Example of fitted values and orthogonal projections

This example will be due as part of your next homework assignment. We're just getting a start on it in class together.

Model Statement

Suppose we use the model

$$\begin{aligned} y_i &= \beta + \varepsilon_i \\ \varepsilon_i &\sim \text{Normal}(0, \sigma^2) \end{aligned}$$

Also suppose we have $n = 2$ observations, and the observed response vector is $y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(a) What is the design matrix X?

$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) Find the hat matrix H.

$$\begin{aligned} H &= X(X'X)^{-1}X' \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{aligned}$$

(c) Find the fitted values $\hat{y} = Hy$.

$$\begin{aligned} \hat{y} &= Hy \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix} \end{aligned}$$

(d) Draw a figure showing $\mathcal{C}(X)$ (it is a line), y, and $\hat{y}$, clearly labelling each. Connect y and $\hat{y}$ with a line segment, and by drawing an appropriate right angle on your figure, illustrate that $\hat{y}$ is the orthogonal projection of y onto $\mathcal{C}(X)$.

The design matrix X has one column, $\begin{bmatrix}1\\1\end{bmatrix}$. $\mathcal{C}(X)$ is the line going through that point and the origin. y is the point $\begin{bmatrix}1\\2\end{bmatrix}$, and as found in part (c), $\hat{y}$ is the point $\begin{bmatrix}‘1.5\\1.5\end{bmatrix}$. $\hat{y}$ is the orthogonal projection of y into $\mathcal{C}(X)$, so the line connecting $\hat{y}$ and y meets $\mathcal{C}(X)$ at a right angle.