Fibonacci Nim

LUMS Students' Mathematics Society

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1 What is Fibonacci Nim?

Fibonacci Nim was invented by Dr. R. E. Gaskell of Oregon State University. The game involves n counters placed in front of 2 players. On any move m, each player can remove at least 1 and at most q_m counters. Let r_m be the counters removed by a player on the mth move. Then,

$$q_m = 2r_{m-1}$$
$$q_1 = n - 1$$

The player who removes the last counter wins.

2 Explanation

2.1 Winning Strategy

The winning strategy is to represent the n counters on the table in any given move as a sum of non-consecutive Fibonacci numbers and removing the smallest such number of counters in the representation. If we begin with a Fibonacci number of counters, the winning strategy cannot be followed by Player 1 on the first move since $q_1 = n - 1$. Therefore, Player 2 always has a winning strategy.

2.2 Zeckendorf Representation

Theorem 1. *For any positive integers, there exists a unique Zeckendorf Representation which consists of a sum of non-consecutive Fibonacci numbers f_n where $f_1 = 1$, $f_2 = 2$ and $f_n = f_{n-1} + f_{n-2}$.*

Proof. To prove that such a representation always exists, we write a base case (and a few further examples to demonstrate the representation).

$$1 = f_1$$
$$2 = f_2$$
$$3 = f_3$$
$$4 = f_3 + f_1$$

Let there exist a Zeckendorf representation for all integers less than or equal to k. If $k+1$ is a Fibonacci number, we are done. Otherwise, we can write $f_n < k+1 < f_{n+1}$ for some n. We now define $a = k+1-f_n$. It is clear that $a \leq k$ so a has a Zeckendorf representation. Furthermore,

$$\begin{aligned} a + f_n &= k+1 < f_{n+1} \\ f_{n+1} &= f_n + f_{n-1} \\ \implies a &< f_{n-1} \end{aligned}$$

Therefore, the Zeckendorf representation of a (denoted by $Z(a)$) does not contain f_{n-1}. Therefore, $Z(k+1) = Z(a)+f_n$ which is itself a Zeckendorf representation. □

The proof for the uniqueness of this representation is left as an exercise to the reader.

Thus, every integer n can be written as:

$$n = \sum_{i=1}^{\infty} c_i f_i$$

where $c_i = 0$ or 1 and c_{i+1} and c_{i-1} are 0 if $c_i = 1$.

2.3 Safe And Unsafe Positions

A safe position is defined as one where no winning moves are possible and every move makes the position unsafe.

An unsafe position is defined as one where there is at least one move to take the game into a safe position. Such a move is a winning move.

Let F_m denote the smallest Fibonacci number in the Zeckendorf representation of n (the number of counters remaining after m moves). Then, if $q_m < F_m$, the game is in a safe position. If $q_m \geq F_m$, the game is in an unsafe position and removing F_m counters will bring the game to a safe position.

Theorem 2. *Any unsafe position can be made safe.*

Proof. Let n_m denote the number of counters left before the mth move and let the game be in an unsafe position. If $n_m = F_m$, the player takes the F_m counters and wins the game. If $n_m > F_m$, we can write

$$n_m = \ldots + f_k + f_i$$

where f_k is the second smallest Fibonacci number in the Zeckendorf representation of n and $f_i = F_m$. We know that there must exist at least one f_j such that $f_i < f_j < f_k$ and $f_i + f_j \leq f_k$.

$$\begin{aligned} f_i &< f_j \\ 2f_i &< f_j + f_i \leq f_k \\ 2f_i &< f_k \\ q_{m+1} &= 2F_m = 2f_i \\ \implies q_{m+1} &< f_k \end{aligned}$$

After the player removes F_m counters on the mth move, the number of counters n_{m+1} remaining is:

$$\begin{aligned} n_{m+1} &= n_m - F_m \\ &= \ldots + f_k + f_i - f_i \\ &= \ldots + f_k \end{aligned}$$

Therefore, $F_{m+1} = f_k$ and

$$q_{m+1} < F_{m+1}$$

where, by definition, we have reached a safe position. □

Theorem 3. *Any move from a safe position must make it unsafe.*

Proof. Let the game be in a safe position i.e. $q_m < F_m$. Therefore, a move must take $r_m < F_m$ counters. Therefore,

$$\begin{aligned} n_m &= c + f_i \\ n_{m+1} &= n_m - r_m \\ &= c + f_i - r_m \\ &= c + c_1 + f_h \end{aligned}$$

where $f_h = F_{m+1} < f_i$.

Lemma 1. *Any move that leaves $f_h = F_{m+1}$ must remove at least f_{h-1} counters.*

Let $c_1 + f_h \leq f_{i-1} + f_{i-3} + \cdots + f_{h+2} + f_h$. Then it can be proven (proof left as an exercise) that

$$\begin{aligned} c_1 + f_h + f_{h-1} &\leq f_{i-1} + f_{i-3} + \cdots + f_{h+2} + f_h + f_{h-1} \\ &= f_i \end{aligned}$$

Therefore,

$$\begin{aligned} r_m &= n_m - n_{m+1} \\ &= c + f_i - c - c_1 - f_h \\ &\geq f_{h-1} \end{aligned}$$

Now, following the same procedure as the previous proof:

$$\begin{aligned} f_{h-1} &\geq f_{h-2} \\ 2f_{h-1} &\geq f_{h-2} + f_{h-1} = f_h \\ 2f_{h-1} &\geq f_h \\ q_{m+1} = 2r_m &\geq 2f_{h-1} \\ \implies q_{m+1} &\geq f_h \end{aligned}$$

But $f_h = F_{m+1}$. Therefore, $q_{m+1} \geq F_{m+1}$ and the game is in an unsafe position. □

3 Further Reading

Whinihan, Michael J. (1963), "Fibonacci Nim" (PDF), *Fibonacci Quarterly*, **1** (4): 9 – 13.

Allen, Cody; Ponomarenko, Vadim (2014), "Fibonacci Nim and a full characterization of winning moves", *Involve*, **7** (6).