# Flip the Dice

LUMS Students' Mathematics Society

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## 1 What is Flip the Dice?

The game consists of a finite number of dice $n$ that are rolled by the player. The dice are placed left to right in the order that they are rolled. A **move** consists of the following steps:

- Choose 2 dice where the left one (call it $A$) has a higher number on top than the right one (call it $B$).
- Flip $A$ upside down.
- Swap the dice.

There will come a point where no more moves are possible. In other words, the game will end in a finite number of moves for any $n$.

## 2 Explanation

Let the number on the top face of the $i$th die be $k_i$, counting from right to left. Then, we can define the sequence of dice as a number in base$-10$ representation.

**Brief digression**: Technically, we can define the number in any base$-m$ representation where $m \geq 7$ is an integer. However, most people are comfortable with the base$-10$ or decimal representation.

The sequence of dice then becomes an integer $s$:

$$\begin{aligned} s &= \underbrace{k_n k_{n-1} \cdots k_2 k_1}_{n \text{ terms}} \\ &= 10^{n-1}k_n + 10^{n-2}k_{n-1} + \cdots + 10k_2 + k_1 \end{aligned}$$

Let's observe what a move does to $n$. Say we choose $A$ and $B$ to be the $(i+1)$th and $i$th dice respectively so that $k_{i+1} > k_i$. Note that if a face of a die has $a$

dots on it, the opposite face has $7 - a$ dots. Let $s'$ be the new integer after the move has been performed. Then,

$$\begin{aligned} s &= 10^{n-1}k_n + 10^{n-2}k_{n-1} + \cdots + 10^i k_{i+1} + 10^{i-1}k_i + \cdots + 10k_2 + k_1 \\ s' &= 10^{n-1}k_n + 10^{n-2}k_{n-1} + \cdots + 10^i k_i + 10^{i-1}(7 - k_{i+1}) + \cdots + 10k_2 + k_1 \end{aligned}$$

Subtracting the two gives:

$$\begin{aligned} s - s' &= 10^i k_{i+1} + 10^{i-1}k_i - 10^i k_i - 10^{i-1}(7 - k_{i+1}) \\ &= 10^i(k_{i+1} - k_i) + 10^{i-1}k_{i+1} + k_i - 7 \\ &= 10^{i-1}(11k_{i+1} - 9k_i - 7) \end{aligned}$$

Now, we know that $k_{i+1} - k_i \geq 1$. Then,

$$\begin{aligned} 11k_{i+1} - 9k_i &> 11k_{i+1} - 11k_i \\ &\geq 11 \\ &> 7 \\ \implies s - s' &> 0 \end{aligned}$$

Therefore, any move definitely decreases $s$. Since $s$ must always be positive (remember, $1 \leq k_i \leq 6$ for all $i$ because it is a die), $s$ continues to decrease until the lower limit is reached in a finite number of moves.