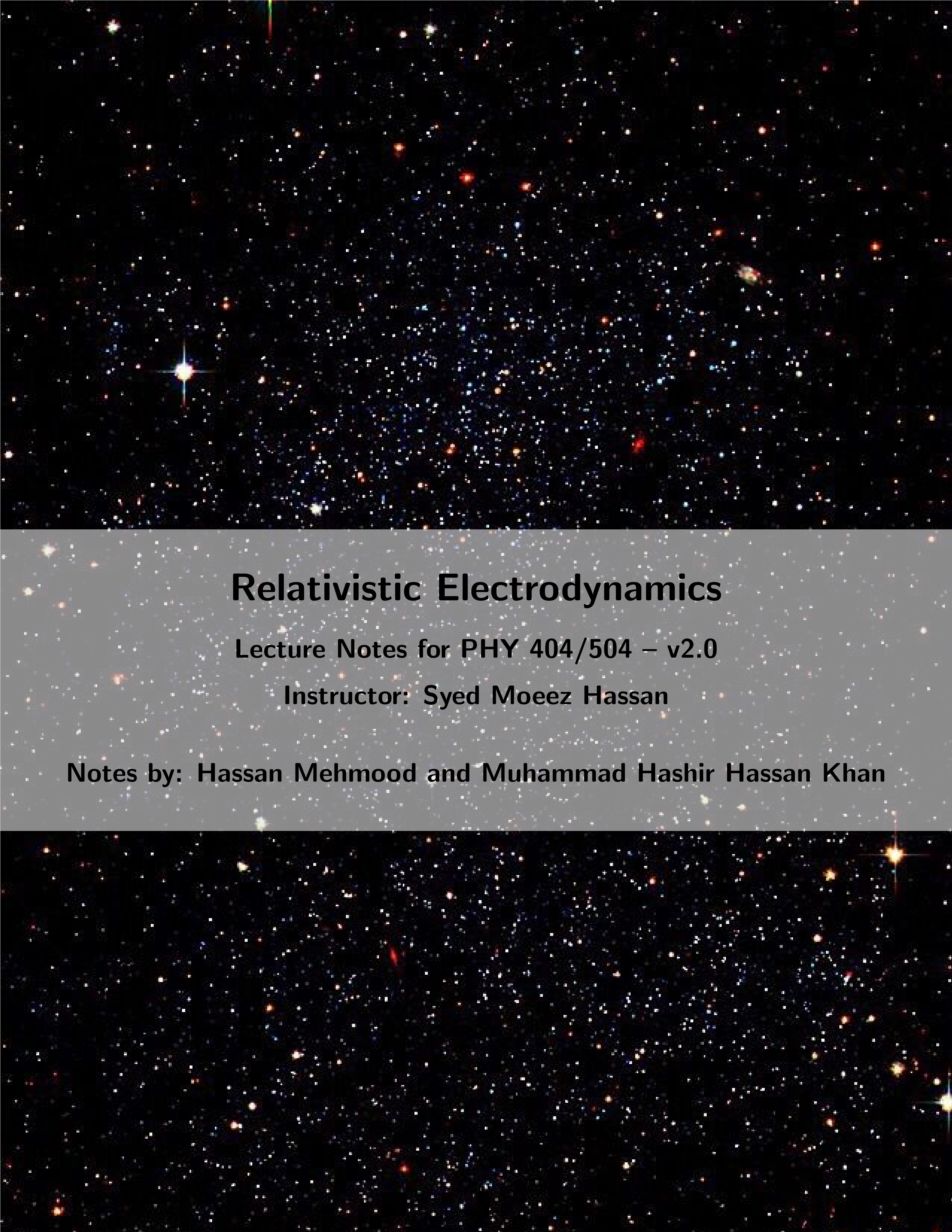

Relativistic Electrodynamics

Lecture Notes for PHY 404/504 – v2.0

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Preface

What you are about to read are lecture notes for the course PHY 404/504 – Relativistic Electrodynamics offered at the Syed Babar Ali School of Science and Engineering at LUMS by Dr. Moeez Hassan. With such books as Andrew Zangwill's *Modern Electrodynamics* and John David Jackson's *Classical Electrodynamics* already providing excellent reading material for the content of the course, one may wonder why there was a need for these lecture notes. The answer lies mainly in the level of difficulty and the vastly different notations and conventions used by the two authors. While Jackson proves to be a great resource for a graduate student, it has not proven so readable for the average undergraduate. Since the course at LUMS is currently designated as a core course for physics undergraduates, Zangwill seems the better book of choice. However, Zangwill's tendency to state hundreds of results (often requiring a page's worth of calculations) without working does drive away the average undergraduate.

Which brings us to these lecture notes. They serve as a bridge between Zangwill and Jackson, explicitly solving most of the steps involved in obtaining the necessary theoretical results which would later be useful for problem solving. The notes also follow the conventions used in class which are different from the ones used in either Zangwill or Jackson. In particular, the spacetime metric throughout these notes is the Minkowski metric with metric signature $(-,+,+,+)$. Everything is calculated in $c=1$ units unless otherwise stated. The conventions related to the electromagnetic field tensor are discussed in Chapter 1 (which has small edits from last year to remove a few inconsistencies and to fully follow the conventions used in class).

Happy reading!

Best,
Muhammad Hashir Hassan Khan

Chapter 1

Covariant Electrodynamics

Maxwell's equations are invariant under Lorentz transformations. In fact, Einstein was led to formulate special relativity precisely because electrodynamics did not obey Galilean relativity – to him, frame-invariance applied first and foremost to electrodynamics; if such a principle of relativity entailed that other laws of physics, such as those of Newtonian mechanics, are not the same in all inertial frames, then so much the worse for those laws, for it is they who must be modified, not electrodynamics.

However, if we write down Maxwell's equations (in $c = 1$ units, which entail $\mu_0 \epsilon_0 = 1$),

$$\nabla \cdot \mathbf{E} = \mu_0 \rho, \tag{1.1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}, \tag{1.4}$$

it is not immediately apparent that they will look the same in another Lorentz frame. To flesh out the manifest invariance of these equations, we need to write them in the language of tensors. How does one achieve that? Well, the first thing that might come to mind is to try to write down the three-vectors **E** and **B** in Euclidean space as four-vectors in Minkowski spacetime. However, it is not at all clear how that can be done. In fact, there is a good reason to believe that such a task is not possible. As we know very well, an observer stationary with respect to a charge will only measure an electric field due to it, whereas an observer drifting past it would observe both an electric and a magnetic field. It thus follows that Lorentz transformations between different frames should mix the components of electric and magnetic fields together just as they mix the space and time components together. But if electric and magnetic fields were each described by distinct four-vectors, they would not be able to mix among each other upon a change of coordinates, since the components of each individual four-vector would just transform among themselves in much the same way as the components of, for instance, the spacetime vector x^α.

It thus seems reasonable to think that there is a single tensorial object that describes both the electric and the magnetic fields. Again, a four-vector won't do, for we have a total of six independent variables (three components of $\mathbf{E}$ and $\mathbf{B}$ each) to cater for. What about a rank-2 tensor, e.g. some $F_{\alpha\beta}$? Well, that would have 16 components and hence be too generous. But we are close: we demand that $F_{\alpha\beta}$ be *antisymmetric*, i.e. $F_{\alpha\beta} = -F_{\beta\alpha}$. This immediately implies that $F_{\alpha\beta}$ has only six independent components! (Prove it.)

Before proceeding further, since the reason of our existence currently lies in writing Maxwell's equations in tensor language, we will find it existentially very meaningful to express them first in index notation. Recall that in this notation,

$$\mathbf{K} \leftrightarrow K_i, \tag{1.5}$$

$$\nabla \leftrightarrow \partial_i, \tag{1.6}$$

where $\mathbf{K}$ is some arbitrary three-vector. Recall also that in the index notation, the dot product and cross product of two vectors $\mathbf{X}$ and $\mathbf{Y}$ are written, respectively, as $\mathbf{X} \cdot \mathbf{Y} = \delta_{ij} X^i Y^i$ and $(\mathbf{X} \times \mathbf{Y})_i = \epsilon_{ijk} X^j Y^k$, ϵ_{ijk} being the Levi-Civita symbol. These facts should be enough to let you arrive at the following gorgeous incarnation of Maxwell's equations:

$$\delta^{ij} \partial_i E_j = \mu_0 \rho, \tag{1.7}$$

$$\delta^{ij} \partial_i B_j = 0, \tag{1.8}$$

$$\epsilon^{ij}{}_k \partial_j E^k = -\partial_0 B^i, \tag{1.9}$$

$$\epsilon^{ij}{}_k \partial_j B^k = \mu_0 J^i + \partial_0 E^i. \tag{1.10}$$

Let us now return to $F^{\alpha\beta}$. It is a $(2,0)$ tensor, and dubbed the *electromagnetic field tensor*. Our central task is to rewrite Maxwell's equations in terms of this tensor. To that end, we first need to consider how to arrange the components of $\mathbf{E}$ and $\mathbf{B}$ in the matrix associated with $F^{\alpha\beta}$. By virtue of the antisymmetry, the diagonal entries must be zero. Now consider the first Maxwell equation. It suggests very naturally that

$$E^i = F^{0i}; \tag{1.11}$$

antisymmetry then fixes F^{i0} as well (Of course, looking at the second Maxwell equation, one could equally well have started with $B^i = F^{0i}$. We will come back to this seeming ambiguity later). Next, consider the third Maxwell equation. Contracting on both sides with $\epsilon_i{}^{mn}$, we obtain

$$\begin{aligned}
\epsilon_i{}^{mn}\epsilon^{ij}{}_k\partial_j E^k &= -\epsilon_i{}^{mn}\partial_0 B^i, \\
(\delta^{jm}\delta^n_k - \delta^m_k\delta^{nj})\partial_j E^k &= -\partial_0(\epsilon_i{}^{mn}B^i), \\
\partial^{[m}E^{n]} &= -\partial_0(\epsilon_i{}^{mn}B^i),
\end{aligned}$$

where in the second line, we used the identity

$$\epsilon_i{}^{mn}\epsilon^{ij}{}_k = \epsilon^{mn}{}_i\epsilon^j{}_k{}^i = \delta^{jm}\delta^n_k - \delta^m_k\delta^{nj}. \tag{1.12}$$

This immediately suggests that $F^{mn} = \epsilon_i{}^{mn}B^i$, whence, by contracting on both sides with $\epsilon^l{}_{mn}$ and using the identity $\epsilon^l{}_{mn}\epsilon_i{}^{mn} = 2\delta^l_i$, we obtain (show this)

$$B^i = \frac{1}{2}\epsilon^i{}_{jk}F^{jk}. \tag{1.13}$$

In view of eq. (1.11) and (1.13), we can write

$$F^{\alpha\beta} := \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{1.14}$$

To write Maxwell's equations in terms of $F^{\alpha\beta}$, simply substitute eq. (1.11) and (1.13) into eq. (1.7)–(1.10). Let us proceed in steps here. eq. (1.7) becomes

$$\partial_i F^{0i} = -\mu_0\rho, \tag{1.15}$$

while eq. (1.10) becomes

$$\begin{aligned}
\frac{1}{2}\epsilon^{ij}{}_k\epsilon^k{}_{mn}\partial_j F^{mn} &= \mu_0 J^i + \partial_0 F^{0i}, \\
-\frac{1}{2}\epsilon_k{}^{ji}\epsilon^k{}_{mn}\partial_j F^{mn} &= \mu_0 J^i + \partial_0 F^{0i}, \\
-\frac{1}{2}(\delta^j_m\delta^i_n - \delta^j_n\delta^i_m)\partial_j F^{mn} &= \mu_0 J^i + \partial_0 F^{0i}, \\
-\frac{1}{2}(\partial_m F^{mi} - \partial_n F^{in}) &= \mu_0 J^i + \partial_0 F^{0i}, \\
-\partial_m F^{mi} - \partial_0 F^{0i} &= \mu_0 J^i \\
\partial_\alpha F^{\alpha i} &= -\mu_0 J^i.
\end{aligned}$$

In the second line above, we used the antisymmetry of $\epsilon^{ij}{}_k$ in all the indices; in the third line, we used eq. (1.12), and finally, in the fifth line, we used the antisymmetry of F^{mn}. Now take a close look at eq. (1.15) and the last equation above. If we define a four-vector $J^\alpha \equiv (\rho, \mathbf{J})$, we can combine these two equations into a single, more elegant one:

$$\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta. \tag{1.16}$$

Since the drill is now clear, I leave it up to you to convince yourself that the remaining Maxwell equations (i.e. eq. (1.8) and (1.10)) are equivalent to the so-called *Bianchi identity*[1]:

$$\partial_{[\alpha} F_{\beta\gamma]} = 0, \tag{1.17}$$

where the square brackets refer to antisymmetrization over all three indices. I now implore you to relish in the utter cuteness of eq. (1.16) and (1.17).

Considerations of beauty notwithstanding, we have achieved what we hoped for in the beginning: the Lorentz invariance of eq. (1.16) and (1.17) is as manifest as the sun in a clear sky. Why? Consider eq. (1.16). Upon a Lorentz transformation, the partial derivative offers an inverse Lorentz matrix, $F^{\alpha\beta}$ furnishes two Lorentz matrices, and J^β embraces a Lorentz matrix. Thus we effectively have one Lorentz matrix on the left-hand side, but its effect is cancelled by the Lorentz matrix on the right-hand side. As for eq. (1.17), the lonely inverse Lorentz matrix that effectively remains doesn't count, for we can with a smirk of impunity multiply both sides of the equation with a Lorentz matrix.

Let me now come to a remark that I made earlier, namely, why not write $B^i = F^{0i}$ instead of $E^i = F^{0i}$? The answer is that we can do that, with no change whatsoever in the resulting Maxwell's equations; the only thing that will change will be the matrix representation in eq. (1.14), with the components of the magnetic and electric fields swapping places. This rearranged tensor is called the *Hodge dual* of $F_{\alpha\beta}$ in technical parlance. The two are related by

$$G^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\lambda} F_{\gamma\lambda}, \tag{1.18}$$

where $\epsilon^{\alpha\beta\gamma\lambda}$ is the Levi-Civita symbol in four dimensions[2]. If this is not enough to convince you of the wonderful properties of $F^{\alpha\beta}$, I urge you to work on the following exercises.

[1] Interesting fact: the origin of the Bianchi identity is thoroughly geometric. It has nothing to do with the fact that $F_{\alpha\beta}$ describes electromagnetic fields, but with the fact that it is an antisymmetric second-rank tensor, which in the language of differential forms is called a 2-form.

[2] Another interesting digression. A tensor is called self dual (anti-self dual) if its Hodge dual is equal to itself (minus itself). Any antisymmetric, rank-2 tensor can be decomposed into a sum of its self dual and anti-self dual parts. The study of these parts, along with the Bianchi identity, is an immensely useful way of finding solutions to Maxwell's equations.

1. $F_{\alpha\beta}F^{\alpha\beta} = 2(B^2 - E^2)$.
2. $G^{\alpha\beta}F_{\alpha\beta} = -4\mathbf{B}\cdot\mathbf{E}$.
3. $\det(F_{\alpha\beta}) = (\mathbf{B}\cdot\mathbf{E})^2$.
4. $\mathrm{Tr}(F_{\alpha\beta}) = F^{\alpha}{}_{\alpha} = 0$.
5. Show that the inhomogenous Maxwell equation (eq. (1.16)) leads to charge conservation: $\partial_\alpha J^\alpha = 0$.

The story is not finished yet, for Maxwell's equations only tell us how electric and magnetic fields behave. How will these fields affect the dynamics of charged particles is a task left to the Lorentz force law,

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}, \tag{1.19}$$

where $\mathbf{v}$ is the velocity of a charged particle and q its charge. How can we write this equation in a tensorial form? First, observe that the equation can be rewritten as

$$\frac{d\mathbf{p}}{d\tau} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B},$$

or better yet, as

$$\frac{dp^i}{d\tau} = qE^i + q\epsilon^{ij}{}_k v_j B^k. \tag{1.20}$$

Here, p^i is the three-momentum of the particle and τ is its proper time. Next, we ask ourselves: can we promote this equation to one written in terms of four-vectors? Indeed, we can, for recall that the rate at which a charged particle gains energy in an electric field is given by

$$\frac{dE}{d\tau} = q\mathbf{E}\cdot\mathbf{v} = qE^i v_i. \tag{1.21}$$

Aha! The left-hand sides of the preceding two equations can be combined into our old friend, the four-momentum $p^\alpha = (E, p^i)$. On the other hand, in view of Eqs (1.11)–(1.13), the right-hand sides of the two equations are, respectively

$$\frac{dp^i}{d\tau} = q(F^{0i} - v_j F^{ij}), \tag{1.22}$$

$$\frac{dp^0}{d\tau} = qF^{0i}v_i. \tag{1.23}$$

It is now straightforward to arrive at the covariant version of the Lorentz force law:

$$\frac{dp^\alpha}{d\tau} = qF^{\alpha\beta}U_\beta, \tag{1.24}$$

where U^{β} is the particle's four-velocity. You can convince yourself that this equation is manifestly Lorentz invariant.

One final thing. Recall that the electric and the magnetic fields can be written in terms of potentials. Specifically,

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{1.25}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{1.26}$$

Now that we have identified $F^{\alpha\beta}$ as the fundamental entity of electrodynamics, it is natural to ask whether it can be written in terms of some potential. To this end, we first use eq. (1.11)–(1.13) to show that the eq. (1.25) above can be recast as

$$\begin{aligned}
B^i &= \epsilon^{ijk}\partial_j A_k, \\
\frac{1}{2}\epsilon^i{}_{pq}F^{pq} &= \epsilon^{ijk}\partial_j A_k, \\
\frac{1}{2}\epsilon_{imn}\epsilon^i{}_{pq}F^{pq} &= \epsilon_{imn}\epsilon^{ijk}\partial_j A_k, \\
\frac{1}{2}(\delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np})F^{pq} &= (\delta^j_m\delta^k_n - \delta^k_m\delta^j_n)\partial_j A_k, \\
\frac{1}{2}(F_{mn} - F_{nm}) &= \partial_m A_n - \partial_n A_m,
\end{aligned}$$

where in the first line, we used eq. (1.13), in the second line, we contracted on both sides with $\epsilon_i{}^{mn}$, in the third line, we used eq. (1.12) and in the last line, we used the anti-symmetry of F_{mn}. An easier calculation for eq. (1.26) (left as an exercise to the reader) gives the following tensorial form for the two equations:

$$F_{mn} = \partial_m A_n - \partial_n A_m, \tag{1.27}$$

$$F_{0i} = \partial_0 A_i - \partial_i A_0. \tag{1.28}$$

Now define $A_\alpha \equiv (-\phi, \mathbf{A})$ and $\partial_\alpha \equiv (\partial_0, \partial_i)$ to show that these equations reduce to

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{1.29}$$

This completes our description of covariant electrodynamics.

Chapter 2

Wave Equation and the Green Functions

The reformulation of Maxwell's equations into their explicitly Lorentz invariant form was the first step on our path to develop an approach to solve them. We shall venture forth by deriving the wave equation. Taking the curl of eq. (1.3), we get

$$\begin{aligned}
\nabla \times \nabla \times \mathbf{E} &= -\nabla \times \frac{\partial \mathbf{B}}{\partial t}, \\
\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}), \\
\nabla(\mu_0 \rho) - \nabla^2 \mathbf{E} &= -\frac{\partial}{\partial t}\left(\mu_0 \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}\right), \\
-\frac{\partial^2 \mathbf{E}}{\partial t^2} + \nabla^2 \mathbf{E} &= \mu_0 \left(\rho + \frac{\partial \mathbf{J}}{\partial t}\right),
\end{aligned}$$

where in the second line, we used the identity $\nabla \times \nabla \times \mathbf{c} = \nabla(\nabla \cdot \mathbf{c}) - \nabla^2 \mathbf{c}$ and in the third line, we used eq. (1.1) and (1.4). An analogous equation can be derived for the magnetic field. Defining the d'Alembertian operator $\Box \equiv \partial_\alpha \partial^\alpha = -\frac{\partial^2}{\partial t^2} + \nabla^2$, we can write down the wave equations:

$$\Box^2 \mathbf{E} = \mu_0 \left(\rho + \frac{\partial \mathbf{J}}{\partial t}\right) \tag{2.1}$$

$$\Box^2 \mathbf{B} = -\mu_0 (\nabla \times \mathbf{J}) \tag{2.2}$$

This seems a right mess and more importantly, difficult to solve. To make the equations more amenable to a solution, we express the fields in terms of the scalar and vector potentials and hope that the resulting wave equations are simpler. Substituting eq. (1.25) and (1.26) into eq. (1.1) and (1.3) and using the aforementioned curl identity, we get

$$\nabla^2 \phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\mu_0 \rho, \tag{2.3}$$

$$\nabla^2 \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left(\nabla \cdot \mathbf{A} + \frac{\partial \phi}{\partial t} \right) = -\mu_0 \mathbf{J}. \tag{2.4}$$

While these equations do not look any less complicated than the previous ones, we do have one important weapon up our sleeve – the gauge invariance of Maxwell's equations. The two most popular gauge choices are the Coulomb and the Lorenz[1] gauges respectively:

$$\partial_i A^i = \nabla \cdot \mathbf{A} = 0, \tag{2.5}$$

$$\partial_\alpha A^\alpha = \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0. \tag{2.6}$$

Using the Lorenz gauge, we can write down the wave equations for the scalar and vector potentials:

$$\Box^2 \phi_L = -\mu_0 \rho, \tag{2.7}$$

$$\Box^2 \mathbf{A}_L = -\mu_0 \mathbf{J}. \tag{2.8}$$

Denoting $A^\alpha \equiv (\phi, \mathbf{A})$ and $J^\alpha \equiv (\rho, \mathbf{J})$, we can rewrite these equations in the manifestly Lorentz invariant form,

$$\Box^2 A^\alpha = -\mu_0 J^\alpha, \tag{2.9}$$

where we have implicitly assumed that the Lorenz gauge choice has been made. The analogous calculation for the Coulomb gauge results in an equation that is not manifestly Lorentz invariant. Its derivation is left as an exercise to the reader.

We now have sufficient machinery to introduce the notion of Green's functions. A **Green's function** $G(x, x_0)$ is a solution to the linear differential equation,

$$\mathrm{L}\, G(x, x_0) = \delta(x - x_0), \tag{2.10}$$

where L is a linear differential operator and $\delta(x - x_0)$ is the Dirac delta function of the appropriate dimension. By finding the Green's function for a given linear differential equation, one can construct a solution for a different source term (i.e. the right-hand side of the differential equation) in the following way:

$$\int \mathrm{L}\, G(x, x_0) f(x_0)\, dx_0 = \int \delta(x - x_0) f(x_0)\, dx_0,$$

$$\mathrm{L} \int G(x, x_0) f(x_0)\, dx_0 = f(x),$$

[1] Named after the Danish physicist Ludvig Lorenz, not to be confused (but often done) with the Dutch physicist Hendrik Lorentz or the American mathematician Edward Norton Lorenz.

where in the second line, we used the fact that L is a differential operator on functions of x and hence can be moved outside the integral. We can now conclude that $u(x) = \int G(x, x_0) f(x_0)\, dx_0$ is the solution to the general linear differential equation,

$$Lu(x) = f(x). \tag{2.11}$$

Let us now see how to compute Green's functions in practice. Consider the premier equation in electrostatics, Poisson's equation,

$$\nabla^2 \phi(\mathbf{x}) = -\mu_0 \rho, \tag{2.12}$$

in a given volume V bounded by a surface S. The Dirac delta function in three dimensions is

$$\begin{gathered} \delta(\mathbf{x} - \mathbf{x_0}) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \\ \iiint_V V\, \delta(\mathbf{x} - \mathbf{x_0}) = 1 \qquad \text{iff } \mathbf{x_0} \in V. \end{gathered} \tag{2.13}$$

We want to solve eq. (2.12) for a general charge distribution $\rho(x, y, z)$ using the Green's function method outlined above. The Green's function for Poisson's equation is the solution to

$$\nabla^2 G(\mathbf{x}, \mathbf{x_0}) = \delta(\mathbf{x} - \mathbf{x_0}), \tag{2.14}$$

where we are considering the charge distribution of a point charge placed at $\mathbf{x_0}$. The reader may point out here that we have conveniently brushed $-\mu_0$ under the carpet. It, however, is only a constant which may be introduced later when building up our solution. Recall that we have the choice of two types of boundary conditions,

$$G = 0 \text{ on } S \qquad \text{(Dirichlet B.C.)}, \tag{2.15}$$

$$\nabla G \cdot \hat{\mathbf{n}} = \frac{\partial G}{\partial n} = \frac{1}{A} \text{ on } S \qquad \text{(Neumann B.C.)}, \tag{2.16}$$

where A is the area of the surface S and $\hat{\mathbf{n}}$ is the unit normal to the surface. Once specified, they ensure that G is unique. Note that if we define our volume to be all of space, then the Dirichlet boundary condition becomes $G(\mathbf{x}) \to 0$ as $\mathbf{x} \to \infty$, Similarly, since $A \to \infty$, the Neumann boundary condition becomes $\frac{\partial G}{\partial n}(\mathbf{x}) \to 0$ as $\mathbf{x} \to \infty$. Returning to eq. (2.14) and setting $\mathbf{x_0} = 0$, we observe that our problem is to find the electric potential for a point charge at the origin. Recall that this should be spherically symmetric and hence, G only depends on the radial distance r i.e. $G(\mathbf{x}) = g(r)$. Therefore, we consider a sphere of radius R as our surface S. We now proceed in steps, beginning from eq. (2.13):

$$
\begin{aligned}
1 &= \iiint_V dV\, \delta(\mathbf{x}), \\
&= \iiint_V dV\, \nabla \cdot (\nabla G), \\
&= \oiint_S dS\, \nabla G \cdot \hat{\mathbf{n}}, \\
&= \oiint_S dS\, \frac{\partial G}{\partial r}\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}, \\
&= \oiint_S dS\, g'(r), \\
&= 4\pi R^2 g'(R),
\end{aligned}
$$

where in the second line, we used eq. (2.14) with $\mathbf{x_0} = 0$, in the third line, we used the Divergence Theorem, in the fourth line, we noted that $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ on the sphere S and in the last line, we solved the surface integral for a sphere of radius R. Note that this is true for all spheres and hence, all R. This allows us to find the function $g(r)$:

$$
\begin{aligned}
g'(r) &= \frac{1}{4\pi r^2}, \\
g(r) &= -\frac{1}{4\pi r} + c.
\end{aligned}
$$

Finally, we apply the Dirichlet boundary conditions: as $r \to \infty$, $g(r) \to 0$. This implies that $c = 0$. Since $r = |\mathbf{x}|$ and making the substitution $\mathbf{x} \to \mathbf{x} - \mathbf{x_0}$, we get the Green's function for Poisson's equation,

$$
G(\mathbf{x}, \mathbf{x_0}) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x_0}|}. \tag{2.17}
$$

Now, if we need to solve for a point charge q at $\mathbf{x_0}$, we just multiply G with $-\mu_0 q$. But we wanted to solve Poisson's equation for a general charge distribution. The important idea here, at least intuitively, is that each charge distribution may be modelled as a collection of point charges. To make things precise, we use Green's second identity,

$$
\iiint_V dV (\phi \nabla^2 \psi - \psi \nabla^2 \phi) = \oiint_S dS(\phi \nabla \psi - \psi \nabla \phi) \cdot \hat{\mathbf{n}}. \tag{2.18}
$$

Let $\psi = G$ and let ϕ satisfy eq. (2.12). Then,

$$\iiint_V dV (\phi \nabla^2 G - G \nabla^2 \phi) = \oiint_S dS (\phi \nabla G \cdot \hat{\mathbf{n}} - G \nabla \phi \cdot \hat{\mathbf{n}}),$$
$$\iiint_V dV (\phi \delta(\mathbf{x} - \mathbf{x_0}) + \mu_0 \rho G) = \oiint_S dS \left(\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n} \right),$$

where in the second line, we used eq. (2.12) and (2.14). Using the Dirac delta function to get rid of the integral on ϕ, we get the general solution for an arbitrary charge distribution,

$$\phi(\mathbf{x_0}) = -\mu_0 \iiint_V dV \, \rho(\mathbf{x}) G(\mathbf{x} - \mathbf{x_0}) + \oiint_S dS \, \phi \frac{\partial G}{\partial n} - \oiint_S dS \, G \frac{\partial \phi}{\partial n}. \tag{2.19}$$

What an extraordinary result! Notice that we need to solve once for the Green's function and then, we *only* need to solve the three integrals to get the solution. This solution is unique once we are provided one of the two choices of boundary conditions,

$$\phi = f \text{ on } S \qquad \text{(Dirichlet B.C.)}, \tag{2.20}$$
$$\nabla \phi \cdot \hat{\mathbf{n}} = \frac{\partial \phi}{\partial n} = h \text{ on } S \qquad \text{(Neumann B.C.)}, \tag{2.21}$$

where f and h are some functions. Note that one of the last two terms in eq. (2.19) vanishes after choosing one of the two boundary conditions in eq. (2.15) and (2.16). The other integral can then be solved by choosing from eq. (2.20) and (2.21) the counterpart of the boundary conditions we chose earlier. The first term does not depend on the boundary condition explicitly and instead depends only on the charge distribution and the Green's function. In fact, if we choose to integrate over all space, then both the surface terms vanish and we need only solve the first term. The resultant solution, using eq. (2.17), is

$$\phi(\mathbf{x_0}) = -\frac{\mu_0}{4\pi} \iiint_V dV \, \frac{\rho(\mathbf{x})}{|\mathbf{x} - \mathbf{x_0}|}. \tag{2.22}$$

Now, Poisson's equation is not only limited to electrostatics. In fact, it pops up in Newtonian gravity as well. There, we want to solve for the gravitational potential given a mass distribution. Why does the Green's function method work here? The attentive reader may be able to point out that this is because Poisson's equation is linear and a superposition of solutions is also a solution. In general, the Green's function method would not work for a non-linear differential equation, with the prime examples being the Einstein field equations of general relativity.

Let us go back to where we started. We want to solve the wave equation for the electric and magnetic fields. In particular, we now have a method to solve an equation similar in form to eq. (2.7). Laying our goal out clearly, we need to solve the following equation:

$$\Box^2\psi(\mathbf{r},t) = -f(\mathbf{r},t). \tag{2.23}$$

The Green's function for eq. (2.23) is the solution to the equation,

$$\Box^2 G(\mathbf{r},t;\mathbf{r}',t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{2.24}$$

where the minus sign is added in hindsight. Here, $\Box^2 \equiv -\frac{\partial^2}{\partial t'^2} + \nabla'^2$ by convention, the sources are located at $(t',\mathbf{r}')$ and we are evaluating the potentials at $(t,\mathbf{r})$. Multiplying both sides by $\psi(\mathbf{r}',t')$ and integrating over the primed variables, we get

$$\int d^3r'\,dt'\,\Box^2 G(\mathbf{r},t;\mathbf{r}',t')\psi(\mathbf{r}',t') = -\int d^3r'\,dt'\,\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\psi(\mathbf{r}',t'),$$

$$\psi(\mathbf{r},t) = \int d^3r'\,dt'\,\psi(\mathbf{r}',t')\frac{\partial^2 G}{\partial t'^2} - \int d^3r'\,dt'\,\psi(\mathbf{r}',t')\nabla'^2 G. \tag{2.25}$$

We now solve the dt' integral on the left by integrating by parts twice.

$$\begin{aligned}\int dt'\,\psi(\mathbf{r}',t')\frac{\partial^2 G}{\partial t'^2} &= \psi\frac{\partial G}{\partial t'}\bigg|_{t_1}^{t_2} - \int dt'\,\frac{\partial G}{\partial t'}\frac{\partial \psi}{\partial t'},\\ &= \psi\frac{\partial G}{\partial t'}\bigg|_{t_1}^{t_2} - G\frac{\partial \psi}{\partial t'}\bigg|_{t_1}^{t_2} + \int dt'\,G\frac{\partial^2\psi}{\partial t'^2}.\end{aligned}$$

Next, we solve the d^3r' integral on the right using eq. (2.18) with $\phi \to \psi$ and $\psi \to G$.

$$\begin{aligned}\int d^3r'(\psi\nabla'^2 G - G\nabla'^2\psi) &= \oiint_S dS(\psi\nabla' G - G\nabla'\psi)\cdot\hat{\mathbf{n}},\\ \int d^3r'\,\psi(\mathbf{r}',t')\nabla'^2 G &= \int d^3r'\,G\nabla'^2\psi + \oiint_S dS(\psi\nabla' G - G\nabla'\psi)\cdot\hat{\mathbf{n}}.\end{aligned}$$

Substituting these solved integrals into eq. (2.25),

$$\begin{aligned}\psi(\mathbf{r},t) &= \int d^3r'\left[\psi\frac{\partial G}{\partial t'} - G\frac{\partial\psi}{\partial t'}\bigg|_{t_1}^{t_2} + \int dt'\,G\frac{\partial^2\psi}{\partial t'^2}\right] + \\ &\quad -\int dt'\left[\int d^3r'\,G\nabla'^2\psi + \oiint_S dS(\psi\nabla' G - G\nabla'\psi)\cdot\hat{\mathbf{n}}\right],\\ &= \int d^3r'\,dt'\,G\left(\frac{\partial^2\psi}{\partial t'^2} - \nabla'^2\psi\right) + \int d^3r'\left[\psi\frac{\partial G}{\partial t'} - G\frac{\partial\psi}{\partial t'}\right]_{t_1}^{t_2}\\ &\quad -\int dt'\oiint_S dS(\psi\nabla' G - G\nabla'\psi)\cdot\hat{\mathbf{n}},\end{aligned}$$

and finally, using eq. (2.23),

$$\begin{aligned}\psi(\mathbf{r},t) = \int d^3r'\,dt'\,G(\mathbf{r},t;\mathbf{r}',t')f(\mathbf{r}',t') + \int d^3r'\left[\psi\frac{\partial G}{\partial t'} - G\frac{\partial \psi}{\partial t'}\right]_{t_1}^{t_2} \\ - \int dt' \oiint_S dS(\psi\nabla' G - G\nabla'\psi)\cdot\hat{\mathbf{n}}.\end{aligned} \tag{2.26}$$

The last two integrals are boundary terms in time and space respectively. To solve the first integral, we need to know the Green's function and the source term. Our next step, therefore, is to find G. Let us simplify our problem by considering the point charge at the origin in spacetime i.e. at $\mathbf{r}' = 0$ and $t' = 0$. This charge distribution is that of a point charge that exists momentarily and then never returns. Hence, eq. (2.24) becomes

$$\Box^2 G(\mathbf{r},t) = -\delta(\mathbf{r})\delta(t). \tag{2.27}$$

There are no primed variables and $\Box^2 \equiv -\frac{\partial^2}{\partial t^2} + \nabla^2$. Again, we use the arguments we used for Poisson's equation to conclude that our solution should be spherically symmetric i.e. $G(\mathbf{r},t) = G(r,t)$, where $r = |\mathbf{r}|$. Using spherical coordinates, the Laplacian becomes

$$\begin{aligned}\nabla^2 G &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right) \\ &= \frac{1}{r^2}\left(2r\frac{\partial G}{\partial r} + r^2\frac{\partial G}{\partial r^2}\right) \\ &= \frac{1}{r}\frac{\partial}{\partial r^2}(rG).\end{aligned}$$

The reader is invited to verify the last line. Denoting $\partial_t \equiv \frac{\partial}{\partial t}$, and noting that $\partial_t^2 G = \frac{1}{r}\partial_t^2(rG)$,

$$\begin{aligned}\frac{1}{r}(-\partial_t^2 + \partial_r^2)(rG) &= -\delta(r)\delta(t), \\ \frac{1}{r}(-\partial_t + \partial_r)(\partial_t + \partial_r)(rG) &= -\delta(r)\delta(t).\end{aligned}$$

Note that the operator in the first equality above is just the differential operator for the 2D wave equation, which we already know the solution to. In the region $r > 0$, $\delta(r) = 0$, and we have two solutions,

$$G_\pm(r,t) = \frac{1}{r}g_\pm(t \pm r). \tag{2.28}$$

where $g_\pm$ are the two solutions to the 2D wave equation. For the full solution, we substitute this Ansatz into eq. (2.27):

$$\begin{aligned}\Box^2 G_\pm &= \Box^2\left(\frac{1}{r}g_\pm\right),\\ &= -\frac{\partial^2}{\partial t^2}\left(\frac{1}{r}g_\pm\right) + \nabla^2\left(\frac{1}{r}g_\pm\right).\end{aligned}$$

Evaluating the Laplacian, we get

$$\begin{aligned}\nabla^2\left(\frac{1}{r}g_\pm\right) &= \nabla\cdot\nabla\left(\frac{1}{r}g_\pm\right),\\ &= \nabla\cdot\left[\frac{1}{r}\nabla g_\pm + g_\pm\nabla\left(\frac{1}{r}\right)\right],\\ &= \frac{1}{r}\nabla^2 g_\pm + \nabla\left(\frac{1}{r}\right)\cdot\nabla g_\pm + \nabla g_\pm\cdot\nabla\left(\frac{1}{r}\right) + g_\pm\nabla^2\left(\frac{1}{r}\right),\\ &= \frac{1}{r}\nabla^2 g_\pm - 2\nabla g_\pm\cdot\frac{\hat{\mathbf{r}}}{r^2} - 4\pi\delta(\mathbf{r})g_\pm,\\ &= \frac{1}{r}\frac{\partial^2 g_\pm}{\partial r^2} - 4\pi\delta(\mathbf{r})g_\pm,\end{aligned}$$

where in the second line, we used the product rule for gradients, in the third line, we used the product rule for divergences and in the last line, we used the gradient and Laplacian of $\frac{1}{r}$. The reader is encouraged to calculate the last two lines explicitly. Working further,

$$\begin{aligned}\Box^2 G_\pm &= -\frac{\partial^2}{\partial t^2}\left(\frac{1}{r}g_\pm\right) + \frac{1}{r}\frac{\partial^2 g_\pm}{\partial r^2} - 4\pi\delta(\mathbf{r})g_\pm,\\ &= \frac{1}{r}\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)g_\pm - 4\pi\delta(\mathbf{r})g_\pm,\\ &= -4\pi\delta(\mathbf{r})g_\pm,\end{aligned}$$

where in the last line, we used the fact that $g_\pm$ are the two solutions to the 2D wave equation. Our aim was to solve eq. (2.27). In that vein, note that if $4\pi g_\pm = \delta(t)$, then we have our solution. There is one subtlety to note here though. $g_\pm(t\pm r)$ is a function of $t\pm r$ so the Dirac delta should also be a function of $t\pm r$ instead of t. Combining it all, we have our solution

$$G_\pm(\mathbf{r},t) = \frac{1}{4\pi r}\delta(t\pm r). \tag{2.29}$$

The more general Green's function is the solution to eq. (2.24) and can, in view of eq. (2.29), be found to be

$$G_{\pm}(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\delta[(t-t') \pm |\mathbf{r}-\mathbf{r}'|]. \tag{2.30}$$

Let us study this solution for a bit. The primed variables are the sources while the unprimed variables are the points at which one is measuring the fields. Note that the Green's functions are only non-zero when $t = t' \mp |\mathbf{r}-\mathbf{r}'|$. In particular, $t < t'$ for G_+ and $t > t'$ for G_-. This is an astonishing result. It states, in no uncertain terms, that for G_+, one measures the effect of the fields *before* they have been produced by the sources! G_+ is therefore called the *advanced Green's function*. Similarly, for G_-, one measures the effect of the fields some finite time after they have been produced by the sources. G_- is therefore, called the *retarded Green's function*. At this point, the reader is invited to pause and ponder over both the results and what they have come to be called.

To make things even clearer, we use the Green's functions in eq. (2.26) to find the solution for a given charge distribution. Note that we can always take an infinite volume so that the spatial boundary terms vanish. Looking at the time boundary term, we observe that the time t at which we measure the fields is between t_1 and t_2. In particular, for G_+, the earlier boundary term vanishes since $t_1 < t < t'$, while for G_-, the later boundary term vanishes since $t_2 > t > t'$. Therefore, our advanced and retarded solutions are

$$\psi_+(\mathbf{r},t) = \int d^3r'\,\frac{f(\mathbf{r}',t+|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|} + \psi_{\text{out}}(\mathbf{r},t_2), \tag{2.31}$$

$$\psi_-(\mathbf{r},t) = \int d^3r'\,\frac{f(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|} + \psi_{\text{in}}(\mathbf{r},t_1), \tag{2.32}$$

where the ψ_{out} and ψ_{in} are the results of the evaluation of the time boundary integrals at times t_2 and t_1 respectively. Now, in the words of Dr. Sheldon Cooper, we "subscribe to a linear understanding of time and causality." Therefore, we discard the advanced solution on physical terms – fields in the present cannot be caused by a future source.

The reader is now encouraged to work through some problems to test their understanding of this material. An example problem is to calculate the fields due to a charge moving on the positive z–axis.

Chapter 3

Liénard-Weichert Potentials

We begin this section with a discussion of the wave equation for potentials in index notation, eq. (2.9). First, we write it out to explicitly show the dependence on the spacetime coordinates x^β:

$$\Box^2 A^\alpha(x^\beta) = -\mu_0 J^\alpha. \tag{3.1}$$

Note that this is just eq.(2.23) with $f(\mathbf{r}, t) = \mu_0 J^\alpha$. The solution to this, in accordance with eq. (2.26) is

$$A^\alpha(x^\beta) = \mu_0 \int d^4x' \, G_-(x^\beta - x'^\beta) J^\alpha(x'^\beta), \tag{3.2}$$

after appropriately setting the boundary terms to 0. We now need to write everything in index notation. The retarded Green's function $G_-(\mathbf{r} - \mathbf{r}', t - t')$ from eq. (2.30) is

$$G_-(\mathbf{r} - \mathbf{r}', t - t') = \frac{1}{4\pi R} \delta(t - t' - R), \tag{3.3}$$

where $R = |\mathbf{r} - \mathbf{r}'|$. Note that G_- only contributes when $t = t' + R$. We can therefore insert a Heaviside step function into the expression without disturbing its meaning:

$$G_-(\mathbf{r} - \mathbf{r}', t - t') = \frac{\Theta(t - t')}{4\pi R} \delta(t - t' - R). \tag{3.4}$$

With benefit of hindsight, we calculate $\Theta(t - t')\delta[(x^\beta - x'^\beta)^2]$:

$$\begin{aligned}
\Theta(t - t')\delta[(x^\beta - x'^\beta)^2] &= \Theta(t - t')\delta[-(t - t')^2 + R^2], \\
&= \Theta(t - t')\delta[(t - t')^2 - R^2], \\
&= \Theta(t - t')\delta[(t - t' + R)(t - t' - R)], \\
&= \frac{1}{2R}[\Theta(t - t')\delta(t - t' + R) + \Theta(t - t')\delta(t - t' - R)], \\
&= \frac{1}{2R}[\Theta(t - t')\delta(t - t' - R)],
\end{aligned}$$

where in the last line, we used the fact that the first delta function contributes at $t = t' - R$, but the Heaviside step function is zero for $t < 0$. In the second last line, we used the delta function identity,

$$\delta(x^2 - a^2) = \frac{1}{2|a|}[\delta(x+a) + \delta(x-a)]. \tag{3.5}$$

Combining the result of this calculation with eq. (3.4), we get

$$G_-(x^\beta - x'^\beta) = \frac{1}{2\pi}\Theta(t-t')\delta[(x^\beta - x'^\beta)^2]. \tag{3.6}$$

Now, for a charged particle in arbitrary motion specified by $r^\alpha(\tau)$, the four-current density is given by

$$J^\alpha(x'^\beta) = q\int d\tau\, V^\alpha(\tau)\delta[x'^\beta - r^\beta(\tau)], \tag{3.7}$$

where $V^\alpha(\tau)$ is the four-velocity of the charge q. The reader is invited to verify that this is equivalent to the following in vector notation:

$$\rho(\mathbf{x}', t') = q\delta(\mathbf{x} - \mathbf{r}(t')), \tag{3.8}$$

$$\mathbf{J}(\mathbf{x}', t') = \mathbf{v}(t')\rho(\mathbf{x}', t'). \tag{3.9}$$

Using these in eq. (3.2) and solving the primed integral,

$$A^\alpha(x^\beta) = \frac{\mu_0 q}{2\pi}\int d\tau\, V^\alpha(\tau)\Theta(t - r^0(\tau))\delta[(x^\beta - r^\beta(\tau))^2]. \tag{3.10}$$

Note that the delta function contributes when $(x^\beta - r^\beta(\tau_0))^2 = 0$. This is just the light cone condition which implies that x^β and $r^\beta(\tau_0)$ are null-separated. Further, note that the Heaviside step function is non-zero only for $t > r^0(\tau)$. These two conditions imply that $r^\beta(\tau_0)$ lies on the past light cone of the observer. To proceed further, we need the delta function identity,

$$\delta[f(x)] = \sum_{i=1}^{\infty}\frac{\delta(x - x_i)}{|f'(x_i)|}, \tag{3.11}$$

where x_i are the roots of $f(x)$. Consider $f(\tau) = (x^\beta - r^\beta(\tau))^2$. Then,

$$\delta[f(\tau)] = \frac{\delta(\tau - \tau_0)}{|f'(\tau_0)|}. \tag{3.12}$$

Evaluating the derivative of $f(\tau)$,

$$\begin{aligned}\frac{d}{d\tau}(x^\beta - r^\beta(\tau))^2 &= \frac{d}{d\tau}[\eta_{\alpha\beta}(x^\alpha - r^\alpha(\tau))(x^\beta - r^\beta(\tau))],\\ &= \eta_{\alpha\beta}\left(-\frac{dr^\alpha}{d\tau}\right)(x^\beta - r^\beta(\tau)) + \eta_{\alpha\beta}\left(-\frac{dr^\beta}{d\tau}\right)(x^\alpha - r^\alpha(\tau)),\\ &= \eta_{\alpha\beta}\left(-\frac{dr^\alpha}{d\tau}\right)(x^\beta - r^\beta(\tau)) + \eta_{\beta\alpha}\left(-\frac{dr^\alpha}{d\tau}\right)(x^\beta - r^\beta(\tau)),\end{aligned}$$

where in the third line, we renamed the indices $\alpha \leftrightarrow \beta$ in the second term. Using the fact that the Minkowski metric is symmetric and that the derivative of the position of the charged particle gives its four-velocity, we get

$$\frac{df}{d\tau} = -2\eta_{\alpha\beta}V^\alpha(\tau)(x^\beta - r^\beta(\tau)), \tag{3.13}$$

Therefore,

$$\delta[(x^\beta - r^\beta(\tau))^2] = \frac{\delta(\tau - \tau_0)}{|2\eta_{\rho\sigma}V^\rho(\tau)(x^\sigma - r^\sigma(\tau))|_{\tau=\tau_0}}. \tag{3.14}$$

Using this in eq. (3.10),

$$\begin{aligned}A^\alpha(x^\beta) &= \frac{\mu_0 q}{2\pi}\int d\tau\, V^\alpha(\tau)\Theta(t - r^0(\tau))\frac{\delta(\tau - \tau_0)}{|2\eta_{\rho\sigma}V^\rho(\tau)(x^\sigma - r^\sigma(\tau))|_{\tau=\tau_0}},\\ &= \frac{\mu_0 q}{4\pi}\frac{V^\alpha(\tau_0)\Theta(t - r^0(\tau_0))}{|\eta_{\rho\sigma}V^\rho(\tau)(x^\sigma - r^\sigma(\tau))|_{\tau=\tau_0}}.\end{aligned}$$

Note that τ_0 is defined to ensure that $t > r^0(\tau_0)$. Hence,

$$A^\alpha(x^\beta) = \frac{\mu_0 q}{4\pi}\frac{V^\alpha(\tau)}{|\eta_{\rho\sigma}V^\rho(\tau)(x^\sigma - r^\sigma(\tau))|}\Bigg|_{\tau=\tau_0}. \tag{3.15}$$

We now proceed to recover the scalar and three-vector potential from this. Recall that $V^\alpha(\tau) = (\gamma, \gamma\mathbf{v})$ and denote $\mathbf{R}(\tau) \equiv R\hat{\mathbf{n}} \equiv \mathbf{x} - \mathbf{r}(\tau)$. Then, $R = |\mathbf{x} - \mathbf{r}(\tau)|$. The light cone condition then ensures that $-(t - r^0(\tau_0))^2 + |\mathbf{x} - \mathbf{r}(\tau_0)|^2 = 0$ and

$$t - r^0(\tau_0) = R. \tag{3.16}$$

Now, we solve the denominator in eq. (3.15):

$$\begin{aligned}|\eta_{\rho\sigma}V^\rho(\tau)(x^\sigma - r^\sigma(\tau))|_{\tau=\tau_0} &= |-\gamma(t - r^0(\tau)) + \gamma\mathbf{v}\cdot(\mathbf{x} - \mathbf{r}(\tau))|_{\tau=\tau_0},\\ &= |-\gamma R + \gamma R\mathbf{v}\cdot\hat{\mathbf{n}}|_{\tau=\tau_0}.\end{aligned}$$

Using the fact that $\mathbf{v}\cdot\hat{\mathbf{n}} < 1$ since the particle's speed $v < 1$,

$$|\eta_{\rho\sigma}V^\rho(\tau)(x^\sigma - r^\sigma(\tau))|_{\tau=\tau_0} = \gamma R(1-\mathbf{v}\cdot\hat{\mathbf{n}})|_{\tau=\tau_0}, \tag{3.17}$$

Putting all of this in eq. (3.15) and noting that $\tau = \tau_0$ is the condition that imposes the retarded time, we get

$$\phi(\mathbf{x},t) = \frac{\mu_0 q}{4\pi}\left[\frac{1}{(1-\mathbf{v}\cdot\hat{\mathbf{n}})R}\right]_{\text{ret}}, \tag{3.18}$$

$$\mathbf{A}(\mathbf{x},t) = \mathbf{v}\phi(\mathbf{x},t). \tag{3.19}$$

Our next target is the evaluation of the fields. Before that, though, we need to find the electromagnetic field strength tensor, $F_{\alpha\beta}$. Recalling eq. (1.29), suppressing the dependences in eq. (3.15), denoting $\Theta \equiv \Theta(t - r^0(\tau))$ and $\delta \equiv \delta[(x^\sigma - r^\sigma(\tau))^2]$ and solving for the derivative,

$$\begin{aligned}\partial_\alpha A_\beta &= \partial_\alpha\left[\frac{\mu_0 q}{4\pi}\int d\tau(V_\beta\Theta\delta)\right],\\ &= \frac{\mu_0 q}{4\pi}\int d\tau[V_\beta\partial_\alpha(\Theta)\delta] + \frac{\mu_0 q}{4\pi}\int d\tau[V_\beta\Theta\partial_\alpha(\delta)],\end{aligned} \tag{3.20}$$

where in the second line, we used the fact that the four-velocity of the particle V^α does not depend on the observer's coordinates x^α. Calculating the two derivatives separately, we begin by noticing that Θ only depends on t. Therefore, $\partial_\alpha(\Theta) = \partial_t(\Theta) = \delta(t - r^0(\tau))$. Next,

$$\begin{aligned}\partial_\alpha(\Theta)\delta &= \delta(t-r^0(\tau))\delta[(x^\beta - r^\beta(\tau))^2],\\ &= \delta(t-r^0(\tau))\delta[-(t-r^0(\tau))^2 + |\mathbf{x}-\mathbf{r}|^2].\end{aligned}$$

Using the fact that the first delta function only contributes at $t = r^0(\tau)$, we get

$$\partial_\alpha(\Theta)\delta = \delta(R^2). \tag{3.21}$$

This only contributes at $R = 0$. This means that the observer and the charge are at the same position and hence, can be excluded on physical grounds. Next, we compute $\partial_\alpha(\delta)$. Denoting $f(\tau) \equiv (x^\sigma - r^\sigma(\tau))^2$ once again and using the chain rule,

$$\begin{aligned}\partial_\alpha[\delta(f)] &= \partial_\alpha f\frac{d}{df}\delta(f),\\ &= \partial_\alpha f\frac{d\tau}{df}\frac{d}{d\tau}\delta(f).\end{aligned}$$

We now evaluate the terms in this expression, beginning with $\partial_\alpha f$:

$$\begin{aligned}
\partial_\alpha f &= \partial_\alpha[\eta_{\rho\sigma}(x^\rho - r^\rho)(x^\sigma - r^\sigma)], \\
&= \eta_{\rho\sigma}(x^\rho - r^\rho)\delta^\sigma_\alpha + \eta_{\rho\sigma}(x^\sigma - r^\sigma)\delta^\rho_\alpha, \\
&= \eta_{\rho\alpha}(x^\rho - r^\rho) + \eta_{\alpha\sigma}(x^\sigma - r^\sigma), \\
&= \eta_{\rho\alpha}(x^\rho - r^\rho) + \eta_{\alpha\rho}(x^\rho - r^\rho), \\
&= 2\eta_{\rho\alpha}(x^\rho - r^\rho), \\
&= 2(x_\alpha - r_\alpha),
\end{aligned}$$

where in the second line, we used the fact that the position of the particle r^α does not depend on the observer's coordinates x^α, in the fourth line, we renamed $\sigma \to \rho$ in the second term and in the fifth line, we used the fact that the Minkowski metric is symmetric. Recalling eq. (3.13), putting everything together and simplifying,

$$\partial_\alpha[\delta(f)] = -\frac{x_\alpha - r_\alpha}{V_\rho(x^\rho - r^\rho)}\frac{d}{d\tau}\delta(f). \tag{3.22}$$

Putting eqs. (3.21) and (3.22) in eq. (3.20),

$$\partial_\alpha A_\beta = -\frac{\mu_0 q}{4\pi}\int d\tau\, \frac{V_\beta(x_\alpha - r_\alpha)}{V_\rho(x^\rho - r^\rho)}\Theta(t - r^0)\frac{d}{d\tau}\delta[(x^\sigma - r^\sigma)^2]. \tag{3.23}$$

We now define $T_{\alpha\beta} \equiv \frac{V_\beta(x_\alpha - r_\alpha)}{V_\rho(x^\rho - r^\rho)}$. Consider the following:

$$\int d\tau\, \frac{d}{d\tau}(T_{\alpha\beta}\Theta\delta) = \int d\tau\left[\frac{d}{d\tau}(T_{\alpha\beta})\Theta\delta + T_{\alpha\beta}\frac{d}{d\tau}(\Theta)\delta + T_{\alpha\beta}\Theta\frac{d}{d\tau}(\delta)\right]. \tag{3.24}$$

Let us dissect this equation. The term on the left is a boundary term which we can set to 0. Using a similar procedure to what we used to obtain eq. (3.21), the reader is invited to verify that the second term on the right is proportional to $\delta(R^2)$ and can therefore be excluded on physical grounds. The last term on the right is proportional to $\partial_\alpha A_\beta$. Hence,

$$\partial_\alpha A_\beta = \frac{\mu_0 q}{4\pi}\int d\tau\, \frac{d}{d\tau}(T_{\alpha\beta})\Theta(t - r^0(\tau))\delta[(x^\beta - r^\beta(\tau))^2]. \tag{3.25}$$

The alert reader may point out that this equation has the same structure as eq. (3.10) with $V^\alpha(\tau) \to \frac{d}{d\tau}(T_{\alpha\beta})$. The only thing left to do then is to make the appropriate replacement in the final answer given by eq. (3.15) to get

$$\partial_\alpha A_\beta = \frac{\mu_0 q}{4\pi}\frac{1}{|\eta_{\rho\sigma}V^\rho(\tau)(x^\sigma - r^\sigma(\tau))|}\frac{d}{d\tau}T_{\alpha\beta}\bigg|_{\tau=\tau_0}. \tag{3.26}$$

Using this in eq. (1.29), renaming and lowering some indices along the way, yields

$$F_{\alpha\beta} = \frac{\mu_0 q}{4\pi} \frac{1}{|V_\rho(\tau)(x^\rho - r^\rho(\tau))|} \frac{d}{d\tau} \left[\frac{(x_\alpha - r_\alpha(\tau))V_\beta(\tau) - (x_\beta - r_\beta(\tau))V_\alpha(\tau)}{V_\sigma(\tau)(x^\sigma - r^\sigma(\tau))} \right] \Bigg|_{\tau=\tau_0}. \tag{3.27}$$

The next step is to find the electric and magnetic fields. To do this, we need to perform some preliminary calculations. First, recall that $V^\alpha = (\gamma, \gamma\mathbf{v})$. Differentiating γ w.r.t. τ gives

$$\begin{aligned} \frac{d}{d\tau}\gamma &= \frac{dt}{d\tau}\frac{d}{dt}\frac{1}{\sqrt{1-v^2}}, \\ &= \gamma\left(-\frac{1}{2}\right)(1-v^2)^{-3/2}\left(-2\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right), \\ &= \gamma^4 \mathbf{v}\cdot\mathbf{a}, \end{aligned}$$

where in the last line, we used the facts that $\gamma = (1-v^2)^{-1/2}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt}$. Now, differentiating $\mathbf{v}$ w.r.t. τ gives

$$\begin{aligned} \frac{d}{d\tau}\mathbf{v} &= \frac{dt}{d\tau}\frac{d}{dt}\mathbf{v}, \\ &= \gamma\mathbf{a}. \end{aligned}$$

Combining these two results,

$$\begin{aligned} \frac{d}{d\tau}(\gamma\mathbf{v}) &= \frac{d}{d\tau}(\gamma)\mathbf{v} + \gamma\frac{d}{d\tau}(\mathbf{v}), \\ &= (\gamma^4\mathbf{v}\cdot\mathbf{a})\mathbf{v} + \gamma(\gamma\mathbf{a}), \\ &= \gamma^2\mathbf{a} + \gamma^4(\mathbf{v}\cdot\mathbf{a})\mathbf{v}. \end{aligned}$$

Therefore, after lowering the index,

$$\frac{dV_\alpha}{d\tau} = (-\gamma^4\mathbf{v}\cdot\mathbf{a}, \gamma^2\mathbf{a} + \gamma^4(\mathbf{v}\cdot\mathbf{a})\mathbf{v}). \tag{3.28}$$

Next, we calculate the derivative of the denominator in eq. (3.27) w.r.t. τ:

$$\begin{aligned} \frac{d}{d\tau}[V_\alpha(\tau)(x^\alpha - r^\alpha(\tau))] &= (x^\alpha - r^\alpha(\tau))\frac{dV_\alpha}{d\tau} + V_\alpha(\tau)[-V^\alpha(\tau)], \\ &= 1 + (x^\alpha - r^\alpha(\tau))\frac{dV_\alpha}{d\tau}, \end{aligned}$$

where in the last line, we used the fact that $V_\alpha V^\alpha = -1$ for any four-velocity V^α. From eq. (3.16) and the preceding discussion, it is easy to see that $x^\alpha - r^\alpha = (R, R\hat{\mathbf{n}})$.

Using this, we can find

$$\begin{aligned}(x^\alpha - r^\alpha(\tau))\frac{dV_\alpha}{d\tau} &= (R, R\hat{\mathbf{n}}) \cdot (-\gamma^4 \mathbf{v}\cdot\mathbf{a}, \gamma^2\mathbf{a} + \gamma^4(\mathbf{v}\cdot\mathbf{a})\mathbf{v}),\\ &= -\gamma^4 R(\mathbf{v}\cdot\mathbf{a}) + \gamma^2 R(\mathbf{a}\cdot\hat{\mathbf{n}}) + \gamma^4 R(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\hat{\mathbf{n}}),\\ &= -\gamma^4 R(\mathbf{v}\cdot\mathbf{a})(1-\mathbf{v}\cdot\hat{\mathbf{n}}) + \gamma^2 R(\mathbf{a}\cdot\hat{\mathbf{n}}).\end{aligned}$$

In particular,

$$\frac{d}{d\tau}[V_\alpha(\tau)(x^\alpha - r^\alpha(\tau))] = 1 - \gamma^4 R(\mathbf{v}\cdot\mathbf{a})(1-\mathbf{v}\cdot\hat{\mathbf{n}}) + \gamma^2 R(\mathbf{a}\cdot\hat{\mathbf{n}}). \tag{3.29}$$

Finally, differentiating the first term in the numerator in eq. (3.27) w.r.t. τ,

$$\begin{aligned}\frac{d}{d\tau}[(x_\alpha - r_\alpha(\tau))V_\beta(\tau)] &= (x_\alpha - r_\alpha(\tau))\frac{dV_\beta}{d\tau} + V_\beta(\tau)\frac{d}{d\tau}(x_\alpha - r_\alpha(\tau)),\\ &= (x_\alpha - r_\alpha(\tau))\frac{dV_\beta}{d\tau} - V_\alpha(\tau)V_\beta(\tau).\end{aligned}$$

Using eq. (3.17) and all the results we have just calculated,

$$\begin{aligned}F_{\alpha\beta} = &\frac{\mu_0 q}{4\pi}\frac{1}{|V_\rho(x^\rho - r^\rho)|[V_\sigma(x^\sigma - r^\sigma)]^2}\Bigg[V_\mu(x^\mu - r^\mu)\Bigg[(x_\alpha - r_\alpha)\frac{dV_\beta}{d\tau} + V_\beta\frac{d}{d\tau}(x_\alpha - r_\alpha) +\\ &- (x_\beta - r_\beta)\frac{dV_\alpha}{d\tau} - V_\alpha\frac{d}{d\tau}(x_\beta - r_\beta)\Bigg] - [(x_\alpha - r_\alpha)V_\beta - (x_\beta - r_\beta)V_\alpha]\frac{d}{d\tau}[V_\nu(x^\nu - r^\nu)]\Bigg],\\ = &\frac{\mu_0 q}{4\pi}\frac{1}{\gamma^3 R^3(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}\Bigg[-\gamma R(1-\mathbf{v}\cdot\hat{\mathbf{n}})\Bigg[(x_\alpha - r_\alpha)\frac{dV_\beta}{d\tau} - V_\beta V_\alpha +\\ &- (x_\beta - r_\beta)\frac{dV_\alpha}{d\tau} + V_\alpha V_\beta\Bigg] - [1 - \gamma^4 R(\mathbf{v}\cdot\mathbf{a})(1-\mathbf{v}\cdot\hat{\mathbf{n}}) + \gamma^2 R(\mathbf{a}\cdot\hat{\mathbf{n}})]\times\\ &\times [(x_\alpha - r_\alpha)V_\beta - (x_\beta - r_\beta)V_\alpha]\Bigg].\end{aligned}$$

Finally, after tidying up the expression, we get

$$\begin{aligned}F_{\alpha\beta} = &\frac{\mu_0 q}{4\pi}\frac{1}{\gamma^2 R^2(1-\mathbf{v}\cdot\hat{\mathbf{n}})^2}\left[(x_\beta - r_\beta)\frac{dV_\alpha}{d\tau} - (x_\alpha - r_\alpha)\frac{dV_\beta}{d\tau}\right] +\\ &+ \frac{\mu_0 q}{4\pi}\frac{1}{\gamma^3 R^3(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}[1 - \gamma^4 R(\mathbf{v}\cdot\mathbf{a})(1-\mathbf{v}\cdot\hat{\mathbf{n}}) + \gamma^2 R(\mathbf{a}\cdot\hat{\mathbf{n}})]\times\\ &\times [(x_\beta - r_\beta)V_\alpha - (x_\alpha - r_\alpha)V_\beta]\Bigg].\end{aligned} \tag{3.30}$$

Notice that we have separated $F_{\alpha\beta}$ into a velocity part (depending on V_α) and an acceleration part $\left(\text{depending on } \frac{dV_\alpha}{d\tau}\right)$. A moment's thought would be enough to convince yourself that this separation is Lorentz invariant i.e. such a separation exists in every reference frame. Now, using eq. (1.14) and appropriately lowering the indices,

$$
\begin{aligned}
E_i &= F_{i0}, \\
&= \frac{\mu_0 q}{4\pi}\frac{1}{\gamma^2 R^2(1-\mathbf{v}\cdot\hat{\mathbf{n}})^2}\left[(x_0-r_0)\frac{dV_i}{d\tau}-(x_i-r_i)\frac{dV_0}{d\tau}\right]+ \\
&\quad+\frac{\mu_0 q}{4\pi}\frac{1}{\gamma^3 R^3(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}[1-\gamma^4 R(\mathbf{v}\cdot\mathbf{a})(1-\mathbf{v}\cdot\hat{\mathbf{n}})+\gamma^2 R(\mathbf{a}\cdot\hat{\mathbf{n}})]\times \\
&\quad\times[(x_0-r_0)V_i-(x_i-r_i)V_0]\Big], \\
&= \frac{\mu_0 q}{4\pi}\frac{1}{\gamma^2 R^2(1-\mathbf{v}\cdot\hat{\mathbf{n}})^2}\left[-R[\gamma^2 a_i+\gamma^4(\mathbf{v}\cdot\mathbf{a})v_i]+\gamma^4(\mathbf{v}\cdot\mathbf{a})Rn_i\right]+ \\
&\quad+\frac{\mu_0 q}{4\pi}\frac{1}{\gamma^3 R^3(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}[1-\gamma^4 R(\mathbf{v}\cdot\mathbf{a})(1-\mathbf{v}\cdot\hat{\mathbf{n}})+\gamma^2 R(\mathbf{a}\cdot\hat{\mathbf{n}})][-\gamma Rv_i+\gamma Rn_i]\Big], \\
&= \frac{\mu_0 q}{4\pi}\frac{1}{\gamma^3 R^3(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}[-\gamma^3 R^2 a_i+\gamma^3 R^2(\mathbf{v}\cdot\hat{\mathbf{n}})a_i-\gamma^5 R^2(\mathbf{v}\cdot\mathbf{a})v_i+ \\
&\quad+\gamma^5 R^2(\mathbf{v}\cdot\hat{\mathbf{n}})(\mathbf{v}\cdot\mathbf{a})v_i+\gamma^5 R^2(\mathbf{v}\cdot\mathbf{a})n_i-\gamma^5 R^2(\mathbf{v}\cdot\hat{\mathbf{n}})(\mathbf{v}\cdot\mathbf{a})n_i-\gamma Rv_i+\gamma Rn_i+ \\
&\quad+\gamma^5 R^2(\mathbf{v}\cdot\mathbf{a})v_i-\gamma^5 R^2(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\hat{\mathbf{n}})v_i-\gamma^5 R^2(\mathbf{v}\cdot\mathbf{a})n_i+\gamma^5 R^2(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\hat{\mathbf{n}})n_i+ \\
&\quad-\gamma^3 R^2(\mathbf{a}\cdot\hat{\mathbf{n}})v_i+\gamma^3 R^2(\mathbf{a}\cdot\hat{\mathbf{n}})n_i], \\
&= \frac{\mu_0 q}{4\pi}\frac{1}{\gamma^2(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}\frac{n_i-v_i}{R^2}+\frac{\mu_0 q}{4\pi}\frac{1}{(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}\frac{-a_i(1-\mathbf{v}\cdot\hat{\mathbf{n}})+(\mathbf{a}\cdot\hat{\mathbf{n}})(n_i-v_i)}{R}.
\end{aligned}
$$

Remember that everything is calculated at the retarded time in the preceding calculation. We now complete a calculation that would appear useless if not for hindsight:

$$
\begin{aligned}
\hat{\mathbf{n}}\times[(\hat{\mathbf{n}}-\mathbf{v})\times\mathbf{a}] &= (\hat{\mathbf{n}}-\mathbf{v})(\hat{\mathbf{n}}\cdot\mathbf{a})-\mathbf{a}[\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}-\mathbf{v})], \\
&= -\mathbf{a}(1-\mathbf{v}\cdot\hat{\mathbf{n}})+(\mathbf{a}\cdot\hat{\mathbf{n}})(\hat{\mathbf{n}}-\mathbf{v}).
\end{aligned}
$$

Using this, we can write our final expression for the electric field:

$$
\mathbf{E}=\frac{\mu_0 q}{4\pi}\left[\frac{\hat{\mathbf{n}}-\mathbf{v}}{\gamma^2(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}\frac{1}{R^2}\right]_{\text{ret}}+\frac{\mu_0 q}{4\pi}\left[\frac{\hat{\mathbf{n}}\times[(\hat{\mathbf{n}}-\mathbf{v})\times\mathbf{a}]}{(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}\frac{1}{R}\right]_{\text{ret}}. \tag{3.31}
$$

Moving onto the magnetic field,

$$\begin{aligned}
B_1 &= F_{23}, \\
&= \frac{\mu_0 q}{4\pi}\frac{1}{\gamma^2 R^2(1-\mathbf{v}\cdot\hat{\mathbf{n}})^2}\left[(x_3-r_3)\frac{dV_2}{d\tau}-(x_2-r_2)\frac{dV_3}{d\tau}\right]+ \\
&\quad+\frac{\mu_0 q}{4\pi}\frac{1}{\gamma^3 R^3(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}[1-\gamma^4 R(\mathbf{v}\cdot\mathbf{a})(1-\mathbf{v}\cdot\hat{\mathbf{n}})+\gamma^2 R(\mathbf{a}\cdot\hat{\mathbf{n}})]\times \\
&\quad\times\left[(x_3-r_3)V_2-(x_2-r_2)V_3]\right], \\
&= \frac{\mu_0 q}{4\pi}\frac{1}{\gamma^2 R^2(1-\mathbf{v}\cdot\hat{\mathbf{n}})^2}\left[Rn_3[\gamma^2 a_2+\gamma^4(\mathbf{v}\cdot\mathbf{a})v_2]-Rn_2[\gamma^2 a_3+\gamma^4(\mathbf{v}\cdot\mathbf{a})v_3]\right]+ \\
&\quad+\frac{\mu_0 q}{4\pi}\frac{1}{\gamma^3 R^3(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}[1-\gamma^4 R(\mathbf{v}\cdot\mathbf{a})(1-\mathbf{v}\cdot\hat{\mathbf{n}})+\gamma^2 R(\mathbf{a}\cdot\hat{\mathbf{n}})]\times \\
&\quad\times\left[Rn_3(\gamma v_2)-Rn_2(\gamma v_3)]\right], \\
&= \frac{\mu_0 q}{4\pi}\frac{1}{\gamma^3 R^3(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}[\gamma^3 R^2 n_3 a_2-\gamma^3 R^2(\mathbf{v}\cdot\hat{\mathbf{n}})n_3 a_2+\gamma^5 R^2(\mathbf{v}\cdot\mathbf{a})n_3 v_2+ \\
&\quad-\gamma^5 R^2(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\hat{\mathbf{n}})n_3 v_2-\gamma^3 R^2 n_2 a_3+\gamma^3 R^2(\mathbf{v}\cdot\hat{\mathbf{n}})n_2 a_3-\gamma^5 R^2(\mathbf{v}\cdot\mathbf{a})n_2 v_3+ \\
&\quad+\gamma^5 R^2(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\hat{\mathbf{n}})n_2 v_3+\gamma R n_3 v_2-\gamma R n_2 v_3-\gamma^5 R^2(\mathbf{v}\cdot\mathbf{a})n_3 v_2+ \\
&\quad+\gamma^5 R^2(\mathbf{v}\cdot\mathbf{a})n_2 v_3+\gamma^5 R^2(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\hat{\mathbf{n}})n_3 v_2-\gamma^5 R^2(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\hat{\mathbf{n}})n_2 v_3+ \\
&\quad+\gamma^3 R^2(\mathbf{a}\cdot\hat{\mathbf{n}})n_3 v_2-\gamma^3 R^2(\mathbf{a}\cdot\hat{\mathbf{n}})n_2 v_3], \\
&= \frac{\mu_0 q}{4\pi}\frac{1}{\gamma^2(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}\frac{n_3 v_2-n_2 v_3}{R^2}+ \\
&\quad+\frac{\mu_0 q}{4\pi}\frac{1}{\gamma^2(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}\frac{(n_3 a_2-n_2 a_3)(1-\mathbf{v}\cdot\hat{\mathbf{n}})+(n_3 v_2-n_2 v_3)(\mathbf{a}\cdot\hat{\mathbf{n}})}{R}. \\
&= \frac{\mu_0 q}{4\pi}\frac{(\mathbf{v}\times\hat{\mathbf{n}})_1}{\gamma^2(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}\frac{1}{R^2}+\frac{\mu_0 q}{4\pi}\frac{(\mathbf{a}\times\hat{\mathbf{n}})_1(1-\mathbf{v}\cdot\hat{\mathbf{n}})+(\mathbf{v}\times\hat{\mathbf{n}})_1(\mathbf{a}\cdot\hat{\mathbf{n}})}{\gamma^2(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}\frac{1}{R}.
\end{aligned}$$

where in the last line, we used that fact that $(\mathbf{v}\times\hat{\mathbf{n}})_1 = v_2 n_3 - v_3 n_2$ and $(\mathbf{a}\times\hat{\mathbf{n}}) = a_2 n_3 - a_3 n_2$. The reader is invited to use the same process for the other two components to get

$$\mathbf{B} = \frac{\mu_0 q}{4\pi}\left[\frac{\mathbf{v}\times\hat{\mathbf{n}}}{\gamma^2(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}\frac{1}{R^2}\right]_{\text{ret}}+\frac{\mu_0 q}{4\pi}\left[\frac{(\mathbf{a}\times\hat{\mathbf{n}})(1-\mathbf{v}\cdot\hat{\mathbf{n}})+(\mathbf{v}\times\hat{\mathbf{n}})(\mathbf{a}\cdot\hat{\mathbf{n}})}{\gamma^2(1-\mathbf{v}\cdot\hat{\mathbf{n}})^3}\frac{1}{R}\right]_{\text{ret}}. \tag{3.32}$$

Notice that eqs. (3.31) and (3.32) show that the electric and magnetic fields are composed of two distinct components added together. These are the velocity fields (denoted by $\mathbf{E_v}$ and $\mathbf{B_v}$) that go as $\frac{1}{R^2}$ and the acceleration fields (denoted by $\mathbf{E_a}$

and $\mathbf{B_a}$) that go as $\frac{1}{R}$. The reader is now invited to verify the following important properties of the fields:

1. $\mathbf{B_v} = [\hat{\mathbf{n}} \times \mathbf{E_v}]_{\mathrm{ret}}$, $\mathbf{B_a} = [\hat{\mathbf{n}} \times \mathbf{E_a}]_{\mathrm{ret}}$ and therefore, $\mathbf{B} = [\hat{\mathbf{n}} \times \mathbf{E}]_{\mathrm{ret}}$;
2. $|\mathbf{B}| \leq |\mathbf{E}|$ and $|\mathbf{B}| = |\mathbf{E}| \Leftrightarrow \hat{\mathbf{n}} \cdot \mathbf{E} = 0$;
3. $(\hat{\mathbf{n}}_{\mathrm{ret}}, \mathbf{E_a}, \mathbf{B_a})$ always form an orthogonal triad.

This last property makes the acceleration fields good candidates as sources of *propagating radiation*.

Chapter 4

Radiation from Moving Charges

We begin this section with the definition of radiation. A compact source radiates into $d\Omega$ if $\mathbf{r}$ lies on the surface of the sphere $r^2 d\Omega$ and the Poynting vector, $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$, gives a finite non-zero value for the following:

$$\frac{dU}{dt} = dP(t) = \lim_{r\to\infty} (\mathbf{S} \cdot \hat{\mathbf{r}} r^2 d\Omega). \tag{4.1}$$

Let us take a closer look at this equation. The Poynting vector represents the energy flux density i.e. the energy flowing through a unit area in unit time. Multiplying it with the area element, $d\mathbf{a} = \hat{\mathbf{r}} r^2 d\Omega$ (not to be confused with the vector potential $\mathbf{A}$ or the three-acceleration $\hat{\mathbf{r}} r^2 d\Omega$), gives energy radiated per unit time (or, in other words, the power radiated) through that area. Taking the limit then gives the power radiated through that area to infinity. Integrating eq. (4.1) gives the total power radiated to infinity,

$$P(t) = \lim_{r\to\infty} \int \mathbf{S} \cdot d\mathbf{A}. \tag{4.2}$$

To get a finite answer for $P(t)$, observe that the dependence on r in $\mathbf{S}$ must exactly cancel out the r^2 dependence in $d\mathbf{A}$. This gives us the following condition:

$$\mathbf{S} \propto \frac{\hat{\mathbf{r}}}{r^2}. \tag{4.3}$$

From the defining equation for $\mathbf{S}$, one can then deduce that the fields must follow their own proportionality "laws":

$$\mathbf{E} \propto \frac{\hat{\mathbf{r}}}{r}, \tag{4.4}$$

$$\mathbf{B} \propto \frac{\hat{\mathbf{r}}}{r}. \tag{4.5}$$

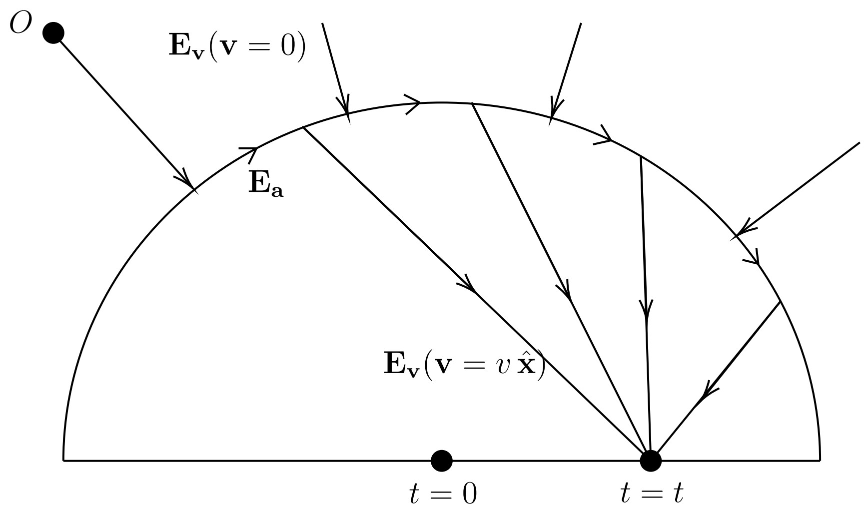

Figure 4.1: A charge undergoing instantaneous acceleration.

But we have already seen in the preceding section that $\mathbf{E_a}$ and $\mathbf{B_a}$ follow these laws. Thus, they are the radiation fields while $\mathbf{E_v}$ and $\mathbf{B_v}$ are the static fields.

Let us now imagine a scenario in which a charge $-q$ undergoes an instantaneous acceleration at $t = 0$ to a final speed v. We now look at what happens at time t. Look at Fig. (4.1). The figure shows the fields at time t. Originally, the charge was at the central dot labelled $t = 0$. After time t, the charge is at the dot labelled $t = t$. In this time, the information that the charge has undergone acceleration has travelled in all directions at speed $v = 1$. Therefore, this information has reached all observers inside a sphere of radius $R = t$ around the original position of the charge. Therefore, the field lines inside the sphere (labelled $\mathbf{E_v}(\mathbf{v} = v\,\hat{\mathbf{x}})$) point towards the charge's current position at time t. Now, consider an observer O outside the sphere. Since this information has not reached O, the field lines outside the sphere (labelled $\mathbf{E_v}(\mathbf{v} = 0)$) are those for a stationary charge at its original position at $t = 0$. Since the electric field is continuous, the $\mathbf{E_a}$ lines lie on the surface of the sphere as expected since $\hat{\mathbf{n}}_{\text{ret}} \cdot \mathbf{E_a} = 0$.

Now, eqs. (4.1) and (4.2) give us the power radiated as a function of the observer's time t. However, it would be better to calculate the power radiated as a function of the charge's proper time t_{ret}. Simple application of the chain rule gives

$$dP(t_{\text{ret}}) = \frac{dU}{dt_{\text{ret}}} = \frac{dt}{dt_{\text{ret}}}\frac{dU}{dt}. \tag{4.6}$$

Recalling that $t = t_{\text{ret}} + R(t_{\text{ret}})$ and differentiating w.r.t. t_{ret} leads to

$$\frac{dt}{dt_{\text{ret}}} = 1 + \frac{dR(t_{\text{ret}})}{dt_{\text{ret}}}. \tag{4.7}$$

Moving on, we differentiate $R = \sqrt{\mathbf{R}\cdot\mathbf{R}}$ w.r.t. to t and recall that $\mathbf{R}(t) = x - \mathbf{r}(t)$ to get

$$\begin{aligned}\frac{dR}{dt} &= \frac{1}{2}(\mathbf{R}\cdot\mathbf{R})^{-1/2}(\mathbf{R}\cdot\frac{d\mathbf{R}}{dt}),\\ &= \frac{1}{R}(R\hat{\mathbf{n}}\cdot(-\mathbf{v})),\\ &= -\mathbf{v}\cdot\hat{\mathbf{n}}.\end{aligned}$$

Now, using eqs. (4.6) and (4.7),

$$dP(t_{\text{ret}}) = (1 - \mathbf{v}\cdot\hat{\mathbf{n}})dP(t). \tag{4.8}$$

Let us now calculate the angular distribution of the power radiated. Using eqs. (4.1) and (4.8),

$$\frac{dP(t_{\text{ret}})}{d\Omega} = \lim_{r\to\infty}(1 - \mathbf{v}\cdot\hat{\mathbf{n}})(\mathbf{S}\cdot\hat{\mathbf{n}})R^2. \tag{4.9}$$

Since the radiation fields are the acceleration fields, we can find that

$$\begin{aligned}\mu_0\mathbf{S} &= \mathbf{E_a}\times\mathbf{B_a},\\ &= \mathbf{E_a}\times(\hat{\mathbf{n}}\times\mathbf{E_a}),\\ &= \hat{\mathbf{n}}(\mathbf{E_a}\cdot\mathbf{E_a}) - \mathbf{E_a}(\mathbf{E_a}\cdot\hat{\mathbf{n}}),\\ &= |\mathbf{E_a}|^2\hat{\mathbf{n}},\\ &= \left(\frac{\mu_0 q}{4\pi}\right)^2\left[\frac{|\hat{\mathbf{n}}\times[(\hat{\mathbf{n}}-\mathbf{v})\times\mathbf{a}]|^2}{(1-\mathbf{v}\cdot\hat{\mathbf{n}})^6}\frac{1}{R^2}\right],\end{aligned}$$

where in the second and fourth line, we used the fact that $(\hat{\mathbf{n}}, \mathbf{E_a}, \mathbf{B_a})$ form an orthogonal triad and in the last line, we used eq. (3.31). Now, using eq. (4.9),

$$\frac{dP(t_{\text{ret}})}{d\Omega} = \mu_0\left(\frac{q}{4\pi}\right)^2\left[\frac{|\hat{\mathbf{n}}\times[(\hat{\mathbf{n}}-\mathbf{v})\times\mathbf{a}]|^2}{(1-\mathbf{v}\cdot\hat{\mathbf{n}})^5}\right]. \tag{4.10}$$

From now onward, we will not explicitly write out the dependence on t_{ret} and it is assumed that the reader understands this. Let us now look at some interesting cases.

The first of these is **non-relativistic motion** i.e. $\mathbf{a} \neq 0$ and $\mathbf{v} \ll 1$. Taking the limit $\mathbf{v} \to 0$,

$$
\begin{aligned}
\lim_{\mathbf{v}\to 0} |\hat{\mathbf{n}} \times [(\hat{\mathbf{n}} - \mathbf{v}) \times \mathbf{a}]|^2 &= \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{a}), \\
&= |\hat{\mathbf{n}}|^2 |\hat{\mathbf{n}} \times \mathbf{a}|^2 \sin^2\left(\frac{\pi}{2}\right), \\
&= |\hat{\mathbf{n}}|^2 |\mathbf{a}|^2 \sin^2\theta, \\
&= |\mathbf{a}|^2 \sin^2\theta,
\end{aligned}
$$

where in the second line, we used the fact that $\hat{\mathbf{n}}$ is perpendicular to $\hat{\mathbf{n}} \times \mathbf{a}$ and in the third line, θ is the angle between $\mathbf{a}$ and $\hat{\mathbf{n}}$. Furthermore, $\lim_{\mathbf{v}\to 0}(1 - \mathbf{v}\cdot\hat{\mathbf{n}}) = 1$. Therefore, the angular distribution of the power radiated in this case is

$$
\frac{dP}{d\Omega} = \mu_0 \left(\frac{q}{4\pi}\right)^2 a^2 \sin^2\theta, \tag{4.11}
$$

where $a = |\mathbf{a}|$. The reader is invited to integrate this over the solid angle to obtain Larmor's formula,

$$
P = \frac{\mu_0}{6\pi} q^2 a^2. \tag{4.12}
$$

The second case is **cyclotron radiation** when acceleration is parallel to velocity. Denoting the angle between $\mathbf{a}$ and $\hat{\mathbf{n}}$ by θ,

$$
\begin{aligned}
|\hat{\mathbf{n}} \times [(\hat{\mathbf{n}} - \mathbf{v}) \times \mathbf{a}]|^2 &= |\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{a}) - \hat{\mathbf{n}} \times (\mathbf{v} \times \mathbf{a})|^2, \\
&= |\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{a})|^2, \\
&= |\mathbf{a}|^2 \sin^2\theta,
\end{aligned}
$$

where in the second line, we used the fact that $\mathbf{v} \parallel \mathbf{a}$ and in the last line, we used our result from the previous case. Furthermore, since $\mathbf{v} \parallel \mathbf{a}$, the angle between $\mathbf{v}$ and $\hat{\mathbf{n}}$ is also θ and $\mathbf{v}\cdot\hat{\mathbf{n}} = |\mathbf{v}|\cos\theta$. The angular distribution of the power radiated in this case is

$$
\frac{dP}{d\Omega} = \mu_0 \left(\frac{q}{4\pi}\right)^2 \frac{a^2 \sin^2\theta}{(1 - v\cos\theta)^5}, \tag{4.13}
$$

where $v = |\mathbf{v}|$. Note that the power only depends on the magnitude of the acceleration and not on whether the charge was accelerating or decelerating. If the radiation is produced by deceleration of charges, it is called *bremsstrahlung* which is German for "braking radiation". Let us now consider the *ultra-relativistic limit* where $v \approx 1$ and

$\theta \ll 1$. Let us first evaluate $1-v$ with these conditions:

$$
\begin{aligned}
1 - v &= \frac{(1-v)(1+v)}{1+v}, \\
&= \frac{1-v^2}{1+v}, \\
&= \frac{1}{\gamma^2(1+v)}, \\
&\approx \frac{1}{2\gamma^2},
\end{aligned}
$$

where we took the limit $v \to 1$ in the last line. Now, for $\theta \to 0$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$ and $\sin\theta \approx \theta$. Therefore,

$$
\begin{aligned}
1 - v\cos\theta &= (1-v)\cos\theta + 1 - \cos\theta, \\
&\approx \frac{1}{2\gamma^2}\left(1 - \frac{\theta^2}{2}\right) + 1 - 1 + \frac{\theta^2}{2}, \\
&= \frac{1 - \frac{\theta^2}{2} + \gamma^2\theta^2}{2\gamma^2},
\end{aligned}
$$

Now, taking the limit $\theta \to 1$ for the θ which was not being multiplied by γ,

$$
1 - v\cos\theta \approx \frac{1+\gamma^2\theta^2}{2\gamma^2}. \tag{4.14}
$$

Using this,

$$
\begin{aligned}
\frac{\sin^2\theta}{(1 - \mathbf{v}\cdot\hat{\mathbf{n}})^5} &\approx \frac{\theta^2}{\left(\frac{1+\gamma^2\theta^2}{2\gamma^2}\right)^5}, \\
&= \frac{32\gamma^{10}\theta^2}{(1+\gamma^2\theta^2)^5}.
\end{aligned}
$$

Finally, in the ultra-relativistic limit, eq. (4.13) reduces to

$$
\frac{dP}{d\Omega} = 2\mu_0 \frac{q^2}{\pi^2} \frac{\gamma^8(\gamma\theta)^2}{[1+(\gamma\theta)^2]^5}. \tag{4.15}
$$

The reader is now invited to maximize this expression w.r.t. $\gamma\theta$ to find the angle at which maximum power is emitted for a given velocity.

The third case is **synchotron radiation** when acceleration is perpendicular to the velocity. For this, we set $\mathbf{v} = v\,\hat{\mathbf{z}}$, $\mathbf{a} = a\,\hat{\mathbf{x}}$ and $\mathbf{n}_{\rm ret} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$. Now,

$$
\begin{aligned}
\hat{\mathbf{n}} \times \mathbf{a} &= (\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}) \times a\,\hat{\mathbf{x}}, \\
&= -a\sin\theta\sin\phi\,\hat{\mathbf{z}} + a\cos\theta\,\hat{\mathbf{y}}, \\
\mathbf{v} \times \mathbf{a} &= v\,\hat{\mathbf{z}} \times a\,\hat{\mathbf{x}}, \\
&= va\,\hat{\mathbf{y}}, \\
\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{a}) &= (\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}) \times (-a\sin\theta\sin\phi\,\hat{\mathbf{z}} + a\cos\theta\,\hat{\mathbf{y}}), \\
&= a\sin^2\theta\sin\phi\cos\phi\,\hat{\mathbf{y}} + a\cos\theta\sin\theta\cos\phi\,\hat{\mathbf{z}} - a\sin^2\theta\sin^2\phi\,\hat{\mathbf{x}} \\
&\quad - a\cos^2\theta\,\hat{\mathbf{x}}, \\
&= -a(\sin^2\theta\sin^2\phi + \cos^2\theta)\,\hat{\mathbf{x}} + a\sin^2\theta\sin\phi\cos\phi\,\hat{\mathbf{y}} + a\sin\theta\cos\theta\cos\phi\,\hat{\mathbf{z}}, \\
\hat{\mathbf{n}} \times (\hat{\mathbf{v}} \times \mathbf{a}) &= (\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}) \times va\,\hat{\mathbf{y}}, \\
&= va\sin\theta\cos\phi\,\hat{\mathbf{z}} - va\cos\theta\,\hat{\mathbf{x}}\,.
\end{aligned}
$$

Combining all this, we find

$$
\begin{aligned}
\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{a}) - \hat{\mathbf{n}} \times (\hat{\mathbf{v}} \times \mathbf{a}) = &-a(-v\cos\theta + \sin^2\theta\sin^2\phi + \cos^2\theta)\,\hat{\mathbf{x}} + \\
&+ a\sin^2\theta\sin\phi\cos\phi\,\hat{\mathbf{y}} + a\sin\theta\cos\phi(-v + \cos\theta)\,\hat{\mathbf{z}}\,.
\end{aligned}
$$

The magnitude squared of this vector is

$$
\begin{aligned}
&|\hat{\mathbf{n}} \times [(\hat{\mathbf{n}} - \mathbf{v}) \times \mathbf{a}]|^2 \\
&= a^2v^2\cos^2\theta + a^2\sin^4\theta\sin^4\phi - 2a^2v\cos\theta\sin^2\theta\sin^2\phi + a^2\cos^4\theta - 2a^2v\cos^3\theta + \\
&\quad + 2a^2\sin^2\theta\cos^2\theta\sin^2\phi + a^2\sin^4\theta\sin^2\phi\cos^2\phi + a^2v^2\sin^2\theta\cos^2\phi + \\
&\quad + a^2\cos^2\theta\sin^2\theta\cos^2\phi - 2a^2v\cos\theta\sin^2\theta\cos^2\phi, \\
&= a^2[v^2\cos^2\theta + \sin^2\theta\sin^2\phi - \sin^2\theta\sin^2\phi\cos^2\theta - \sin^2\theta\sin^2\phi\cos^2\phi + \\
&\quad + \sin^2\theta\sin^2\phi\cos^2\theta\cos^2\phi - 2v\cos\theta(\sin^2\phi + \cos^2\phi) + \cos^2\theta - \cos^2\theta\sin^2\theta + \\
&\quad - 2v\cos\theta + 2v\cos\theta\sin^2\theta + 2\sin^2\theta\cos^2\theta\sin^2\phi + \sin^2\theta\sin^2\phi\cos^2\phi + \\
&\quad - \sin^2\theta\cos^2\theta\sin^2\phi\cos^2\phi + v^2\sin^2\theta\cos^2\phi + \cos^2\theta\sin^2\theta\cos^2\phi], \\
&= 1 - 2v\cos\theta + v^2\cos^2\theta - \sin^2\theta\cos^2\phi + v^2\sin^2\theta\cos^2\phi, \\
&= (1 - v\cos\theta)^2 - (1 - v^2)\sin^2\theta\cos^2\phi, \\
&= (1 - v\cos\theta)^2 - \frac{1}{\gamma^2}\sin^2\theta\cos^2\phi,
\end{aligned}
$$

where in the second line, the author has suppressed the voices in his head and revealed quite a bit of working using trigonometric identities and in the third line, the author has succumbed to the wishes of the voices and has left it to the reader to fill in the steps. Finally, using this result and the fact that $1 - \mathbf{v} \cdot \hat{\mathbf{n}} = 1 - v\cos\theta$ in eq. (4.10),

$$\frac{dP}{d\Omega} = \mu_0 \left(\frac{q}{4\pi}\right)^2 \frac{a^2}{(1 - v\cos\theta)^3} \left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1 - v\cos\theta)^2}\right]. \tag{4.16}$$

As in the previous case, we consider the ultra-relativistic limit. Using eq. (4.14),

$$\begin{aligned} \frac{\sin^2\theta}{\gamma^2(1 - v\cos\theta)^2} &\approx \frac{\theta^2}{\gamma^2\left(\frac{1+\gamma^2\theta^2}{2\gamma^2}\right)^2}, \\ &= \frac{4\gamma^2\theta^2}{(1+\gamma^2\theta^2)^2}; \\ \frac{1}{(1 - v\cos\theta)^3} &\approx \frac{1}{\left(\frac{1+\gamma^2\theta^2}{2\gamma^2}\right)^3}, \\ &= \frac{8\gamma^6}{(1+\gamma^2\theta^2)^3}. \end{aligned}$$

We can now write the final answer for the power radiated in the ultra-relativistic limit,

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2} \frac{8\gamma^6}{(1+\gamma^2\theta^2)^3} \left[1 - \frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}\right]. \tag{4.17}$$

As before, the reader is invited to maximize this expression w.r.t. $\gamma\theta$ to find the angle at which maximum power is emitted for a given velocity.

This discussion of radiation, albeit small, is intended to provide the reader with a taste of what such calculations look like. The reader is now invited to solve as many questions from the problem set (and any other suitable problems they may find) as they can.

Bibliography

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