On the Riemann Rearrangement Theorem

Course Project for MATH 309: Introduction to Analysis II

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The Riemann Rearrangement Theorem remains one of the most astounding results in foundational real analysis. It takes human intuition about that most basic of operations – addition – and turns it around on its head until we do not know what to believe any more.

Before we get to the statement of the theorem, we make the following remark:

Remark 1. *Rearranging the terms of a finite sum does not change the value of the sum.*

This is clear from intuition. As an example, $1 + 2 = 2 + 1$. One then wonders whether this statement holds true for any arbitrary infinite series. Let us first precisely define *rearrangement*.

Definition 2. Let $\sum a_n$ be a series of real numbers. Let $\sigma(n) : \mathbb{N} \to \mathbb{N}$ be a bijection. Then, $\sum a_{\sigma(n)}$ is called a *rearrangement* of $\sum a_n$.

Now, we take a look at a few examples:

Example 3. Consider the series $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \ldots + n + \ldots$. Clearly, this series diverges to ∞. Now, consider changing the order in which we add the numbers so that our new series is $1+3+2+4+5+7+6+8+\ldots$. This new series diverges to ∞ as well so the order did not matter in this example.

Example 4. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} + \ldots$. This series converges to $\frac{\pi^2}{6}$. We will not prove it here, but any new rearrangement does not change the convergence to $\frac{\pi^2}{6}$.

Example 5. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} + \ldots$. Let us find what this series converges to. We begin by writing down the Taylor series for $\ln(1 + x)$:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{(-1)^{n+1} x^n}{n} + \ldots.$$

Now, if we take $x = 1$, we get our original series. Therefore, our series converges to $\ln 2$. Now, consider a rearrangement of the series to get the "new" series,

$$\begin{aligned}
&\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \ldots, \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \ldots + \frac{1}{2(2n-1)} - \frac{1}{4n} + \ldots, \\
&= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots + \frac{(-1)^{n+1}}{n} \ldots\right), \\
&= \frac{1}{2} \ln 2.
\end{aligned}$$

This is astonishing. A rearrangement of the terms has led to a different answer!

As Example 5 shows, when it comes to infinite series, we cannot trust the intuition we have developed using finite sums. A property of the series in Example 5 is that it is *conditionally convergent*.

Definition 6. A series $\sum a_n$ is said to be *conditionally convergent* if $\sum a_n$ converges but $\sum |a_n|$ does not.

Let us now prove a lemma before we move on to our main result.

Lemma 7. Let $\sum a_n$ be a conditionally convergent series of real numbers. Let P_n and Q_n denote the nth positive and negative term of the series respectively. Then, $\sum P_n$ and $\sum -Q_n$ both diverge.

Proof. Let $p_n = \frac{a_n+|a_n|}{2}$ and $q_n = \frac{|a_n|-a_n}{2}$. Clearly, $p_n, q_n > 0$, $p_n - q_n = a_n$ and $p_n + q_n = |a_n|$. Since $\sum a_n$ is conditionally convergent, $\sum |a_n|$ is divergent while $\sum a_n = \sum p_n - q_n$ is convergent. If $\sum p_n$ and $\sum q_n$ were both convergent, then $\sum (p_n + q_n) = \sum |a_n|$ would converge, contradicting our assumption of conditional convergence. Moreover, if one of $\sum p_n$ or $\sum q_n$ was convergent while the other was divergent, then $\sum (p_n - q_n) = \sum a_n$ would diverge, contradictin gour assumption of conditional convergence. Therefore, both $\sum p_n$ and $\sum q_n$ must be divergent. Now,

$$p_n = \begin{cases} a_n & \text{if } a_n > 0; \\ 0 & \text{if } a_n \leq 0, \end{cases}$$

$$q_n = \begin{cases} 0 & \text{if } a_n \geq 0; \\ -a_n & \text{if } a_n < 0. \end{cases}$$

But this just means that p_n and $-q_n$ are the positive and negative terms of the sequence a_n, with some zeroes in between. In particular, $\sum p_n$ and $\sum q_n$ differ from $\sum P_n$ and $\sum -Q_n$ only by an infinite number of zeroes. Since addition of zeroes does not affect convergence or divergence, $\sum P_n$ and $\sum -Q_n$ both diverge. □

We are now in a position to state and prove the *Riemann Rearrangement Theorem*.

Theorem 8. *Let $\sum a_n$ be a conditionally convergent series of real numbers. Let $-\infty \leq \alpha \leq \beta \leq \infty$. Then, there exists a rearrangement $\sum a_{\sigma(n)}$ with partial sums $s_{\sigma(n)}$ such that*

$$\liminf_{n\to\infty} s_{\sigma(n)} = \alpha, \qquad \limsup_{n\to\infty} s_{\sigma(n)} = \beta.$$

Proof. We present a proof by construction. Let P_n and Q_n be defined as in the previous lemma. Let (α_n) and (β_n) be sequences of real numbers which converge to α and β respectively. Further, $\alpha_n < \beta_n$ and $\beta_1 > 0$. We will construct two sequences (i_n) and (j_n) which will form part of our rearrangement. Let i_1 and j_1 be the smallest natural numbers such that

$$P_1 + \ldots + P_{i_1} > \beta_1,$$
$$P_1 + \ldots + P_{i_1} + Q_1 + \ldots + Q_{j_1} < \alpha_1.$$

In other words, we first add precisely enough positive terms (in their original order) from (a_n) to get a sum greater than β_1. To this sum, we add precisely enough negative terms (again in their original order) from (a_n) to get a sum less than α_1. In the next step, we add precisely enough positive terms to get a sum greater than β_2 and then precisely enough negative terms to get a sum less than α_2. In particular, in the nth step, we have

$$P_1 + \ldots + P_{i_1} + Q_1 + \ldots + Q_{j_1} + \ldots + P_{i_{n-1}+1} + \ldots + P_{i_n} > \beta_n,$$
$$P_1 + \ldots + P_{i_1} + Q_1 + \ldots + Q_{j_1} + \ldots + P_{i_{n-1}+1} + \ldots + P_{i_n} + Q_{j_{n-1}+1} + \ldots + Q_{j_n} < \alpha_n.$$

Note that in this construction, we have always added precisely enough terms at each step. In particular, if we remove the last positive (resp. negative) term we added in any step, we would get a sum less than (resp.

greater than) β_n (resp. α_n). Therefore, the absolute difference between the sum and β_n or α_n is at most the last term added. The partial sums in the nth step then satisfy

$$|s_{j_{n-1}+i_n} - \beta_n| \leq P_{i_n}, \qquad |s_{i_n+j_n} - \alpha_n| \leq -Q_{j_n}.$$

Note that a series converges if and only if the sequence of partial sums (s_k) converges. By the Cauchy convergence criterion, this is equivalent to the existence of $N \in \mathbb{N}$ such that for all $\epsilon > 0$ and all $m > n \geq N$,

$$|s_m - s_n| < \epsilon.$$

In particular, $|s_n - s_{n-1}| = |a_n| < \epsilon$ for all $n \geq N$. Therefore, we have that $P_{i_n} < \epsilon$ and $-Q_{j_n} < \epsilon$ for all $i_n, j_n \geq n \geq N$ since $\sum a_n$ is convergent. As a result, denoting $x_n := s_{j_{n-1}+i_n}$ as the partial sum ending in P_{i_n} and $y_n := s_{i_n+j_n}$ as the partial sum ending in Q_{j_n},

$$|x_n - \beta_n| \leq \epsilon, \qquad |y_n - \alpha_n| \leq \epsilon,$$

for all $n \geq N$. Hence, $x_n \to \beta$ and $y_n \to \alpha$ i.e. they are limit points of our rearrangement. In particular, $x_n \leq \beta + \epsilon$ and $y_n \leq \alpha + \epsilon$ for all $\epsilon > 0$ and all $n \geq N$. Therefore, no number less than α or greater than β can be a subsequential limit of the partial sums. Finally, we state our results. The series

$$P_1 + \ldots + P_{i_1} + Q_1 + \ldots + Q_{j_1} + \ldots + P_{i_{n-1}+1} + \ldots + P_{i_n} + Q_{j_{n-1}+1} + \ldots + Q_{j_n} + \ldots$$

is a rearrangement of $\sum a_n$ and has

$$\liminf_{n\to\infty} s_{\sigma(n)} = \alpha, \qquad \limsup_{n\to\infty} s_{\sigma(n)} = \beta,$$

where $s_{\sigma(n)}$ is the nth partial sum of this rearrangement. □

This immediately leads to the following corollary.

Corollary 9. Let $\sum a_n$ be a conditionally convergent series of real numbers. Then, for every $R \in \mathbb{R}$, there exists a rearrangement $\sum a_{\sigma(n)}$ which converges to R. Furthermore, there also exist rearrangements such that the series diverges to ∞, $-\infty$ and becomes eventually bounded but fails to converge.

Proof. Take $\alpha = \beta = R$ in the previous proof. Since $\liminf = \limsup$, the series converges to R. Now, take $\alpha_n = \beta_n = n$ (resp. $= -n$) in the previous proof to get that the series diverges to ∞ (resp. $-\infty$). Finally, take $\alpha = -1$ and $\beta = 1$ to get that the series eventually stays between -1 and 1 but does not converge since $\liminf \neq \limsup$. □

Note that this theorem is only true for conditionally convergent series. We will not prove this here but for an absolutely convergent series, any rearrangement converges to the same limit as the original series.