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# Filtering in the Frequency Domain

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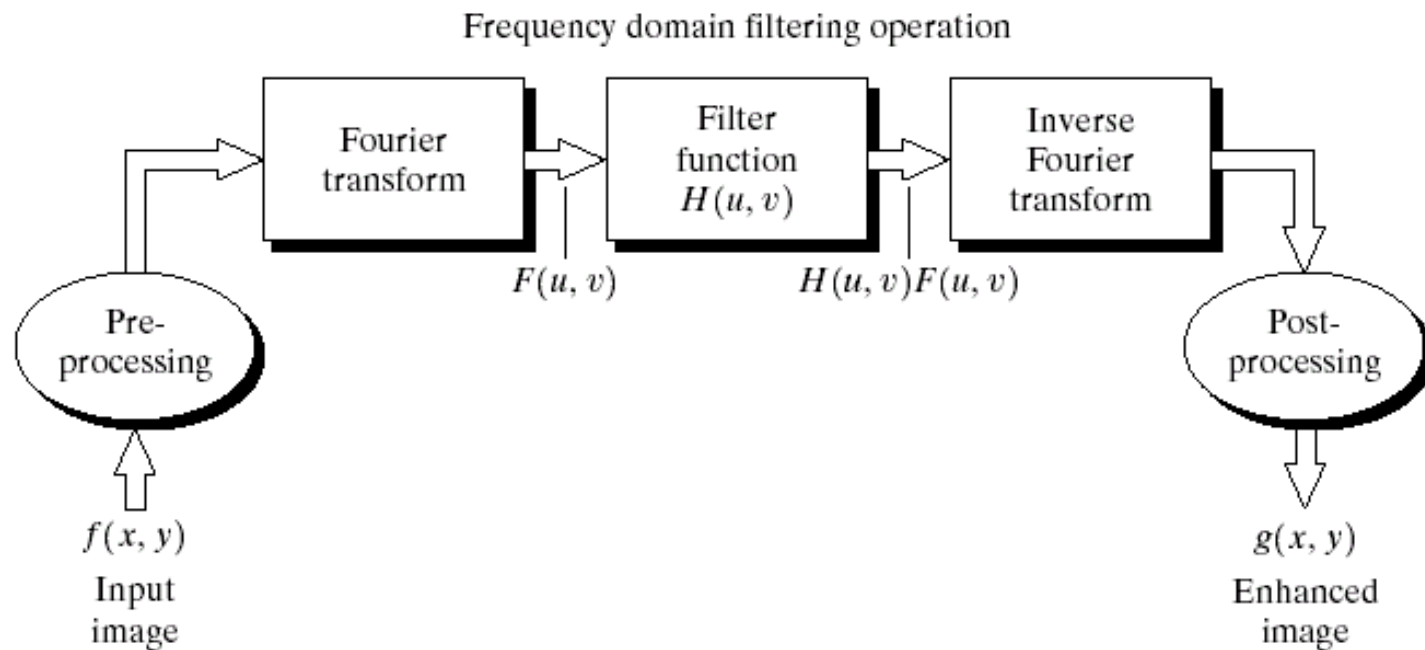
# Objectives

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- This lecture reviews frequency domain filtering.
  - Convolution theorem
  - Frequency bands
  - Lowpass filter
    - Ideal, Butterworth, Gaussian
  - Highpass filter
    - Ideal, Butterworth, Gaussian
  - Notch filter
  - Homomorphic filtering

# Basic Steps

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**FIGURE 4.5** Basic steps for filtering in the frequency domain.

# Basics

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1. Multiply input image by  $(-1)^{x+y}$  to center the transform to  $u = M/2$  and  $v = N/2$  (if  $M$  and  $N$  are even numbers, then shifted coordinates will be integers)
2. Compute  $F(u,v)$ , the DFT of the image from (1)
3. Multiply  $F(u,v)$  by a filter function  $H(u,v)$
4. Compute the inverse DFT of the result in (3)
5. Obtain the real part of the result in (4)
6. Multiply the result in (5) by  $(-1)^{x+y}$  to cancel the multiplication of the input image.

# Convolution Theorem

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“Multiply  $F(u,v)$  by a filter function  $H(u,v)$ ”

This step exploits the convolution theorem:

Convolution in one domain is equivalent to multiplication in the other domain.

$f(x)*g(x)$	$\leftrightarrow F(u) G(u)$
$f(x) g(x)$	$\leftrightarrow F(u)*G(u)$

# Periodicity

$$F(u,v) = F(u+M, v) = F(u, v+N) = F(u+M, v+N)$$

a b  
c d

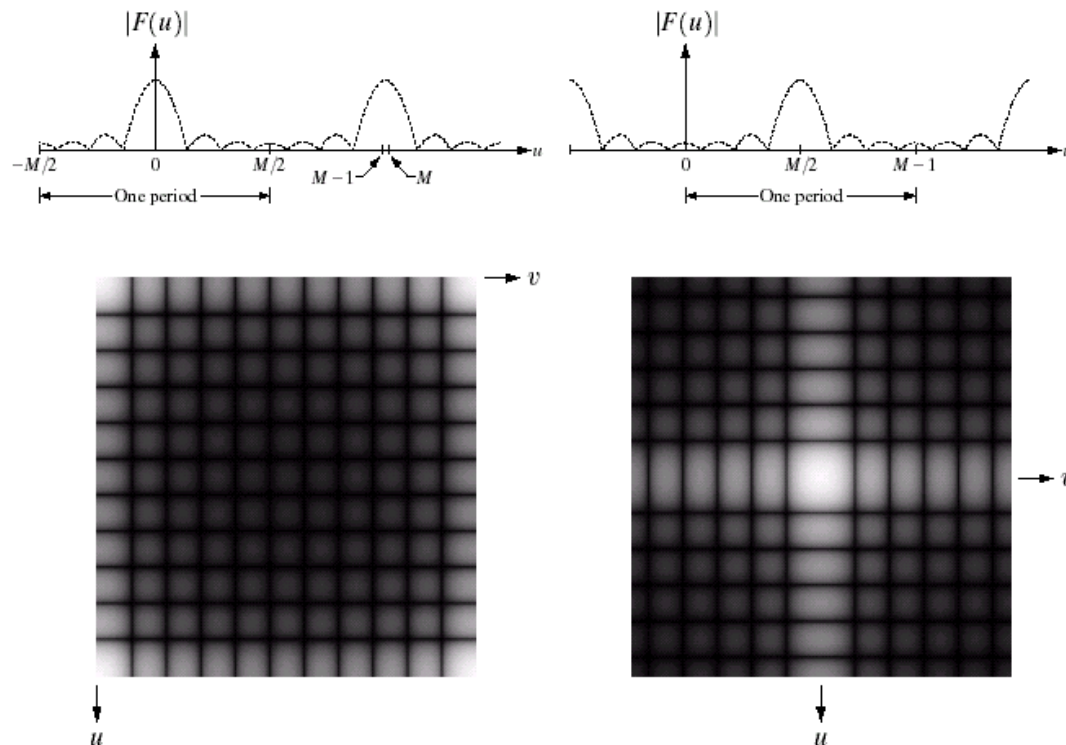
**FIGURE 4.34**

(a) Fourier spectrum showing back-to-back half periods in the interval  $[0, M-1]$ .

(b) Shifted spectrum showing a full period in the same interval.

(c) Fourier spectrum of an image, showing the same back-to-back properties as (a), but in two dimensions.

(d) Centered Fourier spectrum.



# Comments

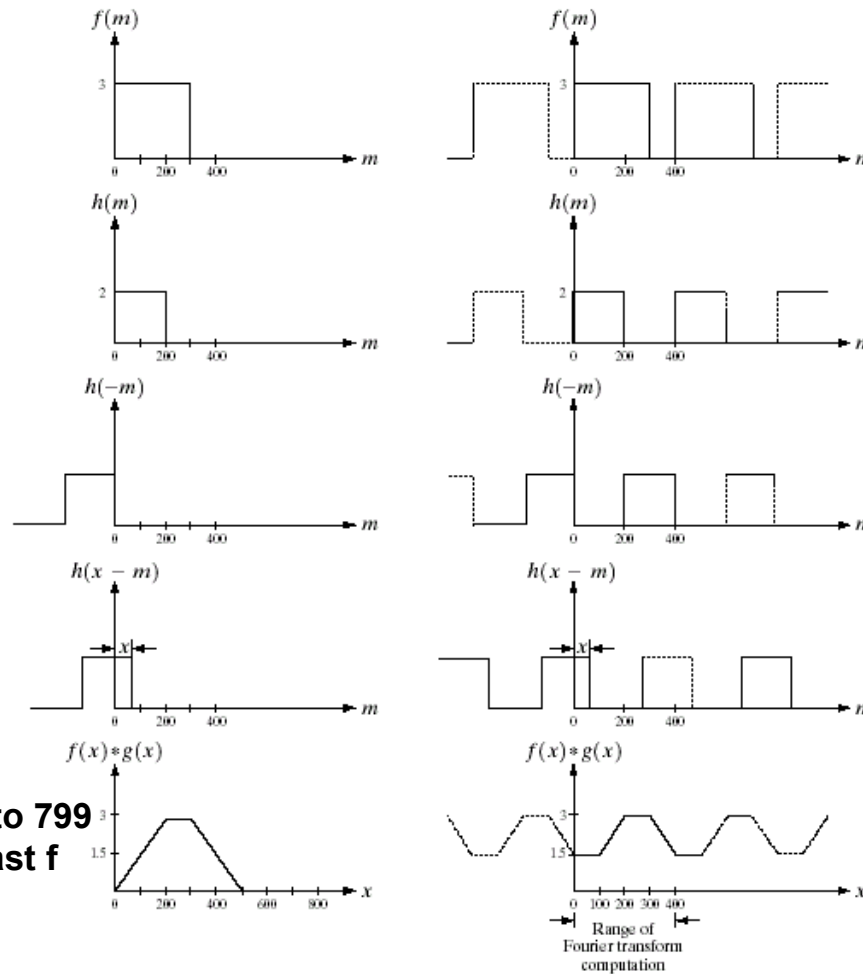
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- The magnitudes from  $(M/2)+1$  to  $M-1$  are reflections of the values in the half period to the left of the origin.
- DFT is formulated for values of  $u$  in the interval  $[0, M-1]$ .
- This yields two back-to-back half periods in this interval.
- For one full period, move origin to  $u=M/2$  by multiplying  $f(x)$  by  $(-1)^x$ .

# Significance of Periodicity

a f  
b g  
c h  
d i  
e j

**FIGURE 4.36** Left: convolution of two discrete functions. Right: convolution of the same functions, taking into account the implied periodicity of the DFT. Note in (j) how data from adjacent periods corrupt the result of convolution.



**x ranges from 0 to 799  
for  $h()$  to slide past f**

**period too short**

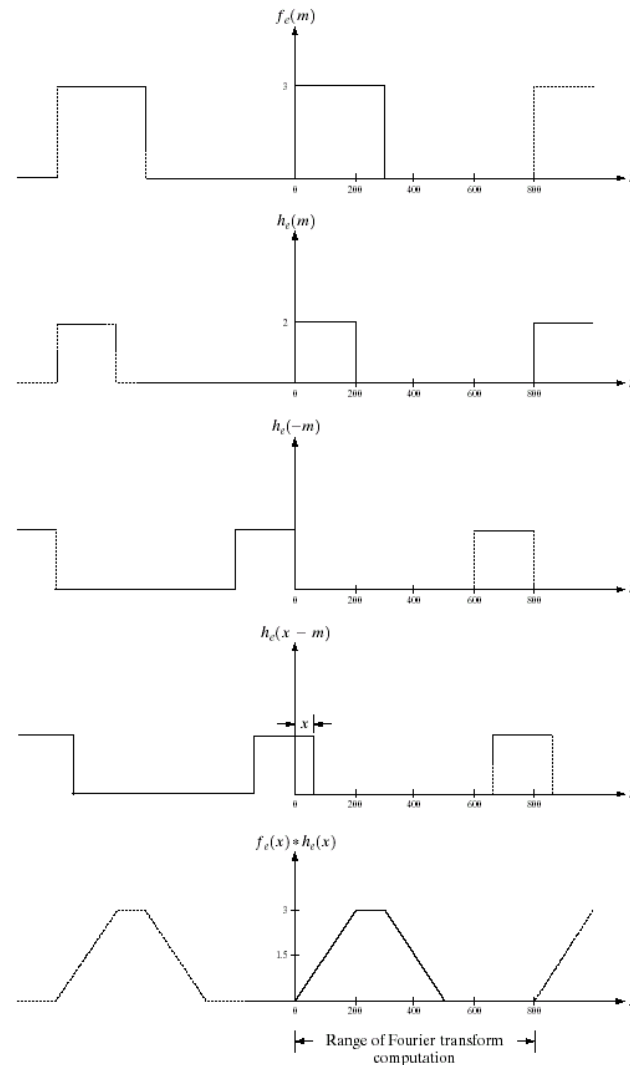


# Padding (1)

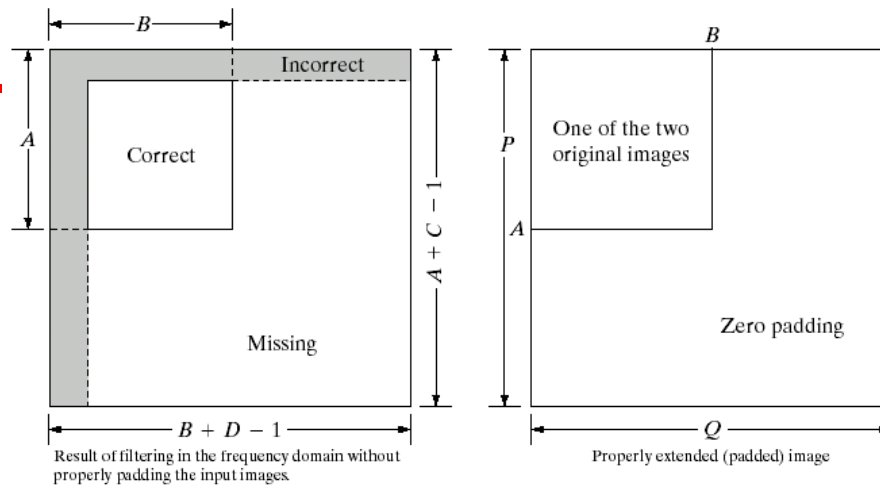
- Corrupted results were due to inadequate period.
- Solution: add padding to increase period.

a  
b  
c  
d  
e

**FIGURE 4.37**  
Result of performing convolution with extended functions. Compare Figs. 4.37(e) and 4.36(e).



# Padding (2)

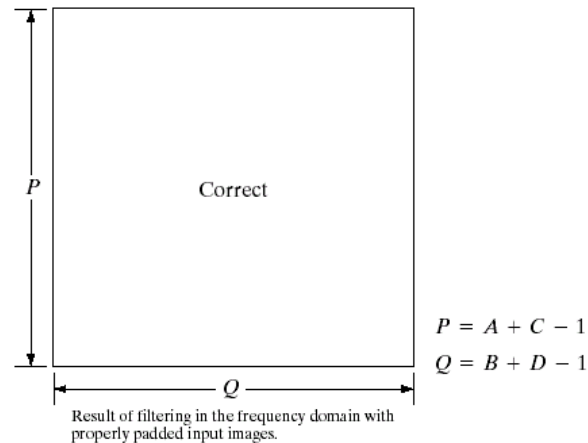


a b  
c

**FIGURE 4.38**

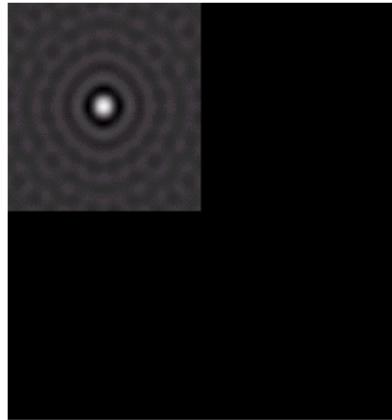
Illustration of the need for function padding.

(a) Result of performing 2-D convolution without padding.  
(b) Proper function padding.  
(c) Correct convolution result.



# Example

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**FIGURE 4.39** Padded lowpass filter in the spatial domain (only the real part is shown).



**FIGURE 4.40** Result of filtering with padding. The image is usually cropped to its original size since there is little valuable information past the image boundaries.

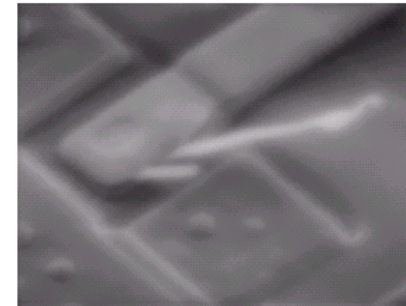
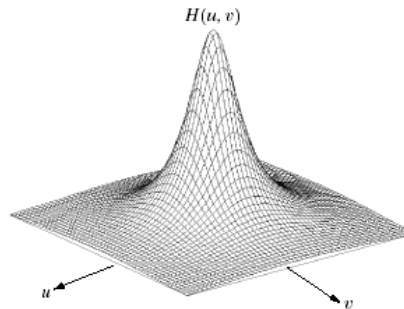
# Frequency Bands

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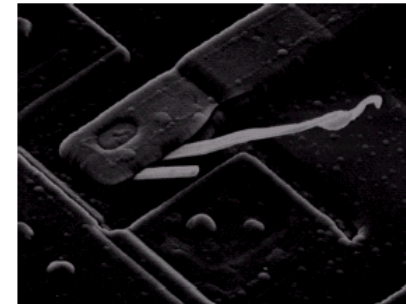
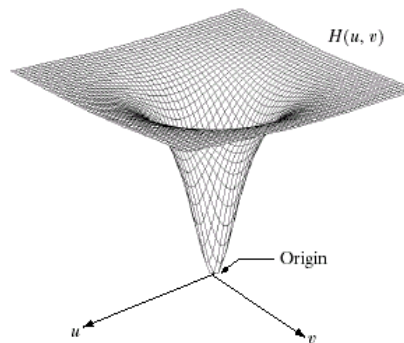
- Low frequencies: general graylevel appearance over smooth areas: slowly varying grayscales.
- High frequencies: responsible for detail, such as edges and noise.
- A filter that attenuates high frequencies while “passing” low frequencies is a *lowpass filter*.
- A filter that attenuates low frequencies while “passing” high frequencies is a *highpass filter*.
- Lowpass filters blur images.
- Highpass filters highlight edges.

# Lowpass and Highpass Filters

Lowpass filter



Highpass filter



a b  
c d

**FIGURE 4.7** (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

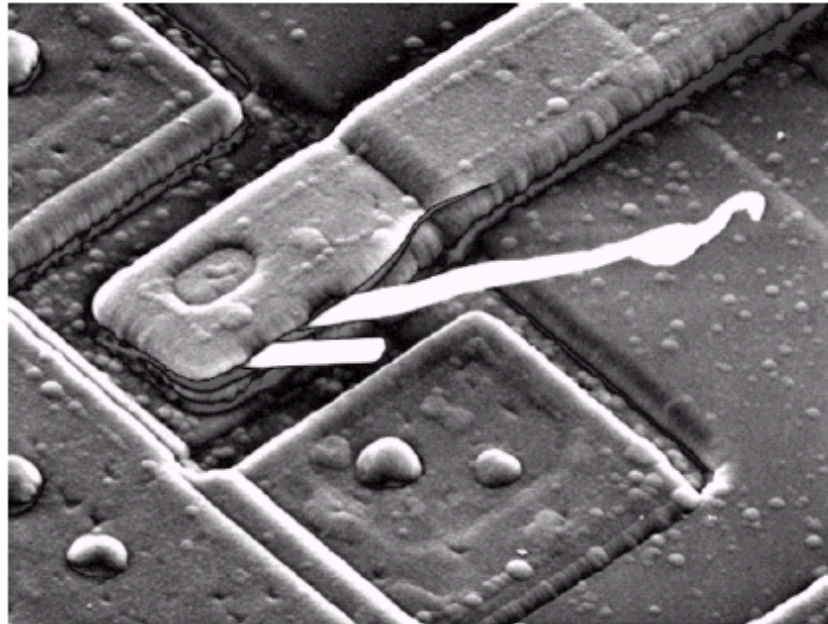
# Improved Highpass Output

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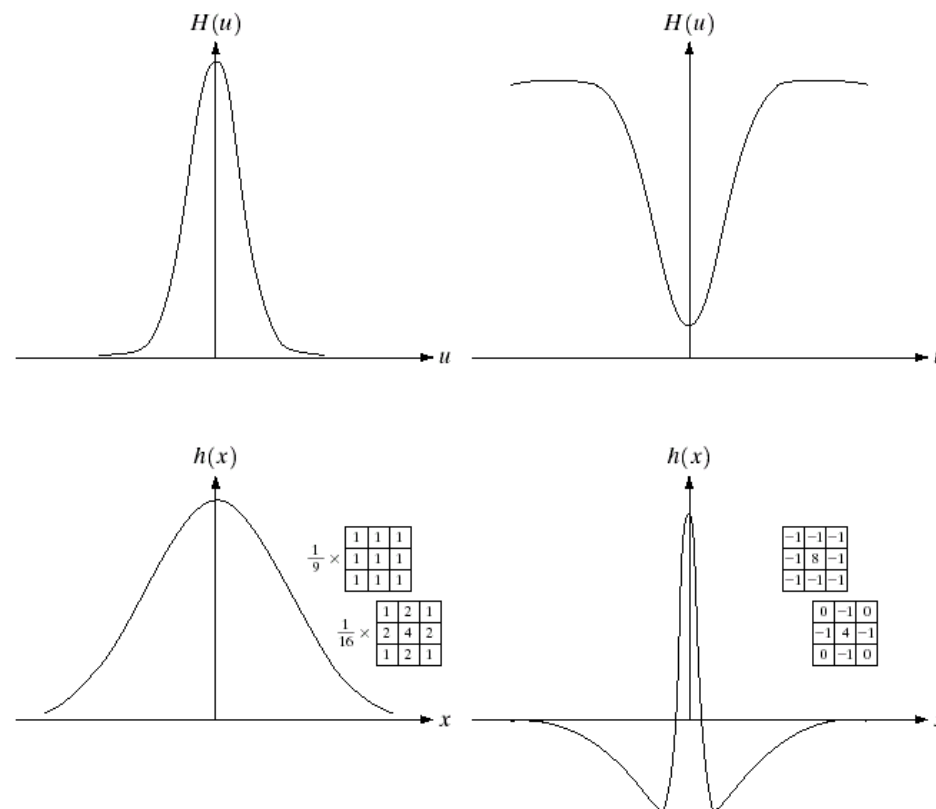
- Add constant to filter so that it will not eliminate  $F(0,0)$

**FIGURE 4.8**

Result of highpass filtering the image in Fig. 4.4(a) with the filter in Fig. 4.7(c), modified by adding a constant of one-half the filter height to the filter function. Compare with Fig. 4.4(a).



# Corresponding Filters in the Spatial and Frequency Domains



a b  
c d

**FIGURE 4.9**

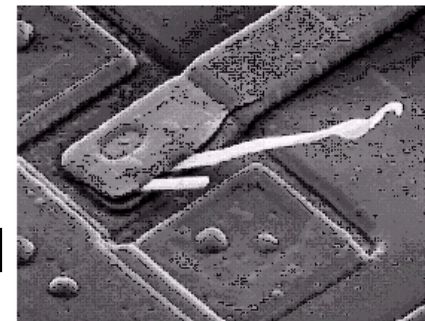
(a) Gaussian frequency domain lowpass filter.  
(b) Gaussian frequency domain highpass filter.  
(c) Corresponding lowpass spatial filter.  
(d) Corresponding highpass spatial filter. The masks shown are used in Chapter 3 for lowpass and highpass filtering.

# Notch Filter

- $F(0,0)$  represents the average value.
- If  $F(0,0) = 0$ , then average will be 0.
- Achieved by applying a notch filter:

$$H(u, v) = \begin{cases} 0 & \text{if } (u, v) = (M/2, N/2) \\ 1 & \text{otherwise} \end{cases}$$

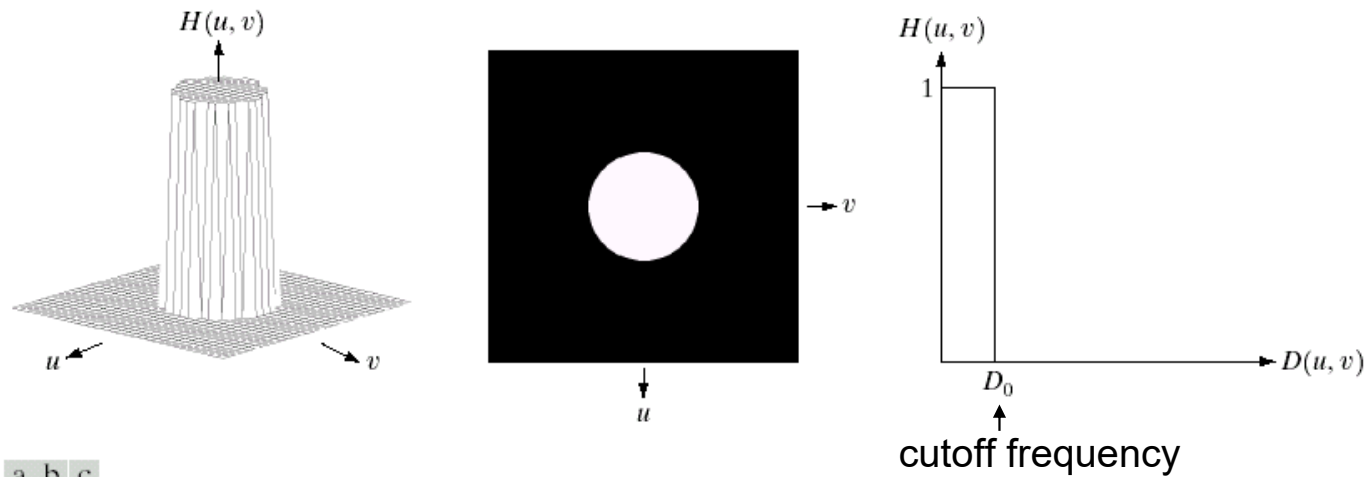
- In reality the average of the displayed image can't be zero as it needs to have negative gray levels. The output image needs to scale the graylevel.
- Filter has notch (hole) at origin.





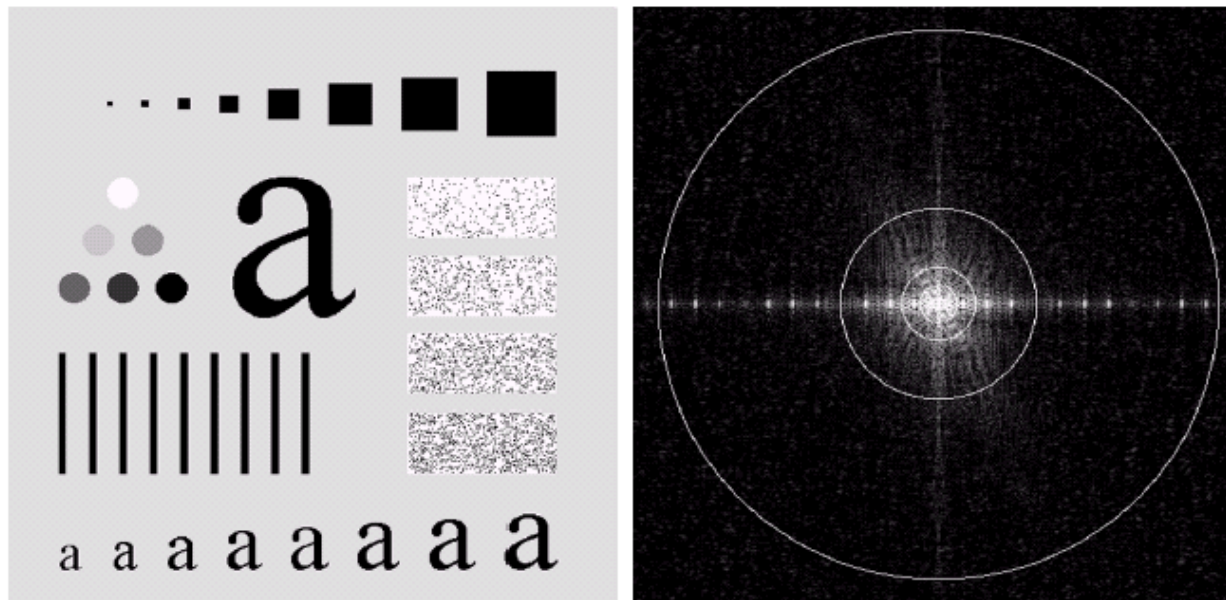
# Ideal Lowpass Filter

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**FIGURE 4.10** (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

# Image Power Circles

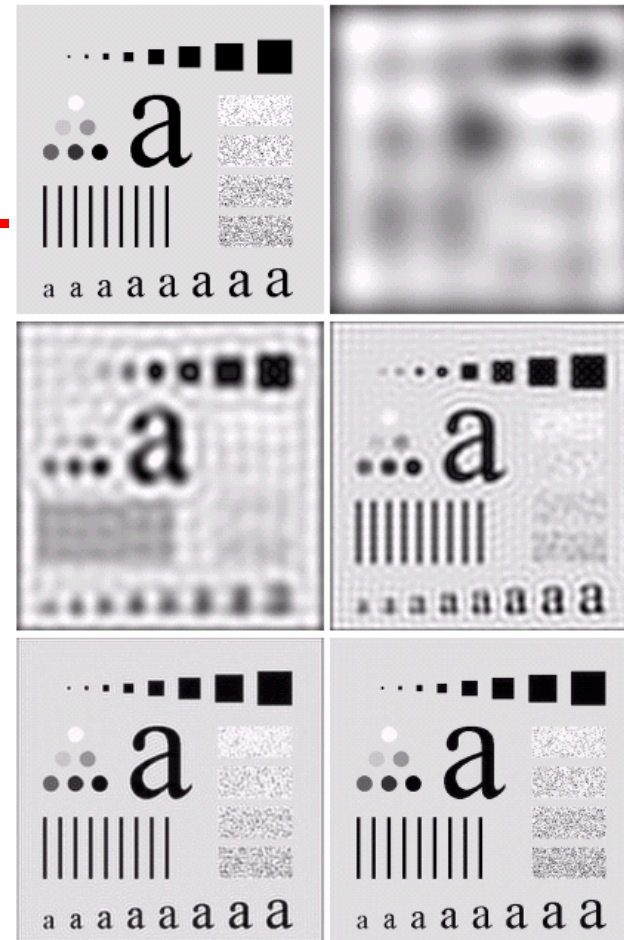


a b

**FIGURE 4.11** (a) An image of size  $500 \times 500$  pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

## Varying ILPF Cutoff Frequencies

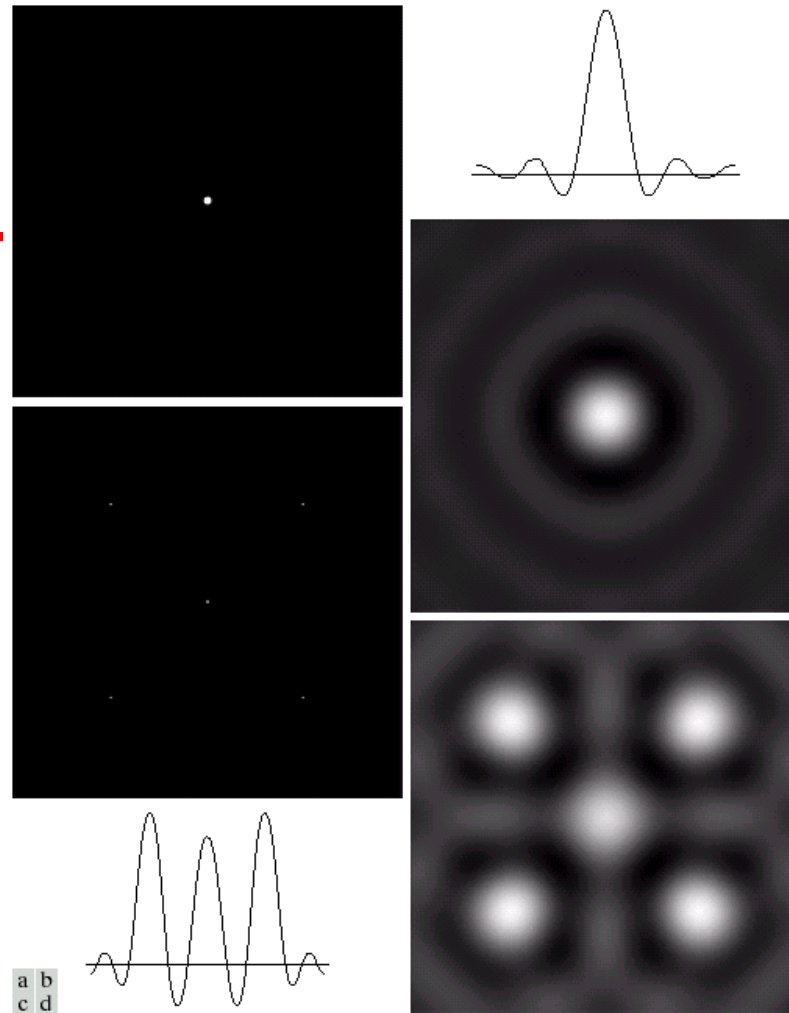
- Most sharp detail in this image is contained in the 8% power removed by filter.
- Ringing behavior is a characteristic of ideal filters.
- Little edge info contained in upper 0.5% of spectrum power in this case.



**FIGURE 4.12** (a) Original image. (b)–(f) Results of ideal lowpass filtering with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). The power removed by these filters was 8, 5.4, 3.6, 2, and 0.5% of the total, respectively.

## ILPF Ringing

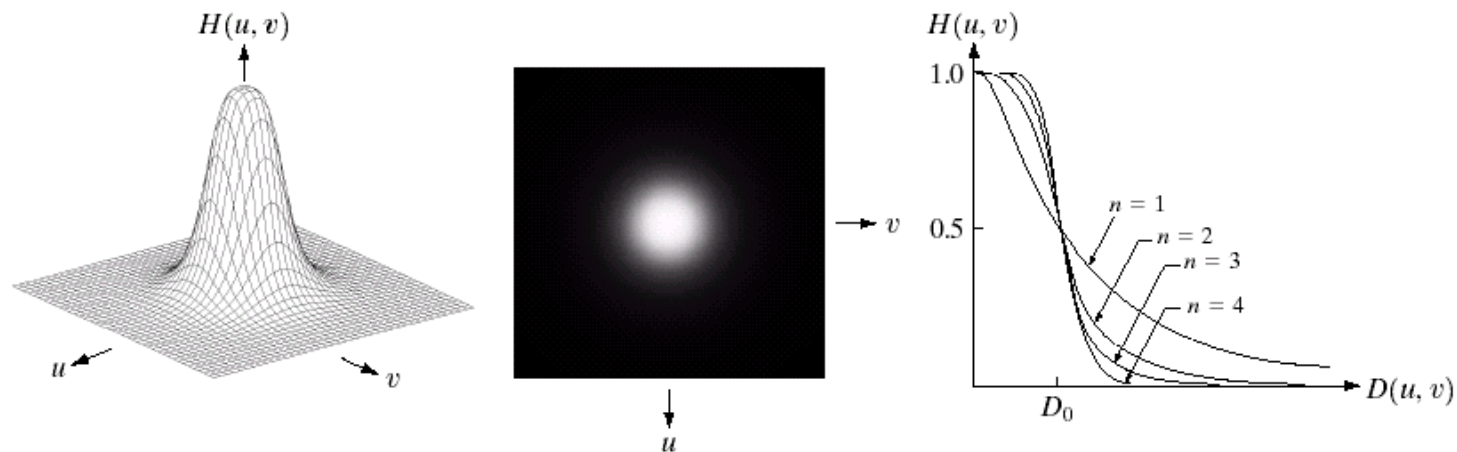
- Ringing in spatial filter has two characteristics: dominant component at origin and concentric circular components.
- Center component responsible for blurring.
- Concentric components responsible for ringing.
- Radius of center comp. and #circles/unit distance are inversely proportional to cutoff frequency.



**FIGURE 4.13** (a) A frequency-domain ILPF of radius 5. (b) Corresponding spatial filter (note the ringing). (c) Five impulses in the spatial domain, simulating the values of five pixels. (d) Convolution of (b) and (c) in the spatial domain.

# Butterworth Lowpass Filter (BLPF)

$$H(u, v) = \frac{1}{1 + [D(u, v) / D_0]^{2n}} \leftarrow n \text{ is filter order}$$

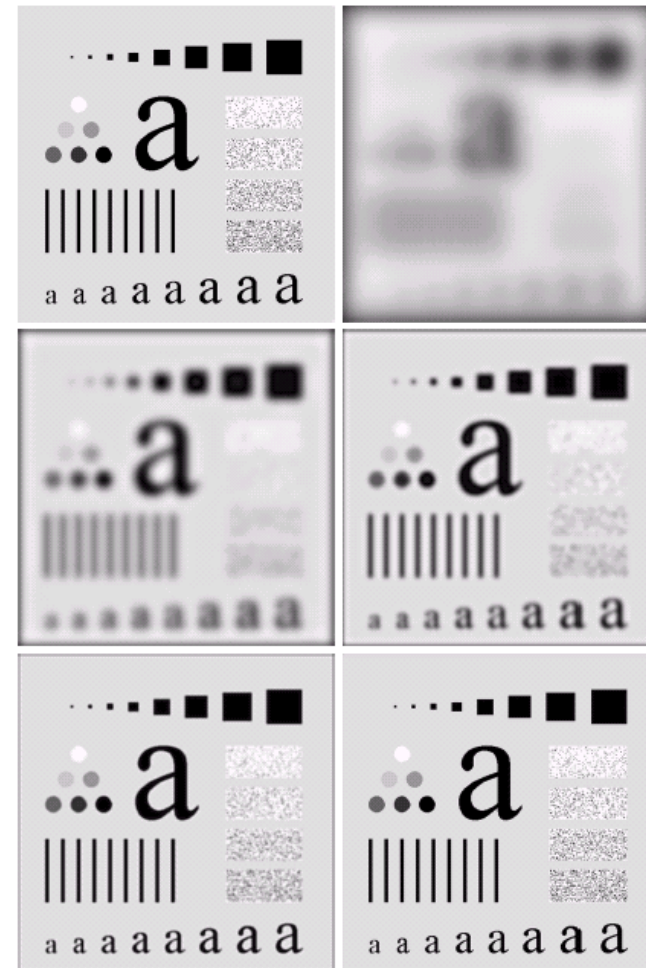


a b c

**FIGURE 4.14** (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

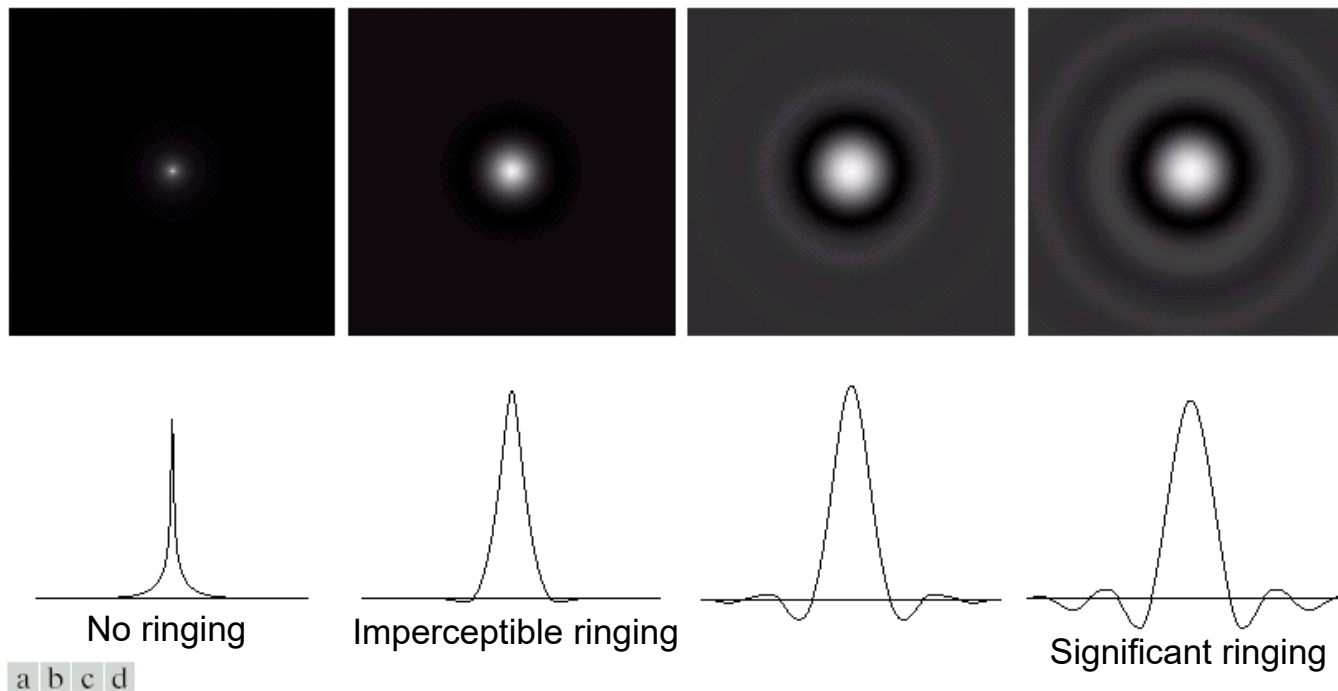
## Varying BLPF Cutoff Frequencies

- Unlike the ILPF, BLPF does not have a sharp discontinuity at the cutoff frequency.
- At  $D_0$ ,  $H(u,v)$  is down 50% from max value of 1.
- Smooth transition in blurring as cutoff frequency increases.
- No ringing due to filter's smooth transition between low and high frequencies.



**FIGURE 4.15** (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

# Spatial Representation of BLPFs

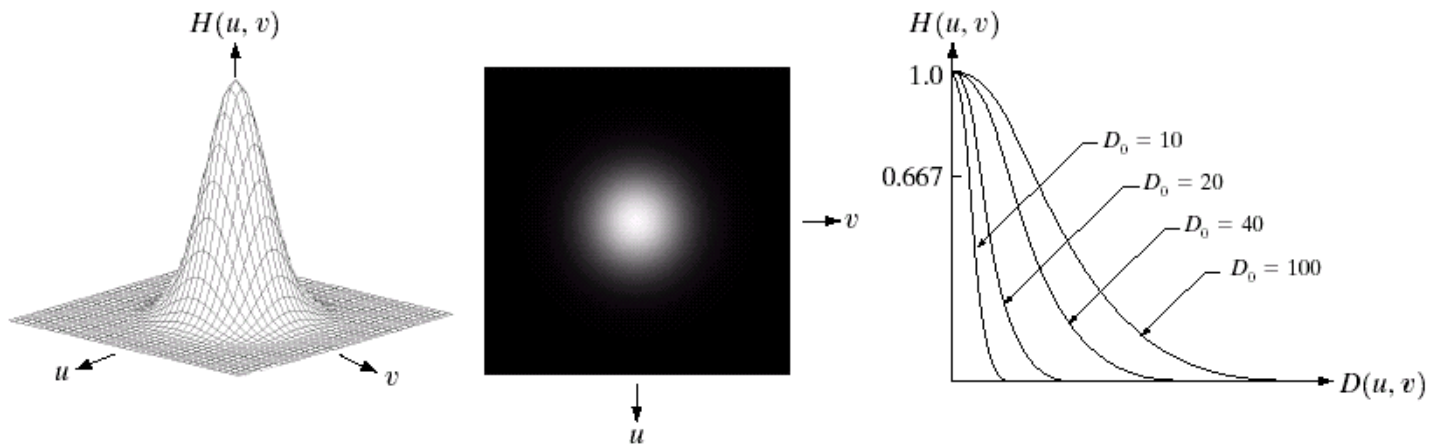


**FIGURE 4.16** (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.



# Gaussian Lowpass Filter (GLPF)

$$H(u, v) = e^{-D^2(u, v) / 2D_0^2} \leftarrow \sigma = D_0 = \text{cutoff freq}$$



a b c

**FIGURE 4.17** (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of  $D_0$ .



## Varying GLPF Cutoff Frequencies

- Smooth transition in blurring as cutoff frequency increases.
- GLPF did not achieve as much smoothing as BLPF of order 2 for same cutoff freq.
- GLPF profile is not as tight as profile of the BLPF of order 2.
- No ringing.
- BLPF is more suitable choice if tight control is needed around cutoff frequency. Only drawback: possible ringing.

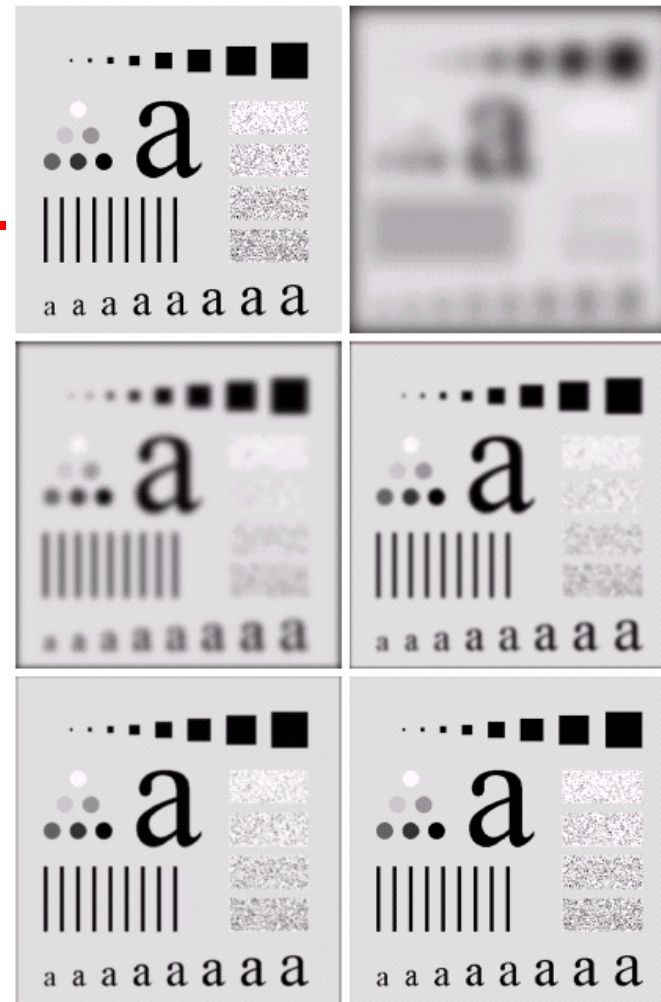


FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

# Example (1)

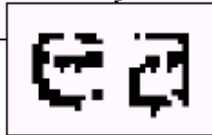
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a b

**FIGURE 4.19**

(a) Sample text of poor resolution (note broken characters in magnified view).  
(b) Result of filtering with a GLPF (broken character segments were joined).

Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.



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## Example (2)

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a b c

**FIGURE 4.20** (a) Original image ( $1028 \times 732$  pixels). (b) Result of filtering with a GLPF with  $D_0 = 100$ . (c) Result of filtering with a GLPF with  $D_0 = 80$ . Note reduction in skin fine lines in the magnified sections of (b) and (c).

## Example (3)

- Crude but simple way of removing prominent scan lines.
- Blurs out detail while leaving large features recognizable.
- Averages out features smaller than the ones of interest.



a b c

**FIGURE 4.21** (a) Image showing prominent scan lines. (b) Result of using a GLPF with  $D_0 = 30$ . (c) Result of using a GLPF with  $D_0 = 10$ . (Original image courtesy of NOAA.)

# Highpass Filters: $1-H_{LP}$

## Ideal highpass filter

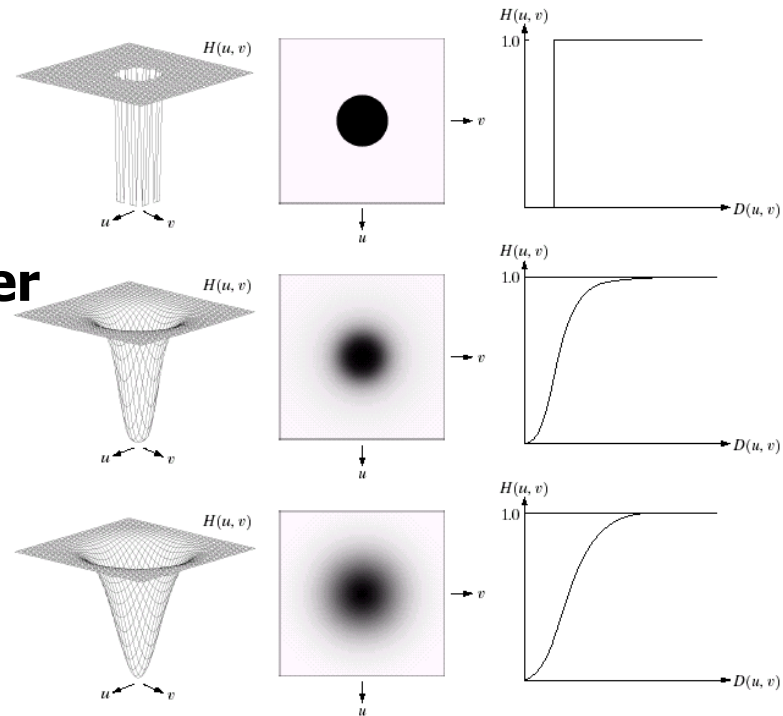
$$H(u, v) = \begin{cases} 0 & \text{if } D(u, v) \leq D_0 \\ 1 & \text{if } D(u, v) > D_0 \end{cases}$$

## Butterworth highpass filter

$$H(u, v) = \frac{1}{1 + [D_0/D(u, v)]^{2n}}$$

## Gaussian highpass filter

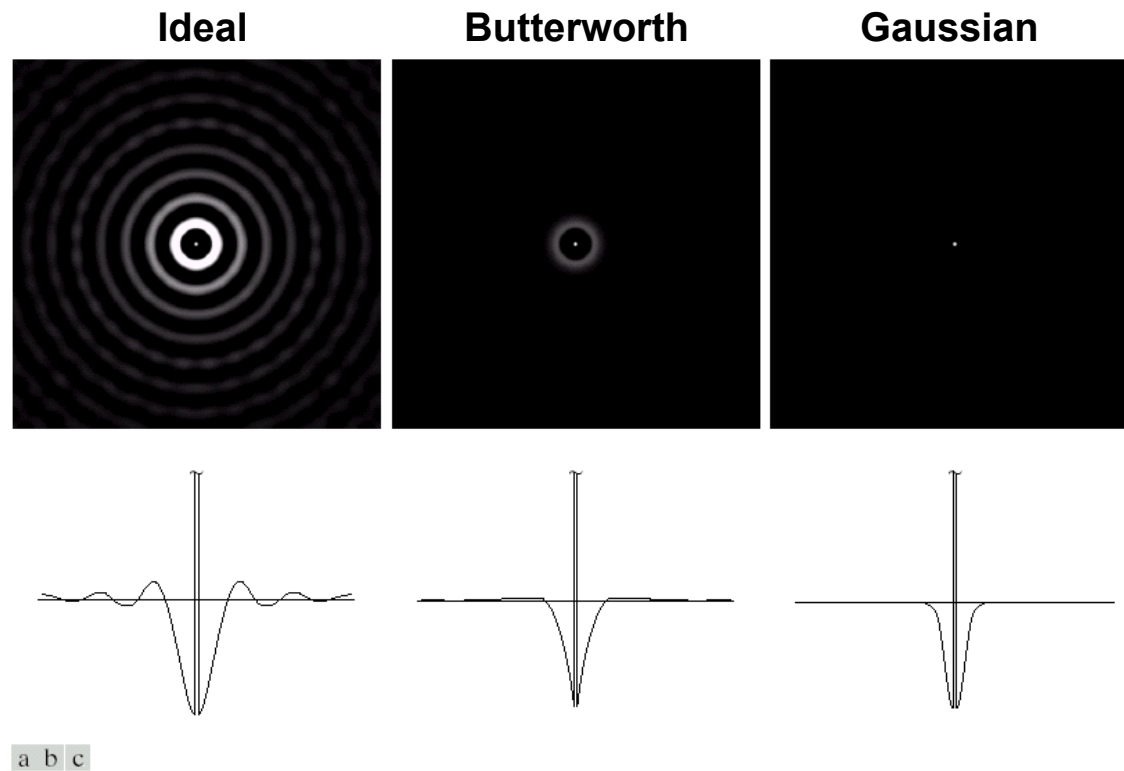
$$H(u, v) = 1 - e^{-D^2(u, v)/2D_0^2}$$



a b c  
d e f  
g h i

FIGURE 4.22 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

# Spatial Representation of Filters

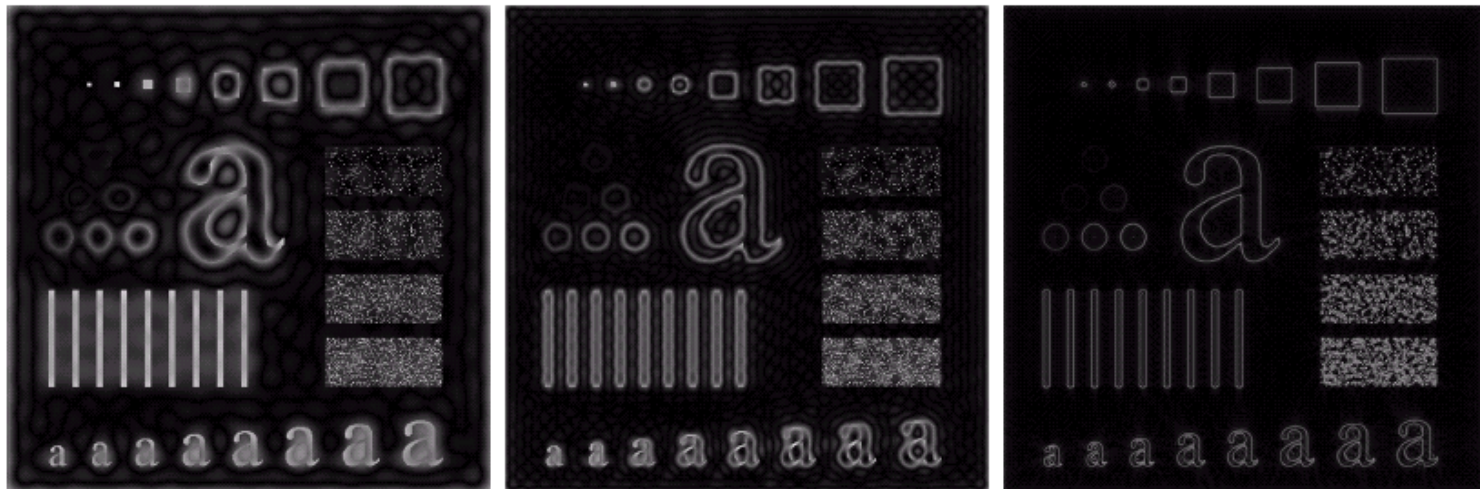


**FIGURE 4.23** Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.



## Varying IHPF Cutoff Frequencies

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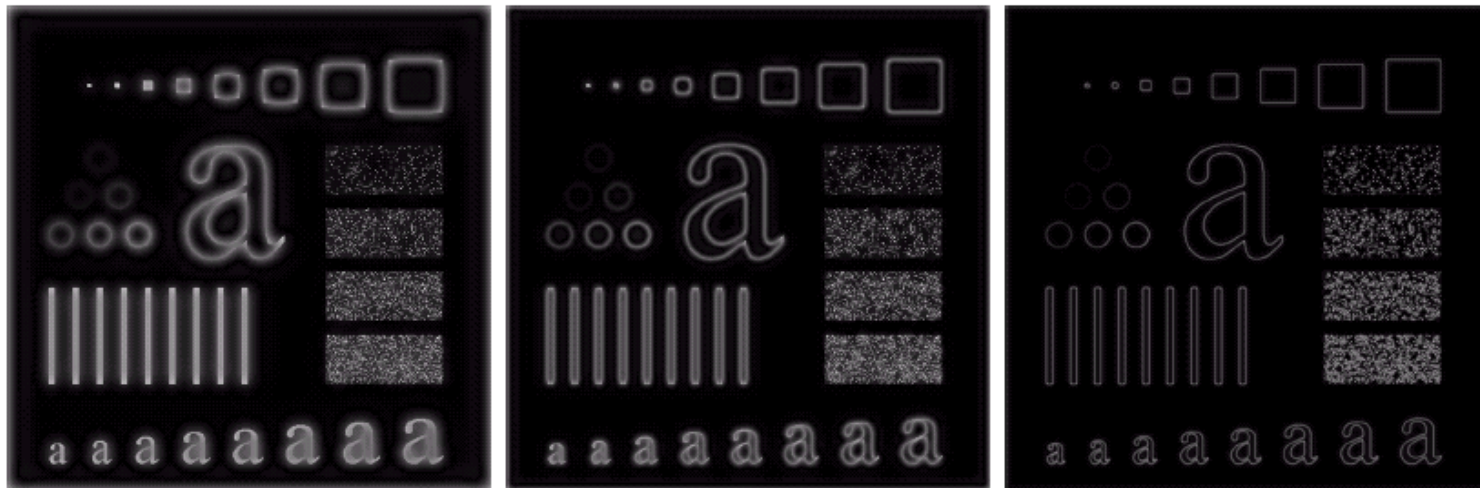


a b c

**FIGURE 4.24** Results of ideal highpass filtering the image in Fig. 4.11(a) with  $D_0 = 15, 30,$  and  $80,$  respectively. Problems with ringing are quite evident in (a) and (b).

## Varying BHPF Cutoff Frequencies

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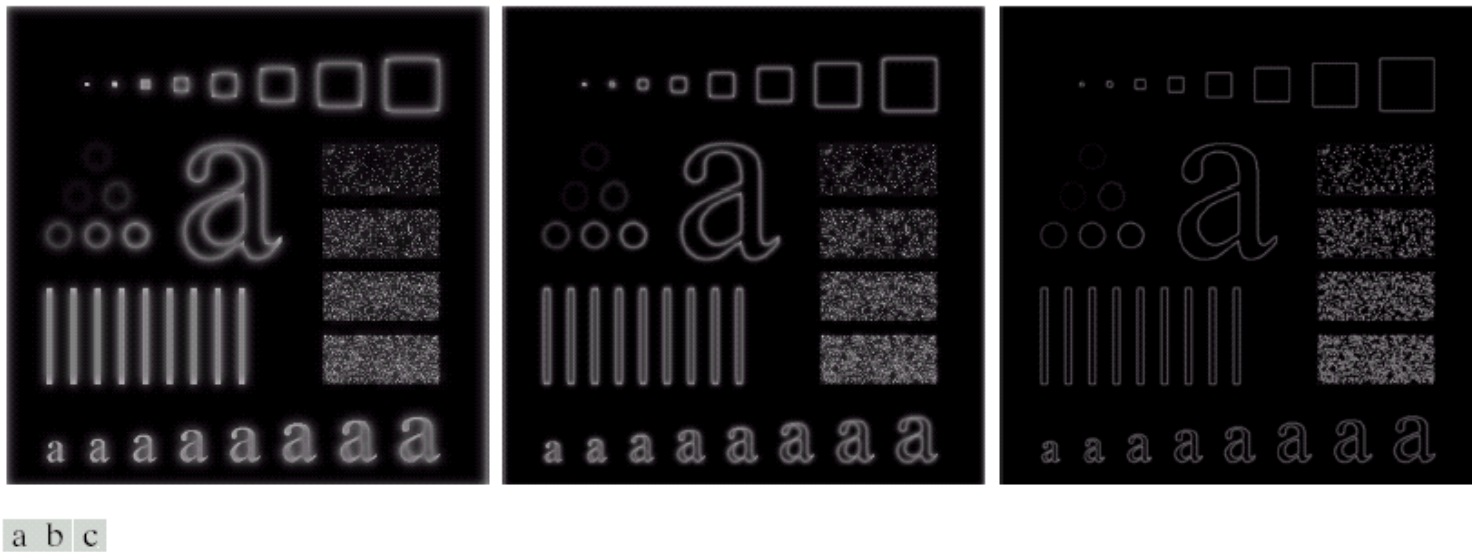
a b c

**FIGURE 4.25** Results of highpass filtering the image in Fig. 4.11(a) using a BHPF of order 2 with  $D_0 = 15$ , 30, and 80, respectively. These results are much smoother than those obtained with an ILPF.



## Varying GHPF Cutoff Frequencies

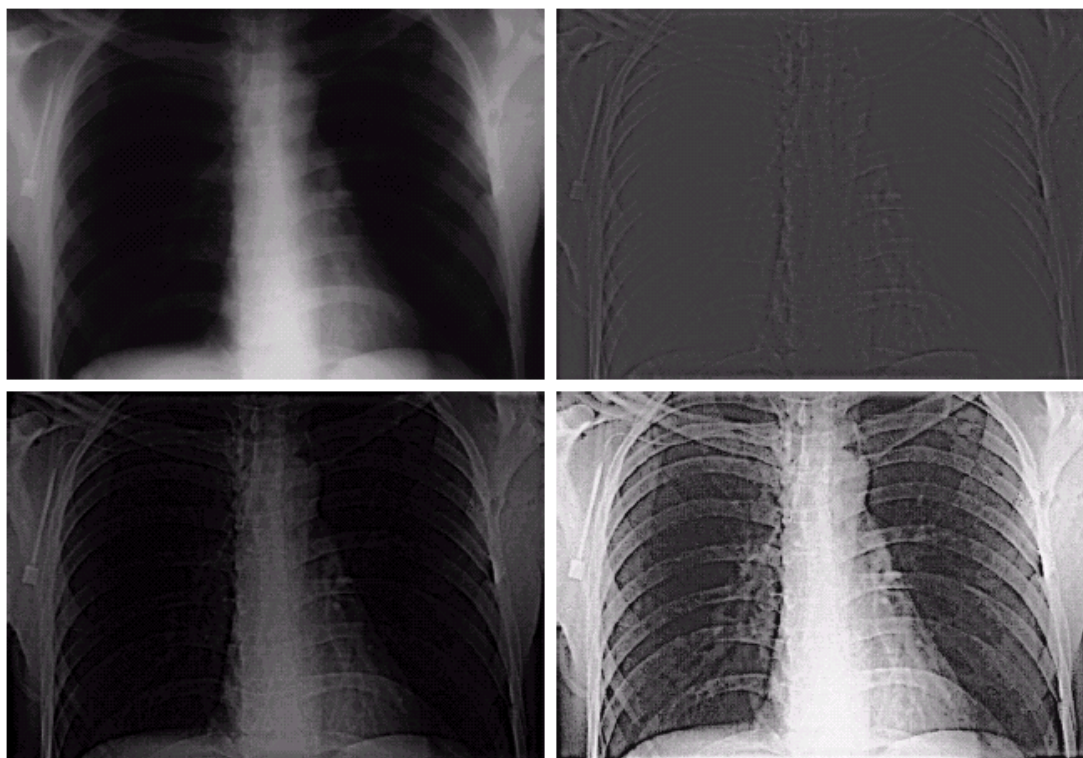
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**FIGURE 4.26** Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with  $D_0 = 15$ , 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.

# Example

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a	b
c	d

**FIGURE 4.30**

(a) A chest X-ray image. (b) Result of Butterworth highpass filtering. (c) Result of high-frequency emphasis filtering. (d) Result of performing histogram equalization on (c). (Original image courtesy Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

# Homomorphic Filtering (1)

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- Consider an image to be expressed as:

$$f(x,y) = i(x,y) r(x,y)$$

↑            ↑  
illumination   reflectance

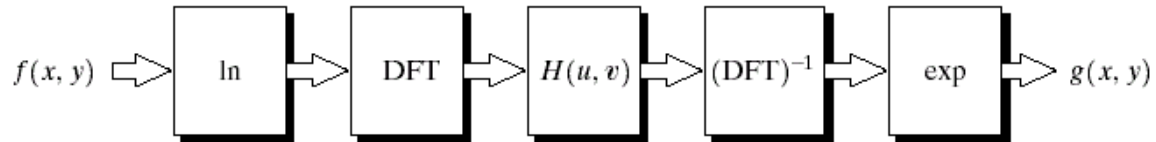
- Take the logarithm of  $f(x,y)$  to separate  $i(x,y)$  and  $r(x,y)$ :

$$\ln f(x,y) = \ln i(x,y) + \ln r(x,y)$$

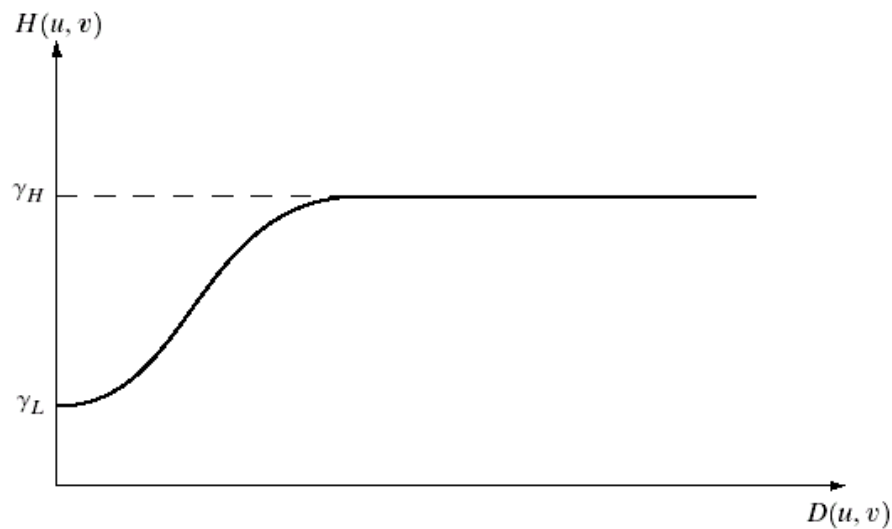
- Illumination component: slow spatial variations
- Reflectance component: tends to vary abruptly
- Filter out low frequencies to remove illumination effects.

# Homomorphic Filtering (2)

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**FIGURE 4.31**  
Homomorphic filtering approach for image enhancement.



**FIGURE 4.32**  
Cross section of a circularly symmetric filter function.  $D(u, v)$  is the distance from the origin of the centered transform.

## Example

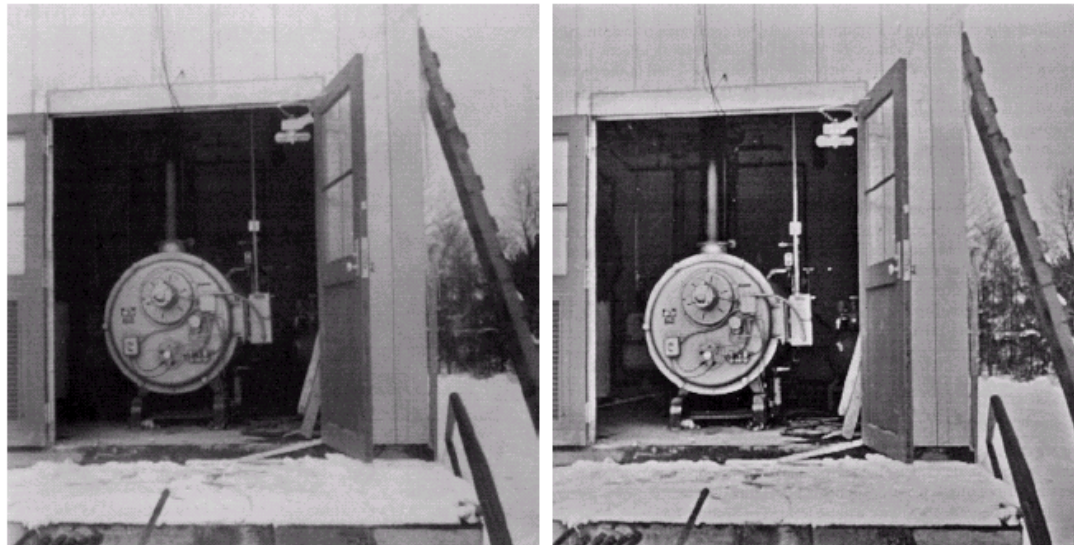
- Interior details are obscured by glare from outside walls.
- Reduction of dynamic range in brightness, together with an increase in contrast, brought out interior details and balanced the graylevels of the outside wall.

a b

**FIGURE 4.33**  
(a) Original image. (b) Image processed by homomorphic filtering (note details inside shelter). (Stockham.)

$$\gamma_L = 0.5$$

$$\gamma_H = 2.0$$



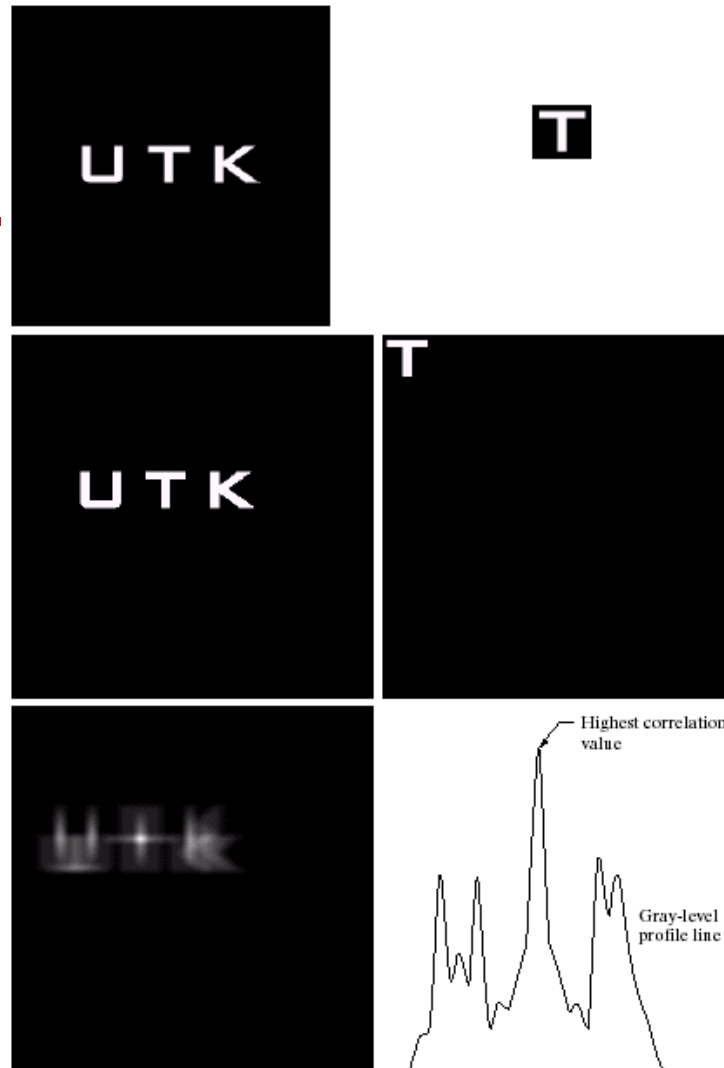
# Correlation

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- Identical to convolution except that kernel is not flipped.
- Kernel is now referred to as the template.
- It is the pattern that we seek to find in the larger image.

$$f(x, y) \circ h(x, y) \Leftrightarrow F^*(u, v)H(u, v)$$

## Example (1)



a	b
c	d
e	f

**FIGURE 4.41**

(a) Image.  
 (b) Template.  
 (c) and  
 (d) Padded  
 images.  
 (e) Correlation  
 function displayed  
 as an image.  
 (f) Horizontal  
 profile line  
 through the  
 highest value in  
 (e), showing the  
 point at which the  
 best match took  
 place.

## Example (2)

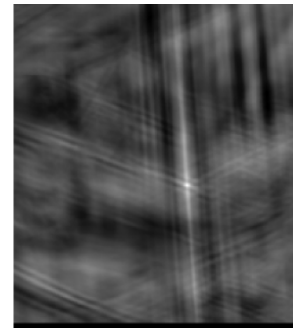
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**Reference image**



**Template**



**Correlation image**

Peak corresponds to best match location