

# PEU 218 Assignment 4

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# 1 Question 1

## 1.1 Problem

Paraboloidal coordinates  $u, v, \varphi$  are defined in terms of Cartesian coordinates

$$x = \alpha\beta \cos \varphi, \quad y = \alpha\beta \sin \varphi, \quad z = \frac{1}{2}(\alpha^2 - \beta^2)$$

where  $0 \leq \alpha \leq \infty, \quad 0 \leq \beta \leq \infty, \quad 0 \leq \varphi \leq 2\pi$

Prove that the  $\alpha$ -component of  $\vec{\nabla} \times \vec{A}$  is

$$\frac{1}{(\alpha^2 + \beta^2)^{1/2}} \left( \frac{A_\varphi}{\beta} + \frac{\partial \varphi}{\partial \beta} \right) - \frac{1}{\alpha\beta} \frac{\partial A_\beta}{\partial \varphi}$$

## 1.2 Solution

$$\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$$

$$\begin{aligned} (\vec{\nabla} \times \vec{A})_\alpha &= \frac{1}{h_\beta h_\varphi} \left( \frac{\partial}{\partial \beta} (A_\varphi h_\varphi) - \frac{\partial}{\partial \varphi} (A_\beta h_\beta) \right) \\ &= \frac{1}{h_\beta h_\varphi} \left( \frac{\partial A_\varphi}{\partial \beta} h_\varphi + \frac{\partial h_\varphi}{\partial \beta} A_\varphi - \frac{\partial A_\beta}{\partial \varphi} h_\beta - \frac{\partial h_\beta}{\partial \varphi} A_\beta \right) \\ h_\beta &= \left| \frac{\partial \vec{r}}{\partial \beta} \right| = \sqrt{\left( \frac{\partial x}{\partial \beta} \right)^2 + \left( \frac{\partial y}{\partial \beta} \right)^2 + \left( \frac{\partial z}{\partial \beta} \right)^2} \\ &= \sqrt{(\alpha \cos \varphi)^2 + (\alpha \sin \varphi)^2 + (\beta)^2} = \sqrt{\alpha^2 + \beta^2} \\ h_\varphi &= \left| \frac{\partial \vec{r}}{\partial \varphi} \right| = \sqrt{\left( \frac{\partial x}{\partial \varphi} \right)^2 + \left( \frac{\partial y}{\partial \varphi} \right)^2 + \left( \frac{\partial z}{\partial \varphi} \right)^2} \end{aligned}$$

$$= \sqrt{(\alpha\beta \sin \varphi)^2 + (\alpha\beta \cos \varphi)^2} = \alpha\beta$$

$$\frac{\partial h_\beta}{\partial \varphi} = \frac{\partial}{\partial \varphi} \left( \sqrt{\alpha^2 + \beta^2} \right) = 0$$

$$\frac{\partial h_\varphi}{\partial \beta} = \frac{\partial}{\partial \beta} (\alpha\beta) = \alpha$$

$$\begin{aligned} (\vec{\nabla} \times \vec{A})_\alpha &= \frac{1}{\alpha\beta\sqrt{\alpha^2 + \beta^2}} \left( \frac{\partial A_\varphi}{\partial \beta} \alpha\beta + \alpha A_\varphi - \frac{\partial A_\beta}{\partial \varphi} \sqrt{\alpha^2 + \beta^2} \right) \\ &= \frac{1}{(\alpha^2 + \beta^2)^{1/2}} \left( \frac{A_\varphi}{\beta} + \frac{\partial A_\varphi}{\partial \beta} \right) - \frac{1}{\alpha\beta} \frac{\partial A_\beta}{\partial \varphi} \end{aligned}$$

## 2 Question 2

### 2.1 Problem

Using spherical coordinates, evaluate

$$\nabla^2 \left( \vec{\nabla} \cdot \frac{\vec{r}}{r^2} \right)$$

### 2.2 Solution

$$\vec{r} = \langle r, 0, 0 \rangle$$

$$\frac{\vec{r}}{r^2} = \vec{r} = \left\langle \frac{1}{r}, 0, 0 \right\rangle$$

$$\vec{\nabla} \cdot \vec{u} = \frac{1}{r^2} \frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\sin \theta u_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial (u_\varphi)}{\partial \varphi}$$

$$\vec{\nabla} \cdot \frac{\vec{r}}{r^2} = \frac{1}{r^2} \frac{\partial r}{\partial r} = \frac{1}{r^2}$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2}$$

$$\nabla^2 \frac{1}{r^2} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \frac{1}{r^2}}{\partial r} \right) = -\frac{2}{r^2} \frac{\partial}{\partial r} (r^{-1}) = \frac{2}{r^4}$$

$$\nabla^2 \left( \vec{\nabla} \cdot \frac{\vec{r}}{r^2} \right) = \frac{2}{r^4}$$

### 3 Question 3

#### 3.1 Problem

We introduce the so-called spheroidal coordinates  $(\eta, \theta, \varphi)$  by the following equations expressed in rectangular coordinates

$$\begin{aligned}x &= a \sinh \eta \sin \theta \cos \varphi, \\y &= a \sinh \eta \sin \theta \sin \varphi, \\z &= a \cosh \eta \cos \theta\end{aligned}$$

where  $0 \leq \eta \leq \infty$ ,  $0 \leq \theta \leq \pi$ ,  $0 \leq \varphi \leq 2\pi$

- a) Show that this coordinate system is orthogonal.
- b) Find the scale factors.
- c) Show that the function  $f(\eta, \theta, \varphi) = \ln \tanh \left( \frac{\eta}{2} \right)$  is a solution of Laplace's equation.

#### 3.2 Solution

a)

$$\vec{r} = a \langle \sinh \eta \sin \theta \cos \varphi, \sinh \eta \sin \theta \sin \varphi, \cosh \eta \cos \theta \rangle$$

$$\vec{e}_\eta = \frac{\partial \vec{r}}{\partial \eta} = a \langle \cosh \eta \sin \theta \cos \varphi, \cosh \eta \sin \theta \sin \varphi, \sinh \eta \cos \theta \rangle$$

$$\vec{e}_\theta = \frac{\partial \vec{r}}{\partial \theta} = a \langle \sinh \eta \cos \theta \cos \varphi, \sinh \eta \cos \theta \sin \varphi, -\cosh \eta \sin \theta \rangle$$

$$\vec{e}_\varphi = \frac{\partial \vec{r}}{\partial \varphi} = a \langle -\sinh \eta \sin \theta \sin \varphi, \sinh \eta \sin \theta \cos \varphi, 0 \rangle$$

$$\frac{\partial \vec{r}}{\partial \eta} \cdot \frac{\partial \vec{r}}{\partial \theta} = a^2 (\sinh \eta \cosh \eta \sin \theta \cos \theta - \sinh \eta \cosh \eta \sin \theta \cos \theta) = 0$$

$$\frac{\partial \vec{r}}{\partial \eta} \cdot \frac{\partial \vec{r}}{\partial \varphi} =$$

$$a^2 \left( -\sinh \eta \cosh \eta \sin \varphi \cos \varphi \sin^2 \theta + \sinh \eta \cosh \eta \sin \varphi \cos \varphi \sin^2 \theta \right) = 0$$

$$\frac{\partial \vec{r}}{\partial \theta} \cdot \frac{\partial \vec{r}}{\partial \varphi} =$$

$$a^2 \left( -\sinh^2 \eta \sin \theta \cos \theta \sin \varphi \cos \varphi + \sinh^2 \eta \sin \theta \cos \theta \sin \varphi \cos \varphi \right) = 0$$

Since the dot product of different basis vectors is zero, the coordinate system is orthogonal.

b)

$$\cosh^2 \eta = 1 + \sinh^2 \eta$$

$$h_\eta = \left| \frac{\partial \vec{r}}{\partial \eta} \right| = a \sqrt{\cosh^2 \eta \sin^2 \theta + \sinh^2 \eta \cos^2 \theta}$$

$$= a \sqrt{(1 + \sinh^2 \eta) \sin^2 \theta + \sinh^2 \eta \cos^2 \theta} = a \sqrt{\sinh^2 \eta + \sin^2 \theta}$$

$$h_\theta = \left| \frac{\partial \vec{r}}{\partial \theta} \right| = a \sqrt{\sinh^2 \eta \cos^2 \theta + \cosh^2 \eta \sin^2 \theta} = h_\eta$$

$$h_\varphi = \left| \frac{\partial \vec{r}}{\partial \varphi} \right| = a \sinh \eta \sin \theta$$

c)

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial q_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial f}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left( \frac{h_3 h_1}{h_2} \frac{\partial f}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial f}{\partial q_3} \right) \right]$$

In this coordinate system,  $h_1 = h_2 = h_\eta$ ,  $h_3 = h_\varphi$ ,

$$\nabla^2 f = \frac{1}{h_\eta^2 h_\varphi} \left[ \frac{\partial}{\partial \eta} \left( h_\varphi \frac{\partial f}{\partial \eta} \right) + \frac{\partial}{\partial \theta} \left( h_\varphi \frac{\partial f}{\partial \theta} \right) + \frac{\partial}{\partial \varphi} \left( \frac{h_\eta^2}{h_\varphi} \frac{\partial f}{\partial \varphi} \right) \right]$$

Since we are solving laplace equation, we need only to focus on what is inside the square brackets.

$$\frac{\partial}{\partial \eta} \left( h_\varphi \frac{\partial f}{\partial \eta} \right) + \frac{\partial}{\partial \theta} \left( h_\varphi \frac{\partial f}{\partial \theta} \right) + \frac{\partial}{\partial \varphi} \left( \frac{h_\eta^2}{h_\varphi} \frac{\partial f}{\partial \varphi} \right)$$

Since  $f$  is a function of  $\eta$  only, we can take our solution to be a function of only  $\eta$  so second and third terms will be zero.

$$\frac{\partial}{\partial \eta} \left( h_\varphi \frac{\partial f}{\partial \eta} \right)$$

We could also induce that,

$$h_\varphi \frac{\partial f}{\partial \eta} = g(\theta, \varphi)$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial}{\partial \eta} \left( \ln \tanh \left( \frac{\eta}{2} \right) \right) = \frac{1}{2} \operatorname{sech}^2 \frac{\eta}{2} \coth \left( \frac{\eta}{2} \right) = \frac{1}{2} \operatorname{csch} \frac{\eta}{2} \operatorname{sech} \frac{\eta}{2}$$

$$h_\varphi \frac{\partial f}{\partial \eta} = a \sin \theta \sinh \eta \left( \frac{1}{2} \operatorname{csch} \frac{\eta}{2} \operatorname{sech} \frac{\eta}{2} \right) = a \sin \theta \sinh \eta \operatorname{csch} \eta = a \sin \theta$$

Since  $g$  is not a function of  $\eta$ , we could deduce that  $f$  is a solution of Laplace's equation.

## References

- [1] M.H. El-Deeb. [PEU-218 Assignments](#).