Homework 0: Written Part

STAT 343: Mathematical Statistics

SOLUTIONS

Details

Due Date

This assignment is due on Wednesday, February 2.

How to Write Up

The written part of this assignment can be either typeset using LaTeX (either via R Markdown or otherwise) or hand written.

Grading

5% of your grade on this assignment is for turning in something legible and organized.

An additional 15% of your grade is for completion. A quick pass will be made to ensure that you've made a reasonable attempt at all problems.

In the range of 1 to 3 problems will be graded more carefully for correctness. In grading these problems, an emphasis will be placed on full explanations of your thought process. You don't need to write more than a few sentences for any given problem, but you should write complete sentences! Understanding and explaining the reasons behind what you are doing is at least as important as solving the problems correctly.

Solutions to all problems will be provided.

Collaboration

You are allowed to work with others on this assignment, but you must complete and submit your own write up. You should not copy large blocks of code or written text from another student.

Sources

You may refer to our text, Wikipedia, and other online sources. All sources you refer to must be cited in the space I have provided at the end of this problem set.

Problem 1:

If $f(x)$ and $g(x)$ are probability density functions, show that $h(x) = \alpha f(x) + (1 - \alpha)g(x)$ is a probability density function for $0 < \alpha < 1$.

Since $f(x)$ and $g(x)$ are probability density functions, they satisfy the following three properties:

(1) $f(x) \geq 0 \ \forall x$ (same for $g(x)$);
(2) f is piecewise continuous (same for g);
(3) $\int_{-\infty}^{\infty} f(x)dx = 1$ (same for $g(x)$).

We need to show that $h(x)$ also satisfies these three properties:

(1) WTS: $h(x) \geq 0 \; \forall x$

$$\begin{aligned} h(x) &= \underbrace{\alpha}_{>0} \underbrace{f(x)}_{\geq 0 \; \forall x} + \underbrace{(1-\alpha)}_{>0} \underbrace{g(x)}_{\geq 0 \; \forall x} \\ &\geq 0 \; \forall x \end{aligned}$$

(2) WTS: h is piecewise continuous

- $f(x)$ is piecewise continuous, and for $0 < \alpha < 1$, $\alpha f(x)$ is also piecewise continuous, since multiplying by a constant does not impact the continuity. The same argument can be made for $(1-\alpha)g(x)$, so both $\alpha f(x)$ and $(1-\alpha)g(x)$ are piecewise continuous.
- The addition of two piecewise continuous functions also does not impact continuity. Therefore, $h(x) = \alpha f(x) + (1-\alpha)g(x)$ is still piecewise continuous.

(3) WTS: $\int_{-\infty}^{\infty} h(x)dx = 1$

$$\begin{aligned} \int_{-\infty}^{\infty} h(x)dx &= \int_{-\infty}^{\infty} \left[\alpha f(x) + (1-\alpha)g(x)\right] dx \\ &= \alpha \underbrace{\int_{-\infty}^{\infty} f(x)dx}_{1} + (1-\alpha) \underbrace{\int_{-\infty}^{\infty} g(x)dx}_{1} \\ &= \alpha + 1 - \alpha \\ &= 1 \end{aligned}$$

Since $h(x)$ satisfies the three properties of a probability density function (pdf), we have shown it is a valid pdf.

Problem 2:

Suppose X_1 and X_2 are continuous random variables with

$$\begin{aligned} X_1 &\sim \text{Unif}(0,1) \\ X_2 | X_1 = x_1 &\sim \text{Unif}(0, X_1) \end{aligned}$$

(a) Find the pdf for the joint distribution of X_1 and X_2

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= f_{X_1}(x_1) f_{X_2|X_1=x_1}(x_2|x_1) \\ &= 1 \times \mathbf{1}_{\{0<x_1<1\}} \times \frac{1}{x_1} \times \mathbf{1}_{\{0<x_2<x_1\}} \\ &= \frac{1}{x_1} \times \mathbf{1}_{\{0<x_2<x_1<1\}} \end{aligned}$$

(b) Find the pdf for the marginal distribution of X_1

$$\begin{aligned}
f_{X_1}(x_1) &= \int_{-\infty}^{\infty} \frac{1}{x_1} \times \mathbf{1}_{\{0<x_2<x_1<1\}} dx_2 \\
&= \frac{1}{x_1} \int_{-\infty}^{\infty} \mathbf{1}_{\{0<x_2<x_1<1\}} dx_2 \\
&= \frac{1}{x_1} \mathbf{1}_{\{0<x_1<1\}} \int_{0}^{x_1} dx_2 \\
&= \frac{1}{x_1} \mathbf{1}_{\{0<x_1<1\}} (x_1 - 0) \\
&= 1 \times \mathbf{1}_{\{0<x_1<1\}}
\end{aligned}$$

In other words,

$$X_1 \sim \text{Unif}(0, 1)$$

(c) Find the pdf for the marginal distribution of X_2

$$\begin{aligned}
f_{X_2}(x_2) &= \int_{-\infty}^{\infty} \frac{1}{x_1} \times \mathbf{1}_{\{0<x_2<x_1<1\}} dx_1 \\
&= \int_{x_2}^{1} \frac{1}{x_1} dx_1 \\
&= \ln(x_1)|_{x_2}^{1} \times \mathbf{1}_{\{0<x_2<1\}} \\
&= [\ln(1) - \ln(x_2)] \times \mathbf{1}_{\{0<x_2<1\}} \\
&= -\ln(x_2) \times \mathbf{1}_{\{0<x_2<1\}}
\end{aligned}$$

(d) Find the pdf for the conditional distribution of $X_1|X_2 = x_2$

$$\begin{aligned}
f_{X_1|X_2}(x_1|x_2) &= \frac{f_{X_1,X_2}(x_1, x_2)}{f_{X_2}(x_2)} \\
&= \frac{\frac{1}{x_1} \times \mathbf{1}_{\{0<x_2<x_1<1\}}}{-\ln(x_2) \times \mathbf{1}_{\{0<x_2<1\}}} \\
&= -\frac{1}{x_1 \ln(x_1)} \times \mathbf{1}_{\{0<x_2<x_1<1\}}
\end{aligned}$$

(e) Write 1 or 2 sentences explaining how this problem relates to Bayes' Rule. This is a direct application of Bayes' Rule. We find the joint distribution in (a) using information about the marginal distribution for X_1 and the conditional distribution for $X_2|X_1$, and we find the marginal distribution for X_2 in (c) - we apply Bayes' Rule in (d) to find the conditional distribution for $X_1|X_2$.

Problem 3:

Let X be a continuous random variable with probability density function

$$f(x) = 2x, \qquad 0 \leq x \leq 1$$

(a) Find $E(X)$

$$\begin{aligned} E(X) &= \int_0^1 x f(x) dx \\ &= \int_0^1 2x^2 dx \\ &= \frac{2}{3} x^3 \Big|_0^1 \\ &= \frac{2}{3}\left(1^3 - 0^3\right) \\ &= \frac{2}{3} \end{aligned}$$

(b) Find $E(X^2)$

$$\begin{aligned} E(X^2) &= \int_0^1 x^2 f(x) dx \\ &= \int_0^1 2x^3 dx \\ &= \frac{2}{4} x^4 \Big|_0^1 \\ &= \frac{1}{2}(1^4 - 0^4) \\ &= \frac{1}{2} \end{aligned}$$

(c) Find $Var(X)$

$$\begin{aligned} Var(X) &= E(X^2) - E(X)^2 \\ &= \frac{1}{2} - \left(\frac{2}{3}\right)^2 \\ &= \frac{1}{2} - \frac{4}{9} \\ &= \frac{1}{18} \end{aligned}$$

Problem 4:

Suppose I have two choices for stocks to invest in. The return for the first stock is modelled as a random variable X_1 with $E(X_1) = \mu_1 = 1$ and $Var(X_1) = \sigma_1^2 = 0.1$; similarly, the return for the second stock is modelled as a random variable X_2 with $E(X_2) = \mu_2 = 0.8$ and $Var(X_2) = \sigma_2^2 = 0.12$. Additionally, suppose that the correlation between X_1 and X_2 as $Cor(X_1, X_2) = -0.8$.

Recall that if X_1 and X_2 are random variables, then

$$Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$$

Additionally, $Cor(X_1, X_2) = \frac{Cov(X_1, X_2)}{[Var(X_1)Var(X_2)]^{0.5}}$.

(a) If you could invest in only one of the stocks, which would you choose and why? The expected return is 1 for the first stock, which is larger than the expected return for the second stock. Also, the variability in the first stock is slightly lower than for the second stock. For these reasons, I would choose to invest in the first stock.

(b) Suppose you invest half of your money in the first stock and half of your money in the second stock. Define $Y = 0.5X_1 + 0.5X_2$ to be the return for this strategy. Find the expected return for your investment and the variance of the return.

- Expected Return:

$$\begin{aligned} E(Y) &= E(0.5X_1 + 0.5X_2) \\ &= E(0.5X_1) + E(0.5X_2) \text{ by linearity of expectations} \\ &= 0.5E(X_1) + 0.5E(X_2) \\ &= 0.5 \times 1 + 0.5 \times 0.8 \\ &= 0.9 \end{aligned}$$

- Variance of Return:

$$\begin{aligned} Var(Y) &= Var(0.5X_1 + 0.5X_2) \\ &= Var(0.5X_1) + Var(0.5X_2) + 2Cov(0.5X_1, 0.5X_2) \\ &= 0.25Var(X_1) + 0.25Var(X_2) + 2 \times 0.25Cov(X_1, X_2) \\ &= 0.25Var(X_1) + 0.25Var(X_2) + 2 \times 0.25 \times Cor(X_1, X_2) \times [Var(X_1)Var(X_2)]^{1/2} \\ &= (0.25 \times 0.1) + (0.25 \times 0.12) + 0.5 \times (-0.8) \times [0.1 \times 0.12]^{1/2} \\ &= 0.0112 \end{aligned}$$

```r
0.25*0.1+0.25*0.12+0.5*(-0.8)*(0.1*0.12)^(1/2)
```

```
## [1] 0.0111822
```

(c) If the proportion invested in the first stock is $\pi = 0.8$ and the proportion invested in the second stock is $(1 - \pi) = 0.2$, what is the expected return and the variance of the return?

- Expected Return:

$$\begin{aligned} E(Y) &= E(0.8X_1 + 0.2X_2) \\ &= E(0.8X_1) + E(0.2X_2) \text{ by linearity of expectations} \\ &= 0.8E(X_1) + 0.2E(X_2) \\ &= 0.8 \times 1 + 0.2 \times 0.8 \\ &= 0.96 \end{aligned}$$

- Variance in Return:

$$\begin{aligned} Var(Y) &= Var(0.8X_1 + 0.2X_2) \\ &= Var(0.8X_1) + Var(0.2X_2) + 2Cov(0.8X_1, 0.2X_2) \\ &= 0.64Var(X_1) + 0.04Var(X_2) + 2 \times 0.16Cov(X_1, X_2) \\ &= 0.64Var(X_1) + 0.04Var(X_2) + 2 \times 0.16 \times Cor(X_1, X_2) \times [Var(X_1)Var(X_2)]^{1/2} \\ &= (0.64 \times 0.1) + (0.04 \times 0.12) + 0.32 \times (-0.8) \times [0.1 \times 0.12]^{1/2} \\ &= 0.0408 \end{aligned}$$

```r
0.64*0.1+0.04*0.12+0.32*(-0.8)*(0.1*0.12)^(1/2)
```

```
## [1] 0.04075661
```

(d) Find an expression for the expected return as a function of the proportion invested in the first stock, π.

$$\begin{aligned} E(Y) &= E(\pi X_1 + (1 - \pi)X_2) \\ &= \pi E(X_1) + (1 - \pi)E(X_2) \\ &= \pi + (1 - \pi)(0.8) \\ &= 0.8 + 0.2\pi \end{aligned}$$

(e) Find an expression for the variance of the return as a function of the proportion invested in the first stock, π.

$$\begin{aligned}
Var(X) &= Var(\pi X_1 + (1-\pi)X_2) \\
&= \pi^2 Var(X_1) + (1-\pi)^2 Var(X_2) + 2 \times \pi \times (1-\pi) \times Cor(X_1, X_2) \times [Var(X_1)Var(X_2)]^{1/2} \\
&= \pi^2(0.1) + (1-\pi)^2(0.12) + 2\pi(1-\pi)(-0.8)\,[0.1 \times 0.12]^{1/2} \\
&= \pi^2(0.1) + (1-\pi)^2(0.12) - 0.16\pi(1-\pi)\,[0.1 \times 0.12]^{1/2}
\end{aligned}$$

(f) Repeat parts d and e if $Cor(X_1, X_2) = 0.1$ Part (d) is unaffected by a change in the correlation between X_1 and X_2.

For Part (e),

$$\begin{aligned}
Var(X) &= Var(\pi X_1 + (1-\pi)X_2) \\
&= \pi^2 Var(X_1) + (1-\pi)^2 Var(X_2) + 2 \times \pi \times (1-\pi) \times Cor(X_1, X_2) \times [Var(X_1)Var(X_2)]^{1/2} \\
&= \pi^2(0.1) + (1-\pi)^2(0.12) + 2\pi(1-\pi)(0.1)\,[0.1 \times 0.12]^{1/2} \\
&= \pi^2(0.1) + (1-\pi)^2(0.12) + 0.2\pi(1-\pi)\,[0.1 \times 0.12]^{1/2}
\end{aligned}$$