Development of an algorithm that uses piecewise cubic functions to approximate a function.

For a cubic function, we will need 4 points $ax^3 + bx^2 + cx + d$
Picking the 2 middle points requires calculating the points between a and b that are ⅓ and ⅔ of the way between the endpoints. For a given interval defined by [a,b], ⅓ of the width of the interval is $\frac{b-a}{3}$.

For m_1, we have $m_1 = a + \frac{b-a}{3} = \frac{3a+b-a}{3} = \frac{2a+b}{3}$

For m_2, we have $m_1 = a + 2 * \frac{b-a}{3} = \frac{3a+2b-2a}{3} = \frac{a+2b}{3}$

Next, we set $\int_a^b f(x)\,dx = A_1 f(a) + A_2 f(m_1) + A_3 f(m_2) + A_4 f(b)$

Now, we calculate the results of integrating some simple functions (1, x, x^2, x^3) from a to b.

$$\int_a^b 1\,dx = [x]_a^b = b - a = A_1 + A_2 + A_3 + A_4$$

$$\int_a^b x\,dx = [\frac{x^2}{2}]_a^b = \frac{b^2-a^2}{2} = aA_1 + \frac{2a+b}{3}A_2 + \frac{a+2b}{3}A_3 + bA_4$$

$$\int_a^b x^2\,dx = [\frac{x^3}{3}]_a^b = \frac{b^3-a^3}{3} = a^2A_1 + (\frac{2a+b}{3})^2A_2 + (\frac{a+2b}{3})^2A_3 + b^2A_4$$

$$\int_a^b x^3\,dx = [\frac{x^4}{4}]_a^b = \frac{b^4-a^4}{4} = a^3A_1 + (\frac{2a+b}{3})^3A_2 + (\frac{a+2b}{3})^3A_3 + b^3A_4$$

Converting these equations into an augmented matrix gives us:

$$\left|\begin{array}{ccccc} 1 & 1 & 1 & 1 & b-a \\ a & \frac{2a+b}{3} & \frac{a+2b}{3} & b & \frac{b^2-a^2}{2} \\ a^2 & (\frac{2a+b}{3})^2 & (\frac{a+2b}{3})^2 & b^2 & \frac{b^3-a^3}{3} \\ a^3 & (\frac{2a+b}{3})^3 & (\frac{a+2b}{3})^3 & b^3 & \frac{b^4-a^4}{3} \end{array}\right|$$

First, we subtract a*R1 from R2, a^2*R1 from R3, and a^3*R1 from R4 to get:

$$\left|\begin{array}{ccccc} 1 & 1 & 1 & 1 & b-a \\ 0 & \frac{b-a}{3} & \frac{2b-2a}{3} & b-a & \frac{(b-a)^2}{2} \\ 0 & \frac{(b-a)(5a+b)}{9} & \frac{(a+b)^2-9a^2}{9} & b^2-a^2 & \frac{(a-b)^3(2a+b)}{3} \\ 0 & \frac{(2a+b)^3-27a^3}{27} & \frac{(a+2b)^3-27a^3}{27} & b^3-a^3 & \frac{3a^4-4a^3b+b^4}{4} \end{array}\right|$$

Next, we simplify R2 by dividing it by $\frac{b-a}{3}$

$$\left|\begin{array}{ccccc} 1 & 1 & 1 & 1 & b-a \\ 0 & 1 & 2 & 3 & \frac{3b-3a}{2} \\ 0 & \frac{(b-a)(5a+b)}{9} & \frac{(a+b)^2-9a^2}{9} & b^2-a^2 & \frac{(a-b)^3(2a+b)}{3} \\ 0 & \frac{(2a+b)^3-27a^3}{27} & \frac{(a+2b)^3-27a^3}{27} & b^3-a^3 & \frac{3a^4-4a^3b+b^4}{4} \end{array}\right|$$

Next we subtract R2 from R1, $\frac{(b-a)(5a+b)}{9}$*R2 from R3, and $\frac{(2a+b)^3-27a^3}{27}$*R2 from R4

$$\left|\begin{array}{ccccc} 1 & 0 & -1 & -2 & \frac{a-b}{2} \\ 0 & 1 & 2 & 3 & \frac{3b-3a}{2} \\ 0 & 0 & \frac{2(a-b)^2}{9} & \frac{2(a-b)^2}{3} & \frac{(b-1)^3}{6} \\ 0 & 0 & \frac{2(a-b)^2(2a+b)}{9} & \frac{2(a-b)^2(5a+4b)}{9} & \frac{-(a-b)^3(11a+7b)}{36} \end{array}\right|$$

Next, we simplify by multiplying R3 by $\frac{9}{2(a-b)^2}$

$$\left|\begin{array}{ccccc} 1 & 0 & -1 & -2 & \frac{a-b}{2} \\ 0 & 1 & 2 & 3 & \frac{3b-3a}{2} \\ 0 & 0 & 1 & 3 & \frac{(3b-3a)}{4} \\ 0 & 0 & \frac{2(a-b)^2(2a+b)}{9} & \frac{2(a-b)^2(5a+4b)}{9} & \frac{-(a-b)^3(11a+7b)}{36} \end{array}\right|$$

Next, we add R3 to R1, and subtract 2*R3 from R2 and $\frac{2(a-b)^2(2a+b)}{9}$*R3 from R4

$$\left|\begin{array}{ccccc} 1 & 0 & 0 & 1 & \frac{b-a}{4} \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 3 & \frac{(3b-3a)}{4} \\ 0 & 0 & 0 & \frac{2(b-a)^3}{9} & \frac{(a-b)^4}{36} \end{array}\right|$$

Next, we simplify R4 by multiplying it by $\frac{9}{2(b-a)^3}$

$$\left|\begin{array}{ccccc} 1 & 0 & 0 & 1 & \frac{b-a}{4} \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 3 & \frac{3(b-a)}{4} \\ 0 & 0 & 0 & 1 & \frac{b-a}{8} \end{array}\right|$$

Finally, we subtract R4 from R1, add 3*R4 to R2 and subtract 3*R4 from R3

$$\left|\begin{array}{ccccc} 1 & 0 & 0 & 0 & \frac{b-a}{8} \\ 0 & 1 & 0 & 0 & \frac{3(b-a)}{8} \\ 0 & 0 & 1 & 0 & \frac{3(b-a)}{8} \\ 0 & 0 & 0 & 1 & \frac{b-a}{8} \end{array}\right|$$

So, $\int_a^b f(x)\,dx = \frac{b-a}{8} f(a) + \frac{3(b-a)}{8} f(m_1) + \frac{3(b-a)}{8} f(m_2) + \frac{b-a}{8} f(b)$

$$= \frac{b-a}{8}[f(a) + 3f(\frac{2a+b}{3}) + 3f(\frac{a+2b}{3}) + f(b)]$$

Example 1

<u>Exact</u>

$$\int_0^{11} x\,dx = [\frac{x^2}{2}]_0^{11} = [\frac{121}{2} - 0] = \frac{121}{2} = 60.5$$

<u>Approximation</u>

$$\int_0^{11} x\,dx = \frac{11-0}{8}[f(0) + 3f(\frac{2(0)+11}{3}) + 3f(\frac{0+2(11)}{3}) + f(11)]$$

$$= \frac{11}{8}[0 + 3f(\frac{11}{3}) + 3f(\frac{22}{3}) + 11]$$

$$= \frac{11}{8}[11 + 11 + 22] = 11/8(44) = \frac{121}{2} = 60.5$$

<u>Error</u>

Exact - Approximation = 60.5 - 60.5 = 0

Example 2

<u>Exact</u>

$$\int_{12}^{13} x^{-2}\,dx = [-\frac{1}{x}]_{12}^{13} = -\frac{1}{13} - (-\frac{1}{12}) = -\frac{12}{156} + \frac{13}{156} = \frac{1}{156} \approx .0064102564102564$$

<u>Approximation</u>

$$\int_{12}^{13} x^{-2}\,dx = \frac{13-12}{8}[f(12) + 3f(\frac{2(12)+13}{3}) + 3f(\frac{12+2(13)}{3}) + f(13)]$$

$$= \frac{1}{8}[\frac{1}{144} + 3f(\frac{37}{3}) + 3f(\frac{38}{3}) + \frac{1}{169}] = \frac{1}{8}[\frac{313}{24336} + \frac{27}{1369} + \frac{27}{1444}]$$

$$= \frac{1}{8}(\frac{616773301}{12027070224}) = \frac{616773301}{96216561792} \approx .0064102612846771$$

Error

Exact - Approximation = $\frac{1}{156} - \frac{616773301}{96216561792} = -\frac{469}{96216561792} \approx -4.87442 * 10^{-9}$

Example 3

Exact

$$\int_3^6 3x^2 + \frac{x}{7}\, dx = [x^3 + \frac{x^2}{14}]_3^6 = [(216 + \frac{18}{7}) - (27 + \frac{9}{14}) = \frac{1530}{7} - \frac{387}{14} = \frac{2673}{14}$$

≈190.9285714

Approximation

$$\int_3^6 3x^2 + \frac{x}{7}\, dx = \frac{6-3}{8}[f(3) + 3f(\frac{2(3)+6}{3}) + 3f(\frac{3+2(6)}{3}) + f(6)]$$

$$= \frac{3}{8}[\frac{192}{7} + 3f(4) + 3f(5) + \frac{762}{7}] = \frac{3}{8}[\frac{954}{7} + \frac{1020}{7} + \frac{1590}{7}]$$

$$= \frac{3}{8}(\frac{3564}{7}) = \frac{2673}{14} \approx 190.9285714$$

Error

Exact - Approximation = $\frac{2673}{14} - \frac{2673}{14} = 0$

Example 4

Exact

$$\int_{-2}^{5} x^3 - 5\, dx = [\frac{x^4}{4} - 5x]_{-2}^{5} = [(\frac{5^4}{4} - 25) - (\frac{-2^4}{4} + 10)] = \frac{525}{4} - 14 = \frac{469}{4} = 117.25$$

Approximation

$$\int_{-2}^{5} x^3 - 5\, dx = \frac{5-(-2)}{8}[f(-2) + 3f(\frac{2(-2)+5}{3}) + 3f(\frac{-2+2(5)}{3}) + f(5)]$$

$$= \frac{7}{8}[f(-2) + 3f(\frac{1}{3}) + 3f(\frac{8}{3}) + f(5)]$$

$$= \frac{7}{8}[-13 + 3(\frac{-134}{27}) + 3(\frac{377}{27}) + 120]$$

$$= \frac{7}{8}[107 + 27] = \frac{938}{8} = \frac{469}{4} = 117.25$$

Error

Exact - Approximation = 117.25 - 117.25 = 0

However, this algorithm is only exact for powers up to 3. It is not exact for powers higher than 3.

Example 5 (order of accuracy)

Exact (width = 4)

$$\int_1^5 x^4\,dx = [\frac{x^5}{5}]_1^5 = [\frac{3125}{5} - \frac{1}{5}] = \frac{3124}{5} = 624.8$$

Approximation

$$\int_1^5 x^4\,dx = \frac{5-(1)}{8}[f(1) + 3f(\frac{2(1)+5}{3}) + 3f(\frac{1+2(5)}{3}) + f(5)]$$

$$= \frac{1}{2}[1 + 3f(\frac{7}{3}) + 3f(\frac{11}{3}) + 625]$$

$$= \frac{1}{2}[626 + \frac{7^4}{3^3} + \frac{11^4}{3^3}]$$

$$= \frac{1}{2}[626 + \frac{17042}{27}]$$

$$= \frac{16972}{27} \approx 628.\overline{592}$$

Error

Exact - Approximation = $\frac{3124}{5} - \frac{16972}{27} \approx -\ 3.7\overline{925}$

Exact (width = 2)

$$\int_1^3 x^4\,dx = [\frac{x^5}{5}]_1^3 = [\frac{243}{5} - \frac{1}{5}] = \frac{242}{5} = 48.4$$

Approximation

$$\int_1^3 x^4\,dx = \frac{3-(1)}{8}[f(1) + 3f(\frac{2(1)+3}{3}) + 3f(\frac{1+2(3)}{3}) + f(3)]$$

$$= \frac{1}{4}[1 + 3f(\frac{5}{3}) + 3f(\frac{7}{3}) + 81]$$

$$= \frac{1}{4}[82 + \frac{5^4}{3^3} + \frac{7^4}{3^3}]$$

$$= \frac{1}{4}[82 + \frac{3026}{27}]$$

$$= \frac{1310}{27} \approx 48.\overline{518}$$

Error

Exact - Approximation = $\frac{242}{5} - \frac{1310}{27} = \frac{-16}{135} \approx -\ 0.118\overline{518}$

Exact (width = 1)

$$\int_1^2 x^4\,dx = [\frac{x^5}{5}]_1^2 = [\frac{32}{5} - \frac{1}{5}] = \frac{31}{5} = 6.2$$

Approximation

$$\int_1^2 x^4\,dx = \frac{2-(1)}{8}[f(1) + 3f(\frac{2(1)+2}{3}) + 3f(\frac{1+2(2)}{3}) + f(2)]$$

$$= \frac{1}{8}[1 + 3f(\frac{4}{3}) + 3f(\frac{5}{3}) + 16]$$

$$= \frac{1}{8}[17 + \frac{4^4}{3^3} + \frac{5^4}{3^3}]$$

$$= \frac{1}{8}[17 + \frac{881}{27}]$$

$$= \frac{335}{54} \approx 6.2\overline{037}$$

Error

Exact - Approximation = $\frac{31}{5} - \frac{335}{54} \approx -.00\overline{370}$

So for a 4th degree function, the width of the interval and absolute value of the error are as follows:

Width	|Error|	Remarks
4	3.7925925925	
2	0.1185185185	$\approx\frac{1}{32} *\ \ error\ of\ width\ 4$
1	0.0037037037	$\approx\frac{1}{32} *\ \ error\ of\ width\ 2$

As we halve the width of the interval, the error decreases by a factor of 32, so the order of accuracy is 5 (2^5=32).