Buffon's Needle

LUMS Students' Mathematics Society

14 March 2022

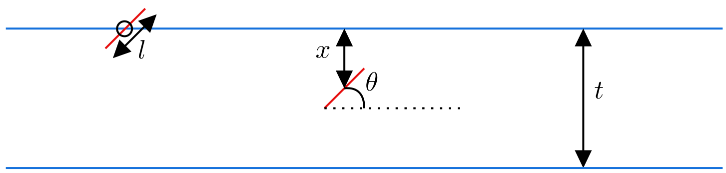
1 What is Buffon's Needle?

Buffon's needle is an experiment designed to approximate π. It is based on a question posed by Georges-Louis Leclerc, Comte de Buffon. The question amounts to finding the probability of a needle of length l landing across a line when dropped onto a plane where parallel lines are drawn t units apart. Here, t and l are specified in the same units. The probability (for $l < t$) comes out to be:

$$P(\text{needle lying across a line}) = \frac{2}{\pi}\frac{l}{t}$$

2 Explanation

Assume that the needle can fall anywhere on the plane with equal probability. It must be true that the needle lies across at most 1 line since $l < t$.



2.1 PDF of X

Let X denote the random variable that takes on the values x corresponding to the perpendicular distance between the center of the needle and the line closest to it as depicted in the figure. Since the distance between any 2 lines is t, the center of a needle can lie at most $\frac{t}{2}$ units away from a given line. Then, assuming that each x is equally probable, we get the probability density function (PDF) of X:

$$p_X(x) = \begin{cases} \frac{2}{t} & 0 \leq x \leq \frac{t}{2} \\ 0 & \text{otherwise} \end{cases}$$

The interested reader should integrate the given probability density function over all values of x to convince themselves that it is correct.

2.2 PDF of Θ

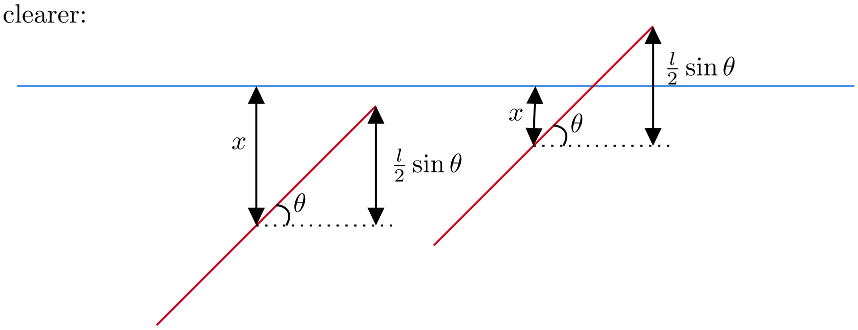
Similarly, let Θ denote the random variable that takes on the values θ corresponding to the acute angle between the needle and the horizontal as shown in the figure. It can easily be seen that θ can be at most $\frac{\pi}{2}$. Then, assuming that each θ is equally probable, we get the PDF of Θ:

$$p_\Theta(\theta) = \begin{cases} \frac{2}{\pi} & 0 \leq \theta \leq \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

Again, the interested reader is encouraged to integrate over all values of θ to convince themselves that the uniform probability distribution is correct.

2.3 Condition on values of X

Now, note that X and Θ are independent, so $P\{X = x \cup \Theta = \theta\} = P\{X = x\}P\{\Theta = \theta\}$. For the needle to lie across a line, for any x, the distance from the top end of the needle to the center in the direction perpendicular to the drawn lines must be less than or equal to x. The following magnified diagram makes it clearer:



2.4 Properties of Continuous Random Variables

Here, we take a small detour to explain some properties of probability distributions. Let Y be some continuous random variable and $p_Y(y)$ be the PDF of Y. Then,

$$P(y_1 \leq Y \leq y_2) = \int_{y_1}^{y_2} p_Y(y)dy$$

Let Z be another continuous random variable independent from Y. Then, the joint PDF $p_{Y,Z}(y, z)$ becomes:

$$p_{Y,Z}(y, z) = p_Y(y)p_Z(z)$$

2.5 Finding the Probability

Returning to the problem at hand, the joint PDF of X and Θ can be integrated to find the probability of a needle lying across a line:

$$\begin{aligned}
P\left(0 \leq \Theta \leq \frac{\pi}{2}, 0 \leq X \leq \frac{l}{2}\sin\theta\right) &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{l}{2}\sin\theta} \frac{2}{\pi}\frac{2}{t}\,dx\,d\theta \\
&= \frac{4}{\pi t}\int_0^{\frac{\pi}{2}} x\Big|_0^{\frac{l}{2}\sin\theta} d\theta \\
&= \frac{4}{\pi t}\int_0^{\frac{\pi}{2}} \frac{l}{2}\sin\theta\,d\theta \\
&= \frac{4}{\pi t}\frac{l}{2}(-\cos\theta)\Big|_0^{\frac{pi}{2}} \\
&= \frac{2l}{\pi t}\left(-\cos\frac{\pi}{2} + \cos 0\right) \\
&= \frac{2l}{\pi t}
\end{aligned}$$

2.6 Approximation of π

The Law of Large Numbers, more or less, argues that if the experiment is conducted N times with $N \to \infty$ and the event occurs n times ($n \leq N$), then the probability of the event can be approximated by:

$$P = \frac{n}{N}$$

Let n be the number of needles that lie across the line when N needles are dropped on the plane. Let $l : t = 1 : 2$. Then,

$$\begin{aligned}
\frac{n}{N} &\approx \frac{2l}{\pi t} \\
\implies \pi &\approx \frac{N}{n}
\end{aligned}$$

3 Further Reading

For the case $l > t$ and more discussion, see

L. Schroeder (1974), "Buffon's needle problem: An exciting application of many mathematical concepts." *Mathematics Teacher*, **67** (2), p. 183 – 186.