Problem 10.2.3.

Show that the form under the integral sign is exact in the plane and evaluate the integral. Show the details of your work.

$$\int_{(\pi/2,\pi)}^{(\pi,0)} \left(\frac{1}{2}\cos\frac{1}{2}x\cos 2y\ dx - 2\sin\frac{1}{2}x\sin 2y\ dy\right)$$

Solution.

The form is exact iff $\nabla\times\mathbf{F}=\mathbf{0}$, where $\mathbf{F}=\left[\frac{1}{2}\cos\frac{1}{2}x\cos 2y, -2\sin\frac{1}{2}x\sin 2y\right]$. For 2D $\mathbf{F}$,

$$\begin{aligned}\nabla\times\mathbf{F} &= (F_2)_x - (F_1)_y\\ &= \left(-\cos\frac{1}{2}x\sin 2y\right) - \left(-\cos\frac{1}{2}x\sin 2y\right)\\ &= 0\end{aligned}$$

Therefore the form is **exact**. The integral takes the form,

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = \int_A^B df = f(B) - f(A)$$

Finding f, such that $\nabla f = \mathbf{F}$,

$$f = \int F_1 dx = \int \frac{1}{2}\cos\frac{1}{2}x\cos 2y\ dx = \sin\frac{1}{2}x\cos 2y + g(x)$$

$$f_y = -2\sin\frac{1}{2}x\sin 2y + g_y = F_2 = -2\sin\frac{1}{2}x\sin 2y \implies g_y = 0 \implies g = 0$$

$$f = \sin\frac{1}{2}x\cos 2y$$

The integral is then,

$$\begin{aligned}\int_{(\pi/2,\pi)}^{(\pi,0)} df &= f(\pi,0) - f(\pi/2,\pi)\\ &= \boxed{1-\frac{\sqrt{2}}{2}}\end{aligned}$$

Problem 10.2.5.

Show that the form under the integral sign is exact in space and evaluate the integral. Show the details of your work.

$$\int_{(0,0,\pi)}^{(2,1/2,\pi/2)} e^{xy}(y\sin z\ dx + x\sin z\ dy + \cos z\ dz)$$

Solution.

The form is exact iff $\nabla\times\mathbf{F}=\mathbf{0}$, where $\mathbf{F}=e^{xy}\,[y\sin z, x\sin z, \cos z]$.

$$\begin{aligned}\nabla\times\mathbf{F} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}\\ &= \begin{bmatrix}(F_3)_y-(F_2)_z\\ (F_1)_z-(F_3)_x\\ (F_2)_x-(F_1)_y\end{bmatrix} = \begin{bmatrix}(xe^{xy}\cos z)-(xe^{xy}\cos z)\\ (ye^{xy}\cos z)-(ye^{xy}\cos z)\\ (xye^{xy}\sin z+e^{xy}\sin z)-(xye^{xy}\sin z+e^{xy}\sin z)\end{bmatrix}=\mathbf{0}\end{aligned}$$

Therefore the form is **exact**. The integral takes the form,

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = \int_A^B df = f(B)-f(A)$$

Finding f, such that $\nabla f=\mathbf{F}$,

$$f=\int F_1 dx = \int e^{xy} y\sin z\ dx = e^{xy}\sin z + g(y,z)$$

$$f_y = xe^{xy}\sin z + g_y = F_2 = xe^{xy}\sin z \implies g_y = 0 \implies g = h(z)$$

$$f_z = e^{xy}\cos z + h'(z) = F_3 = e^{xy}\cos z \implies h'(z)=0 \implies h(z)=0$$

$$f = e^{xy}\sin z$$

The integral is then,

$$\begin{aligned}\int_{(0,0,\pi)}^{(2,1/2,\pi/2)} df &= f(2,1/2,\pi/2)-f(0,0,\pi)\\ &= e\sin(\pi/2)-e^0\sin(\pi)\\ &= \boxed{e}\end{aligned}$$

Problem 10.2.13.

Check, and if independent, integrate from $(0,0,0)$ to (a,b,c).

$$2e^{x^2}(x\cos 2y\ dx - \sin 2y\ dy)$$

Solution.

The integral is independent iff $\nabla\times\mathbf{F}=\mathbf{0}$, where $\mathbf{F}=2e^{x^2}\,[x\cos 2y, -\sin 2y]$. For 2D $\mathbf{F}$,

$$\begin{aligned}\nabla\times\mathbf{F} &= (F_2)_x-(F_1)_y\\ &= \left(-4xe^{x^2}\sin 2y\right)-\left(-4xe^{x^2}\sin 2y\right)\\ &= 0\end{aligned}$$

Therefore the integral is **independent of path**. The integral takes the form,

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B df = f(B) - f(A)$$

Finding f, such that $\nabla f = \mathbf{F}$,

$$f = \int F_1 dx = \int 2e^{x^2} x \cos 2y \; dx = e^{x^2} \cos 2y + g(y)$$

$$f_y = -2e^{x^2} \sin 2y + g_y = F_2 = -2e^{x^2} \sin 2y \implies g_y = 0 \implies g(y) = 0$$

$$f = e^{x^2} \cos 2y$$

The integral is then,

$$\begin{aligned}\int_{(0,0,0)}^{(a,b,c)} df &= f(a,b,c) - f(0,0,0) \\ &= e^{a^2} \cos 2b - e^0 \cos 0 \\ &= \boxed{e^{a^2} \cos 2b - 1}\end{aligned}$$

Problem 10.2.16.

Check, and if independent, integrate from $(0,0,0)$ to (a,b,c).

$$e^y \; dx + (xe^y - e^z) \; dy - ye^z \; dz$$

Solution.

The integral is independent iff $\nabla \times \mathbf{F} = \mathbf{0}$, where $\mathbf{F} = [e^y, (xe^y - e^z), -ye^z]$.

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & (xe^y - e^z) & -ye^z \end{vmatrix} \\ &= \left[\left(\frac{\partial}{\partial y}(-ye^z) - \frac{\partial}{\partial z}(xe^y - e^z) \right), \left(\frac{\partial}{\partial z}(e^y) - \frac{\partial}{\partial x}(-ye^z) \right), \left(\frac{\partial}{\partial x}(xe^y - e^z) - \frac{\partial}{\partial y}(e^y) \right) \right] \\ &= \left[\left((-e^z) - (-e^y) \right), (0 - 0), (e^y - e^y) \right] \\ &= \mathbf{0}\end{aligned}$$

Therefore the integral is **independent of path**. The integral takes the form,

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B df = f(B) - f(A)$$

Finding f, such that $\nabla f = \mathbf{F}$,

$$f = \int F_2 dy = \int (xe^y - e^z) dy = xe^y - ye^z + g(x,z)$$

$$f_x = e^y + g_x = F_1 = e^y \implies g_x = 0 \implies g(x,z) = h(z)$$

$$f_z = -ye^z + h_z = F_3 = -ye^z \implies h_z = 0 \implies h(z) = 0$$

$$f = xe^y - ye^z$$

The integral is then,

$$\begin{aligned} \int_{(0,0,0)}^{(a,b,c)} df &= f(a,b,c) - f(0,0,0) \\ &= (ae^b - be^c) - (0 - 0) \\ &= \boxed{ae^b - be^c} \end{aligned}$$

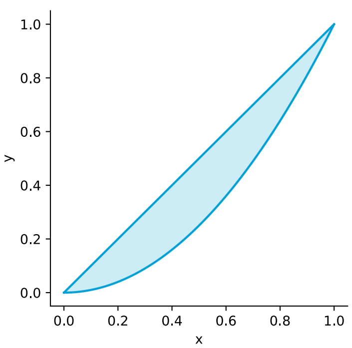
Problem 10.3.5.

Describe the region of integration and evaluate.

$$\int_0^1 \int_{x^2}^x (1 - 2xy)\ dy\ dx$$

Solution.

The region of integration is the volume under the surface $z = 1 - 2xy$ for $y \in [x^2, x]$ and $x \in [0,1]$. The region on the xy-plane looks like a slim leaf-like shape.



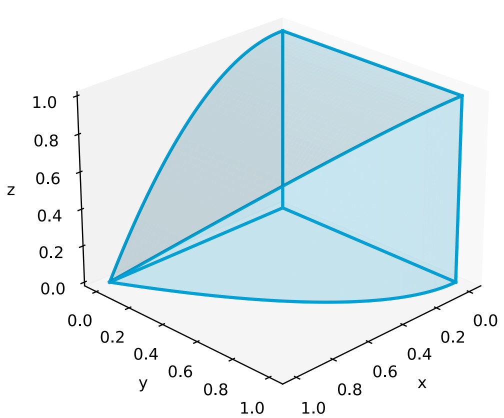
$$\begin{aligned} \int_0^1 \int_{x^2}^x (1 - 2xy)\ dy\ dx &= \int_0^1 \left[y - xy^2\right]_{x^2}^x\ dx \\ &= \int_0^1 \left[x - x^2 - x^3 + x^5\right]\ dx \\ &= \left[\frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^6}{6}\right]_0^1 \\ &= \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{6} \\ &= \boxed{\frac{1}{12}} \end{aligned}$$

Problem 10.3.10.

Find the volume of the first octant region bounded by the coordinate planes and the surfaces $y = 1 - x^2$ and $z = 1 - x^2$. Sketch it.

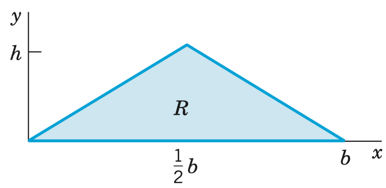
Solution.

$$
\begin{aligned}
V &= \int_0^1 \int_0^{1-x^2} \int_0^{1-x^2} dz\ dy\ dx \\
&= \int_0^1 \int_0^{1-x^2} [z]_{z=0}^{z=1-x^2}\ dy\ dx \\
&= \int_0^1 \int_0^{1-x^2} (1-x^2)\ dy\ dx \\
&= \int_0^1 \left[y(1-x^2)\right]_{y=0}^{y=1-x^2}\ dx \\
&= \int_0^1 (1-x^2)(1-x^2)\ dx \\
&= \int_0^1 (1-2x^2+x^4)\ dx \\
&= x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \Big|_0^1 \\
&= 1 - \frac{2}{3} + \frac{1}{5} \\
&= \boxed{\frac{8}{15}}
\end{aligned}
$$



Problem 10.3.12.

Find the center of gravity $(\bar{x}, \bar{y})$ of a mass of density $f(x, y) = 1$ in the given region R.



Solution.

The center of gravity is found at the coordinates $(\bar{x}, \bar{y})$ where,

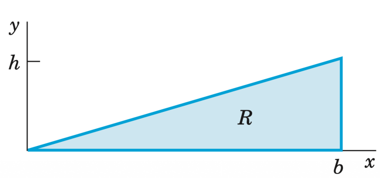
$$\bar{x} = \frac{1}{M}\iint_R xf(x,y)\ dx\ dy \qquad \bar{y} = \frac{1}{M}\iint_R yf(x,y)\ dx\ dy \qquad M = \iint_R f(x,y)\ dx\ dy$$

$$\begin{aligned} M &= \iint_R (1)\ dx\ dy \\ &= \frac{hb}{2} \end{aligned}$$

$$\begin{aligned} \bar{x} &= \frac{2}{hb}\iint_R x\ dx\ dy \\ &= \frac{2}{hb}\left[\int_0^{b/2}\int_0^{(2h/b)x} x\ dy\ dx + \int_{b/2}^{b}\int_0^{2h-(2h/b)x} x\ dy\ dx\right] \\ &= \frac{2}{hb}\left[\int_0^{b/2} [xy]_{y=0}^{y=(2h/b)x}\ dx + \int_{b/2}^{b} [xy]_{y=0}^{y=2h-(2h/b)x}\ dx\right] \\ &= \frac{2}{hb}\left[\int_0^{b/2}\left(\frac{2h}{b}x^2\right)\ dx + \int_{b/2}^{b}\left(2hx - \frac{2h}{b}x^2\right)\ dx\right] \\ &= \frac{2}{hb}\left[\left[\frac{2h}{3b}x^3\right]_0^{b/2} + \left[hx^2 - \frac{2h}{3b}x^3\right]_{b/2}^{b}\right] \\ &= \frac{2}{hb}\left[\left[\frac{2h}{3b}\left(\frac{b}{2}\right)^3\right] + \left[hb^2 - \frac{2h}{3b}(b)^3 - h\left(\frac{b}{2}\right)^2 + \frac{2h}{3b}\left(\frac{b}{2}\right)^3\right]\right] \\ &= \frac{2}{hb}\left[\frac{hb^2}{6} + hb^2 - \frac{2hb^2}{3} - \frac{hb^2}{4}\right] \\ &= b\left[\frac{1}{3} + 2 - \frac{4}{3} - \frac{1}{2}\right] \\ \bar{x} &= \boxed{\frac{b}{2}} \end{aligned}$$

Problem 10.3.17.

Find I_x, I_y, I_0 of a mass of density $f(x,y) = 1$ in the region R.



Solution.