On Chaos in Simple Electrical Circuits

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Chaos is ubiquitous in the physical world from the weather to the dripping faucet. The theory of non-linear, chaotic systems is discussed. Three simple electrical circuits (the RLC circuit, the RLD circuit and the double-scroll Chua's circuit), easily constructed in the laboratory, are used to demonstrate non-linear behaviour and chaos qualitatively. First, the differential equation of the RLC circuit is studied to show that it does not display non-linear behaviour. Then, a Proteus simulation of the RLD circuit is used to show that increasing the source potential causes the circuit to follow a period-doubling bifurcation path from order to chaos. Finally, the resistance in the non-linear Chua's circuit is varied experimentally to find its time-series and phase plots using an oscilloscope. The familiar Lorenz attractor is seen.

I. NON-LINEAR DYNAMICS AND CHAOS: AN INTRODUCTION

The major part of the physical world is made up of non-linear systems. These systems exhibit complex behaviour. Despite being described by deterministic equations, the long-term behaviour of the system is impossible to predict. At one time, they might exhibit periodic behaviour reminiscent of linear systems while at another, they may descend into increasingly complex, non-repeating, chaotic behaviour as the value of a parameter is changed. This is because the systems are described by *non-linear* differential equations which do not obey the superposition principle.

Examples of non-linear dynamical systems include the weather, a raging water body or the population dynamics of fish. In the laboratory, though, there are simpler systems that lend themselves to the observation of chaos, for example the double pendulum. One such system is the electrical circuit (consisting of a combination of inductors, resistors, capacitors and diodes), the discussion of which we have devoted this report to.

To understand any non-linear system, one can plot time-series plots, phase portraits, Poincaré sections and bifurcation diagrams. We shall employ the first two in our quest to observe chaos in simple electrical circuits.

II. THE DRIVEN RLC CIRCUIT

The driven RLC ciruit is a prototypical example of a linear electrical circuit - one that does not display chaotic behaviour. This is apparent if we study the differential equation for the circuit.

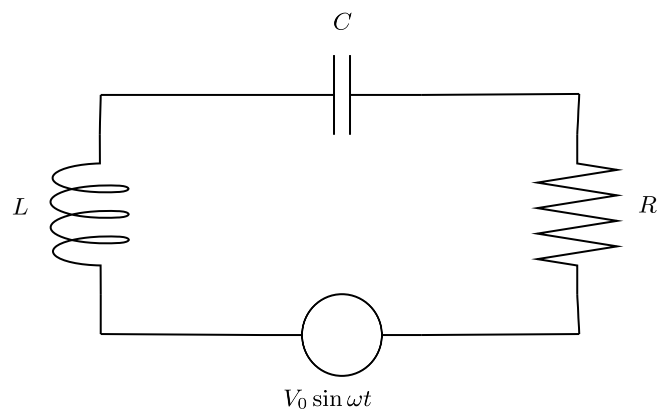

FIG. 1: The Driven RLC Circuit

A. The Schematic of the Circuit

The driven RLC circuit consists of a resistor (R), an inductor (L), a capacitor (C) and a voltage source which in this case is sinusoidal.

B. The Differential Equation

There is only one loop around the circuit to which we can easily apply Kirchhoff's voltage law:

$$-\frac{Q}{C} - L\,\frac{dI}{dt} - IR + V_0 \sin \omega t = 0 \tag{1}$$

where Q is the total charge that passes through the capacitor in some time t i.e. $Q = \int^t I\, dt'$. Differentiating with respect to t gives us the following second-order ordinary differential equation:

$$L\,\frac{d^2 I}{dt^2} + R\,\frac{dI}{dt} + \frac{1}{C} I = V_0 \omega \cos \omega t \tag{2}$$

This can be readily solved using standard methods of solving for the complementary solution (using the homogeneous equation) followed by the particular solution

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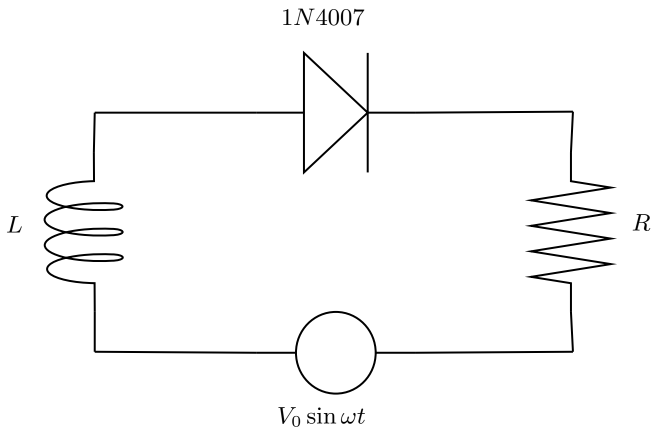


FIG. 2: The Driven RLD Circuit

(including the driving term)[1]. The final answer, after suitable algebraic manipulation, can be written as:

$$I(t) = \frac{V_0}{Z_b} \cos(\omega t - \theta_b) + B e^{-\frac{Rt}{2L}} \cos(\omega_b t - \phi) \tag{3}$$

where $\omega_0^2 = \frac{1}{LC}$, $Z_b = \frac{L}{\omega}\sqrt{\omega_0^2 - \omega^2 + \left(\frac{R\omega}{L}\right)^2}$, $\theta_b = \tan^{-1}\left(\frac{\omega R}{L(\omega_0^2 - \omega^2)}\right)$ and $\omega_b = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}$. B and ϕ are integration constants and depend on the boundary conditions.

As the equation and its solution clearly shows, there is no non-linear term. We are interested in the long-term behaviour of this circuit, so we can ignore the second term in equation (3) since that corresponds to the transient behaviour (which dies off relatively quickly). The (long-term) current in the circuit varies in a periodic manner, specifically as a sinusoidal wave. Different values of V_0 only serve to change the amplitude of the wave but the period remains the same. Therefore, there are no bifurcations and no switch to chaotic behaviour.

III. THE DRIVEN RLD CIRCUIT

To create non-linearity in the RLC circuit, we replace the capacitor with a diode to get the driven RLD circuit. This is one of the simplest circuits which follows a period-doubling path to chaos.

A. The Schematic of the Circuit

The driven RLD circuit consists of a resistor (R), an inductor (L), a diode (D) and a voltage source which in this case is sinusoidal as well. The circuit can be understood as switching between two simpler circuit combinations over time. This is due to the conducting and non-conducting cycles of the diode where it behaves as a battery with a fixed voltage drop V_f and a capacitor

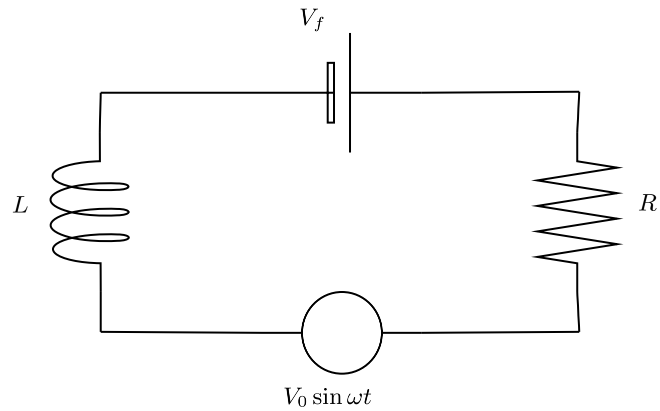


FIG. 3: The driven RLD circuit in the conducting cycle

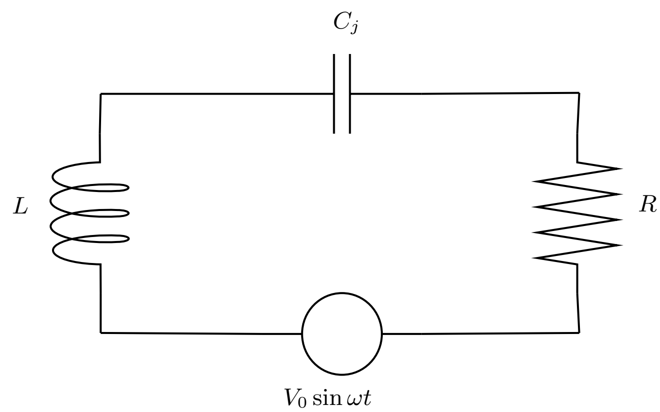


FIG. 4: The driven RLD circuit in the non-conducting cycle

with junction capacitance C_j respectively as FIG. 3 and 4 show.

It is this switching of the circuit dynamics from the conducting to the non-conducting cycles that plays a big role in introducing non-linearity and hence chaotic behaviour in the circuit.

B. The Differential Equation

In the conducting cycle, there is only one loop around the circuit to which we can apply Kirchhoff's voltage law:

$$-L\frac{dI}{dt} + V_f - RI + V_0 \sin \omega t = 0 \tag{4}$$

This is a first-order ordinary linear differential equation and can be solved using the standard methods[1]:

$$I(t) = \frac{V_0}{Z_a} \cos(\omega t - \theta) + \frac{V_f}{R} + A e^{-\frac{Rt}{L}} \tag{5}$$

where $Z_a = \sqrt{R^2 + L^2\omega^2}$ and $\theta = \tan^{-1}\left(-\frac{R}{L\omega}\right)$. A is an integration constant which depends on boundary conditions.

In the non-conducting cycle, since the diode behaves like a capacitor with junction capacitance C_j, we can use equation (3) and replace C with C_j to get:

$$I(t) = \frac{V_0}{Z_b}\cos(\omega t - \theta_b) + Be^{-\frac{Rt}{2L}}\cos(\omega_b t - \phi) \qquad (6)$$

where the only change is $\omega_0^2 = \frac{1}{LC_j}$.

C. The Junction Capacitance and Reverse Recovery Time of the Diode

By far, the most interesting change from the RLC equation is the introduction of the **junction capacitance** C_j of a diode. When a reverse bias is applied to a p-n junction diode, the "holes" (in the p-type region) and the electrons (in the n-type region) are pulled further apart from each other, causing the region between the two (called the depletion region) to be widened. This leads to a case where positive charge is stored on one side and negative charge on the other with an electric field between them in the depletion region. Acting as the positive and negative plates and the dielectric medium in a capacitor respectively, the three parts of the diode combine to produce the junction capacitance.

The **reverse recovery time** τ_r of a diode is the time taken for the accumulated charge to be completely removed. Since the junction capacitance increases with the accumulated charge, τ_r must increase with increases in C_j. In particular, τ_r depends on the recent most maximum forward current $I_{\max}$ as[2]:

$$\tau_r = \tau_m \left[1 - \exp\left(-\frac{|I_{\max}|}{I_c}\right)\right] \qquad (7)$$

This ensures that τ_r (and hence C_j) change as the potential difference across the diode changes. This means that not only the switching time between the conducting and non-conducting cycles changes with changing voltage but C_j also varies with time in equation (6). Therefore, by varying the input voltage, we can expect to observe non-linear behaviour.

D. The Proteus Simulation

To clearly see the period-doubling route to chaos, we used the circuit simulating software, Proteus. We chose the 1N4007 diode. The values for the inductance, resistance and frequency were chosen to be $L = 15.054$ mH, $R = 100\,\Omega$ and $f = 2\pi\omega = 50$ kHz. This particular frequency is the resonant frequency f_r for the circuit and can be calculated as[3]:

$$f_r = \frac{1}{2\pi\sqrt{LC_j}} \qquad (8)$$

The peak-to-peak voltage V_{pp} of the source was changed and time-series plots for the voltage across the resistor and the diode were obtained using a virtual oscilloscope. Figures 5 and 6 show the results.

The voltages across the resistor and diode both showed periodic behaviour of $1T$, $2T$ and $4T$ for $V_{pp} = 1$ V, 3 V and 7 V respectively where T denotes the time period corresponding to 1 V. This can be seen in FIG. 5 and 6 where the repeating waveform's crests are seen to be further and further apart as V_{pp} is increased. Specifically, FIG. 5 is clearer in this regard because the oscilloscope scale in FIG. 7(c) was adjusted to show the dips between the voltage drops more accurately.

As V_{pp} was increased to 13 V for the diode and 14 V for the resistor, the respective voltages across them showed a non-repeating pattern and had hence become chaotic. FIG. 5(d) and 6(d) show clearly how no part of the waveform is repeated on the oscilloscope and the time-series plot is no longer periodic.

To make the transition from period-doubling to chaos clearer, phase portraits were drawn via the virtual oscilloscope. One channel was connected to the resistor and the other to the diode. The results are summarized in FIG. 7.

In a phase portrait, repeating loops signify periodic motion and increase in the number of loops indicates period doubling. For $V_{pp} = 1$ V, even the maximum resolution of the oscilloscope was not able to show a clear loop. Even at $V_{pp} = 3$ V, the problem persisted but 2 relatively distinct loops (ignoring the noise) in FIG. 7(a) indicate a period of $2T$. FIG. 7(b) shows 4 distinct loops and hence a period of $4T$ for $V_{pp} = 7$ V. FIG. 7(c), on the other hand, shows no distinct loops, indicating that the voltage follows a non-periodic, non-repeating pattern. This indicates chaotic behaviour for $V_{pp} = 13$ V.

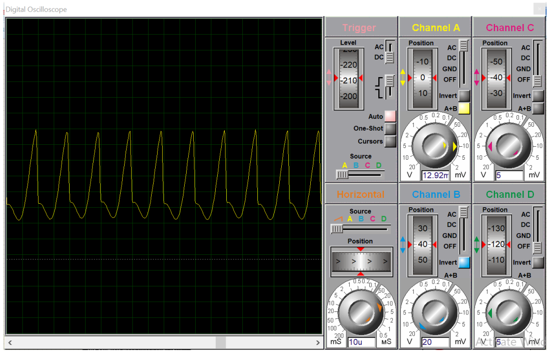
The time series plots and phase portraits showed that the driven RLD circuit follows a period-doubling route to chaos, despite only differing from the linear RLC circuit in one component. The reason for this non-linearity is speculated to be the time delay due to the finite reverse recovery time of the diode[2].

We now experimentally study a circuit built to display chaos: the double scroll Chua's circuit.

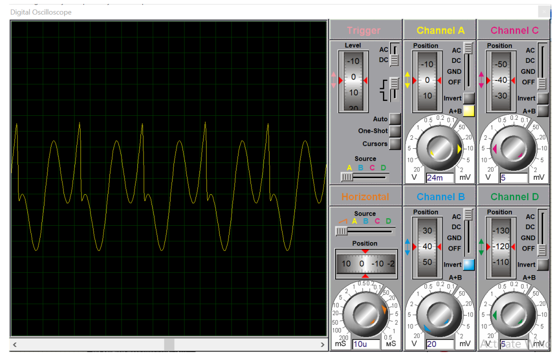
IV. THE DOUBLE SCROLL CHUA'S CIRCUIT

The double scroll Chua's circuit is the simplest electrical circuit which has been mathematically shown to display chaos since it fulfills Kennedy's three necessary conditions for an autonomous circuit to do so[4]:

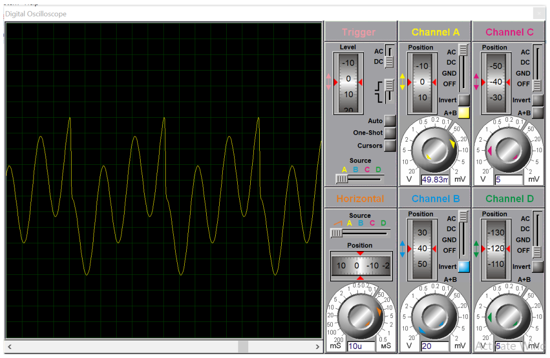
1. one or more nonlinear elements
2. one or more locally active resistors
3. three or more energy storage elements



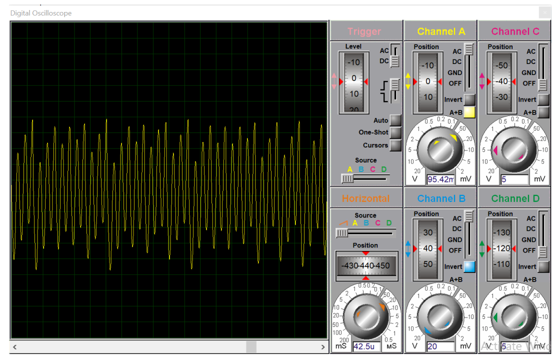
(a) 1 V



(b) 3 V

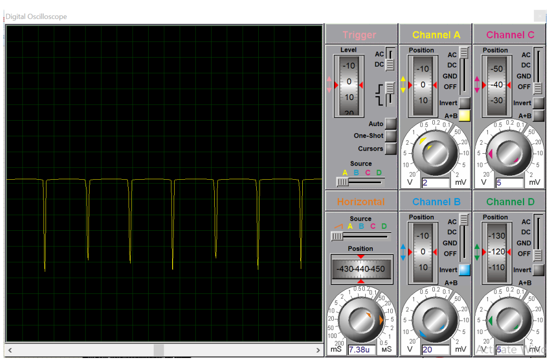


(c) 7 V

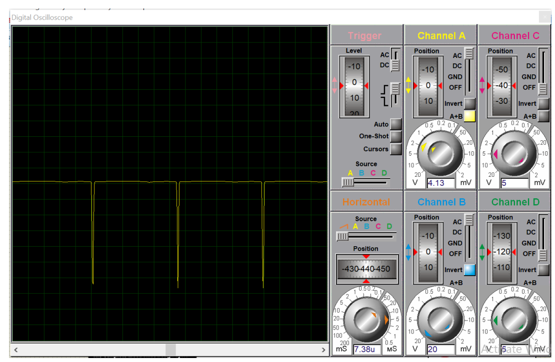


(d) 14 V

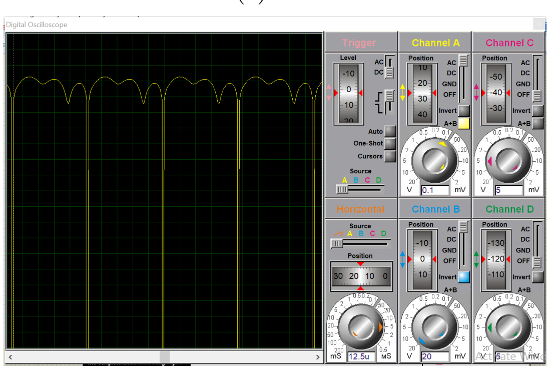
FIG. 5: Time series plots of voltage across the resistor



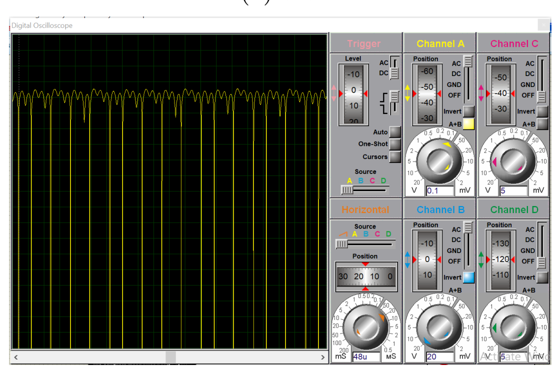
(a) 1 V



(b) 3 V

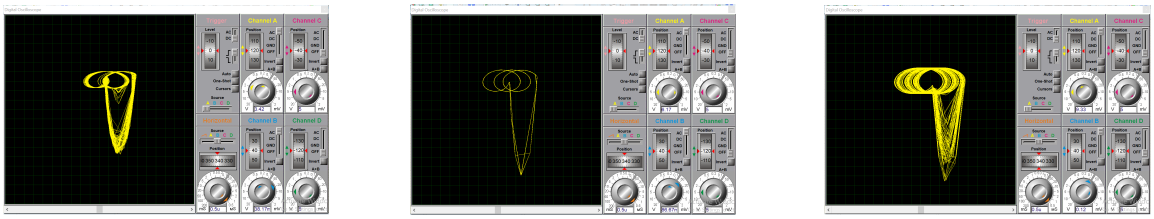


(c) 7 V



(d) 13 V

FIG. 6: Time series plots of voltage across the diode



(a) 3 V (b) 7 V (c) 13 V

FIG. 7: Phase portraits for voltage across resistor and diode

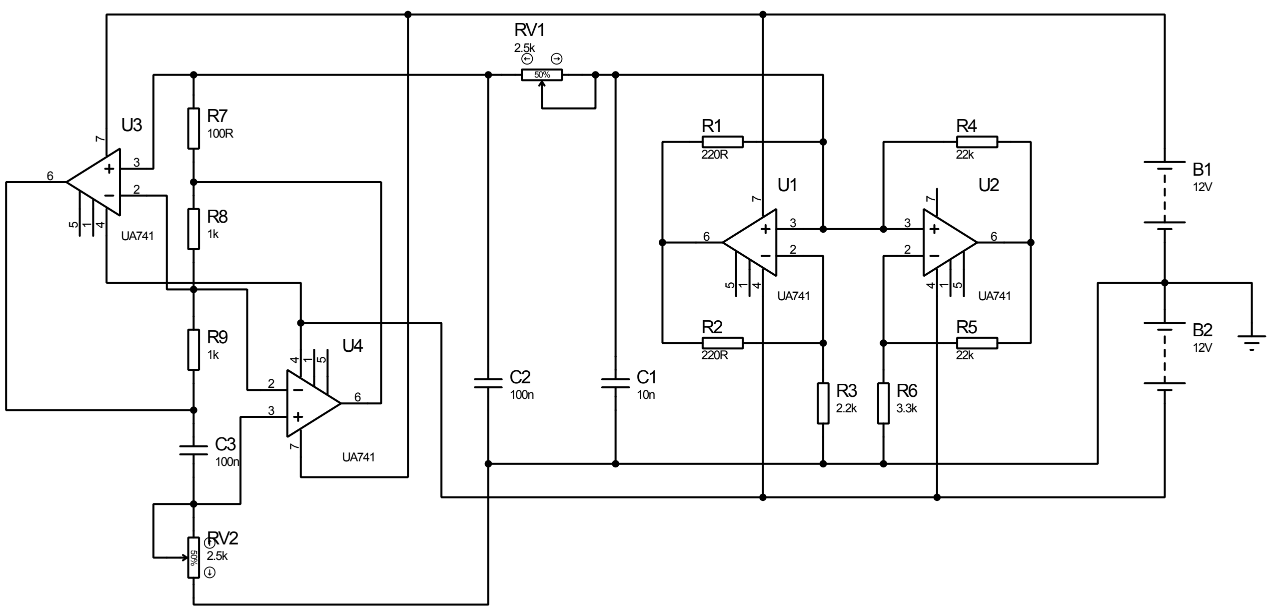


FIG. 8: The Double Scroll Chua's Circuit

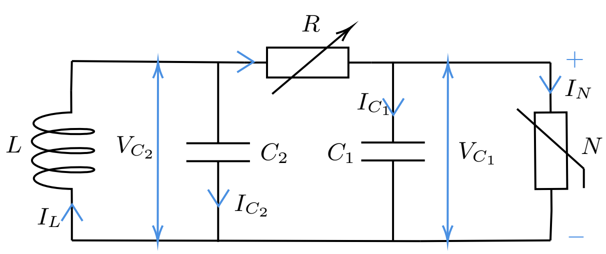


FIG. 9: The simplified double scroll Chua's circuit

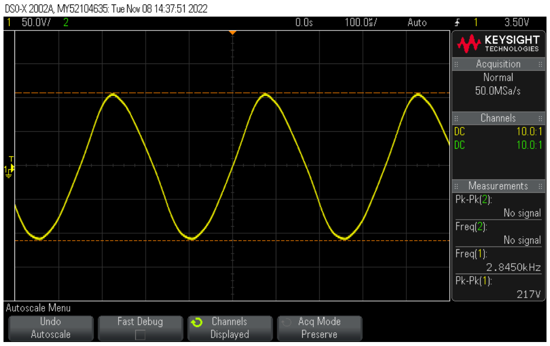
A. The Schematic of the Circuit

The double scroll Chua's circuit looks a lot more complicated than the RLC and the RLD circuit. As FIG. 8 shows, it consists of a collection of diodes, resistors (both variable and fixed), capacitors and a power supply connected in various series and parallel combinations. This entire collection, however, can be understood in terms of 5 components - the variable resistor, 2 capacitors, a simulated inductance (the part to the left of C_2) and a non-linear resistance (the part to the right of C_1), thereby reducing the circuit to the one shown in FIG. 9.

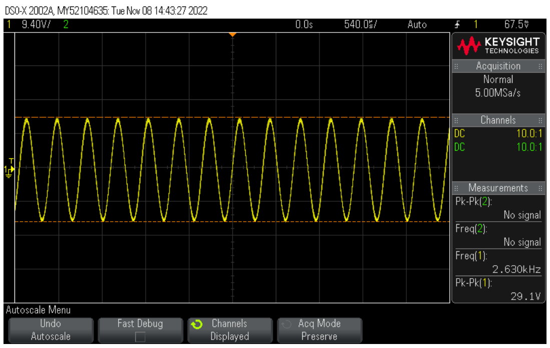
B. The Differential Equation

To arrive at a system of differential equations describing Chua's circuit, we need to first choose our phase space variables. Denoting the current and potential difference across a component A by I_A and V_A respectively, we choose our variables to be I_L, V_{C_1} and V_{C_2}.

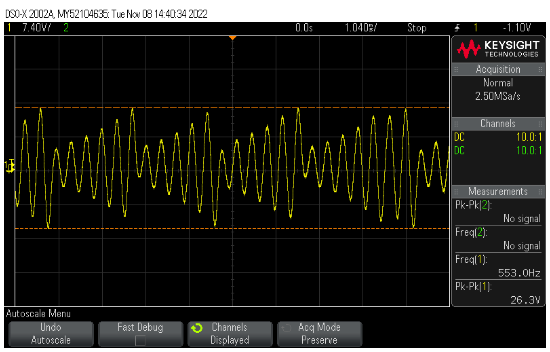
We shall first rewrite all our constants in terms of our chosen variables using Kirchhoff's current law and the



(a) Minimum resistance

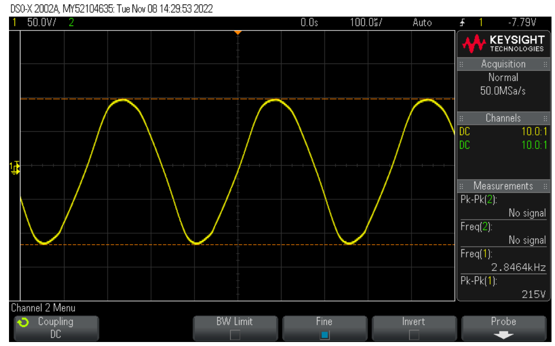


(b) Maximum resistance before chaos

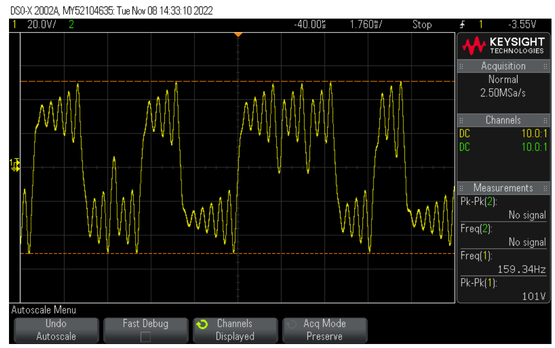


(c) Chaos

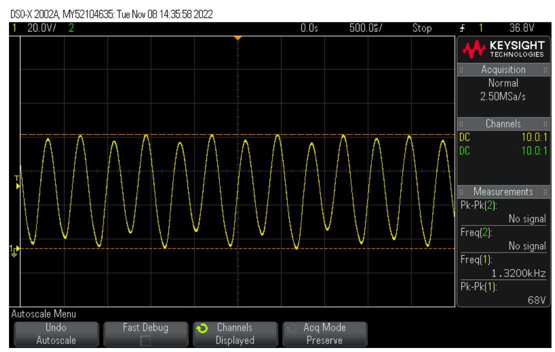
FIG. 10: Time series plots of voltage across the inductor part of Chua's circuit



(a) Minimum resistance



(b) Chaos



(c) Maximum resistance

FIG. 11: Time series plots of voltage across the non-linear part of Chua's circuit

definition of capacitance appropriately:

$$I_{C_2} = C_2 \frac{dV_{C_2}}{dt} \tag{9}$$

$$I_R = I_L - C_2 \frac{dV_{C_2}}{dt} \tag{10}$$

$$I_{C_1} = C_1 \frac{dV_{C_1}}{dt} \tag{11}$$

$$I_N = I_L - C_2 \frac{dV_{C_2}}{dt} - C_1 \frac{dV_{C_1}}{dt} \tag{12}$$

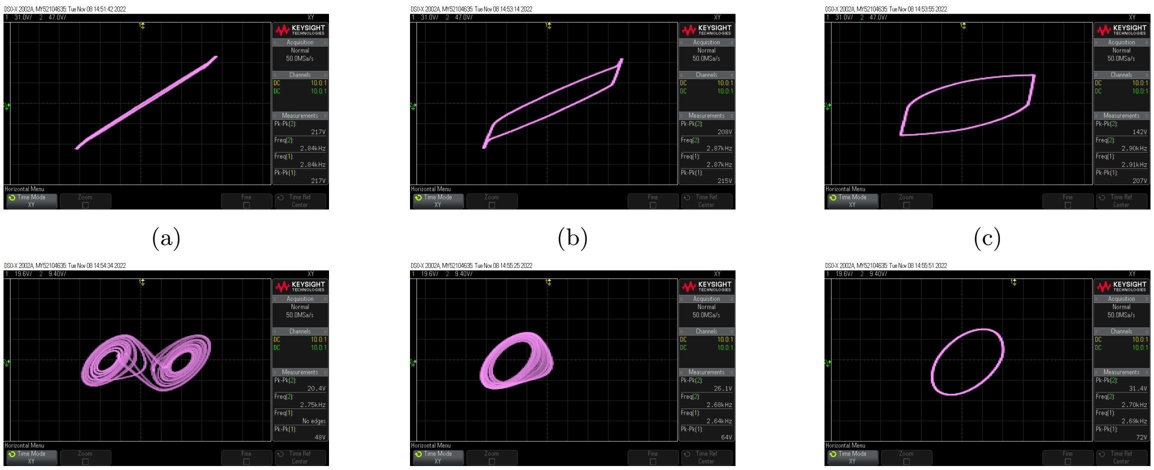
We are now in the position to write down the differential equations for the circuit by applying Kirchhoff's voltage law to each of the three loops in the circuit:

$$\frac{dI_L}{dt} = -\frac{V_{C_2}}{L} \tag{13}$$

$$\frac{dV_{C_2}}{dt} = \frac{1}{RC_2}(V_{C_1} - V_{C_2}) + \frac{I_L}{C_2} \tag{14}$$

$$\frac{dV_{C_1}}{dt} = \frac{1}{RC_2}(V_{C_2} - V_{C_1}) - \frac{I_N}{C_2} \tag{15}$$

where the current across the non-linear part $I_N = f(V_N)$ is, in general, a function of the potential difference across it. $f(V_N)$ can be written as a piecewise function, thereby clearly showing that the third equation is non-linear. Since the equations are coupled, all three must be non-linear differential equations and will therefore show



(a) (b) (c) (d) (e) (f)

FIG. 12: Phase portraits for Chua's circuit - resistance was increased from left to right

chaotic behaviour for appropriate values of the resistance R.

C. The Experimental Observation of Chaos

The circuit shown in FIG. 8 was used in the laboratory to observe chaos. A 220 V AC voltage was applied to the circuit. The variable resistance R was varied and the voltage across the inductor (the left side) and the non-linear (the right side) part of the circuit was recorded using an oscilloscope. The results are summarized in figures 9 and 10.

FIG. 10(a) and 11(a) show that with the variable resistance set to its minimum value, the voltage across the inductor part and the non-linear part show periodic patterns. As the resistance was increased, the peak-to-peak voltage V_{pp} across the two parts decreased.

For the inductor part, the peak-to-peak voltage across it continued to decrease with increasing resistance but the frequency remained the same as FIG. 10(b) shows. At $V_{pp} = 26.3$ V, chaos was observed. In FIG. 10(c), there is no repeating pattern. While the overall profile of the plot shows four complete sets of a sinusoidal wave modulated by $x = |y|$, the period of each of these sets is different. Therefore, the time series is plot is chaotic.

At $V_{pp} = 101$ V across the non-linear part, the time-series plot descended into chaos. As FIG. 11(b) shows, there are no repeating patterns in the plot. As the resistance was increased further, at $V_{pp} = 68$ V, the chaotic behaviour of the non-linear part gave way to a periodic motion with twice the time period of the original minimum-resistance time-series plot.

To look at the order to chaos dynamics more clearly, we plotted the phase portraits of the voltage across the inductor part against the voltage across the non-linear part. The results for varying resistance values are summarized in FIG. 12.

As the resistance was increased, the single loop in the phase portrait became wider until suddenly (at $V_{pp} = 48$ V across the non-linear part and 20.4 V across the linear part), the oscilloscope showed the familiar image of Lorenz's strange attractor (FIG. 12(d)). This attractor is the classic indicator of chaos in any dynamical system, alongside the Feigenbaum constant for the period-doubling route to chaos[5]. The attractor never crosses itself and therefore, there are no repeated patterns and the system does not show periodic behaviour.

Upon increasing the resistance further, one "wing" of the strange attractor collapsed to give the display shown in FIG. 12(e). Further increases in resistance led to a single loop remaining (FIG. 12(f)), showing that the circuit had returned to periodic behaviour.

V. CONCLUDING REMARKS

Linear differential equations simply cannot capture the beautiful complexity in nature. Chaos abounds there and the order beneath it must be understood to fully comprehend nature. Even seemingly simple electrical circuits such as the RLD circuit can show chaos for varying values of the parameter (in this case, the driving voltage). These circuits are described by deterministic equations which show a sensitive dependence on initial conditions. Being constructed using simple electrical components, these circuits are easy to study in the laboratory and learn the techniques to study chaotic systems.

[1] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists*, 6th ed. (Elsevier Academic Press, 2005).

[2] R. W. Rollins and E. R. Hunt, Exactly Solvable Model of a Physical System Exhibiting Universal Chaotic Behavior,

Phys. Rev. Lett. **49**, 1295 (1982).

[3] M. P. Hanias, Z. Avgerinos, and G. S. Tombras, Period doubling, Feigenbaum constant and time series prediction in an experimental chaotic RLD circuit, Chaos, Solitons and Fractals **40**, 1050 (2007).

[4] M. P. Kennedy, Three Steps to Chaos−Part I: Evolution, IEEE Transactions on Circuits and Systems−I: Fundamental Theory and Applications **40**, 640 (1993).

[5] J. Gleick, *Chaos: Making a New Science* (Penguin Books, 1987).

[6] J. R. Taylor, *Classical Mechanics* (University Science Books, 2005).

[7] T. Matsumoto, Chaos in electrical circuits, Proceedings of the IEEE **75**, 1033 (1987).