The Strategic Game of Nim

Name: ______________________________ **Date:** ____________________

The game involves the following 10 "sticks":

I I I I I I I I I I

The rules of the game are as follows:

1. Players take turns to "take" the sticks by crossing them out.
2. At each turn, a player may take either 1 or 2 sticks, but not 0 or more than 2 sticks.
3. The player who takes the last stick loses.

Here is an example game:

Player 1: ~~I~~ I I I I I I I I I

Player 2: ~~I~~ ~~I~~ ~~I~~ I I I I I I I

Player 1: ~~I~~ ~~I~~ ~~I~~ ~~I~~ ~~I~~ I I I I I

Player 2: ~~I~~ ~~I~~ ~~I~~ ~~I~~ ~~I~~ ~~I~~ ~~I~~ I I I

Player 1: ~~I~~ ~~I~~ ~~I~~ ~~I~~ ~~I~~ ~~I~~ ~~I~~ ~~I~~ ~~I~~ I

Note that Player 2 now must take the last stick and therefore loses.

Player 2: ~~I~~ ~~I~~ ~~I~~ ~~I~~ ~~I~~ ~~I~~ ~~I~~ ~~I~~ ~~I~~ ~~I~~

Q1. Play the game with your friend and mention which player wins each time:

Game 1: I I I I I I I I I I **Winner:** ____________________

Game 2: I I I I I I I I I I **Winner:** ____________________

Game 3: I I I I I I I I I I **Winner:** ____________________

Game 4: I I I I I I I I I I **Winner:** ____________________

Game 5: I I I I I I I I I I **Winner:** ____________________

Q2. Is there a losing position in the game? That is, is there a number of sticks after which the player who has the next turn is certain to lose?

Q3. Are there more losing positions? If yes, list all of them down.

Q4. Can you find a winning strategy for Player 1?

As you may have found, the winning strategy is to force the game into a losing position. We played the game with 10 sticks (call this the *total*) and in one turn, a player could take a maximum number of 2 sticks (call this the *quota*). What if we change the total and the quota?

Q5. Write out all the losing positions for the following games. The first one has been done for you.

		Quota				
		2	**3**	**4**	**5**	**6**
Total	**10**	1, 3, 5, 7, 9				
	11					
	12					
	13					
	14					

We now take a detour to Fibonacci numbers. Here is a list of the first 10 Fibonacci numbers. Use these to answer the following questions:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55

The **Zeckendorf representation** of a number is to write it as a sum of non-repeating, non-consecutive Fibonacci numbers. For example, the Zeckendorf representation of 10 is 8 + 2. It can be found by repeatedly subtracting the largest Fibonacci number less than or equal to your given number. Let us work through the steps for 33:

1. The largest Fibonacci number less than 33 is 21. We subtract it from 33 to get 33 – 21 = 12. In this step, we have found 33 = 21 + …

2. The largest Fibonacci number less than 12 is 8. We subtract it from 12 to get 12 – 8 = 4. In this step, we have found 33 = 21 + 8 + …
3. The largest Fibonacci number less than 4 is 3. We subtract it from 4 to get 4 – 3 = 1. In this step, we have found 33 = 21 + 8 + 4 + …
4. The largest Fibonacci number equal to 1 is 1. We subtract it from 1 to get 1 – 1 = 0. In this step, we have found 33 = 21 + 8 + 4 + 1. This is the Zeckendorf representation of 33.

Q6. Find the Zeckendorf representation of the following numbers:

a) 20

b) 45

c) 7

We now return to our game of Nim and change the rules. There are a total of n sticks. The new rules are as follows:

1. Players take turns to "take" the sticks by crossing them out.
2. In the first turn, a player takes at least 1 and a maximum of $n-1$ sticks.
3. In every subsequent turn, if a player took m sticks in the previous turn, the other player can take a maximum of $2m$ sticks in the current turn.
4. The player who takes the last stick wins.

Play the game with 10 sticks and note down the quota at the end of every move.

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Now, consider the following strategy for Player 1:

1. Find the Zeckendorf representation of the remaining sticks in the pile.
2. Take the number of sticks equal to the smallest number in the Zeckendorf representation.
3. Repeat this until you take the last stick and win.

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Sticks taken in first move: _________ **Quota after first move: _________**

Sticks taken in second move: _________ **Quota after second move: _________**

Sticks taken in third move: _________ **Quota after third move: _________**

Sticks taken in fourth move: _________ **Quota after fourth move: _________**

Sticks taken in fifth move: _________ **Quota after fifth move: _________**

Sticks taken in sixth move: _________ **Quota after sixth move: _________**

Sticks taken in seventh move: _________ **Quota after seventh move: _________**

Sticks taken in eighth move: _________ **Quota after eighth move: _________**

Sticks taken in ninth move: _________ **Quota after ninth move: _________**

Q7. Does this strategy work if there are 11 sticks? What if there are 13 sticks?