

### Problem 10.2.3.

Show that the form under the integral sign is exact in the plane and evaluate the integral. Show the details of your work.

$$\int_{(\pi/2, \pi)}^{(\pi, 0)} \left( \frac{1}{2} \cos \frac{1}{2}x \cos 2y \, dx - 2 \sin \frac{1}{2}x \sin 2y \, dy \right)$$

#### Solution.

The form is exact iff  $\nabla \times \mathbf{F} = \mathbf{0}$ , where  $\mathbf{F} = \left[ \frac{1}{2} \cos \frac{1}{2}x \cos 2y, -2 \sin \frac{1}{2}x \sin 2y \right]$ . For 2D  $\mathbf{F}$ ,

$$\begin{aligned} \nabla \times \mathbf{F} &= (F_2)_x - (F_1)_y \\ &= \left( -\cos \frac{1}{2}x \sin 2y \right) - \left( -\cos \frac{1}{2}x \sin 2y \right) \\ &= 0 \end{aligned}$$

Therefore the form is **exact**. The integral takes the form,

$$\int_A^B \mathbf{F} \cdot d\mathbf{x} = \int_A^B df = f(B) - f(A)$$

Finding  $f$ , such that  $\nabla f = \mathbf{F}$ ,

$$f = \int F_1 dx = \int \frac{1}{2} \cos \frac{1}{2}x \cos 2y \, dx = \sin \frac{1}{2}x \cos 2y + g(y)$$

$$f_y = -2 \sin \frac{1}{2}x \sin 2y + g_y = F_2 = -2 \sin \frac{1}{2}x \sin 2y \implies g_y = 0 \implies g = 0$$

$$f = \sin \frac{1}{2}x \cos 2y$$

The integral is then,

$$\begin{aligned} \int_{(\pi/2, \pi)}^{(\pi, 0)} df &= f(\pi, 0) - f(\pi/2, \pi) \\ &= \boxed{1 - \frac{\sqrt{2}}{2}} \end{aligned}$$

### Problem 10.2.5.

Show that the form under the integral sign is exact in space and evaluate the integral. Show the details of your work.

$$\int_{(0,0,\pi)}^{(2,1/2,\pi/2)} e^{xy} (y \sin z \, dx + x \sin z \, dy + \cos z \, dz)$$

**Solution.**

The form is exact iff  $\nabla \times \mathbf{F} = \mathbf{0}$ , where  $\mathbf{F} = e^{xy} [y \sin z, x \sin z, \cos z]$ .

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \begin{bmatrix} (F_3)_y - (F_2)_z \\ (F_1)_z - (F_3)_x \\ (F_2)_x - (F_1)_y \end{bmatrix} = \begin{bmatrix} (xe^{xy} \cos z) - (xe^{xy} \cos z) \\ (ye^{xy} \cos z) - (ye^{xy} \cos z) \\ (xye^{xy} \sin z + e^{xy} \sin z) - (xye^{xy} \sin z + e^{xy} \sin z) \end{bmatrix} = \mathbf{0} \end{aligned}$$

Therefore the form is **exact**. The integral takes the form,

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B df = f(B) - f(A)$$

Finding  $f$ , such that  $\nabla f = \mathbf{F}$ ,

$$f = \int F_1 dx = \int e^{xy} y \sin z \, dx = e^{xy} \sin z + g(y, z)$$

$$f_y = xe^{xy} \sin z + g_y = F_2 = xe^{xy} \sin z \implies g_y = 0 \implies g = h(z)$$

$$f_z = e^{xy} \cos z + h'(z) = F_3 = e^{xy} \cos z \implies h'(z) = 0 \implies h(z) = 0$$

$$f = e^{xy} \sin z$$

The integral is then,

$$\begin{aligned} \int_{(0,0,\pi)}^{(2,1/2,\pi/2)} df &= f(2, 1/2, \pi/2) - f(0, 0, \pi) \\ &= e \sin(\pi/2) - e^0 \sin(\pi) \\ &= \boxed{e} \end{aligned}$$

**Problem 10.2.13.**

Check, and if independent, integrate from  $(0, 0, 0)$  to  $(a, b, c)$ .

$$2e^{x^2} (x \cos 2y \, dx - \sin 2y \, dy)$$

**Solution.**

The integral is independent iff  $\nabla \times \mathbf{F} = \mathbf{0}$ , where  $\mathbf{F} = 2e^{x^2} [x \cos 2y, -\sin 2y]$ . For 2D  $\mathbf{F}$ ,

$$\begin{aligned} \nabla \times \mathbf{F} &= (F_2)_x - (F_1)_y \\ &= (-4xe^{x^2} \sin 2y) - (-4xe^{x^2} \sin 2y) \\ &= 0 \end{aligned}$$

Therefore the integral is **independent of path**. The integral takes the form,

$$\int_A^B \mathbf{F} \cdot d\mathbf{x} = \int_A^B df = f(B) - f(A)$$

Finding  $f$ , such that  $\nabla f = \mathbf{F}$ ,

$$f = \int F_1 dx = \int 2e^{x^2} x \cos 2y \, dx = e^{x^2} \cos 2y + g(y)$$

$$f_y = -2e^{x^2} \sin 2y + g_y = F_2 = -2e^{x^2} \sin 2y \implies g_y = 0 \implies g(y) = 0$$

$$f = e^{x^2} \cos 2y$$

The integral is then,

$$\begin{aligned} \int_{(0,0,0)}^{(a,b,c)} df &= f(a, b, c) - f(0, 0, 0) \\ &= e^{a^2} \cos 2b - e^0 \cos 0 \\ &= \boxed{e^{a^2} \cos 2b - 1} \end{aligned}$$

### Problem 10.2.16.

Check, and if independent, integrate from  $(0, 0, 0)$  to  $(a, b, c)$ .

$$e^y \, dx + (xe^y - e^z) \, dy - ye^z \, dz$$

### Solution.

The integral is independent iff  $\nabla \times \mathbf{F} = \mathbf{0}$ , where  $\mathbf{F} = [e^y, (xe^y - e^z), -ye^z]$ .

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & (xe^y - e^z) & -ye^z \end{vmatrix} \\ &= \left[ \left( \frac{\partial}{\partial y}(-ye^z) - \frac{\partial}{\partial z}(xe^y - e^z) \right), \left( \frac{\partial}{\partial z}(e^y) - \frac{\partial}{\partial x}(-ye^z) \right), \left( \frac{\partial}{\partial x}(xe^y - e^z) - \frac{\partial}{\partial y}(e^y) \right) \right] \\ &= [((-e^z) - (-e^y)), (0 - 0), (e^y - e^y)] \\ &= \mathbf{0} \end{aligned}$$

Therefore the integral is **independent of path**. The integral takes the form,

$$\int_A^B \mathbf{F} \cdot d\mathbf{x} = \int_A^B df = f(B) - f(A)$$

Finding  $f$ , such that  $\nabla f = \mathbf{F}$ ,

$$f = \int F_2 dy = \int (xe^y - e^z) dy = xe^y - ye^z + g(x, z)$$

$$f_x = e^y + g_x = F_1 = e^y \implies g_x = 0 \implies g(x, z) = h(z)$$

$$f_z = -ye^z + h_z = F_3 = -ye^z \implies h_z = 0 \implies h(z) = 0$$

$$f = xe^y - ye^z$$

The integral is then,

$$\begin{aligned} \int_{(0,0,0)}^{(a,b,c)} df &= f(a, b, c) - f(0, 0, 0) \\ &= (ae^b - be^c) - (0 - 0) \\ &= \boxed{ae^b - be^c} \end{aligned}$$

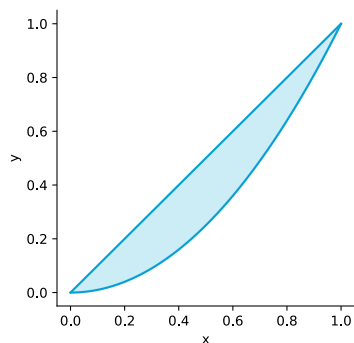
### Problem 10.3.5.

Describe the region of integration and evaluate.

$$\int_0^1 \int_{x^2}^x (1 - 2xy) \, dy \, dx$$

### Solution.

The region of integration is the volume under the surface  $z = 1 - 2xy$  for  $y \in [x^2, x]$  and  $x \in [0, 1]$ . The region on the  $xy$ -plane looks like a slim leaf-like shape.



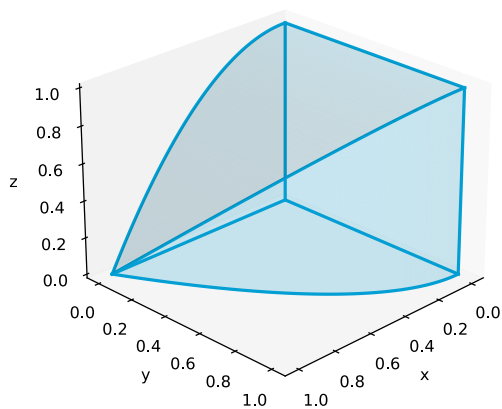
$$\begin{aligned} \int_0^1 \int_{x^2}^x (1 - 2xy) \, dy \, dx &= \int_0^1 [y - xy^2]_{x^2}^x \, dx \\ &= \int_0^1 [x - x^2 - x^3 + x^5] \, dx \\ &= \left[ \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^6}{6} \right]_0^1 \\ &= \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{6} \\ &= \boxed{\frac{1}{12}} \end{aligned}$$

**Problem 10.3.10.**

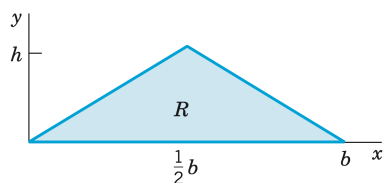
Find the volume of the first octant region bounded by the coordinate planes and the surfaces  $y = 1 - x^2$  and  $z = 1 - x^2$ . Sketch it.

**Solution.**

$$\begin{aligned}
 V &= \int_0^1 \int_0^{1-x^2} \int_0^{1-x^2} dz \, dy \, dx \\
 &= \int_0^1 \int_0^{1-x^2} [z]_{z=0}^{z=1-x^2} dy \, dx \\
 &= \int_0^1 \int_0^{1-x^2} (1-x^2) dy \, dx \\
 &= \int_0^1 [y(1-x^2)]_{y=0}^{y=1-x^2} dx \\
 &= \int_0^1 (1-x^2)(1-x^2) dx \\
 &= \int_0^1 (1-2x^2+x^4) dx \\
 &= x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \Big|_0^1 \\
 &= 1 - \frac{2}{3} + \frac{1}{5} \\
 &= \boxed{\frac{8}{15}}
 \end{aligned}$$

**Problem 10.3.12.**

Find the center of gravity  $(\bar{x}, \bar{y})$  of a mass of density  $f(x, y) = 1$  in the given region  $R$ .



### Equations.

The center of gravity is found at the coordinates  $(\bar{x}, \bar{y})$  where,

$$\bar{x} = \frac{1}{M} \iint_R x f(x, y) \, dx \, dy \quad \bar{y} = \frac{1}{M} \iint_R y f(x, y) \, dx \, dy \quad M = \iint_R f(x, y) \, dx \, dy$$

### Solution.

$$\begin{aligned} M &= \iint_R (1) \, dx \, dy \\ &= \boxed{\frac{hb}{2}} \end{aligned}$$

$$\begin{aligned} \bar{x} &= \frac{2}{hb} \iint_R x \, dx \, dy \\ &= \frac{2}{hb} \left[ \int_0^{b/2} \int_0^{(2h/b)x} x \, dy \, dx + \int_{b/2}^b \int_0^{2h-(2h/b)x} x \, dy \, dx \right] \\ &= \frac{2}{hb} \left[ \int_0^{b/2} [xy]_{y=0}^{y=(2h/b)x} \, dx + \int_{b/2}^b [xy]_{y=0}^{y=2h-(2h/b)x} \, dx \right] \\ &= \frac{2}{hb} \left[ \int_0^{b/2} \left( \frac{2h}{b} x^2 \right) \, dx + \int_{b/2}^b \left( 2hx - \frac{2h}{b} x^2 \right) \, dx \right] \\ &= \frac{2}{hb} \left[ \left[ \frac{2h}{3b} x^3 \right]_0^{b/2} + \left[ hx^2 - \frac{2h}{3b} x^3 \right]_{b/2}^b \right] \\ &= \frac{2}{hb} \left[ \left[ \frac{2h}{3b} \left( \frac{b}{2} \right)^3 \right] + \left[ hb^2 - \frac{2h}{3b} (b)^3 - h \left( \frac{b}{2} \right)^2 + \frac{2h}{3b} \left( \frac{b}{2} \right)^3 \right] \right] \\ &= \frac{2}{hb} \left[ \frac{hb^2}{6} + hb^2 - \frac{2hb^2}{3} - \frac{hb^2}{4} \right] \\ &= b \left[ \frac{1}{3} + 2 - \frac{4}{3} - \frac{1}{2} \right] \\ \bar{x} &= \boxed{\frac{1}{2}b} \end{aligned}$$

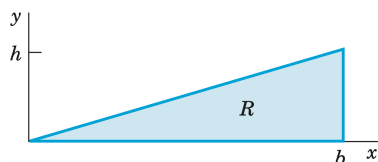
$$\begin{aligned}
\bar{y} &= \frac{2}{hb} \iint_R y \, dx \, dy \\
&= \frac{2}{hb} \left[ \int_0^{b/2} \int_0^{(2h/b)x} y \, dy \, dx + \int_{b/2}^b \int_0^{2h-(2h/b)x} y \, dy \, dx \right] \\
&= \frac{2}{hb} \cdot \frac{1}{2} \left[ \int_0^{b/2} [y^2]_{y=0}^{y=(2h/b)x} \, dx + \int_{b/2}^b [y^2]_{y=0}^{y=2h-(2h/b)x} \, dx \right] \\
&= \frac{1}{hb} \left[ \int_0^{b/2} \left( \frac{2h}{b} x \right)^2 \, dx + \int_{b/2}^b \left( 2h - \frac{2h}{b} x \right)^2 \, dx \right] \\
&= \frac{1}{hb} \left[ \int_0^{b/2} \left( \frac{4h^2}{b^2} x^2 \right) \, dx + \int_{b/2}^b \left( 4h^2 - \frac{8h^2}{b} x + \frac{4h^2}{b^2} x^2 \right) \, dx \right] \\
&= \frac{4h}{b} \left[ \int_0^{b/2} \left( \frac{1}{b^2} x^2 \right) \, dx + \int_{b/2}^b \left( 1 - \frac{2}{b} x + \frac{1}{b^2} x^2 \right) \, dx \right] \\
&= \frac{4h}{b} \left[ \left[ \frac{1}{3b^2} x^3 \right]_0^{b/2} + \left[ x - \frac{1}{b} x^2 + \frac{1}{3b^2} x^3 \right]_{b/2}^b \right] \\
&= \frac{4h}{b} \left[ \left[ \frac{1}{3b^2} \left( \frac{b}{2} \right)^3 \right] + \left[ (b) - \frac{1}{b} (b)^2 + \frac{1}{3b^2} (b)^3 - \left( \frac{b}{2} \right) + \frac{1}{b} \left( \frac{b}{2} \right)^2 - \frac{1}{3b^2} \left( \frac{b}{2} \right)^3 \right] \right] \\
&= \frac{4h}{b} \left[ \frac{b}{3} - \frac{b}{2} + \frac{b}{4} \right] \\
\bar{y} &= \boxed{\frac{h}{3}}
\end{aligned}$$

**Summary.**

$$(\bar{x}, \bar{y}) = \left( \frac{b}{2}, \frac{h}{3} \right)$$

### Problem 10.3.17.

Find  $I_x$ ,  $I_y$ ,  $I_0$  of a mass of density  $f(x, y) = 1$  in the region  $R$ .



**Equations.**

The moments of inertia  $I_x$  and  $I_y$  of the mass in  $R$  about the  $x$ - and  $y$ -axes, respectively, are

$$I_x = \iint_R x^2 f(x, y) \, dx \, dy = \quad I_y = \iint_R y^2 f(x, y) \, dx \, dy$$

and the polar moment of inertia  $I_0$  about the origin of the mass in  $R$  is

$$I_0 = I_x + I_y = \iint_R (x^2 + y^2) f(x, y) \, dx \, dy$$

**Solution.**

$$\begin{aligned}
 I_x &= \iint_R x^2 \, dx \, dy \\
 &= \int_0^b \int_0^{(h/b)x} x^2 \, dy \, dx \\
 &= \int_0^b [x^2 y]_{y=0}^{y=(h/b)x} \, dx \\
 &= \int_0^b x^2 \left( \frac{h}{b} x \right) \, dx \\
 &= \frac{h}{b} \int_0^b x^3 \, dx \\
 &= \frac{h}{4b} x^4 \Big|_0^b \\
 I_x &= \boxed{\frac{1}{4} h b^3}
 \end{aligned}$$

$$\begin{aligned}
 I_y &= \iint_R y^2 \, dx \, dy \\
 &= \int_0^b \int_0^{(h/b)x} y^2 \, dy \, dx \\
 &= \frac{1}{3} \int_0^b [y^3]_{y=0}^{y=(h/b)x} \, dx \\
 &= \frac{1}{3} \int_0^b \left( \frac{h}{b} x \right)^3 \, dx \\
 &= \frac{h^3}{3b^3} \int_0^b x^3 \, dx \\
 &= \frac{h^3}{12b^3} x^4 \Big|_0^b \\
 I_y &= \boxed{\frac{1}{12} h^3 b}
 \end{aligned}$$

$$\begin{aligned}
 I_0 &= I_x + I_y \\
 &= \frac{1}{4} h b^3 + \frac{1}{12} h^3 b \\
 I_0 &= \boxed{\frac{1}{3} h b^3 + \frac{1}{12} h^3 b}
 \end{aligned}$$

**Summary.**

$$\begin{aligned}
 I_x &= \frac{1}{4} h b^3 \\
 I_y &= \frac{1}{12} h^3 b \\
 I_0 &= \frac{1}{3} h b^3 + \frac{1}{12} h^3 b
 \end{aligned}$$