Probability Measures on the Logistic Map

Course Project for MATH 4102: Qualitative Differential Equation

Abderrahmane Labane *

Munawar Khan †

Muhammad Hashir Hassan Khan ‡

December 8, 2023

Abstract

A study of the logistic map is presented. First, a topological conjugacy between the logistic map and the tent map is proved. This is then used to find three invariant and ergodic measures. Finally, Birkhoff's Ergodic Theorem is used to evaluate the mean sojourn time for the set $E = [1/2, 1)$.

Contents

*2023-07-0011

†2023-07-0007

‡2024-10-0111

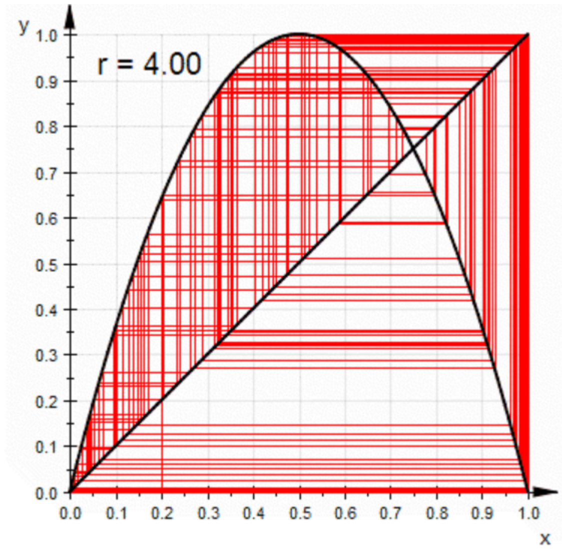

Figure 1.1: The red lines show the orbit of the point $x = 0.2$ for the logistic map (the parabola). The diagonal is also shown for convenience.

1 The Logistic Map

The logistic map is one of the most widely studied examples in the theory of dynamical systems. We begin by defining the map.

Definition. Let $X = [0, 1]$ and let $f : X \to X$ be given by

$$f(x) = rx(1 - x). \tag{1.1}$$

Then, f is called the *logistic map* with parameter r.

For the purposes of this paper, $r = 4$ and therefore, eq. (1.1) becomes

$$f(x) = 4x(1 - x). \tag{1.2}$$

To visualize this map and the orbit of a point $x = 0.2$, see fig. (1.1).

2 A Topological Conjugacy with the Tent Map

We begin with a definition of the tent map with parameter $r = 2$.

Definition. Let $X = [0, 1]$ and let $t : X \to X$ be given by

$$t(x) = \begin{cases} 2x & \text{for } 0 \le x < \frac{1}{2}; \\ 2 - 2x & \text{for } \frac{1}{2} \le x < 1. \end{cases} \tag{2.1}$$

Then, t is called the *tent map* with parameter $r = 2$.

Proposition 2.1. *Let $X = [0, 1]$ and let $h : X \to X$ be given by*

$$h(x) = \sin^2\left(\frac{\pi x}{2}\right). \tag{2.2}$$

Then, h is a topological conjugacy between the logistic map and the tent map.

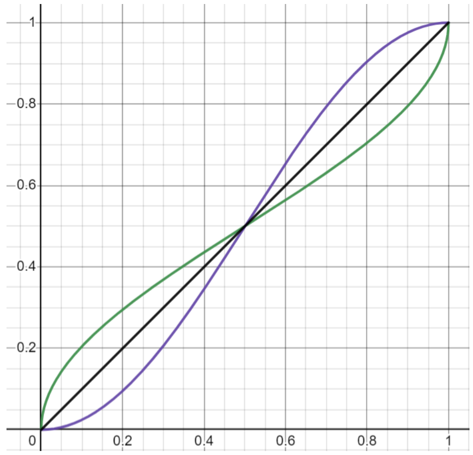

Figure 2.1: The purple line is the graph of $h(x)$. The diagonal is drawn in black. Reflecting across the diagonal gives the green line which is the graph of $h^{-1}(x)$.

Proof. We first compute $f \circ h$:

$$\begin{aligned}(f \circ h)(x) &= 4\sin^2\left(\frac{\pi x}{2}\right)\left(1 - \sin^2\left(\frac{\pi x}{2}\right)\right), \\ &= \sin^2(\pi x),\end{aligned} \tag{2.3}$$

where we used the trigonometric double angle identities in the second line. Next, we compute $h \circ t$:

$$(h \circ t)(x) = \begin{cases} \sin^2(\pi x) & \text{for } 0 \le x < \frac{1}{2}; \\ \sin^2(\pi - \pi x) & \text{for } \frac{1}{2} \le x < 1. \end{cases} \tag{2.4}$$

Using trigonometric identities, we see that $\sin^2(\pi - \pi x) = \sin^2(\pi x)$. Therefore, using eqs. (2.3) and (2.4), $f \circ h = h \circ t$. Next, we seek to prove that h is a homeomorphism with its image. Note that h is continuous and monotone on X as can be seen from its graph in fig. (2.1). Therefore, h is bijective on X and its inverse h^{-1} exists and is continuous. This concludes our proof. □

3 The First Invariant and Ergodic Measure

We begin by noting that t is a piecewise affine full branch map. Therefore, the Lebesgue measure m is invariant and ergodic for t. Moreover, since f and t are topologically conjugate, they are also measurably conjugate. Hence, we can *push-backward* the Lebesgue measure to define a new measure μ on X which is invariant and ergodic for f:

$$\mu(A) \coloneqq h_* m(A) = m(h^{-1}(A)) = \int_a^b |(h^{-1})'(x)|\, dx, \tag{3.1}$$

where $A = (a, b) \subseteq X$. Here, $|(h^{-1})'(x)|$ is called the *density* of μ with respect to the Lebesgue measure m. Let us now evaluate the inverse h^{-1} explicitly. Using $x = \sin^2(\pi y/2)$ and solving for y, we get

$$h^{-1}(x) = \frac{2}{\pi}\arcsin(\sqrt{x}). \tag{3.2}$$

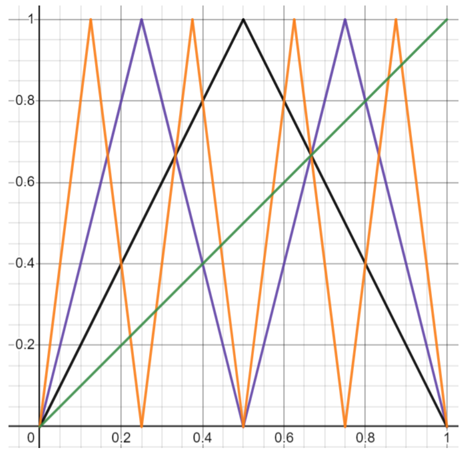

Figure 5.1: The black, purple and orange lines are the graphs for $t(x)$, $t^2(x)$ and $t^3(x)$ respectively. The diagonal is drawn in green.

Using this, we can find the density of μ:

$$|(h^{-1})'(x)| = \frac{1}{\pi\sqrt{x(1-x)}}. \tag{3.3}$$

Using this result in eq. (3.1), the invariant and ergodic measure on f that is absolutely continuous with the Lebesgue measure is

$$\mu(A) = \frac{1}{\pi}\int_a^b \frac{1}{\sqrt{x(1-x)}}\,dx. \tag{3.4}$$

4 A Fixed Point Measure

We now attempt to find a fixed point measure for f. To do this, we first find the fixed points of f using $4x(1-x) = x$. After some algebraic manipulation we obtain

$$x\,(3-4x) = 0. \tag{4.1}$$

Hence, we have two fixed points $x_0 \in \{0, 3/4\}$. The fixed points can also be found by looking at the intersection of the diagonal with f in fig. (1.1). We can use these fixed points to construct the fixed point measure δ_p with $p \in \{0, 3/4\}$, where δ_x is the usual Dirac-delta measure:

$$\delta_x(A) := \begin{cases} 0 & \text{if } x \notin A; \\ 1 & \text{if } x \in A. \end{cases} \tag{4.2}$$

5 A Periodic Point Measure

Next, we attempt to find a periodic point measure for f. To do this, we need to find the periodic points of f using $f^k(x) = x$. This does not seem to admit a simple solution. Luckily, we have already proved a topological conjugacy between the tent map and the logistic map. Therefore, the fixed and periodic points of the tent map are mapped onto the fixed and periodic points of the logistic map. Our task has now been reduced to finding the periodic points of the tent map. To find periodic points with period 2, we compose

$t(x)$ with itself to obtain

$$t^2(x) = \begin{cases} 2t(x) & \text{for } 0 \le x < \frac{1}{2} \text{ and } 0 \le t(x) < \frac{1}{2}; \\ 2 - 2t(x) & \text{for } \frac{1}{2} \le x \le 1 \text{ and } 0 \le t(x) < \frac{1}{2}; \\ 2t(x) & \text{for } 0 \le x < \frac{1}{2} \text{ and } \frac{1}{2} \le t(x) \le 1; \\ 2 - 2t(x) & \text{for } \frac{1}{2} \le x \le 1 \text{ and } \frac{1}{2} \le t(x) \le 1. \end{cases} \tag{5.1}$$

This is equal to

$$t^2(x) = \begin{cases} 2^2 x & 0 \le x < 1/4; \\ 2 - 2^2 x & 1/4 \le x \le 1/2; \\ 2^2 x - 2 & 1/2 \le x \le 3/4; \\ 2^2 (1 - x) & 3/4 \le x < 1. \end{cases} \tag{5.2}$$

Therefore, the fixed points for t^2 (which are periodic points of order 2) are given by $t^2(x) = x$, $p \in \{0, \frac{2}{5}, \frac{2}{3}, \frac{4}{5}\}$. Using the same procedure we obtain the periodic points of period 3,

$$t^3(x) = \begin{cases} 2^3 x & 0 \le x < 1/2^3; \\ 2 - 2^3 x & 1/2^3 \le x \le 2/2^3; \\ 2^3 x - 2 & 2/2^3 \le x \le 3/2^3; \\ 2^2 - 2^3 x & 3/2^3 \le x < 4/2^3; \\ 2^3 x - 2^2 & 4/2^3 \le x < 5/2^3; \\ 6 - 2^3 x & 5/2^3 \le x < 6/2^3; \\ 2^3 x - 6 & 6/2^3 \le x < 7/2^3; \\ 2^3 - 2^3 x & 7/2^3 \le x < 1. \end{cases} \tag{5.3}$$

The periodic points of period 3 are solutions to the equation $t^3(x) = x$. They are $p \in \left\{0, \frac{2}{9}, \frac{2}{7}, \frac{4}{9}, \frac{4}{7}, \frac{6}{9}, \frac{6}{7}, \frac{8}{9}\right\}$. To visualize these examples, see fig. (5.1). Looking at the previous examples, we make the following claim:

Proposition 5.1. *The general form of $t^n(x)$ is*

$$t^n(x) = \begin{cases} 2^n x - \frac{2^n m}{2^{n-1}} & \frac{2m}{2^n} \le x \le \frac{2m+1}{2^n}; \\ -2^n x + \frac{2^n (m+1)}{2^{n-1}} & \frac{2m+1}{2^n} \le x \le \frac{2m+2}{2^n}. \end{cases} \tag{5.4}$$

for $m = 0, 1, 2, \ldots, 2^{n-1} - 1$.

Proof. The tent map $t(x)$ has two branches $x \in [0, 1/2]$ and $x \in [1/2, 1]$. When we compose $t(x)$ with itself, we split every branch into other similar branches. In the case of t^2, we have $x \in [0, 1/4] \cup [1/4, 2/4] \cup [2/4, 3/4] \cup [3/4, 1]$. Here, we have two copies of the original tent map $t(x)$ as is clear from eq. (5.2). We observe that the function in the branch $x \in [0, 1/4]$ and $x \in [2/4, 3/4]$, have the same slope and are shifted by $-2 = -\frac{2^2}{2}$. Define $t^n_m(x)$ to be the mth branch of t^n. Then, we have $t^2_3(x) = t^2_1\left(x - \frac{1}{2}\right)$. Here, $\frac{1}{2}$ is the distance between the first branch and the second branch. We observe the same behavior in t^3. The interval $[0, 1]$ is decomposed into 8 equal parts of length $\frac{1}{2^3}$. Furthermore, we have $t^3_3(x) = t^3_1\left(x - \frac{1}{2^2}\right)$, $t^3_5(x) = t^3_1\left(x - \frac{2}{2^2}\right)$ and $t^3_7(x) = t^3_1\left(x - \frac{3}{2^2}\right)$. Therefore, we get

$$t^n_m(x) = 2^n \left(x - \frac{m-1}{2^n}\right). \tag{5.5}$$

The same argument can be made for the "even" branches. Combining this with the definition for each branch, we get eq. (5.4). ∎

Another observation we can make is that all the periodic points that we have calculated so far are rational numbers with an even numerator and an odd denominator. We can therefore make the following claim.

Proposition 5.2. *The periodic points of period n for $t(x)$ are rational numbers with an even numerator and an odd denominator.*

Proof. We show this using the general formula for $t^n(x)$. The periodic points are solutions to $t^n(x) = x$. Hence, we have, for $\frac{2m}{2^n} \leq x \leq \frac{2m+1}{2^n}$,

$$\begin{aligned} 2^n p - \frac{2^n m}{2^{n-1}} &= p, \\ \implies p &= \frac{2m}{2^n - 1}, \end{aligned} \tag{5.6}$$

and for $\frac{2m+1}{2^n} \leq x \leq \frac{2m+2}{2^n}$,

$$\begin{aligned} -2^n p + \frac{2^n (m+1)}{2^{n-1}} &= p, \\ \implies p &= \frac{2(m+1)}{2^n + 1}. \end{aligned} \tag{5.7}$$

□

Having obtained the periodic point of period n for the tent map, we can use the conjugation map h to find the periodic points for the logistic map. The periodic points of period n for $f(x)$ are

$$p = \begin{cases} \sin^2\left(\frac{\pi m}{2^n - 1}\right) & \text{for } p \in [0, 1/2]\,; \\ \sin^2\left(\frac{\pi(m+1)}{2^n+1}\right) & \text{for } p \in [1/2, 1]\,, \end{cases} \tag{5.8}$$

where $m = 0, 1, 2, \ldots, 2^{n-1} - 1$. In particular, we have 2^n periodic points of period n. Half of them are in the interval $[0, 1/2]$ and the rest are in $[1/2, 1]$.

6 The Mean Sojourn Time

We begin with a definition.

Definition. The *mean sojourn time* τ for a set E and an initial condition x is the a.e. limit

$$\tau(E, x) = \lim_{n\to\infty} \frac{1}{n} \#\{0 \leq i < n : f^i(x) \in E\}. \tag{6.1}$$

For this paper, $E = [1/2, 1)$. All the measures that we are dealing with are invariant and ergodic, hence, we can use the Birkhoff's Ergodic Theorem to compute the mean sojourn time,

$$\tau(E, x) = \mu(E). \tag{6.2}$$

6.1 Fixed Point Measure

We have two fixed points, $p \in \{0, 2/3\}$. For $p = 0$, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \#\left\{0 \leq i < n : f^i(x) \in [1/2, 1)\right\} &= \delta_0\left([1/2, 1)\right) \\ &= 0. \end{aligned}$$

For $p = 2/3$, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \#\left\{0 \leq i < n : f^i(x) \in [1/2, 1)\right\} &= \delta_{2/3}\left([1/2, 1)\right) \\ &= 1 \end{aligned}$$

6.2 Periodic Point Measure

Let $P \coloneqq \{p_0, p_1, ...p_{n-1}\}$, be the set of periodic points of period n. Thus, the periodic measure is

$$\mu_{\text{per}} = \frac{\left(\delta_{p_0} + \delta_{p_1} + ... + \delta_{p_{n-1}}\right)}{n}. \tag{6.3}$$

From the previous question, we know that there are 2^n periodic points of period n. In particular, there are $2^n/2$ points in $[0, 1/2)$ and $2^n/2$ points in $[1/2, 1)$, thus,

$$\mu_{\text{per}}\left([1/2, 1)\right) = 1/2.$$

Hence,

$$\lim_{n\to\infty} \frac{1}{n} \# \left\{0 \leq i < n : f^i(x) \in [1/2, 1)\right\} = 1/2.$$

6.3 The First Invariant and Ergodic Measure

Recall that μ is the absolutely continuous measure with respect to the Lebesgue measure as defined in eq. (3.4). Then, using the substitution $u = \sqrt{x}$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \# \left\{0 \leq i < n : f^i(x) \in [1/2, 1)\right\} &= \frac{1}{\pi} \int_{1/2}^{1} \frac{1}{\sqrt{x\left(1-x\right)}}\, dx, \\
&= \frac{2}{\pi} \int_{1/\sqrt{2}}^{1} \frac{1}{\sqrt{1-u^2}}\, dx, \\
&= \frac{2}{\pi} \left(\arcsin\left(1\right) - \arcsin\left(\frac{1}{\sqrt{2}}\right)\right), \\
&= 1/2.
\end{aligned}$$