PEU 416 Assignment 4

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1 Problem 1

$$\sin\theta \approx \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120}$$

$$\theta - \sin\theta \approx \frac{\theta^3}{6} - \frac{\theta^5}{120}$$

$$9(\theta - \sin\theta)^2 \approx \frac{\theta^6}{4} - \frac{\theta^8}{40}$$

$$2(1 - \cos\theta)^3 \approx \frac{\theta^6}{4} - \frac{\theta^8}{16}$$

$$1 + \delta = \frac{\frac{\theta^6}{6} - \frac{\theta^8}{120}}{\frac{\theta^6}{4} - \frac{\theta^8}{16}}$$

For small θ

$$1 + \delta \approx \left(1 - \frac{\theta^2}{10}\right)\left(1 + \frac{\theta^2}{4}\right) \approx 1 + \frac{3}{20}\theta^2$$

$$\delta \approx \frac{3}{20}\theta^2$$

$$t = B(\theta - \sin\theta) \approx B\frac{\theta^3}{6}$$

$$\theta \approx \left(6\frac{t}{B}\right)^{\frac{1}{3}}$$

$$t_{i,\ \mathrm{max}} = B(\pi - \sin\pi) = B\pi$$

$$t \approx B\frac{\theta^3}{6}$$

$$\theta^2 = (6\pi)^{\frac{2}{3}}\left(\frac{t_i}{t_{i,\ \mathrm{max}}}\right)^{\frac{2}{3}}$$

$$\delta_i = \frac{3}{20}(6\pi)^{\frac{2}{3}}\left(\frac{t_i}{t_{i,\ \mathrm{max}}}\right)^{\frac{2}{3}}$$

2 Problem 2

(a)
$$\sigma^2(M) = \frac{3}{R^3}\int_0^R \xi(r) r^2 dr$$

$$\langle M^2\rangle = \langle M\rangle^2 + \frac{\langle M\rangle^2}{V^2}\int_V\int_V \xi\Big(\big|(x)_1 - (x)_2\big|\Big) d^3(x)_1 d^3(x)_2$$

$$\sigma^2(M) = \frac{\langle M\rangle^2}{V^2}\int_V\int_V \xi\Big(\big|(x)_1 - (x)_2\big|\Big) d^3(x)_1 d^3(x)_2$$

$$\sigma_R^2 = \frac{\sigma^2(M)}{\langle M\rangle^2} = \frac{1}{V^2}\int_V\int_V \xi\Big(\big|(x)_1 - (x)_2\big|\Big) d^3(x)_1 d^3(x)_2$$

$$\frac{1}{V^2}\int_V\int_V \xi\Big(\big|(x)_1 - (x)_2\big|\Big) d^3(x)_1 d^3(x)_2 \approx \frac{1}{V}\int_V \xi(|(x)|) d^3(x)$$

$$\sigma_R^2 \approx \frac{1}{(\frac{4}{3})\pi R^3}\int_0^R \xi(r)(4\pi r^2 dr)$$

$$\sigma_R^2 = \frac{3}{R^3}\int_0^R \xi(r) r^2 dr$$

(b)
$$\xi(r) = \left(\frac{r}{r_0}\right)^\gamma$$

with $r_0 = 5h^{-1}$ Mpc and $\gamma = -1.8$.

$$\sigma_R^2 = \frac{3}{R^3}\int_0^R \left(\frac{r}{r_0}\right)^\gamma r^2 dr = \frac{3}{R^3 r_0^\gamma}\int_0^R r^{\gamma+2} dr$$

$$\sigma_R^2 = \frac{3}{R^3 r_0^\gamma}\left[\frac{r^{\gamma+3}}{\gamma+3}\right]_0^R = \frac{3}{\gamma+3}\left(\frac{R}{r_0}\right)^\gamma$$

$$\sigma_8^2 = \frac{3}{-1.8+3}\left(\frac{8h^{-1}\text{ Mpc}}{5h^{-1}\text{ Mpc}}\right)^{-1.8}$$

$$\sigma_8^2 = \frac{3}{1.2}\left(\frac{8}{5}\right)^{-1.8} = 2.5\times(1.6)^{-1.8}$$

$$\sigma_8^2 \approx 2.5\times 0.4295 \approx 1.0738$$

$$\sigma_8 = \sqrt{1.0738} \approx 1.036$$

The implied value is $\boldsymbol{\sigma_8 \approx 1.04}$.

3 Problem 3

(a)
$$\sigma^2(R) = \int_0^\infty \frac{k^2 dk}{2\pi^2} P(k) \left| W.\widetilde{\text{kR}} \right|^2$$

$$W.\widetilde{\text{kR}} = \begin{cases} 1 \text{ if kR} \le 1 \\ 0 \text{ if kR} > 1 \end{cases}$$

This sets the integral's upper limit to $k = \frac{1}{R}$. With $P(k) = Ak^n$:

$$\sigma^2(R) = \frac{A}{2\pi^2} \int_0^{\frac{1}{R}} k^{n+2} dk = \frac{A}{2\pi^2(n+3)} \left(\frac{1}{R}\right)^{n+3}$$

Thus, $\sigma^2(R) \propto R^{-(n+3)}$. Since mass $M \propto R^3$, we have $R \propto M^{\frac{1}{3}}$. Substituting this:

$$\sigma^2(M) \propto \left(M^{\frac{1}{3}}\right)^{-(n+3)} = M^{-\frac{n+3}{3}}$$

Comparing this to $\sigma^2(M) \propto M^\gamma$, we find the relation:

$$\gamma = -\frac{n+3}{3}$$

(b) For a **Gaussian filter**, the window function is $W.\widetilde{\text{kR}} = e^{-\frac{(\text{kR})^2}{2}}$, so $\left| W.\widetilde{\text{kR}} \right|^2 = e^{-(\text{kR})^2}$. The variance integral becomes:

$$\sigma^2(R) = \frac{A}{2\pi^2} \int_0^\infty k^{n+2} e^{-(\text{kR})^2} dk$$

We use the substitution $t = (\text{kR})^2$:

$$\sigma^2(R) = \frac{A}{4\pi^2 R^{n+3}} \int_0^\infty t^{\frac{n+1}{2}} e^{-t} dt$$

The integral is the definition of the Gamma function, $\Gamma(z)$, with $z = \frac{n+3}{2}$.

$$\sigma^2(R) = \frac{A\Gamma\left(\frac{n+3}{2}\right)}{4\pi^2 R^{n+3}}$$

The variance has the same dependence on scale, $\sigma^2(R) \propto R^{-(n+3)}$. This leads to the **same relationship** between γ and n as in part a:

$$\gamma = -\frac{n+3}{3}$$

(c) We compute the ratio $\frac{\sigma_G^2}{\sigma_S^2}$ for $n = 1$.

Sharp k-space filter (σ_S^2):

$$\sigma_S^2(R) = \frac{A}{2\pi^2(1+3)}\left(\frac{1}{R}\right)^{1+3} = \frac{A}{8\pi^2 R^4}$$

Gaussian filter (σ_G^2):

$$\sigma_G^2(R) = \frac{A\Gamma\left(\frac{1+3}{2}\right)}{4\pi^2 R^{1+3}} = \frac{A\Gamma(2)}{4\pi^2 R^4}$$

The ratio is:

$$\frac{\sigma_G^2(R)}{\sigma_S^2(R)} = \frac{\frac{A\Gamma(2)}{4\pi^2 R^4}}{\frac{A}{8\pi^2 R^4}}$$

The terms A, π^2, and R^4 cancel, leaving:

$$\text{Ratio} = \frac{\frac{\Gamma(2)}{4}}{\frac{1}{8}} = \left(\frac{8}{4}\right)\Gamma(2) = 2\Gamma(2)$$

This expresses the answer in terms of the special function, **Gamma** (Γ). To find the numerical value, we use $\Gamma(z) = (z-1)!$ for integers:

$$\Gamma(2) = (2-1)! = 1! = 1$$

The numerical value of the ratio is:

$$\text{Ratio} = 2 \times 1 = *2*$$

4 Problem 4

(a)

$$M = \frac{4}{3}\pi r_i^3 \rho_i = \frac{4}{3}\pi r_i^3 \bar{\rho}_i (1+\delta_i)$$

$$\ddot{r} = -\frac{G \cdot \frac{4}{3}\pi r_i^3 \bar{\rho}_i (1+\delta_i)}{r^2}$$

$$\frac{\ddot{r}}{r} = -\frac{4\pi G \bar{\rho}_i}{3}(1+\delta_i)\left(\frac{r_i}{r}\right)^3$$

(b)

$$\frac{1}{2}\dot{r}^2 - \frac{GM}{r} = E$$

$$\frac{1}{2}\dot{r}^2 - \frac{4\pi G \bar{\rho}_i}{3}(1+\delta_i)\frac{r_i^3}{r} = E$$

$$\begin{aligned} E &= K(r_i) + V(r_i) \\ &= \frac{1}{2}\dot{r}_i^2 - \frac{4\pi G \bar{\rho}_i}{3}(1+\delta_i) r_i^2 \\ &= \frac{1}{2}\left(H_i r_i \left(1 - \frac{\delta_i}{3}\right)\right)^2 - \frac{4\pi G \bar{\rho}_i}{3}(1+\delta_i) r_i^2 \end{aligned}$$

At the turn-around radius r_{ta}, $\dot{r} = 0$,

$$E = -\frac{4\pi G \bar{\rho}_i}{3}(1+\delta_i)\frac{r_i^3}{r_{\text{ta}}}$$

$$\therefore -\frac{4\pi G \bar{\rho}_i}{3}(1+\delta_i)\frac{r_i^3}{r_{\text{ta}}} = \frac{1}{2}\left(H_i r_i \left(1 - \frac{\delta_i}{3}\right)\right)^2 - \frac{4\pi G \bar{\rho}_i}{3}(1+\delta_i) r_i^2$$

$$-\frac{4\pi G \bar{\rho}_i}{3}(1+\delta_i)\frac{r_i}{r_{\text{ta}}} = \frac{1}{2}H_i^2\left(1 - \frac{\delta_i}{3}\right)^2 - \frac{4\pi G \bar{\rho}_i}{3}(1+\delta_i)$$

$$H_i^2 = \frac{8\pi G \bar{\rho}_i}{3}$$

$$-(1+\delta_i)\frac{r_i}{r_{\text{ta}}} = \left(1 - \frac{\delta_i}{3}\right)^2 - (1+\delta_i)$$

$$-(1+\delta_i)\frac{r_i}{r_{\text{ta}}} = -\frac{5}{3}\delta_i + \frac{1}{9}\delta_i \approx -\frac{5}{3}\delta_i \quad \text{for } \delta_i \ll 1$$

$$\therefore r_{\text{ta}} = \frac{3}{5}\left(\frac{1+\delta_i}{\delta_i}\right) r_i$$

(c)

$$\frac{dr}{d\theta} = A\sin\theta, \quad \frac{dt}{d\theta} = B(1-\cos\theta)$$

$$\dot{r} = \frac{dr}{dt} = \frac{\frac{dr}{d\theta}}{\frac{dt}{d\theta}} = \frac{A\sin\theta}{B(1-\cos\theta)}$$

$$\begin{aligned}
\frac{d\dot{r}}{d\theta} &= \frac{d}{d\theta}\left(\frac{A\sin\theta}{B(1-\cos\theta)}\right) \\
&= \frac{A}{B} * \frac{\cos\theta(1-\cos\theta) - (\sin\theta)(-\sin\theta)}{(1-\cos\theta)^2} \\
&= \frac{A}{B} * \frac{\cos\theta - \cos^2\theta + \sin^2\theta}{(1-\cos\theta)^2} \\
&= \frac{A}{B} * \frac{\cos\theta - \left(1-\sin^2\theta\right) + \sin^2\theta}{(1-\cos\theta)^2} \\
&= \frac{A}{B} * \frac{\cos\theta - 1 + 2\sin^2\theta}{(1-\cos\theta)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{d\dot{r}}{d\theta} &= \frac{A}{B} * \frac{\cos\theta(1-\cos\theta) + \sin^2\theta}{(1-\cos\theta)^2} \\
&= \frac{A}{B} * \frac{\cos\theta - \cos^2\theta + 1 - \cos^2\theta}{(1-\cos\theta)^2} \\
&= \frac{A}{B} * \frac{1 + \cos\theta - 2\cos^2\theta}{(1-\cos\theta)^2} \\
&= \frac{A}{B} * \frac{(1-\cos\theta)(1+2\cos\theta)}{(1-\cos\theta)^2} \\
&= \frac{A}{B} * \frac{1+2\cos\theta}{1-\cos\theta}
\end{aligned}$$

$$\begin{aligned}
\frac{d\dot{r}}{d\theta} &= \frac{A}{B} * \frac{\cos\theta - \cos^2\theta + \sin^2\theta}{(1-\cos\theta)^2} \\
&= \frac{A}{B} * \frac{\cos\theta - \cos^2\theta + \left(1-\cos^2\theta\right)}{(1-\cos\theta)^2} \\
&= \frac{A}{B} * \frac{1 + \cos\theta - 2\cos^2\theta}{(1-\cos\theta)^2} \\
&= \frac{A}{B} * \frac{(1-\cos\theta)(1+2\cos\theta)}{(1-\cos\theta)^2} \\
&= \frac{A}{B} * \frac{1+2\cos\theta}{1-\cos\theta}
\end{aligned}$$

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{\frac{d\dot{r}}{d\theta}}{\frac{dt}{d\theta}}$$

$$\ddot{r} = \frac{\frac{A}{B} * \frac{1+2\cos\theta}{1-\cos\theta}}{B(1-\cos\theta)} = \frac{A}{B^2} * \frac{1+2\cos\theta}{(1-\cos\theta)^2}$$

Since $r = A(1-\cos\theta)$, we have $1-\cos\theta = \frac{r}{A}$, and $\cos\theta = 1-\frac{r}{A}$.

$$\ddot{r} = \frac{A}{B^2} * \frac{1+2(1-\frac{r}{A})}{\left(\frac{r}{A}\right)^2} = \frac{A}{B^2} * \frac{3-2\frac{r}{A}}{\frac{r^2}{A^2}} = \frac{A^3}{B^2 r^2} * \left(3-2\frac{r}{A}\right) = \frac{A^2}{B^2 r^2}(3A-2r)$$

$$\ddot{r} = \frac{\frac{A}{B} * \frac{1+\cos\theta}{1-\cos\theta}}{B(1-\cos\theta)} = \frac{A}{B^2} * \frac{1+\cos\theta}{(1-\cos\theta)^2}$$

With $1-\cos\theta = \frac{r}{A}$ and $1+\cos\theta = 2-(1-\cos\theta) = 2-\frac{r}{A}$:

$$\ddot{r} = \frac{A}{B^2} * \frac{2-\frac{r}{A}}{\left(\frac{r}{A}\right)^2} = \frac{A}{B^2} * \frac{\frac{2A-r}{A}}{\frac{r^2}{A^2}} = \frac{A^3}{B^2} * \frac{2A-r}{r^2}$$

Apply Initial Conditions

At turn-around, $\dot{r} = 0$, which is at $\theta = \pi$.

$$r_{\text{ta}} = r(\theta = \pi) = A(1-\cos\pi) = 2A$$

(d)

$$\delta(\theta) = \frac{9(\theta-\sin\theta)^2}{2(1-\cos\theta)^3} - 1$$

At turn-around, $\theta = \pi$:

$$\delta(\pi) = \frac{9(\pi-\sin\pi)^2}{2(1-\cos\pi)^3} - 1$$

$$\pi - \sin\pi = \pi - 0 = \pi$$

$$1-\cos\pi = 1-(-1) = 2$$

$$\delta(\pi) = \frac{9\pi^2}{2\cdot 2^3} - 1 = \frac{9\pi^2}{16} - 1 \approx 5.55$$

(e) The virial theorem states $2T + U = 0$. The potential energy is $U = -\frac{3}{5}\frac{GM^2}{r}$. At turn-around ($r_{\text{ta}}$), kinetic energy is zero ($T_{\text{ta}} = 0$), so the total energy is:

$$E = T_{\text{ta}} + U_{\text{ta}} = 0 - \frac{3}{5}\frac{GM^2}{r_{\text{ta}}}$$

After virialization, the system is in equilibrium.

$$E = T_{\text{vir}} + U_{\text{vir}}$$

$$U_{\text{vir}} = -\frac{3}{5}\frac{GM^2}{r_{\text{vir}}}$$

From the virial theorem, $T_{\text{vir}} = -\frac{U_{\text{vir}}}{2} = \frac{1}{2}\left(\frac{3}{5}\frac{GM^2}{r_{\text{vir}}}\right) = \left(\frac{3}{10}\right)\frac{GM^2}{r_{\text{vir}}}$. So, the total energy is:

$$E = \left(\frac{3}{10}\right)\frac{GM^2}{r_{\text{vir}}} - \left(\frac{3}{5}\right)\frac{GM^2}{r_{\text{vir}}} = -\left(\frac{3}{10}\right)\frac{GM^2}{r_{\text{vir}}}$$

By conservation of energy:

$$-\frac{3}{5}\frac{GM^2}{r_{\text{ta}}} = -\frac{3}{10}\frac{GM^2}{r_{\text{vir}}}$$

$$r_{\text{vir}} = \frac{r_{\text{ta}}}{2}$$

The density contrast at virialization, Δ_{vir}:

$$\Delta_{\text{vir}} = \frac{\rho_{\text{vir}}}{\bar{\rho}(t_{\text{vir}})}$$

We can find this by relating densities and times at turn-around and virialization. The mass density of the collapsed object changes as $\rho \propto r^{-3}$.

$$\frac{\rho_{\text{vir}}}{\rho_{\text{ta}}} = \left(\frac{r_{\text{ta}}}{r_{\text{vir}}}\right)^3 = \left(\frac{r_{\text{ta}}}{\frac{r_{\text{ta}}}{2}}\right)^3 = 2^3 = 8$$

The background density evolves as $\bar{\rho} \propto t^{-2}$ (in a matter-dominated universe). The time to virialization is twice the time to turn-around, $t_{\text{vir}} = 2t_{\text{ta}}$.

$$\frac{\bar{\rho}(t_{\text{ta}})}{\bar{\rho}(t_{\text{vir}})} = \left(\frac{t_{\text{vir}}}{t_{\text{ta}}}\right)^2 = 2^2 = 4$$

Combining these results:

$$\Delta_{\text{vir}} = \frac{\rho_{\text{vir}}}{\bar{\rho}(t_{\text{vir}})} = \frac{\rho_{\text{vir}}}{\rho_{\text{ta}}} \cdot \frac{\rho_{\text{ta}}}{\bar{\rho}(t_{\text{ta}})} \cdot \frac{\bar{\rho}(t_{\text{ta}})}{\bar{\rho}(t_{\text{vir}})}$$

The density contrast at turn-around is $\frac{\rho_{\text{ta}}}{\bar{\rho}(t_{\text{ta}})} = 1 + \delta(\pi) = \frac{9\pi^2}{16}$.

$$\Delta_{\text{vir}} = 8 \cdot \frac{9\pi^2}{16} \cdot 4 = 18\pi^2 \approx 178$$

References

[1] M. El-Deeb, "PEU-405 Assignments." [Online]. Available: https://github.com/mhdeeb/peu-assignments/tree/main/peu-405