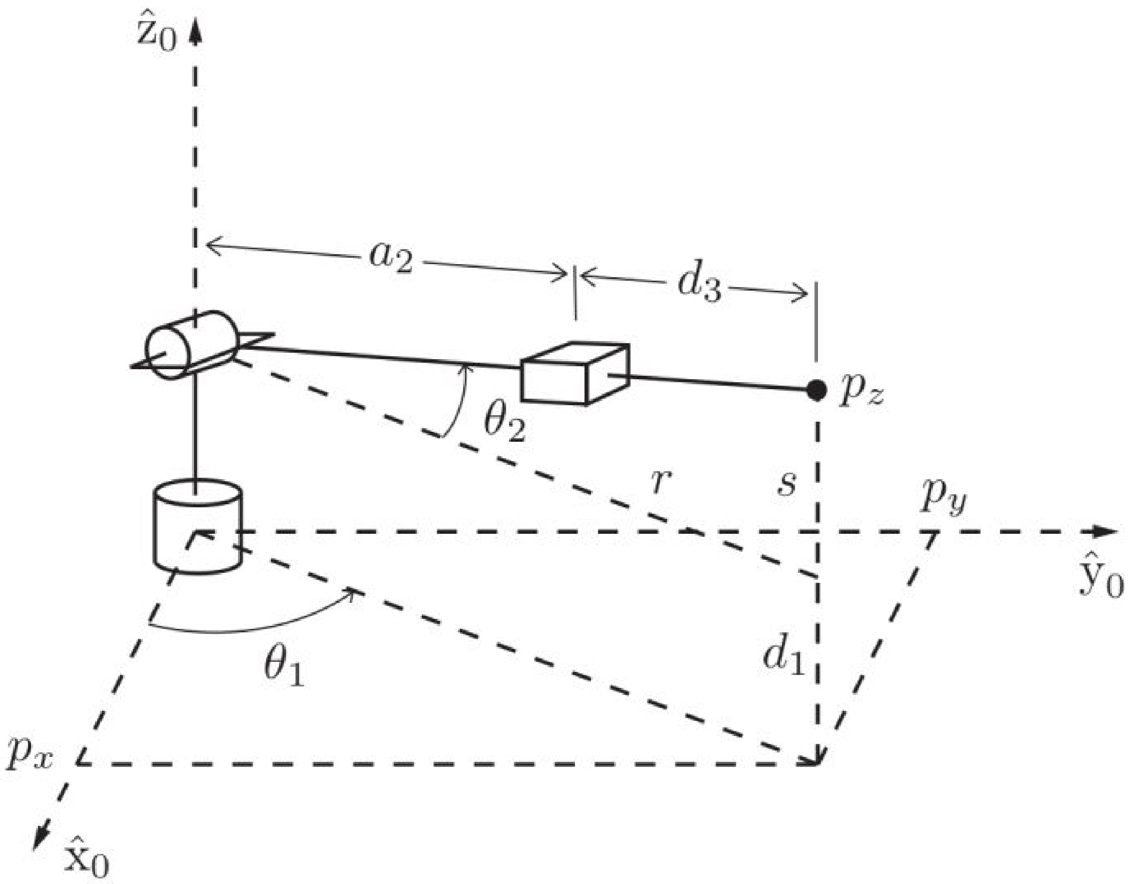
Homework Assignment 3

GitHub link: https://github.com/mhd-medfa/RRP_Robot_Study



Tasks:

1. Derive FK equations for the robot depicted in Fig. 1. Use θ1, θ2, d3 as joint space variables, px, py, pz as operational space variables. Parameters d1, a2 are known (assign them some positive values for succeeding tasks).

px = r.cos(θ1) …(1)
py = r.sin(θ1) …(2)

Where:

r = (a2+d3).cos(θ2) …(3)

pz = d1 + s …(4)

Where:

s = (a2 + d3). sin(θ2) …(5)

As a result the position of the end-effector can be described by the vector:
$O = [px\ py\ pz]^T$ **…(6)**

Where:
px = r.cos(θ1) = (a2+d3).cos(θ1).cos(θ2)
py = r.sin(θ1) = (a2+d3).sin(θ1).cos(θ2) ....(7)
pz = d1 + s = d1 + (a2 + d3). sin(θ2)

$$\begin{bmatrix} px \\ py \\ pz \end{bmatrix} = \begin{bmatrix} (a2 + d3).\ cos(\theta_1).\ cos(\theta_2) \\ (a2 + d3).\ sin(\theta_1).\ cos(\theta_2) \\ d1 + (a2 + d3).\ sin(\theta_2) \end{bmatrix}$$

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2. Derive IK equations (closed form solution). How many IK solutions does this robot have? Show all of them.

From (1) & (2) we can find two solutions for θ1:

θ1 = atan2(py, px) …(8)
or
θ1 = π + atan2(py, px) …(9)

Now to determine θ2 we have:

$r = \sqrt{px^2 + py^2}$ …(10)
s = pz - d1 …(11)

Thus:

In case of (8) ⇒

$2 = \frac{\pi}{2} + atan2(s,\ r)$ …(12)

Or

In case of (9) ⇒

$2 = \frac{3\pi}{2} - atan2(s,\ r)$ …(13)

Finally, in order to find d3 we have:

In case of d3 is positive:

$a2 + d3 = \sqrt{r^2 + s^2}$

$\Rightarrow d3 = \sqrt{r^2 + s^2} - a2$ …(14)

Or

In case of d3 is negative:
$a2 - d3 = \sqrt{r^2 + s^2}$
$\Rightarrow d3 = a2 - \sqrt{r^2 + s^2}$ …(15)

As a result, we find that we have four different solutions:

$$q = \begin{bmatrix} atan2(py,\ px) \\ \frac{\pi}{2} + atan2(s,\ r) \\ \sqrt{r^2 + s^2} - a2 \end{bmatrix} = \begin{bmatrix} atan2(py,\ px) \\ \frac{\pi}{2} + atan2\left(p2 - d1,\ \sqrt{px^2 + py^2}\right) \\ \sqrt{px^2 + py^2 + (pz - d1)^2} - a2 \end{bmatrix}$$

Or

$$q = \begin{bmatrix} \pi + atan2(py,\ px) \\ \frac{3\pi}{2} - atan2(s,\ r) \\ \sqrt{r^2 + s^2} - a2 \end{bmatrix} = \begin{bmatrix} \pi + atan2(py,\ px) \\ \frac{3\pi}{2} - atan2\left(p2 - d1,\ \sqrt{px^2 + py^2}\right) \\ \sqrt{px^2 + py^2 + (pz - d1)^2} - a2 \end{bmatrix}$$

Or

$$q = \begin{bmatrix} \pi + atan2(py,\ px) \\ \frac{\pi}{2} - atan2(s,\ r) \\ a2 - \sqrt{r^2 + s^2} \end{bmatrix} = \begin{bmatrix} \pi + atan2(py,\ px) \\ \frac{\pi}{2} - atan2\left(p2 - d1,\ \sqrt{px^2 + py^2}\right) \\ a2 - \sqrt{px^2 + py^2 + (pz - d1)^2} \end{bmatrix}$$

Or

$$q = \begin{bmatrix} atan2(py,\ px) \\ \frac{3\pi}{2} + atan2(s,\ r) \\ a2 - \sqrt{r^2 + s^2} \end{bmatrix} = \begin{bmatrix} atan2(py,\ px) \\ \frac{3\pi}{2} + atan2\left(p2 - d1,\ \sqrt{px^2 + py^2}\right) \\ a2 - \sqrt{px^2 + py^2 + (pz - d1)^2} \end{bmatrix}$$

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3. Compute the manipulator Jacobian for representation of linear and angular velocity of point p.
• Use classical approach (partial derivatives).
• Use geometric approach (cross products).
• Use numerical derivatives approach.

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4. Analyze the Jacobian for singularities. Characterize each singular configuration if any.

Singularities are those manipulator configurations where it loses one or more degrees-of-freedom, i.e., it can not move further in specific direction(s). Physically, it means a manipulator would be in a singular configuration if its end-effector cannot move in one or more directions. In other words, infinite joint velocities are required to move the end-effector in that particular direction(s).

Mathematically, singularities relate to the manipulator configuration(s) for which its Jacobian matrix becomes singular, i.e.,

det J = 0

We need to take the first three rows in the Jacobian which represents the linear part and calculate the determinant:

det Jv = $-1.0(a_2 + d_3)^2 \cos(\theta_2)$

It means that we have a singular case when:

1. $\theta 2 = \pm \frac{\pi}{2}$

In this case, either px and py will be equal to zero. And this means we lost one DoF because the end-effector intersects with the base frame's Z-axis. Thus any rotation around the base frame won't affect the position of the end-effector.

2. d3 = - a2

   In this case and depending on the equation (14):

   $$d3 = \sqrt{r^2 + s^2} - a2$$

   So d3 will be equal to (-a2) in case of both r and s are equal to zero.

   From (10) we have:

   $$r = \sqrt{px^2 + py^2}$$

   So r will be equal to zero if and only if px and py are both equal to zero which is the same as the first case.

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5. Compute the velocity of the tool frame when joint variables are changing with time as follows:
θ1(t) = sin(t); θ2(t) = cos(2t); d3(t) = sin(3t):
Add some fancy graphs showing evolution of all variables.

Formally, a Jacobian is a set of partial differential equations:

$$\mathbf{J} = \frac{\partial \mathbf{x}}{\partial \mathbf{q}}$$

With a bit of manipulation we can get a neat result:

$$\mathbf{J} = \frac{\partial \mathbf{x}}{\partial t} \frac{\partial t}{\partial \mathbf{q}} \rightarrow \frac{\partial \mathbf{x}}{\partial t} = \mathbf{J} \frac{\partial \mathbf{q}}{\partial t}$$

Or

$$\dot{\mathbf{x}} = \mathbf{J}\, \dot{\mathbf{q}}$$

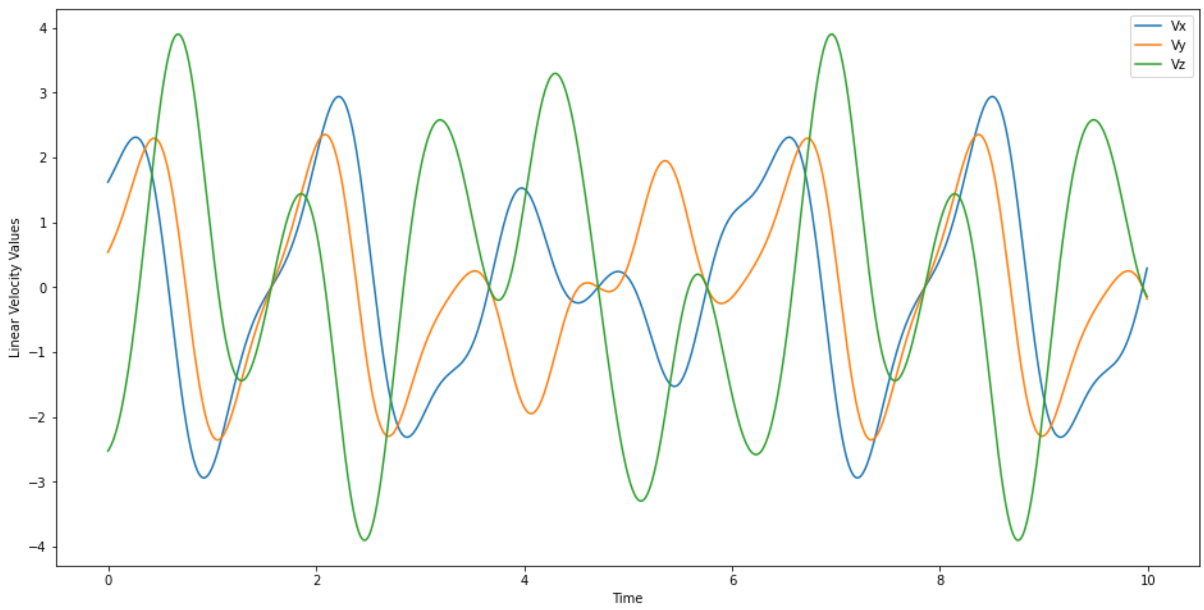
This tells us that the end-effector velocity is equal to the Jacobian J multiplied by the joint angle velocity.

We can substitute variables into the Jacobian multiplied by joint velocities (*) in order to get the velocities of the tool frame. Joint velocities are, respectively:

$$\begin{bmatrix} \cos(t) \\ -2\sin(2t) \\ 3\cos(3t) \end{bmatrix}$$

The following two plots are representing the tool frame velocities where:

1-The first figure shows the linear velocities of the tool frame



2-And the second figure represents the angular velocities