aligned}$$

# Problem 10.7.22

Given a mass of density 1 in a region $T$ of space, find the moment of intertia about the x-axis,

$$I_x = \iiint_T \left(y^2 + z^2\right)\ dV$$

$$T : \text{The paraboloid } y^2 + z^2 \leqq x,\ 0 \leqq x \leqq h$$

## Solution

# Problem 10.8.1. Harmonic functions.

Verify Theorem 1 for $f = 2z^2 - x^2 - y^2$ and $S$ the surface of the box $0 \leqq x \leqq a,\ 0 \leqq y \leqq b,\ 0 \leqq z \leqq c$.

**Theorem 1. A Basic Property of Harmonic Functions**

Let $f(x, y, z)$ be a harmonic function in some domain $D$ is space. Let $S$ be any piecewise smooth closed orientable surface in $D$ whose entire region it encloses belongs to $D$. Then the integral of the normal derivative of $f$ taken over $S$ is zero.

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### Solution

## Problem 10.8.3. Green's First Identity

$$\boxed{\iiint_T (f\,\Delta\,g + \nabla\,f \cdot \nabla\,g)\ dV = \iint_S f\frac{\partial g}{\partial n}\ dA}$$

Verify for $f = 4y^2$, $g = x^2$, $S$ the surface of the "unit cube" $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, $0 \leqq z \leqq 1$. What are the assumptions of $f$ and $g$? Must $f$ and $g$ be harmonic?

### Solution

## Problem 10.8.5. Green's Second Identity

$$\boxed{\iiint_T (f\,\Delta\,g - g\,\Delta\,f)\ dV = \iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)\ dA}$$

Verify for $f = 6y^2$, $g = 2x^2$, $S$ the surface of the "unit cube" $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, $0 \leqq z \leqq 1$.

### Solution

## Problem 10.8.7

Use the divergence theorem, assuming that the assumptions on $T$ and $S$ are satisfied.

Show that a region $T$ with boundary surface S has the volume

$$V = \iint_S x\ dy\ dz = \iint_S y\ dz\ dx = \iint_S z\ dx\ dy = \frac{1}{3}\iint_S (x\ dy\ dz + y\ dz\ dx + z\ dx\ dy)$$

### Solution

## Problem 10.9.3

Evaluate the surface integral $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dA$ for the given $\mathbf{F}$ and $S$.

$$\mathbf{F} = [e^{-z}, e^{-z}\cos y, e^{-z}\sin y], \quad S : z = \frac{y^2}{2},\ -1 \leqq x \leqq 1,\ 0 \leqq y \leqq 1$$

### Solution

## Problem 10.9.5

Evaluate the surface integral $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dA$ for the given $\mathbf{F}$ and $S$.

$$\mathbf{F} = [z^2, \frac{3}{2}x, 0], \quad S: 0 \leqq x \leqq a,\ 0 \leqq y \leqq a,\ z = 1$$

### Solution

## Problem 10.9.13

Calculate $\oint_C \mathbf{F} \cdot \mathbf{r}'(s)\ ds$ by Stokes's theorem,

$$\boxed{\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dA = \oint_C \mathbf{F} \cdot \mathbf{r}'(s)\ ds}$$

for the given $\mathbf{F}$ and $C$. Assume the Cartesian coordinates to be right-handed and the $z$-component of the surface normal to be nonnegative.

$$\mathbf{F} = [-5y, 4x, z], \quad C \text{ the circle } x^2 + y^2 = 16,\ z = 4$$

### Solution

## Problem 10.9.15

Calculate $\oint_C \mathbf{F} \cdot \mathbf{r}'(s)\ ds$ by Stokes's theorem for the given $\mathbf{F}$ and $C$. Assume the Cartesian coordinates to be right-handed and the $z$-component of the surface normal to be nonnegative.

$$\mathbf{F} = [z^3, x^3, y^3], \text{ around the triangle with vertices } (0,0,0),\ (1,0,0),\ (1,1,0)$$

### Solution

## Problem 10.9.19

Calculate $\oint_C \mathbf{F} \cdot \mathbf{r}'(s)\ ds$ by Stokes's theorem for the given $\mathbf{F}$ and $C$. Assume the Cartesian coordinates to be right-handed and the $z$-component of the surface normal to be nonnegative.

$$\mathbf{F} = [z, e^z, 0], \quad C \text{ the boundary curve of the portion of the cone } z = \sqrt{x^2 + y^2},\ x \geqq 0,\ y \geqq 0, 0 \leqq z \leqq 1$$

### Solution