PEU 218 Assignment 5

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1 Question 1

1.1 Problem

Use the Divergence theorem to evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \sin(\pi x)\hat{\imath} + zy^3\hat{\jmath} + (z^2 + 4x)\,\hat{k}$ and S is the surface of the box with $-1 \leq x \leq 2, 0 \leq y \leq 1$ and $1 \leq z \leq 4$. Note that all six sides of the box are included in S.

1.2 Solution

$$\iiint_V \vec{\nabla} \cdot \overrightarrow{\boldsymbol{u}} dV = \oiint_S \overrightarrow{\boldsymbol{u}} \cdot \widehat{\boldsymbol{n}} dS$$

$$\vec{\nabla} \cdot \vec{F} = +\pi \cos(\pi x) + 3zy^2 + 2z$$

$$\iiint_V \vec{\nabla} \cdot \vec{F} dV = \int_1^4 \int_0^1 \int_{-1}^2 \left(\pi \cos(\pi x) + 3zy^2 + 2z\right) dx dy dz$$

$$= \int_1^4 \int_0^1 \left[\sin(\pi x) + 3zy^2 x + 2zx\right]_{-1}^2 dy dz$$

$$= \int_1^4 \int_0^1 \left(6zy^2 + 4z\right) dy dz$$

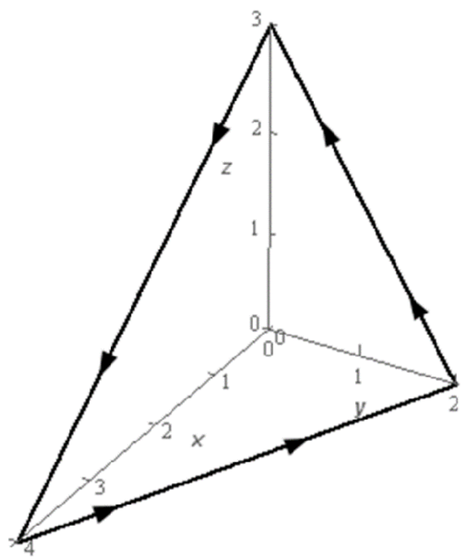
$$= \int_1^4 \left[2zy^3 + 4zy\right]_0^1 dz$$

$$= 6 \int_1^4 z dz = 3\left[z^2\right]_1^4 = 3\,(16 - 1) = 45$$

2 Question 2

2.1 Problem

Use Stokes' theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (3yx^2 + z^3)\,\hat{\imath} + y^2\hat{\jmath} + 4yx^2\hat{k}$ and C is the triangle with vertices $(0,0,3), (0,2,0)$ and $(4,0,0)$. C has a counter clockwise rotation if you are above the triangle and looking towards the xy-plane. See the figure below for a sketch of the curve



2.2 Solution

$$\iint_S \vec{\nabla} \times \boldsymbol{\vec{u}} \cdot \boldsymbol{\hat{n}} dS = \oint_C \boldsymbol{\vec{u}} \cdot d\vec{r}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3yx^2 + z^3 & y^2 & 4yx^2 \end{vmatrix} = 4x^2\hat{\imath} + \left(3z^2 - 8xy\right)\hat{\jmath} - 3x^2\hat{k}$$

The normal of the triangle is same as the normal of the plane that passes through the three vertices of the triangle.

The equation of the plane is

$$\frac{x}{4}+\frac{y}{2}+\frac{z}{3}=1$$

The normal of the plane and the triangle is

$$\vec{n}=\left(\frac{1}{4},\frac{1}{2},\frac{1}{3}\right)$$

$$\hat{n}=\frac{\vec{n}}{|\vec{n}|}=\frac{\frac{1}{4}\hat{\imath}+\frac{1}{2}\hat{\jmath}+\frac{1}{3}\hat{k}}{\sqrt{\frac{1}{16}+\frac{1}{4}+\frac{1}{9}}}=\frac{3\hat{\imath}+6\hat{\jmath}+4\hat{k}}{\sqrt{61}}$$

$$\vec{\nabla}\times\vec{F}\cdot\hat{n}=\left(4x^2\hat{\imath}+\left(3z^2-8xy\right)\hat{\jmath}-3x^2\hat{k}\right)\cdot\frac{3\hat{\imath}+6\hat{\jmath}+4\hat{k}}{\sqrt{61}}$$

$$=\frac{12x^2+18z^2-48xy-12x^2}{\sqrt{61}}=\frac{18z^2-48xy}{\sqrt{61}}$$

$$\iint_S\vec{\nabla}\times\vec{F}\cdot\hat{n}dS=\frac{6}{\sqrt{61}}\iint_S\left(3z^2-8xy\right)dS$$

$$\iint_S f(x,y,z)dS=\iint_D f(x,y,g(x,y))|\vec{\nabla}\emptyset|dxdy$$

$$g(x,y)=3-\frac{3}{4}x-\frac{3}{2}y$$

$$\emptyset=z-3+\frac{3}{4}x+\frac{3}{2}y$$

$$\vec{\nabla}\emptyset=\left(\frac{3}{4},\frac{3}{2},1\right)$$

$$|\vec{\nabla}\emptyset|=\sqrt{\left(\frac{3}{4}\right)^2+\left(\frac{3}{2}\right)^2+1^2}=\sqrt{\frac{9}{16}+\frac{9}{4}+1}=\sqrt{\frac{9+36+16}{16}}$$

$$= \sqrt{\frac{61}{16}} = \frac{\sqrt{61}}{4}$$

$$\iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n} dS = \frac{3}{2} \int_0^2 \int_0^{4-2y} \left(\frac{27}{16}(x + 2y - 4)^2 - 8xy \right) dxdy$$

$$= \frac{3}{2} \int_0^2 \left[\frac{27}{48}(x + 2y - 4)^3 - 4x^2 y \right]_0^{4-2y} dy$$

$$= -\frac{3}{2} \int_0^2 \left(16(y-2)^2 y + \frac{27}{6}(y-2)^3 \right) dy$$

$$= -\frac{3}{2} \int_0^2 \left(16 \left(y^3 - 4y^2 + 4y \right) + \frac{27}{6}(y-2)^3 \right) dy$$

$$= -\frac{3}{2} \left[16 \left(\frac{y^4}{4} - \frac{4}{3} y^3 + 2y^2 \right) + \frac{27}{24}(y-2)^4 \right]_0^2$$

$$= -\frac{3}{2} \left(16 \left(\frac{16}{4} - \frac{32}{3} + 8 \right) - \frac{27 * 16}{24} \right)$$

$$= -5$$

3 Question 3

3.1 Problem

Evaluate the line integral

$$I = \oint_C \left[y\left(4x^2 + y^2\right) dx + x\left(2x^2 + 3y^2\right) dy \right]$$

around the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

3.2 Solution

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$P = y\left(4x^2 + y^2\right), \quad Q = x\left(2x^2 + 3y^2\right)$$

$$\frac{\partial Q}{\partial x} = 6x^2 + 3y^2, \quad \frac{\partial P}{\partial y} = 4x^2 + 3y^2$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x^2$$

$$\iint_R 2x^2 dx dy = 2 \int_{-b}^{b} \int_{-a\sqrt{1-\frac{y^2}{b^2}}}^{a\sqrt{1-\frac{y^2}{b^2}}} x^2 dx dy$$

$$= \frac{2}{3} \int_{-b}^{b} \left[x^3\right]_{-a\sqrt{1-\frac{y^2}{b^2}}}^{a\sqrt{1-\frac{y^2}{b^2}}} dy = \frac{4}{3} a^3 \int_{-b}^{b} \left(1 - \frac{y^2}{b^2}\right)^{\frac{3}{2}} dy = \frac{\pi}{2} a^3 b$$

References

[1] M.H. El-Deeb. PEU-218 Assignments.