# Colour the Graph

LUMS Students' Mathematics Society

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## 1 What is Colour the Graph?

The game consists of a planar graph drawn on a chart paper and a palette of counters of 5 different colours. Each player can place a counter on a vertex with the condition that counters of the same colour cannot be placed on neighbouring vertices. Player $A$ tries to colour all vertices while Player $B$ tries to disrupt this by ensuring that at least one vertex remains uncoloured. $B$ will always have a strategy to win.

## 2 Explanation

### 2.1 Some Definitions

A **simple connected graph** is an ordered pair $G = (V, E)$ containing members of a set of vertices $V$ and a set of edges $E$.
An **edge** $e$ connects 2 **vertices** $v_1$ and $v_2$. $v_1$ and $v_2$ are then referred to as **neighbouring vertices**.
A **face** $f$ is a region bounded by edges, including the outer infinitely large region.
A simple connected **planar** graph is one that can be drawn on a plane such that its edges don't cross over one another.
The **degree** of a vertex $\deg(v)$ is the number of edges associated with $v$.
The **chromatic number** $\chi(G)$ of a graph $G$ is the minimum number of different colours required to colour its vertices such that no two neighbouring vertices have the same colour.

### 2.2 A Different Game

Suppose that only one player was colouring the vertices of a planar graph. Then, the following theorem has been proved (with help from a computer):

**Theorem 1.** *A simple connected planar graph $G$ has $\chi(G) \leq 4$.*

Therefore, every such graph $G$ can be coloured with at most 4 different colours and hence, the name "Four Colour Theorem".

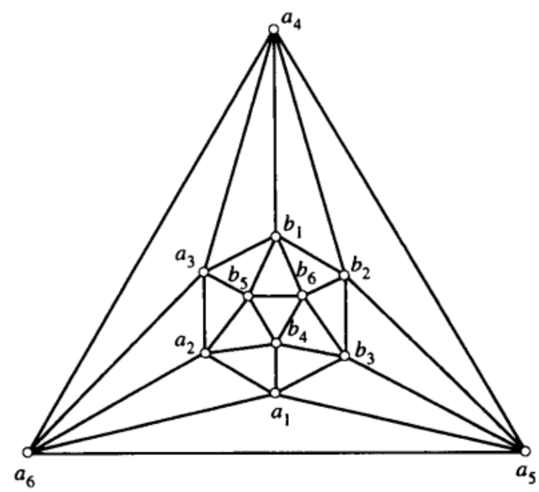


Figure 1: The graph in question

### 2.3 The Original Game Revisited

In terms of the stated definitions, Player $A$'s target is to colour all vertices of the simple connected planar graph $G$. Player $B$ must stop this from happening. Let us label the vertices of the graph as shown in Figure 1. Let $X$ be the set of 5 colours we have in our palette, denoting each colour with an integer.

$$X = \{1, 2, 3, 4, 5\}$$

Now, the labelled vertices form a **dominating set** $\{a_i, b_i\}$ where $i \in \{1, 2, 3, 4, 5, 6\}$. This means that for the set $\{a_i, b_i\}$, every other vertex is a neighbouring vertex for one of the elements in the set. Therefore, $B$ follows a strategy where they colour $a_i$ or $b_i$ with the same colour that $A$ just coloured $b_i$ or $a_i$ with, for every $i$. Therefore, no other vertex can be coloured with the same colour as $a_i$ and $b_i$. It follows that a palette of at least 6 colours is required for $A$ to win. (We say that the **game chromatic number** $\chi_g(G)$ of this graph is at least 6.) But $A$ only has 5 different colours in their palette. Therefore, $B$ will always win (provided they follow this strategy).

## 3 Further Reading

For further discussions on the game chromatic number, see
Kierstead, H.A.; Trotter W.T. (1994), "Planar Graph Coloring with an Uncooperative Partner", *Journal of Graph Theory*, **18** (6): 569 − 584.