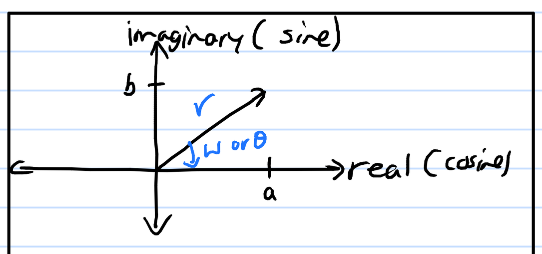
Fourier Transform

- applied to aperiodic signals
- represented as: $\int$(continuum of freqs)?

$e^{j\theta} = \cos\theta + i\sin\theta$ ← eulers formula complex #

$e^{j2\pi\omega t} = \cos(2\pi\omega t) + i\sin(2\pi\omega t)$



$a + ib = r\cos\omega + i r\sin\omega$

- $r = \sqrt{a^2+b^2}$
- ω or θ = (sine, cos) → $\tan^{-1}\left(\frac{\text{sine}}{\cos} \text{ or } \frac{b}{a}\right)$

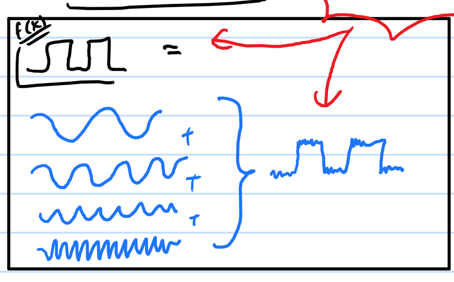
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\text{sinc}\,\theta = \frac{\sin(\pi\theta)}{\theta}$$

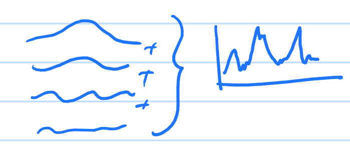
Fourier Series

periodic func = Σ sinusoids of diff freq * some coefficient



forier Transf

aperiodic func (no repeats) = $\int$ some sinusoids $\cdot$ weight func $e^{j2\pi ft}$ area under



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forward Fourier Transform: $\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi\omega t}\, dt$ ← spacial to freq domain

set of sinusoid weights

Inverse Fourier Transform: $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(\omega) e^{+j2\pi\omega t}\, d\omega$ ← freq to spacal domain

or the period T

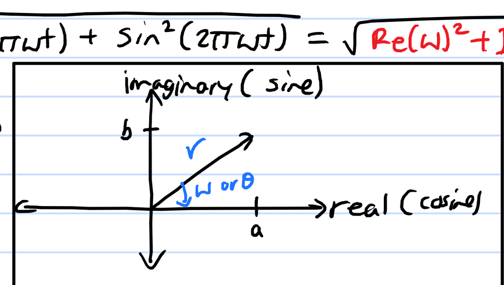
$e^{j2\pi\omega t} = \cos(2\pi\omega t) + i\sin(2\pi\omega t)$

Lets say that: $\mathcal{F}(\omega) = Re(\omega) + i\,Im(\omega)$

Magnitude: $|\mathcal{F}(\omega)| = \sqrt{\cos^2(2\pi\omega t) + \sin^2(2\pi\omega t)} = \sqrt{Re(\omega)^2 + Im^2(\omega)}$

phase spectrum: $= \tan^{-1}\left(\frac{\sin(2\pi\omega t)}{\cos(2\pi\omega t)}\right)$ →

$= \tan^{-1}\left(\frac{Im(\omega)}{Re(\omega)}\right)$



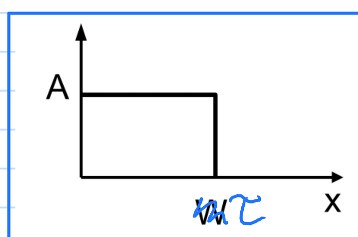
$a + ib = r\cos\omega + i r\sin\omega$

- $r = \sqrt{a^2+b^2}$
- ω or θ = (sine, cos) → $\tan^{-1}\left(\frac{\text{sine}}{\cos} \text{ or } \frac{b}{a}\right)$

Spectral Density $= |\mathcal{F}(\omega)|^2$

$= Re^2(\omega) + Im^2(\omega)$

Ex1: 1D



$$f(x) = \begin{cases} A & 0 \le x \le W \\ 0 & x > W \end{cases}$$

- $\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi\omega t}\, dt$
- eulers formula: $\int e^{at}\, dt = \frac{1}{a} e^{at}$ ← simple integral

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi\omega t}\, dt$$

$$= \int_0^{\tau} A e^{-j2\pi\omega t}\, dt \quad \text{) integrate}$$

$$= \frac{A}{-j2\pi\omega}\left[e^{-j2\pi\omega t}\right]_0^{\tau}$$

$$\left(e^{-j2\pi\omega\tau} - e^{-j2\pi\omega \cdot 0}\right)$$

$$-\left(1 - e^{-j2\pi\omega\tau}\right)$$

split

$$-e^{-j\pi\omega\tau}\underbrace{\left(\frac{e^{j\pi\omega\tau} - e^{-j\pi\omega\tau}}{2i}\right)}_{} \cdot 2i$$

$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta$

$$-2j\, e^{-j\pi\omega\tau} \cdot \sin(\pi\omega\tau)$$

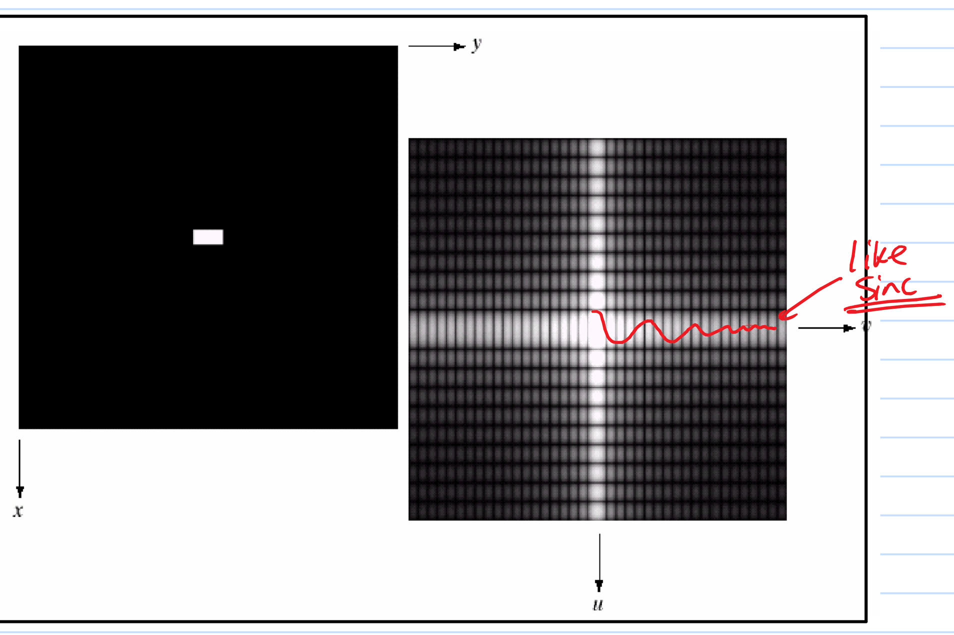
$$\frac{A}{\cancel{-j2}\pi\omega} \cdot \left(\cancel{-2j}\, e^{-j\pi\omega\tau} \cdot \sin(\pi\omega\tau)\right)$$

ans

$$\boxed{\mathcal{F}(\omega) = \frac{A}{\pi\omega} \cdot e^{-j\pi\omega\tau} \cdot \sin \pi\omega\tau}$$

$\frac{\sin\pi\theta}{\pi\theta} = \text{sinc}(\theta)$

Magnitude $= |\mathcal{F}(\omega)|$



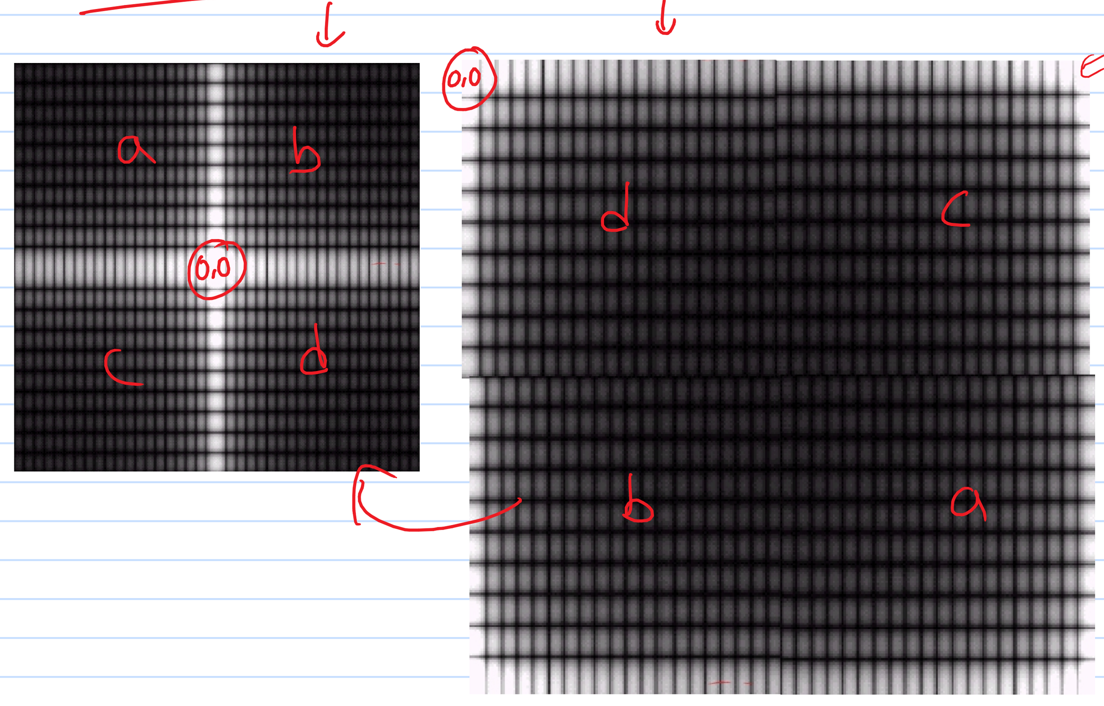
a b

FIGURE 4.3 (a) Image of a 20 × 40 white rectangle on a black background of size 512 × 512 pixels. (b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.

Magnitude

Placed (0,0) in center for visualization

This how it actually looks!



the corners is where the most energy & (low freq)

better: use logrithm bec it squashes large #s to small #s

① $\log_{10}(10^5) = 5$ ← say 10^5 is largest #

② can then scale the 5 to 255 ← map

Fourier Series

· applied to periodic signals

Σ (freq components that are integer multiples of some fundamental freq)

periodic ⊂ series
aperiodic ⊂ transform

(2)

fourier series for Periodic Signals

$$\boxed{f(x) = \sum_{n=-\infty}^{n=\infty} c(nu_0) e^{i2\pi nu_0 x}}$$ where $c(nu_0)$ is the n^{th} Fourier coefficient

$$\boxed{c(nu_0) = \frac{1}{x_0} \int_{-x_0/2}^{x_0/2} f(x) e^{-i2\pi nu_0 x} dx}$$

Periodic signals contain all freq that are harmonics of the fundamental freq

$$c(nu_0) = \frac{1}{x_0} \int_{-x_0/2}^{x_0/2} f(x) e^{-i2\pi nu_0 x} dx \Rightarrow \frac{1}{x_0} \int_{-W/2}^{W/2} A e^{-i2\pi nu_0 x} dx$$

$x_0 \to W$

integrate

$$c(nu_0) = \frac{A}{-i2\pi nu_0 x_0} \left(e^{-i\pi nu_0 W} - e^{+i\pi nu_0 W}\right)$$

exp to sine

$$c(nu_0) = \frac{A}{\pi n} \sin(\pi nu_0 W) \leftarrow \sin x = \frac{e^{ix} - e^{-ix}}{2i};\ u_0 x_0 = 1$$

sin to sinc

$$c(nu_0) = \frac{Au_0 W}{\pi nu_0 W} \sin(\pi nu_0 W) = Au_0 W \operatorname{sinc}(\pi nu_0 W)$$

Note that if $\frac{W}{2} = \frac{x_0}{2}$, then we have a square wave and

$$c(nu_0) = Au_0 x_0 \operatorname{sinc}(\pi nu_0 x_0)$$

$$c(nu_0) = \begin{cases} A \operatorname{sinc}(n) & n = \pm 1, \pm 3, \cdots \\ 0 & n = 0, \pm 2, \pm 4, \cdots \end{cases}$$

(3)

Discrete fourier form

$x = t$
$u = w$

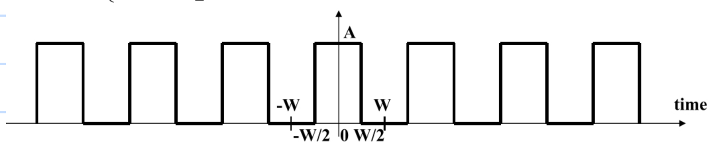
$$\boxed{F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-i2\pi \frac{ux}{N}}}$$ forward DFT

$$\boxed{f(x) = \sum_{u=0}^{N-1} F(u) e^{+i2\pi \frac{ux}{N}}}$$ inverse DFT

for $0 \le u \le N-1$ and $0 \le x \le N-1$ where N is the number of equi-spaced input samples.

Ex : Rectangular Pulse Train

$$f(x) = \begin{cases} A & |x| < \frac{W}{2} \\ 0 & |x| > \frac{W}{2} \end{cases} \quad \text{in interval}[-W/2, W/2]$$



$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-i2\pi \frac{ux}{N}}$$ forward DFT

(a) DFT e to cos

$$\Sigma\, f(x) \left[\cos\left(\frac{2\pi ux}{N}\right) + i \sin\left(\frac{2\pi ux}{N}\right) \right]$$

(b) split

$$\sum_{0}^{N} f(x) \cos\left(\frac{2\pi ux}{N}\right) + f(x) i \sin\left(\frac{2\pi ux}{N}\right)$$

(c) find all real and img for 0 to N−1

save the coeff to real and img

code

for (x=0 ; x<N ; x++){

real += f[x] · cos $\left(\frac{2\pi ux}{N}\right)$

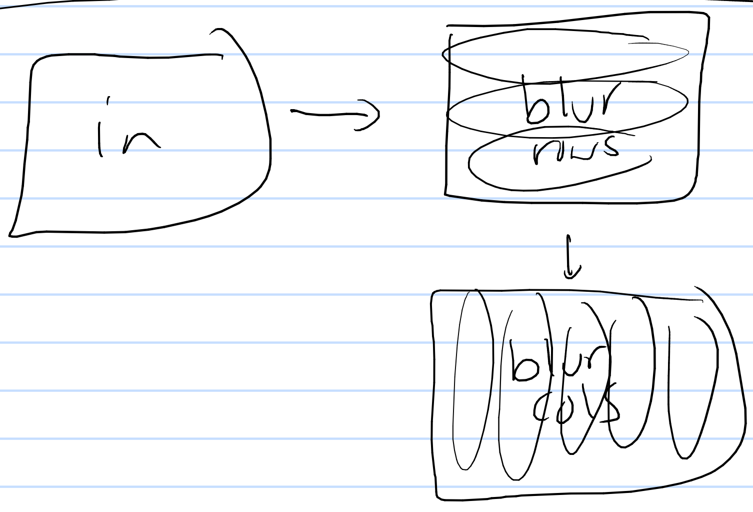
img += f[x] · sin $\left(\frac{2\pi ux}{N}\right)$

freq u has a real and img coeff

$F_{real}[u] = real$

$F_{imaginary}[u] = img$

like blur will do for fourier



$$f[x] \cdot g[x]$$

$$[f_{real} + i f_{img}] \cdot [g_{real} + i g_{img}]$$

$$f_{real} \cos + i f_{real} \sin$$
$$+\ i f_{img} \cos - f_{img} \sin$$

did he make a mistake?

```
for(u=0; u<N; u++) { /*compute spectrum over all freq. u */
  real = imag = 0;    /*reset real, imag component of F(u)*/
  for(x=0; x<N; x++) { /* visit each input pixel */
       real += (f[x]*cos(-2*PI*u*x/N));
       imag += (f[x]*sin(-2*PI*u*x/N));
       /* Note: if f is complex, then
       real += (fr[x]*cos()-fi[x]*sin());
       imag += (fr[x]*sin()+fi[x]*cos());
       because (fr+ifi)(gr+igi)=(frgr-figi)+i(figr+frgi)
       */
  }
  Fr[u] = real / N;
  Fi[u] = imag / N;
}
```


$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-i2\pi \frac{ux}{N}}$ forward DFT

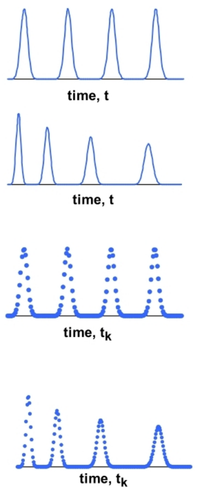
$\Sigma\, f(x) [\cos(\frac{2\pi ux}{N}) + i \sin(\frac{2\pi ux}{N})]$

$\sum_{0}^{N} f(x) \cos(\frac{2\pi ux}{N}) + f(x) i \sin(\frac{2\pi ux}{N})$

$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-i2\pi \frac{ux}{N}}$ forward DFT

$f(x) = \sum_{u=0}^{N-1} F(u) e^{+i2\pi \frac{ux}{N}}$ inverse DFT

Summary



Continuous	Periodic *(period T)*	**FS**	Discrete	$c_k = \frac{1}{T}\cdot\int_0^T s(t)\cdot e^{-ik\omega t}\,dt$
	Aperiodic	**FT**	Continuous	$S(f) = \int_{-\infty}^{+\infty} s(t)\cdot e^{-i2\pi f t}\,dt$
Discrete	Periodic *(period T)*	**DFS**	Discrete	$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-i\frac{2\pi k n}{N}}$
	Aperiodic	**DTFT**	Continuous	$S(f) = \sum_{n=-\infty}^{+\infty} s[n]\cdot e^{-i2\pi f n}$
		DFT	Discrete	$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-i\frac{2\pi k n}{N}}$

Note: *$i=\sqrt{-1}$, $\omega = 2\pi/T$, $s[n]=s(t_n)$, N = # of samples*

just FT or inverse FT
spacial → freq, or freq → spacial

Summary

continous graph: periodic → FS
aperiodic → FT

N = # of samples

Discrete (cant do integrals now)
periodic → DFS
aperiodic → continuous → FT but $\int$ is Σ
→ discrete → DFS