Kepler's Laws and Selected Problems
A PHY 100 Project

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Abstract: I derive Kepler's laws of planetary motion from Newton's second law and the law of gravitation. Ellipses and their properties are discussed in connection with Kepler's first law. The energy distribution in an elliptical orbit is outlined after proving why energy is conserved there. Finally, some assorted problems (including the Earth-Moon system, Halley's comet and a meteoroid approaching Earth) are solved.

I. KEPLER'S FIRST LAW AND PROPERTIES OF ELLIPSES

Johannes Kepler first published his laws of planetary motion in the early 17th century. They describe the motion og the planets around the sun with surprising accuracy. The first law states:
The orbit of every planet is an ellipse with the Sun at one focus.
We attempt to derive it from Newton's law of universal gravitation.

Consider a system of a planet and the sun. We convert the two-body problem to a one-body problem by using the reduced mass μ:

$$\mu = \frac{Mm}{M+m} \tag{1}$$

where m is the mass of the planet and M is the mass of the sun.

The only force acting on the planet is the gravitational force of the sun which is directed towards the center of the sun. Therefore, according to Newton's second law:

$$\mu \frac{d^2 r}{dt^2} = -\frac{GMm}{r^2} \tag{2}$$

where r is the distance between the centres of the two bodies and G is the gravitational constant.

Note that the line of action of this force passes through the center of the planet as can be seen in Fig. 1. This implies that there is no torque on the planet and hence, the angular momentum $L = \mu r^2 \frac{d\theta}{dt}$ of the system must remain constant.

Since there is no external force acting on the system, the law of conservation of energy holds. The total energy E is:

$$E = \frac{1}{2}\mu v^2 - \frac{GMm}{r} \tag{3}$$

The velocity v has a radial component $\frac{dr}{dt}$ and a tangential component $r\frac{d\theta}{dt}$. Therefore, we can write the velocity as $v^2 = (\frac{dr}{dt})^2 + (r\frac{d\theta}{dt})^2$ and angular velocity as $\frac{d\theta}{dt} = \frac{L}{\mu r^2}$. The energy equation then becomes:

$$E = \frac{1}{2}\mu \left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2\mu r^2} - \frac{GMm}{r} \tag{4}$$

Solving for $\frac{dr}{dt}$, we get:

$$\frac{dr}{dt} = \sqrt{\frac{2E}{\mu} - \frac{L^2}{\mu^2 r^2} + \frac{2GMm}{\mu r}} \tag{5}$$

We need to find r in terms of θ, so we use $\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}$ and the equation reduces to:

$$\frac{dr}{d\theta} = \frac{r}{L}\sqrt{2\mu E r^2 + 2\mu GMmr^2 - L^2} \tag{6}$$

All we need to do now is solve the integral

$$\int d\theta = L \int \frac{1}{r\sqrt{2\mu E r^2 + 2\mu GMmr^2 - L^2}} dr \tag{7}$$

By convention, we set the constant of integration $\theta_0 \equiv$

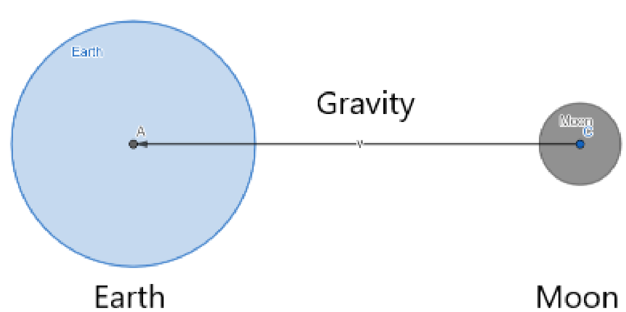

FIG. 1. Gravity acting through the centers of the bodies

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$-\frac{\pi}{2}$. Solving and rearranging, we get

$$r = \frac{L^2/(\mu GMm)}{1 - \sqrt{1 + 2EL^2/(\mu G^2 M^2 m^2)} \cos\theta} \tag{8}$$

This is equivalent to the standard ellipse equation in polar coordinates $r = \frac{r_0}{1-\epsilon\cos\theta}$ and we get

$$r_0 = \frac{L^2}{\mu GMm} \tag{9}$$

$$\epsilon = \sqrt{1 + \frac{2EL^2}{\mu G^2 M^2 m^2}} \tag{10}$$

While Kepler based his laws on empirical observation, we have proved them via Newton's theory. There are some interesting consequences of the law. Since the ellipse is a two-dimensional figure, all planetary orbits are confined to one plane. Most of the planets have nearly circular orbits (the circle is a form of an ellipse) because their eccentricity is close to 0.

Kepler's first law describes the planetary motions very well. However, we have made some approximations by neglecting the effects of the gravity of other planets to convert a many-body problem into a one-body problem. Moreover, even if we take those forces into account, Kepler's first law, like all Newtonian motion, remains an approximation. Einstein's general relativity comes into play on the larger scales with the precession of Mercury's perihelion an example. Furthermore, the planetary orbits are capable of chaotic motion. That, however, is beyond the scope of our report.

A. Mathematics of Ellipses

An ellipse is a closed curve in 2-dimensional Euclidean space defined by the polar and Cartesian equations (with the origin as its center):

$$r = \frac{r_0}{1 - \epsilon\cos\theta} \tag{11}$$

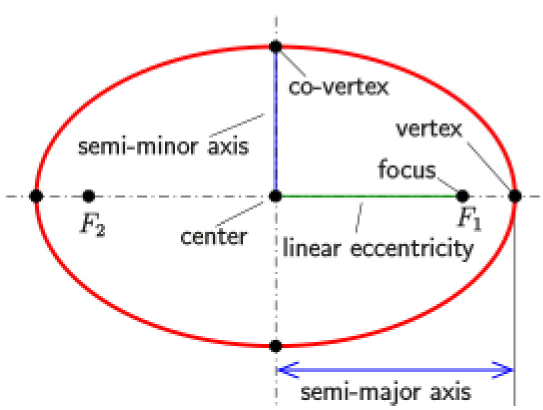

FIG. 2. Features of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{12}$$

Using the figure, we explain some of the important properties of ellipses. All expressions are written using the standard Cartesian equation.

Semi-major axis: It is the distance from the center to the vertex which is further away. It is denoted by a.

Semi-minor axis: It is the distance from the center to the vertex which is closer. It is denoted by b.

Semi-latus rectum: It is half the length of the chord through one focus perpendicular to the major axis. It can be expressed as $l = \frac{b^2}{a}$.

Linear eccentricity: It is the distance between the center and one focus. It can be expressed as $c = \sqrt{a^2 - b^2}$.

Eccentricity: It is the ratio between the linear eccentricity and the semi-major axis. It is a measure of the closeness between the ellipse and a circle. It can be expressed as $\epsilon = \frac{c}{a}$.

Perihelion: It is the point of closest approach in an orbit and the distance at perihelion is $r_p = \frac{r_0}{1+\epsilon}$.

Aphelion: It is the point of furthest approach in an orbit and the distance at aphelion is $r_p = \frac{r_0}{1-\epsilon}$.

II. DERIVATION OF KEPLER'S SECOND AND THIRD LAWS

Kepler's second law states:

A line between the Sun and a planet sweeps out equal areas in equal intervals of time.

Consider a system of the planet and the sun as before. The only force on the planet is the gravitational force from the sun. The position vector of the planet is at 180° to the force vector. Thus, the torque about the sun $\vec{\tau} = \vec{r} \times \vec{F}$ is 0. It follows that the angular momentum L of the system remains constant.

Let us look at the area dA swept out by the line as it moves through a small angle $d\theta$. As Fig. 3 shows, this small element is, in fact, the same as a sector of a circle with radius r and angle $d\theta$. The area of this sector is:

$$dA = \frac{1}{2} r^2 d\theta \tag{13}$$

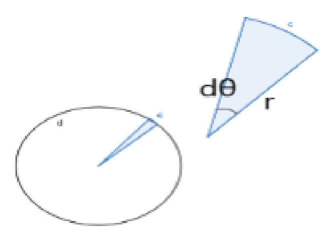

FIG. 3. Features of an ellipse

Dividing throughout by dt, we get

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} \tag{14}$$

But we know from the definition of angular momentum that $\frac{d\theta}{dt} = \frac{L}{\mu r^2}$. The equation then becomes

$$\frac{dA}{dt} = \frac{L}{2\mu} \tag{15}$$

Since both L and μ are constant, it follows that in equal intervals of time dt, equal areas dA will be swept out.

Kepler's third law states:

The square of a planet's orbital period is directly proportional to the cube of the semi-major axis of its orbit.

First, let us write the semi-major axis in polar coordinates:

$$a = \frac{r_{\text{min}} + r_{\text{max}}}{2} \tag{16}$$

$$a = \frac{r_0}{2}\left(\frac{1}{1+\epsilon} + \frac{1}{1-\epsilon}\right) \tag{17}$$

$$a = \frac{r_0}{1-\epsilon^2} \tag{18}$$

Putting in the values of r_0 and ϵ, we get

$$a = \frac{L^2/(\mu GMm)}{1 - 1 + 2EL^2/(\mu(GMm)^2)} \tag{19}$$

And finally,

$$a = -\frac{GMm}{2E} \tag{20}$$

To prove this law, we solve equation 5 using limits such that the time period is found. We, therefore, solve this integral:

$$\int_0^T dt = \mu \int_r^r \frac{r}{\sqrt{2\mu E r^2 + 2\mu GMmr - L^2}} dr \tag{21}$$

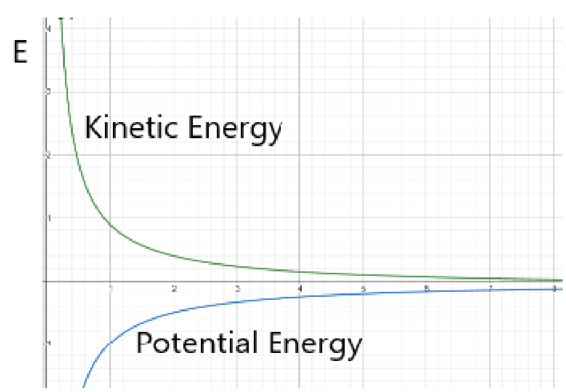

FIG. 4. Distribution of E_k and E_p with increasing r

For an ellipse, $E < 0$. Solving the integral by parts yields

$$t\Big|_0^T = \frac{\sqrt{2\mu E r^2 + 2\mu GMmr - L^2}}{2E}\Bigg|_r^r$$
$$- \left(\frac{\mu GMm}{2E\sqrt{-2\mu E}}\right) \sin^{-1}\left(\frac{-2\mu E r - \mu GMm}{\sqrt{\mu^2 (GMm)^2 + 2\mu E L^2}}\right)\Bigg|_r^r$$

And we finally get

$$T = \left(\frac{\mu GMm}{2E\sqrt{-2\mu E}}\right)(2\pi) \tag{22}$$

Squaring both sides yields

$$T^2 = -\frac{\pi^2 \mu (GMm)^2}{2E^3} \tag{23}$$

Finally, combining equations (20) and (23), we get

$$T^2 = \frac{4\pi^2 \mu}{GMm} a^3 \tag{24}$$

Hence, we have proved that $T^2 \propto a^3$.

III. ENERGY CONSERVATION IN ELLIPTICAL ORBITS

The total mechanical energy is conserved in elliptical orbits. According to the Law of Conservation of Energy, energy is conserved in **isolated** systems. In an elliptical orbit, we define our system to be object A which is revolving around object B. We do not take into consideration any gravitational forces that may be acting on the objects inside our system from objects outside it. Since there is no external force acting on our defined system and the only force doing work is internal, the energy is conserved in elliptical orbits.

Using equation (20), we find an expression for E in terms of a:

$$E = -\frac{GMm}{2a} \tag{25}$$

This will allow us to get an expression for kinetic energy in terms of r:

$$E_k = GMm\left(\frac{1}{r} - \frac{1}{2a}\right) \tag{26}$$

The potential energy in terms of r is:

$$E_p = -\frac{GMm}{r} \tag{27}$$

As Fig. 4 and equations show, when the distance r from the focus increases, E_k decreases while E_p increases (becomes less negative). As r decreases, the opposite happens. However, the total energy E of the system does not depend on the distance from the focus. Instead, E remains constant throughout since it is only dependent on the semi-major axis of the orbit.

IV. SELECTED PROBLEMS

Below we solve some selected problems to get us better acquainted with Kepler's laws and planetary orbits.

A. The Earth-Moon System

Calculate the velocity of the Moon in a circular orbit around the Earth. What kind of trajectory does the theory predict for velocity greater than or less than the velocity of this circular orbit? Calculate the velocity for which the Moon will escape the orbit of the Earth and the Earth-Moon system will become unbound.

In the Earth-Moon system, let M be the mass of the Earth and m be the mass of the moon. Let v_0 be the velocity for the circular orbit. The angular momentum of the system is conserved and is given by $L = \mu v_0 r$. We know the eccentricity of the orbit via equation (10), the total energy of the orbit via equation (25) and the reduced mass by equation 1. For a circular orbit, $a = r$ and $\epsilon = 0$. We therefore get:

$$1 + \frac{2EL^2}{\mu(GMm)^2} = 0 \tag{28}$$

$$\left(-\frac{GMm}{2r}\right)\left(\frac{Mm}{M+m}v_0 r\right)^2 = -\frac{\frac{Mm}{M+m}(GMm)^2}{2} \tag{29}$$

$$v_0 = \sqrt{\frac{G(M+m)}{r}} \tag{30}$$

Putting in the values in SI units, we get

$$v_0 = \sqrt{\frac{6.67\times10^{-11}\times(5.97\times10^{24}+7.35\times10^{22})}{3.84\times10^8}} \tag{31}$$

$$v_0 = 1025ms^{-1} \tag{32}$$

From equation (26), we can derive an expression for the semi-major axis in terms of velocity:

$$\frac{1}{a} = \frac{2}{r} - \frac{\mu v^2}{GMm} \tag{33}$$

At $v = v_0$, $a = r$. When v is increased or decreased, the semi-major axis deviates from r and hence, the eccentricity of the orbit increases, leading to an elliptical orbit.

For the Earth-Moon system to become unbound, $\epsilon \geq 1$. The point at which it becomes unbound will be given by the equality:

$$1 + \frac{2EL^2}{\mu(GMm)^2} = 1 \tag{34}$$

$$EL^2 = 0 \tag{35}$$

The angular momentum of the system is constant so $E = 0$.

$$\frac{1}{2}\left(\frac{Mm}{M+m}\right)v^2 - \frac{GMm}{r} = 0 \tag{36}$$

$$v = \sqrt{\frac{2G(M+m)}{r}} \tag{37}$$

$$v = \sqrt{\frac{2\times6.67\times10^{-11}\times(5.97\times10^{24}+7.35\times10^{22})}{3.84\times10^8}} \tag{38}$$

$$v = 1449ms^{-1} \tag{39}$$

B. Halley's Comet

Halley's comet orbits the Sun in an elliptical orbit (the comet reached perihelion in 1986). When the comet is at perihelion, its distance from the Sun is $8.78\times10^{10}m$, and its speed is $5.45\times10^4 ms^{-1}$. When the comet is at aphelion, its distance is $5.28\times10^{12}m$. What is the speed at aphelion?

Consider a system of Halley's comet and the Sun. Let the mass of the comet be m. The total angular momentum L about the Sun remains constant. We have $r_p = 8.78\times10^{10}m$, $v_p = 5.45\times10^4 ms^{-1}$ and $r_a = 5.28\times10^{12}m$. Since the velocity is perpendicular to the distance at both the aphelion and the perihelion, the angular momentum is:

$$L = mv_p r_p = m\times8.78\times10^{10}\times5.45\times10^4 \tag{40}$$

By the conservation of angular momentum,

$$L = mv_a r_a = m\times v_a\times5.28\times10^{12} \tag{41}$$

And finally,

$$v_a = \frac{m\times8.78\times10^{10}\times5.45\times10^4}{m\times5.28\times10^{12}} \tag{42}$$

$$v_a = 9.06\times10^2 ms^{-1} \tag{43}$$

C. Meteorite Approaching Earth

NASA has detected a distant meteorite which is moving at a fast rate towards Earth. Still far away from the Earth, it is moving along a straight line, that if extended, would pass at a distance $3R_E$ from the center of the Earth, where R_E is the Earth's radius. NASA has means to detect the speed of the meteorite. What minimum speed

must the meteorite have if it is not to collide with the Earth's surface? Write down the approximate trajectory of the meteorite.

Consider a system of the Earth and the meteorite (see figure). Assume that the meteorite is currently at a radial distance ∞ from the center of the Earth. Let the mass of the meteorite be m, its original velocity be v_1, its final velocity be v_2 and the mass of the Earth be M. Assume $M >> m$ so that the reduced mass $\mu = m$. The perpendicular distance from the center of the Earth to the meteorite is $3R_E$. At the point of contact, this distance is R_E. The angular momentum of the system $L = mvr_\perp$ is conserved so:

$$mv_1(3R_E) = mv_2(R_E) \tag{44}$$

$$v_2 = 3v_1 \tag{45}$$

Since the only force acting on the meteorite is the gravitational pull of the Earth, total energy is conserved.

$$\frac{1}{2}mv_1^2 - \frac{GMm}{\infty} = \frac{1}{2}m(3v_1)^2 - \frac{GMm}{R_E} \tag{46}$$

$$v_1 = \sqrt{\frac{GM}{4R_E}} \tag{47}$$

$$v_1 = \sqrt{\frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{4 \times 6.38 \times 10^6}} = 3.95 \times 10^3 ms^{-1} \tag{48}$$

With this velocity, the meteorite will only graze the Earth. Therefore, this must be its minimum velocity. The trajectory followed by the meteorite can be found by calculating r_0 and ϵ from equations (9) and (10) respectively:

$$r_0 = \frac{GM}{4R}m^2(3R_E)^2\frac{1}{GMm^2} = \frac{9R}{4} \tag{49}$$

$$\epsilon = \sqrt{1 + 2\frac{\frac{1}{2}m\left(\frac{GM}{4R_E}\right)\left(\frac{GM}{4R}m^2(3R_E)^2\right)}{G^2M^2m^3}} = \frac{5}{4} \tag{50}$$

The trajectory of the meteorite is thus a hyperbola (since $\epsilon > 1$ and it can be defined by the polar equation:

$$r = \frac{9R}{4\left(1 - \frac{5}{4}\cos\theta\right)} \tag{51}$$