On the Causal Structure of Spacetime

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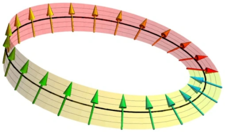
Procedure

- Impose physically reasonable conditions on a spacetime.
- Formulate them in precise mathematical terms.
- Derive important properties and *physical* consequences.

Time Orientation

Definition

A *time-orientable* Lorentzian manifold if a continuous, non-contradictory choice of a future (or equivalently, past) light cone can be made throughout it. It is equivalent to the existence of a non-vanishing timelike vector field t^α on M.



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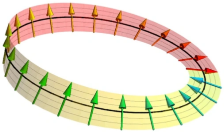

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Theorem

Every Lorentzian manifold (M, g) *has a time-orientable double cover* $(\tilde{M}, \tilde{g})$.

Definition

A differentiable curve $\gamma : \mathbb{R} \to M$ is called *future-directed timelike* (resp. *causal*) if the tangent vector at every point $p \in \gamma$ is a future-directed timelike vector (resp. timelike or null vector).

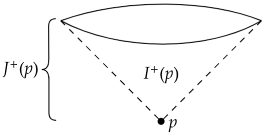

Figure 2. $I^+(p)$ denotes the interior of the light cone at p. $J^+(p)$ includes the boundaries as well.

Domains of Influence

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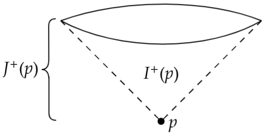
Definition

The *chronological future* $I^+(p)$ (resp. *causal future* $J^+(p)$) of a point $p \in M$ is the set of points $q \in M$ such that a future-directed timelike (resp. causal) curve exists between p and q.



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A set $S \subset M$ is called *achronal* if no two points in S can be connected by a timelike curve.

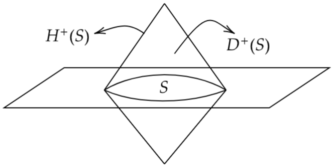


Figure 3. $D^+(S)$ denotes the interior of the cone. $H^+(S)$ is the boundary of said cone.

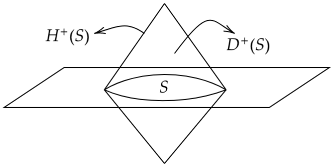
Domains of Dependence

Definition

A set $S \subset M$ is called *achronal* if no two points in S can be connected by a timelike curve.

Definition

The *future domain of dependence* of S, $D^+(S)$, of a closed achronal set $S \subset M$ is the set of points $p \in M$ such that every past inextendible causal curve through p intersects S. The boundary of $D^+(S)$ is called the *future Cauchy horizon* $H^+(S)$.



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A Hierarchy of Increasingly Causal Spacetimes

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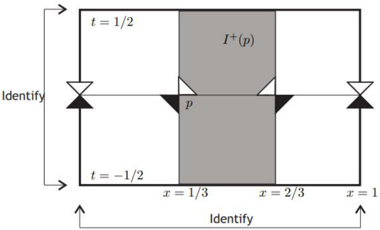

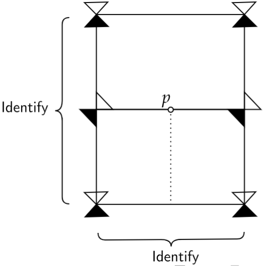

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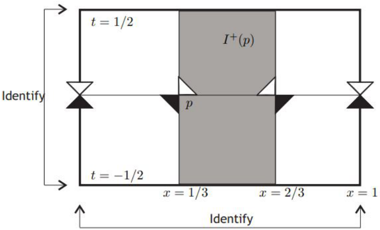

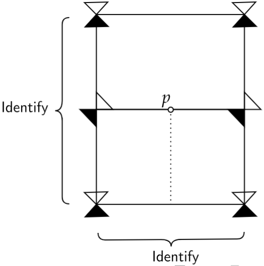

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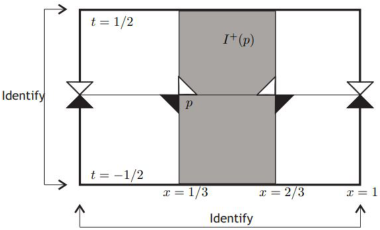

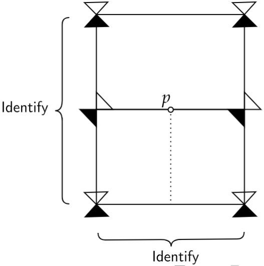

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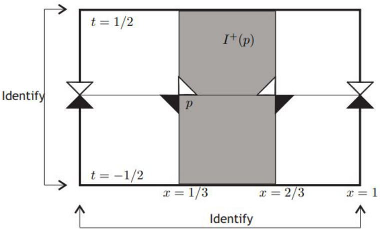
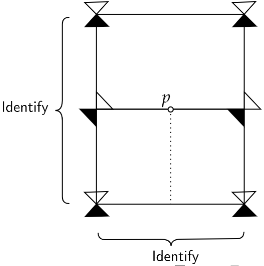
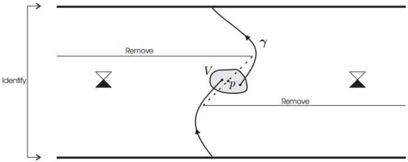
- A *chronological* spacetime has no closed timelike curves.
- A *causal* spacetime has no closed causal curves.
- A *future-distinguishing* spacetime has the property that $I^+(p) = I^+(q) \implies p = q$.

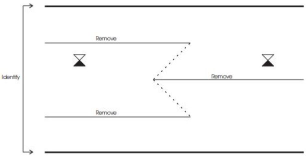




The Hierarchy Continued

- A *strongly causal* spacetime has a neighbourhood $V \subset O$ for every neighbourhood O of every point p such that any causal curve with endpoints in V lies completely in O. There are no causal curves that come arbitrarily close to intersecting themselves.





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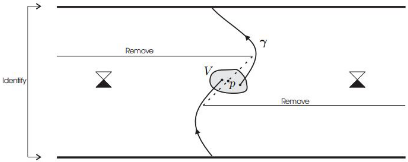
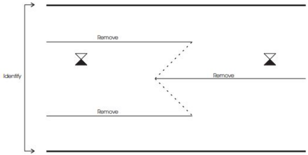
- A *strongly causal* spacetime has a neighbourhood $V \subset O$ for every neighbourhood O of every point p such that any causal curve with endpoints in V lies completely in O. There are no causal curves that come arbitrarily close to intersecting themselves.
- A *stably causal* spacetime has a cosmic time function i.e. a function whose gradient is everywhere timelike. Intuitively, if you slightly open the light cones, there will still be no closed timelike curves.





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- Global hyperbolicity is the strongest causality condition.

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Consequences

- Global hyperbolicity is the strongest causality condition.
- The existence of a Cauchy surface Σ allows one to hypothetically evaluate the entire past and future evolution of the manifold given initial conditions on Σ.

References

- Minguzzi, E.; Sanchez, M. “The causal heirarchy of spacetimes”, in H. Baum and D. Alekseevsky (eds.), vol. Recent developments in pseudo-Riemannian geometry, ESI Lect. Math. Phys., (Eur. Math. Soc. Publ. House, Zurich, 2008), p. 299 – 358, ISBN=978-3-03719-051-7.
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