STAT 340: Linear Models in Matrix Form

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Example: Simple linear regression with n observations

Familiar way to express this model:

For $i = 1, ..., n$,

$$Y_i = \beta_0 + \beta_1 x_{i1} + \epsilon_i,$$

where $\epsilon_i \sim Normal(0,\ \sigma_\epsilon^2)$.

Matrix formulation:

Step 1: Set up system of equations

$$\begin{aligned} y_1 &= \beta_0 + x_{11}\beta_1 + \epsilon_1 \\ y_2 &= \beta_0 + x_{21}\beta_1 + \epsilon_2 \\ &\vdots \\ y_n &= \beta_0 + x_{n1}\beta_1 + \epsilon_n \end{aligned}$$

Step 2: Reorganize system of equations into matrix form

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_{11} \\ 1 & x_{21} \\ \vdots & \vdots \\ 1 & x_{n1} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix},$$

where $\begin{bmatrix} \epsilon_1 & \epsilon_2 & \cdots & \epsilon_n \end{bmatrix}' \sim Normal_n \left(\begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}', \sigma_\epsilon^2 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \right)$.

Step 3: Rewrite in compact matrix form

$$\underbrace{\mathbf{y}}_{(n\times 1)} = \begin{bmatrix} \mathbf{1} & \mathbf{x}_1' \end{bmatrix} \boldsymbol{\beta} = \underbrace{\mathbf{X}}_{n\times 2} \underbrace{\boldsymbol{\beta}}_{2\times 1} + \underbrace{\boldsymbol{\epsilon}}_{n\times 1},$$

where $\epsilon \sim Normal_n(\mathbf{0}_n, \sigma_\epsilon^2 \mathbf{I}_n)$.

Practice

Rewrite the following models in matrix notation. Please show all three steps as above for at least the first practice problem. If you feel sufficiently comfortable after that, you do not need to show all of your steps. Make sure to use clear and consistent notation, and be sure to specify the dimensions of your vectors and matrices (this helps you to keep track of your notation).

Practice 1

For $i = 1, ..., 5$,

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i,$$

where $\epsilon_i \sim Normal(0,\ \sigma_\epsilon^2)$.

Practice 2

Something with indicator variables – an ANOVA model

Practice 3

Recall in ANOVA, we often pose questions that lead