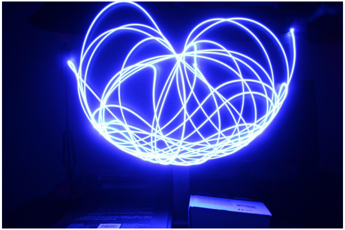
Our Universe is Chaotic (Probably)

Hashir H. Khan

13 May 2023

Our World is Chaotic

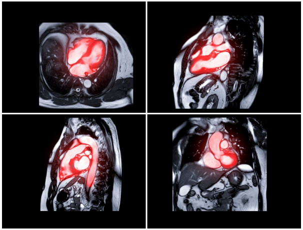


(a) Double Pendulum

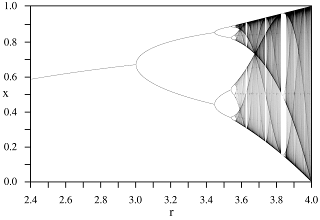
(b) Turbulent Flow

Figure: Chaos in mechanical and fluid motion.

Our World is Chaotic



(a) Cardiac Arrhythmia



(b) Logistic Map

Figure: Chaos in the human body and population models.

Our World is Chaotic

Our world, at large, is characterized by nonlinear, chaotic systems.

Question

Answer

Our World is Chaotic

Our world, at large, is characterized by nonlinear, chaotic systems. In fact, it is hard to find macroscopic systems which do not have a capacity for chaos.

Question

Answer

Our World is Chaotic

Our world, at large, is characterized by nonlinear, chaotic systems. In fact, it is hard to find macroscopic systems which do not have a capacity for chaos.

Question

But what is chaos? Is the universe itself chaotic?

Answer

Our World is Chaotic

Our world, at large, is characterized by nonlinear, chaotic systems. In fact, it is hard to find macroscopic systems which do not have a capacity for chaos.

Question

But what is chaos? Is the universe itself chaotic?

Answer

This and lots more to follow in the talk.

Structure of the Talk

1. The Bianchi IX Cosmology

Structure of the Talk

1. The Bianchi IX Cosmology
2. Measures of Chaos

Structure of the Talk

1. The Bianchi IX Cosmology
2. Measures of Chaos
    1. The BKL Approximation

Structure of the Talk

1. The Bianchi IX Cosmology
2. Measures of Chaos
   1. The BKL Approximation
   2. Painleve Analysis

Structure of the Talk

1. The Bianchi IX Cosmology
2. Measures of Chaos
    i. The BKL Approximation
    ii. Painleve Analysis
    iii. Lyapunov Exponents

Structure of the Talk

1. The Bianchi IX Cosmology
2. Measures of Chaos
   1. The BKL Approximation
   2. Painleve Analysis
   3. Lyapunov Exponents
   4. Fractals

Structure of the Talk

1. The Bianchi IX Cosmology
2. Measures of Chaos
   1. The BKL Approximation
   2. Painleve Analysis
   3. Lyapunov Exponents
   4. Fractals
   5. Curvature

Structure of the Talk

1. The Bianchi IX Cosmology
2. Measures of Chaos
   1. The BKL Approximation
   2. Painleve Analysis
   3. Lyapunov Exponents
   4. Fractals
   5. Curvature
3. Implications For Our Universe

The Bianchi IX Cosmology

- Characterized by homogeneity and anisotropy.

Figure: Example of a homogeneous, anisotropic shape.

The Bianchi IX Cosmology

- Characterized by homogeneity and anisotropy.
- Homogeneous: Every point location looks the same.

Figure: Example of a homogeneous, anisotropic shape.

The Bianchi IX Cosmology

- Characterized by homogeneity and anisotropy.
- Homogeneous: Every point location looks the same.
- Anisotropic: Does not look the same in all directions.

Figure: Example of a homogeneous, anisotropic shape.

The Bianchi IX Cosmology

- The metric:

$$ds^2 = -\,dt^2 + \sum_k (l_k)^2 \sigma_k^2.$$

The Bianchi IX Cosmology

- The metric:
$$ds^2 = -\,dt^2 + \sum_k (l_k)^2 \sigma_k^2.$$
- Here, $k = 1, 2, 3$ where, (in Euler angle coordinates: $0 \leq \psi \leq 4\pi$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$)
  - $\sigma_1 = (0, \sin\psi, -\cos\psi \sin\theta)$,
  - $\sigma_2 = (0, \cos\psi, \sin\psi \sin\theta)$,
  - $\sigma_3 = (-1, 0, \cos\theta)$,

The Bianchi IX Cosmology

- The metric:
$$ds^2 = -\,dt^2 + \sum_k (l_k)^2 \sigma_k^2.$$
- Here, $k = 1, 2, 3$ where, (in Euler angle coordinates: $0 \leq \psi \leq 4\pi$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$)
  - $\sigma_1 = (0, \sin\psi, -\cos\psi \sin\theta)$,
  - $\sigma_2 = (0, \cos\psi, \sin\psi \sin\theta)$,
  - $\sigma_3 = (-1, 0, \cos\theta)$,
- The l_k's, henceforth denoted by $a(t), b(t), c(t)$, are the scale factors.

The Bianchi IX Cosmology

- The three equations of evolution are

$$\frac{d^2}{d\tau^2}(\ln a^2) = (b^2 - c^2)^2 - a^4$$

and cyclically, $a \to b \to c \to a$.

The Bianchi IX Cosmology

- The three equations of evolution are

$$\frac{d^2}{d\tau^2}(\ln a^2) = (b^2 - c^2)^2 - a^4$$

  and cyclically, $a \to b \to c \to a$.
- Here, the new coordinate time τ is related to t by $dt = abc\, d\tau$.

The Bianchi IX Cosmology

- The three equations of evolution are

$$\frac{d^2}{d\tau^2}(\ln a^2) = (b^2 - c^2)^2 - a^4$$

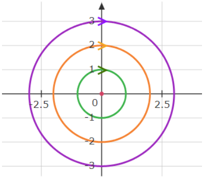
  and cyclically, $a \to b \to c \to a$.
- Here, the new coordinate time τ is related to t by $dt = abc\, d\tau$.
- Aim is to go back in time to the initial singularity ($t = 0$, $\tau = -\infty$) and observe chaos in the scale factors.

Measures of Chaos

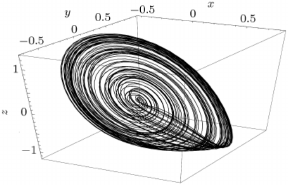
Question

What is chaos?

Answer



(a) No Chaos



(b) Chaos

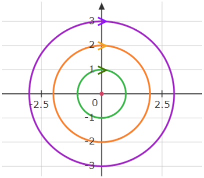
Measures of Chaos

Question

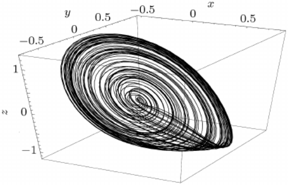
What is chaos?

Answer

Roughly speaking, chaos is a sensitive dependence on initial conditions.



(a) No Chaos



(b) Chaos

Measures of Chaos

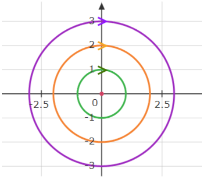
Question

What is chaos?

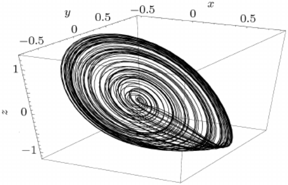
Answer

Roughly speaking, chaos is a sensitive dependence on initial conditions.

A system is chaotic if, given initial conditions with a small uncertainty, we are unable to predict the state of the system after a (long enough) finite time interval.



(a) No Chaos



(b) Chaos

Break for Remembering Stuff

If you forgot everything up till now, just remember the following:

- Our universe is characterized by homogeneous, anisotropic Bianchi IX cosmology.

Break for Remembering Stuff

If you forgot everything up till now, just remember the following:

- Our universe is characterized by homogeneous, anisotropic Bianchi IX cosmology.
- Chaos is a sensitive dependence on initial conditions.

The BKL Approximation

- Belinski–Khalatnikov–Lifshitz (BKL) at the Landau Institute for Theoretical Physics.

The BKL Approximation

- Belinski–Khalatnikov–Lifshitz (BKL) at the Landau Institute for Theoretical Physics.
- Start with the Kasner (Bianchi I) solution:

$$ds^2 = -\,dt^2 + \sum_k t^{2p_k}\, dx_k^2,$$

where $-\frac{1}{3} \leq p_1 \leq 0 \leq p_2 \leq \frac{2}{3} \leq p_3 \leq 1$ and $\sum_k p_k = \sum_k p_k^2 = 1$.

The BKL Approximation

- Belinski–Khalatnikov–Lifshitz (BKL) at the Landau Institute for Theoretical Physics.
- Start with the Kasner (Bianchi I) solution:

$$ds^2 = -\,dt^2 + \sum_k t^{2p_k}\, dx_k^2,$$

  where $-\frac{1}{3} \leq p_1 \leq 0 \leq p_2 \leq \frac{2}{3} \leq p_3 \leq 1$ and $\sum_k p_k = \sum_k p_k^2 = 1$.
- This universe expands in one direction and contracts in two as we go back in time.

The BKL Approximation

- Belinski–Khalatnikov–Lifshitz (BKL) at the Landau Institute for Theoretical Physics.
- Start with the Kasner (Bianchi I) solution:

$$ds^2 = -\,dt^2 + \sum_k t^{2p_k}\, dx_k^2,$$

where $-\frac{1}{3} \le p_1 \le 0 \le p_2 \le \frac{2}{3} \le p_3 \le 1$ and $\sum_k p_k = \sum_k p_k^2 = 1$.

- This universe expands in one direction and contracts in two as we go back in time.
- Introduce the BKL parameter u:

$$p_1 = -\frac{u}{u^2+u+1}, \quad p_2 = \frac{u+1}{u^2+u+1}, \quad p_3 = \frac{u(u+1)}{u^2+u+1}$$

The BKL Approximation

- Describe Bianchi IX using a series of Kasner epochs.

The BKL Approximation

- Describe Bianchi IX using a series of Kasner epochs.
- The epoch-change map (called the Gauss Map) is

$$u_{n+1} = \frac{1}{u_n - [u_n]},$$

where $[u_n]$ is the fractional part of u_n.

The BKL Approximation

- Describe Bianchi IX using a series of Kasner epochs.
- The epoch-change map (called the Gauss Map) is

$$u_{n+1} = \frac{1}{u_n - [u_n]},$$

  where $[u_n]$ is the fractional part of u_n.
- Slight aside: the metric entropy h of a discrete map characterizes its chaoticity by describing how nearby trajectories diverge on average.

$$h = \frac{\langle \ln(\Delta_t/\Delta_0) \rangle}{t}$$

The BKL Approximation

- Describe Bianchi IX using a series of Kasner epochs.
- The epoch-change map (called the Gauss Map) is

$$u_{n+1} = \frac{1}{u_n - [u_n]},$$

where $[u_n]$ is the fractional part of u_n.

- Slight aside: the metric entropy h of a discrete map characterizes its chaoticity by describing how nearby trajectories diverge on average.

$$h = \frac{\langle \ln\left(\Delta_t / \Delta_0\right) \rangle}{t}$$

- Using some measure theory, it can be found that

$$h = \frac{\pi^2}{6(\ln 2)^2}$$

The BKL Approximation

- Describe Bianchi IX using a series of Kasner epochs.
- The epoch-change map (called the Gauss Map) is

$$u_{n+1} = \frac{1}{u_n - [u_n]},$$

  where $[u_n]$ is the fractional part of u_n.
- Slight aside: the metric entropy h of a discrete map characterizes its chaoticity by describing how nearby trajectories diverge on average.

$$h = \frac{\langle \ln(\Delta_t/\Delta_0) \rangle}{t}$$

- Using some measure theory, it can be found that

$$h = \frac{\pi^2}{6(\ln 2)^2}$$

- Since $h > 0$ implies exponential divergence, the Gauss Map is chaotic.

The BKL Approximation

- Numerical estimates show that the BKL approximation describes Bianchi IX very accurately near the singularity.

The BKL Approximation

- Numerical estimates show that the BKL approximation describes Bianchi IX very accurately near the singularity.
- There is a drawback however. For all its accuracy, it remains an approximation and tells nothing about the chaoticity of Bianchi IX itself.

Painleve Analysis

- A test to characterize whether a system is integrable.

Painleve Analysis

- A test to characterize whether a system is integrable.
- Integrability: System has enough first integrals.

Painleve Analysis

- A test to characterize whether a system is integrable.
- Integrability: System has enough first integrals.
- The only movable singularities of the general solution are poles $\implies$ system is integrable.

Painleve Analysis

- A test to characterize whether a system is integrable.
- Integrability: System has enough first integrals.
- The only movable singularities of the general solution are poles $\implies$ system is integrable.
- Movable singularity: a singularity in the solution which depends on the value of the arbitrary coefficient.

Painleve Analysis

- A test to characterize whether a system is integrable.
- Integrability: System has enough first integrals.
- The only movable singularities of the general solution are poles $\implies$ system is integrable.
- Movable singularity: a singularity in the solution which depends on the value of the arbitrary coefficient.
- Example:

$$\frac{du}{dx} + \frac{u^2}{x} = 0, \quad u = \frac{1}{c + \ln x}$$

Painleve Analysis

- After performing a Painleve test, we find that the general solution of the Bianchi IX model has essential singularities of the form $\sim e^{1/\tau}$.

Painleve Analysis

- After performing a Painleve test, we find that the general solution of the Bianchi IX model has essential singularities of the form $\sim e^{1/\tau}$.
- Bianchi IX is therefore non-integrable.

Painleve Analysis

- After performing a Painleve test, we find that the general solution of the Bianchi IX model has essential singularities of the form $\sim e^{1/\tau}$.
- Bianchi IX is therefore non-integrable.
- Drawback: Non-integrability does not imply chaos.

Lyapunov Exponents

- Lyapunov exponents are the metric entropy for continuous systems.

Lyapunov Exponents

- Lyapunov exponents are the metric entropy for continuous systems.
- Certain numerical estimates show that there is a positive Lyapunov exponent in Bianchi IX.

Lyapunov Exponents

- Lyapunov exponents are the metric entropy for continuous systems.
- Certain numerical estimates show that there is a positive Lyapunov exponent in Bianchi IX.
- Other estimates however show that the largest Lyapunov exponent is 0.

Lyapunov Exponents

- Lyapunov exponents are the metric entropy for continuous systems.
- Certain numerical estimates show that there is a positive Lyapunov exponent in Bianchi IX.
- Other estimates however show that the largest Lyapunov exponent is 0.
- These contradictory results therefore do not shed light on any chaos present in Bianchi IX.

Fractals

- Fractals are self-similar structures.

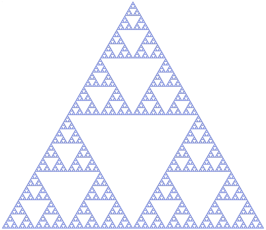


Figure: The Sierpinski Triangle.

Fractals

- Fractals are self-similar structures.
- For each fractal, we can specify a fractal dimension.

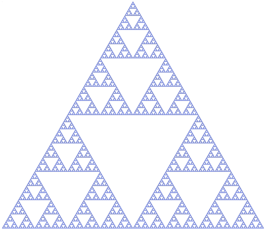


Figure: The Sierpinski Triangle.

Fractals

- Fractals are self-similar structures.
- For each fractal, we can specify a fractal dimension.
- An example is the box-counting dimension which describes the rate at which the number of boxes required to completely cover a fractal grows compared to the rate at which the size of each box shrinks.

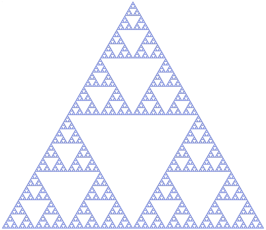


Figure: The Sierpinski Triangle.

Fractals

- By identifying a set of initial conditions in phase space, which when evolved backwards in time, show an oscillatory behaviour close to the singularity, we can identify our fractal set.

Fractals

- By identifying a set of initial conditions in phase space, which when evolved backwards in time, show an oscillatory behaviour close to the singularity, we can identify our fractal set.
- If evolution of these states causes them to remain in this set, we have an invariant set.

Fractals

- By identifying a set of initial conditions in phase space, which when evolved backwards in time, show an oscillatory behaviour close to the singularity, we can identify our fractal set.
- If evolution of these states causes them to remain in this set, we have an invariant set.
- Numerical estimates have shown that the fractal dimension of this invariant set is 1.86 ± 0.01.

Fractals

- By identifying a set of initial conditions in phase space, which when evolved backwards in time, show an oscillatory behaviour close to the singularity, we can identify our fractal set.
- If evolution of these states causes them to remain in this set, we have an invariant set.
- Numerical estimates have shown that the fractal dimension of this invariant set is 1.86 ± 0.01.
- Dimension is a fraction $\implies$ invariant set is a fractal $\implies$ chaos exists.

Fractals

- By identifying a set of initial conditions in phase space, which when evolved backwards in time, show an oscillatory behaviour close to the singularity, we can identify our fractal set.
- If evolution of these states causes them to remain in this set, we have an invariant set.
- Numerical estimates have shown that the fractal dimension of this invariant set is 1.86 ± 0.01.
- Dimension is a fraction $\implies$ invariant set is a fractal $\implies$ chaos exists.
- Drawback: As with all other numerical methods, this remains analytically unproven.

Curvature

- Introduce a Hamiltonian for Bianchi IX:

$$2H = -p_t^2 + p_\zeta^2 + \left(\frac{p_\phi}{\sinh\zeta}\right)^2$$

Curvature

- Introduce a Hamiltonian for Bianchi IX:

$$2H = -p_t^2 + p_\zeta^2 + \left(\frac{p_\phi}{\sinh \zeta}\right)^2$$

- Here, we have ignored the potential term in the Hamiltonian.

Curvature

- Introduce a Hamiltonian for Bianchi IX:

$$2H = -p_t^2 + p_\zeta^2 + \left(\frac{p_\phi}{\sinh\zeta}\right)^2$$

- Here, we have ignored the potential term in the Hamiltonian.
- Note that the spatial part of the Hamiltonian includes the inverse of the hyperbolic plane metric:

$$ds^2 = dx^2 + (\sinh^2 x)\, dy^2$$

Curvature

- Introduce a Hamiltonian for Bianchi IX:

$$2H = -p_t^2 + p_\zeta^2 + \left(\frac{p_\phi}{\sinh\zeta}\right)^2$$

- Here, we have ignored the potential term in the Hamiltonian.
- Note that the spatial part of the Hamiltonian includes the inverse of the hyperbolic plane metric:

$$ds^2 = dx^2 + (\sinh^2 x)\, dy^2$$

- Solutions are geodesics on the hyperbolic plane.

Curvature

- Using results from measure theory and differential geometry, it has been shown that this geodesic motion is chaotic.

Curvature

- Using results from measure theory and differential geometry, it has been shown that this geodesic motion is chaotic.
- In particular, a small finite set will spread almost everywhere on the hyperbolic plane under the equations of motion.

Curvature

- Using results from measure theory and differential geometry, it has been shown that this geodesic motion is chaotic.
- In particular, a small finite set will spread almost everywhere on the hyperbolic plane under the equations of motion.
- Therefore, the evolution of Bianchi IX is chaotic.

Curvature

- Using results from measure theory and differential geometry, it has been shown that this geodesic motion is chaotic.
- In particular, a small finite set will spread almost everywhere on the hyperbolic plane under the equations of motion.
- Therefore, the evolution of Bianchi IX is chaotic.
- Drawback: Ignoring the potential term is an approximation which may not be valid.

Implications For Our Universe

- If the Bianchi IX model describes our universe and is indeed chaotic, we cannot specify a set of initial conditions which brought us to our universe today.

Implications For Our Universe

- If the Bianchi IX model describes our universe and is indeed chaotic, we cannot specify a set of initial conditions which brought us to our universe today.
- In particular, we CANNOT tell how the universe began.

The End

Our universe is (probably) chaotic.

$\sim$ Hashir H. Khan
(the H stands for Hashir H. Khan)