Thermalization from quenching in coupled oscillators

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This notebook presents the codes used for the calculations and plots in the work *Thermalization from Quenching in Coupled Oscillators*. The study employs a simple model consisting of two coupled quantum harmonic oscillators, initially prepared in their ground state, to demonstrate quench-induced finite-time thermalization. Specifically, we examine how the reduced density matrix of one oscillator evolves towards a thermal state under this setup. Owing to the Gaussian character of the dynamics, the evolution is greatly simplified, reducing the problem to just three equations. These equations determine three tunable parameters, enabling the approximation of any desired target temperature with arbitrary precision.

A. Section II: REVIEW OF TWO-MODE GAUSSIAN PURE STATES

In[1]:=
```
ClearAll["Global`*"]
```

Undisplaced two mode Gaussian pure state

Here we consider a pure, undisplaced Gaussian wavefunction of a two-oscillator system of the form

In[2]:=
```
ψ[x1_, x2_] = N1 Exp[-1/2 (A11 x1^2 + A22 x2^2 + 2 A12 x1 x2)]
```

Out[2]= $e^{\frac{1}{2}\left(-A11\, x1^2 - 2\, A12\, x1\, x2 - A22\, x2^2\right)}\, N1$

In[3]:=
```
ψst[x1_, x2_] = N1 Exp[-1/2 (A11st x1^2 + A22st x2^2 + 2 A12st x1 x2)]
```

Out[3]= $e^{\frac{1}{2}\left(-A11st\, x1^2 - 2\, A12st\, x1\, x2 - A22st\, x2^2\right)}\, N1$

To get the normalization, we compute the norm^2 as

In[4]:= `Integrate[ψ[x1, x2] × ψst[x1, x2], {x1, -Infinity, Infinity}, {x2, -Infinity, Infinity}]`

Out[4]=
$$\frac{2\,\text{N1}^2\,\pi}{\sqrt{\text{A22}+\text{A22st}}\,\sqrt{\text{A11}+\text{A11st}-\frac{(\text{A12}+\text{A12st})^2}{\text{A22}+\text{A22st}}}}\quad\text{if}\quad \text{Re}\left[\frac{\text{A12}^2+2\,\text{A12}\,\text{A12st}+\text{A12st}^2-(\text{A11}+\text{A11st})\,(\text{A22}+\text{A22st})}{\text{A22}+\text{A22st}}\right]<0$$

This leads to

In[5]:=
$$\text{N1} = \text{FullSimplify}\left[\left(\frac{\left(\sqrt{\text{A22}+\text{A22st}}\,\sqrt{\text{A11}+\text{A11st}-\frac{(\text{A12}+\text{A12st})^2}{\text{A22}+\text{A22st}}}\right)}{2\,\pi}\right)^{\frac{1}{2}},\ \text{Assumptions}\rightarrow \text{A11}+\text{A11st}>0\,\&\&\,\text{A22}+\text{A22st}>0\right]$$

Out[5]=
$$\frac{\left((\text{A22}+\text{A22st})\left(\text{A11}+\text{A11st}-\frac{(\text{A12}+\text{A12st})^2}{\text{A22}+\text{A22st}}\right)\right)^{1/4}}{\sqrt{2\,\pi}}$$

Reduced density matrix

The reduced density matrix can be found by partial tracing, which leads to

In[6]:= `ρ1[x1p_, x1_] = Integrate[ψ[x1, x2] × ψst[x1p, x2], {x2, -Infinity, Infinity}]`

Out[6]=
$$\frac{\sqrt{(\text{A22}+\text{A22st})\left(\text{A11}+\text{A11st}-\frac{(\text{A12}+\text{A12st})^2}{\text{A22}+\text{A22st}}\right)}\ e^{\frac{\text{A12}^2\,\text{x1}^2-\text{A11}\,(\text{A22}+\text{A22st})\,\text{x1}^2+2\,\text{A12}\,\text{A12st}\,\text{x1}\,\text{x1p}+\left(\text{A12st}^2-\text{A11st}\,(\text{A22}+\text{A22st})\right)\,\text{x1p}^2}{2\,(\text{A22}+\text{A22st})}}}{\sqrt{\text{A22}+\text{A22st}}\,\sqrt{2\,\pi}}\quad \text{if}\ \text{Re}[\text{A22}+\text{A22st}]>0$$

To read-off X,Y,Z variables collect the different terms in the exponential

In[7]:=
$$\text{Collect}\left[\frac{-2}{\text{m}\,\omega}\left(\frac{1}{2\,(\text{A22}+\text{A22st})}\left(\text{A12}^2\,\text{x1}^2-\text{A11}\,(\text{A22}+\text{A22st})\,\text{x1}^2+2\,\text{A12}\,\text{A12st}\,\text{x1}\,\text{x1p}+\left(\text{A12st}^2-\text{A11st}\,(\text{A22}+\text{A22st})\right)\,\text{x1p}^2\right)\right),\ \{\text{x1}^2,\text{x1p}^2,\text{x1}\,\text{x1p}\},\ \text{FullSimplify}\right]$$

Out[7]=
$$\frac{\left(-\text{A12}^2+\text{A11}\,(\text{A22}+\text{A22st})\right)\,\text{x1}^2}{(\text{A22}+\text{A22st})\,\text{m}\,\omega}+\frac{\left(-\text{A12st}^2+\text{A11st}\,(\text{A22}+\text{A22st})\right)\,\text{x1p}^2}{(\text{A22}+\text{A22st})\,\text{m}\,\omega}-\frac{2\,\text{A12}\,\text{A12st}\,\text{x1}\,\text{x1p}}{\text{A22}\,\text{m}\,\omega+\text{A22st}\,\text{m}\,\omega}$$

Hence, X - i Y is

In[8]:=
$$\frac{\left(-A12^2 + A11\ (A22 + A22st)\right)}{(A22 + A22st)\ m\,\omega}$$

Out[8]=
$$\frac{-A12^2 + A11\ (A22 + A22st)}{(A22 + A22st)\ m\,\omega}$$

or

In[9]:=
$$\{X,\ Y\} = \left\{\frac{-ReA12sqr + 2\,ReA11\ (ReA22)}{2\ (ReA22)\ m\,\omega},\ -\frac{-ImA12sqr + 2\,ImA11\ (ReA22)}{2\ (ReA22)\ m\,\omega}\right\}\ //\ FullSimplify$$

Out[9]=
$$\left\{-\frac{ReA12sqr - 2\,ReA11\,ReA22}{2\,m\,ReA22\,\omega},\ \frac{ImA12sqr - 2\,ImA11\,ReA22}{2\,m\,ReA22\,\omega}\right\}$$

In[10]:=

and Z is

In[11]:=
$$Z = -\frac{2\left(ReA12^2 + ImA12^2\right)}{2\ (ReA22\,m\,\omega)} * \frac{1}{2}\ //\ FullSimplify$$

Out[11]=
$$-\frac{ImA12^2 + ReA12^2}{2\,m\,ReA22\,\omega}$$

Let us rewrite the density matrix now as

In[12]:=
$$\rho[x1p\_,\ x1\_] = N2\,Exp\left[-m\,\frac{\omega}{2}\,\left((X1 + I\,Y1)\ x1p^2 + (X1 - I\,Y1)\ x1^2 + 2\,Z1\,x1p\,x1\right)\right]$$

Out[12]=
$$e^{-\frac{1}{2}\,m\,\left(x1^2\,(X1 - i\,Y1) + x1p^2\,(X1 + i\,Y1) + 2\,x1\,x1p\,Z1\right)\,\omega}\ N2$$

The normalization condition requires

In[13]:=
```
Integrate[ρ[x, x], {x, -Infinity, Infinity}]
```

Out[13]=
$$\boxed{\frac{N2\,\sqrt{\pi}}{\sqrt{m\ (X1 + Z1)\ \omega}}\ \text{ if } Re[m\ (X1 + Z1)\ \omega] > 0}$$

be unity, which implies

In[14]:=
$$N2 = \left(\frac{\sqrt{\pi}}{\sqrt{m\ (X1 + Z1)\ \omega}}\right)^{-1}$$

Out[14]=
$$\frac{\sqrt{m\ (X1 + Z1)\ \omega}}{\sqrt{\pi}}$$

Covariance matrix

To find $< x1^2 >$ in terms of X, Y and Z, we compute

In[15]:=
```
Σxx = Integrate[x^2 ρ[x, x], {x, -Infinity, Infinity}]
```

Out[15]= $\frac{1}{2\, m\, (X1 + Z1)\, \omega}$ if $\mathrm{Re}[m\, (X1 + Z1)\, \omega] > 0$

To find $< p1^2>$, given by $\int$ ***ψ*(x) (-ℏ^2 ψ''(x))dx***, in terms of X, Y and Z, we compute the differential, and then integrate the expression.

In[16]:=
```
-D[ρ[xp, x], {x, 2}] /. {xp → x} // FullSimplify
```

Out[16]= $$\frac{e^{-m x^2 (X1+Z1)\,\omega}\, m\,\omega\, \sqrt{m\,(X1 + Z1)\,\omega}\, \left(X1 - i\, Y1 - m\, x^2\, (X1 - i\, Y1 + Z1)^2\, \omega\right)}{\sqrt{\pi}}$$

Now, we integrate the full expression :

In[17]:=
```
Σpp = Integrate[
  (e^(-m x^2 (X1+Z1) ω) m ω Sqrt[m (X1 + Z1) ω] (X1 - i Y1 - m x^2 (X1 - i Y1 + Z1)^2 ω))/Sqrt[π], {x, -Infinity, Infinity}]
```

Out[17]= $\frac{m\, \left(X1^2 + Y1^2 - Z1^2\right)\, \omega}{2\, (X1 + Z1)}$ if $\mathrm{Re}[m\, (X1 + Z1)\, \omega] > 0$

To find $< x1\, p1>$, given by $\int$ ***ψ*(x) (x (-i ℏ ψ'(x)))dx***, in terms of X, Y and Z, we compute

In[18]:=
```
xp D[ρ[xp, x], x] /. {xp → x} // FullSimplify
```

Out[18]= $$-\frac{e^{-m x^2 (X1+Z1)\,\omega}\, m\, x^2\, (X1 - i\, Y1 + Z1)\, \omega\, \sqrt{m\,(X1 + Z1)\,\omega}}{\sqrt{\pi}}$$

Now, we integrate the full expression:

In[19]:=
```
Σxp = Integrate[-I (-(e^(-m x^2 (X1+Z1) ω) m x^2 (X1 - i Y1 + Z1) ω Sqrt[m (X1 + Z1) ω])/Sqrt[π]),
   {x, -Infinity, Infinity}] // FullSimplify
```

Out[19]= $\frac{1}{2}\left(i + \frac{Y1}{X1 + Z1}\right)$ if $\mathrm{Re}[m\, (X1 + Z1)\, \omega] > 0$

To find $< p1\, x1>$, given by $\int$ ***ψ*(x) ((-i ℏ) (x ψ(x))')dx***, in terms of X, Y and Z, we compute:

In[20]:=
```
D[ x ρ[xp, x], x] /. {xp → x } // FullSimplify
```

Out[20]= $$\frac{e^{-m x^2 (X1+Z1)\,\omega}\, \sqrt{m\,(X1 + Z1)\,\omega}\, \left(1 - m\, x^2\, (X1 - i\, Y1 + Z1)\, \omega\right)}{\sqrt{\pi}}$$

Now, we integrate the full expression:

In[21]:=
```
Σpx = Integrate[-I ((E^(-m x^2 (X1+Z1) ω) Sqrt[m (X1 + Z1) ω] (1 - m x^2 (X1 - I Y1 + Z1) ω))/Sqrt[π]),
   {x, -Infinity, Infinity}] // FullSimplify
```

Out[21]= $\frac{1}{2}\left(-i + \frac{Y1}{X1+Z1}\right)$ if $\text{Re}[m\,(X1+Z1)\,\omega] > 0$

So the covariance matrix of our system becomes,

In[22]:=
```
Σ = {{Σxx, Σxp}, {Σpx, Σpp}} // FullSimplify
```

Out[22]= $\left\{\left\{\frac{1}{2\,m\,X1\,\omega + 2\,m\,Z1\,\omega} \text{ if } \text{Re}[m\,(X1+Z1)\,\omega] > 0,\ \frac{1}{2}\left(i + \frac{Y1}{X1+Z1}\right) \text{ if } \text{Re}[m\,(X1+Z1)\,\omega] > 0\right\},\right.$
$\left.\left\{\frac{1}{2}\left(-i + \frac{Y1}{X1+Z1}\right) \text{ if } \text{Re}[m\,(X1+Z1)\,\omega] > 0,\ \frac{m\,(X1^2+Y1^2-Z1^2)\,\omega}{2\,(X1+Z1)} \text{ if } \text{Re}[m\,(X1+Z1)\,\omega] > 0\right\}\right\}$

In[23]:=
```
MatrixForm[Σ]
```

Out[23]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2\,m\,X1\,\omega+2\,m\,Z1\,\omega} \text{ if } \text{Re}[m\,(X1+Z1)\,\omega] > 0 & \frac{1}{2}\left(i + \frac{Y1}{X1+Z1}\right) \text{ if } \text{Re}[m\,(X1+Z1)\,\omega] > 0 \\ \frac{1}{2}\left(-i + \frac{Y1}{X1+Z1}\right) \text{ if } \text{Re}[m\,(X1+Z1)\,\omega] > 0 & \frac{m\,(X1^2+Y1^2-Z1^2)\,\omega}{2\,(X1+Z1)} \text{ if } \text{Re}[m\,(X1+Z1)\,\omega] > 0 \end{pmatrix}$$

R-Space Variables

The dynamics of the density matrix can fully be described by these three parameters X, Y, and Z. These parameters vary with time, so the evolution of the density matrix can be represented conveniently by a three-dimensional curve **R(t)= (X(t), Y(t), Z(t))**.

The general thermal state of a harmonic oscillator, at an inverse temperature β, is given by the equation,

In[24]:=
```
ρβ[x1p_, x1_] = Nβ Exp[(-m ω)/2 ((x1^2 + x1p^2) Coth[β ω] - 2 x1 x1p Csch[β ω])] // FullSimplify
```

Out[24]= $e^{-\frac{1}{2} m\,(x1^2+x1p^2)\,\omega\,\text{Coth}[\beta\,\omega] + m\,x1\,x1p\,\omega\,\text{Csch}[\beta\,\omega]}\ N\beta$

As the oscillator reaches thermal equilibrium, ρ[x1p_,x1_] approaches $\rho\beta$[x1p_,x1_]. Comparing, we get $\boldsymbol{R_\beta}$ **= (Coth[$\beta\,\omega$], 0, -Csch[$\beta\,\omega$])**. We can visualize the set of density matrices, for temperatures, in R space with the following codes:

Figure 1:

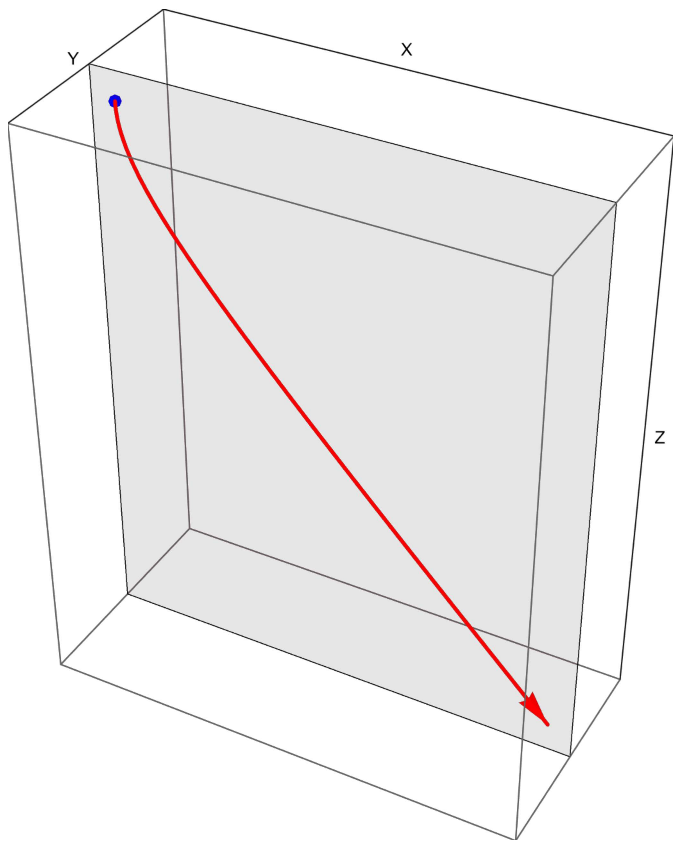
Here we will plot the collection of possible thermal states for oscillator-1 at different temperatures. For the purpose of plotting, we set the frequency to be unity.

In[25]:=
```
ωplot1 = 1
```

Out[25]= 1

In[26]:=
```
Show[ParametricPlot3D[{Coth[ωplot1 / T], 0, -Csch[ωplot1 / T]}, {T, 0, 5},
  Ticks → None, PlotStyle → {Thick, Red}, AxesLabel → {"X", "Y", "Z"}],
 Graphics3D[{Blue, PointSize[Large], Point[{1, 0, 0}]}],
 Graphics3D[{Opacity[0.1, Red], InfinitePlane[{{0, 0, 0}, {1, 0, 0}, {0, 0, 1}}]}],
 Graphics3D[{Red, Arrow[{{Coth[ωplot1 / 4.9], 0, -Csch[ωplot1 / 4.9]},
     {Coth[ωplot1 / 5], 0, -Csch[ωplot1 / 5]}}]}], LabelStyle → Black]
```

Out[26]=



The family of thermal density matrices is represented by the curve $\boldsymbol{R_\beta}$ **= (Coth[$\boldsymbol{\beta\,\omega}$], 0, –Csch[$\boldsymbol{\beta\,\omega}$])**. The curve lies entirely in the X–Z plane, shown as the shaded region.The blue dot indicates the ground state, and the arrow shows the direction of increasing temperature.

B. Section III: THE SET UP

Hamiltonian of the System

We consider the system of two coupled oscillators is described by a Hamiltonian of the form:

In[27]:= $H = \frac{p1^2}{2\,m} + \frac{p2^2}{2\,m} + \frac{1}{2}\,m\,\Omega^2\,x1^2 + \frac{1}{2}\,m\,\Omega^2\,x2^2 + \frac{1}{2}\,K[t]\,(x1 - x2)^2$

Out[27]= $\frac{p1^2}{2\,m} + \frac{p2^2}{2\,m} + \frac{1}{2}\,m\,x1^2\,\Omega^2 + \frac{1}{2}\,m\,x2^2\,\Omega^2 + \frac{1}{2}\,(x1 - x2)^2\,K[t]$

where K is the coupling, and Ω is the frequency of the oscillators. Our coupling is turned on from t=0 to t-T, and is off otherwise. So the time dependence of both coupling and frequency is as follows:

```
In[28]:= K[t_] = Piecewise[{{k, 0 < t < τ}}, 0]
```

Out[28]= $\begin{cases} k & 0 < t < \tau \\ 0 & \text{True} \end{cases}$

```
In[29]:= Ω[t_] = Piecewise[{{ω, t < 0 || t > τ}, {ωp, 0 < t < τ}}]
```

Out[29]= $\begin{cases} \omega & t < 0 \,||\, t > \tau \\ \omega p & 0 < t < \tau \\ 0 & \text{True} \end{cases}$

Figure 2: Variation of equipotential surface of the system during passive and active phase

Here, we will look at the For the purpose of plotting, we set the parameters to arbitrary constants:

```
In[30]:= {Ω1plot2, Ω2plot2, Kplot2} = {1, 1.1, 2}
Out[30]= {1, 1.1, 2}
```

The potentials before and coupling are:

In[31]:= $V0[x1\_, x2\_] = \frac{1}{2}\,\Omega1plot2^2\,(x1^2 + x2^2)$

Out[31]= $\frac{1}{2}\,(x1^2 + x2^2)$

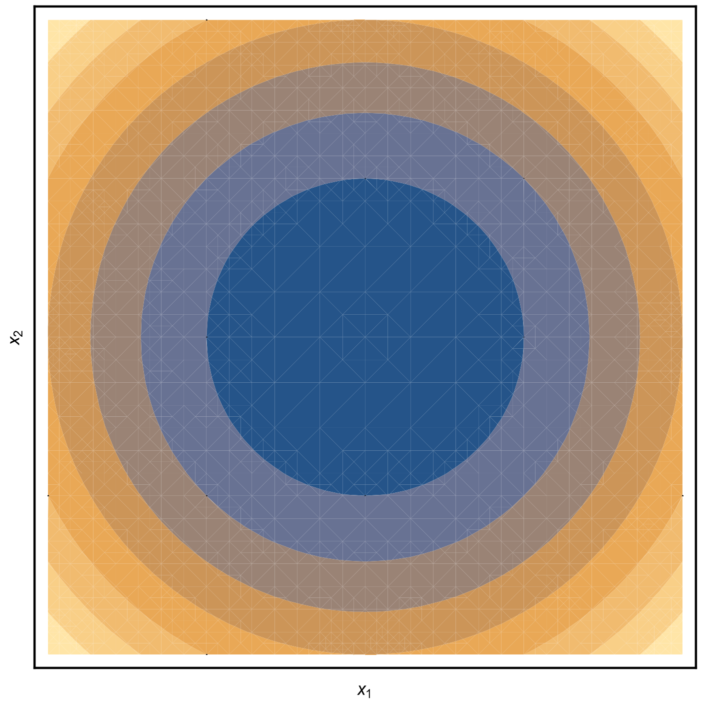
In[32]:= $V[x1\_, x2\_] = \frac{1}{2}\,\Omega2plot2^2\,(x1^2 + x2^2) + \frac{1}{2}\,Kplot2\,(x1 - x2)^2$

Out[32]= $(x1 - x2)^2 + 0.605\,(x1^2 + x2^2)$

Let us call the plots for the two potentials as p1 and p2. Then,

In[33]:=
```
p1 = ContourPlot[V0[x1, x2], {x1, -2, 2}, {x2, -2, 2}, ContourStyle → None,
   FrameLabel → {"x1", "x2"}, LabelStyle → {Black}, FrameTicks → False]
```

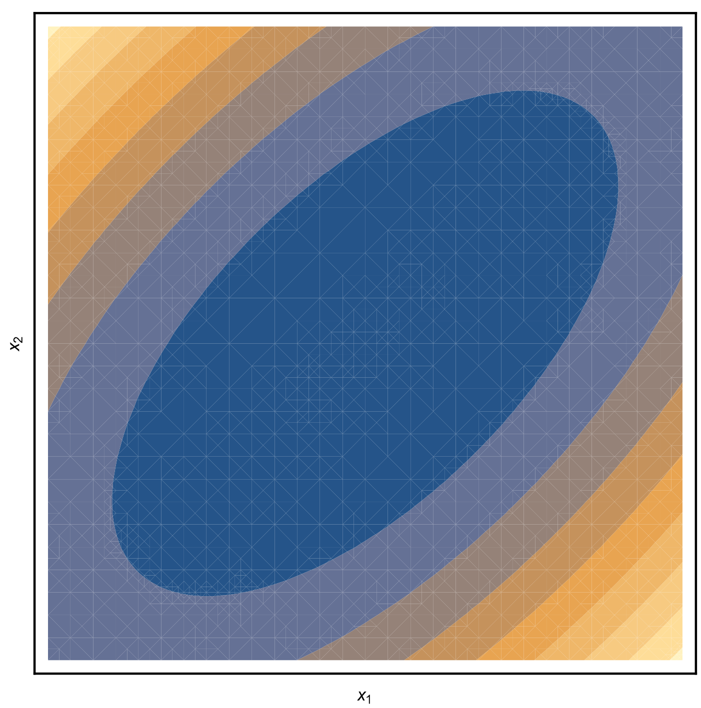
Out[33]=



This is the contour representation of the effective 2D potential of the coupled oscillator system, for typical values of ω'/ω and k/ω^2, when the system is in the uncoupled phase.

In[34]:=
```
p1 = ContourPlot[V[x1, x2], {x1, -2, 2}, {x2, -2, 2}, ContourStyle → None,
   FrameLabel → {"x1", "x2"}, LabelStyle → {Black}, FrameTicks → False]
```

Out[34]=



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The Ermakov Equations

Let us consider a Gaussian wavefunction $\boldsymbol{\psi}$**Diag = Exp**$\left[\frac{-m}{2\,\text{hbar}}\left(\text{Aplus} * \text{xplus}^2 + \text{Aminus} * \text{xminus}^2\right)\right]$ of the system. We define the diagonalized quadratic phase coefficients of the wavefunction **Aplus =** $\frac{-i\,m}{\text{hbar}}\left(\frac{\mu\text{plus}'}{\mu\text{plus}}\right)$ and **Aminus =** $\frac{-i\,m}{\text{hbar}}\left(\frac{\mu\text{minus}'}{\mu\text{minus}}\right)$. Substituting this Gaussian ansatz into the Schrodinger equation and simplifying, we find that μplus and μminus satisfies the classical harmonic oscillator equation $\boldsymbol{\mu'' + \omega^2\,\mu = 0}$.

```
In[35]:= μ[t] = b[t] Exp[I θ[t]]
Out[35]= ⅇ^(ⅈ θ[t]) b[t]
```

Putting this value into the classical oscillator equation,

```
In[36]:= D[μ[t], {t, 2}] + ω^2 μ[t] == 0 // FullSimplify
Out[36]= ⅇ^(ⅈ θ[t]) (2 ⅈ b′[t] θ′[t] + b″[t] + b[t] (ω^2 - θ′[t]^2 + ⅈ θ″[t])) == 0
```

Let us divide this equation by $e^{i\,\theta[t]}$ and then separate the real and imaginary parts. The real part becomes,

```
In[37]:= b″[t] + b[t] (ω^2 - θ′[t]^2) == 0 // FullSimplify
Out[37]= ω^2 b[t] + b″[t] == b[t] θ′[t]^2
```

and imaginary part is

```
In[38]:= 2 b′[t] θ′[t] + b[t] θ″[t] == 0 // FullSimplify
Out[38]= 2 b′[t] θ′[t] + b[t] θ″[t] == 0
```

The LHS of the equation is nothing but $\frac{1}{b}\frac{d}{dt}\left(\theta'\,b^2\right)$. This gives $\theta' = \frac{c}{b^2}$, where c is a constant. Substituting this value of θ' into the real part, we get $\omega^2\, b[t] + b''[t] - \frac{c^2}{b[t]^3} = 0$. This is the Ermakov equation. Let us now demand that the wavefunction approaches the local ground state at $t = 0$ as $t \to 0$. This implies,

```
In[39]:= {b[0], b'[0], θ'[0], c} = {1, 0, ω[0], ω[0]}
Out[39]= {1, 0, ω[0], ω[0]}
```

So the Ermakov equation becomes $b'' + b\,\omega^2 = \frac{\omega(0)^2}{b^3}$.

Expressions for bplus and bminus

The solution of μ when the system is initially at ground state, when $t \to -\infty$, can be called μin and when the system approaches ground state, as $t \to +$, can be called μout. So the solutions take the values:

In[40]:=
```
{μleft, μright} = {Exp[I ωminus t], a1 Exp[I ωplus t] + b1 Exp[-I ωplus t]}
```

Out[40]= $\{e^{i\,t\,\omega minus},\ b1\,e^{-i\,t\,\omega plus} + a1\,e^{i\,t\,\omega plus}\}$

In[41]:=
```
{μleftdot, μrightdot} = {D[μleft, t], D[μright, t]}
```

Out[41]= $\{i\,e^{i\,t\,\omega minus}\,\omega minus,\ -i\,b1\,e^{-i\,t\,\omega plus}\,\omega plus + i\,a1\,e^{i\,t\,\omega plus}\,\omega plus\}$

Using the continuity of μ and μdot, we can find the values of a1 and b1.

In[42]:=
```
Solve[{μleft == μright, μleftdot == μrightdot}, {a1, b1}] /. t → 0
```

Out[42]= $\left\{\left\{a1 \to \frac{\omega minus + \omega plus}{2\,\omega plus},\ b1 \to \frac{-\omega minus + \omega plus}{2\,\omega plus}\right\}\right\}$

Furthermore, we define $\boldsymbol{\mu}$**(t) = b(t)** $\boldsymbol{e^{i\theta}}$. Equating the condition $|\ \mu\ |^2 = |\ b\ |^2$, we solve for b.

In[43]:=
$$\{\mu real,\ \mu imaginary\} = \text{ComplexExpand}\left[\text{ReIm}\left[\frac{e^{-i\,t\,\omega plus}\,(-\omega minus + \omega plus)}{2\,\omega plus} + \frac{e^{i\,t\,\omega plus}\,(\omega minus + \omega plus)}{2\,\omega plus}\right]\right]\ //\ \text{FullSimplify}$$

Out[43]= $\left\{\text{Cos}[t\,\omega plus],\ \frac{\omega minus\ \text{Sin}[t\,\omega plus]}{\omega plus}\right\}$

Note that we have μplus and μminus for the two normal modes. For μplus, **ωminus = ω**, and **ωplus = ωp**. For μminus, **ωminus = ω** and and **ωplus =** $\sqrt{\omega p^2 + 2\,k}$. If we define $\eta = \sqrt{1 + \frac{2\,k}{\omega p^2}}$, we can write

In[44]:= $\eta = \sqrt{1 + \frac{2\,k}{\omega p^2}}$

Out[44]= $\sqrt{1 + \frac{2\,k}{\omega p^2}}$

From μplus and μminus, we will have the corresponding values for b, They can be called bplus and bminus for the two normal modes. **bplus =** $\sqrt{|\ \mu plus\ |^2}$ and **bminus =** $\sqrt{|\ \mu minus\ |^2}$:

In[45]:=
$$\{bplus,\ bminus\} = \left\{\sqrt{\text{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}},\ \sqrt{\text{Cos}[\eta\,t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[\eta\,t\,\omega p]^2}{\eta^2\ \omega p^2}}\right\}\ //\ \text{FullSimplify}$$

Out[45]= $$\left\{\sqrt{\text{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}},\ \sqrt{\text{Cos}\left[t\,\sqrt{1 + \frac{2\,k}{\omega p^2}}\ \omega p\right]^2 + \frac{\omega^2\,\text{Sin}\left[t\,\sqrt{1 + \frac{2\,k}{\omega p^2}}\ \omega p\right]^2}{2\,k + \omega p^2}}\right\}$$

Taking their time derivatives, we get bplusdot and bminusdot.

In[46]:=
```
{bplusdot, bminusdot} = {D[bplus, t], D[bminus, t]} // FullSimplify
```

Out[46]= $$\left\{\frac{(\omega-\omega p)\,(\omega+\omega p)\,\mathrm{Sin}[2\,t\,\omega p]}{2\,\omega p\,\sqrt{\mathrm{Cos}[t\,\omega p]^2+\frac{\omega^2\,\mathrm{Sin}[t\,\omega p]^2}{\omega p^2}}},\ \frac{\left(-2\,k+\omega^2-\omega p^2\right)\,\mathrm{Sin}\left[2\,t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]}{2\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\,\sqrt{\mathrm{Cos}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2+\frac{\omega^2\,\mathrm{Sin}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2}{2\,k+\omega p^2}}}\right\}$$

Now, the argument of the exponential of the wavefunction can be written as

In[47]:=
$$\mathrm{SInitial} = -\frac{1}{2}\left(\mathrm{A11}\,\mathrm{x1}^2+\mathrm{A22}\,\mathrm{x2}^2+2\,\mathrm{A12}\,\mathrm{x1}\,\mathrm{x2}\right)\ /.\ \left\{\mathrm{x1}\to\frac{\mathrm{xPlus}+\mathrm{xMinus}}{\sqrt{2}},\ \mathrm{x2}\to\frac{\mathrm{xPlus}-\mathrm{xMinus}}{\sqrt{2}}\right\}\ //\ \mathrm{FullSimplify}$$

Out[47]= $$\frac{1}{4}\left(-\mathrm{A11}\,(\mathrm{xMinus}+\mathrm{xPlus})^2+(\mathrm{xMinus}-\mathrm{xPlus})\,(\mathrm{A22}\,(-\mathrm{xMinus}+\mathrm{xPlus})+2\,\mathrm{A12}\,(\mathrm{xMinus}+\mathrm{xPlus}))\right)$$

The argument of the exponential in the diagonalized wavefunction ***ψ*Diag** can be defined as,

In[48]:=
$$\mathrm{SDiag} = -\frac{\mathrm{APlus}}{2}\,\mathrm{xPlus}^2-\frac{\mathrm{AMinus}}{2}\,\mathrm{xMinus}^2\ //\ \mathrm{FullSimplify}$$

Out[48]= $$\frac{1}{2}\left(-\mathrm{AMinus}\,\mathrm{xMinus}^2-\mathrm{APlus}\,\mathrm{xPlus}^2\right)$$

In[49]:=
$$\mathrm{Collect}\left[\mathrm{SInitial},\ \left\{\mathrm{xPlus}^2,\ \mathrm{xMinus}^2,\ \mathrm{x1Plus}\ \mathrm{xMinus}\right\},\ \mathrm{FullSimplify}\right]$$

Out[49]= $$\frac{1}{4}\,(-\mathrm{A11}+2\,\mathrm{A12}-\mathrm{A22})\,\mathrm{xMinus}^2+\frac{1}{2}\,(-\mathrm{A11}+\mathrm{A22})\,\mathrm{xMinus}\,\mathrm{xPlus}+\frac{1}{4}\,(-\mathrm{A11}-2\,\mathrm{A12}-\mathrm{A22})\,\mathrm{xPlus}^2$$

Comparing the coefficients, we get,

In[50]:=
$$\mathrm{Solve}\left[\left\{\frac{-\mathrm{AMinus}}{2}==\frac{1}{4}\,(-\mathrm{A11}+2\,\mathrm{A12}-\mathrm{A22}),\ -\frac{\mathrm{APlus}}{2}==\frac{1}{4}\,(-\mathrm{A11}-2\,\mathrm{A12}-\mathrm{A22})\right\},\ \{\mathrm{APlus},\ \mathrm{AMinus}\}\right]\ /.\ \mathrm{A22}\to\mathrm{A11}\ //\ \mathrm{FullSimplify}$$

Out[50]= $\{\{\mathrm{APlus}\to\mathrm{A11}+\mathrm{A12},\ \mathrm{AMinus}\to\mathrm{A11}-\mathrm{A12}\}\}$

But **Aplus =** $\frac{-i\,m}{\mathrm{hbar}}\left(\frac{\mu\mathrm{plus}'}{\mu\mathrm{plus}}\right)$ and **Aminus =** $\frac{-i\,m}{\mathrm{hbar}}\left(\frac{\mu\mathrm{minus}'}{\mu\mathrm{minus}}\right)$.

In[51]:=
```
Clear[μplus, μminus, bplus, bminus, θ, t]
```

In[52]:=
```
{μplus, μminus} = {bplus[t] Exp[I θ[t]], bminus[t] Exp[I θ[t]]}
```

Out[52]= $\left\{e^{i\,\theta[t]}\,\mathrm{bplus}[t],\ e^{i\,\theta[t]}\,\mathrm{bminus}[t]\right\}$

In[53]:=
```
{μplusdot, μminusdot} = {D[μplus, t], D[μminus, t]}
```

Out[53]= $\left\{e^{i\,\theta[t]}\,\mathrm{bplus}'[t]+i\,e^{i\,\theta[t]}\,\mathrm{bplus}[t]\,\theta'[t],\ e^{i\,\theta[t]}\,\mathrm{bminus}'[t]+i\,e^{i\,\theta[t]}\,\mathrm{bminus}[t]\,\theta'[t]\right\}$

In[54]:=
```
{Aplus, Aminus} = {-I μplusdot/μplus, -I μminusdot/μminus} // FullSimplify
```

Out[54]= $\left\{-\frac{i\, \text{bplus}'[t]}{\text{bplus}[t]}+\theta'[t],\ -\frac{i\, \text{bminus}'[t]}{\text{bminus}[t]}+\theta'[t]\right\}$

Now we have Aplus and Aminus. But $\theta' = \omega$. So,

In[55]:=
```
Solve[{-I bplus'[t]/bplus[t] + ω == A11 + A12, -I bminus'[t]/bminus[t] + ω == A11 - A12}, {A11, A12}] //
  FullSimplify
```

Out[55]= $\left\{\left\{\text{A11}\to\omega-\frac{i\, \text{bminus}'[t]}{2\, \text{bminus}[t]}-\frac{i\, \text{bplus}'[t]}{2\, \text{bplus}[t]},\ \text{A12}\to\frac{1}{2}\, i\left(\frac{\text{bminus}'[t]}{\text{bminus}[t]}-\frac{\text{bplus}'[t]}{\text{bplus}[t]}\right)\right\}\right\}$

Or equivalently we can write,

In[56]:=
$$\{\mathbf{A11},\ \mathbf{A12},\ \mathbf{A22}\} = \left\{\left(\frac{\omega}{2\,\mathrm{bplus}^2} - \mathrm{I}\,\frac{\mathrm{bplusdot}}{2\,\mathrm{bplus}} + \frac{\omega}{2\,\mathrm{bminus}^2} - \mathrm{I}\,\frac{\mathrm{bminusdot}}{2\,\mathrm{bminus}}\right),\right.$$
$$\left(\frac{\omega}{2\,\mathrm{bplus}^2} - \mathrm{I}\,\frac{\mathrm{bplusdot}}{2\,\mathrm{bplus}} - \left(\frac{\omega}{2\,\mathrm{bminus}^2} - \mathrm{I}\,\frac{\mathrm{bminusdot}}{2\,\mathrm{bminus}}\right)\right),$$
$$\left.\left(\frac{\omega}{2\,\mathrm{bplus}^2} - \mathrm{I}\,\frac{\mathrm{bplusdot}}{2\,\mathrm{bplus}} + \frac{\omega}{2\,\mathrm{bminus}^2} - \mathrm{I}\,\frac{\mathrm{bminusdot}}{2\,\mathrm{bminus}}\right)\right\}\ //\ \mathrm{FullSimplify}$$

Out[56]=
$$\left\{\frac{\omega}{2\,\mathrm{bminus}^2} + \frac{\omega}{2\,\mathrm{bplus}^2} + \frac{i\,(-\omega+\omega p)\,(\omega+\omega p)\,\mathrm{Sin}[2\,t\,\omega p]}{4\,\mathrm{bplus}\,\omega p\,\sqrt{\mathrm{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\mathrm{Sin}[t\,\omega p]^2}{\omega p^2}}} + \frac{i\,\left(2\,k-\omega^2+\omega p^2\right)\,\mathrm{Sin}\left[2\,t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]}{4\,\mathrm{bminus}\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\,\sqrt{\mathrm{Cos}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2 + \frac{\omega^2\,\mathrm{Sin}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2}{2\,k+\omega p^2}}},\right.$$
$$-\frac{\omega}{2\,\mathrm{bminus}^2} + \frac{\omega}{2\,\mathrm{bplus}^2} + \frac{i\,(-\omega+\omega p)\,(\omega+\omega p)\,\mathrm{Sin}[2\,t\,\omega p]}{4\,\mathrm{bplus}\,\omega p\,\sqrt{\mathrm{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\mathrm{Sin}[t\,\omega p]^2}{\omega p^2}}} + \frac{i\,\left(-2\,k+\omega^2-\omega p^2\right)\,\mathrm{Sin}\left[2\,t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]}{4\,\mathrm{bminus}\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\,\sqrt{\mathrm{Cos}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2 + \frac{\omega^2\,\mathrm{Sin}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2}{2\,k+\omega p^2}}},$$
$$\left.\frac{\omega}{2\,\mathrm{bminus}^2} + \frac{\omega}{2\,\mathrm{bplus}^2} + \frac{i\,(-\omega+\omega p)\,(\omega+\omega p)\,\mathrm{Sin}[2\,t\,\omega p]}{4\,\mathrm{bplus}\,\omega p\,\sqrt{\mathrm{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\mathrm{Sin}[t\,\omega p]^2}{\omega p^2}}} + \frac{i\,\left(2\,k-\omega^2+\omega p^2\right)\,\mathrm{Sin}\left[2\,t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]}{4\,\mathrm{bminus}\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\,\sqrt{\mathrm{Cos}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2 + \frac{\omega^2\,\mathrm{Sin}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2}{2\,k+\omega p^2}}}\right\}$$

And their complex conjugates as:

In[57]:=

$$\{\text{A11st}, \text{A22st}, \text{A12st}\} = \left\{\frac{\omega}{2\,\text{bplus}^2} + \text{I}\,\frac{\text{bplusdot}}{2\,\text{bplus}} + \frac{\omega}{2\,\text{bminus}^2} + \text{I}\,\frac{\text{bminusdot}}{2\,\text{bminus}},\ \frac{\omega}{2\,\text{bplus}^2} + \text{I}\,\frac{\text{bplusdot}}{2\,\text{bplus}} + \frac{\omega}{2\,\text{bminus}^2} + \text{I}\,\frac{\text{bminusdot}}{2\,\text{bminus}},\ \frac{\omega}{2\,\text{bplus}^2} + \text{I}\,\frac{\text{bplusdot}}{2\,\text{bplus}} - \left(\frac{\omega}{2\,\text{bminus}^2} + \text{I}\,\frac{\text{bminusdot}}{2\,\text{bminus}}\right)\right\} \ //\ \text{FullSimplify}$$

Out[57]=

$$\Bigg\{\frac{\omega}{2\,\text{bminus}^2} + \frac{\omega}{2\,\text{bplus}^2} + \frac{i\,(\omega-\omega p)\,(\omega+\omega p)\,\text{Sin}[2\,t\,\omega p]}{4\,\text{bplus}\,\omega p\,\sqrt{\text{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}}} + \frac{i\,\left(-2\,k+\omega^2-\omega p^2\right)\,\text{Sin}\left[2\,t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]}{4\,\text{bminus}\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\,\sqrt{\text{Cos}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2 + \frac{\omega^2\,\text{Sin}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2}{2\,k+\omega p^2}}},$$

$$\frac{\omega}{2\,\text{bminus}^2} + \frac{\omega}{2\,\text{bplus}^2} + \frac{i\,(\omega-\omega p)\,(\omega+\omega p)\,\text{Sin}[2\,t\,\omega p]}{4\,\text{bplus}\,\omega p\,\sqrt{\text{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}}} + \frac{i\,\left(-2\,k+\omega^2-\omega p^2\right)\,\text{Sin}\left[2\,t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]}{4\,\text{bminus}\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\,\sqrt{\text{Cos}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2 + \frac{\omega^2\,\text{Sin}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2}{2\,k+\omega p^2}}},$$

$$-\frac{\omega}{2\,\text{bminus}^2} + \frac{\omega}{2\,\text{bplus}^2} + \frac{i\,(\omega-\omega p)\,(\omega+\omega p)\,\text{Sin}[2\,t\,\omega p]}{4\,\text{bplus}\,\omega p\,\sqrt{\text{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}}} + \frac{i\,\left(2\,k-\omega^2+\omega p^2\right)\,\text{Sin}\left[2\,t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]}{4\,\text{bminus}\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\,\sqrt{\text{Cos}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2 + \frac{\omega^2\,\text{Sin}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2}{2\,k+\omega p^2}}}\Bigg\}$$

Figure 3: Evolution of the density matrix for randomly chosen values

Here we plot an arbitrary (uncontrolled) evolution of the oscillator, for arbitrary values of k, ω, and ωp.

Recall:

In[58]:=
$$\{bplus, bminus, bplusdot, bminusdot\} = \left\{\sqrt{\text{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}},\ \sqrt{\text{Cos}[\eta\,t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[\eta\,t\,\omega p]^2}{\eta^2\ \omega p^2}},\ \frac{(\omega - \omega p)\ (\omega + \omega p)\ \text{Sin}[2\,t\,\omega p]}{2\,\omega p\,\sqrt{\text{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}}},\ \frac{\left(-2\,k + \omega^2 - \omega p^2\right)\,\text{Sin}\left[2\,t\,\sqrt{1 + \frac{2\,k}{\omega p^2}}\ \omega p\right]}{2\,\sqrt{1 + \frac{2\,k}{\omega p^2}}\ \omega p\,\sqrt{\text{Cos}\left[t\,\sqrt{1 + \frac{2\,k}{\omega p^2}}\ \omega p\right]^2 + \frac{\omega^2\,\text{Sin}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\ \omega p\right]^2}{2\,k+\omega p^2}}}\right\} \text{// FullSimplify}$$

Out[58]=
$$\left\{\sqrt{\text{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}},\ \sqrt{\text{Cos}\left[t\,\sqrt{1 + \frac{2\,k}{\omega p^2}}\ \omega p\right]^2 + \frac{\omega^2\,\text{Sin}\left[t\,\sqrt{1 + \frac{2\,k}{\omega p^2}}\ \omega p\right]^2}{2\,k + \omega p^2}},\ \frac{(\omega - \omega p)\ (\omega + \omega p)\ \text{Sin}[2\,t\,\omega p]}{2\,\omega p\,\sqrt{\text{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}}},\ \frac{\left(-2\,k + \omega^2 - \omega p^2\right)\,\text{Sin}\left[2\,t\,\sqrt{1 + \frac{2\,k}{\omega p^2}}\ \omega p\right]}{2\,\sqrt{1 + \frac{2\,k}{\omega p^2}}\ \omega p\,\sqrt{\text{Cos}\left[t\,\sqrt{1 + \frac{2\,k}{\omega p^2}}\ \omega p\right]^2 + \frac{\omega^2\,\text{Sin}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\ \omega p\right]^2}{2\,k+\omega p^2}}}\right\}$$

Now we can obtain the real and imaginary parts of A_{ij} for the particular values k=1, ω=1, and ωp=2.

In[59]:=
$$\{ReA12, ImA12\} = \left\{\left(\frac{\omega}{2\,bplus^2} - \frac{\omega}{2\,bminus^2}\right),\ \left(-\frac{bplusdot}{2\,bplus} + \frac{bminusdot}{2\,bminus}\right)\right\} \text{/. } \{k \to 1,\ \omega \to 1,\ \omega p \to 2\} \text{ // FullSimplify}$$

Out[59]=
$$\left\{\frac{4}{5 + 3\,\text{Cos}[4\,t]} - \frac{6}{7 + 5\,\text{Cos}\left[2\,\sqrt{6}\,t\right]},\ \frac{3\,\text{Sin}[4\,t]}{5 + 3\,\text{Cos}[4\,t]} - \frac{5\,\sqrt{\frac{3}{2}}\,\text{Sin}\left[2\,\sqrt{6}\,t\right]}{7 + 5\,\text{Cos}\left[2\,\sqrt{6}\,t\right]}\right\}$$

In[60]:=
$$\{ReA11, ImA11, ReA22, ImA22\} = \left\{\frac{\omega}{2\,bplus^2} + \frac{\omega}{2\,bminus^2},\ -\frac{bplusdot}{2\,bplus} - \frac{bminusdot}{2\,bminus},\ \frac{\omega}{2\,bplus^2} + \frac{\omega}{2\,bminus^2},\ -\frac{bplusdot}{2\,bplus} - \frac{bminusdot}{2\,bminus}\right\} \text{/. } \{k \to 1,\ \omega \to 1,\ \omega p \to 2\} \text{ // FullSimplify}$$

Out[60]=
$$\left\{\frac{4}{5 + 3\,\text{Cos}[4\,t]} + \frac{6}{7 + 5\,\text{Cos}\left[2\,\sqrt{6}\,t\right]},\ \frac{3\,\text{Sin}[4\,t]}{5 + 3\,\text{Cos}[4\,t]} + \frac{5\,\sqrt{\frac{3}{2}}\,\text{Sin}\left[2\,\sqrt{6}\,t\right]}{7 + 5\,\text{Cos}\left[2\,\sqrt{6}\,t\right]},\ \frac{4}{5 + 3\,\text{Cos}[4\,t]} + \frac{6}{7 + 5\,\text{Cos}\left[2\,\sqrt{6}\,t\right]},\ \frac{3\,\text{Sin}[4\,t]}{5 + 3\,\text{Cos}[4\,t]} + \frac{5\,\sqrt{\frac{3}{2}}\,\text{Sin}\left[2\,\sqrt{6}\,t\right]}{7 + 5\,\text{Cos}\left[2\,\sqrt{6}\,t\right]}\right\}$$

The real and imaginary terms of $A_{12}{}^2$ is given by:

In[61]:=
$$\text{ReA12sqr} = \frac{-\text{bminus}^2\ \text{bplus}^2\ (\text{bminusdot bplus} - \text{bminus bplusdot})^2 + \left(\text{bminus}^2 - \text{bplus}^2\right)^2 \omega^2}{4\ \text{bminus}^4\ \text{bplus}^4}\ /.$$

```
{k → 1, ω → 1, ωp → 2} // FullSimplify
```

Out[61]=
$$\left(2 \times \left(2 \left(1 + 9 \operatorname{Cos}[4 t] - 10 \operatorname{Cos}\left[2 \sqrt{6} t\right]\right)^2 - \frac{3}{4} \left(\sqrt{6} \left(7 + 5 \operatorname{Cos}\left[2 \sqrt{6} t\right]\right) \operatorname{Sin}[4 t] - 5 \times (5 + 3 \operatorname{Cos}[4 t]) \operatorname{Sin}\left[2 \sqrt{6} t\right]\right)^2\right)\right) \Big/ \left((5 + 3 \operatorname{Cos}[4 t])^2 \left(7 + 5 \operatorname{Cos}\left[2 \sqrt{6} t\right]\right)^2\right)$$

In[62]:=
$$\text{ImA12sqr} = -\frac{(\text{bminus} - \text{bplus})\ (\text{bminus} + \text{bplus})\ (-\text{bminusdot bplus} + \text{bminus bplusdot})\ \omega}{2\ \text{bminus}^3\ \text{bplus}^3}\ /.$$

```
{k → 1, ω → 1, ωp → 2} // FullSimplify
```

Out[62]=
$$-\left(\left(\left(1 + 9 \operatorname{Cos}[4 t] - 10 \operatorname{Cos}\left[2 \sqrt{6} t\right]\right) \times \left(84 \operatorname{Sin}[4 t] - 50 \sqrt{6} \operatorname{Sin}\left[2 \sqrt{6} t\right] - 15 \times \left(-2 + \sqrt{6}\right) \operatorname{Sin}\left[2 \times \left(2 + \sqrt{6}\right) t\right] + 15 \times \left(2 + \sqrt{6}\right) \operatorname{Sin}\left[4 t - 2 \sqrt{6} t\right]\right)\right) \Big/ \left((5 + 3 \operatorname{Cos}[4 t])^2 \left(7 + 5 \operatorname{Cos}\left[2 \sqrt{6} t\right]\right)^2\right)\right)$$

Now, as we have obtained the expressions of real and imaginary parts of A_{ij}, we can find the corresponding X, Y, and Z.

In[63]:=
$$\{X, Y, Z\} = \left\{\text{ReA11} - \frac{\text{ReA12sqr}}{2\ \text{ReA11}}, \text{ImA11} - \frac{\text{ImA12sqr}}{2\ \text{ReA11}}, \frac{-(\text{ReA12}^{\wedge}2 + \text{ImA12}^{\wedge}2)}{2\ \text{ReA11}}\right\}\ //\ \text{FullSimplify}$$

Out[63]=
$$\left\{\left(926 + 42 \operatorname{Cos}[4 t] - 50 \operatorname{Cos}\left[2 \sqrt{6} t\right] - 15 \times \left(5 + 2 \sqrt{6}\right) \operatorname{Cos}\left[2 \times \left(-2 + \sqrt{6}\right) t\right] + 15 \times \left(-5 + 2 \sqrt{6}\right) \operatorname{Cos}\left[2 \times \left(2 + \sqrt{6}\right) t\right]\right) \Big/ \left(16 \times \left(29 + 9 \operatorname{Cos}[4 t] + 10 \operatorname{Cos}\left[2 \sqrt{6} t\right]\right)\right), \frac{2 \times \left(9 \operatorname{Sin}[4 t] + 5 \sqrt{6} \operatorname{Sin}\left[2 \sqrt{6} t\right]\right)}{29 + 9 \operatorname{Cos}[4 t] + 10 \operatorname{Cos}\left[2 \sqrt{6} t\right]}, \left(-158 - 42 \operatorname{Cos}[4 t] + 50 \operatorname{Cos}\left[2 \sqrt{6} t\right] + 15 \times \left(5 + 2 \sqrt{6}\right) \operatorname{Cos}\left[2 \times \left(-2 + \sqrt{6}\right) t\right] + 15 \times \left(5 - 2 \sqrt{6}\right) \operatorname{Cos}\left[2 \times \left(2 + \sqrt{6}\right) t\right]\right) \Big/ \left(16 \times \left(29 + 9 \operatorname{Cos}[4 t] + 10 \operatorname{Cos}\left[2 \sqrt{6} t\right]\right)\right)\right\}$$

Here, we consider the evolution for a time interval T = 20.

In[64]:=
```
T = 20
```

Out[64]= 20

Now the initial values of X, Y, and Z, when when t=0 be Xi, Yi, and Zi.

In[65]:=
```
{Xi, Yi, Zi} = {X, Y, Z} /. t → 0 // FullSimplify
```

Out[65]= {1, 0, 0}

And the final values of X, Y, and Z, when when t=T be Xf, Yf, and Zf.

In[66]:=
```
{Xf, Yf, Zf} = {X, Y, Z} /. t → T // FullSimplify
```

Out[66]=

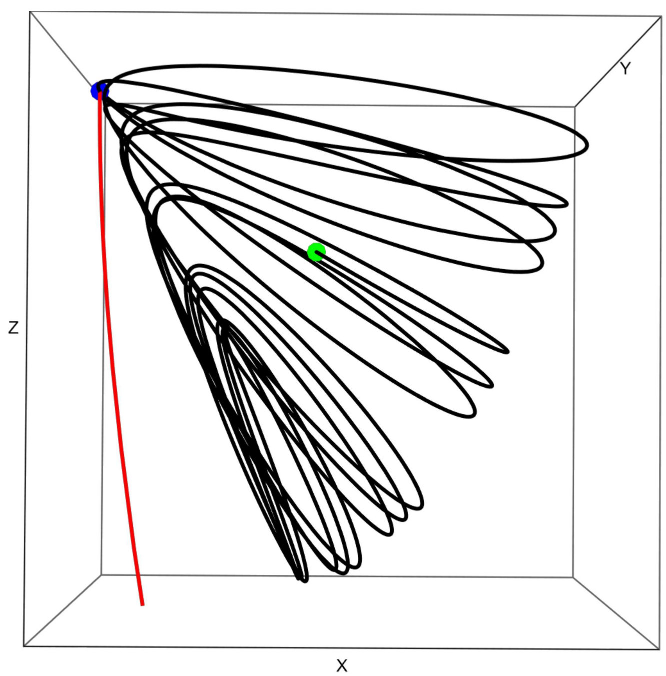
$$\left\{\frac{463+21\,\mathrm{Cos}[80]-25\times(1+3\,\mathrm{Cos}[80])\,\mathrm{Cos}\left[40\sqrt{6}\right]-30\sqrt{6}\,\mathrm{Sin}[80]\,\mathrm{Sin}\left[40\sqrt{6}\right]}{8\times\left(29+9\,\mathrm{Cos}[80]+10\,\mathrm{Cos}\left[40\sqrt{6}\right]\right)},\right.$$

$$\frac{2\times\left(9\,\mathrm{Sin}[80]+5\sqrt{6}\,\mathrm{Sin}\left[40\sqrt{6}\right]\right)}{29+9\,\mathrm{Cos}[80]+10\,\mathrm{Cos}\left[40\sqrt{6}\right]},$$

$$\left.\frac{-79-21\,\mathrm{Cos}[80]+25\times(1+3\,\mathrm{Cos}[80])\,\mathrm{Cos}\left[40\sqrt{6}\right]+30\sqrt{6}\,\mathrm{Sin}[80]\,\mathrm{Sin}\left[40\sqrt{6}\right]}{8\times\left(29+9\,\mathrm{Cos}[80]+10\,\mathrm{Cos}\left[40\sqrt{6}\right]\right)}\right\}$$

In[67]:=
```
Show[{ParametricPlot3D[{X, Y, Z}, {t, 0, T}, PlotStyle → Black,
   BoxRatios → {1, 1, 1}, Ticks → None, AxesLabel → {"X", "Y", "Z"}]},
  {ParametricPlot3D[{Coth[θ], 0, -Csch[θ]}, {θ, 0, 10}, PlotStyle → Red],
   Graphics3D[{Blue, PointSize[0.03], Point[{Xi, Yi, Zi}], Green,
     PointSize[0.03], Point[{Xf, Yf, Zf}]}]}, LabelStyle → Black]
```

Out[67]=



This represents the evolution of $\rho^{(1)}$(x1 x'1) represented in the R-space, assuming a randomly chosen set of the tunable parameters ω', k and τ. The oscillator-1 is initially in the ground state (blue dot) and evolves, at t = τ, to the green point. The red curve is the family of thermal states.

C. SECTION IV: THERMALIZATION FROM QUENCHING

Figure 4: The Energy-Frequency diagram

Now let us look at the energy frequency diagram of the oscillator when it undergoes our protocol. We

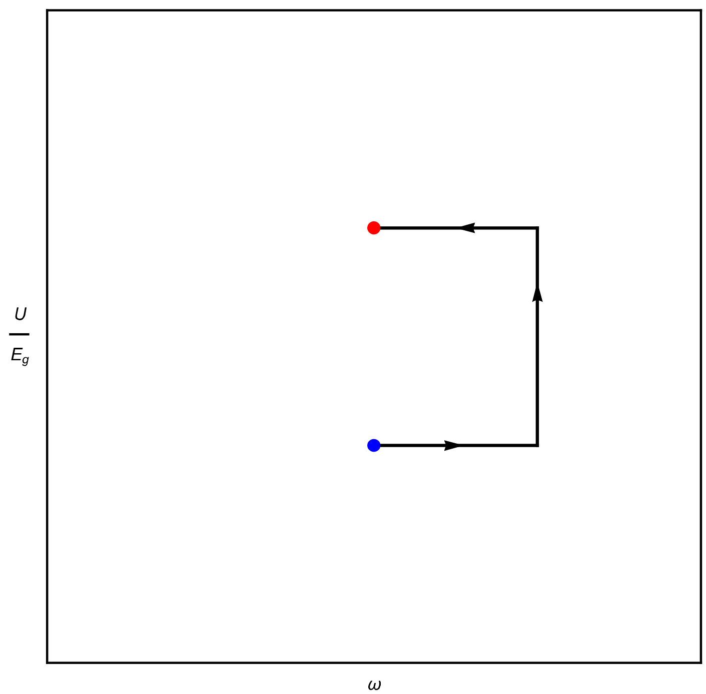
ωtilde = $\frac{\omega'}{\omega}$

$\frac{U_\beta}{E_g}$

will represent the dimensionless frequency parameter ωtilde = $\frac{\omega'}{\omega}$ on x-axis, and the ratio of average energy and ground state, $\frac{U_\beta}{E_g}$ on y-axis.

In[68]:=
```
Show[{ContourPlot[y == 1, {x, 0, 1},
   {y, -3, 3}, ContourStyle → Black, PlotRange → {{-2, 2}, {-3, 3}},
   FrameLabel → {"ω", Rotate["U/E_g", -π/2]}, LabelStyle → {Black}, FrameTicks → False],
  ContourPlot[x == 1, {x, -1, 2}, {y, 1, -1}, ContourStyle → Black], ContourPlot[
   y == -1, {x, 0, 1}, {y, -3, 3}, ContourStyle → Black, PlotRange → {{-2, 2}, {-3, 3}}],
  Graphics[{Black, Arrowheads[0.03], Arrow[{{0.6, 1}, {0.5, 1}}],
    Arrow[{{1, 0}, {1, 0.5}}], Arrow[{{0.5, -1}, {0.55, -1}}]}],
  Graphics[{Blue, PointSize[0.02], Point[{0, -1}]}],
  Graphics[{Red, PointSize[0.02], Point[{0, 1}]}]}]
```

Out[68]=



This is the energy-frequency diagram for oscillator-1 as it undergoes the protocol.

Defining Dimensionless Parameters

In[69]:=
```
ClearAll[ω, ωp, k, τ]
```

For convenience, we introduce the dimensionless parameters ωptilde, ktilde, and τtilde, such that:

In[70]:=
```
{ωp, k, τ} = {ω ωtilde, ktilde ω^2, 2 π τtilde / ω}
```

Out[70]= $\left\{\omega\ \omega\text{tilde},\ \text{ktilde}\ \omega^2,\ \frac{2\pi\ \tau\text{tilde}}{\omega}\right\}$

A. Special discrete set of thermal states

The three equations that describe the dynamics of the oscillator are analytically unsolvable for arbitrary temperatures. However, exact analytic solutions exist for a special discrete set (SDS) of temperatures. This SDS temperatures require the condition that the two normal mode frequencies are commensurate (p/q, where p and q are integers). Furthermore, if p and q are odd integers, we get:

In[71]:= ClearAll[η, ω, ωp, k, bplus, bminus, bminusdot, bplusdot]

In[72]:= $\tau\text{tilde} == \frac{2\,l+1}{4\,\omega\text{tilde}} == \frac{2\,n+1}{4\sqrt{\omega\text{tilde}^2 + 2\,\text{ktilde}}}$

Out[72]= $\tau\text{tilde} == \frac{1+2\,l}{4\,\omega\text{tilde}} == \frac{1+2\,n}{4\sqrt{2\,\text{ktilde} + \omega\text{tilde}^2}}$

In[73]:= $\eta\text{dimensionless} == \frac{\sqrt{\omega\text{tilde}^2 + 2\,\text{ktilde}}}{\omega\text{tilde}} == \frac{2\,n+1}{2\,l+1}$

Out[73]= $\eta\text{dimensionless} == \frac{\sqrt{2\,\text{ktilde} + \omega\text{tilde}^2}}{\omega\text{tilde}} == \frac{1+2\,n}{1+2\,l}$

Now, we redefine the real and imaginary parts of A_{ij} .

In[74]:= {ReA12sqr, ReA11, ImA11, ReA22, ImA22} =

$$\left\{\frac{-\text{bminus}^2\,\text{bplus}^2\,(\text{bminusdot bplus} - \text{bminus bplusdot})^2 + \left(\text{bminus}^2 - \text{bplus}^2\right)^2 \omega^2}{4\,\text{bminus}^4\,\text{bplus}^4}, \frac{\omega}{2\,\text{bplus}^2} + \frac{\omega}{2\,\text{bminus}^2}, -\frac{\text{bplusdot}}{2\,\text{bplus}} - \frac{\text{bminusdot}}{2\,\text{bminus}}, \frac{\omega}{2\,\text{bplus}^2} + \frac{\omega}{2\,\text{bminus}^2}, -\frac{\text{bplusdot}}{2\,\text{bplus}} - \frac{\text{bminusdot}}{2\,\text{bminus}}\right\}$$

Out[74]= $$\left\{\frac{-\text{bminus}^2\,\text{bplus}^2\,(\text{bminusdot bplus} - \text{bminus bplusdot})^2 + \left(\text{bminus}^2 - \text{bplus}^2\right)^2 \omega^2}{4\,\text{bminus}^4\,\text{bplus}^4}, \frac{\omega}{2\,\text{bminus}^2} + \frac{\omega}{2\,\text{bplus}^2}, -\frac{\text{bminusdot}}{2\,\text{bminus}} - \frac{\text{bplusdot}}{2\,\text{bplus}}, \frac{\omega}{2\,\text{bminus}^2} + \frac{\omega}{2\,\text{bplus}^2}, -\frac{\text{bminusdot}}{2\,\text{bminus}} - \frac{\text{bplusdot}}{2\,\text{bplus}}\right\}$$

In[75]:= $\frac{\text{ReA12sqr}}{2\,\text{ReA22}} - \text{ReA11}$ // FullSimplify

Out[75]= $$-\frac{\text{bminus}^2\,\text{bplus}^2\,(\text{bminusdot bplus} - \text{bminus bplusdot})^2 + \left(\text{bminus}^4 + 6\,\text{bminus}^2\,\text{bplus}^2 + \text{bplus}^4\right)\omega^2}{4\,\text{bminus}^2\,\text{bplus}^2\left(\text{bminus}^2 + \text{bplus}^2\right)\omega}$$

If we write the density matrix as rho1 ∝ Exp$\left[-\frac{\gamma}{2}\left(\text{x1}^2 + \text{x1}'^2\right) + \delta\,\text{x1 x1}'\right]$, the coefficient of x^2 in the Gaussian density matrix is $\frac{1}{2}\left[\frac{\text{Re}\left[\text{A12}^2\right]}{2\,\text{Re}\,[\text{A22}]} - \text{Re}\,[\text{A11}]\right]$. Then $-\gamma/2$ would be equal to

$$\frac{1}{2}\left(-\left(\left(\text{bminus}^2\,\text{bplus}^2\,(\text{bminusdot bplus} - \text{bminus bplusdot})^2 + \left(\text{bminus}^4 + 6\,\text{bminus}^2\,\text{bplus}^2 + \text{bplus}^4\right)\omega^2\right) \Big/ \left(4\,\text{bminus}^2\,\text{bplus}^2\left(\text{bminus}^2 + \text{bplus}^2\right)\omega\right)\right)\right)$$

In[76]:= $\{\text{bminus}, \text{bplus}\} = \left\{\frac{\omega}{\omega p}, \frac{\omega}{\omega p\, \eta}\right\}$ // FullSimplify

Out[76]= $\left\{\frac{\omega}{\omega p}, \frac{\omega}{\eta\, \omega p}\right\}$

For the SDS, bplus and bminus are constants in time.

In[77]:= {bminusdot, bplusdot} = {0, 0}

Out[77]= {0, 0}

In[78]:= $\gamma = \left(\text{bminus}^2\, \text{bplus}^2\, (\text{bminusdot bplus} - \text{bminus bplusdot})^2 + \left(\text{bminus}^4 + 6\, \text{bminus}^2\, \text{bplus}^2 + \text{bplus}^4\right) \omega^2\right) / \left(4\, \text{bminus}^2\, \text{bplus}^2 \left(\text{bminus}^2 + \text{bplus}^2\right) \omega\right)$ // FullSimplify

Out[78]= $\frac{\left(1 + 6\, \eta^2 + \eta^4\right) \omega p^2}{4 \times \left(1 + \eta^2\right) \omega}$

Similarly, the coefficient of x x’ δ is equal to Re[A12^2]/(2 Re[A11])

In[79]:= {ReA12sqr, ReA11} // FullSimplify

Out[79]= $\left\{\frac{\left(-1 + \eta^2\right)^2 \omega p^4}{4\, \omega^2}, \frac{\left(1 + \eta^2\right) \omega p^2}{2\, \omega}\right\}$

In[80]:= $\delta = \frac{\text{ReA12sqr}}{2\, \text{ReA11}}$ // FullSimplify

Out[80]= $\frac{\left(-1 + \eta^2\right)^2 \omega p^2}{4 \times \left(1 + \eta^2\right) \omega}$

Comparing to the general thermal state of a harmonic oscillator, we get $\frac{\gamma}{\delta}$ = Cosh[$\beta\omega$] and δ = Cosech[$\beta\omega$]. Consequently, we get:

In[81]:= $\left\{\frac{\left(\eta^4 + 6\, \eta^2 + 1\right)}{\left(\eta^2 - 1\right)^2}, \frac{\left(4\, \left(\eta^2 + 1\right)\right)}{\left(\omega\text{tilde}^2 \left(\eta^2 - 1\right)^2\right)}\right\}$ == {Cosh[ω β], Sinh[ω β]}

Out[81]= $\left\{\frac{1 + 6\, \eta^2 + \eta^4}{\left(-1 + \eta^2\right)^2}, \frac{4 \times \left(1 + \eta^2\right)}{\left(-1 + \eta^2\right)^2 \omega\text{tilde}^2}\right\}$ == {Cosh[β ω], Sinh[β ω]}

Now we can obtain the expression for η

In[82]:= $\text{Solve}\left[\frac{1+6\eta^2+\eta^4}{(-1+\eta^2)^2} == \text{Cosh}[\beta\,\omega],\ \eta\right]$

Out[82]= $\left\{\left\{\eta \to -\text{Coth}\left[\frac{\beta\,\omega}{4}\right]\right\}, \left\{\eta \to \text{Coth}\left[\frac{\beta\,\omega}{4}\right]\right\}, \left\{\eta \to -\text{Tanh}\left[\frac{\beta\,\omega}{4}\right]\right\}, \left\{\eta \to \text{Tanh}\left[\frac{\beta\,\omega}{4}\right]\right\}\right\}$

So $\eta = \frac{1}{\omega\text{tilde}^2} = [\text{Tanh}[\beta\,\omega]]^{\pm 1} = \frac{2n+1}{2l+1}$. Here, $\beta\,\omega = \frac{E_g}{k_B\,T}$. Therefore : $\boldsymbol{\frac{E_g}{k_B\,T} = \{2\,\text{ArcTanh}\left[\frac{2n+1}{2l+1}\right],\ 2\,\text{ArcCoth}\left[\frac{2n+1}{2l+1}\right]\} = \log\left(\frac{n+l+1}{|n-l|}\right)}$. We can call this special discrete set of inverse temperatures βSDS. We also get $\boldsymbol{\omega\text{ptilde} = \sqrt{\frac{2l+1}{2n+1}}}$. Since $\eta = \sqrt{\omega' + 2k}\Big/\omega'$, we get:

In[83]:= $\eta = \left\{-\text{Coth}\left[\frac{\beta\,\omega}{4}\right], \text{Coth}\left[\frac{\beta\,\omega}{4}\right], -\text{Tanh}\left[\frac{\beta\,\omega}{4}\right], \text{Tanh}\left[\frac{\beta\,\omega}{4}\right]\right\}$

Out[83]= $\left\{-\text{Coth}\left[\frac{\beta\,\omega}{4}\right], \text{Coth}\left[\frac{\beta\,\omega}{4}\right], -\text{Tanh}\left[\frac{\beta\,\omega}{4}\right], \text{Tanh}\left[\frac{\beta\,\omega}{4}\right]\right\}$

In[84]:= $\text{k} = \left(\frac{\omega^2}{2}\right)\left(\eta - \frac{1}{\eta}\right)$ // Simplify

Out[84]= $\left\{-\omega^2\,\text{Csch}\left[\frac{\beta\,\omega}{2}\right], \omega^2\,\text{Csch}\left[\frac{\beta\,\omega}{2}\right], \omega^2\,\text{Csch}\left[\frac{\beta\,\omega}{2}\right], -\omega^2\,\text{Csch}\left[\frac{\beta\,\omega}{2}\right]\right\}$

In[85]:= $\beta\text{SDS} = \frac{1}{\text{Eg}}\,\text{Log}\left[\frac{\text{n}+\text{l}+1}{\text{Abs}[\text{n}-\text{l}]}\right]$

Out[85]= $\frac{\text{Log}\left[\frac{1+\text{l}+\text{n}}{\text{Abs}[-\text{l}+\text{n}]}\right]}{\text{Eg}}$

In[86]:= $\omega\text{tilde} = \text{Sqrt}\left[\frac{(2\,\text{l}+1)}{(2\,\text{n}+1)}\right]$

Out[86]= $\sqrt{\frac{1+2\,\text{l}}{1+2\,\text{n}}}$

In[87]:= $\text{Solve}\left[\frac{(\omega\text{tilde}^2 + 2\,\text{ktilde})}{\omega\text{tilde}^2} == \frac{1}{\omega\text{tilde}^4},\ \text{ktilde}\right]$ // FullSimplify

Out[87]= $\left\{\left\{\text{ktilde} \to \frac{2\,(-\text{l}\,(1+\text{l}) + \text{n} + \text{n}^2)}{(1+2\,\text{l}) \times (1+2\,\text{n})}\right\}\right\}$

So $\text{ktilde} = \frac{2\,(\text{n}-\text{l})\,(\text{n}+\text{l}+1)}{(2\,\text{l}+1) \times (2\,\text{n}+1)}$. Also, from the relation $\tau\text{tilde} = \frac{2\,\text{l}+1}{4\,\omega\text{ptilde}} = \frac{2\,\text{n}+1}{4\,\sqrt{\omega\text{ptilde}^2 + 2\,\text{ktilde}}}$, we get $\boldsymbol{\tau\text{tilde} = \frac{1}{4}\sqrt{(2\,l+1) \times (2\,n+1)}}$.

In[88]:= ClearAll["Global`*"]

So now we have:

In[89]:=
```
Framed[Column[{Eg/(kB Tnl) == Piecewise[{{2 ArcTanh[(2 n + 1) / (2 l + 1)], ";   l<n"},
    {2 ArcCoth[(2 n + 1) / (2 l + 1)], ";   l>n"}}] ==
   Log[(n + l + 1) / Abs[n - l]], ωtilde == Sqrt[(2 l + 1)/(2 n + 1)], ktilde == (2 (n - l) (n + l + 1))/((2 l + 1) × (2 n + 1)),
  τtilde == (1/4) Sqrt[(2 l + 1) × (2 n + 1)]}], FrameStyle → Black]
```

Out[89]=

$$\boxed{\begin{array}{l} \frac{Eg}{k_B\, T_{nl}} == \begin{cases} 2\,\text{ArcTanh}\left[\frac{1+2n}{1+2l}\right] & ;\quad l<n \\ 2\,\text{ArcCoth}\left[\frac{1+2n}{1+2l}\right] & ;\quad l>n \\ 0 & \text{True} \end{cases} == \text{Log}\left[\frac{1+l+n}{\text{Abs}[-l+n]}\right] \\ \omega\text{tilde} == \sqrt{\frac{1+2l}{1+2n}} \\ \text{ktilde} == \frac{2\,(-l+n)\,(1+l+n)}{(1+2l)\times(1+2n)} \\ \tau\text{tilde} == \frac{1}{4}\sqrt{(1+2l)\times(1+2n)} \end{array}}$$

Let Z(β) be the partition function, and it is equal to $\frac{1}{2\,\text{Sinh}[\beta\,Eg]}$. Then the average energy U_β is given by,

In[90]:= `Z = 1/(2 Sinh[β Eg])`

Out[90]= $\frac{1}{2}\,\text{Csch}[Eg\,\beta]$

In[91]:= `Uβ = - D[Log[Z], β]`

Out[91]= $Eg\,\text{Coth}[Eg\,\beta]$

Now substitute βSDS for β,

In[92]:= `Uβ = Eg Coth[Eg β] /. β → βSDS // FullSimplify`

Out[92]= $Eg\,\text{Coth}[Eg\,\beta\text{SDS}]$

The above equation is equal to Eg $\left(\frac{(2l+1)^2+(2n+1)^2}{2\times(2l+1)\times(2n+1)}\right)$. This is the expression for average energy in terms of n and l.

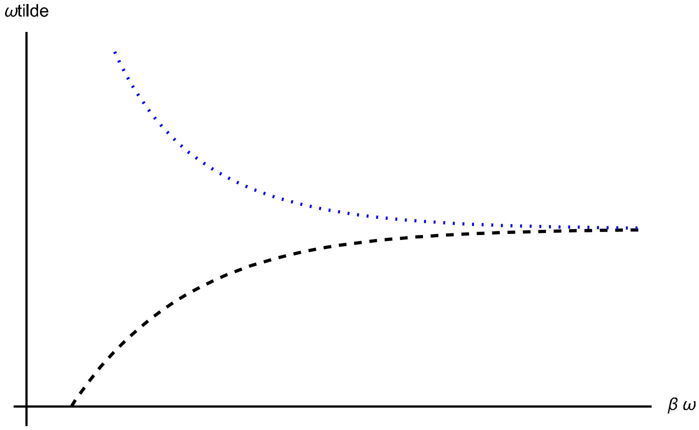
Figure 5: (a) omega-tilde and (b) k-tilde

For the purpose of plotting, we take $\frac{\beta\omega}{2}$ = θ5. We have obtained the expressions

$\omega = \left\{\sqrt{\text{Tanh}\left[\frac{\beta\omega}{2}\right]}, \sqrt{\text{Coth}\left[\frac{\beta\omega}{2}\right]}\right\}$ and $k = \{\omega^2\,\text{Csch}\left[\frac{\beta\omega}{2}\right], -\omega^2\,\text{Csch}\left[\frac{\beta\omega}{2}\right]\}$.

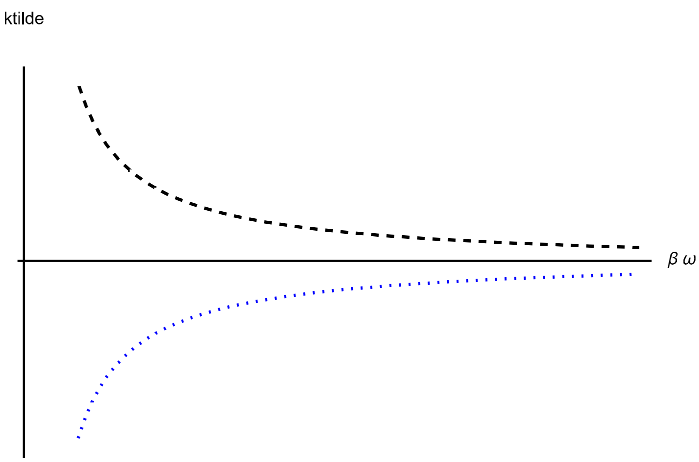
In[93]:=
```
Plot[{√Tanh[θ5] , √Coth[θ5]}, {θ5, 0.1, 3}, AxesLabel → {"β ω", "ωtilde"},
 PlotStyle → {{Black, Dashed}, {Blue, DotDashed}}, LabelStyle → Black, Ticks → None]
```

Out[93]=



In[94]:=
```
Plot[{Csch[θ5], -Csch[θ5]}, {θ5, 0.01, 1}, AxesLabel → {"β ω", "ktilde
"}, PlotStyle → {{Black, Dashed}, {Blue, DotDashed}}, LabelStyle → Black, Ticks → None]
```

Out[94]=



The ωptilde and ktilde as a function of the SDS temperatures . The dashed and dot-dashed curves correspond to l<n and l>n cases,respectively.

Figure 6: The Quickest Case in SDS

In[95]:=
```
ClearAll[ω, ωp, k, τ, bminus, bplus, bplusdot, bminusdot, A11, A12, A22,
 ReA11, ReA12, ReA22, ReA12sqr, ImA12, ImA12sqr, ωtilde, ktilde, τtilde, η]
```

The quickest case of thermalization occurs when the values of ωtilde, ktilde,τtilde of our special discrete set have the values:

In[96]:=
```
{ωtilde, ktilde, τtilde} = {1/ √3 , 4/3, √3/4}
```

Out[96]= $\left\{\frac{1}{\sqrt{3}}, \frac{4}{3}, \frac{\sqrt{3}}{4}\right\}$

For convenience, let ω=1.

In[97]:= $\omega = 1$

Out[97]= 1

Then

In[98]:= $\{\omega p, k, \tau\} = \left\{\omega\,\omega\text{tilde}, \text{ktilde}\,\omega^2, 2\pi\frac{\tau\text{tilde}}{\omega}\right\}$

Out[98]= $\left\{\frac{1}{\sqrt{3}}, \frac{4}{3}, \frac{\sqrt{3}\,\pi}{2}\right\}$

In[99]:= $\eta = \sqrt{1+\frac{2\,k}{\omega p^2}}$

Out[99]= 3

Now we can obtain bplus, bminus, bplusdot, and bminusdot in terms of these values.

In[100]:= $\{\text{bplus}, \text{bminus}, \text{bplusdot}, \text{bminusdot}\} = \left\{\sqrt{\text{Cos}[t\,\omega p]^2+\frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}}, \sqrt{\text{Cos}[\eta\,t\,\omega p]^2+\frac{\omega^2\,\text{Sin}[\eta\,t\,\omega p]^2}{\eta^2\,\omega p^2}}, \frac{(\omega-\omega p)\,(\omega+\omega p)\,\text{Sin}[2\,t\,\omega p]}{2\,\omega p\,\sqrt{\text{Cos}[t\,\omega p]^2+\frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}}}, \frac{\left(-2\,k+\omega^2-\omega p^2\right)\text{Sin}\left[2\,t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]}{2\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\,\sqrt{\text{Cos}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2+\frac{\omega^2\,\text{Sin}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2}{2\,k+\omega p^2}}}\right\}$ // FullSimplify

Out[100]= $\left\{\sqrt{2-\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right]}, \frac{\sqrt{2+\text{Cos}\left[2\,\sqrt{3}\,t\right]}}{\sqrt{3}}, \frac{\text{Sin}\left[\frac{2\,t}{\sqrt{3}}\right]}{\sqrt{6-3\,\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right]}}, -\frac{\text{Sin}\left[2\,\sqrt{3}\,t\right]}{\sqrt{2+\text{Cos}\left[2\,\sqrt{3}\,t\right]}}\right\}$

Now we can separate the real and imaginary parts of A_{ij}.

In[101]:= $\{\text{ReA12}, \text{ImA12}\} = \left\{\left(\frac{\omega}{2\,\text{bplus}^2}-\frac{\omega}{2\,\text{bminus}^2}\right), \left(-\frac{\text{bplusdot}}{2\,\text{bplus}}+\frac{\text{bminusdot}}{2\,\text{bminus}}\right)\right\}$ // FullSimplify

Out[101]= $\left\{\frac{1}{2}\left(\frac{1}{2-\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right]}-\frac{3}{2+\text{Cos}\left[2\,\sqrt{3}\,t\right]}\right), \frac{\frac{\text{Sin}\left[\frac{2\,t}{\sqrt{3}}\right]}{-2+\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right]}-\frac{3\,\text{Sin}\left[2\,\sqrt{3}\,t\right]}{2+\text{Cos}\left[2\,\sqrt{3}\,t\right]}}{2\,\sqrt{3}}\right\}$

In[102]:= $\{\text{ReA11}, \text{ImA11}, \text{ReA22}, \text{ImA22}\} = \left\{\frac{\omega}{2\,\text{bplus}^2} + \frac{\omega}{2\,\text{bminus}^2}, -\frac{\text{bplusdot}}{2\,\text{bplus}} - \frac{\text{bminusdot}}{2\,\text{bminus}}, \frac{\omega}{2\,\text{bplus}^2} + \frac{\omega}{2\,\text{bminus}^2}, -\frac{\text{bplusdot}}{2\,\text{bplus}} - \frac{\text{bminusdot}}{2\,\text{bminus}}\right\}$ // FullSimplify

Out[102]= $$\left\{\frac{1}{2}\left(\frac{1}{2-\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right]} + \frac{3}{2+\text{Cos}\left[2\,\sqrt{3}\,t\right]}\right), \frac{\frac{\text{Sin}\left[\frac{2\,t}{\sqrt{3}}\right]}{-2+\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right]} + \frac{3\,\text{Sin}\left[2\,\sqrt{3}\,t\right]}{2+\text{Cos}\left[2\,\sqrt{3}\,t\right]}}{2\,\sqrt{3}}, \frac{1}{2}\left(\frac{1}{2-\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right]} + \frac{3}{2+\text{Cos}\left[2\,\sqrt{3}\,t\right]}\right), \frac{\frac{\text{Sin}\left[\frac{2\,t}{\sqrt{3}}\right]}{-2+\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right]} + \frac{3\,\text{Sin}\left[2\,\sqrt{3}\,t\right]}{2+\text{Cos}\left[2\,\sqrt{3}\,t\right]}}{2\,\sqrt{3}}\right\}$$

The real and imaginary parts of the $A_{12}{}^2$ are obtained by

In[103]:= $\text{ReA12sqr} = \frac{1}{4\,\text{bminus}^4\,\text{bplus}^4}\left(-\text{bminus}^2\,\text{bplus}^2\,(\text{bminusdot bplus} - \text{bminus bplusdot})^2 + \left(\text{bminus}^2 - \text{bplus}^2\right)^2\omega^2\right)$ // FullSimplify

Out[103]= $$\frac{1}{12}\times\left(4 + \frac{6}{\left(-2+\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right]\right)^2} - \frac{5}{-2+\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right]} + \frac{54}{\left(2+\text{Cos}\left[2\,\sqrt{3}\,t\right]\right)^2} - \frac{3\times\left(15 + 4\,\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right] + 2\,\text{Cos}\left[\frac{4\,t}{\sqrt{3}}\right]\right)}{2+\text{Cos}\left[2\,\sqrt{3}\,t\right]}\right)$$

In[104]:= $\text{ImA12sqr} = -\left(((\text{bminus} - \text{bplus})\,(\text{bminus} + \text{bplus})\,(-\text{bminusdot bplus} + \text{bminus bplusdot})\,\omega) \Big/ \left(2\,\text{bminus}^3\,\text{bplus}^3\right)\right)$ // FullSimplify

Out[104]= $$-\frac{\left(-4 + 3\,\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right] + \text{Cos}\left[2\,\sqrt{3}\,t\right]\right)\times\left(2\,\text{Sin}\left[\frac{2\,t}{\sqrt{3}}\right] - 2\,\text{Sin}\left[\frac{4\,t}{\sqrt{3}}\right] - \text{Sin}\left[\frac{8\,t}{\sqrt{3}}\right] + 6\,\text{Sin}\left[2\,\sqrt{3}\,t\right]\right)}{2\,\sqrt{3}\,\left(-2+\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right]\right)^2\left(2+\text{Cos}\left[2\,\sqrt{3}\,t\right]\right)^2}$$

Now, as we have obtained the expressions of real and imaginary parts of A_{ij}, we can find the corresponding X, Y, and Z.

In[105]:= $\{X, Y, Z\} = \left\{\text{ReA11} - \frac{\text{ReA12sqr}}{2\,\text{ReA11}}, \text{ImA11} - \frac{\text{ImA12sqr}}{2\,\text{ReA11}}, \frac{-(\text{ReA12^2} + \text{ImA12^2})}{2\,\text{ReA11}}\right\}$ // FullSimplify

Out[105]= $$\left\{\frac{1}{3}\times\left(-4 + \text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right] + \frac{3\times\left(19 - 8\,\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right] + \text{Cos}\left[\frac{4\,t}{\sqrt{3}}\right]\right)}{8 - 3\,\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right] + \text{Cos}\left[2\,\sqrt{3}\,t\right]}\right), \frac{\sqrt{3}\,\left(-\text{Sin}\left[\frac{2\,t}{\sqrt{3}}\right] + \text{Sin}\left[2\,\sqrt{3}\,t\right]\right)}{8 - 3\,\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right] + \text{Cos}\left[2\,\sqrt{3}\,t\right]}, \frac{4}{3} - \frac{1}{3}\,\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right] - \frac{13 - 8\,\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right] + \text{Cos}\left[\frac{4\,t}{\sqrt{3}}\right]}{8 - 3\,\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right] + \text{Cos}\left[2\,\sqrt{3}\,t\right]}\right\}$$

Now the initial values of X, Y, and Z, when when t=0 be Xi, Yi, and Zi.

In[106]:=
```
{Xi, Yi, Zi} = {X, Y, Z} /. t → 0 // FullSimplify
```

Out[106]= {1, 0, 0}

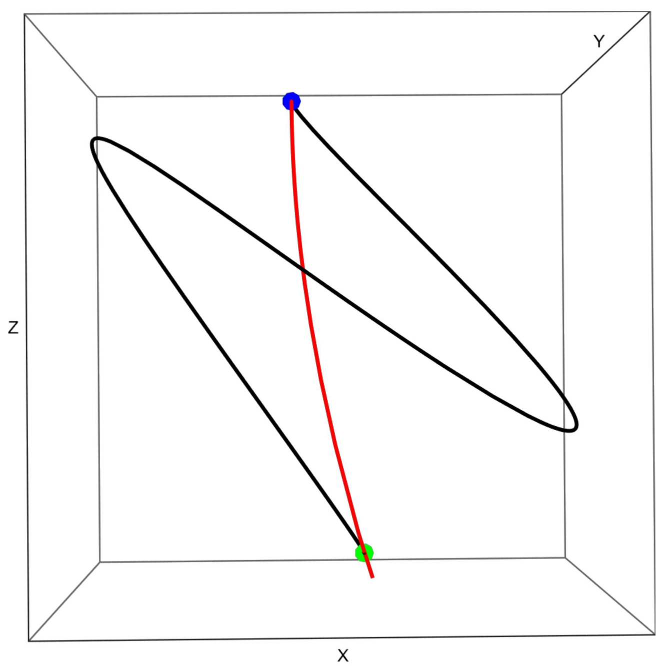
And the final values of X, Y, and Z, when when t = T be Xf, Yf, and Zf .

In[107]:=
```
{Xf, Yf, Zf} = {X, Y, Z} /. t → τ // FullSimplify
```

Out[107]= $\left\{\frac{17}{15}, 0, -\frac{8}{15}\right\}$

In[108]:=
```
Show[{ParametricPlot3D[{X, Y, Z}, {t, 0, τ}, PlotStyle → Black,
    BoxRatios → {1, 1, 1}, Ticks → None, AxesLabel → {"X", "Y", "Z"}]},
  {ParametricPlot3D[{Coth[θ], 0, -Csch[θ]}, {θ, 0, 10}, PlotStyle → Red],
   Graphics3D[{Blue, PointSize[0.03], Point[{Xi, Yi, Zi}], Green,
     PointSize[0.03], Point[{Xf, Yf, Zf}]}]}, LabelStyle → Black]
```

Out[108]=



This illustrates the finite-time thermalization of oscillator 1 to the thermal state at $\beta = \frac{k_B}{E_g} \log 2$, starting from the ground state. The blue and green dots denote the initial and final states of the system.

Figure 6:

The SDS does not exactly thermalize our system to arbitrary temperatures. But we can always find elements of SDS arbitrarily close to the target temperature. We can understand this by plotting the evolution of the density matrix for different values of {n,l}.

In[109]:=
```
ClearAll[ω, ωp, k, τ, bminus, bplus, bplusdot, bminusdot, A11, A12, A22,
 ReA11, ReA12, ReA22, ReA12sqr, ImA12, ImA12sqr, ωtilde, ktilde, τtilde, η]
```

In[110]:=
```
ω = 1
```
Out[110]= 1

(a) {n, l} = {12, 11}

Here we will visualize the evolution toward the thermal state for (n,l) = (12,11) to the target inverse temperature $\beta = \frac{k_B}{E_g}\pi$.

In[111]:=
```
{n, l} = {12, 11}
```
Out[111]= {12, 11}

The corresponding values of β and the dimensionless parameters become:

In[112]:= $\{\beta, \omega\text{tilde}, \text{ktilde}, \tau\text{tilde}\} = \left\{\text{Log}\left[\frac{n+l+1}{n-l}\right], \sqrt{\frac{2\,l+1}{2\,n+1}}, \frac{2\,(n-l)\,(n+l+1)}{(2\,l+1)\times(2\,n+1)}, \frac{1}{4}\sqrt{(2\,l+1)\times(2\,n+1)}\right\}$

Out[112]= $\left\{\text{Log}[24], \frac{\sqrt{23}}{5}, \frac{48}{575}, \frac{5\sqrt{23}}{4}\right\}$

In[113]:= $\{\omega p, k, \tau\} = \left\{\omega\,\omega\text{tilde}, \text{ktilde}\,\omega^2, 2\pi\frac{\tau\text{tilde}}{\omega}\right\}$

Out[113]= $\left\{\frac{\sqrt{23}}{5}, \frac{48}{575}, \frac{5\sqrt{23}\,\pi}{2}\right\}$

In[114]:= $\eta = \sqrt{1 + \frac{2\,k}{\omega p^2}}$

Out[114]= $\frac{25}{23}$

The corresponding values of bplus, bminus, bplusdot, and bminusdot :

In[115]:=
$$\{bplus, bminus, bplusdot, bminusdot\} = \left\{\sqrt{\text{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}},\ \sqrt{\text{Cos}[\eta\,t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[\eta\,t\,\omega p]^2}{\eta^2\ \omega p^2}},\ \frac{(\omega - \omega p)\ (\omega + \omega p)\ \text{Sin}[2\,t\,\omega p]}{2\,\omega p\,\sqrt{\text{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}}},\ \frac{\left(-2\,k + \omega^2 - \omega p^2\right)\,\text{Sin}\left[2\,t\,\sqrt{1 + \frac{2\,k}{\omega p^2}}\ \omega p\right]}{2\,\sqrt{1 + \frac{2\,k}{\omega p^2}}\ \omega p\,\sqrt{\text{Cos}\left[t\,\sqrt{1 + \frac{2\,k}{\omega p^2}}\ \omega p\right]^2 + \frac{\omega^2\,\text{Sin}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\ \omega p\right]^2}{2\,k+\omega p^2}}}\right\} \text{ // FullSimplify}$$

Out[115]=
$$\left\{\frac{\sqrt{24 - \text{Cos}\left[\frac{2\,\sqrt{23}\,t}{5}\right]}}{\sqrt{23}},\ \frac{1}{5}\,\sqrt{24 + \text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]},\ \frac{\text{Sin}\left[\frac{2\,\sqrt{23}\,t}{5}\right]}{5\,\sqrt{24 - \text{Cos}\left[\frac{2\,\sqrt{23}\,t}{5}\right]}},\ -\frac{\text{Sin}\left[\frac{10\,t}{\sqrt{23}}\right]}{\sqrt{23}\,\sqrt{24 + \text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]}}\right\}$$

The real and imaginary parts of A_{ij} :

In[116]:=
$$\{ReA12, ImA12\} = \left\{\left(\frac{\omega}{2\,bplus^2} - \frac{\omega}{2\,bminus^2}\right),\ \left(-\frac{bplusdot}{2\,bplus} + \frac{bminusdot}{2\,bminus}\right)\right\} \text{ // FullSimplify}$$

Out[116]=
$$\left\{-\frac{25}{2\times\left(24 + \text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right)} - \frac{23}{2\times\left(-24 + \text{Cos}\left[\frac{2\,\sqrt{23}\,t}{5}\right]\right)},\ \frac{-\frac{25\,\text{Sin}\left[\frac{10\,t}{\sqrt{23}}\right]}{24+\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]} + \frac{23\,\text{Sin}\left[\frac{2\,\sqrt{23}\,t}{5}\right]}{-24+\text{Cos}\left[\frac{2\,\sqrt{23}\,t}{5}\right]}}{10\,\sqrt{23}}\right\}$$

In[117]:=
$$\{ReA11, ImA11, ReA22, ImA22\} = \left\{\frac{\omega}{2\,bplus^2} + \frac{\omega}{2\,bminus^2},\ -\frac{bplusdot}{2\,bplus} - \frac{bminusdot}{2\,bminus},\ \frac{\omega}{2\,bplus^2} + \frac{\omega}{2\,bminus^2},\ -\frac{bplusdot}{2\,bplus} - \frac{bminusdot}{2\,bminus}\right\} \text{ // FullSimplify}$$

Out[117]=
$$\left\{\frac{25}{2\times\left(24 + \text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right)} - \frac{23}{2\times\left(-24 + \text{Cos}\left[\frac{2\,\sqrt{23}\,t}{5}\right]\right)},\ \frac{\frac{25\,\text{Sin}\left[\frac{10\,t}{\sqrt{23}}\right]}{24+\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]} + \frac{23\,\text{Sin}\left[\frac{2\,\sqrt{23}\,t}{5}\right]}{-24+\text{Cos}\left[\frac{2\,\sqrt{23}\,t}{5}\right]}}{10\,\sqrt{23}},\ \frac{25}{2\times\left(24 + \text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right)} - \frac{23}{2\times\left(-24 + \text{Cos}\left[\frac{2\,\sqrt{23}\,t}{5}\right]\right)},\ \frac{\frac{25\,\text{Sin}\left[\frac{10\,t}{\sqrt{23}}\right]}{24+\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]} + \frac{23\,\text{Sin}\left[\frac{2\,\sqrt{23}\,t}{5}\right]}{-24+\text{Cos}\left[\frac{2\,\sqrt{23}\,t}{5}\right]}}{10\,\sqrt{23}}\right\}$$

The real and imaginary parts of (A_{12})^2 are given by

In[118]:=
```
ReA12sqr = 1/(4 bminus^4 bplus^4)
  (-bminus^2 bplus^2 (bminusdot bplus - bminus bplusdot)^2 + (bminus^2 - bplus^2)^2 ω^2) // Simplify
```

Out[118]=
$$-\left(\left(-575\left(-48+23\,\mathrm{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]+25\,\mathrm{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)^2+\left(25\times\left(-24+\mathrm{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)\mathrm{Sin}\left[\frac{10\,t}{\sqrt{23}}\right]-23\times\left(24+\mathrm{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right)\mathrm{Sin}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)^2\right)\Big/\left(2300\left(24+\mathrm{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right)^2\left(-24+\mathrm{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)^2\right)\right)$$

In[119]:=
```
ImA12sqr = -(((bminus - bplus) (bminus + bplus) (-bminusdot bplus + bminus bplusdot) ω) /
  (2 bminus^3 bplus^3)) // FullSimplify
```

Out[119]=
$$\left(\left(-48+23\,\mathrm{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]+25\,\mathrm{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)\times\left(24\,\mathrm{Sin}\left[\frac{4\,t}{5\sqrt{23}}\right]-600\,\mathrm{Sin}\left[\frac{10\,t}{\sqrt{23}}\right]+\mathrm{Sin}\left[\frac{96\,t}{5\sqrt{23}}\right]-552\,\mathrm{Sin}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)\right)\Big/\left(10\sqrt{23}\left(24+\mathrm{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right)^2\left(-24+\mathrm{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)^2\right)$$

Now, as we have obtained the expressions of real and imaginary parts of A_{ij}, we can find the corresponding X, Y, and Z.

In[120]:=
$$\{X, Y, Z\} = \left\{\text{ReA11} - \frac{\text{ReA12sqr}}{2\,\text{ReA11}},\ \text{ImA11} - \frac{\text{ImA12sqr}}{2\,\text{ReA11}},\ \frac{-(\text{ReA12}^\wedge 2 + \text{ImA12}^\wedge 2)}{2\,\text{ReA11}}\right\} \text{ // Simplify}$$

Out[120]=
$$\left\{\frac{1324227 + 576\,\text{Cos}\left[\frac{4\,t}{5\sqrt{23}}\right] - 1152\,\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right] + \text{Cos}\left[\frac{96\,t}{5\sqrt{23}}\right] - 1152\,\text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]}{1150 \times \left(1152 + 23\,\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right] - 25\,\text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)},\right.$$

$$-\left(\left(5\sqrt{23}\left(\text{Sin}\left[\frac{42\,t}{5\sqrt{23}}\right] + 2304\,\text{Sin}\left[\frac{10\,t}{\sqrt{23}}\right] - \text{Sin}\left[\frac{54\,t}{5\sqrt{23}}\right] - 96\,\text{Sin}\left[\frac{96\,t}{5\sqrt{23}}\right] + 48\,\text{Sin}\left[\frac{20\,t}{\sqrt{23}}\right] + \text{Sin}\left[\frac{142\,t}{5\sqrt{23}}\right] - \text{Sin}\left[\frac{146\,t}{5\sqrt{23}}\right] - 2304\,\text{Sin}\left[\frac{2\sqrt{23}\,t}{5}\right] + 48\,\text{Sin}\left[\frac{4\sqrt{23}\,t}{5}\right]\right)\right)\Big/\right.$$

$$\left.\left(4 \times \left(24 + \text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right) \times \left(1152 + 23\,\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right] - 25\,\text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right) \times \left(-24 + \text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)\right)\right),$$

$$\left.-\frac{\left(\frac{25}{2\times\left(24+\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right)} + \frac{23}{2\times\left(-24+\text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)}\right)^2 + \frac{\left(\frac{25\,\text{Sin}\left[\frac{10\,t}{\sqrt{23}}\right]}{24+\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]} - \frac{23\,\text{Sin}\left[\frac{2\sqrt{23}\,t}{5}\right]}{-24+\text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]}\right)^2}{2300}}{2\left(\frac{25}{2\times\left(24+\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right)} - \frac{23}{2\times\left(-24+\text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)}\right)}\right\}$$

Now the initial values of X, Y, and Z, when when t = 0 be Xi, Yi, and Zi .

In[121]:=
```
{Xi, Yi, Zi} = {X, Y, Z} /. t → 0 // FullSimplify
```

Out[121]= {1, 0, 0}

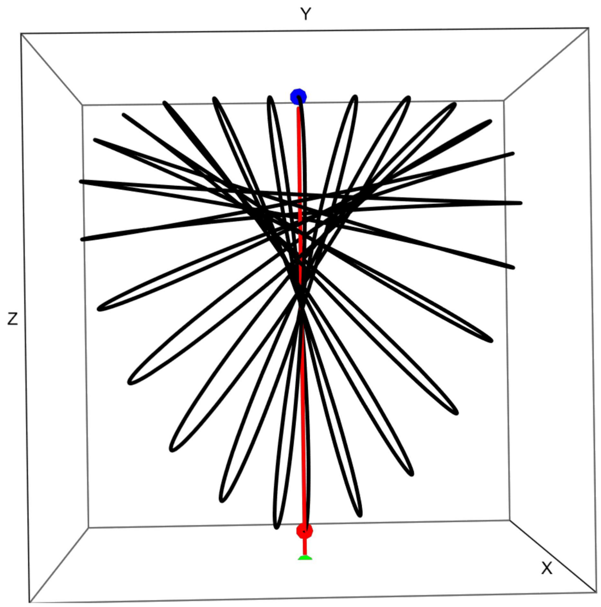
And the final values of X, Y, and Z, when when t=T be Xf, Yf, and Zf.

In[122]:=
```
{Xf, Yf, Zf} = {X, Y, Z} /. t → τ // FullSimplify
```

Out[122]=
$$\left\{\frac{331777}{331775},\ 0,\ -\frac{1152}{331775}\right\}$$

In[123]:=
```
Show[{ParametricPlot3D[{X, Y, Z}, {t, 0, τ}, PlotStyle → Black,
    BoxRatios → {1, 1, 1}, Ticks → None, AxesLabel → {"X", "Y", "Z"}]},
  {ParametricPlot3D[{Coth[θ], 0, -Csch[θ]}, {θ, 0, 10}, PlotStyle → Red],
   Graphics3D[{Blue, PointSize[0.03], Point[{Xi, Yi, Zi}],
     Red, PointSize[0.03], Point[{Xf, Yf, Zf}]}], Graphics3D[
    {Green, PointSize[0.03], Point[{Coth[2 π], 0, -Csch[2 π]}]}]}, LabelStyle → Black]
```

Out[123]=



Evolution toward the thermal state for (n,l) = (12,11) to the target inverse temperature $\beta = \frac{k_B}{E_g}\pi$. The exact thermal state corresponding to this value is marked by a green dot, while the approximate thermal state is indicated by a red dot; both lie on the thermal curve.

(a) {n, l} = {24, 22}

Now let us visualize the evolution toward the thermal state for (n, l) = (24, 22) to the target inverse temperature $\beta = \frac{k_B}{E_g}\pi$.

In[124]:=
```
{n, l} = {24, 22}
```

Out[124]= {24, 22}

The corresponding values of β and the dimensionless parameters become:

In[125]:=
```
{β, ωtilde, ktilde, τtilde} =
```

$$\left\{\text{Log}\left[\frac{n+l+1}{n-l}\right], \sqrt{\frac{2l+1}{2n+1}}, \frac{2(n-l)(n+l+1)}{(2l+1)\times(2n+1)}, \frac{1}{4}\sqrt{(2l+1)\times(2n+1)}\right\}$$

Out[125]=

$$\left\{\text{Log}\left[\frac{47}{2}\right], \frac{3\sqrt{5}}{7}, \frac{188}{2205}, \frac{21\sqrt{5}}{4}\right\}$$

In[126]:= $\{\omega p, k, \tau\} = \left\{\omega\, \omega\text{tilde}, \text{ktilde}\, \omega^2, 2\pi \frac{\tau\text{tilde}}{\omega}\right\}$

Out[126]= $\left\{\frac{3\sqrt{5}}{7}, \frac{188}{2205}, \frac{21\sqrt{5}\,\pi}{2}\right\}$

In[127]:= $\eta = \sqrt{1 + \frac{2\,k}{\omega p^2}}$

Out[127]= $\frac{49}{45}$

The corresponding values of bplus, bminus, bplusdot, and bminusdot are :

In[128]:= $\{\text{bplus}, \text{bminus}, \text{bplusdot}, \text{bminusdot}\} = \left\{\sqrt{\text{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}}, \sqrt{\text{Cos}[\eta\,t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[\eta\,t\,\omega p]^2}{\eta^2\,\omega p^2}}, \frac{(\omega - \omega p)\,(\omega + \omega p)\,\text{Sin}[2\,t\,\omega p]}{2\,\omega p\sqrt{\text{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}}}, \frac{\left(-2\,k + \omega^2 - \omega p^2\right)\text{Sin}\left[2\,t\sqrt{1 + \frac{2\,k}{\omega p^2}}\,\omega p\right]}{2\sqrt{1 + \frac{2\,k}{\omega p^2}}\,\omega p\sqrt{\text{Cos}\left[t\sqrt{1 + \frac{2\,k}{\omega p^2}}\,\omega p\right]^2 + \frac{\omega^2\,\text{Sin}\left[t\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2}{2\,k + \omega p^2}}}\right\}$ // FullSimplify

Out[128]= $\left\{\frac{\sqrt{47 - 2\,\text{Cos}\left[\frac{6\sqrt{5}\,t}{7}\right]}}{3\sqrt{5}}, \frac{1}{7}\sqrt{47 + 2\,\text{Cos}\left[\frac{14\,t}{3\sqrt{5}}\right]}, \frac{2\,\text{Sin}\left[\frac{6\sqrt{5}\,t}{7}\right]}{7\sqrt{47 - 2\,\text{Cos}\left[\frac{6\sqrt{5}\,t}{7}\right]}}, -\frac{2\,\text{Sin}\left[\frac{14\,t}{3\sqrt{5}}\right]}{3\sqrt{5}\sqrt{47 + 2\,\text{Cos}\left[\frac{14\,t}{3\sqrt{5}}\right]}}\right\}$

The real and imaginary parts of A_{ij} are:

In[129]:= $\{\text{ReA12}, \text{ImA12}\} = \left\{\left(\frac{\omega}{2\,\text{bplus}^2} - \frac{\omega}{2\,\text{bminus}^2}\right), \left(-\frac{\text{bplusdot}}{2\,\text{bplus}} + \frac{\text{bminusdot}}{2\,\text{bminus}}\right)\right\}$ // FullSimplify

Out[129]= $\left\{-\frac{49}{94 + 4\,\text{Cos}\left[\frac{14\,t}{3\sqrt{5}}\right]} + \frac{45}{94 - 4\,\text{Cos}\left[\frac{6\sqrt{5}\,t}{7}\right]}, \frac{-\frac{49\,\text{Sin}\left[\frac{14\,t}{3\sqrt{5}}\right]}{47 + 2\,\text{Cos}\left[\frac{14\,t}{3\sqrt{5}}\right]} + \frac{45\,\text{Sin}\left[\frac{6\sqrt{5}\,t}{7}\right]}{-47 + 2\,\text{Cos}\left[\frac{6\sqrt{5}\,t}{7}\right]}}{21\sqrt{5}}\right\}$

In[130]:=
$$\{\text{ReA11},\ \text{ImA11},\ \text{ReA22},\ \text{ImA22}\} = \left\{\frac{\omega}{2\,\text{bplus}^2} + \frac{\omega}{2\,\text{bminus}^2},\ -\frac{\text{bplusdot}}{2\,\text{bplus}} - \frac{\text{bminusdot}}{2\,\text{bminus}},\ \frac{\omega}{2\,\text{bplus}^2} + \frac{\omega}{2\,\text{bminus}^2},\ -\frac{\text{bplusdot}}{2\,\text{bplus}} - \frac{\text{bminusdot}}{2\,\text{bminus}}\right\}\ \text{// FullSimplify}$$

Out[130]=
$$\left\{\frac{49}{94+4\,\text{Cos}\left[\frac{14\,t}{3\sqrt{5}}\right]} + \frac{45}{94-4\,\text{Cos}\left[\frac{6\sqrt{5}\,t}{7}\right]},\ \frac{\frac{49\,\text{Sin}\left[\frac{14\,t}{3\sqrt{5}}\right]}{47+2\,\text{Cos}\left[\frac{14\,t}{3\sqrt{5}}\right]} + \frac{45\,\text{Sin}\left[\frac{6\sqrt{5}\,t}{7}\right]}{-47+2\,\text{Cos}\left[\frac{6\sqrt{5}\,t}{7}\right]}}{21\sqrt{5}},\ \frac{49}{94+4\,\text{Cos}\left[\frac{14\,t}{3\sqrt{5}}\right]} + \frac{45}{94-4\,\text{Cos}\left[\frac{6\sqrt{5}\,t}{7}\right]},\ \frac{\frac{49\,\text{Sin}\left[\frac{14\,t}{3\sqrt{5}}\right]}{47+2\,\text{Cos}\left[\frac{14\,t}{3\sqrt{5}}\right]} + \frac{45\,\text{Sin}\left[\frac{6\sqrt{5}\,t}{7}\right]}{-47+2\,\text{Cos}\left[\frac{6\sqrt{5}\,t}{7}\right]}}{21\sqrt{5}}\right\}$$

The real and imaginary parts of (A_{12})^2 are given by

In[131]:=
$$\text{ReA12sqr} = \frac{1}{4\,\text{bminus}^4\,\text{bplus}^4}\left(-\text{bminus}^2\,\text{bplus}^2\,(\text{bminusdot bplus} - \text{bminus bplusdot})^2 + \left(\text{bminus}^2 - \text{bplus}^2\right)^2\omega^2\right)\ \text{// Simplify}$$

Out[131]=
$$\left(8820\left(-94+45\,\text{Cos}\left[\frac{14\,t}{3\sqrt{5}}\right]+49\,\text{Cos}\left[\frac{6\sqrt{5}\,t}{7}\right]\right)^2 - 4\left(49\times\left(47-2\,\text{Cos}\left[\frac{6\sqrt{5}\,t}{7}\right]\right)\text{Sin}\left[\frac{14\,t}{3\sqrt{5}}\right]+45\times\left(47+2\,\text{Cos}\left[\frac{14\,t}{3\sqrt{5}}\right]\right)\text{Sin}\left[\frac{6\sqrt{5}\,t}{7}\right]\right)^2\right)\Big/\left(8820\left(47+2\,\text{Cos}\left[\frac{14\,t}{3\sqrt{5}}\right]\right)^2\left(47-2\,\text{Cos}\left[\frac{6\sqrt{5}\,t}{7}\right]\right)^2\right)$$

In[132]:=
$$\text{ImA12sqr} = -\left(((\text{bminus}-\text{bplus})\,(\text{bminus}+\text{bplus})\,(-\text{bminusdot bplus}+\text{bminus bplusdot})\,\omega)\Big/\left(2\,\text{bminus}^3\,\text{bplus}^3\right)\right)\ \text{// Simplify}$$

Out[132]=
$$\left(\left(15\sqrt{47+2\,\text{Cos}\left[\frac{14\,t}{3\sqrt{5}}\right]}-7\sqrt{5}\sqrt{47-2\,\text{Cos}\left[\frac{6\sqrt{5}\,t}{7}\right]}\right)\times\left(15\sqrt{47+2\,\text{Cos}\left[\frac{14\,t}{3\sqrt{5}}\right]}+7\sqrt{5}\sqrt{47-2\,\text{Cos}\left[\frac{6\sqrt{5}\,t}{7}\right]}\right)\times\left(94\,\text{Sin}\left[\frac{8\,t}{21\sqrt{5}}\right]-2303\,\text{Sin}\left[\frac{14\,t}{3\sqrt{5}}\right]+4\,\text{Sin}\left[\frac{188\,t}{21\sqrt{5}}\right]-2115\,\text{Sin}\left[\frac{6\sqrt{5}\,t}{7}\right]\right)\right)\Big/\left(105\sqrt{5}\left(47+2\,\text{Cos}\left[\frac{14\,t}{3\sqrt{5}}\right]\right)^2\left(47-2\,\text{Cos}\left[\frac{6\sqrt{5}\,t}{7}\right]\right)^2\right)$$

Now, as we have obtained the expressions of real and imaginary parts of A_{ij}, we can find the corresponding X, Y, and Z.

In[133]:= $$\{X,\ Y,\ Z\} = \left\{\mathbf{ReA11} - \frac{\mathbf{ReA12sqr}}{2\,\mathbf{ReA11}},\ \mathbf{ImA11} - \frac{\mathbf{ImA12sqr}}{2\,\mathbf{ReA11}},\ \frac{-(\mathbf{ReA12}\verb|^|2 + \mathbf{ImA12}\verb|^|2)}{2\,\mathbf{ReA11}}\right\} \ \verb|//| \ \mathbf{Simplify}$$

Out[133]= $$\left\{\frac{4\,868\,648 + 2209\,\mathrm{Cos}\left[\frac{8\,t}{21\,\sqrt{5}}\right] - 4418\,\mathrm{Cos}\left[\frac{14\,t}{3\,\sqrt{5}}\right] + 4\,\mathrm{Cos}\left[\frac{188\,t}{21\,\sqrt{5}}\right] - 4418\,\mathrm{Cos}\left[\frac{6\,\sqrt{5}\,t}{7}\right]}{2205 \times \left(2209 + 45\,\mathrm{Cos}\left[\frac{14\,t}{3\,\sqrt{5}}\right] - 49\,\mathrm{Cos}\left[\frac{6\,\sqrt{5}\,t}{7}\right]\right)},\right.$$

$$\frac{1}{210\,\sqrt{5}}\left(\frac{490\,\mathrm{Sin}\left[\frac{14\,t}{3\,\sqrt{5}}\right]}{47 + 2\,\mathrm{Cos}\left[\frac{14\,t}{3\,\sqrt{5}}\right]} + \left(\left(15\,\sqrt{47 + 2\,\mathrm{Cos}\left[\frac{14\,t}{3\,\sqrt{5}}\right]} - 7\,\sqrt{5}\,\sqrt{47 - 2\,\mathrm{Cos}\left[\frac{6\,\sqrt{5}\,t}{7}\right]}\right) \times\right.\right.$$

$$\left(15\,\sqrt{47 + 2\,\mathrm{Cos}\left[\frac{14\,t}{3\,\sqrt{5}}\right]} + 7\,\sqrt{5}\,\sqrt{47 - 2\,\mathrm{Cos}\left[\frac{6\,\sqrt{5}\,t}{7}\right]}\right) \times$$

$$\left.\left(94\,\mathrm{Sin}\left[\frac{8\,t}{21\,\sqrt{5}}\right] - 2303\,\mathrm{Sin}\left[\frac{14\,t}{3\,\sqrt{5}}\right] + 4\,\mathrm{Sin}\left[\frac{188\,t}{21\,\sqrt{5}}\right] - 2115\,\mathrm{Sin}\left[\frac{6\,\sqrt{5}\,t}{7}\right]\right)\right) \Big/$$

$$\left(\left(47 + 2\,\mathrm{Cos}\left[\frac{14\,t}{3\,\sqrt{5}}\right]\right) \times \left(2209 + 45\,\mathrm{Cos}\left[\frac{14\,t}{3\,\sqrt{5}}\right] - 49\,\mathrm{Cos}\left[\frac{6\,\sqrt{5}\,t}{7}\right]\right) \times \left(-47 + 2\,\mathrm{Cos}\left[\frac{6\,\sqrt{5}\,t}{7}\right]\right)\right) +$$

$$\left.\left.\frac{450\,\mathrm{Sin}\left[\frac{6\,\sqrt{5}\,t}{7}\right]}{-47 + 2\,\mathrm{Cos}\left[\frac{6\,\sqrt{5}\,t}{7}\right]}\right),\ -\frac{\left(\frac{49}{94+4\,\mathrm{Cos}\left[\frac{14\,t}{3\,\sqrt{5}}\right]} + \frac{45}{-94+4\,\mathrm{Cos}\left[\frac{6\,\sqrt{5}\,t}{7}\right]}\right)^2 + \frac{\left(\frac{49\,\mathrm{Sin}\left[\frac{14\,t}{3\,\sqrt{5}}\right]}{47+2\,\mathrm{Cos}\left[\frac{14\,t}{3\,\sqrt{5}}\right]} - \frac{45\,\mathrm{Sin}\left[\frac{6\,\sqrt{5}\,t}{7}\right]}{-47+2\,\mathrm{Cos}\left[\frac{6\,\sqrt{5}\,t}{7}\right]}\right)^2}{2205}}{2\left(\frac{49}{94+4\,\mathrm{Cos}\left[\frac{14\,t}{3\,\sqrt{5}}\right]} + \frac{45}{94-4\,\mathrm{Cos}\left[\frac{6\,\sqrt{5}\,t}{7}\right]}\right)}\right\}$$

Now the initial values of X, Y, and Z, when when t=0 be Xi, Yi, and Zi.

In[134]:= $$\{\mathbf{Xi},\ \mathbf{Yi},\ \mathbf{Zi}\} = \{X,\ Y,\ Z\}\ \verb|/.|\ \mathbf{t} \to \mathbf{0}\ \verb|//|\ \mathbf{FullSimplify}$$

Out[134]= $$\{1,\ 0,\ 0\}$$

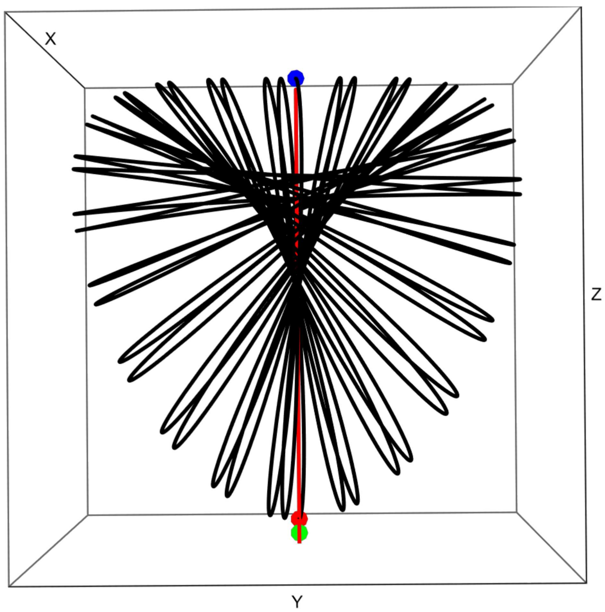
And the final values of X, Y, and Z, when t=τ be Xf, Yf, and Zf.

In[135]:= $$\{\mathbf{Xf},\ \mathbf{Yf},\ \mathbf{Zf}\} = \{X,\ Y,\ Z\}\ \verb|/.|\ \mathbf{t} \to \tau\ \verb|//|\ \mathbf{FullSimplify}$$

Out[135]= $$\left\{\frac{4\,879\,697}{4\,879\,665},\ 0,\ -\frac{17\,672}{4\,879\,665}\right\}$$

In[136]:=
```
Show[{ParametricPlot3D[{X, Y, Z}, {t, 0, τ}, PlotStyle → Black,
   BoxRatios → {1, 1, 1}, Ticks → None, AxesLabel → {"X", "Y", "Z"}]},
  {ParametricPlot3D[{Coth[θ], 0, -Csch[θ]}, {θ, 0, 10}, PlotStyle → Red],
   Graphics3D[{Blue, PointSize[0.03], Point[{Xi, Yi, Zi}],
     Red, PointSize[0.03], Point[{Xf, Yf, Zf}]}], Graphics3D[
    {Green, PointSize[0.03], Point[{Coth[2 π], 0, -Csch[2 π]}]}]}, LabelStyle → Black]
```

Out[136]=



Evolution toward the thermal state for (n,l) = (24,22) to the target inverse temperature $\beta = \frac{k_B}{E_g}\pi$. The exact thermal state corresponding to this value is marked by a green dot, while the approximate thermal state is indicated by a red dot; both lie on the thermal curve.

(c) {n, l} = {36, 33}

Now let us visualize the evolution toward the thermal state for (n, l) = (36, 33) to the target inverse temperature $\beta = \frac{k_B}{E_g}\pi$.

In[137]:=
```
{n, l} = {36, 33}
```

Out[137]= {36, 33}

The corresponding values of β and the dimensionless parameters become:

In[138]:= {β, ωtilde, ktilde, τtilde} =

$$\left\{\text{Log}\left[\frac{n+l+1}{n-l}\right], \sqrt{\frac{2l+1}{2n+1}}, \frac{2(n-l)(n+l+1)}{(2l+1)\times(2n+1)}, \frac{1}{4}\sqrt{(2l+1)\times(2n+1)}\right\}$$

Out[138]=
$$\left\{\text{Log}\left[\frac{70}{3}\right], \sqrt{\frac{67}{73}}, \frac{420}{4891}, \frac{\sqrt{4891}}{4}\right\}$$

In[139]:= $\{\omega p, k, \tau\} = \left\{\omega\,\omega\text{tilde}, \text{ktilde}\,\omega^2, 2\pi\frac{\tau\text{tilde}}{\omega}\right\}$

Out[139]= $\left\{\sqrt{\frac{67}{73}}, \frac{420}{4891}, \frac{\sqrt{4891}\,\pi}{2}\right\}$

In[140]:= $\eta = \sqrt{1+\frac{2\,k}{\omega p^2}}$

Out[140]= $\frac{73}{67}$

The corresponding values of bplus,bminus,bplusdot,bminusdot, and the real and imaginary parts of A_{ij} :

In[141]:= $\{\text{bplus}, \text{bminus}, \text{bplusdot}, \text{bminusdot}\} = \left\{\sqrt{\text{Cos}[t\,\omega p]^2+\frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}}, \sqrt{\text{Cos}[\eta\,t\,\omega p]^2+\frac{\omega^2\,\text{Sin}[\eta\,t\,\omega p]^2}{\eta^2\,\omega p^2}}, \frac{(\omega-\omega p)\,(\omega+\omega p)\,\text{Sin}[2\,t\,\omega p]}{2\,\omega p\,\sqrt{\text{Cos}[t\,\omega p]^2+\frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}}}, \frac{\left(-2\,k+\omega^2-\omega p^2\right)\,\text{Sin}\left[2\,t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]}{2\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\,\sqrt{\text{Cos}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2+\frac{\omega^2\,\text{Sin}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2}{2\,k+\omega p^2}}}\right\}$ // FullSimplify

Out[141]= $\left\{\frac{\sqrt{70-3\,\text{Cos}\left[2\,\sqrt{\frac{67}{73}}\,t\right]}}{\sqrt{67}}, \frac{\sqrt{70+3\,\text{Cos}\left[2\,\sqrt{\frac{73}{67}}\,t\right]}}{\sqrt{73}}, \frac{3\,\text{Sin}\left[2\,\sqrt{\frac{67}{73}}\,t\right]}{\sqrt{73}\,\sqrt{70-3\,\text{Cos}\left[2\,\sqrt{\frac{67}{73}}\,t\right]}}, -\frac{3\,\text{Sin}\left[2\,\sqrt{\frac{73}{67}}\,t\right]}{\sqrt{67}\,\sqrt{70+3\,\text{Cos}\left[2\,\sqrt{\frac{73}{67}}\,t\right]}}\right\}$

In[142]:= $\{\text{ReA12}, \text{ImA12}\} = \left\{\left(\frac{\omega}{2\,\text{bplus}^2}-\frac{\omega}{2\,\text{bminus}^2}\right), \left(-\frac{\text{bplusdot}}{2\,\text{bplus}}+\frac{\text{bminusdot}}{2\,\text{bminus}}\right)\right\}$ // FullSimplify

Out[142]= $\left\{\frac{67}{140-6\,\text{Cos}\left[2\,\sqrt{\frac{67}{73}}\,t\right]}-\frac{73}{140+6\,\text{Cos}\left[2\,\sqrt{\frac{73}{67}}\,t\right]}, \frac{3\left(\frac{67\,\text{Sin}\left[2\,\sqrt{\frac{67}{73}}\,t\right]}{-70+3\,\text{Cos}\left[2\,\sqrt{\frac{67}{73}}\,t\right]}-\frac{73\,\text{Sin}\left[2\,\sqrt{\frac{73}{67}}\,t\right]}{70+3\,\text{Cos}\left[2\,\sqrt{\frac{73}{67}}\,t\right]}\right)}{2\,\sqrt{4891}}\right\}$

In[143]:=

$$\{\mathbf{ReA11},\ \mathbf{ImA11},\ \mathbf{ReA22},\ \mathbf{ImA22}\} = \left\{\frac{\omega}{2\,\mathbf{bplus}^2} + \frac{\omega}{2\,\mathbf{bminus}^2},\ -\frac{\mathbf{bplusdot}}{2\,\mathbf{bplus}} - \frac{\mathbf{bminusdot}}{2\,\mathbf{bminus}},\ \frac{\omega}{2\,\mathbf{bplus}^2} + \frac{\omega}{2\,\mathbf{bminus}^2},\ -\frac{\mathbf{bplusdot}}{2\,\mathbf{bplus}} - \frac{\mathbf{bminusdot}}{2\,\mathbf{bminus}}\right\}\ \texttt{// Simplify}$$

Out[143]=

$$\left\{\frac{67}{140 - 6\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right]} + \frac{73}{140 + 6\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]},\ \frac{3\left(\frac{67\,\mathrm{Sin}\left[2\sqrt{\frac{67}{73}}\,t\right]}{-70+3\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right]} + \frac{73\,\mathrm{Sin}\left[2\sqrt{\frac{73}{67}}\,t\right]}{70+3\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]}\right)}{2\sqrt{4891}},\right.$$

$$\left.\frac{67}{140 - 6\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right]} + \frac{73}{140 + 6\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]},\ \frac{3\left(\frac{67\,\mathrm{Sin}\left[2\sqrt{\frac{67}{73}}\,t\right]}{-70+3\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right]} + \frac{73\,\mathrm{Sin}\left[2\sqrt{\frac{73}{67}}\,t\right]}{70+3\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]}\right)}{2\sqrt{4891}}\right\}$$

The real and imaginary parts of (A_{12})^2 are given by

In[144]:=

$$\mathbf{ReA12sqr} = \frac{1}{4\,\mathbf{bminus}^4\,\mathbf{bplus}^4}\left(-\mathbf{bminus}^2\,\mathbf{bplus}^2\,(\mathbf{bminusdot\ bplus} - \mathbf{bminus\ bplusdot})^2 + \left(\mathbf{bminus}^2 - \mathbf{bplus}^2\right)^2\omega^2\right)\ \texttt{// Simplify}$$

Out[144]=

$$\left(44\,019\left(-140 + 73\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right] + 67\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]\right)^2 - 9\left(67\times\left(70 + 3\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]\right)\mathrm{Sin}\left[2\sqrt{\frac{67}{73}}\,t\right] + 73\times\left(70 - 3\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right]\right)\mathrm{Sin}\left[2\sqrt{\frac{73}{67}}\,t\right]\right)^2\right) \Big/ \left(19\,564\left(70 - 3\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right]\right)^2\left(70 + 3\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]\right)^2\right)$$

In[145]:=

$$\mathbf{ImA12sqr} = -\left(((\mathbf{bminus} - \mathbf{bplus})\ (\mathbf{bminus} + \mathbf{bplus})\ (-\mathbf{bminusdot\ bplus} + \mathbf{bminus\ bplusdot})\ \omega) \Big/ \left(2\,\mathbf{bminus}^3\,\mathbf{bplus}^3\right)\right)\ \texttt{// Simplify}$$

Out[145]=

$$\left(3\times\left(73\sqrt{67}\sqrt{70 - 3\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right]} - 67\sqrt{73}\sqrt{70 + 3\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]}\right)\times\left(73\sqrt{67}\sqrt{70 - 3\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right]} + 67\sqrt{73}\sqrt{70 + 3\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]}\right)\times\left(4690\,\mathrm{Sin}\left[2\sqrt{\frac{67}{73}}\,t\right] + 5110\,\mathrm{Sin}\left[2\sqrt{\frac{73}{67}}\,t\right] - 210\,\mathrm{Sin}\left[\frac{12\,t}{\sqrt{4891}}\right] - 9\,\mathrm{Sin}\left[\frac{280\,t}{\sqrt{4891}}\right]\right)\right) \Big/ \left(9782\sqrt{4891}\left(70 - 3\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right]\right)^2\left(70 + 3\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]\right)^2\right)$$

Now, as we have obtained the expressions of real and imaginary parts of A_{ij}, we can find the corresponding X, Y, and Z.

In[146]:=
```
{X, Y, Z} = {ReA11 - ReA12sqr/(2 ReA11), ImA11 - ImA12sqr/(2 ReA11), -(ReA12^2 + ImA12^2)/(2 ReA11)} // Simplify
```

Out[146]=

$$\Bigg\{\frac{95819743 - 88200\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right] - 88200\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right] + 44100\,\mathrm{Cos}\left[\frac{12\,t}{\sqrt{4891}}\right] + 81\,\mathrm{Cos}\left[\frac{280\,t}{\sqrt{4891}}\right]}{9782\times\left(9800 - 219\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right] + 201\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]\right)},$$

$$\frac{1}{19564\sqrt{4891}}\,3\times\Bigg(9782\left(\frac{67\,\mathrm{Sin}\left[2\sqrt{\frac{67}{73}}\,t\right]}{-70+3\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right]} + \frac{73\,\mathrm{Sin}\left[2\sqrt{\frac{73}{67}}\,t\right]}{70+3\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]}\right) -$$

$$\Bigg(2\times\left(73\sqrt{67}\sqrt{70-3\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right]} - 67\sqrt{73}\sqrt{70+3\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]}\right)\times$$

$$\left(73\sqrt{67}\sqrt{70-3\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right]} + 67\sqrt{73}\sqrt{70+3\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]}\right)\times$$

$$\left(4690\,\mathrm{Sin}\left[2\sqrt{\frac{67}{73}}\,t\right] + 5110\,\mathrm{Sin}\left[2\sqrt{\frac{73}{67}}\,t\right] - 210\,\mathrm{Sin}\left[\frac{12\,t}{\sqrt{4891}}\right] - 9\,\mathrm{Sin}\left[\frac{280\,t}{\sqrt{4891}}\right]\right)\Bigg)\Big/$$

$$\Bigg(\left(-70+3\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right]\right)\times\left(-9800+219\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right] - 201\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]\right)\times$$

$$\left(70+3\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]\right)\Bigg)\Bigg),$$

$$-\frac{\left(\frac{67}{140-6\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right]} - \frac{73}{140+6\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]}\right)^2 + \frac{9\left(\frac{67\,\mathrm{Sin}\left[2\sqrt{\frac{67}{73}}\,t\right]}{-70+3\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right]} - \frac{73\,\mathrm{Sin}\left[2\sqrt{\frac{73}{67}}\,t\right]}{70+3\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]}\right)^2}{19564}}{2\left(\frac{67}{140-6\,\mathrm{Cos}\left[2\sqrt{\frac{67}{73}}\,t\right]} + \frac{73}{140+6\,\mathrm{Cos}\left[2\sqrt{\frac{73}{67}}\,t\right]}\right)}\Bigg\}$$

Now the initial values of X, Y, and Z, when when t=0 be Xi, Yi, and Zi.

In[147]:=
```
{Xi, Yi, Zi} = {X, Y, Z} /. t → 0 // FullSimplify
```

Out[147]= {1, 0, 0}

And the final values of X, Y, and Z, when t=τ be Xf, Yf, and Zf.

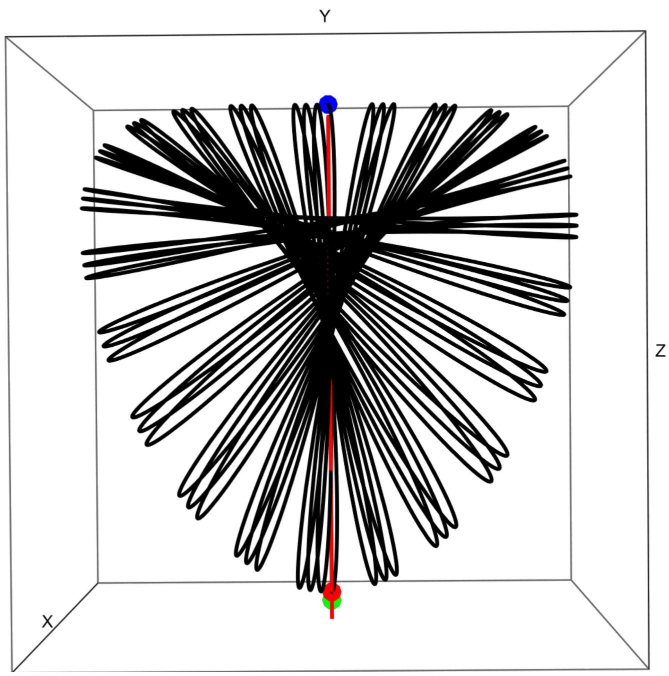
In[148]:=
```
{Xf, Yf, Zf} = {X, Y, Z} /. t → τ // FullSimplify
```

Out[148]=

$$\left\{\frac{24010081}{24009919},\ 0,\ -\frac{88200}{24009919}\right\}$$

In[149]:=
```
Show[{ParametricPlot3D[{X, Y, Z}, {t, 0, τ}, PlotStyle → Black,
   BoxRatios → {1, 1, 1}, Ticks → None, AxesLabel → {"X", "Y", "Z"}]},
 {ParametricPlot3D[{Coth[θ], 0, -Csch[θ]}, {θ, 0, 10}, PlotStyle → Red],
  Graphics3D[{Blue, PointSize[0.03], Point[{Xi, Yi, Zi}],
    Red, PointSize[0.03], Point[{Xf, Yf, Zf}]}], Graphics3D[
   {Green, PointSize[0.03], Point[{Coth[2 π], 0, -Csch[2 π]}]}]}], LabelStyle → Black]
```

Out[149]=



Evolution toward the thermal state for (n,l) = (36,33) to the target inverse temperature $\beta = \frac{k_B}{E_g}\pi$. The exact thermal state corresponding to this value is marked by a green dot, while the approximate thermal state is indicated by a red dot; both lie on the thermal curve.

(d) {n, l} = {48, 44}

Now let us visualize the evolution toward the thermal state for (n, l) = (48, 44) to the target inverse temperature $\beta = \frac{k_B}{E_g}\pi$.

In[150]:= `{n, l} = {48, 44}`

Out[150]= {48, 44}

The corresponding values of β and the dimensionless parameters become:

In[151]:= `{β, ωtilde, ktilde, τtilde} =`

$$\left\{\mathrm{Log}\left[\frac{n+l+1}{n-l}\right], \sqrt{\frac{2\,l+1}{2\,n+1}}, \frac{2\,(n-l)\,(n+l+1)}{(2\,l+1)\times(2\,n+1)}, \frac{1}{4}\sqrt{(2\,l+1)\times(2\,n+1)}\right\}$$

Out[151]=
$$\left\{\mathrm{Log}\left[\frac{93}{4}\right], \sqrt{\frac{89}{97}}, \frac{744}{8633}, \frac{\sqrt{8633}}{4}\right\}$$

In[152]:= $\{\omega p, k, \tau\} = \left\{\omega\, \omega\text{tilde}, \text{ktilde}\, \omega^2, 2\pi \frac{\tau\text{tilde}}{\omega}\right\}$

Out[152]= $\left\{\sqrt{\frac{89}{97}}, \frac{744}{8633}, \frac{\sqrt{8633}\,\pi}{2}\right\}$

In[153]:= $\eta = \sqrt{1 + \frac{2\,k}{\omega p^2}}$

Out[153]= $\frac{97}{89}$

The corresponding values of bplus, bminus, bplusdot, bminusdot, and the real and imaginary parts of A_{ij} :

In[154]:= $\{\text{bplus}, \text{bminus}, \text{bplusdot}, \text{bminusdot}\} = \left\{\sqrt{\text{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}}, \sqrt{\text{Cos}[\eta\,t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[\eta\,t\,\omega p]^2}{\eta^2\,\omega p^2}}, \frac{(\omega - \omega p)\,(\omega + \omega p)\,\text{Sin}[2\,t\,\omega p]}{2\,\omega p\,\sqrt{\text{Cos}[t\,\omega p]^2 + \frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}}}, \frac{\left(-2\,k + \omega^2 - \omega p^2\right)\,\text{Sin}\left[2\,t\,\sqrt{1 + \frac{2\,k}{\omega p^2}}\,\omega p\right]}{2\,\sqrt{1 + \frac{2\,k}{\omega p^2}}\,\omega p\,\sqrt{\text{Cos}\left[t\,\sqrt{1 + \frac{2\,k}{\omega p^2}}\,\omega p\right]^2 + \frac{\omega^2\,\text{Sin}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\,\omega p\right]^2}{2\,k+\omega p^2}}}\right\}$ // FullSimplify

Out[154]= $\left\{\frac{\sqrt{93 - 4\,\text{Cos}\left[2\,\sqrt{\frac{89}{97}}\,t\right]}}{\sqrt{89}}, \frac{\sqrt{93 + 4\,\text{Cos}\left[2\,\sqrt{\frac{97}{89}}\,t\right]}}{\sqrt{97}}, \frac{4\,\text{Sin}\left[2\,\sqrt{\frac{89}{97}}\,t\right]}{\sqrt{97}\,\sqrt{93 - 4\,\text{Cos}\left[2\,\sqrt{\frac{89}{97}}\,t\right]}}, -\frac{4\,\text{Sin}\left[2\,\sqrt{\frac{97}{89}}\,t\right]}{\sqrt{89}\,\sqrt{93 + 4\,\text{Cos}\left[2\,\sqrt{\frac{97}{89}}\,t\right]}}\right\}$

In[155]:= $\{\text{ReA12}, \text{ImA12}\} = \left\{\left(\frac{\omega}{2\,\text{bplus}^2} - \frac{\omega}{2\,\text{bminus}^2}\right), \left(-\frac{\text{bplusdot}}{2\,\text{bplus}} + \frac{\text{bminusdot}}{2\,\text{bminus}}\right)\right\}$ // Simplify

Out[155]= $\left\{\frac{89}{186 - 8\,\text{Cos}\left[2\,\sqrt{\frac{89}{97}}\,t\right]} - \frac{97}{186 + 8\,\text{Cos}\left[2\,\sqrt{\frac{97}{89}}\,t\right]}, \frac{2\left(\frac{89\,\text{Sin}\left[2\,\sqrt{\frac{89}{97}}\,t\right]}{-93+4\,\text{Cos}\left[2\,\sqrt{\frac{89}{97}}\,t\right]} - \frac{97\,\text{Sin}\left[2\,\sqrt{\frac{97}{89}}\,t\right]}{93+4\,\text{Cos}\left[2\,\sqrt{\frac{97}{89}}\,t\right]}\right)}{\sqrt{8633}}\right\}$

In[156]:=
$$\{\text{ReA11}, \text{ImA11}, \text{ReA22}, \text{ImA22}\} = \left\{\frac{\omega}{2\,\text{bplus}^2} + \frac{\omega}{2\,\text{bminus}^2}, -\frac{\text{bplusdot}}{2\,\text{bplus}} - \frac{\text{bminusdot}}{2\,\text{bminus}}, \frac{\omega}{2\,\text{bplus}^2} + \frac{\omega}{2\,\text{bminus}^2}, -\frac{\text{bplusdot}}{2\,\text{bplus}} - \frac{\text{bminusdot}}{2\,\text{bminus}}\right\} \;\text{// Simplify}$$

Out[156]=
$$\left\{\frac{89}{186 - 8\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right]} + \frac{97}{186 + 8\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]}, \frac{2\left(\frac{89\,\text{Sin}\left[2\sqrt{\frac{89}{97}}\,t\right]}{-93+4\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right]} + \frac{97\,\text{Sin}\left[2\sqrt{\frac{97}{89}}\,t\right]}{93+4\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]}\right)}{\sqrt{8633}},\right.$$
$$\left.\frac{89}{186 - 8\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right]} + \frac{97}{186 + 8\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]}, \frac{2\left(\frac{89\,\text{Sin}\left[2\sqrt{\frac{89}{97}}\,t\right]}{-93+4\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right]} + \frac{97\,\text{Sin}\left[2\sqrt{\frac{97}{89}}\,t\right]}{93+4\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]}\right)}{\sqrt{8633}}\right\}$$

The real and imaginary parts of (A_{12})^2 are given by

In[157]:=
$$\text{ReA12sqr} = \frac{1}{4\,\text{bminus}^4\,\text{bplus}^4}\left(-\text{bminus}^2\,\text{bplus}^2\,(\text{bminusdot bplus} - \text{bminus bplusdot})^2 + \left(\text{bminus}^2 - \text{bplus}^2\right)^2\omega^2\right) \;\text{// Simplify}$$

Out[157]=
$$\left(138\,128\left(-186 + 97\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right] + 89\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]\right)^2 - 16\left(89\times\left(93 + 4\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]\right)\text{Sin}\left[2\sqrt{\frac{89}{97}}\,t\right] + 97\times\left(93 - 4\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right]\right)\text{Sin}\left[2\sqrt{\frac{97}{89}}\,t\right]\right)^2\right) \Big/ \left(34\,532\left(93 - 4\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right]\right)^2\left(93 + 4\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]\right)^2\right)$$

In[158]:=
$$\text{ImA12sqr} = -\left(((\text{bminus} - \text{bplus})\,(\text{bminus} + \text{bplus})\,(-\text{bminusdot bplus} + \text{bminus bplusdot})\,\omega) \Big/ \left(2\,\text{bminus}^3\,\text{bplus}^3\right)\right) \;\text{// Simplify}$$

Out[158]=
$$\left(2\times\left(97\sqrt{89}\sqrt{93 - 4\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right]} - 89\sqrt{97}\sqrt{93 + 4\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]}\right)\times\left(97\sqrt{89}\sqrt{93 - 4\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right]} + 89\sqrt{97}\sqrt{93 + 4\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]}\right)\times\left(8277\,\text{Sin}\left[2\sqrt{\frac{89}{97}}\,t\right] + 9021\,\text{Sin}\left[2\sqrt{\frac{97}{89}}\,t\right] - 4\times\left(93\,\text{Sin}\left[\frac{16\,t}{\sqrt{8633}}\right] + 4\,\text{Sin}\left[\frac{372\,t}{\sqrt{8633}}\right]\right)\right)\right) \Big/ \left(8633\sqrt{8633}\left(93 - 4\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right]\right)^2\left(93 + 4\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]\right)^2\right)$$

Now, as we have obtained the expressions of real and imaginary parts of A_{ij}, we can find the corresponding X, Y, and Z.

In[159]:=
```
{X, Y, Z} = {ReA11 - ReA12sqr/(2 ReA11), ImA11 - ImA12sqr/(2 ReA11), -(ReA12^2 + ImA12^2)/(2 ReA11)} // Simplify
```

Out[159]=
$$\left\{\frac{74632413 - 69192\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right] - 69192\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right] + 34596\,\text{Cos}\left[\frac{16\,t}{\sqrt{8633}}\right] + 64\,\text{Cos}\left[\frac{372\,t}{\sqrt{8633}}\right]}{8633\times\left(8649 - 194\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right] + 178\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]\right)},\right.$$

$$\frac{1}{8633\sqrt{8633}}\left(17266\left(\frac{89\,\text{Sin}\left[2\sqrt{\frac{89}{97}}\,t\right]}{-93+4\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right]} + \frac{97\,\text{Sin}\left[2\sqrt{\frac{97}{89}}\,t\right]}{93+4\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]}\right) - \right.$$
$$\left(\left(97\sqrt{89}\sqrt{93-4\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right]} - 89\sqrt{97}\sqrt{93+4\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]}\right)\times\right.$$
$$\left(97\sqrt{89}\sqrt{93-4\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right]} + 89\sqrt{97}\sqrt{93+4\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]}\right)\times$$
$$\left.\left(8277\,\text{Sin}\left[2\sqrt{\frac{89}{97}}\,t\right] + 9021\,\text{Sin}\left[2\sqrt{\frac{97}{89}}\,t\right] - 4\times\left(93\,\text{Sin}\left[\frac{16\,t}{\sqrt{8633}}\right] + 4\,\text{Sin}\left[\frac{372\,t}{\sqrt{8633}}\right]\right)\right)\right)\Big/$$
$$\left(\left(-93+4\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right]\right)\times\left(-8649+194\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right] - 178\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]\right)\times\right.$$
$$\left.\left.\left(93+4\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]\right)\right)\right),$$

$$\left.-\frac{\left(\frac{89}{186-8\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right]} - \frac{97}{186+8\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]}\right)^2 + \frac{4\left(\frac{89\,\text{Sin}\left[2\sqrt{\frac{89}{97}}\,t\right]}{-93+4\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right]} - \frac{97\,\text{Sin}\left[2\sqrt{\frac{97}{89}}\,t\right]}{93+4\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]}\right)^2}{8633}}{2\left(\frac{89}{186-8\,\text{Cos}\left[2\sqrt{\frac{89}{97}}\,t\right]} + \frac{97}{186+8\,\text{Cos}\left[2\sqrt{\frac{97}{89}}\,t\right]}\right)}\right\}$$

Now the initial values of X, Y, and Z, when when t=0 be Xi, Yi, and Zi.

In[160]:=
```
{Xi, Yi, Zi} = {X, Y, Z} /. t → 0 // FullSimplify
```

Out[160]= {1, 0, 0}

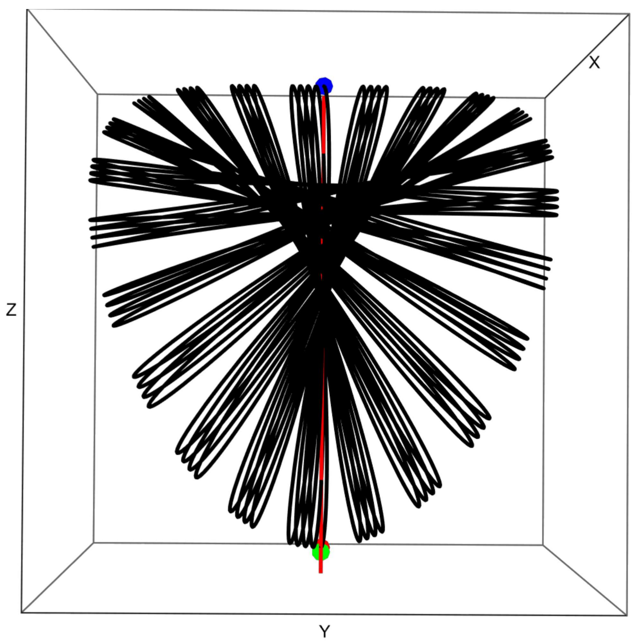
And the final values of X, Y, and Z, when t=τ be Xf, Yf, and Zf.

In[161]:=
```
{Xf, Yf, Zf} = {X, Y, Z} /. t → τ // FullSimplify
```

Out[161]=
$$\left\{\frac{74805457}{74804945},\ 0,\ -\frac{276768}{74804945}\right\}$$

In[162]:=
```
Show[{ParametricPlot3D[{X, Y, Z}, {t, 0, τ}, PlotStyle → Black,
    BoxRatios → {1, 1, 1}, Ticks → None, AxesLabel → {"X", "Y", "Z"}]},
  {ParametricPlot3D[{Coth[θ], 0, -Csch[θ]}, {θ, 0, 10}, PlotStyle → Red],
   Graphics3D[{Blue, PointSize[0.03], Point[{Xi, Yi, Zi}],
     Red, PointSize[0.03], Point[{Xf, Yf, Zf}]}], Graphics3D[
    {Green, PointSize[0.03], Point[{Coth[2 π], 0, -Csch[2 π]}]}]}}, LabelStyle → Black]
```

Out[162]=



Evolution toward the thermal state for (n,l) = (48,44) to the target inverse temperature $\beta = \frac{k_B}{E_g}\pi$. The exact thermal state corresponding to this value is marked by a green dot, while the approximate thermal state is indicated by a red dot; both lie on the thermal curve.

Accuracy and Thermalization time for {n,l}={12,11}

In[163]:=
```
ClearAll["Global`*"]
```

In[164]:=
```
{n, l} = {12, 11}
```

Out[164]= {12, 11}

We have $\frac{Eg}{k_B T_{nl}} = \frac{Eg\,\beta}{k_B} = \mathrm{Log}\left[\frac{1+l+n}{\mathrm{Abs}[-l+n]}\right]$. So β_{nl} becomes:

In[165]:=
$$N\left[\beta nl = \frac{kB}{Eg}\,\mathrm{Log}\left[\frac{1+l+n}{\mathrm{Abs}[-l+n]}\right]\right]$$

Out[165]=
$$\frac{3.17805\ kB}{Eg}$$

The target temperature here is $\beta = \frac{kB}{Eg}\pi$.

In[166]:= $\beta = \frac{kB}{Eg}\pi$

Out[166]= $\frac{kB\,\pi}{Eg}$

The difference between target temperature and achieved temperature is:

In[167]:= $\delta\beta = \beta - \beta nl$

Out[167]= $\frac{kB\,\pi}{Eg} - \frac{kB\,\mathrm{Log}[24]}{Eg}$

The accuracy, in percentage, can be calculated by:

In[168]:= $N\left[\frac{\delta\beta}{\beta}\,100\right]$ // FullSimplify

Out[168]= -1.1606

The dimensionless time parameter τtilde takes the value:

In[169]:= $N\left[\tau tilde = \frac{1}{4}\sqrt{(1+2\,l)\times(1+2\,n)}\right]$

Out[169]= 5.99479

Accuracy and Thermalization time for {n,l}={24,22}

In[170]:=
```
ClearAll["Global`*"]
```

In[171]:= {n, l} = {24, 22}

Out[171]= {24, 22}

We have $\frac{Eg}{k_B\,T_{nl}} = \frac{Eg\,\beta}{k_B} = \mathrm{Log}\left[\frac{1+l+n}{\mathrm{Abs}[-l+n]}\right]$. So β_{nl} becomes:

In[172]:= $N\left[\beta nl = \frac{kB}{Eg}\,\mathrm{Log}\left[\frac{1+l+n}{\mathrm{Abs}[-l+n]}\right]\right]$

Out[172]= $\frac{3.157\,kB}{Eg}$

The target temperature here is $\beta = \frac{kB}{Eg}\pi$.

In[173]:= $\beta = \frac{kB}{Eg}\pi$

Out[173]= $\frac{kB\,\pi}{Eg}$

The difference between target temperature and achieved temperature is:

In[174]:= `δβ = β - βnl`

Out[174]= $\frac{\text{kB}\,\pi}{\text{Eg}} - \frac{\text{kB}\,\text{Log}\left[\frac{47}{2}\right]}{\text{Eg}}$

The accuracy, in percentage, can be calculated by:

In[175]:= $\text{N}\left[\frac{\delta\beta}{\beta}\,100\right]$ `// FullSimplify`

Out[175]= `-0.490444`

The dimensionless time parameter τtilde takes the value:

In[176]:= $\text{N}\left[\tau\text{tilde} = \frac{1}{4}\sqrt{(1+2\,\text{l})\times(1+2\,\text{n})}\right]$

Out[176]= `11.7394`

Accuracy and Thermalization time for {n,l}={36,33}

In[177]:= `ClearAll["Global`*"]`

In[178]:= `{n, l} = {36, 33}`

Out[178]= `{36, 33}`

We have $\frac{\text{Eg}}{k_B\,T_{nl}} = \frac{\text{Eg}\,\beta}{k_B} = \text{Log}\left[\frac{1+l+n}{\text{Abs}[-l+n]}\right]$. So β_{nl} becomes:

In[179]:= $\text{N}\left[\beta\text{nl} = \frac{\text{kB}}{\text{Eg}}\,\text{Log}\left[\frac{1+\text{l}+\text{n}}{\text{Abs}[-\text{l}+\text{n}]}\right]\right]$

Out[179]= $\frac{3.14988\,\text{kB}}{\text{Eg}}$

The target temperature here is $\beta = \frac{\text{kB}}{\text{Eg}}\,\pi$.

In[180]:= $\beta = \frac{\text{kB}}{\text{Eg}}\,\pi$

Out[180]= $\frac{\text{kB}\,\pi}{\text{Eg}}$

The difference between target temperature and achieved temperature is:

In[181]:= `δβ = β - βnl`

Out[181]= $\frac{\text{kB}\,\pi}{\text{Eg}} - \frac{\text{kB}\,\text{Log}\left[\frac{70}{3}\right]}{\text{Eg}}$

The accuracy, in percentage, can be calculated by:

In[182]:= $N\left[\frac{\delta\beta}{\beta}\,100\right]$ // FullSimplify

Out[182]= -0.263888

The dimensionless time parameter τtilde takes the value:

In[183]:= $N\left[\tau\text{tilde} = \frac{1}{4}\sqrt{(1+2\,l)\times(1+2\,n)}\right]$

Out[183]= 17.4839

Accuracy and Thermalization time for {n,l}={48,44}

In[184]:=
```
ClearAll["Global`*"]
```

In[185]:=
```
{n, l} = {48, 44}
```

Out[185]= {48, 44}

We have $\frac{Eg}{k_B\,T_{nl}} = \frac{Eg\,\beta}{k_B} = \text{Log}\left[\frac{1+l+n}{\text{Abs}[-l+n]}\right]$. So β_{nl} becomes:

In[186]:= $N\left[\beta nl = \frac{kB}{Eg}\,\text{Log}\left[\frac{1+l+n}{\text{Abs}[-l+n]}\right]\right]$

Out[186]= $\frac{3.14631\,kB}{Eg}$

The target temperature here is $\beta = \frac{kB}{Eg}\,\pi$.

In[187]:= $\beta = \frac{kB}{Eg}\,\pi$

Out[187]= $\frac{kB\,\pi}{Eg}$

The difference between target temperature and achieved temperature is:

In[188]:= $\delta\beta = \beta - \beta nl$

Out[188]= $\frac{kB\,\pi}{Eg} - \frac{kB\,\text{Log}\left[\frac{93}{4}\right]}{Eg}$

The accuracy, in percentage, can be calculated by:

In[189]:= $N\left[\frac{\delta\beta}{\beta}\,100\right]$ // FullSimplify

Out[189]= -0.150003

The dimensionless time parameter τtilde takes the value:

In[190]:= $N\left[\tau\text{tilde} = \frac{1}{4}\sqrt{(1+2\,l)\times(1+2\,n)}\right]$

Out[190]= 23.2285

Accuracy and Thermalization time for {n,l}={60,55}

In[191]:=
```
ClearAll["Global`*"]
```

In[192]:= {n, l} = {60, 55}

Out[192]= {60, 55}

We have $\frac{Eg}{k_B\,T_{nl}} = \frac{Eg\,\beta}{k_B} = \text{Log}\left[\frac{1+l+n}{\text{Abs}[-l+n]}\right]$. So β_{nl} becomes:

In[193]:= $N\left[\beta nl = \frac{kB}{Eg}\,\text{Log}\left[\frac{1+l+n}{\text{Abs}[-l+n]}\right]\right]$

Out[193]= $\frac{3.14415\,kB}{Eg}$

The target temperature here is $\beta = \frac{kB}{Eg}\,\pi$.

In[194]:= $\beta = \frac{kB}{Eg}\,\pi$

Out[194]= $\frac{kB\,\pi}{Eg}$

The difference between target temperature and achieved temperature is:

In[195]:= $\delta\beta = \beta - \beta nl$

Out[195]= $\frac{kB\,\pi}{Eg} - \frac{kB\,\text{Log}\left[\frac{116}{5}\right]}{Eg}$

The accuracy, in percentage, can be calculated by:

In[196]:= $N\left[\frac{\delta\beta}{\beta}\,100\right]$ // FullSimplify

Out[196]= -0.0814754

The dimensionless time parameter τtilde takes the value:

In[197]:= $N\left[\tau\text{tilde} = \frac{1}{4}\sqrt{(1+2\,l)\times(1+2\,n)}\right]$

Out[197]= 28.973

Table 1:

So the SDS temperatures don’t include every possible temperature. But they do form a countably dense subset of the positive real numbers, which represent all possible temperatures.

In[198]:=
```
Column[{Grid[{{Style["(l,n)", Bold], Style["Egβnl/kB", Bold],
    Style["δβ/β", Bold], Style["τtilde", Bold]}, {"(12,11)", 3.178, "1.16%", 5.99},
   {"(24,22)", 3.157, ".49%", 11.74}, {"(36,33)", 3.149, ".26%", 17.48},
   {"(48,44)", 3.146, ".15%", 23.23}, {"(60,55)", 3.144, ".08%", 28.97}},
  Frame → All, Spacings → {3, 1}, ItemStyle → {FontFamily → "Times", FontSize → 20}]}]
```

Out[198]=

(l,n)	$E_g\beta_{nl}/k_B$	$\delta\beta/\beta$	τtilde
(12,11)	3.178	1.16%	5.99
(24,22)	3.157	.49%	11.74
(36,33)	3.149	.26%	17.48
(48,44)	3.146	.15%	23.23
(60,55)	3.144	.08%	28.97

TABLE 1 : List of SDS approximations to $\beta = \pi \frac{k_B}{E_g}$. As one moves down the list, the approximation error in temperature decreases by roughly a factor of two with each step. However, this improved accuracy comes at the cost of longer thermalization times, which in this example is scaling roughly inversely with the %-error in temperature.

Average Energy

Now we will look at the average energy of the oscillator.

In[199]:= `m = 1`

Out[199]= 1

Average energy is given by the expectation value of Hamiltonian, U(t) = <H0>

In[200]:= $\text{H0Exp} = \frac{\Sigma pp}{2} + \frac{\omega^2\,\Sigma xx}{2}$ // FullSimplify

Out[200]= $\frac{1}{2}\left(\Sigma pp + \Sigma xx\,\omega^2\right)$

So the average energy is equal to $\frac{E_g\left(1+X1^2+Y1^2-Z1^2\right)}{2\,(X1+Z1)}$.

Figure 8(a) : Evolution of U(t) when {n,l} = {1,0}

We shall see the dynamics of average energy U(t) for the case of (n,l)=(1,0).

In[201]:= `ω = 1`

Out[201]= 1

In[202]:= $\{n, l\} = \{1, 0\}$

Out[202]= $\{1, 0\}$

The corresponding values of β and the dimensionless parameters become:

In[203]:= $\{\beta, \omega\text{tilde}, \text{ktilde}, \tau\text{tilde}\} =$

$$\left\{\text{Log}\left[\frac{n+l+1}{n-l}\right], \sqrt{\frac{2\,l+1}{2\,n+1}}, \frac{2\,(n-l)\,(n+l+1)}{(2\,l+1)\times(2\,n+1)}, \frac{1}{4}\sqrt{(2\,l+1)\times(2\,n+1)}\right\}$$

Out[203]= $\left\{\text{Log}[2], \frac{1}{\sqrt{3}}, \frac{4}{3}, \frac{\sqrt{3}}{4}\right\}$

In[204]:= $\{\omega p, k, \tau\} = \left\{\omega\,\omega\text{tilde}, \text{ktilde}\,\omega^2, 2\,\pi\,\frac{\tau\text{tilde}}{\omega}\right\}$

Out[204]= $\left\{\frac{1}{\sqrt{3}}, \frac{4}{3}, \frac{\sqrt{3}\,\pi}{2}\right\}$

In[205]:= $\eta = \sqrt{1+\frac{2\,k}{\omega p^2}}$

Out[205]= 3

The corresponding values of bplus, bminus, bplusdot, and bminusdot are:

In[206]:= $\{\text{bplus}, \text{bminus}, \text{bplusdot}, \text{bminusdot}\} = \left\{\sqrt{\text{Cos}[t\,\omega p]^2+\frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}},\right.$

$$\sqrt{\text{Cos}[\eta\,t\,\omega p]^2+\frac{\omega^2\,\text{Sin}[\eta\,t\,\omega p]^2}{\eta^2\;\omega p^2}}, \frac{(\omega-\omega p)\,(\omega+\omega p)\,\text{Sin}[2\,t\,\omega p]}{2\,\omega p\,\sqrt{\text{Cos}[t\,\omega p]^2+\frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}}},$$

$$\left.\frac{\left(-2\,k+\omega^2-\omega p^2\right)\,\text{Sin}\left[2\,t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\;\omega p\right]}{2\,\sqrt{1+\frac{2\,k}{\omega p^2}}\;\omega p\,\sqrt{\text{Cos}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\;\omega p\right]^2+\frac{\omega^2\,\text{Sin}\left[t\,\sqrt{1+\frac{2\,k}{\omega p^2}}\;\omega p\right]^2}{2\,k+\omega p^2}}}\right\}\text{ // FullSimplify}$$

Out[206]= $\left\{\sqrt{2-\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right]}, \frac{\sqrt{2+\text{Cos}\left[2\,\sqrt{3}\,t\right]}}{\sqrt{3}}, \frac{\text{Sin}\left[\frac{2\,t}{\sqrt{3}}\right]}{\sqrt{6-3\,\text{Cos}\left[\frac{2\,t}{\sqrt{3}}\right]}}, -\frac{\text{Sin}\left[2\,\sqrt{3}\,t\right]}{\sqrt{2+\text{Cos}\left[2\,\sqrt{3}\,t\right]}}\right\}$

Separating the real and imaginary parts of A_{ij}:

In[207]:=
$$\{ReA12, ImA12\} = \left\{\left(\frac{\omega}{2\,bplus^2} - \frac{\omega}{2\,bminus^2}\right), \left(-\frac{bplusdot}{2\,bplus} + \frac{bminusdot}{2\,bminus}\right)\right\} \text{ // Simplify}$$

Out[207]=
$$\left\{\frac{1}{2}\left(\frac{1}{2 - Cos\left[\frac{2\,t}{\sqrt{3}}\right]} - \frac{3}{2 + Cos\left[2\,\sqrt{3}\,t\right]}\right), \frac{\frac{Sin\left[\frac{2\,t}{\sqrt{3}}\right]}{-2+Cos\left[\frac{2\,t}{\sqrt{3}}\right]} - \frac{3\,Sin\left[2\,\sqrt{3}\,t\right]}{2+Cos\left[2\,\sqrt{3}\,t\right]}}{2\,\sqrt{3}}\right\}$$

In[208]:=
$$\{ReA11, ImA11, ReA22, ImA22\} = \left\{\frac{\omega}{2\,bplus^2} + \frac{\omega}{2\,bminus^2}, -\frac{bplusdot}{2\,bplus} - \frac{bminusdot}{2\,bminus}, \frac{\omega}{2\,bplus^2} + \frac{\omega}{2\,bminus^2}, -\frac{bplusdot}{2\,bplus} - \frac{bminusdot}{2\,bminus}\right\} \text{ // Simplify}$$

Out[208]=
$$\left\{\frac{1}{2}\left(\frac{1}{2 - Cos\left[\frac{2\,t}{\sqrt{3}}\right]} + \frac{3}{2 + Cos\left[2\,\sqrt{3}\,t\right]}\right), \frac{2\,Sin\left[\frac{2\,t}{\sqrt{3}}\right] + Sin\left[\frac{4\,t}{\sqrt{3}}\right] + 2\,Sin\left[\frac{8\,t}{\sqrt{3}}\right] - 6\,Sin\left[2\,\sqrt{3}\,t\right]}{2\,\sqrt{3}\,\left(-2 + Cos\left[\frac{2\,t}{\sqrt{3}}\right]\right) \times \left(2 + Cos\left[2\,\sqrt{3}\,t\right]\right)},\right.$$
$$\left.\frac{1}{2}\left(\frac{1}{2 - Cos\left[\frac{2\,t}{\sqrt{3}}\right]} + \frac{3}{2 + Cos\left[2\,\sqrt{3}\,t\right]}\right), \frac{2\,Sin\left[\frac{2\,t}{\sqrt{3}}\right] + Sin\left[\frac{4\,t}{\sqrt{3}}\right] + 2\,Sin\left[\frac{8\,t}{\sqrt{3}}\right] - 6\,Sin\left[2\,\sqrt{3}\,t\right]}{2\,\sqrt{3}\,\left(-2 + Cos\left[\frac{2\,t}{\sqrt{3}}\right]\right) \times \left(2 + Cos\left[2\,\sqrt{3}\,t\right]\right)}\right\}$$

The real and imaginary parts of (A_{12})^2 are given by

In[209]:=
$$ReA12sqr = \frac{1}{4\,bminus^4\,bplus^4}\left(-bminus^2\,bplus^2\,(bminusdot\,bplus - bminus\,bplusdot)^2 + \left(bminus^2 - bplus^2\right)^2\omega^2\right) \text{ // Simplify}$$

Out[209]=
$$-\left(\left(\left(6 + 93\,Cos\left[\frac{2\,t}{\sqrt{3}}\right] + 74\,Cos\left[\frac{4\,t}{\sqrt{3}}\right] + 26\,Cos\left[\frac{8\,t}{\sqrt{3}}\right] + 22\,Cos\left[\frac{10\,t}{\sqrt{3}}\right] + Cos\left[\frac{14\,t}{\sqrt{3}}\right] + 76\,Cos\left[2\,\sqrt{3}\,t\right] - 10\,Cos\left[4\,\sqrt{3}\,t\right]\right) Sin\left[\frac{t}{\sqrt{3}}\right]^2\right) \Big/ \left(6\left(-2 + Cos\left[\frac{2\,t}{\sqrt{3}}\right]\right)^2 \left(2 + Cos\left[2\,\sqrt{3}\,t\right]\right)^2\right)\right)$$

In[210]:=
$$ImA12sqr = -\left(((bminus - bplus)\,(bminus + bplus)\,(-bminusdot\,bplus + bminus\,bplusdot)\,\omega) \Big/ \left(2\,bminus^3\,bplus^3\right)\right) \text{ // Simplify}$$

Out[210]=
$$\left(\left(-3\,\sqrt{2 - Cos\left[\frac{2\,t}{\sqrt{3}}\right]} + \sqrt{3}\,\sqrt{2 + Cos\left[2\,\sqrt{3}\,t\right]}\right) \times \left(3\,\sqrt{2 - Cos\left[\frac{2\,t}{\sqrt{3}}\right]} + \sqrt{3}\,\sqrt{2 + Cos\left[2\,\sqrt{3}\,t\right]}\right) \times \left(-2\,Sin\left[\frac{2\,t}{\sqrt{3}}\right] + 2\,Sin\left[\frac{4\,t}{\sqrt{3}}\right] + Sin\left[\frac{8\,t}{\sqrt{3}}\right] - 6\,Sin\left[2\,\sqrt{3}\,t\right]\right)\right) \Big/ \left(6\,\sqrt{3}\left(-2 + Cos\left[\frac{2\,t}{\sqrt{3}}\right]\right)^2 \left(2 + Cos\left[2\,\sqrt{3}\,t\right]\right)^2\right)$$

Now, as we have obtained the expressions of real and imaginary parts of A_{ij}, we can find the corresponding X, Y, and Z.

In[211]:= `{X, Y, Z} = {ReA11 - ReA12sqr/(2 ReA11), ImA11 - ImA12sqr/(2 ReA11), -(ReA12^2 + ImA12^2)/(2 ReA11)} // Simplify`

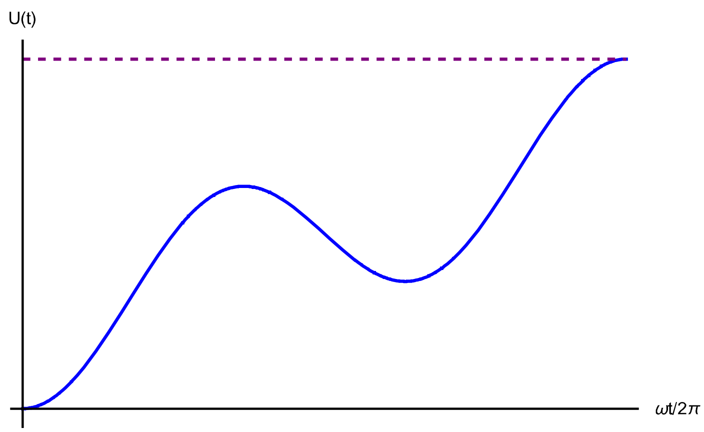
Out[211]=
$$\left\{\frac{47-8\,\mathrm{Cos}\left[\frac{2t}{\sqrt{3}}\right]+4\,\mathrm{Cos}\left[\frac{4t}{\sqrt{3}}\right]+\mathrm{Cos}\left[\frac{8t}{\sqrt{3}}\right]-8\,\mathrm{Cos}\left[2\sqrt{3}\,t\right]}{6\times\left(8-3\,\mathrm{Cos}\left[\frac{2t}{\sqrt{3}}\right]+\mathrm{Cos}\left[2\sqrt{3}\,t\right]\right)},\right.$$
$$\left.\frac{\sqrt{3}\left(-\mathrm{Sin}\left[\frac{2t}{\sqrt{3}}\right]+\mathrm{Sin}\left[2\sqrt{3}\,t\right]\right)}{8-3\,\mathrm{Cos}\left[\frac{2t}{\sqrt{3}}\right]+\mathrm{Cos}\left[2\sqrt{3}\,t\right]},\frac{2\times\left(-10-9\,\mathrm{Cos}\left[\frac{2t}{\sqrt{3}}\right]-6\,\mathrm{Cos}\left[\frac{4t}{\sqrt{3}}\right]+\mathrm{Cos}\left[2\sqrt{3}\,t\right]\right)\mathrm{Sin}\left[\frac{t}{\sqrt{3}}\right]^2}{3\times\left(8-3\,\mathrm{Cos}\left[\frac{2t}{\sqrt{3}}\right]+\mathrm{Cos}\left[2\sqrt{3}\,t\right]\right)}\right\}$$

The final values of X, Y, and Z, when t = τ be Xf, Yf, and Zf .

In[212]:= `{Xf, Yf, Zf} = {X, Y, Z} /. t → τ // FullSimplify`

Out[212]= $\left\{\frac{17}{15}, 0, -\frac{8}{15}\right\}$

In[213]:=
```
Plot[{(1 + X^2 + Y^2 - Z^2)/(4 (X + Z)), (1 + Xf^2 + Yf^2 - Zf^2)/(4 (Xf + Zf))}, {t, 0, τ},
 PlotStyle → {Blue, {Purple, Dashed}}, LabelStyle → Black,
 Ticks → None, AxesLabel → {"ωt/2π", "U(t)"}, PlotRange → All]
```

Out[213]=


$\beta = \frac{k_B}{E_g}$ log2 corresponding to (l,n) = (1,0)

In[214]:=

Figure 8(b): Evolution of U(t) for {n,l} = {12,11}

Now we will look at the dynamics of average energy U(t) for the case of (n,l)=(12,11).

In[215]:= `ω = 1`

Out[215]= 1

In[216]:= `{n, l} = {12, 11}`

Out[216]= {12, 11}

The corresponding values of β and the dimensionless parameters become:

In[217]:= $\{\beta, \omega\text{tilde}, \text{ktilde}, \tau\text{tilde}\} = \left\{\text{Log}\left[\frac{n+l+1}{n-l}\right], \sqrt{\frac{2l+1}{2n+1}}, \frac{2(n-l)(n+l+1)}{(2l+1)\times(2n+1)}, \frac{1}{4}\sqrt{(2l+1)\times(2n+1)}\right\}$

Out[217]= $\left\{\text{Log}[24], \frac{\sqrt{23}}{5}, \frac{48}{575}, \frac{5\sqrt{23}}{4}\right\}$

In[218]:= $\{\omega p, k, \tau\} = \left\{\omega\,\omega\text{tilde}, \text{ktilde}\,\omega^2, 2\pi\frac{\tau\text{tilde}}{\omega}\right\}$

Out[218]= $\left\{\frac{\sqrt{23}}{5}, \frac{48}{575}, \frac{5\sqrt{23}\,\pi}{2}\right\}$

In[219]:= $\eta = \sqrt{1+\frac{2k}{\omega p^2}}$

Out[219]= $\frac{25}{23}$

The corresponding values of bplus, bminus, bplusdot, and bminusdot are:

In[220]:= $\{\text{bplus}, \text{bminus}, \text{bplusdot}, \text{bminusdot}\} = \left\{\sqrt{\text{Cos}[t\,\omega p]^2+\frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}}, \sqrt{\text{Cos}[\eta\,t\,\omega p]^2+\frac{\omega^2\,\text{Sin}[\eta\,t\,\omega p]^2}{\eta^2\,\omega p^2}}, \frac{(\omega-\omega p)(\omega+\omega p)\,\text{Sin}[2\,t\,\omega p]}{2\,\omega p\sqrt{\text{Cos}[t\,\omega p]^2+\frac{\omega^2\,\text{Sin}[t\,\omega p]^2}{\omega p^2}}}, \frac{\left(-2k+\omega^2-\omega p^2\right)\text{Sin}\left[2\,t\sqrt{1+\frac{2k}{\omega p^2}}\,\omega p\right]}{2\sqrt{1+\frac{2k}{\omega p^2}}\,\omega p\sqrt{\text{Cos}\left[t\sqrt{1+\frac{2k}{\omega p^2}}\,\omega p\right]^2+\frac{\omega^2\,\text{Sin}\left[t\sqrt{1+\frac{2k}{\omega p^2}}\,\omega p\right]^2}{2k+\omega p^2}}}\right\}$ // FullSimplify

Out[220]= $\left\{\frac{\sqrt{24-\text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]}}{\sqrt{23}}, \frac{1}{5}\sqrt{24+\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]}, \frac{\text{Sin}\left[\frac{2\sqrt{23}\,t}{5}\right]}{5\sqrt{24-\text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]}}, -\frac{\text{Sin}\left[\frac{10\,t}{\sqrt{23}}\right]}{\sqrt{23}\sqrt{24+\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]}}\right\}$

Separating the real and imaginary parts of A_{ij}:

In[221]:= $\{\text{ReA12}, \text{ImA12}\} = \left\{\left(\frac{\omega}{2\,\text{bplus}^2}-\frac{\omega}{2\,\text{bminus}^2}\right), \left(-\frac{\text{bplusdot}}{2\,\text{bplus}}+\frac{\text{bminusdot}}{2\,\text{bminus}}\right)\right\}$ // Simplify

Out[221]= $\left\{-\frac{25}{2\times\left(24+\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right)}-\frac{23}{2\times\left(-24+\text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)}, \frac{-\frac{25\,\text{Sin}\left[\frac{10\,t}{\sqrt{23}}\right]}{24+\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]}+\frac{23\,\text{Sin}\left[\frac{2\sqrt{23}\,t}{5}\right]}{-24+\text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]}}{10\sqrt{23}}\right\}$

In[222]:=

$$\{\mathtt{ReA11},\ \mathtt{ImA11},\ \mathtt{ReA22},\ \mathtt{ImA22}\} = \left\{\frac{\omega}{2\ \mathtt{bplus}^2} + \frac{\omega}{2\ \mathtt{bminus}^2},\ -\frac{\mathtt{bplusdot}}{2\ \mathtt{bplus}} - \frac{\mathtt{bminusdot}}{2\ \mathtt{bminus}},\ \frac{\omega}{2\ \mathtt{bplus}^2} + \frac{\omega}{2\ \mathtt{bminus}^2},\ -\frac{\mathtt{bplusdot}}{2\ \mathtt{bplus}} - \frac{\mathtt{bminusdot}}{2\ \mathtt{bminus}}\right\}\ \texttt{// Simplify}$$

Out[222]=

$$\left\{\frac{25}{2\times\left(24+\mathrm{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right)} - \frac{23}{2\times\left(-24+\mathrm{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)},\ \frac{\frac{25\,\mathrm{Sin}\left[\frac{10\,t}{\sqrt{23}}\right]}{24+\mathrm{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]} + \frac{23\,\mathrm{Sin}\left[\frac{2\sqrt{23}\,t}{5}\right]}{-24+\mathrm{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]}}{10\sqrt{23}},\right.$$

$$\left.\frac{25}{2\times\left(24+\mathrm{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right)} - \frac{23}{2\times\left(-24+\mathrm{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)},\ \frac{\frac{25\,\mathrm{Sin}\left[\frac{10\,t}{\sqrt{23}}\right]}{24+\mathrm{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]} + \frac{23\,\mathrm{Sin}\left[\frac{2\sqrt{23}\,t}{5}\right]}{-24+\mathrm{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]}}{10\sqrt{23}}\right\}$$

The real and imaginary parts of (A_{12})^2 are given by

In[223]:=

$$\mathtt{ReA12sqr} = \frac{1}{4\ \mathtt{bminus}^4\ \mathtt{bplus}^4}\left(-\mathtt{bminus}^2\ \mathtt{bplus}^2\ (\mathtt{bminusdot\ bplus} - \mathtt{bminus\ bplusdot})^2 + \left(\mathtt{bminus}^2 - \mathtt{bplus}^2\right)^2 \omega^2\right)\ \texttt{// Simplify}$$

Out[223]=

$$-\left(\left(\left(-575\left(-48+23\,\mathrm{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]+25\,\mathrm{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)\right)^2 + \left(25\times\left(-24+\mathrm{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)\mathrm{Sin}\left[\frac{10\,t}{\sqrt{23}}\right] - 23\times\left(24+\mathrm{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right)\mathrm{Sin}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)^2\right)\Big/\left(2300\left(24+\mathrm{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right)^2\left(-24+\mathrm{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)^2\right)\right)$$

In[224]:=

$$\mathtt{ImA12sqr} = -\left(((\mathtt{bminus} - \mathtt{bplus})\ (\mathtt{bminus} + \mathtt{bplus})\ (-\mathtt{bminusdot\ bplus} + \mathtt{bminus\ bplusdot})\ \omega)\Big/\left(2\ \mathtt{bminus}^3\ \mathtt{bplus}^3\right)\right)\ \texttt{// Simplify}$$

Out[224]=

$$\left(\left(23\sqrt{24+\mathrm{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]} - 5\sqrt{23}\sqrt{24-\mathrm{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]}\right)\times\left(23\sqrt{24+\mathrm{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]} + 5\sqrt{23}\sqrt{24-\mathrm{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]}\right)\times\left(24\,\mathrm{Sin}\left[\frac{4\,t}{5\sqrt{23}}\right] - 600\,\mathrm{Sin}\left[\frac{10\,t}{\sqrt{23}}\right] + \mathrm{Sin}\left[\frac{96\,t}{5\sqrt{23}}\right] - 552\,\mathrm{Sin}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)\right)\Big/\left(230\sqrt{23}\left(24+\mathrm{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right)^2\left(-24+\mathrm{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)^2\right)$$

Now, as we have obtained the expressions of real and imaginary parts of A_{ij}, we can find the corresponding X, Y, and Z.

In[225]:=
$$\{X, Y, Z\} = \left\{\text{ReA11} - \frac{\text{ReA12sqr}}{2\,\text{ReA11}},\ \text{ImA11} - \frac{\text{ImA12sqr}}{2\,\text{ReA11}},\ \frac{-(\text{ReA12}^{\wedge}2 + \text{ImA12}^{\wedge}2)}{2\,\text{ReA11}}\right\} \mathbin{//} \text{Simplify}$$

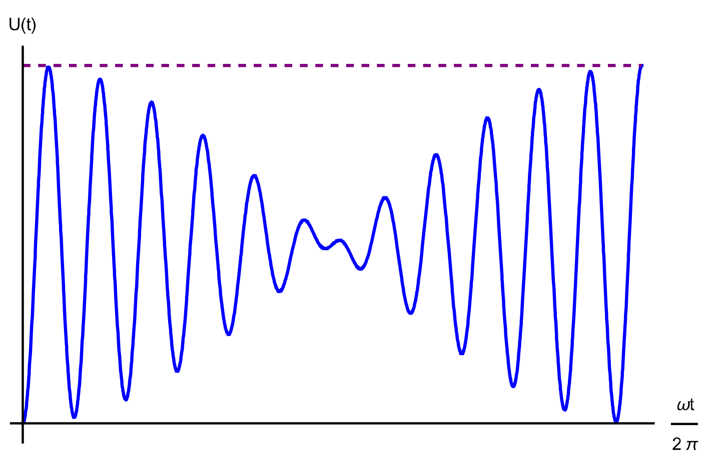
Out[225]=
$$\left\{\frac{1324227 + 576\,\text{Cos}\left[\frac{4\,t}{5\sqrt{23}}\right] - 1152\,\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right] + \text{Cos}\left[\frac{96\,t}{5\sqrt{23}}\right] - 1152\,\text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]}{1150\times\left(1152 + 23\,\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right] - 25\,\text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)},\right.$$
$$-\left(\left(5\sqrt{23}\left(\text{Sin}\left[\frac{42\,t}{5\sqrt{23}}\right] + 2304\,\text{Sin}\left[\frac{10\,t}{\sqrt{23}}\right] - \text{Sin}\left[\frac{54\,t}{5\sqrt{23}}\right] - 96\,\text{Sin}\left[\frac{96\,t}{5\sqrt{23}}\right] + 48\,\text{Sin}\left[\frac{20\,t}{\sqrt{23}}\right] + \right.\right.\right.$$
$$\left.\left.\text{Sin}\left[\frac{142\,t}{5\sqrt{23}}\right] - \text{Sin}\left[\frac{146\,t}{5\sqrt{23}}\right] - 2304\,\text{Sin}\left[\frac{2\sqrt{23}\,t}{5}\right] + 48\,\text{Sin}\left[\frac{4\sqrt{23}\,t}{5}\right]\right)\right)\Big/$$
$$\left.\left(4\times\left(24 + \text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right)\times\left(1152 + 23\,\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right] - 25\,\text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)\times\left(-24 + \text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)\right)\right),$$
$$\left.-\frac{\left(\frac{25}{2\times\left(24+\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right)} + \frac{23}{2\times\left(-24+\text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)}\right)^2 + \frac{\left(\frac{25\,\text{Sin}\left[\frac{10\,t}{\sqrt{23}}\right]}{24+\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]} - \frac{23\,\text{Sin}\left[\frac{2\sqrt{23}\,t}{5}\right]}{-24+\text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]}\right)^2}{2300}}{2\left(\frac{25}{2\times\left(24+\text{Cos}\left[\frac{10\,t}{\sqrt{23}}\right]\right)} - \frac{23}{2\times\left(-24+\text{Cos}\left[\frac{2\sqrt{23}\,t}{5}\right]\right)}\right)}\right\}$$

The final values of X, Y, and Z, when $t = \tau$ be Xf, Yf, and Zf .

In[226]:=
```
{Xf, Yf, Zf} = {X, Y, Z} /. t → τ // FullSimplify
```

Out[226]=
$$\left\{\frac{331777}{331775},\ 0,\ -\frac{1152}{331775}\right\}$$

In[227]:=
$$\text{Plot}\left[\left\{\frac{\left(1 + X^2 + Y^2 - Z^2\right)}{4\,(X + Z)},\ \frac{\left(1 + \text{Xf}^2 + \text{Yf}^2 - \text{Zf}^2\right)}{4\,(\text{Xf} + \text{Zf})}\right\},\ \{t, 0, \tau\},\right.$$
$$\text{PlotStyle} \to \{\text{Blue}, \{\text{Purple}, \text{Dashed}\}\},\ \text{LabelStyle} \to \text{Black},$$
$$\left.\text{Ticks} \to \text{None},\ \text{AxesLabel} \to \left\{\text{"}\frac{\omega t}{2\pi}\text{"},\ \text{"U(t)"}\right\},\ \text{PlotRange} \to \text{All}\right]$$

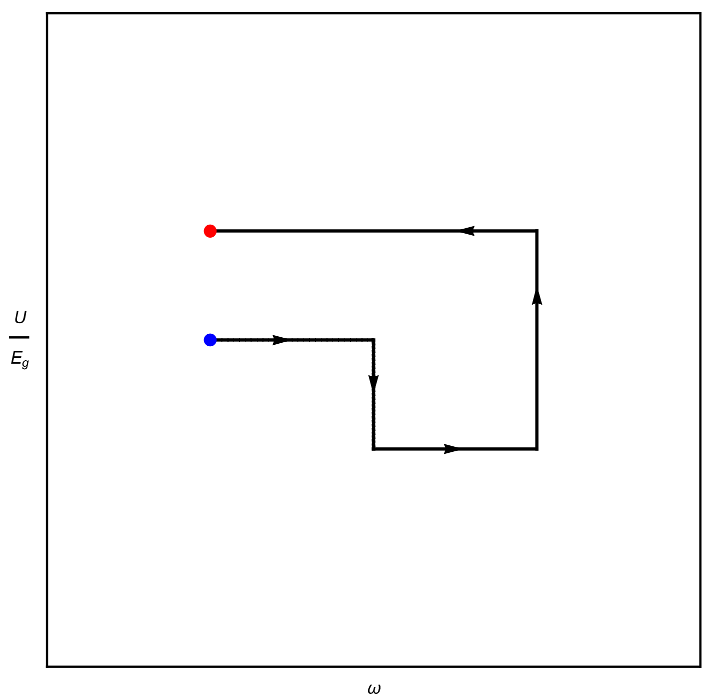
Out[227]=


$\beta = \frac{k_B}{E_g}$ 3.178 corresponding to (l,n) = (12, 11)

Figure 9(b): Heating and Cooling from Quenching

In[228]:=
```
Show[{ContourPlot[y == 1, {x, -1, 1},
   {y, -3, 3}, ContourStyle → Black, PlotRange → {{-2, 2}, {-3, 3}},
   FrameLabel → {"ω", Rotate["U/E_g", -π/2]}, LabelStyle → {Black}, FrameTicks → None],
  ContourPlot[x == 1, {x, -1, 2}, {y, 1, -1}, ContourStyle → Black],
  ContourPlot[x == 0, {x, -3, 3}, {y, -1, 0}, ContourStyle → Black,
   PlotRange → {{-2, 2}, {-3, 3}}], ContourPlot[y == 0, {x, -1, 0}, {y, -3, 3},
   ContourStyle → Black, PlotRange → {{-2, 2}, {-3, 3}}], ContourPlot[y == -1, {x, 0, 1},
   {y, -3, 3}, ContourStyle → Black, PlotRange → {{-2, 2}, {-3, 3}}], Graphics[
   {Black, Arrowheads[0.03], Arrow[{{0.6, 1}, {0.5, 1}}], Arrow[{{-1, 0}, {-0.5, 0}}],
    Arrow[{{0, 0}, {0, -0.5}}], Arrow[{{1, 0}, {1, 0.5}}], Arrow[{{0.5, -1}, {0.55, -1}}],
    Blue, PointSize[0.02], Point[{-1, 0}], Red, PointSize[0.02], Point[{-1, 1}]}]
}]
```

Out[228]=



The energy–frequency diagram of oscillator-1, for either heating or cooling, is shown here for the case of heating.