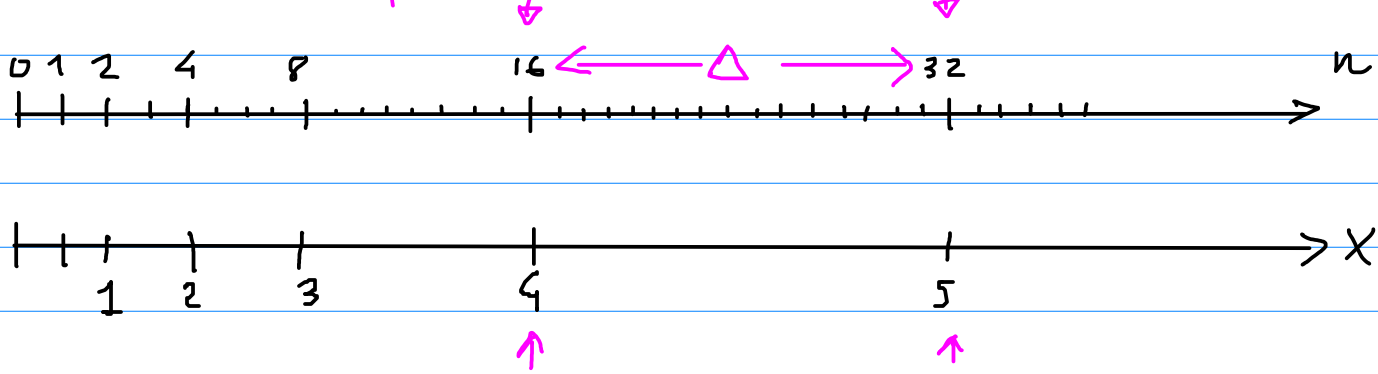
Problem: Count a large nb. of events n with a small counter register

Keep track of n in log scale $(X \simeq \lg n)$

As n moves from here to here



X moves from here to here

How to know n has moved Δ steps?

⚠ can't count because = original problem

⇨ increment X after Δ steps of n

$\approx$

increment X at each step of n w/ prob $1/\Delta$

But Also want "small" relative error:

$$\mathbb{P}\left[\,|\tilde{n} - n| > \varepsilon n\,\right] < \delta$$

Does this idea ensure that?

Algorithm Morris

Input: n nb. of events

Output: $\tilde{n}$ approx. count

begin

$X \leftarrow 0$

for $i \leftarrow 1, \dots, n$ do

$$X \leftarrow \begin{cases} X+1 & \text{w. prob. } 1/2^X \\ X & \text{w. prob } 1 - 1/2^X \end{cases}$$

return $2^X - 1$

end

Impl. convenience: have $X \approx \lg(n+1)$ s.t. $X \geq 0$

Thm 1 Alg. Morris is correct on average, i.e.

$$\mathbb{E}\left[\tilde{n} := 2^X - 1\right] = n \quad \text{for all } n.$$

Proof $\mathbb{E}[2^X - 1] = n \iff \mathbb{E}[2^X] = n+1$

by induction on n

BASE) $n = 0 \Rightarrow \tilde{n} = 0$ always ✓

INDUCTION) Let X_k be the r.v. corresponding to the value of the register X at the end of the k^{th} iteration of Morris

$$\mathbb{E}\left[2^{X_{n+1}}\right] = \sum_{j=0}^{\infty} \mathbb{P}(X_n = j)\, \mathbb{E}\left[2^{X_{n+1}} \mid X_n = j\right] \quad \text{Part. Thm}$$

$$= \sum_j \mathbb{P}[X_n = j] \left[2^{\cancel{j}+1} \cdot \frac{1}{\cancel{2^j}} + 2^j\left(1 - \frac{1}{2^j}\right)\right]$$

$$= \sum_j \mathbb{P}[X_n = j]\left[2 + 2^j - 1\right]$$

$$= \sum_j \mathbb{P}[X_n = j]\,(2^j + 1) \quad \text{Def } \mathbb{E}[2^{X_n}]$$

$$= \mathbb{E}\left[2^{X_n} + 1\right] \quad \text{linearity } \mathbb{E}$$

$$= \mathbb{E}\left[2^{X_n}\right] + 1 \quad \text{i.h.}$$

$$= (n+1) + 1 \quad \blacksquare$$

Thm 2 Let $\tilde{n} := 2^{X_n} - 1$ be the output of Morris Alg. on input n. Then, for all n,

$$\mathbb{P}[\,|\tilde{n} - n| > \varepsilon n\,] < \delta$$

where $\delta = 1/2\varepsilon^2$.

Proof Recall Chebyshev ineq:

$$\mathbb{P}[\,|X - \mathbb{E}[X]| > \lambda\,] < \mathbb{E}[(X - \mathbb{E}[X])^2] / \lambda^2$$

$$\mathbb{P}[\,|2^{X_n} - 1 - \mathbb{E}[2^{X_n} - 1]| > \varepsilon n\,] < \mathbb{E}[(2^{X_n} - 1 - \mathbb{E}(2^{X_n} - 1))^2] / \varepsilon^2 n^2$$

$$\therefore \mathbb{P}[\,|2^{X_n} \cancel{-1} - \mathbb{E}[2^{X_n}] \cancel{+1}| > \varepsilon n\,] < \mathbb{E}[(2^{X_n} \cancel{-1} - \mathbb{E}[2^{X_n}] \cancel{+1})^2] / \varepsilon^2 n^2$$

$$\therefore \mathbb{P}[\,|2^{X_n} - \mathbb{E}[2^{X_n}]| > \varepsilon n\,] < \mathbb{E}[(2^{X_n} - \mathbb{E}(2^{X_n}))^2] / \varepsilon^2 n^2$$

$$= \mathbb{E}[2^{2X_n} - 2 \cdot 2^{X_n} \cdot (n+1) + (n+1)^2] / \varepsilon^2 n^2$$

$$= \left[\mathbb{E}[2^{2X_n}] - 2(n+1)\,\mathbb{E}[2^{X_n}] + (n+1)^2\right] / \varepsilon^2 n^2$$

$$= \left[\mathbb{E}[2^{2X_n}] - (n+1)^2\right] / \varepsilon^2 n^2.$$

Claim $\mathbb{E}[2^{2X_n}] = 3n^2/2 + 3n/2 + 1$

$$= \left[\frac{3n^2}{2} + \frac{3}{2}n \cancel{+1} - n^2 - 2n \cancel{-1}\right] / \varepsilon^2 n^2$$

$$= \left[\frac{n^2}{2} - \frac{n}{2}\right] / \varepsilon^2 n^2 = \frac{1}{2\varepsilon^2} - \frac{1}{2n\varepsilon^2}$$

$$\therefore \mathbb{P}[\,|\tilde{n} - n| > \varepsilon n\,] < \frac{1}{2\varepsilon^2}$$

Proof of Claim induction on n.

Base $(n=0)$: $\mathbb{E}[2^{2X_n}] = \mathbb{E}[2^{2\cdot 0}] = 1$ ✓

Step: use i.h. $\mathbb{E}[2^{2X_n}] = \sum \mathbb{P}[2^{X_n}=j]\cdot j^2 = 3n^2/2 + 3n/2 + 1$

$$\mathbb{E}[2^{2X_{n+1}}] = \sum_j \mathbb{P}[2^{X_n}=j]\ \mathbb{E}[2^{2X_{n+1}} \mid 2^{X_n}=j] \quad \text{Part. Thm}$$

Morris: $X_{n+1} = \begin{cases} X_n+1, & \text{w/p } 1/2^{X_n} \\ X_n, & \text{w/p } 1-1/2^{X_n} \end{cases}$ + Def $\mathbb{E}$

$$= \sum_j \mathbb{P}[2^{X_n}=j]\left[\frac{1}{j}\left(2^{X_n+1}\right)^2 + \left(1-\frac{1}{j}\right)\left(2^{X_n}\right)^2\right]$$

$$= \sum_j \mathbb{P}[2^{X_n}=j]\left[\frac{1}{j}(2j)^2 + \left(1-\frac{1}{j}\right)j^2\right]$$

$$= \sum_j \mathbb{P}[2^{X_n}=j]\left[4j + j^2 - j\right]$$

$$= \sum_j \mathbb{P}[2^{X_n}=j]\, j^2 + 3\sum_j \mathbb{P}[2^{X_n}=j]\cdot j \qquad \text{i.h.}$$

$$= \frac{3n^2}{2} + \frac{3n}{2} + 1 + 3\,\mathbb{E}[2^{X_n}]$$

$$= \frac{3n^2}{2} + \frac{3n}{2} + 1 + 3(n+1)$$

$$= \frac{3n^2 + 3n + 2 + 6n + 6}{2}$$

$$= \frac{3n^2 + 6n + 3 + 3n + 3 + 2}{2}$$

$$= \frac{3(n+1)^2 + 3(n+1) + 2}{2} = \frac{3}{2}(n+1)^2 + \frac{3}{2}(n+1) + 1 \quad \blacksquare$$

⚠ - Let's call ε-risk of an algorithm the probability of not giving an estimate with relative error of at most a fixed ε : $\mathbb{P}(\text{not error} \leq \varepsilon) = \mathbb{P}(\text{error} > \varepsilon)$.

- Thm 2 shows that the ε-risk of Morris is $> \frac{1}{2\varepsilon^2} =: \delta$.

- Conversely, the ε-accuracy is the prob. of giving an estimate within error ε.

- For ex. By Thm 2, a result with relative error > 1 occurs with prob < 0.5, or the risk for $\varepsilon = 1$ is < 0.5.
For $\sim 70\%$ error ($\varepsilon = 1/\sqrt{2}$) or lower, there is basically no guarantees since the risk upper bound is $\delta = 1$.

Run s independent copies and average results to lower the risk, or equivalently to improve the average accuracy.

Algorithm Morris+
Input: n, s
Output: Approximate count
begin
    $X \leftarrow 0$
    for $i \leftarrow 0, \dots, s-1$ do
        $X \leftarrow X + \text{Morris}(n)$
    return X/s
end

Thm 3 Let $\tilde{n}_s := \text{Morris}^+(n,s)$ be the output of Morris+ on input (n,s). Then, for all (n,s),

$$\mathbb{P}[\,|\tilde{n}_s - n| > \varepsilon n\,] < \delta$$

for $\delta = 1/2s\varepsilon^2$.

Proof $\tilde{n}_s$ is an avg. of s i.i.d. r.v $\sim \text{Morris}(n)$

$$\tilde{n}_s = \frac{1}{s}\sum_{i=1}^{s} \tilde{n}.$$

Hence $\mathbb{E}[\tilde{n}_s] = \frac{1}{\cancel{s}}\cancel{s}\cdot\mathbb{E}[\tilde{n}] = \mathbb{E}[\tilde{n}] = n$

and $\text{Var}[\tilde{n}_s] = \frac{1}{s^2}\sum_i \text{Var}(\tilde{n}) = \frac{\text{Var}(\tilde{n})}{s}$

By Chebyshev Ineq.

$$\mathbb{P}[\,|\tilde{n}_s - \mathbb{E}[\tilde{n}_s]\,| > \varepsilon n] =$$

$$\mathbb{P}[\,|\tilde{n}_s - n| > \varepsilon n] < \frac{\text{Var}(\tilde{n}_s)}{\varepsilon^2 n^2}$$

$$= \frac{1}{s}\cdot\frac{\text{Var}(\tilde{n})}{\varepsilon^2 n^2}$$ as in Proof Thm 2

$$< \frac{1}{s}\cdot\frac{1}{2\varepsilon^2} = \frac{1}{2s\varepsilon^2} \qquad \blacksquare$$

⚠ Thm 3 says that we can <u>Linearly</u> decrease the ε-risk for fixed ε by running multiple copies of Morris.

OTOH for a fixed risk threshold δ, the error rate decreases with the square root of the # copies