

Problem 10.7.11

Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} \, dA$ by the divergence theorem:

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \iiint_T \nabla \cdot \mathbf{F} \, dV}$$

$$\mathbf{F} = [e^x, e^y, e^z], \quad S \text{ the surface of the cube } |x| \leq 1, |y| \leq 1, |z| \leq 1$$

Solution

$$\nabla \cdot \mathbf{F} = e^x + e^y + e^z$$

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dA &= \iiint_T \nabla \cdot \mathbf{F} \, dV \\ &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (e^x + e^y + e^z) \, dx \, dy \, dz \\ &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 e^x \, dx \, dy \, dz + \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 e^y \, dx \, dy \, dz + \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 e^z \, dx \, dy \, dz \\ &= 3 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 e^x \, dx \, dy \, dz \\ &= 12 \int_{-1}^1 e^x \, dx \\ &= 12e^x \Big|_{-1}^1 \\ &= \boxed{12(e - e^{-1})} \end{aligned}$$

Problem 10.7.13

Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} \, dA$ by the divergence theorem.

$$\mathbf{F} = [\sin y, \cos x, \cos z], \quad S \text{ the surface of } x^2 + y^2 \leq 4, |z| \leq 2 \text{ (a cylinder and two disks!)}$$

Solution

$$\begin{aligned} \nabla \cdot \mathbf{F} &= 0 + 0 - \sin z \\ &= -\sin z \end{aligned}$$

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dA &= \iiint_T \nabla \cdot \mathbf{F} \, dV \\ &= \int_{-2}^2 \int_0^{2\pi} \int_0^2 (-\sin z) r \, dr \, d\theta \, dz \\ &= \int_0^2 r \, dr \cdot \int_0^{2\pi} d\theta \cdot \int_{-2}^2 (-\sin z) \, dz \\ &= \left[\frac{r^2}{2} \right]_0^2 \cdot (2\pi) \cdot [\cos z]_{-2}^2 \\ &= 4\pi(\cos(2) - \cos(-2)) \\ &= \boxed{0} \end{aligned}$$

Problem 10.7.17

Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} \, dA$ by the divergence theorem.

$$\mathbf{F} = [x^2, y^2, z^2], \quad S \text{ the surface of the cone } x^2 + y^2 \leq z^2, \quad 0 \leq z \leq h$$

Solution

$$\nabla \cdot \mathbf{F} = 2x + 2y + 2z$$

In cylindrical coordinates,

$$\nabla \cdot \mathbf{F} = 2(r \cos \theta + r \sin \theta + z)$$

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dA &= \iiint_T \nabla \cdot \mathbf{F} \, dV \\ &= 2 \int_0^h \int_0^{2\pi} \int_0^z (r \cos \theta + r \sin \theta + z) r \, dr \, d\theta \, dz \\ &= 2 \int_0^h \int_0^{2\pi} \int_0^z (r^2 \cos \theta + r^2 \sin \theta + rz) \, dr \, d\theta \, dz \\ &= 2 \int_0^h \int_0^{2\pi} \left[\frac{r^3}{3} \cos \theta + \frac{r^3}{3} \sin \theta + \frac{r^2}{2} z \right]_{r=0}^{r=z} d\theta \, dz \\ &= 2 \int_0^h \int_0^{2\pi} z^3 \left(\frac{1}{3} \cos \theta + \frac{1}{3} \sin \theta + \frac{1}{2} \right) d\theta \, dz \\ &= 2 \int_0^h z^3 \, dz \cdot \int_0^{2\pi} \left(\frac{1}{3} \cos \theta + \frac{1}{3} \sin \theta + \frac{1}{2} \right) d\theta \\ &= \frac{1}{2} [z^4]_0^h \cdot \left[\frac{1}{3} \sin \theta - \frac{1}{3} \cos \theta + \frac{1}{2} \theta \right]_0^{2\pi} \\ &= \boxed{\frac{\pi h^4}{2}} \end{aligned}$$

Problem 10.7.22

Given a mass of density 1 in a region T of space, find the moment of inertia about the x-axis,

$$I_x = \iiint_T (y^2 + z^2) \, dV$$

$$T : \text{The paraboloid } y^2 + z^2 \leq x, \quad 0 \leq x \leq h$$

Solution

In cylindrical coordinates along the x -axis, the integrand becomes r^2 and

$$T : \text{The paraboloid } r^2 \leq x, \quad 0 \leq x \leq h$$

$$\begin{aligned}
I_x &= \iiint_T (y^2 + z^2) \, dV \\
&= \iiint_T (r^2) r \, dr \, d\theta \, dx \\
&= \int_0^h \int_0^{2\pi} \int_0^{\sqrt{x}} (r^2) r \, dr \, d\theta \, dx \\
&= \int_0^h \int_0^{2\pi} \int_0^{\sqrt{x}} (r^3) \, dr \, d\theta \, dx \\
&= \frac{1}{4} \int_0^h \int_0^{2\pi} (x^2) \, d\theta \, dx \\
&= \frac{1}{4} \int_0^h (x^2) \, dx \cdot \int_0^{2\pi} d\theta \\
&= \frac{\pi}{6} x^3 \Big|_0^h \\
&= \boxed{\frac{\pi h^3}{6}}
\end{aligned}$$

Problem 10.8.1. Harmonic functions.

Theorem 1. A Basic Property of Harmonic Functions

Let $f(x, y, z)$ be a harmonic function in some domain D in space. Let S be any piecewise smooth closed orientable surface in D whose entire region it encloses belongs to D . Then the integral of the normal derivative of f taken over S is zero.

Verify Theorem 1 for $f = 2z^2 - x^2 - y^2$ and S the surface of the box $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$.

Solution

$$\begin{aligned}
\Delta f &= f_{xx} + f_{yy} + f_{zz} \\
&= (-2) + (-2) + (4) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\iint_S \frac{\partial f}{\partial n} \, dA &= \iiint_T \Delta f \, dV \\
&= \boxed{0}
\end{aligned}$$

Problem 10.8.3. Green's First Identity

$$\boxed{\iiint_T (f \Delta g + \nabla f \cdot \nabla g) \, dV = \iint_S f \frac{\partial g}{\partial n} \, dA}$$

Verify for $f = 4y^2$, $g = x^2$, S the surface of the “unit cube” $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$. What are the assumptions of f and g ? Must f and g be harmonic?

Solution

	f	g
	$4y^2$	x^2
∇	$[0, 8y]$	$[2x, 0]$
Δ	8	2

$$\begin{aligned}
 \iiint_T (f \Delta g + \nabla f \cdot \nabla g) \, dV &= \int_0^1 \int_0^1 \int_0^1 (8y^2) \, dx \, dy \, dz \\
 &= \left. \frac{8y^3}{3} \right|_0^1 \\
 &= \boxed{\frac{8}{3}}
 \end{aligned}$$

$$\iint_S f \frac{\partial g}{\partial n} \, dA = \iint_S f (\nabla g \cdot \mathbf{n}) \, dA$$

Integrating over the cube surface, only the $x = 1$ face is non-zero,

$$\begin{aligned}
 \iint_S f (\nabla g \cdot \mathbf{n}) \, dA &= \int_0^1 f g_x(x=1) \, dy \\
 &= \int_0^1 (4y^2)(2) \, dy \\
 &= 8 \int_0^1 y^2 \, dy \\
 &= 8 \left[\frac{y^3}{3} \right]_0^1 \\
 &= \boxed{\frac{8}{3}}
 \end{aligned}$$

The functions do not need to be harmonic, but the divergence theorem assumptions must hold. The functions need to be twice differentiable and S must be piecewise smooth.

Problem 10.8.5. Green's Second Identity

$$\boxed{\iiint_T (f \Delta g - g \Delta f) \, dV = \iint_S \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) \, dA}$$

Verify for $f = 6y^2$, $g = 2x^2$, S the surface of the “unit cube” $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Solution

	f	g
	$6y^2$	$2x^2$
∇	$[0, 12y]$	$[4x, 0]$
Δ	12	4

$$\begin{aligned}
\iiint_T (f \Delta g - g \Delta f) \, dV &= 24 \int_0^1 \int_0^1 \int_0^1 (y^2 - x^2) \, dx \, dy \, dz \\
&= 24 \left(\int_0^1 y^2 \, dy - \int_0^1 x^2 \, dx \right) \cdot \int_0^1 dz \\
&= 8 \left(y^3 \Big|_0^1 - x^3 \Big|_0^1 \right) \\
&= \boxed{0}
\end{aligned}$$

$$\iint_S \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) dA = \iint_S f (\nabla g \cdot \mathbf{n}) - g (\nabla f \cdot \mathbf{n}) \, dA$$

Integrating over the cube surface, only the $x = 1$ face (first term) and $y = 1$ face (second term) are non-zero,

$$\begin{aligned}
\iint_S f (\nabla g \cdot \mathbf{n}) \, dA &= \int_0^1 f g_x(x=1) \, dy - \int_0^1 g f_y(y=1) \, dx \\
&= \int_0^1 (6y^2)(4) \, dy - \int_0^1 (2x^2)(12) \, dx \\
&= 24 \int_0^1 y^2 \, dy - 24 \int_0^1 x^2 \, dx \\
&= 8 \left(y^3 \Big|_0^1 - x^3 \Big|_0^1 \right) \\
&= \boxed{0}
\end{aligned}$$

Problem 10.8.7

Use the divergence theorem, assuming that the assumptions on T and S are satisfied.

Show that a region T with boundary surface S has the volume

$$V = \iiint_S x \, dy \, dz = \iiint_S y \, dz \, dx = \iiint_S z \, dx \, dy = \frac{1}{3} \iint_S (x \, dy \, dz + y \, dz \, dx + z \, dx \, dy)$$

Solution

Take $\mathbf{F} = [x, 0, 0]$, therefore $\nabla \cdot \mathbf{F} = 1$ and

$$\begin{aligned}
V &= \iiint_T \nabla \cdot \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \mathbf{n} \, dA \\
V &= \iint_S x \, dy \, dz
\end{aligned}$$

Take $\mathbf{F} = [0, y, 0]$, therefore $\nabla \cdot \mathbf{F} = 1$ and

$$\begin{aligned}
V &= \iiint_T \nabla \cdot \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \mathbf{n} \, dA \\
V &= \iint_S y \, dx \, dz
\end{aligned}$$

Take $\mathbf{F} = [0, 0, z]$,

$$V = \iiint_T \nabla \cdot \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \mathbf{n} \, dA$$

$$V = \iint_S z \, dx \, dy$$

Summing the above,

$$3V = \iint_S (x \, dy \, dz + y \, dx \, dz + z \, dx \, dy)$$

$$V = \frac{1}{3} \iint_S (x \, dy \, dz + y \, dz \, dx + z \, dx \, dy)$$

Problem 10.9.3

Evaluate the surface integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dA$ directly for the given \mathbf{F} and S .

$$\mathbf{F} = [e^{-z}, e^{-z} \cos y, e^{-z} \sin y], \quad S : z = \frac{y^2}{2}, \quad -1 \leq x \leq 1, \quad 0 \leq y \leq 1$$

Solution

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{-z} & e^{-z} \cos y & e^{-z} \sin y \end{vmatrix} \\ &= [(e^{-z} \cos y + e^{-z} \cos y), -(0 + e^{-z}), (0 - 0)] \\ &= [2e^{-z} \cos y, -e^{-z}, 0] \end{aligned}$$

$$S : \mathbf{r}(x, y) = [x, y, \frac{y^2}{2}]$$

$$\begin{aligned} \mathbf{n} &= \mathbf{r}_x \times \mathbf{r}_y \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & y \end{vmatrix} \\ &= [0, -y, 1] \end{aligned}$$

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dA &= \iint_S [2e^{-z} \cos y, -e^{-z}, 0] \cdot [0, -y, 1] \, dx \, dy \\ &= \iint_S (ye^{-z} + 0) \, dx \, dy \\ &= \int_{-1}^1 dx \cdot \int_0^1 ye^{-\frac{y^2}{2}} \, dy \\ &= 2 \int_0^1 ye^{-\frac{y^2}{2}} \, dy \\ &= -2e^{-\frac{y^2}{2}} \Big|_0^1 \\ &= \boxed{2(1 - e^{-\frac{1}{2}})} \end{aligned}$$

Problem 10.9.5

Evaluate the surface integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dA$ directly for the given \mathbf{F} and S .

$$\mathbf{F} = [z^2, \frac{3}{2}x, 0], \quad S : 0 \leq x \leq a, \quad 0 \leq y \leq a, \quad z = 1$$

Solution

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & \frac{3}{2}x & 0 \end{vmatrix} \\ &= \left[(0 - 0), -(0 - 2z), \left(\frac{3}{2} - 0 \right) \right] \\ &= \left[0, 2z, \frac{3}{2} \right] \end{aligned}$$

The surface is a flat plane and therefore $\mathbf{n} = [0, 0, 1]$.

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dA &= \iint_S \left[0, 2z, \frac{3}{2} \right] \cdot [0, 0, 1] \, dx \, dy \\ &= \iint_S \frac{3}{2} \, dx \, dy \\ &= \frac{3}{2} \int_0^a dx \cdot \int_0^a dy \\ &= \boxed{\frac{3a^2}{2}} \end{aligned}$$

Problem 10.9.13

Calculate $\oint_C \mathbf{F} \cdot \mathbf{r}'(s) \, ds$ by Stokes's theorem,

$$\boxed{\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dA = \oint_C \mathbf{F} \cdot \mathbf{r}'(s) \, ds}$$

for the given \mathbf{F} and C . Assume the Cartesian coordinates to be right-handed and the z -component of the surface normal to be nonnegative.

$$\mathbf{F} = [-5y, 4x, z], \quad C \text{ the circle } x^2 + y^2 = 16, \quad z = 4$$

Solution

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -5y & 4x & z \end{vmatrix} \\ &= [(0 - 0), -(0 - 0), (4 + 5)] = [0, 0, 9] \end{aligned}$$

C is the boundary of a flat disc and therefore $\mathbf{n} = [0, 0, 1]$.

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathbf{r}'(s) \, ds &= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dA \\
&= 9 \iint_S dA \\
&= 9 \cdot \text{Area}(S) \\
&= 9 \cdot 16\pi \\
&= \boxed{144\pi}
\end{aligned}$$

Problem 10.9.15

Calculate $\oint_C \mathbf{F} \cdot \mathbf{r}'(s) \, ds$ by Stokes's theorem for the given \mathbf{F} and C . Assume the Cartesian coordinates to be right-handed and the z -component of the surface normal to be nonnegative.

$$\mathbf{F} = [z^3, x^3, y^3], \text{ around the triangle with vertices } (0, 0, 0), (1, 0, 0), (1, 1, 0)$$

Solution

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^3 & x^3 & y^3 \end{vmatrix} \\
&= [(3y^2 - 0), -(0 - 3z^2), (3x^2 - 0),] \\
&= 3[y^2, z^2, x^2]
\end{aligned}$$

C is the boundary of a flat triangle and therefore $\mathbf{n} = [0, 0, 1]$.

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathbf{r}'(s) \, ds &= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dA \\
&= 3 \int_0^1 \int_0^x [y^2, z^2, x^2] \cdot [0, 0, 1] \, dy \, dx \\
&= 3 \int_0^1 \int_0^x x^2 \, dy \, dx \\
&= 3 \int_0^1 x^3 \, dx \\
&= \boxed{\frac{3}{4}}
\end{aligned}$$

Problem 10.9.19

Calculate $\oint_C \mathbf{F} \cdot \mathbf{r}'(s) \, ds$ by Stokes's theorem for the given \mathbf{F} and C . Assume the Cartesian coordinates to be right-handed and the z -component of the surface normal to be nonnegative.

$$\mathbf{F} = [z, e^z, 0], \quad C \text{ the boundary curve of the portion of the cone } z = \sqrt{x^2 + y^2}, \quad x \geq 0, \quad y \geq 0, \quad 0 \leq z \leq 1$$

Solution