Homework 5: Written Part
STAT 343: Mathematical Statistics

Details

How to Write Up

The written part of this assignment can be either typeset using latex or hand written.

Grading

5% of your grade on this assignment is for turning in something legible and organized

An additional 15% of your grade is for completion. A quick pass will be made to ensure that you've made a reasonable attempt at all problems.

Across both the written part and the R part, in the range of 1 to 3 problems will be graded more carefully for correctness. In grading these problems, an emphasis will be placed on full explanations of your thought process. You don't need to write more than a few sentences for any given problem, but you should write complete sentences! Understanding and explaining the reasons behind what you are doing is at least as important as solving the problems correctly.

Collaboration

You are allowed to work with others on this assignment, but you must complete and submit your own write up. You should not copy large blocks of code or written text from another student.

Sources

You may refer to our text, Wikipedia, and other online sources. All sources you refer to must be cited.

Problem I: Binomial Distribution

We have previously found that if $X \sim \text{Binomial}(n, \theta)$ then the maximum likelihood estimator of θ is $\hat{\theta}^{MLE} = \frac{X}{n}$. We showed that $E(\hat{\theta}^{MLE}) = \theta$, $Var(\hat{\theta}^{MLE}) = \frac{\theta(1-\theta)}{n}$, and $MSE(\hat{\theta}^{MLE}) = \frac{\theta(1-\theta)}{n}$.

(a) Is it possible for another estimator $\tilde{\Theta}$ to have lower variance than the MLE? If so, give an example of such an estimator. If not, explain why not with reference to a theorem from class.

(b) Is it possible for another *unbiased* estimator $\tilde{\Theta}$ to have lower variance than the MLE? If so, give an example of such an estimator. If not, explain why not with reference to a theorem from class.

Problem II: Exponential Distribution

Suppose $X_1, \ldots, X_n \sim \text{Exp}(\lambda)$, where the X_i are independent.

Here are some facts about the exponential distribution (please use this parameterization of the exponential distribution for this problem):

If $X \sim Exp(\lambda)$ then the density function is given by $f(x|\lambda) = \lambda e^{-\lambda x}$

The mean and variance are given by $E(X) = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$.

Based on a sample $x_1, \ldots, x_n$, the log-likelihood function is:

$$L(\lambda|x_1, \ldots, x_n) = n\log(\lambda) - \lambda \sum_{i=1}^{n} x_i$$

The first derivative of the log-likelihood function is:

$$\frac{d}{d\lambda} L(\lambda|x_1, \ldots, x_n) = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

The second derivative of the log-likelihood function is:

$$\frac{d^2}{d\lambda^2} L(\lambda|x_1, \ldots, x_n) = -\frac{n}{\lambda^2}$$

The maximum likelihood estimator can be shown to be: $\hat{\lambda}_{MLE} = 1/\bar{X}$.

Find a normal approximation to the sampling distribution of the maximum likelihood estimator. This will depend on the unknown parameter λ.

Problem III:

Suppose $X_1, ..., X_n \overset{i.i.d.}{\sim} Normal(\mu, \sigma^2)$ with μ and σ^2 unknown. Show that the 90% confidence interval for μ is

$$\left[\bar{X} - t_{n-1}(0.95) \frac{S}{\sqrt{n}}, \bar{X} + t_{n-1}(0.95) \frac{S}{\sqrt{n}} \right]$$

where $t_{n-1}(0.95)$ is the 95th percentile of a t_{n-1} distribution. You must explain every step for full credit.