Kruskal's Count

LUMS Students' Mathematics Society

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1 What is Kruskal's Count?

Martin Kruskal was an American mathematician and physicist who devised a game now called Kruskal's Count. Our variation of this game involves a deck of 52 playing cards. The player is asked to shuffle the deck and choose a number between 1 and 10 inclusive. They are then asked to follow the following algorithm:

- As the magician reveals the cards one-by-one, the player should count (not aloud!) to their chosen number with each revealed card.
- The card that corresponds to their chosen number is now called the tapped card.
- Each card has a value v equal to the number of alphabets in its denomination, so $v = 4$ for **Jack** and $v = 5$ for **three**.
- The player should choose the value of the tapped card as their chosen number.
- The player should then begin the count again to their chosen number with each revealed card, leading to another tapped card.
- This process should continue until all cards are revealed.
- The magician will then tell, with considerably high probability, the final tapped card of the player.

2 Explanation

2.1 The Strategy

Unbeknownst to the player, the magician also selects a number and follows the same algorithm as the player. With a high probability (which we will attempt to calculate), the magician's and the player's tapped cards will coincide at some point in the deck and eventually follow the same path as each other to reach the final tapped card.

2.2 Assumptions

Now, to calculate the probability, we begin by making a large number of simplifying assumptions which are not true in general. As mentioned previously, each card has a value. We assume that for a given value v, the probability p_v that the card would have the value v is:

$$p_v = p^{v-1}(1-p)$$

Therefore, we are assuming that the values of the cards are taken from a geometric distribution of the random variable X with probability of success $1-p$. We further assume that the player chooses their first number (which we also denote by v) from the same distribution.

Now, the eagle-eyed reader may have noticed that this means that there is a considerable non-zero probability (the calculation of which we leave as an exercise for the reader) of selecting 11 or more as the player's first number when actually, the player can't do that. However, we must make these simplifying assumptions to make our calculations easier.

2.3 Expected Card Value

The expected card value can be found as follows:

$$\begin{aligned} E[X] &= \frac{v_{\text{ACE}} + v_{\text{TWO}} + v_{\text{THREE}} + \cdots + + v_{\text{JACK}} + v_{\text{QUEEN}} + v_{\text{KING}}}{13} \\ &= \frac{3+3+5+4+4+3+5+5+4+3+4+5+4}{13} \\ &= 4 \end{aligned}$$

But we know from the properties of the geometric distribution that $E[X] = \frac{1}{1-p}$. This implies that $p = \frac{3}{4}$. Now, we can rewrite the probability p_v that the card has a value v:

$$p_v = \left(\frac{3}{4}\right)^{v-1}\left(\frac{1}{4}\right)$$

2.4 Coupling Time

We now define the coupling time t as the position in the deck (the first card has position 1) where the magician and the player's tapped cards first coincide. Then, it is easy to see that

$$P\{t=1\} = p_1 \times p_1 = \frac{1}{16}$$

Now, the trick will fail if the coupling time is greater than the number of cards in the deck. Therefore,

$$\begin{aligned}
P(\text{the trick fails}) &= P\{t > 52\} \\
&= P\{t > 52 | t = 1\} P\{t = 1\} + P\{t > 52 | t \neq 1\} P\{t \neq 1\} \\
&= \left(0 \times \frac{1}{16}\right) + P\{t > 52 | t \neq 1\} \left(1 - \frac{1}{16}\right) \\
&= P\{t > 52 | t > 1\} \left(\frac{15}{16}\right)
\end{aligned}$$

We now use the property of memorylessness of geometric distributions which implies that

$$P\{X > m + n | X > m\} = P\{X > n\}$$

Therefore,

$$\begin{aligned}
P\{t > 52\} &= P\{t > 51\} \left(\frac{15}{16}\right) \\
&= P\{t > 51 | t \neq 1\} \left(1 - \frac{1}{16}\right) \left(\frac{15}{16}\right) \\
&= P\{t > 51 | t > 1\} \left(\frac{15}{16}\right)^2 \\
&= P\{t > 50\} \left(\frac{15}{16}\right)^2 \\
&\vdots \\
&= \left(\frac{15}{16}\right)^{52}
\end{aligned}$$

The probability of success is therefore:

$$\begin{aligned}
P(\text{trick is successful}) &= 1 - P\{t > 52\} \\
&= 1 - \left(\frac{15}{16}\right)^{52} \\
&\approx 0.9651
\end{aligned}$$

In our much simplified model, we find the probability that the magician successfully tells the final tapped card of the participant as more than 96.51%.

3 Further Reading

James Grime. “Kruskal’s Count”.

Lagarias, Jeffrey C.; Rains, Eric; Vanderbei, Robert J. (2009) [2001-10-13]. “The Kruskal Count” (PDF). *The Mathematics of Preference, Choice and Order. Essays in Honor of Peter J. Fishburn.* Berlin / Heidelberg, Germany: Springer-Verlag. pp. 371–391.