The Weighted OWA Operator

Vicenç Torra*
Dpt. d'Enginyeria Informatica, Escola Tecnica Superior d'Enginyeria (ETSE), Universitat Rovira i Virgili, Carretera de Salou, s/n, E-43006 Tarragona (Spain)

One of the properties that the OWA operator satisfies is commutativity. This condition, that is not satisfied by the weighted mean, stands for equal reliability of all the information sources that supply the data. In this article we define a new combination function, the WOWA (Weighted OWA), that combines the advantages of the OWA operator and the ones of the weighted mean. We study some of its properties and show how it can be extended to deal with linguistic labels.

I. INTRODUCTION

Nowadays, in several fields of human knowledge the need for data fusion methods and techniques is increasing due to the fact that data is, gradually, obtained in an easier way. Some of the fields that use combination functions are, e.g., mathematics,[1,2] economics,[3,4] biology,[5] and education[6]). Within Artificial Intelligence, synthesis of information is used in several fields, e.g., robotics,[7] vision,[8] fuzzy logic controllers,[9] and knowledge acquisition.[10]

In Artificial Intelligence, these techniques are mainly used when a system has to make a decision or when it needs a comprehensive representation of its domain.

In the first case, for instance, it is possible that instead of a single criteria for each alternative, the system has several ones. This case, that corresponds to a multicriteria decision-making problem, is usually solved[11] in a two-phase process: (i) the aggregation of the degree of satisfaction for all criteria, per decision alternative; and (ii) the ranking of the alternatives with respect to the global aggregated degree of satisfaction.

On the other hand, to have a good representation of an environment, a system needs the knowledge supplied by information sources (or the knowledge embedded in it) to be reliable and extend on the whole domain of actuation. However, the information supplied by a single information source (by a single expert or sensor) is often not reliable enough and/or too narrow in relation to

*E-mail: vtorra@etse.urv.es.

 CCC 0884-8173/97/020153-14

the working domain. In this case, the information provided from several sensors (or experts) can be combined to improve data reliability and accuracy and to include some features that are impossible to perceive with individual sensors.

As combination functions are defined according to the objects to synthesize, they depend on the knowledge representation formalism. This dependence provokes, as pointed out in Ref. 2, that the problems related to knowledge representation (e.g., selection of the best representation for a given fact) are also present in combination functions. Therefore, there are functions in several formalisms, e.g., to combine probability distributions,[2] mass functions,[12] and fuzzy sets,[13–15] preference relations (both quantitative[16,17] and qualitative[18,19]), classifications (in several formalisms as dendrograms[20] or partitions[21]), data matrices[10] or rules[22].

When the objects to synthesize are numeric values (e.g., numbers in the [0, 1] interval) two classical aggregation functions are considered: the arithmetic mean and the weighted mean. These functions, together with the quasi-arithmetic mean (a general form), are studied in Refs. 23–25. Recently Yager[26,27] defined an alternative combination function to synthesize also numeric values—the OWA operator. This function has attracted the interest of several researchers. Several articles have been published where its properties are studied[28,29] or where it is applied.[3,19,30]

Both functions, the weighted mean and the OWA operator, are to combine values according to a set of weights. The main difference is the meaning that the set of weights have in each function. On one hand, the weighted mean allows the system to compute an aggregate value from the ones corresponding to several sources, taking into account the reliability of each information source. In fact, each source has an attached weight that measures its reliability. Alternatively, the OWA operator permits weighting the values in relation to their ordering. In this way, a system can give more importance to a subset of the input values than to another subset. For instance, the influence of extreme values to the result can be diminished, increasing the influence of central values. The function, however, is commutative, so any permutation of the arguments gives the same result.

Therefore, in the weighted mean, weights measure the importance of an information source with independence of the value that the source has captured. On the other hand, in the OWA, weights measure the importance of a value (in relation to other values) with independence of the information source that has captured it. In this article we define a new combination function, the WOWA (Weighted OWA), that combines the advantages of the OWA operator and the ones of the weighted mean. The new function allows the user to weight the reliability of the information source, as the weighted mean does, and the values in relation to their relative position, as the OWA operator.

The article begins (Section II) with a review of the OWA operator and the weighted mean. Then (Section III) it presents the WOWA operator and studies some of their properties. Section IV reviews the process of calculating the output through the operator. Section V gives some examples of the new operator introduced. Finally, the article finishes with some extensions of the WOWA operator (Sec. VI) and with the conclusions (Sec. VII).

II. PRELIMINARIES

DEFINITION 1.[26,27] *Let* $\mathbf{w}$ *be a weighting vector of dimension* n ($\mathbf{w} = [w_1 w_2 \ldots w_n]$) *such that*

$$\text{(i)}\ w_i \in [0, 1]$$

$$\text{(ii)}\ \Sigma_i w_i = 1$$

In this case, a mapping $f_{owa}: \mathbb{R}^n \to \mathbb{R}$ *is an* **Ordered Weighted Averaging (OWA) operator** *of dimension* n *if*

$$f_{owa}(a_1, \ldots, a_n) = \sum_i w_i a_{\sigma(i)}$$

where $\{\sigma(1), \ldots, \sigma(n)\}$ *is a permutation of* $\{1, \ldots, n\}$ *such that* $a_{\sigma(i-1)} \geq a_{\sigma(i)}$ *for all* $i = 2, \ldots, n$. (*i.e.,* $a_{\sigma(i)}$ *is the ith largest element in the collection* $a_1, \ldots, a_n$).

PROPOSITION 1. *The OWA operator satisfies the following properties:*[27]

(1) *It is an aggregation operator which remains between the minimum and the maximum of the arguments:*

$$\min\{a_1, \ldots, a_n\} \leq f_{owa}(a_1, \ldots, a_n) \leq \max\{a_1, \ldots, a_n\} \text{ for any set of values } a_1, \ldots, a_n$$

(2) *It satisfies idempotency (unanimity):*

$$f_{owa}(a_1, \ldots, a_n) = a \text{ when } a_i = a \text{ for all } i = 1, \ldots, n$$

(3) *It is commutative:*

$$f_{owa}(a_1, \ldots, a_n) = f_{owa}(a_{\pi(1)}, \ldots, a_{\pi(n)}) \text{ for any permutation } \pi$$

(4) *It is monotone in relation to the values* a_i:

$$f_{owa}(a_1, \ldots, a_n) \geq f_{owa}(b_1, \ldots, b_n) \text{ when } a_i \geq b_i \text{ for all } i = 1, \ldots, n$$

(5) *It leads to the arithmetic mean when* $w_i = 1/n$ *for all* $i = 1, \ldots, n$

$$f_{owa}(a_1, \ldots, a_n) = \sum_i (1/n) a_{\sigma(i)} = (1/n) \sum_i a_{\sigma(i)}$$

(6) *It leads to the maximum when* $w_1 = 1$ *and* $w_i = 0$ *for all* $n \geq i > 1$.
(7) *It leads to the minimum when* $w_n = 1$ *and* $w_i = 0$ *for all* $n > i \geq 1$.

DEFINITION 2. *Let* $\mathbf{p}$ *be a weighting vector of dimension* n ($\mathbf{p} = [p_1 p_2 \ldots p_n]$) *such that*

$$\text{(i)}\ p_i \in [0, 1]$$

$$\text{(ii)}\ \Sigma_i p_i = 1$$

In this case, a mapping f_{wm}: $\mathbb{R}^n \rightarrow \mathbb{R}$ is a **Weighted Mean (WM) operator** *of dimension n if*

$$f_{wm}(a_1, \ldots, a_n) = \sum_i p_i a_i$$

PROPOSITION 2. *The weighted mean satisfies the following properties:*[13]

(1) *It is an aggregation operator which remains between the minimum and the maximum.*
(2) *It satisfies idempotency (unanimity).*
(3) *It is commutative if and only if $p_i = 1/n$ for all $i = 1, \ldots, n$.*
(4) *It is monotone in relation to the values a_i.*
(5) *It leads to the arithmetic mean when $p_i = 1/n$ for all $i = 1, \ldots, n$*
(6) *It leads to dictatorship of the ith value when $p_i = 1$ and $p_j = 0$ for all $j = 1, \ldots, n$ but $j \neq 1$.*

Definition 2 shows that in the weighted mean the value of the ith information source (or expert) is weighted according to the weight w_i. In fact, not all information sources are equally reliable—sensors, due to its hardware capabilities and the information they capture, and human experts due to their background. Besides that, intelligent systems, as in Ref. 8, take into account data recorded in previous measurements. The weights, as it is pointed out in Ref. 31 cannot only be used to account for differences between sensors but also to fuse together a sequence of measurements from a single sensor so that the more recent measurements are given a greater weight. In general, in a weighted mean we express with $p_i = 0$ that the ith information source is not, at all, relevant for the output (or, that there is no confidence on the value corresponding to that source); with $p_i \geq p_j$ that the reliability of the ith expert is greater than the one of jth expert; and with $p_i = 1$ that the output should only consider the ith information source (this is the dictatorship condition above, the ith expert gives the correct answer).

On the other hand, the OWA operator weights the values instead of weighting the experts (or the information sources). This is so because each w_i is attached to the ith value in decreasing order without considering from which information source the value comes from. Notice that the OWA operator is commutative. This is, all information sources (or experts) have an equal contribution to the final solution. With this kind of weight, the OWA operator calculates the output, for example, without considering extreme values ($w_1 = w_n = 0$); or considering only the values that *most* of the experts give where *most* is a fuzzy quantifier[32] (the definition of the set of weights from linguistic quantifiers is described in Ref. 27. Similarly, it is also possible to define the OWA operator as the median value ($p_{(n+1)/2} = 1$ if n is odd, $p_{n/2} = p_{n/2+1} = 1/2$ if n is even).

Therefore, weights represent different aspects in both combination functions. However, although both points of view are meaningful in a single problem, both combination functions present the drawback of considering only one of them. To solve this drawback, we have defined a combination function that allows a system to consider both the relevance of the information sources (as the weighted mean) and the relevance of the values (as the OWA operator). In

Section III we define this new function, the WOWA (Weighted OWA), that considers two weight vectors: **p** corresponding to the relevance of the sources, and **w** corresponding to the relevance of the values.

The function is defined in a way that when all the experts are equally reliable (i.e., $p_i = 1/n$) it reduces to the OWA operator and when all the values have equal relevance (i.e., $w_i = 1/n$), it reduces to the weighted mean.

III. THE WOWA OPERATOR

DEFINITION 3. *Let* **p** *and* **w** *be weighting vectors of dimension n* ($\mathbf{p} = [p_1 \ldots p_n]$, $\mathbf{w} = [w_1 \ldots w_n]$) *such that:*

$$\text{(i)}\ \ p_i \in [0, 1] \quad \text{and} \quad \sum_i p_i = 1$$

$$\text{(ii)}\ \ w_i \in [0, 1] \quad \text{and} \quad \sum_i w_i = 1$$

In this case, a mapping f_{wowa}: $\mathbb{R}^n \to \mathbb{R}$ *is a* **Weighted Ordered Weighted Averaging (WOWA) operator** *of dimension n if*

$$f_{wowa}(a_1, \ldots, a_n) = \sum_i \omega_i a_{\sigma(i)}$$

where $\{\sigma(1), \ldots, \sigma(n)\}$ *is a permutation of* $\{1, \ldots, n\}$ *such that* $a_{\sigma(i-1)} \geq a_{\sigma(i)}$ *for all* $i = 2, \ldots, n$. *(i.e.,* $a_{\sigma(i)}$ *is the ith largest element in the collection* $a_1, \ldots, a_n$*), and the weight* ω_i *is defined as*

$$\omega_i = w^*\left(\sum_{j \leq i} p_{\sigma(j)}\right) - w^*\left(\sum_{j < i} p_{\sigma(j)}\right)$$

with w^* *a monotone increasing function that interpolates the points* $(i/n, \Sigma_{j \leq i} w_j)$ *together with the point* $(0, 0)$. w^* *is required to be a straight line when the points can be interpolated in this way. This later condition is required to prove Proposition* 4. *From now on,* ω *represent the set of weights* $\{\omega_i\}$, *i.e.,* $\omega = [\omega_1, \ldots, \omega_n]$.

In the rest of this section we study some of the properties of the WOWA operator. First we establish that the summation of the weights ω_i equals 1. Then, we consider the relation between the WOWA and the OWA and with the weighted mean. We show first that the weighted mean is a special case of the WOWA operator, and second, that the OWA is also a special case of the WOWA. This implies that the WOWA operator generalizes both the weighted mean and the OWA. The section finishes with some other results.

PROPOSITION 3. *Let* f_{wowa} *be a WOWA operator and let* **p** *and* **w** *be its weighting vectors of dimension n. In this case, the weighting vector* $\omega = [\omega_1, \ldots, \omega_n]$ *satisfies* $\Sigma_{i=1,n}\ \omega_i = 1$.

Proof. According to the definition of ω_i, we have that $\Sigma_{i=1,n}\, \omega_i = \Sigma_{i=1,n}\, [w^*(\Sigma_{j \leq i} p_{\sigma(j)}) - w^*(\Sigma_{j<i}\, p_{\sigma(j)})]$. Therefore

$$\sum_{i=1,n} \omega_i = \sum_i w^*\left(\sum_{j\le i} p_{\sigma(j)}\right) - \sum_i w^*\left(\sum_{j<i} p_{\sigma(j)}\right)$$

So, as $w^*(\Sigma_{j<i}\, p_{\sigma(j)}) = w^*(\Sigma_{j\le i-1}\, p_{\sigma(j)})$:

$$\sum_{i=1,n} \omega_i = w^*\left(\sum_{j\le n} p_{\sigma(j)}\right) - w^*\left(\sum_{j<1} p_{\sigma(j)}\right) = w^*(1) - w^*(0)$$

that is equal 1 because w interpolates $(1 = n/n,\ 1 = \Sigma_{j\le n}\, w_j)$ together with $(0, 0)$. ■

PROPOSITION 4. *Let f_{wowa} be a WOWA operator and let* **p** *and* **w** *be its weighting vectors of dimension n. In this case, if* **w** *is defined as $w_i = 1/n$ for all $i = 1, \ldots, n$ then the WOWA operator reduces to a weighted mean with a weighting vector* **p**.

Proof. First of all notice that f_{wowa} is a weighted mean if

$$f_{wowa}(a_1, \ldots, a_n) = f_{wm}(a_1, \ldots, a_n)$$

holds. As, $f_{wm}(a_1, \ldots, a_n) = \Sigma_i\, p_i a_i = \Sigma_i\, p_{\sigma(i)} a_{\sigma(i)}$, and $f_{wowa}(a_1, \ldots, a_n) = \Sigma_i\, \omega_i a_{\sigma(i)}$, the equality holds if and only if $\Sigma_i\, p_{\sigma(i)} a_{\sigma(i)} = \Sigma_i\, \omega_i a_{\sigma(i)}$. This is equivalent to $\omega_i = p_{\sigma(i)}$. Therefore, the f_{wowa} is a weighted mean iff $\omega_i = p_{\sigma(i)}$.

If $w_i = 1/n$ for all $i = 1, \ldots, n$ we have that w interpolates $(0, 0)$ and $(i/n, \Sigma_{j\le i}\, w_j) = (i/n, i/n)$. As these points are all over a straight line, w^* should also be a straight line. Therefore, as $w^*(x) = x$, we have:

$$\omega_i = w^*\left(\sum_{j\le i} p_{\sigma(j)}\right) - w^*\left(\sum_{j<i} p_{\sigma(j)}\right) = \sum_{j\le i} p_{\sigma(j)} - \sum_{j<i} p_{\sigma(j)}$$

It is clear now that $\omega_i = p_{\sigma(i)}$ and therefore f_{wowa} is a weighted mean with weights p. ■

PROPOSITION 5. *Let f_{wowa} be a WOWA operator and let* **p** *and* **w** *be its weighting vectors of dimension n. In this case, if* **p** *is defined as $p_i = 1/n$ for all $i = 1, \ldots, n$ then the WOWA operator reduces to an OWA operator with a weighting vector* **w**.

Proof. Notice that f_{wowa} is an OWA operator if

$$f_{wowa}(a_1, \ldots, a_n) = f_{owa}(a_1, \ldots, a_n)$$

holds. As, $f_{owa}(a_1, \ldots, a_n) = \Sigma_i\, w_i a_{\sigma(i)}$, and $f_{wowa}(a_1, \ldots, a_n) = \Sigma_i\, \omega_i a_{\sigma(i)}$, the equality holds if and only if $\omega_i = w_i$ for all i. Therefore, the f_{wowa} is an OWA operator iff $\omega_i = p_{\sigma(i)}$.

If $p_i = 1/n$ for all i, then

$$\omega_i = w^*\left(\sum_{j\le i} p_{\sigma(j)}\right) - w^*\left(\sum_{j<i} p_{\sigma(j)}\right) = w^*(i/n) - w^*((i-1)/n)$$

As w^* interpolates $(i/n, \Sigma_{j\leq i} w_i)$ and (0, 0) we have that $w^*(i/n) = \Sigma_{j\leq i} w_j$ and $w^*(0) = 0$. This implies that, $\omega_i = w_i$. Therefore f_{wowa} is an OWA operator with weights w. ■

LEMMA 1. *Let f_{wowa} be a WOWA operator and let* **p** *and* **w** *be its weighting vectors of dimension n. In this case, the f_{wowa} generalizes both the weighted mean and the OWA operator.*

LEMMA 2. *Let f_{wowa} be a WOWA operator and let* **p** *and* **w** *be its weighting vectors of dimension n. In these conditions, the WOWA operator is an OWA operator with weights ω.*

PROPOSITION 6. *Let f_{wowa} be a WOWA operator and let* **p** *and* **w** *be its weighting vectors of dimension n. In these conditions f_{wowa} satisfies the following properties:*

(1) *It is an aggregation operator which remains between the minimum and the maximum.*
(2) *It satisfies idempotency.*
(3) *It is commutative if and only if $p_i = 1/n$ for all $i = 1, \ldots, n$ such that $w_i \neq 0$.*
(4) *It is monotone in relation to the input values a_i.*
(5) *It leads to dictatorship of the ith value when $p_i = 1$ and $p_j = 0$ for all $j = 1, \ldots, n$ but $j \neq i$.*
(6) *It leads to the arithmetic mean when $p_i = 1/n$ and $w_i = 1/n$ for all $i = 1, \ldots, n$*

$$f_{wowa}(a_1, \ldots, a_n) = \sum_i (1/n)a_{\sigma(i)} = (1/n)\sum_i a_{\sigma(i)}$$

PROPOSITION 7. *Let f_{wowa} be a WOWA operator and let* **p** *and* **w** *be its weighting vectors of dimension n. Let ω be the set of weights calculated from* **p** *and* **w** *according to Definition 3. Let f'_{wowa} be a WOWA operator with weights ω' instead of ω, where $\omega'_j > \omega_j$ and $\omega'_k < \omega_k$ and for some $1 \leq j < k \leq n$ and $\omega'_i = \omega_i$ for all $i \neq j, k$. In these conditions:*

$$f'_{wowa}(\mathbf{a}) \geq f_{wowa}(\mathbf{a}) \quad \textit{for all } \mathbf{a} = [a_1, \ldots, a_n]$$

Proof. As $\omega'_i = \omega_i$ for all $i \neq j, k$ we have to prove

$$\omega'_j a_{\sigma(j)} + \omega'_k a_{\sigma(k)} = \omega_j a_{\sigma(j)} + \omega_k a_{\sigma(k)}$$

that is equivalent to

$$(\omega'_j - \omega_j)a_{\sigma(j)} > (\omega_k - \omega'_k)a_{\sigma(k)}$$

As $\omega'_k < \omega_k$, then $(\omega_k - \omega'_k) > 0$ holds. Therefore this expression equals

$$(\omega'_j - \omega_j)/(\omega_k - \omega'_k)a_{\sigma(j)} > a_{\sigma(k)}$$

That holds because $(\omega'_j - \omega_j) = (\omega_k - \omega'_k)$ as $\Sigma_i \omega_i = \Sigma_i \omega'_i = 1$, and $a_{\sigma(j)} > a_{\sigma(k)}$ as $j < k$. Therefore the proposition is proven. ■

In general, we can say that if $w'^* > w^*$, then $f'_{wowa}(\mathbf{a}) \geq f_{wowa}(\mathbf{a})$.

IV. THE DETERMINATION OF THE WOWA OPERATOR

Given a set of weights **p** and **w**, and the data vector **a**, the procedure to calculate the f_{wowa} is as follows:

```
procedure f_wowa (p, w: weight vectors; a: data vector) is
      Define S={(i/n, Σ_{j≤i} w_j) | i=1, . . . ,n} ∪ {(0,0)}
      Define W* as the function that interpolates S
      Order the vector a and determine the permutation s.
      Calculate w_i according to definition 3
      return (Σ_i w_i a_{s(i)})
end f_wowa.
```

In fact, to define the WOWA operator there are, mainly, two possible ways to proceed.

(i) To establish the weighting vector **w** and then to interpolate W^* (as it was previously stated in Definition 3). To interpolate the weighting points, it is adequate any method that from monotone data and bounded in the unit interval defines a monotone and bounded function. To define this interpolation method we can follow the guidelines of the method described in Ref. 33 for membership function definition. This method, based on the algorithm of McAllister and Roulier for producing a second-degree Bernstein polynomial, obtains from a set of monotone or convex observations a monotone or convex function. The method has to be adapted to satisfy the statement in Definition 3 that w^* is required to be a straight line when the points can be interpolated in this way.

(ii) On the other hand, it is also possible to begin with the definition of w^* (without considering the initial step of defining the weight vector **w**). In this way, we can derive the set of weights ω from w^*, where w^* is any monotone increasing function within the $[0, 1]$ interval with $w^*(0) = 0$ and $w^*(1) = 1$. In fact, a similar procedure is presented in Ref. 27 and 34. In Ref. 27 Yager defines a way to extract the weights from fuzzy quantifiers when they are *regular monotonically nondecreasing* (i.e., they are defined as functions $Q(x)$ that satisfy $Q(0) = 0$, $Q(1) = 1$ and if $x > y$ then $Q(x) \geq Q(y)$. Notice that w^* is a regular monotonically nondecreasing function. In Ref. 34, Yager uses these latter functions to extract the weights when there is an importance measure associated to each criteria (each information source is our case). The results he obtains are analogous to ours.

In the examples given in Section V we use the second approach.

V. EXAMPLES

In this section we show the results that would be obtained for different input values and different weighting vectors and functions. In the examples we define the set of input values already ordered from the maximum to the minimum value (i.e., $\mathbf{a} = (a_1, a_2, a_3, a_4)$ with $a_i \geq a_{i+1}$ for $i = 1, 2, 3$). In this way we avoid the ordering process and the results are easier to interpret. Notice that the results

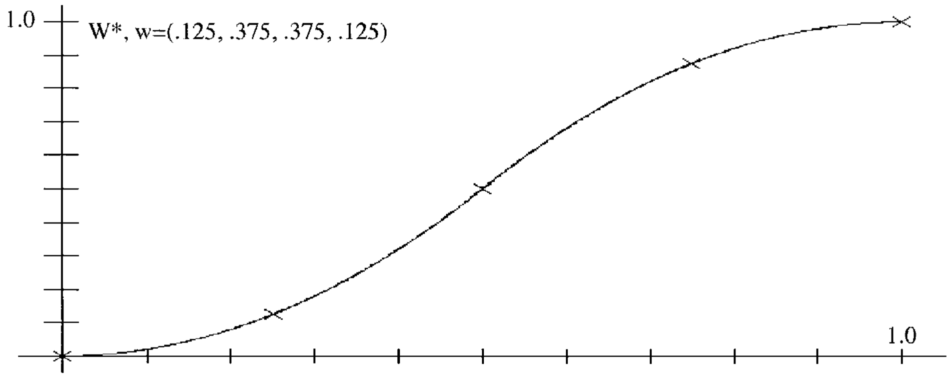

Figure 1. The interpolation function W*.

have been obtained using the second of the approaches described above. This is, we have defined two weighting functions W^* and W'^* [Figs. 1 and 2]. The first one, that corresponds to a weighting vector $w = (.125, .375, .375, .125)$ considers more important central values than extreme ones. The second one, $w' = (.375, .125, .125, .375)$, sets the importance to extreme values. These functions are defined as follows:

```
w*
     y = (x / .5) * (x / .5) * .5                      if x<0.5
     y = 1 - ((1 - x) / .5) * ((1 - x) / .5) * .5      if x≥0.5
w'*
     y = SQR(x / .5) * .5)                              if x<0.5
     y = 1 - SQR((1 - x) / .5) * .5                     if x≥0.5
```

In the tables we can see that the final weight of a certain value depends both the reliability attached to its corresponding information source and the

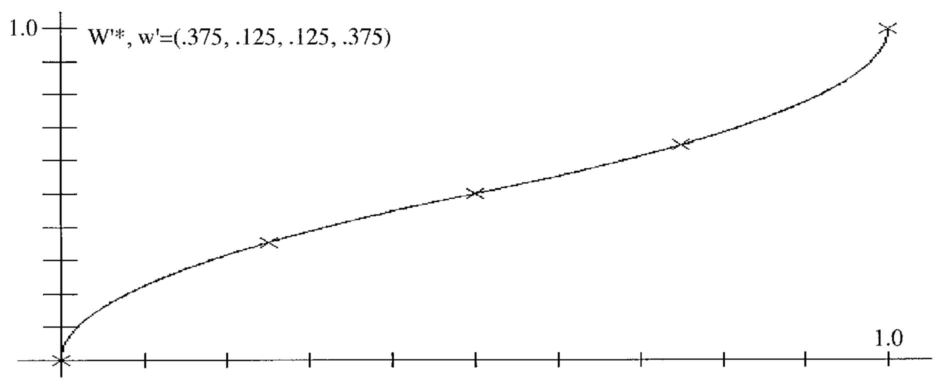

Figure 2. The interpolation function W′*.

importance of its value. See, for instance in Table I, that when we consider w^* and $\mathbf{p} = (.1\ .4\ .3\ .2)$, the weights ω_2 and ω_3 (corresponding to a_2 and a_3), that are already the greatest, are increased in relation to w_i because the function w^* gives more importance to central values. On the other hand, also in Table I, when $\mathbf{p} = (.4\ .1\ .2\ .3)$, due to the fact that in this case central values are the less reliable ones, their corresponding weights ω_i decrease. It can be observed that the results corresponding to w'^* are analogous (in this case, weights ω_i increase in extreme values and decrease in central ones).

In Table II we display the results that are obtained when the weighting function together with the input data is fixed. We also fix two of the weights p_i leaving the others change according to a parameter λ. It can be seen that when the weight p_i (that correspond to the highest value) is decreased, the result also decreases. Notice also that the weight ω_1 and ω_2 are not equal with p because they are influenced by the function W^*. For instance, when $p_1 = .4$, then $\omega_1 = .32$ while when $p_2 = .4$ then $\omega_2 = .48$. This is due to the fact that w^* gives more importance to central values and in the cases displayed in Table II, the first value is the maximum one, and therefore it should have less importance in the final result.

VI. EXTENSIONS OF THE WOWA OPERATOR

The OWA operator has been used as the basis of some other operators: the family of quasi-OWA operators[29] and the linguistic-OWA.[35] In this section we see that both extensions can be also applied to the WOWA operator. We define according to this the quasi-WOWA operators and the linguistic WOWA.

A1. The Quasi-WOWA

The quasi-arithmetic mean generalizes most of the averaging operators and it is defined as:[25]

$$f_m^{\varphi}(a_1, \ldots, a_n) = \varphi^{-1}\left(\sum_{i=1,n} \varphi(a_i)/n\right)$$

For instance, it generalizes the arithmetic mean when $\varphi(x) = Kx + K'$, the geometric mean ($\varphi(x) = K \ln x + K'$), and the harmonic mean ($\varphi(x) = Kx^{-1} + K'$). Its weighted counterpart allows us to weight the information sources:

Definition 4. *Let* $\mathbf{p}$ *be a weighting vector of dimension* n ($\mathbf{p} = [p_1, \ldots, p_n]$) *such that* $p_i \in [0, 1]$, $\Sigma\, p_i = 1$. *Let* φ *be any continuous strictly monotone function. In this case* f_{wm}^{φ} *is a* **quasi-arithmetic mean** *of dimension* n *if:*

$$f_{wm}^{\varphi}(a_1, \ldots, a_n) = \varphi^{-1}\left(\sum_{i=1,n} p_i \varphi(a_i)\right)$$

In a similar way, Ref. 29 defines the quasi-OWA and we define here this quasi-WOWA operator:

Table I. Examples with weighting function w^* and w'^*.

$\mathbf{p}$	ω	$f(.7, .6, .4, .3)$	$f(.9, .7, .5, .3)$	ω'	$f'(.7, .6, .4, .3)$	$f'(.9, .7, .5, .3)$
.25 .25 .25 .25	.13 .37 .37 .13	.5	.6	.35 .15 .15 .35	.5	.6
.4 .1 .2 .3	.32 .18 .32 .18	.514	.628	.45 .05 .11 .39	.51	.61
.1 .4 .3 .2	.02 .48 .42 .08	.494	.588	.22 .28 .18 .32	.49	.58

Table II. Examples with $\mathbf{p} = (\lambda, .5 - \lambda, .2\ .3)$ and weighting function w^*.

	λ=.5	λ = .4	λ = .3	λ = .2	λ = .1	λ = .0
$f(.7, .6, .4, .3)$	.532	.514	.5	.49	.484	.482
$f(.9, .7, .5, .3)$	.664	.628	.6	.58	.568	.564
ω	.5 .0 .32 .18	.32 .18 .32 .18	.18 .32 .32 .18	.08 .42 .32 .18	.02 .48 .32 .18	.0 .5 .32 .18

DEFINITION 5. *Let* **p** *and* **w** *be weighting vectors of dimension* n ($\mathbf{p} = [p_i \ldots p_n]$, $\mathbf{w} = [w_1 \ldots w_n]$) *such that* $p_i \in [0, 1]$ *and* $\Sigma_i p_i = 1$, $w_i \in [0, 1]$ *and* $\Sigma_i w_i = 1$. *Let* φ *be any continuous strictly monotone function.*

In this case, a mapping $f^{\varphi}_{wowa}: \mathbb{R}^n \to \mathbb{R}$ is a **quasi-WOWA operator** of dimension n if:

$$f^{\varphi}_{wowa}(a_1, \ldots, a_n) = \varphi^{-1}\left(\sum_{i=1,n} \omega_i \varphi(a_{\sigma(i)})\right)$$

where $\{\sigma(1), \ldots, \sigma(n)\}$ is a permutation of $\{1, \ldots, n\}$ such that $a_{\sigma(i-1)} \geq a_{\sigma(i)}$ for all $i = 2, \ldots, n$, and the weight ω_i is defined according to Definition 3.

B. The Linguistic-WOWA

Most of the systems in Artificial Intelligence deal with some kind of qualitative information. Often, when several information sources are used, there is a need for combination of this qualitative information. Among the synthesis methods[10,21] there is the linguistic OWA[35] based on the OWA operator and on the convex combination of linguistic labels defined by Delgado et al. in Ref. 36.

It is assumed that there is a set of n labels to aggregate ($A = \{a_1, \ldots, a_n\}$) that belong to a certain domain L. L is assumed to be a set of ordered linguistic labels $L = \{l_1, \ldots, l_m\}$, i.e., $l_i < l_{i+1}$ for all $i \in \{1, \ldots, m-1\}$.

DEFINITION 6.[19,35] *Let* **w** *be a weighted vector of dimension* n *such that* $w_i \in [0, 1]$ *and* $\Sigma_i w_i = 1$. *Let* L *be a set of ordered linguistic labels. In this case, a mapping* $f_{L\text{-}owa}: L^n \to L$ *is a linguistic-OWA (L-OWA) operator of dimension* n *if:*

$$\begin{aligned} f_{L\text{-}owa}(a_1, \ldots, a_n) &= \mathbb{C}^n\{w_i, a_{\sigma(i)}\} \\ &= w_1 \odot b_1 \oplus (1 - w_1) \odot \mathbb{C}^{n-1}\left\{w_h \Big/ \sum_{k=2,n} w_k, b_h, h = 2, \ldots, n\right\} \end{aligned}$$

where $\{\sigma(1), \ldots, \sigma(n)\}$ is a permutation of $\{1, \ldots, n\}$ according to Definition 3, and $\mathbb{C}^n$ is the convex combination operator[36] of n labels and if $n = 2$ it is defined as:

$$\mathbb{C}^2\{w_i, b_i, i = 1, .2\} = w_1 \odot l_j \oplus (1 - w_1) \odot l_i$$
$$\text{with } l_j = \max(b_1, b_2) \text{ and } l_i = \min(b_1, b_2)$$

and its results is

$$\mathbb{C}^2\{w_i, b_i, i = 1, .2\} = l_k$$

where $k = \min\{n, i + \text{round}\,(w_i * (j - i))\}$ where ''round'' is the usual round operation.

The same approach can be used to define the linguistic WOWA operator, as the convex combination can also be used. The difference is the calculation of the weights.

DEFINITION 7. *Let* **p** *and* **w** *be weighting vectors of dimension n such that* p_i, $w_i \in [0, 1]$ *and* $\Sigma_i\, p_i = \Sigma_i\, w_i = 1$. *Let L be a set of ordered linguistic labels. In this case, a mapping* $f_{L\text{-}wowa}$: $L^n \rightarrow L$ *is a* **linguistic-WOWA (L-WOWA) operator** *of dimension n if:*

$$f_{L\text{-}wowa}(a_1, \ldots, a_n) = \mathbb{C}^n\{\omega_i, a_{\sigma(i)}\}$$
$$= \omega_1 \odot b_1 \oplus (1 - \omega_1) \odot \mathbb{C}^{n-1}\left\{\omega_h \Big/ \sum_{k=2,n} \omega_k, b_h, h = 2, \ldots, n\right\}$$

where the permutation σ and the weighting vector ω is defined according to Definition 3 and the convex combination $\mathbb{C}$ is defined according to Definition 6.

VII. CONCLUSIONS

In this article we have analyzed some of the drawbacks of the weighted mean and the OWA operator. We have defined the WOWA operator that considers two weight vectors: **p** corresponding to the relevance of the sources, and **w** corresponding to the relevance of the values. We have seen that it solves the drawbacks pointed out. We have analyzed their properties and given an example of its application. We have also shown how the operator can be extended to linguistic labels.

Part of this work was completed at the Institut d'Investigació en Intel.ligència Artificial (IIIA, CSIC). The support of the European Community, under the contract VIM: CHRX-CT93-0401, is acknowledged.

References

1. S. Roberts, "On the indicator function of the plurity function," *Math. Soc. Sci.* **22,** 163–174 (1991).
2. C. Genest and J.V. Zidek, "Combining probability distributions: A critique and an annotated bibliography," *Statistical Science,* **1**(1), 114–118 (1986).
3. R.R. Yager, L.S. Goldstein, and E. Mendels, "FUZMAR: An approach to aggregating market research data based on fuzzy reasoning," *Fuzzy Sets Syst.* **68,** 1–11 (1994).
4. H. Raiffa, *Decision Analysis: Introduction Lectures on Choices Under Uncertainty,* Addison-Wesley, 1968.
5. W.H.E. Day and F.R. McMorris, "Critical comparison of consensus methods for molecular sequences," *Nucleic Acids Research,* **20**(5), 1093–1099 (1992).
6. C. Fourali, "Fuzzy logic and the quality of assessment of portfolios," *Fuzzy Sets Syst.* **68,** 123–139 (1994).
7. J. Amat, F. Esteva, and R. López de Mántaras, "Autonomous navigation troup for cooperative modelling of unknown environments," in *Proceedings of the Int. Conference on Advanced Robotics* (ICAR'95), Sant Feliu de Guixols, Catalunya, Spain, 1995, pp. 383–389.
8. B. López, S. Alvarez, P. Millán, D. Puig, D. Riaño, and V. Torra, "Multistage vision system for road lane markings and obstacle detection," Proceedings of *Euriscon '94,* Malaga, 1994, pp. 489–497.

9. R.R. Yager and D.P. Filev, *Essentials of Fuzzy Modeling and Control,* Wiley, New York, 1994.
10. V. Torra and U. Cortes, "Towards an automatic consensus generator tool: EGAC," *IEEE Trans. Syst. Man Cybern.,* **25**(5), 888–894 (1995).
11. R.A. Ribeiro, "Fuzzy multiple attribute decision making: A review and new preference elicitation techniques," *Fuzzy Sets Syst.,* **78**(2), 155–181 (1996).
12. V. Torra, "A new combination function in evidence theory," *Int. J Intell. Syst.,* **10**(12), 1021–1033 (1995).
13. D. Dubois and J-L. Koning, "Social choice axioms for fuzzy set aggregation," *Fuzzy Sets Syst.,* **43,** 257–274 (1991).
14. D. Dubois, and H. Prade, "Combination of information in the framework of possibility theory," in *Data Fusion in Robotics and Machine Intelligence,* (M.Al Abidi on R.C. Gonzalez, Eds., Academic Press, 1992, pp. 481–505.
15. V. Torra, "Combining fuzzy sets: The geometric consensus function family," *Fuzzy Sets Syst.,* **74,** 335–342 (1995).
16. J. Kacprzyk, "Group decision making with a fuzzy linguistic majority," *Fuzzy Sets Syst.,* **18,** 105–118 (1986).
17. J. Kacprzyk and M. Fedrizzi, *Multiperson Decision Making Models Using Fuzzy Sets and Possibility Theory,* Kluwer Academic Publishers, Netherlands, 1990.
18. F. Herrera, E. Herrera-Viedma, and J.L. Verdegay, "A sequential selection process in group decision making with a linguistic assessment approach," *Int. J. Inform. Sci.,* **80,** 1–17 (1995).
19. F. Herrera, E. Herrera-Viedma, and J.L. Verdegay, "A model of consensus in group decision making under linguistic assessments," *Fuzzy Sets Syst.,* **78,** 73–87 (1996).
20. J.P. Barthelemy, B. Leclerc, and B. Monjardet, "On the use of comparison and consensus of classifications," *J. Classification,* **3,** 187–224 (1986).
21. P.C. Fishburn and A. Rubistein, "Aggregation of equivalence relations," *J. Classification,* **3,** 61–65 (1986).
22. A. Trice and R. Davis, *Consensus Knowledge Acquisition,* MIT, AI Memo No 1183, December 1989.
23. J. Aczel and C. Alsina, "On synthesis of judgments," *Socio-Econom Plann. Sci.,* (1986).
24. J. Aczel, "On weighted synthesis of judgements," *Aequationes Math.,* **27,** 288–307 (1984).
25. J. Aczel, *A Short Course on Functional Equations,* D. Reidel, 1987.
26. R.R. Yager, "On ordered weighted averaging aggregation operators in multi-criteria decision making," *IEEE Trans. SMC,* **18,** 183–190 (1988).
27. R.R. Yager, "Families of OWA operators," *Fuzzy Sets Syst.,* **59,** 125–148 (1993).
28. J. Fodor, J.-L. Marichal, and M. Roubens, "Characterization of the ordered weighted averaging operators," *IEEE Trans. Fuzzy Syst.,* **3**(2), 236–240 (1995).
29. M. Grabisch, "Fuzzy integral in multicriteria decision making," *Fuzzy Sets Syst.,* **69,** 279–298 (1995).
30. P. Sobrevilla, D. Puig, and E. Montseny, "Segmentation of textured aerial images using OWA operators," *Proceedings of the IPMU '96*, pp. 1513–1518.
31. R.C. Luo and M.G. Kay, "Data fusion and sensor integration: State-of-the-art 1990s," in *Data Fusion in Robotics and Machine Intelligence,* M. Al Abidi and R.C. Gonzalez, Eds., Academic Press, 1992, pp. 7–135.
32. L.A. Zadeh, "A computational approach to fuzzy quantifiers in natural languages," *Computing Math. Appl.,* **9,** 149–184 (1983).
33. J.E. Chen and K.N. Otto, "Constructing membership functions using interpolation and measurement theory," *Fuzzy Sets Syst.,* **73**(3), 313–327 (1995).
34. R.R. Yager, "Quantifier guided aggregation using OWA operators," *Int. J. Intell. Syst.,* **11,** 49–73 (1996).
35. F. Herrera and J.L. Verdegay, "Linguistic assessments in group decision," *Proc. of First European Congress on Fuzzy and Intelligent Technologies,* Aachen, 1993, pp. 941–948.
36. M. Delgado, J.L. Verdegay, and M.A. Vila, "On aggregation operations of linguistic labels," *Int. J. Intell. Syst.,* **8,** 351–370 (1993).