PEU 416 Assignment 3

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1 Problem 1

(a)

$$\rho(t) = \rho_0 \left(\frac{a_0}{a(t)} \right)^3$$

$$\dot{\rho}(t) = -3\rho_0 \frac{a_0^3}{a^4}\dot{a} = -3\frac{\dot{a}}{a}\rho_0 \left(\frac{a_0}{a} \right)^3 = -3H\rho(t)$$

(b)

$$g = \ddot{r} = -\frac{GM(r_i)}{r^2} = -\frac{4\pi}{3}\frac{Gr_i^3\rho_i}{r^2}$$

$$a \equiv \frac{r}{r_i} \quad \rightarrow \quad r = ar_i, \quad \ddot{r} = \ddot{a}r_i$$

$$\ddot{a}r_i = -\frac{4\pi}{3}\frac{Gr_i\rho_i}{a^2}$$

$$\ddot{a} = -\frac{4\pi}{3}\frac{G\rho_i}{a^2} = -\frac{4\pi}{3}\frac{G(\rho(t)a^3)}{a^2} = -\frac{4\pi}{3}G\rho a \quad \text{for } a_0 = 1$$

$$\ddot{a} + \frac{4\pi}{3}G\rho a = 0$$

$$\dot{a}\left(\ddot{a} + \frac{4\pi}{3}\frac{G\rho_i}{a^2} \right) = 0$$

$$\frac{dE}{dt} = 0 \quad \rightarrow \quad E = \text{const}$$

$$\frac{\dot{a}^2}{2} - \frac{4\pi}{3}\frac{G\rho_i}{a} = E$$

$$\frac{\dot{a}^2}{2} - \frac{4\pi G\rho(t)a^2}{3} = E \equiv -\frac{K}{2}$$

$$\frac{\dot{a}^2}{2} - \frac{4\pi G\rho(t)a^3}{3a} = -\frac{K}{2}$$

$$\frac{\dot{a}^2}{2} - \frac{4\pi G\rho_i a_i^3}{3a} = -\frac{K}{2}$$

$$V = -\frac{4\pi G\rho_i a_i^3}{3a}$$

$$\frac{\dot{a}^2}{2} + V = -\frac{K}{2}$$

(c)

$$
\begin{aligned}
\frac{\dot{a}^2}{2} + V &= 0 \\
\dot{a} &= \sqrt{-2V} \\
\sqrt{a}\dot{a} &= \sqrt{\frac{8\pi G\rho_i a_i^3}{3}} \\
\sqrt{a}\ \mathrm{da} &= \sqrt{\frac{8\pi G\rho_i a_i^3}{3}}\ \mathrm{dt} \\
\frac{2}{3}a^{\frac{3}{2}} &= \sqrt{\frac{8\pi G\rho_i a_i^3}{3}}t \\
a &= a_i\sqrt[3]{12\pi G\rho_i}t^{\frac{2}{3}}
\end{aligned}
$$

(d)

$$
\frac{\dot{a}^2}{2} - \frac{4\pi G\rho(t)a^2}{3} = -\frac{K}{2}
$$

$$
\begin{aligned}
K &= \frac{8\pi G\rho(t)a^2}{3} - \dot{a}^2 \\
&= \frac{8\pi G\rho(t)a^2}{3} - a^2\frac{\dot{a}^2}{a^2} \\
&= \frac{8\pi G\rho(t)a^2}{3} - a^2H^2 \\
&= \frac{8\pi G(\rho_c\Omega)a^2}{3} - a^2\left(\frac{8}{3}\pi G\rho_c\right) \\
&= \frac{8\pi G}{3}a^2\rho_c(\Omega - 1)
\end{aligned}
$$

Ω value indications

Since by measuring Ω, we have the K value

$$
\begin{aligned}
\frac{\dot{a}^2}{2} - \frac{4\pi G\rho_i a_i^3}{3a} &= -\frac{K}{2} \\
\dot{a}^2 &= -K + \frac{8\pi G\rho_i a_i^3}{3a}
\end{aligned}
$$

If $\Omega < 1$ then $K < 0$, $\dot{a}^2 > 0$: the universe expands forever.

If $\Omega = 1$ then $K = 0$: the universe is flat

If $\Omega > 1$ then $K > 0$, $a_{\max} \geq \dot{a} \geq 0$: the universe approaches maximum size and then contracts to a Big Crunch.

2 Problem 2

$$\Phi = \phi + a\ddot{a}\frac{x^2}{2}$$

$$\nabla_x^2 \Phi = \nabla_x^2 \phi + \nabla_x^2 \left(a\ddot{a}\frac{x^2}{2} \right)$$

$$\nabla_r^2 \phi = 4\pi G \rho$$

$$\nabla_r^2 = \frac{1}{a^2}\nabla_x^2 \quad \rightarrow \quad \nabla_x^2 \phi = 4\pi G a^2 \rho$$

$$\nabla_x^2 \left(a\ddot{a}\frac{x^2}{2} \right) = \sum_{i=1,2,3} \frac{\partial^2}{\partial x_i^2} \left(a\ddot{a}\frac{x_i^2}{2} \right) = 3a\ddot{a} = -4\pi G \bar{\rho} a^2$$

$$\therefore \nabla_x^2 \Phi = 4\pi G a^2 (\rho - \bar{\rho}) = 4\pi G a^2 \bar{\rho} \delta$$

3 Problem 3

(a)

$$\ddot{\delta}_m + 2H\dot{\delta}_m = 4\pi G\bar{\rho}_m\delta_m$$

$$\ddot{\delta}_m + 2H\dot{\delta}_m = 4\pi G\bar{\rho}_{m,0}a^{-3}\delta_m$$

$$\frac{d}{dt} = \frac{da}{dt}\cdot\frac{d}{da} = \dot{a}\frac{d}{da} = Ha\frac{d}{da}$$

$$\dot{\delta}_m = Ha\frac{d\delta_m}{da}, \quad \ddot{\delta}_m = \frac{d}{dt}\big(\dot{\delta}_m\big) = \frac{d}{dt}\left(Ha\frac{d\delta_m}{da}\right) = Ha\frac{d}{da}\left(Ha\frac{d\delta_m}{da}\right)$$

$$\therefore Ha\frac{d}{da}\left(Ha\frac{d\delta_m}{da}\right) + 2H^2a\frac{d\delta_m}{da} = 4\pi G\bar{\rho}_{m,0}a^{-3}\delta_m$$

multiply both sides by $\frac{a}{H}$

$$a^2\frac{d}{da}\left(Ha\frac{d\delta_m}{da}\right) + 2Ha^2\frac{d\delta_m}{da} = 4\pi G\bar{\rho}_{m,0}\frac{\delta_m}{Ha^2}$$

$$\frac{d}{da}\left(a^3H\frac{d\delta_m}{da}\right) = 4\pi G\bar{\rho}_{m,0}\frac{\delta_m}{Ha^2}$$

(b)

$$\delta_m \propto \begin{cases} H \\ H\int\frac{\mathrm{da}}{(aH)^3} \end{cases}$$

For $\delta_m = H$

$$\frac{d}{da}\left(a^3H\frac{d\delta_m}{da}\right) = 4\pi G\bar{\rho}_{m,0}\frac{\delta_m}{Ha^2} = \frac{3}{2}aH^2$$

$$\frac{d}{da}\left(a^3H\frac{dH}{da}\right) = \frac{3}{2}aH^2$$

$$\frac{1}{2}\frac{d}{da}\left(a^3\frac{d}{da}(H^2)\right) = \frac{3}{2}aH^2$$

$$\frac{1}{2}\left(3a^2\dot{a}\frac{d}{da}(H^2) + a^3\frac{d^2H^2}{da^2}\right) = \frac{3}{2}aH^2$$

$$\frac{3}{2}a^3\left(H\frac{d}{da}(H^2) + \frac{1}{3}\frac{d^2H^2}{da^2}\right) = \frac{3}{2}aH^2$$

$$\frac{2}{3}\left((3\dot{a}-1)\dot{H} + \frac{\ddot{H}}{H}\right)$$

$$
\begin{aligned}
&\frac{d}{da}\left(a^3 H \frac{dH}{da}\right) \\
&= \frac{1}{a^2}\left(3\dot{a}a^4 H \frac{dH}{da} + a^5\left(\frac{dH}{da}\right)^2 + a^5 H \frac{d^2 H}{da^2}\right) \\
&= \frac{1}{a^2}\left(3\dot{a}^2 a^3 \frac{d\left(\frac{\dot{a}}{a}\right)}{da} + a^5\left(\frac{d\left(\frac{\dot{a}}{a}\right)}{da}\right)^2 + \dot{a}a^4 \frac{d^2\left(\frac{\dot{a}}{a}\right)}{da^2}\right) \\
&= \frac{1}{a^2}\left(3\dot{a}^2 a^3 \frac{a\ddot{a}-\dot{a}^2}{a^2} + a^5\left(\frac{a\ddot{a}-\dot{a}^2}{a^2}\right)^2 + \dot{a}a^4 \frac{d\left(\frac{a\ddot{a}-\dot{a}^2}{a^2}\right)}{da}\right) \\
&= \frac{1}{a^2}\left(3\dot{a}^2 a\big(a\ddot{a}-\dot{a}^2\big) + a\big(a\ddot{a}-\dot{a}^2\big)^2 + \dot{a}a^4\left(\frac{a^2(\ddot{a}^2+a\dddot{a}-2\dot{a}\ddot{a})-(a\ddot{a}-\dot{a}^2)(2a\dot{a})}{a^4}\right)\right) \\
&= \frac{1}{a^2}\Big(3\dot{a}^2 a\big(a\ddot{a}-\dot{a}^2\big) + a\big(a\ddot{a}-\dot{a}^2\big)^2 + \dot{a}\big(a^2\big(\ddot{a}^2+a\dddot{a}-2\dot{a}\ddot{a}\big) - 2a\dot{a}\big(a\ddot{a}-\dot{a}^2\big)\big)\Big) \\
&= \frac{1}{a^2}\Big(a\dot{a}^2\big(a\ddot{a}-\dot{a}^2\big) + a\big(a\ddot{a}-\dot{a}^2\big)^2 + \dot{a}a^2\big(\ddot{a}^2+a\dddot{a}-2\dot{a}\ddot{a}\big)\Big) \\
&= \frac{1}{a^2}(a^2\dot{a}^2\ddot{a} - a\dot{a}^4 + a^3\ddot{a}^2 + a\dot{a}^4 - 2a^2\dot{a}^2\ddot{a} + \dot{a}a^2\ddot{a}^2 + \dot{a}a^3\dddot{a} - 2a^2\dot{a}^2\ddot{a}) \\
&= \frac{1}{a^2}(a^3\ddot{a}^2 - 3a^2\dot{a}^2\ddot{a} + a^2\dot{a}\ddot{a}^2 + a^3\dot{a}\dddot{a}) \\
&= a\ddot{a}^2 - 3\dot{a}^2\ddot{a} + \dot{a}\ddot{a}^2 + a\dot{a}\dddot{a}
\end{aligned}
$$

$$H^2 = \frac{8\pi G}{3}\rho_m$$

$$H = \frac{\dot{a}}{a}$$

$$\dot{H} = \frac{a\ddot{a}-\dot{a}^2}{a^2}$$

$$\ddot{H} = \frac{a^2(\ddot{a}^2+a\dddot{a}-2\dot{a}\ddot{a})-(a\ddot{a}-\dot{a}^2)(2a\dot{a})}{a^4}$$

$$\ddot{\delta}_m + 2H\dot{\delta}_m = 4\pi G\bar{\rho}_m\delta_m$$

$$\ddot{H} + 2H\dot{H} = 4\pi G\bar{\rho}_m H$$

$$\frac{a^2(\ddot{a}^2 + a\dddot{a} - 2\dot{a}\ddot{a}) - (a\ddot{a} - \dot{a}^2)(2a\dot{a})}{a^4} + 2\left(\frac{\dot{a}}{a}\right)\left(\frac{a\ddot{a} - \dot{a}^2}{a^2}\right) = 4\pi G\bar{\rho}_m\left(\frac{\dot{a}}{a}\right)$$

$$\frac{a^2(\ddot{a}^2 + a\dddot{a} - 2\dot{a}\ddot{a}) - (a\ddot{a} - \dot{a}^2)(2a\dot{a})}{a^3} + 2\frac{a\dot{a}\ddot{a} - \dot{a}^3}{a^2} = 4\pi G\bar{\rho}_m\dot{a}$$

$$a(\ddot{a}^2 + a\dddot{a} - 2\dot{a}\ddot{a}) - 2\dot{a}(a\ddot{a} - \dot{a}^2) + 2(a\dot{a}\ddot{a} - \dot{a}^3) = 4\pi G\bar{\rho}_m\dot{a}a^2$$

$$a\ddot{a}^2 + a^2\dddot{a} - 2a\dot{a}\ddot{a} = 4\pi G\bar{\rho}_m a^2\dot{a}$$

$$\frac{\ddot{a}^2}{a\dot{a}} + \frac{\dddot{a}}{a} - \frac{2\ddot{a}}{a} = 4\pi G\bar{\rho}_m$$

$$\frac{\ddot{a}^2 - 2\dot{a}\ddot{a} + \dot{a}\dddot{a}}{a\dot{a}} = 4\pi G\bar{\rho}_m$$

$$\frac{d(\dot{a}\ddot{a} - \dot{a}^2)}{dt}\frac{1}{a\dot{a}} = 4\pi G\bar{\rho}_m$$

For $\delta_m = H\int \frac{\mathrm{da}}{(aH)^3}$

(c)

$$H^2 = \frac{8\pi G}{3}\rho_{\text{total}} = \frac{8\pi G}{3}(\rho_r + \rho_m)$$

$$\rho_r \propto a^{-4}, \quad \rho_m \propto a^{-3}$$

$$\rho_r = \rho_{\text{eq}}\left(\frac{a_{\text{eq}}}{a}\right)^4, \quad \rho_m = \rho_{\text{eq}}\left(\frac{a_{\text{eq}}}{a}\right)^3 \quad \text{where } \rho_{\text{eq}} = \rho_{r,\text{ eq}} = \rho_{m,\text{ eq}}$$

$$y \equiv \frac{a}{a_{\text{eq}}} \quad \to \quad \rho_r = \rho_{\text{eq}}y^{-4}, \quad \rho_m = \rho_{\text{eq}}y^{-3}$$

$$H^2 = \frac{8\pi G}{3}\rho_{\text{eq}}(y^{-4} + y^{-3}) = \frac{8\pi G}{3}\frac{\rho_{\text{eq}}}{y^4}(1+y)$$

$$\therefore H = \frac{A}{y^2}\sqrt{1+y} \quad \text{where } A = \sqrt{\frac{8\pi G}{3}\rho_{\text{eq}}}$$

$$\frac{d}{da}\left(a^3 H \frac{d\delta_m}{da}\right) = 4\pi G \bar{\rho}_{m,0} \frac{\delta_m}{Ha^2}$$

$$a = a_{\text{eq}} y, \quad \frac{d}{da} = \frac{1}{a_{\text{eq}}}\frac{d}{dy}$$

$$\frac{1}{a_{\text{eq}}}\frac{d}{dy}\left(\left(a_{\text{eq}} y\right)^3 H \frac{1}{a_{\text{eq}}}\frac{d\delta_m}{dy}\right) = 4\pi G \bar{\rho}_{m,0} \frac{\delta_m}{H\left(a_{\text{eq}} y\right)^2}$$

$$\frac{d}{dy}\left(y^3 H \frac{d\delta_m}{dy}\right) = 4\pi G \frac{\bar{\rho}_{m,0}}{a_{\text{eq}}^3} \frac{\delta_m}{Hy^2}$$

$$\frac{d}{dy}\left(y^3 H \frac{d\delta_m}{dy}\right) = 4\pi G \rho_{\text{eq}} \frac{\delta_m}{Hy^2}$$

Substituting with $H = \frac{A}{y^2}\sqrt{1+y}$

$$\frac{d}{dy}\left(Ay\sqrt{1+y}\frac{d\delta_m}{dy}\right) = 4\pi G \rho_{\text{eq}} \frac{\delta_m}{A\sqrt{1+y}}$$

$$\frac{d}{dy}\left(y\sqrt{1+y}\frac{d\delta_m}{dy}\right) = \frac{4\pi G \rho_{\text{eq}}}{A^2} \frac{\delta_m}{\sqrt{1+y}}$$

$$\frac{d}{dy}\left(y\sqrt{1+y}\frac{d\delta_m}{dy}\right) = \frac{3}{2} \frac{\delta_m}{\sqrt{1+y}}$$

$$y\sqrt{1+y}\frac{d^2\delta_m}{dy^2} + \left(\sqrt{1+y} + \frac{y}{2\sqrt{1+y}}\right)\frac{d\delta_m}{dy} = \frac{3}{2} \frac{\delta_m}{\sqrt{1+y}}$$

$$\frac{d^2\delta_m}{dy^2} + \left(\frac{1}{y} + \frac{1}{2(1+y)}\right)\frac{d\delta_m}{dy} = \frac{3}{2} \frac{\delta_m}{y(1+y)}$$

$$\frac{d^2\delta_m}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\delta_m}{dy} = \frac{3}{2} \frac{\delta_m}{y(1+y)}$$

Verifying the solutions

for $\delta_{m1} = 1 + \frac{3}{2}y$:

$$\frac{d\delta_m}{dy} = \frac{3}{2}, \quad \frac{d^2\delta_m}{dy^2} = 0$$

$$0 + \frac{2+3y}{2y(1+y)}\frac{3}{2} = \frac{3}{2}\frac{1+\frac{3}{2}y}{y(1+y)}$$

$$\frac{2+3y}{2y(1+y)}\frac{3}{2} = \frac{3}{2}\frac{2+3y}{2y(1+y)} \quad \text{(verified)}$$

for $\delta_{m2} = \left(1+\frac{3}{2}y\right)\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 3\sqrt{1+y}$:

$$
\begin{aligned}
\frac{d\delta_m}{dy} &= \frac{3}{2}\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - \left(1+\frac{3}{2}y\right)\frac{1}{y\sqrt{y+1}} - \frac{3}{2\sqrt{1+y}} \\
&= \frac{3}{2}\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - \frac{1}{y\sqrt{y+1}} - \frac{3}{\sqrt{1+y}} \\
&= \frac{3}{2}\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - \left(\frac{1}{y}+3\right)\frac{1}{\sqrt{y+1}} \\
\frac{d^2\delta_m}{dy^2} &= -\frac{3}{2}\frac{1}{y\sqrt{y+1}} + \frac{1}{y^2\sqrt{y+1}} + \frac{1}{2}\left(\frac{1}{y}+3\right)\frac{1}{(y+1)^{\frac{3}{2}}} \\
&= \left(-\frac{3}{2}+\frac{1}{y}\right)\frac{1}{y\sqrt{y+1}} + \frac{1}{2}\left(\frac{1}{y}+3\right)\frac{1}{(y+1)^{\frac{3}{2}}} \\
&= \frac{2-3y}{2y}\frac{1}{y\sqrt{y+1}} + \frac{3y+1}{2y}\frac{1}{(y+1)^{\frac{3}{2}}}
\end{aligned}
$$

$$
\begin{aligned}
\text{LHS} &= \left[\frac{2-3y}{2y}\frac{1}{y\sqrt{y+1}} + \frac{3y+1}{2y}\frac{1}{(y+1)^{\frac{3}{2}}}\right] + \frac{2+3y}{2y(1+y)}\left[\frac{3}{2}\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - \frac{3y+1}{y}\frac{1}{\sqrt{y+1}}\right] \\
&= \left[\frac{2-3y}{2y}\frac{1}{y\sqrt{y+1}} + \frac{3y+1}{2y}\frac{1}{(y+1)^{\frac{3}{2}}}\right] + \frac{3}{2}\frac{2+3y}{2y(1+y)}\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - \frac{(2+3y)(3y+1)}{2y^2}\frac{1}{(y+1)^{\frac{3}{2}}} \\
&= \left[\frac{2-3y}{2y}\frac{1}{y\sqrt{y+1}} + \frac{3y+1}{2y}\frac{1}{(y+1)^{\frac{3}{2}}}\right] + \frac{3}{2}\frac{2+3y}{2y(1+y)}\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - \frac{9y^2+9y+2}{2y^2}\frac{1}{(y+1)^{\frac{3}{2}}} \\
&= \left[\frac{2-3y}{2y}\frac{1}{y\sqrt{y+1}} + \cancel{\frac{3y+1}{2y}\frac{1}{(y+1)^{\frac{3}{2}}}}\right] + \frac{3}{2}\frac{2+3y}{2y(1+y)}\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - \left[\cancel{\frac{3y^2+y}{2y^2}} + \frac{6y^2+8y+2}{2y^2}\right]\frac{1}{(y+1)^{\frac{3}{2}}} \\
&= \left[\frac{2-3y}{2y}\frac{1}{y\sqrt{y+1}}\right] + \frac{3}{2}\frac{2+3y}{2y(1+y)}\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - \left[\frac{2(3y+1)(y+1)}{2y^2}\right]\frac{1}{(y+1)^{\frac{3}{2}}} \\
&= \left[\frac{2-3y}{2y}\frac{1}{y\sqrt{y+1}}\right] + \frac{3}{2}\frac{2+3y}{2y(1+y)}\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - \left[\frac{2(3y+1)}{2y}\right]\frac{1}{y\sqrt{y+1}} \\
&= \frac{3}{2}\frac{2+3y}{2y(1+y)}\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - \frac{9}{2}\frac{1}{y\sqrt{1+y}} \\
\text{RHS} &= \frac{3}{2}\frac{1}{y(1+y)}\left[\left(1+\frac{3}{2}y\right)\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 3\sqrt{1+y}\right] \\
&= \frac{3}{2}\frac{2+3y}{2y(1+y)}\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - \frac{9}{2}\frac{1}{y\sqrt{1+y}} \quad \text{(verified)}
\end{aligned}
$$

The growing and decaying modes

at early times, $a \ll a_{\text{eq}}$, $y \ll 1$,

$$
\delta_{m1} \to 1, \quad \delta_{m2} \to \infty
$$

The growing mode: δ_{m2}
The decaying mode: δ_{m1}

at late times, $a \gg a_{\rm eq}$, $y \gg 1$,

$$\delta_{m1} \to \frac{3}{2}y, \quad \delta_{m2} \to -3\sqrt{y}$$

The growing mode: δ_{m1}
The decaying mode: δ_{m2}

4 Problem 4

(a)

$$c_s^2 = \frac{\partial P}{\partial \rho} = \frac{4}{3}\alpha\rho^{\frac{1}{3}}$$

$$t \propto a^{\frac{3}{2}}, \quad \rho \propto a^{-3} \quad \text{for a matter-dominated universe}$$

$$dt = a d\eta$$

$$d\eta = \frac{1}{a}dt \quad \rightarrow \quad \eta \propto a^{\frac{1}{2}} \quad \rightarrow \quad a \propto \eta^2$$

$$\therefore \rho \propto \eta^{-6} \quad \rightarrow \quad c_s^2 \propto \eta^{-2}$$

$$\therefore c_s^2 = c_0^2\left(\frac{\eta_0}{\eta}\right)^2$$

(b) **Small Scales (large k, where $c_s^2k^2 \gg \frac{3}{2}H^2$)**:
the term $c_s^2k^2$ dominates the equation because k^2 (related to the wavelength $\lambda = \frac{2\pi}{k}$) is large, and the sound speed C_s allows pressure to resist gravitational collapse. The equation approximates to $\delta'' + c_s^2k^2\delta \approx 0$, which resembles an oscillatory solution due to pressure waves. This indicates that density perturbations oscillate as acoustic waves rather than growing, as pressure prevents collapse on scales smaller than the Jeans length (where $C_s k \sim H$).

Large Scales (small k, where $c_s^2k^2 \ll \frac{3}{2}H^2$):
the term $\frac{3}{2}H^2$ dominates, as k^2 is small. The equation simplifies to $\delta'' + H\delta' - \frac{3}{2}H^2\delta \approx 0$. In a matter-dominated universe, $H \propto \eta^{-1}$, leading to a growing mode $\delta \propto \eta^2 \propto a$. This reflects gravitational instability, where perturbations grow over time due to self-gravity, unopposed by pressure on scales larger than the Jeans length.

(c)

$$a \propto \eta^2 \quad \rightarrow \quad H = \frac{2}{\eta}$$

$$\delta'' + \frac{2}{\eta}\delta' + \left(c_0^2\left(\frac{\eta_0}{\eta}\right)^2 k^2 - \frac{6}{\eta^2}\right)\delta = 0$$

$$\eta^2\delta'' + 2\eta\delta' + (c_0^2\eta_0^2k^2 - 6)\delta = 0$$

Let $\delta = C\eta^\beta$,

$$\delta' = C\beta\eta^{\beta-1}, \quad \delta'' = C\beta(\beta-1)\eta^{\beta-2}$$

$$C\beta(\beta-1)\eta^\beta + 2C\beta\eta^\beta + (c_0^2\eta_0^2k^2 - 6)C\eta^\beta = 0$$

$$\beta(\beta-1) + 2\beta + (c_0^2\eta_0^2k^2 - 6) = 0$$

$$\beta^2 + \beta + (c_0^2\eta_0^2k^2 - 6) = 0$$

$$
\begin{aligned}
\beta &= \frac{1}{2}\Big(-1 \pm \sqrt{1 - 4(c_0^2\eta_0^2k^2 - 6)}\Big) \\
&= \frac{1}{2}\Big(-1 \pm \sqrt{25 - 4c_0^2\eta_0^2k^2}\Big)
\end{aligned}
$$

The transition occurs at

$$
\begin{aligned}
25 - 4c_0^2\eta_0^2k^2 &= 0 \\
k &= \frac{5}{2c_0\eta_0}
\end{aligned}
$$

The growing mode is η^{β_+}, where $\beta_+ = \frac{1}{2}\big(-1 + \sqrt{25 - 4c_0^2\eta_0^2k^2}\big)$ it is constant when $\beta_+ = 0$, or

$$
\begin{aligned}
-1 + \sqrt{25 - 4c_0^2\eta_0^2k^2} &= 0 \\
25 - 4c_0^2\eta_0^2k^2 &= 1 \\
24 - 4c_0^2\eta_0^2k^2 &= 0 \\
k &= \frac{\sqrt{6}}{c_0\eta_0}
\end{aligned}
$$

References

[1] M. El-Deeb, "PEU-405 Assignments." [Online]. Available: https://github.com/mhdeeb/peu-assignments/tree/main/peu-405