Value at Risk: A Brief Review

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Variance-Covariance Approach

For a portfolio with open position (OP), we would like to derive value at risk (VaR) of portfolio in a time interval T. The OP is assumed to be constituted of hedged energy products (base and peak) such that

$$OP = OP_b + OP_p, \tag{1}$$

where OP_b and OP_b are respectively the open positions of base and peak products. Assuming correlation between the base and peak product prices, VaR is derived as following

$$\text{VaR} = -z_{1-\alpha}\sqrt{T}\sqrt{\left(OP_b\ p_b\right)^2\ \sigma_b^2 + \left(OP_p\ p_p\right)^2\ \sigma_p^2 + 2\rho\ \sigma_b\sigma_p OP_b\ p_b OP_p\ p_p}, \tag{2}$$

where $z_{1-\alpha}$ is the $1-\alpha$ quantile of the standard normal distribution. Furthermore, p_{base} and p_{peak} are respectively the settlement prices of base and peak products, and σ_{base} and σ_{base} are respectively the volatility of product prices, and ρ is the correlation between base and peak prices [3, 2]. The latter is estimated from historical data (usually using the prices in the last 60 days). The VaR is calculated as

$$\text{VaR}^{(e)} = -z_{1-\alpha}\sqrt{T}\Big(OP_b\ p_b\ \sigma_b + OP_p\ p_p\ \sigma_p\Big). \tag{3}$$

This is derived by assuming $\rho = 1$, i.e., instead of computing ρ from historical prices.

Note, in PFÜ the VaR for a portfolio with multiple sub-portfolios is calculated as the sum of VaRs calculated for each sub-portfolio, i.e., assuming that the VaR for such a portfolio is subadditive. The question is if this holds generally. This does not hold generally if the returns are not Gaussian. If the returns are Gaussian, then the VaR is subadditive (see [1] **Remark1** and **Example 12**).

Monte Carlo Approach

To apply this method, first, we need to create assumptions for the underlying stochastic process corresponding to the market prices. This is explained next.

Geometric Brownian Motion

We assume that the random process S_t associated with price values as a function of time is a geometric Brownian motion (GBM; Note that a GBM is also an Itô process).

No sub-portfolio

- We assume S_t to follow a GBM with constant volatility σ and constant drift μ which satisfies the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma dB_t, \tag{4}$$

  where B_t is a wiener process (or equivalently a Brownian motion; https://en.wikipedia.org/wiki/Wiener_process).

- In practice, μ and σ are estimated from historical prices. They correspond to the mean and standard deviation of returns (price changes).

- Given that S_t satisfies (4), by Itô's lemma (https://en.wikipedia.org/wiki/It%C3%B4%27s_lemma), if $X_t = f(t, S_t)$ is a twice differentiable function, in the limit $dt \to 0$ we derive

$$dX_t = \left(\frac{\partial X_t}{\partial t} + \mu S_t \frac{\partial X_t}{\partial S_t} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 X_t}{\partial S_t^2}\right) dt + \sigma \frac{\partial X_t}{\partial S_t} dB_t. \tag{5}$$

- Since we are interested in returns (price changes), we let $f(t, S_t) = \log(S_t)$. It is easy to see that $\frac{\partial X_t}{\partial S_t} = \frac{1}{S_t}$ and $\frac{\partial^2 X_t}{\partial S_t^2} = -\frac{1}{S_t^2}$. By letting $Z_t =$ Then we will have

$$\log(\frac{S_t}{S_0}) = \big(\mu - \frac{\sigma^2}{2}\big) dt + \sigma d(B_t - B_0). \tag{6}$$

  It follows that

$$\log(\frac{S_t}{S_0}) = \big(\mu - \frac{\sigma^2}{2}\big) t + \sigma (B_t - B_0). \tag{7}$$

  Finally, we will have

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma (B_t - B_0)}. \tag{8}$$

  where $B_t - B_0$ is an increment in the Wiener process and has standard normal distribution (see below).

- Since B_t is a Wiener process, it has Gaussian increments, i.e.,

$$Z_t = B_{s+t} - B_s \sim \mathcal{N}(0, t). \tag{9}$$

Hence, we can rewrite (8) as

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z_t}. \tag{10}$$

or equivalently

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z\sqrt{t}}. \tag{11}$$

where $Z \sim \mathcal{N}(0,1)$. For "$T :=$ "$timetoholdandnotbuy/sell$"" this becomes

$$S_T = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma Z\sqrt{T}}. \tag{12}$$

Multiple sub-portfolios

- For a portfolio with m sub-portfolios, let $\mu_1, \mu_2, \ldots, \mu_m$ be the mean (constant drifts) and $\sigma_1, \sigma_2, \ldots, \sigma_m$ be the volatilities of these sub-portfolios price values. In vector form, let $\boldsymbol{\mu} := (\mu_1, \mu_2, \ldots, \mu_m)$ and $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \ldots, \sigma_m)$. Additionally, let

$$\Sigma := \begin{pmatrix} \sigma_1^2 & \sigma_{1,2} & \ldots & \sigma_{1,m} \\ \sigma_{2,1} & \sigma_1^2 & \ldots & \sigma_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m,1} & \sigma_m^2 & \ldots & \sigma_m^2 \end{pmatrix} \tag{13}$$

be the covariance matrix of portfolio's price value. The matrix Σ is symmetric which means $\sigma_{i,j} = \sigma_{j,i}$ for all i and j.

- Furthermore, let $\boldsymbol{S}_T = (S_{1,T}, S_{2,T}, \ldots, S_{m,T})$ be the random (vector) process associated with the portfolio's price value. Then we can rewrite (12) as follows

$$\boldsymbol{S}_T = \boldsymbol{S}_0' \, e^{(\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{\sigma}'\boldsymbol{\sigma})T + \boldsymbol{Z}\sqrt{T}} \tag{14}$$

where $\boldsymbol{Z}$ has multivariate normal distribution as follows

$$\boldsymbol{Z} \sim \mathcal{N}(0, \Sigma) \tag{15}$$

with $\boldsymbol{Z} = \left(Z_1, \ldots, Z_m\right)$ being correlated Gaussian random numbers. The matrix Σ is usually estimated from historical data. Note that $\boldsymbol{X}'$ is the transpose of $\boldsymbol{X}$.

Simulation

- The Monte Carlo simulation procedure is as follows:

1. Specify the number of desired simulated scenarios K, e.g., $K = 1000$.
2. Generate random numbers $\boldsymbol{Z}^{(1)}, \boldsymbol{Z}^{(2)}, \ldots, \boldsymbol{Z}^{(K)}$ are according to (15).

3. Derive $\boldsymbol{S}_T^{(1)}, \boldsymbol{S}_T^{(2)}, \ldots, \boldsymbol{S}_T^{(K)}$ from (14).

4. Calculate the portfolio value for each scenario by

$$V^{(i)} = \boldsymbol{S}_T^{(i)} \boldsymbol{Q}, \tag{16}$$

where $\boldsymbol{Q} = \big(Q_1, \ldots, Q_m\big)'$ is the vector of sub-portfolio quantities.

5. Sort portfolio scenario increasingly.

6. Finally, the value at risk (with $1 - \alpha$ confidence level) is the α percentile of the sorted portfolio scenario values.

Notes

- The variance-covaraince method considers that the profit & loss distribution is symmetric. That results in having the same value at risk in both cases that we need to buy and sell.

- On the other hand, MC method does not assume symmetry and empirically estimates the distribution of profit and loss.

- Note that, if mean returns is positive then the prices are increasing. Hence, the value at risk needs to be higher if we need to buy and lower if we need to sell in order to close the open position. If mean returns are negative then the prices are decreasing, which leads to the exact opposite scenario mentioned above. Among the two aforementioned methods, the MC method generates results which resemble the above considerations. Hence, MC can be considered a better method to calculate VaR.

References

[1] Laura Ballotta and Gianluca Fusai. A gentle introduction to value at risk, 04 2017.

[2] M. Choudhry and C. Alexander. *An Introduction to Value-at-Risk*. Securities Institute. Wiley, 2013.

[3] Otso Ojanen. Otso johannes ojanen comparative analysis of risk management strategies for electricity retailers. *https://api.semanticscholar.org/CorpusID:17547719*, 2002.