Problem 9.7.3.

Find ∇f. Graph some level curves $f = $ const. Indicate ∇f by arrows at some points of these curves.
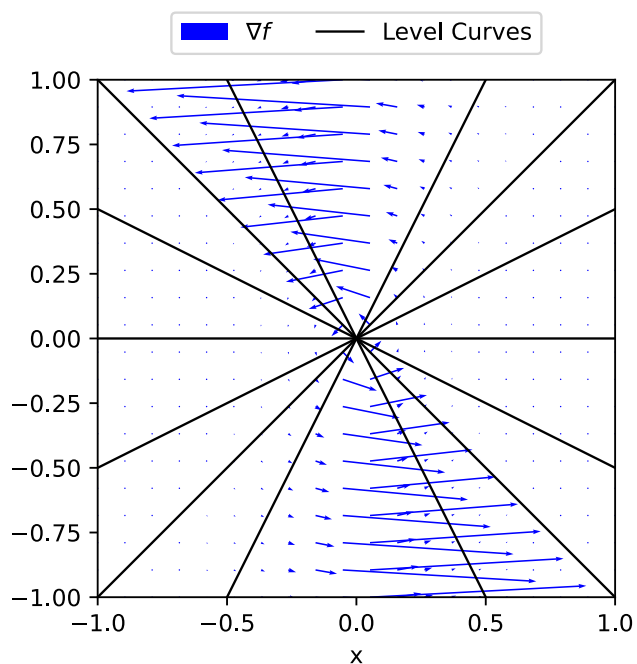
$$f = \frac{y}{x}$$

Solution.

$$\nabla f = [f_x, f_y]$$

$$\boxed{\nabla f = \left[-\frac{y}{x^2}, \frac{1}{x} \right]}$$

Level Surfaces:

$$f(x, y) = \frac{y}{x} = c$$

$$\boxed{y = cx}$$

Problem 9.7.8.

Prove and illustrate by an example

$$\nabla(fg) = f \nabla g + g \nabla f$$

Proof.

$$\nabla(fg) = \left[\frac{\partial}{\partial x_1}(fg), \dots, \frac{\partial}{\partial x_n}(fg) \right]$$

The i-th component can be expanded by chain rule,

$$\frac{\partial}{\partial x_i}(fg) = f\frac{\partial g}{\partial x_i} + g\frac{\partial f}{\partial x_i}$$

Therefore in general,

$$\nabla(fg) = \left[\frac{\partial}{\partial x_1}(fg), \ldots, \frac{\partial}{\partial x_n}(fg)\right]$$

$$= f\left[\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n}\right] + g\left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right]$$

$$= \boxed{f\,\nabla\,g + g\,\nabla\,f}$$

Example.

$$\nabla(xy) = x\,\nabla\,y + y\,\nabla\,x$$

$$\nabla(xy) = x\,[0,1] + y\,[1,0] = [y,x]$$

Problem 9.7.12.

The force in an electrostatic field given by $f(x,y,z)$ has the direction of the gradient. Find $\nabla\,f$ and its value at P.

$$f = \frac{x}{x^2 + y^2}, \quad P : (1,1)$$

Solution.

$$\nabla\,f = \left[\frac{\partial}{\partial x}(f), \frac{\partial}{\partial y}(f)\right]$$

$$= \boxed{\left[\frac{y^2 - x^2}{(x^2 + y^2)^2}, \frac{-2xy}{(x^2 + y^2)^2}\right]}$$

$$\nabla\,f(P) = \boxed{\left[0, -\frac{1}{2}\right]}$$

Problem 9.7.15.

The force in an electrostatic field given by $f(x,y,z)$ has the direction of the gradient. Find $\nabla\,f$ and its value at P.

$$f = 4x^2 + 9y^2 + z^2, \quad P : (5,1,11)$$

Solution.

$$\nabla\,f = \left[\frac{\partial}{\partial x}(f), \frac{\partial}{\partial y}(f), \frac{\partial}{\partial z}(f)\right]$$

$$= \boxed{[8x, 18y, 2z]}$$

$$\nabla\,f(P) = \boxed{[40, 18, 22]}$$

Problem 9.7.21.

Given the velocity potential f of a flow, find the velocity $\mathbf{v} = \nabla f$ of the field and its value $\mathbf{v}(P)$ at P. Sketch $\mathbf{v}(P)$ and the curve $f = \text{const}$ passing through P

$$f = e^x \cos y, \quad P : (1,1)$$

Solution.

$$\mathbf{v} = \nabla(e^x \cos y)$$
$$= \boxed{[e^x \cos y, -e^x \sin y]}$$

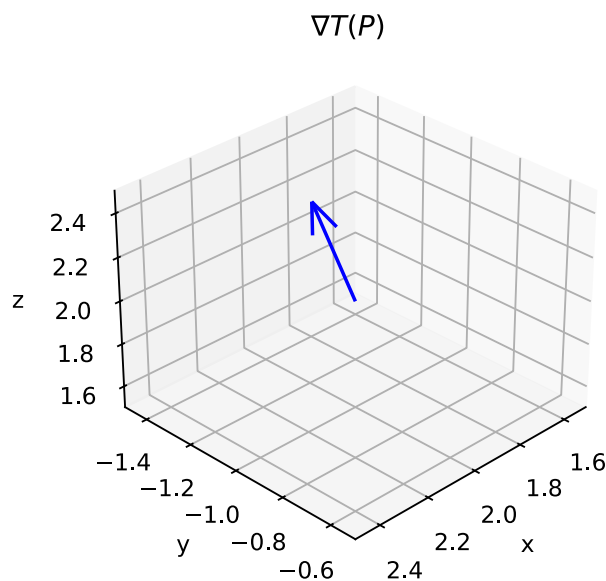$$\boxed{\mathbf{v}(P) = [e \cos 1, -e \sin 1]}$$

Problem 9.7.26.

Experiments show that in a temperature field, heat flows in the direction of maximum decrease of temperature T. Find this direction in general and at the given point P. Sketch that direction at P as an arrow.

$$T = x^2 + y^2 + 4z^2, \quad P : (2, -1, 2)$$

Solution.

$$\boxed{\nabla T = [2x, 2y, 8z]}$$

$$\boxed{\nabla T(P) = [4, -2, 16]}$$

$\nabla T(P)$

Problem 9.8.3.

Find $\nabla \cdot \mathbf{v}$ and its value at P:

$$\mathbf{v} = (x^2 + y^2)^{-1}[x, y]$$

Solution.

$$\mathbf{v} = \left[\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right]$$

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right)$$

$$= \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$= \boxed{0}$$

Problem 9.8.5.

Find $\nabla \cdot \mathbf{v}$ and its value at P:

$$\mathbf{v} = x^2 y^2 z^2 [x, y, z], \quad P : (3, -1, 4)$$

Solution.

$$\mathbf{v} = \left[x^3 y^2 z^2, x^2 y^3 z^2, x^2 y^2 z^3 \right]$$

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(x^3 y^2 z^2) + \frac{\partial}{\partial y}(x^2 y^3 z^2) + \frac{\partial}{\partial z}(x^2 y^2 z^3)$$

$$= \boxed{9 x^2 y^2 z^2}$$

$$\boxed{\nabla \cdot \mathbf{v}(\mathbf{P}) = 1296}$$

Problem 9.8.9. PROJECT. Useful Formulas for the Divergence.

Part a.

Prove $\nabla \cdot (k\mathbf{v}) = k \, \nabla \cdot \mathbf{v}$, ($k$ **constant**)

$$k\mathbf{v} = [kv_1, \ldots, kv_n]$$

$$\nabla \cdot (k\mathbf{v}) = \frac{\partial}{\partial x_1}(kv_1) + \cdots + \frac{\partial}{\partial x_n}(kv_n)$$

$$= k \left(\frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n} \right)$$

$$= \boxed{k \, \nabla \cdot \mathbf{v}}$$

Part b.

Prove $\nabla\cdot(f\mathbf{v}) = f(\nabla\cdot\mathbf{v}) + \mathbf{v}\cdot\nabla f$

$$f\mathbf{v} = [fv_1, \ldots, fv_n]$$

$$\nabla\cdot(f\mathbf{v}) = \frac{\partial}{\partial x_1}(fv_1) + \cdots + \frac{\partial}{\partial x_n}(fv_n)$$

The i-th component can be expanded by chain rule,

$$\frac{\partial}{\partial x_i}(fv_i) = f\frac{\partial v_i}{\partial x_i} + \frac{\partial f}{\partial x_i}v_i$$

Therefore in general,

$$\begin{aligned}
\nabla\cdot(f\mathbf{v}) &= \frac{\partial}{\partial x_1}(fv_1) + \cdots + \frac{\partial}{\partial x_n}(fv_n) \\
&= f\left(\frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n}\right) + \left(v_1\frac{\partial f}{\partial x_1} + \cdots + v_n\frac{\partial f}{\partial x_n}\right) \\
&= f\left(\frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n}\right) + \mathbf{v}\cdot\left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right] \\
&= \boxed{f\,\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla f}
\end{aligned}$$

Verify for $f = e^{xyz}$ **and** $\mathbf{v} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}.$

$$\begin{aligned}
\nabla\cdot(f\mathbf{v}) &= \nabla\cdot[axe^{xyz}, bye^{xyz}, cze^{xyz}] \\
&= (ae^{xyz} + axyze^{xyz}) + (be^{xyz} + bxyze^{xyz}) + (ce^{xyz} + cxyze^{xyz}) \\
&= \boxed{(a+b+c)e^{xyz} + (a+b+c)xyze^{xyz}}
\end{aligned}$$

$$\begin{aligned}
\nabla\cdot(f\mathbf{v}) &= e^{xyz}\,\nabla\cdot[ax, by, cz] + [ax, by, cz]\cdot\nabla e^{xyz} \\
&= (a+b+c)e^{xyz} + [ax, by, cz]\cdot[yze^{xyz}, xze^{xyz}, xye^{xyz}] \\
&= \boxed{(a+b+c)e^{xyz} + (a+b+c)xyze^{xyz}}
\end{aligned}$$

Obtain the answer to Prob. 6.

Find $\nabla\cdot(x^2 + y^2 + z^2)^{-3/2}[x, y, z].$

$$\begin{aligned}
\mathbf{v} = [x, y, z], \qquad\qquad \nabla\cdot\mathbf{v} &= \nabla\cdot[x, y, z] \\
&= 3 \\
f = (x^2 + y^2 + z^2)^{-3/2}, \quad \nabla f &= \nabla(x^2 + y^2 + z^2)^{-3/2} \\
&= -3(x^2 + y^2 + z^2)^{-5/2}[x, y, z]
\end{aligned}$$

$$\begin{aligned}
\nabla\cdot(f\mathbf{v}) &= f\,\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla f \\
&= (x^2 + y^2 + z^2)^{-3/2}\cdot 3 + [x, y, z]\cdot -3(x^2 + y^2 + z^2)^{-5/2}[x, y, z] \\
&= \boxed{0}
\end{aligned}$$

Part c.

Prove $\nabla\cdot(f\,\nabla\,g) = f\,\Delta\,g + \nabla\,f\cdot\nabla\,g$

$$f\,\nabla\,g = \left[f\frac{\partial g}{\partial x_1}, \ldots, f\frac{\partial g}{\partial x_n}\right]$$

$$\nabla\cdot(f\,\nabla\,g) = \frac{\partial}{\partial x_1}\left(f\frac{\partial g}{\partial x_1}\right) + \cdots + \frac{\partial}{\partial x_n}\left(f\frac{\partial g}{\partial x_n}\right)$$

The i-th component can be expanded by chain rule,

$$\frac{\partial}{\partial x_i}\left(f\frac{\partial g}{\partial x_i}\right) = f\frac{\partial^2 g}{\partial x_i^2} + \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i}$$

Therefore in general,

$$\begin{aligned}
\nabla\cdot(f\,\nabla\,g) &= \frac{\partial}{\partial x_1}\left(f\frac{\partial g}{\partial x_1}\right) + \cdots + \frac{\partial}{\partial x_n}\left(f\frac{\partial g}{\partial x_n}\right) \\
&= f\left(\frac{\partial^2 g}{\partial x_1^2} + \cdots + \frac{\partial^2 g}{\partial x_n^2}\right) + \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right]\cdot\left[\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n}\right] \\
&= \boxed{f\,\Delta\,g + \nabla\,f\cdot\nabla\,g}
\end{aligned}$$

Verify for $f = x^2 - y^2$ **and** $g = e^{x+y}$.

	∇	Δ
f	$[2x, -2y]$	
g	$[e^{x+y}, e^{x+y}]$	$[e^{x+y}, e^{x+y}]$

$$\begin{aligned}
\nabla\cdot(f\,\nabla\,g) &= \nabla\cdot\left((x^2 - y^2)\left[e^{x+y}, e^{x+y}\right]\right) \\
&= \nabla\cdot\left[x^2 e^{x+y} - y^2 e^{x+y}, x^2 e^{x+y} - y^2 e^{x+y}\right] \\
&= (2x + x^2 - y^2)e^{x+y} + (x^2 - 2y - y^2)e^{x+y} \\
&= \boxed{2(x + x^2 - y - y^2)e^{x+y}}
\end{aligned}$$

$$\begin{aligned}
\nabla\cdot(f\,\nabla\,g) &= f\,\Delta\,g + \nabla\,f\cdot\nabla\,g \\
&= (x^2 - y^2)[e^{x+y}, e^{x+y}] + [2x, -2y]\cdot[e^{x+y}, e^{x+y}] \\
&= \boxed{2(x + x^2 - y - y^2)e^{x+y}}
\end{aligned}$$

Part d.

Prove $\nabla\cdot(f\,\nabla\,g) - \nabla\cdot(g\,\nabla\,f) = f\,\Delta\,g - g\,\Delta\,f$

From **Part c**,

$$\nabla\cdot(f\,\nabla\,g) - \nabla\cdot(g\,\nabla\,f) = (f\,\Delta\,g + \nabla\,f\cdot\nabla\,g) - (g\,\Delta\,f + \nabla\,g\cdot\nabla\,f)$$

Therefore,

$$\nabla\cdot(f\,\nabla\,g) - \nabla\cdot(g\,\nabla\,f) = f\,\Delta\,g - g\,\Delta\,f$$

Problem 9.8.17.

Calculate Δf by Eq. (3). Check by direct differentiation. Indicate when (3) is simpler. Show the details of your work.

Eq. (3):

$$\nabla \cdot (\nabla f) = \Delta f$$

$$f = \ln(x^2 + y^2)$$

Solution.

$$\begin{aligned}
\Delta f &= \nabla \cdot (\nabla f) \\
&= \nabla \cdot \left[\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right] \\
&= \frac{2(x^2 + y^2) - 4x^2}{(x^2 + y^2)^2} + \frac{2(x^2 + y^2) - 4y^2}{(x^2 + y^2)^2} \\
&= \frac{4(x^2 + y^2) - 4(x^2 + y^2)}{(x^2 + y^2)^2} \\
&= \boxed{0}
\end{aligned}$$

$$\begin{aligned}
\Delta f &= f_{xx} + f_{yy} \\
&= \frac{2(x^2 + y^2) - 4x^2}{(x^2 + y^2)^2} + \frac{2(x^2 + y^2) - 4y^2}{(x^2 + y^2)^2} \\
&= \frac{4(x^2 + y^2) - 4(x^2 + y^2)}{(x^2 + y^2)^2} \\
&= \boxed{0}
\end{aligned}$$

Problem 9.9.4.

Find $\nabla \times \mathbf{v}$ for $\mathbf{v}$ given with respect to right-handed Cartesian coordinates. Show the details of your work.

$$\mathbf{v} = [2y^2, 5x, 0]$$

Solution.

$$\begin{aligned}
\nabla \times \mathbf{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^2 & 5x & 0 \end{vmatrix} \\
&= \hat{i} \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(5x) \right) - \hat{j} \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(2y^2) \right) + \hat{k} \left(\frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(2y^2) \right) \\
&= \hat{k}(5 - 4y) \\
&= \boxed{[0, 0, 5 - 4y]}
\end{aligned}$$

Problem 9.9.9.

Let $\mathbf{v}$ be the velocity vector of a steady fluid flow.

$$\mathbf{v} = [0, 3z^2, 0]$$

Is the flow irrotational?

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 3z^2 & 0 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(3z^2)\right) - \hat{j}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(0)\right) + \hat{k}\left(\frac{\partial}{\partial x}(3z^2) - \frac{\partial}{\partial y}(0)\right)$$

$$= \hat{i}(-6z)$$

$$= \boxed{[-6z, 0, 0]}$$

$\nabla \times \mathbf{v} \neq 0$ therefore the flow is **rotational**.

Is the flow incompressible?

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(3z^2) + \frac{\partial}{\partial z}(0) = \boxed{0}$$

$\nabla \cdot \mathbf{v} = 0$ therefore the flow is **incompressible**.

Find the streamlines (the paths of the particles).

$$\mathbf{v} = \mathbf{r}' = [x', y', z']$$

$$\begin{cases} x' = 0 \\ y' = 3z^2 \\ z' = 0 \end{cases}$$

Integrating, we get:

$$\begin{cases} x = x_0 \\ y = y_0 + 3z_0^2 t \\ z = z_0 \end{cases}$$

Streamlines are in the form $\boxed{\mathbf{r}(t) = [x_0, y_0 + 3z_0^2 t, z_0]}$.

Problem 9.9.14. PROJECT. Useful Formulas for the Curl.

Assuming sufficient differentiability, show that

Part a.

$\nabla \times (\mathbf{u} + \mathbf{v}) = \nabla \times \mathbf{u} + \nabla \times \mathbf{v}$

Solution.

For 3D vectors $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{u} + \mathbf{v} = [u_x + v_x, u_y + v_y, u_z + v_z]$$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x + v_x & u_y + v_y & u_z + v_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(u_z + v_z) - \frac{\partial}{\partial z}(u_y + v_y)\right) - \hat{j}\left(\frac{\partial}{\partial x}(u_z + v_z) - \frac{\partial}{\partial z}(u_x + v_x)\right) + \hat{k}\left(\frac{\partial}{\partial x}(u_y + v_y) - \frac{\partial}{\partial y}(u_x + v_x)\right)$$

$$= \left[\left(\frac{\partial}{\partial y}(u_z + v_z) - \frac{\partial}{\partial z}(u_y + v_y)\right), \left(\frac{\partial}{\partial z}(u_x + v_x) - \frac{\partial}{\partial x}(u_z + v_z)\right), \left(\frac{\partial}{\partial x}(u_y + v_y) - \frac{\partial}{\partial y}(u_x + v_x)\right)\right]$$

$$= \left[\left(\frac{\partial u_z}{\partial y} + \frac{\partial v_z}{\partial y} - \frac{\partial u_y}{\partial z} - \frac{\partial v_y}{\partial z}\right), \left(\frac{\partial u_x}{\partial z} + \frac{\partial v_x}{\partial z} - \frac{\partial u_z}{\partial x} - \frac{\partial v_z}{\partial x}\right), \left(\frac{\partial u_y}{\partial x} + \frac{\partial v_y}{\partial x} - \frac{\partial u_x}{\partial y} - \frac{\partial v_x}{\partial y}\right)\right]$$

$$= \left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right), \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right), \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\right] + \left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right), \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right), \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right]$$

$$= \boxed{\nabla \times \mathbf{u} + \nabla \times \mathbf{v}}$$

Part b

$\nabla \cdot (\nabla \times \mathbf{v}) = 0$

Solution.

For 3D vector $\mathbf{v}$,

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \hat{j}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \hat{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$= \left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right), \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right), \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right]$$

$$\nabla \cdot (\nabla \times \mathbf{v}) = \nabla \cdot \left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right), \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right), \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$= \frac{\partial^2 v_z}{\partial x \partial y} - \frac{\partial^2 v_y}{\partial x \partial z} + \frac{\partial^2 v_x}{\partial y \partial z} - \frac{\partial^2 v_z}{\partial x \partial y} + \frac{\partial^2 v_y}{\partial x \partial z} - \frac{\partial^2 v_x}{\partial y \partial z}$$

$$= \boxed{0}$$

Part c

$\nabla \times (f\mathbf{v}) = (\nabla f) \times \mathbf{v} + f\, \nabla \times \mathbf{v}$

Solution.

For 3D vector $\mathbf{v}$,

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fv_x & fv_y & fv_z \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial}{\partial y}(fv_z) - \frac{\partial}{\partial z}(fv_y) \right) - \hat{j} \left(\frac{\partial}{\partial z}(fv_x) - \frac{\partial}{\partial x}(fv_z) \right) + \hat{k} \left(\frac{\partial}{\partial y}(fv_x) - \frac{\partial}{\partial x}(fv_y) \right)$$

$$= \left[\left(\frac{\partial}{\partial y}(fv_z) - \frac{\partial}{\partial z}(fv_y) \right), \left(\frac{\partial}{\partial x}(fv_z) - \frac{\partial}{\partial z}(fv_x) \right), \left(\frac{\partial}{\partial y}(fv_x) - \frac{\partial}{\partial x}(fv_y) \right) \right]$$

Expanding with chain rule,

$$= \left[\left(f\frac{\partial v_z}{\partial y} + v_z\frac{\partial f}{\partial y} - f\frac{\partial v_y}{\partial z} - v_y\frac{\partial f}{\partial z} \right), \left(f\frac{\partial v_z}{\partial x} - v_z\frac{\partial f}{\partial x} - f\frac{\partial v_x}{\partial z} + v_x\frac{\partial f}{\partial z} \right), \left(f\frac{\partial v_y}{\partial x} + v_y\frac{\partial f}{\partial x} - f\frac{\partial v_x}{\partial y} - v_x\frac{\partial f}{\partial y} \right) \right]$$

$$= \left[\left(v_z\frac{\partial f}{\partial y} - v_y\frac{\partial f}{\partial z} \right), \left(v_x\frac{\partial f}{\partial z} - v_z\frac{\partial f}{\partial x} \right), \left(v_y\frac{\partial f}{\partial x} - v_x\frac{\partial f}{\partial y} \right) \right] + \left[f\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right), f\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right), f\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \right]$$

$$= \boxed{(\nabla f) \times \mathbf{v} + f\,\nabla \times \mathbf{v}}$$

Part d

$\nabla \times (\nabla f) = \mathbf{0}$

Solution.

For $f(x, y, z)$,

$$\nabla \times (\nabla f) = \nabla \times \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right]$$

$$= \left[\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) \right), \left(\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) \right), \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \right) \right]$$

$$= [0, 0, 0]$$

$$= \boxed{\mathbf{0}}$$

Part e

$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \nabla \times \mathbf{u} - \mathbf{u} \cdot \nabla \times \mathbf{v}$

Solution.

For 3D vectors $\mathbf{u}$ and $\mathbf{v}$,

$$(\mathbf{u} \times \mathbf{v}) = \left[\left(u_y v_z - u_z v_y\right), \left(u_z v_x - u_x v_z\right), \left(u_x v_y - u_y v_x\right) \right]$$

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \nabla \cdot \left[(u_y v_z - u_z v_y), (u_z v_x - u_x v_z), (u_x v_y - u_y v_x) \right]$$

$$= \frac{\partial}{\partial x} (u_y v_z - u_z v_y) + \frac{\partial}{\partial y} (u_z v_x - u_x v_z) + \frac{\partial}{\partial z} (u_x v_y - u_y v_x)$$

$$= \left(v_z \frac{\partial u_y}{\partial x} - v_y \frac{\partial u_z}{\partial x} \right) + \left(v_x \frac{\partial u_z}{\partial y} - v_z \frac{\partial u_x}{\partial y} \right) + \left(v_y \frac{\partial u_x}{\partial z} - v_x \frac{\partial u_y}{\partial z} \right) +$$

$$\left(u_y \frac{\partial v_z}{\partial x} - u_z \frac{\partial v_y}{\partial x} \right) + \left(u_z \frac{\partial v_x}{\partial y} - u_x \frac{\partial v_z}{\partial y} \right) + \left(u_x \frac{\partial v_y}{\partial z} - u_y \frac{\partial v_x}{\partial z} \right)$$

$$= v_x \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right) + v_y \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right) + v_z \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) +$$

$$u_x \left(\frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y} \right) + u_y \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right) + u_z \left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right)$$

$$= [v_x, v_y, v_z] \cdot \left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right), \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right), \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \right] +$$

$$[u_x, u_y, u_z] \cdot \left[\left(\frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y} \right), \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right), \left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right) \right]$$

$$= \boxed{\mathbf{v} \cdot \nabla \times \mathbf{u} - \mathbf{u} \cdot \nabla \times \mathbf{v}}$$

Problem 10.1.2.

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If $\mathbf{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

$$\mathbf{F} = [y^2, -x^2], \quad C : y = 4x^2 \text{ from } (0, 0) \text{ to } (1, 4)$$

Solution.

$$\mathbf{r} = [t, 4t^2], \quad t \in [0, 1]$$

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

$$= \int_0^1 [16t^4, -t^2] \cdot [1, 8t] \, dt$$

$$= \int_0^1 (16t^4 - 8t^3) \, dt$$

$$= \left[\frac{16}{5} t^5 - 2t^4 \right]_0^1$$

$$= \frac{16}{5} - 2$$

$$= \boxed{\frac{6}{5}}$$

Problem 10.1.3.

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If $\mathbf{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

$$\mathbf{F} = [y^2, -x^2], \quad C \text{ from } (0, 0) \text{ straight to } (1, 4)$$

Solution.

$$\mathbf{r} = [t, 4t], \quad t \in [0, 1]$$

$$
\begin{aligned}
\int_C \mathbf{F(r)} \cdot d\mathbf{r} = \int_a^b \mathbf{F(r}(t)) \cdot \mathbf{r}'(t) \ dt \\
= \int_0^1 [16t^2, -t^2] \cdot [1, 4] \ dt \\
= \int_0^1 (16t^2 - 4t^2) \ dt \\
= \int_0^1 12t^2 \ dt \\
= \left[4t^3\right]_0^1 \\
= \boxed{4}
\end{aligned}
$$

Problem 10.1.5.

Calculate $\int_C \mathbf{F(r)} \cdot d\mathbf{r}$ for the given data. If $\mathbf{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

$$\mathbf{F} = [xy, x^2 y^2], \quad C \text{ the quarter-circle from } (2, 0) \text{ to } (0, 2) \text{ with center } (0, 0)$$

Solution.

$$\mathbf{r} = 2[\cos t, \sin t], \quad t \in \left[0, \frac{\pi}{2}\right]$$

$$
\begin{aligned}
\int_C \mathbf{F(r)} \cdot d\mathbf{r} = \int_a^b \mathbf{F(r}(t)) \cdot \mathbf{r}'(t) \ dt \\
= \int_0^1 [4\cos t \sin t, 16\cos^2 t \sin^2 t] \cdot 2[-\sin t, \cos t] \ dt \\
= \int_0^1 \left(-8\cos t \sin^2 t + 32\cos^3 t \sin^2 t\right) \ dt
\end{aligned}
$$

Substituting $u = \sin t$, $du = \cos t \ dt$ and $\cos^2 t = 1 - \sin^2 t$:

$$
\begin{aligned}
= \int_0^1 \left(-8u^2 + 32u^2(1 - u^2)\right) \ du \\
= \int_0^1 \left(-8u^2 + 32u^2 - 32u^4\right) \ du \\
= \int_0^1 \left(24u^2 - 32u^4\right) \ du \\
= \left[8u^3 - \frac{32}{5}u^5\right]_0^1 \\
= \boxed{\frac{8}{5}}
\end{aligned}
$$

Problem 10.1.6.

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If $\mathbf{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

$$\mathbf{F} = [x - y, y - z, z - x], \quad C : \mathbf{r} = [2\cos t, t, 2\sin t] \text{ from } (2, 0, 0) \text{ to } (2, 2\pi, 0)$$

Solution.

$$
\begin{aligned}
\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \\
&= \int_0^{2\pi} [2\cos t - t, t - 2\sin t, 2\sin t - 2\cos t] \cdot [-2\sin t, 1, 2\cos t] \, dt \\
&= \int_0^{2\pi} \left(-4\cos t \sin t + 2t\sin t + t - 2\sin t + 4\cos t \sin t - 4\cos^2 t \right) \, dt \\
&= \int_0^{2\pi} \left(2t\sin t + t - 2\sin t - 4\cos^2 t \right) \, dt
\end{aligned}
$$

Integrate Term 1 by parts ($\int u \, dv = uv - \int v \, du$):

$$
\begin{aligned}
u &= t & du &= dt \\
v &= -\cos t & dv &= \sin t \, dt
\end{aligned}
$$

$$\int_0^{2\pi} 2t\sin t \, dt = 2\left[-t\cos t - \int_0^{2\pi} \cos t \, dt \right] = 2\left[-t\cos t - \sin t \right]_0^{2\pi} = -4\pi$$

Term 2:

$$\int_0^{2\pi} t \, dt = \frac{t^2}{2}\bigg|_0^{2\pi} = 2\pi^2$$

Term 3:

$$\int_0^{2\pi} (-2\sin t) \, dt = 2\cos t \bigg|_0^{2\pi} = 0$$

In Term 4, substitute $\cos^2 t = \frac{1 + \cos 2t}{2}$:

$$\int_0^{2\pi} \left(-4\cos^2 t \right) \, dt = -2 \int_0^{2\pi} (1 + \cos 2t) \, dt = -2\left[t + \frac{1}{2}\sin 2t \right]_0^{2\pi} dt = -4\pi$$

Adding all terms together:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = -4\pi + 2\pi^2 + 0 - 4\pi = \boxed{2\pi^2 - 8\pi}$$

Problem 10.1.10.

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If $\mathbf{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

$$\mathbf{F} = [x, -z, 2y] \text{ from } (0, 0, 0) \text{ straight to } (1, 1, 0), \text{ then to } (1, 1, 1), \text{ back to } (0, 0, 0)$$

Solution.

$$
\begin{array}{llll}
(0,\,0,\,0) \text{ to } (1,\,1,\,0) & \Rightarrow & \mathbf{r}_1(t) = [t,t,0] & t \in [0,1] \\
(1,\,1,\,0) \text{ to } (1,\,1,\,1) & \Rightarrow & \mathbf{r}_2(t) = [1,1,t] & t \in [0,1] \\
(1,\,1,\,1) \text{ to } (0,\,0,\,0) & \Rightarrow & \mathbf{r}_3(t) = [1-t,1-t,1-t] & t \in [0,1]
\end{array}
$$

$$
\begin{aligned}
\mathbf{F}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t) &= [t,0,2t] \cdot [1,1,0] \\
&= t
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}(\mathbf{r}_2(t)) \cdot \mathbf{r}_2'(t) &= [1,-t,2] \cdot [0,0,1] \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}(\mathbf{r}_3(t)) \cdot \mathbf{r}_3'(t) &= [1-t,t-1,2(1-t)] \cdot [-1,-1,-1] \\
&= -1+t-t+1-2+2t \\
&= 2t-2
\end{aligned}
$$

$$
\begin{aligned}
\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \ dt \\
&= \int_0^1 (t) + (2) + (2t-2) \ dt \\
&= \int_0^1 3t \ dt \\
&= \left[\frac{3}{2}t^2 \right]_0^1 \\
&= \boxed{\frac{3}{2}}
\end{aligned}
$$