Some PDEs can be recast as integral equations, where instead of the unknown being acted on by a differential operator, the unknown is acted on by an integral operator. Integral formulations can be advantageous from a computational point of view by reducing the dimensionality of the unknown (in the case of boundary integral equations) and by improving the conditioning of the problem. Differential operators are generally unbounded. This means that if we apply a spectral theorem to them, we may be able to find a set of eigenfunctions and a sequence of eigenvalues that tends towards infinity. From a computational point of view, discretizing unbounded operators may result in matrices with a large condition number, and therefore will result in a loss of accuracy during computation. However, differential operators have the advantage of being local operators. The influence of a given point is felt only by nearby points, and this is reflected in the sparse matrix structure that arises from their discretization. Sparse matrices are easy to store and work well with iterative solvers (possibly after some preconditioning). Integral equations, and specifically boundary integral equations (BIEs), are global operators, so the influence of a single point is felt at every other point in the domain of interest. This means that upon discretization of the integral equation, we end up with a dense matrix. But this dense matrix will usually be well-conditioned. One way to overcome the problem of dense matrices in the discretization is to realize that many BIEs are "effectively low rank," that is, while the influence of a single point is felt by every other point, the magnitude of that influence is larger for nearby points, and quite small as the distance increases. This observation is the basis for the fast multipole method. With the FMM, BIEs scale much more favorably and can be used to solve enormous problems that would be inaccessible with a traditional discretization approach.

Here's a simple example of using a BIE to solve a PDE. Consider the exterior Laplace Dirichlet problem

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } \Omega_+ \\ u &= g \qquad \text{on } \partial\Omega_+ \end{aligned}$$

Laplace's equation has the fundamental solution (in 2d) $G(x,y) = -\frac{1}{2\pi}\log(|x-y|)$, and it's normal derivative is $\partial_\nu(y)G(x,y) = \frac{1}{2\pi}\frac{1}{|x-y|}\frac{(x-y)\cdot\nu(y)}{|x-y|}$. Using these two kernels, we can define two boundary integral operators on $\Gamma := \partial\Omega_+$:

$$\begin{aligned} S\Lambda(x) &:= \int_\Gamma G(x,y)\Lambda(y)d\Gamma_y \qquad\qquad x \notin \Gamma, \\ D\Phi(x) &:= \int_\Gamma \partial_\nu(y)G(x,y)\Phi(y)d\Gamma(y) \quad x \notin \Gamma. \end{aligned}$$

These are the single- and double-layer potentials for the Laplacian. From these, we can take traces and normal derivatives to find four related boundary integral operators (they have similar forms, but are instead only evaluated on the boundary). These are the

operators of the Calderòn projector. Now, we can propose a solution to the Neumann problem using only a double-layer potential:

$$u(x) = D\Phi(x)$$

where Φ is an unknown density defined on Γ. To find Φ, we take the restriction to the boundary and impose the boundary condition:

$$-\frac{1}{2}\Phi(x) + \int_\Gamma \frac{1}{2\pi}\frac{1}{|x-y|}\frac{(x-y)\cdot\nu(y)}{|x-y|}\Phi(y)d\Gamma_y = g(x). \tag{1}$$

If we suppose that Γ is parameterized by a 1-periodic smooth function $s(t)$, then the integrand is smooth. One approach we can take to solving this integral equation is apply a quadrature rule, such as the trapezoid rule, to the integral and enforce the equation at the quadrature nodes. If the domain Γ is parameterized by a smooth 1-period function, then the trapezoidal rule will have super-algebraic convergence. We then substitute this parameterization into the integral equation and arrive at

$$-\frac{1}{2}\Phi(s(t)) + \int_0^1 \frac{1}{2\pi}\frac{1}{|s(t)-s(\tau)|}\frac{(s(t)-s(\tau))\cdot\nu(s(\tau))}{|s(t)-s(\tau)|}\Phi(s(\tau))d\tau = g(s(t)). \tag{2}$$

We approximate (**??**) by

$$-\frac{1}{2}\Phi(s(t_m)) + \frac{h}{2}\sum_{n=1}^{N}\left(f(s(t_m), s(\tau_n)) + f(s(t_m), s(\tau_{n+1}))\right) = g(s(t_m))$$

for $m = 1 \ldots N$, where

$$f(x_m, y_n) = \frac{1}{2\pi}\frac{1}{|x_m - y_n|}\frac{(x_m - y_n)\cdot\nu(y_n)}{|x_m - y_n|}\Phi(y_n)$$

and $h = 1/n$. This results in an $N \times N$ linear system that is well-conditioned. We can then use our favorite linear solver to compute the boundary unknown $\Phi(y)$. Once we have this, we come back to the potential representation and simply evaluate the integral

$$u(x) = \int_\Gamma \frac{1}{2\pi}\frac{1}{|x-y|}\frac{(x-y)\cdot\nu(y)}{|x-y|}\Phi(y)d\Gamma_y$$

at chosen target points. The integral is again discretized with the periodic trapezoid rule.