PEU 218 Assignment 2

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Contents

1 Question 1

1.1 Problem

Evaluate

$$\int_C \left(2yx^2 - 4x\right) dr$$

Where C is lower half of the circle centered at the origin of radius 3 with clockwise rotation.

1.2 Solution

$$x = 3\cos(\theta)$$

$$y = 3\sin(\theta)$$

$$\vec{r} = \langle x, y \rangle = \langle 3\cos(\theta), 3\sin(\theta) \rangle$$

$$d\vec{r} = 3\,\langle -\sin(\theta), \cos(\theta) \rangle\, d\theta$$

$$dr = 3d\theta$$

The limits of integration are $\theta = \pi$ to $\theta = 0$.

$$\int_C \left(2yx^2 - 4x\right) dr = 18 \int_\pi^0 \left(9\sin(\theta)\cos^2(\theta) - 2\cos(\theta)\right) d\theta$$

$$= 18\Big[3\cos^3(\theta) + 2\sin(\theta)\Big]_0^\pi = -108$$

2 Question 2

2.1 Problem

Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \langle y, 3y^3 - x, z \rangle$ and the path C is defined by $C(t) = \langle t, t^n, 0 \rangle$, $0 \leq t \leq 1$ where $n = 1, 2, 3, ..$

2.2 Solution

$$\vec{r} = \langle t, t^n, 0 \rangle$$

$$d\vec{r} = \left\langle 1, nt^{n-1}, 0 \right\rangle dt$$

$$\vec{F} = \left\langle t^n, 3t^{3n} - t, 0 \right\rangle$$

$$\vec{F} \cdot d\vec{r} = 3nt^{4n-1} + (1-n)t^n dt$$

$$\int_0^1 \left(3nt^{4n-1} + (1-n)t^n\right) dt$$

$$= \left[\frac{3}{4}t^{4n} + \frac{1-n}{1+n}t^{n+1}\right]_0^1 = \frac{3}{4} + \frac{1-n}{1+n}$$

3 Question 3

3.1 Problem

Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \langle xy, 1+3y, 0\rangle$ and C is the line segment from $(0,-4)$ to $(-2,-4)$ followed by portion of $y = -x^2$ from $x = -2$ to $x = 2$ which is in turn followed by the line segment from $(2,-4)$ to $(5,1)$.

3.2 Solution

I will reduce the dimension of the problem to 2D since $\vec{F}$ is the only 3D vector and its z component is 0.

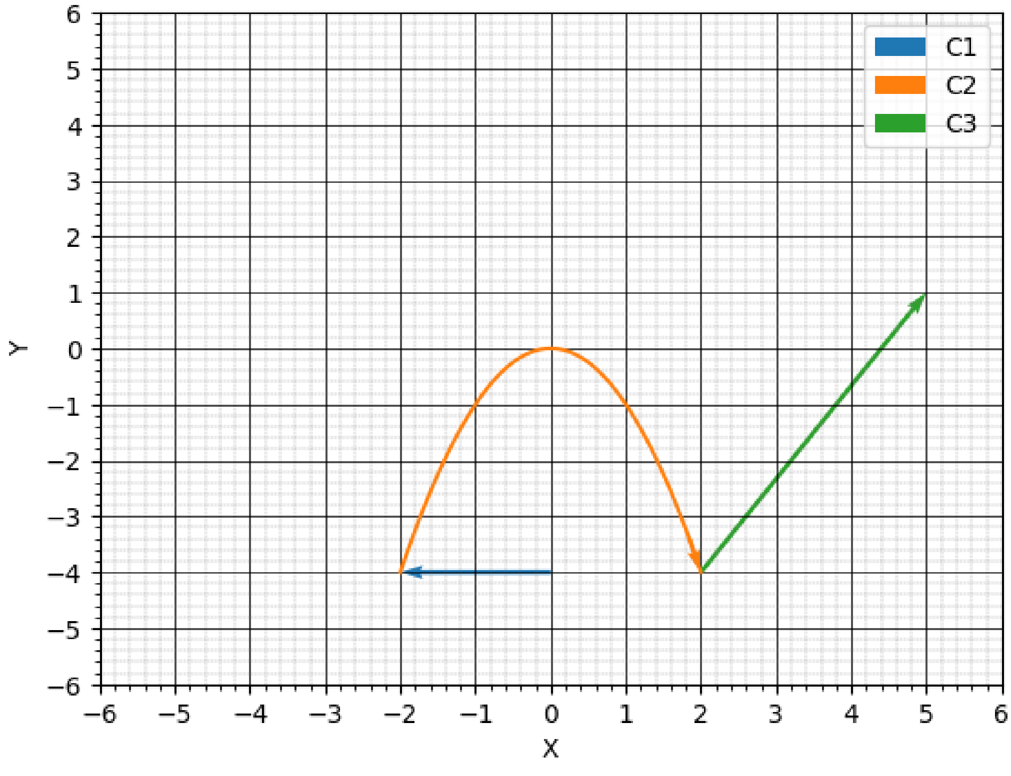

Figure 1: Path equations for the three contours.[2]

$$\vec{F} = \langle xy, 1+3y \rangle$$

$$C1 = \langle -2t, -4 \rangle, \quad t = [0,1]$$

$$\vec{F_1} = \langle 8t, -11 \rangle$$

$$d\vec{r_1} = \langle -2, 0 \rangle \, dt$$

$$C2 = \left\langle t, -t^2 \right\rangle, \quad t = [-2,2]$$

$$\vec{F_2} = \left\langle -t^3, 1-3t^2 \right\rangle$$

$$d\vec{r_2} = \langle 1, -2t \rangle \, dt$$

$$C3 = \langle 3t+2, 5t-4 \rangle, \quad t = [0,1]$$

$$\vec{F_3} = \left\langle 15t^2 - 2t - 8, 15t - 11 \right\rangle$$

$$d\vec{r_3} = \langle 3, 5 \rangle \, dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C1} \vec{F_1} \cdot d\vec{r_1} + \int_{C2} \vec{F_2} \cdot d\vec{r_2} + \int_{C3} \vec{F_3} \cdot d\vec{r_3}$$

$$= \int_{-2}^{2} \left(5t^3 - 2t\right) dt + \int_0^1 \left(45t^2 + 53t - 79\right) dt$$

$$= \left[\frac{5}{4}t^4 - t^2\right]_{-2}^{2} + \left[15t^3 + \frac{53}{2}t^2 - 79t\right]_0^1 = -37.5$$

4 Question 4

4.1 Problem

Evaluate $\iint_S 2y \; dS$ where S is the portion $y^2 + z^2 = 4$ between $x = 0$ and $x = 3 - z$.

4.2 Solution

$$x = x, \quad y = 2\cos(\theta), \quad z = 2\sin(\theta)$$

$$d\vec{S} = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial \theta}$$

$$\vec{r} = \langle x, 2\cos(\theta), 2\sin(\theta) \rangle$$

$$\frac{\partial \vec{r}}{\partial x} = \langle 1, 0, 0 \rangle \, dx, \quad \frac{\partial \vec{r}}{\partial \theta} = 2 \langle 0, -\sin(\theta), \cos(\theta) \rangle \, d\theta$$

$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial \theta} = 2 \langle 0, \cos(\theta), \sin(\theta) \rangle \, d\theta dx$$

$$dS = \left\| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial \theta} \right\| = 2 dx d\theta$$

$$\iint_S 2y \; dS = 8 \int_0^{2\pi} \int_0^{3-2\sin(\theta)} \cos(\theta) dx d\theta$$

$$= 8 \int_0^{2\pi} [3 - 2\sin(\theta)] \cos(\theta) d\theta$$

$$= 8 \int_0^{2\pi} [3\cos(\theta) - 2\sin(\theta)\cos(\theta)] \, d\theta$$

$$= 8 \left[3\sin(\theta) + \cos^2(\theta) \right]_0^{2\pi} = 0$$

5 Question 5

5.1 Problem

Let the temperature of a point in space be given by $T(x, y, z) = 3x^2 + 3z^2$. Compute the heat flux across the surface $x^2 + z^2 = 2, 0 \leq y \leq 2$, if $k = 1$. (Give a physical explanation to justify the sign of your result?)

5.2 Solution

The heat flux is given by $-k\nabla T$.

$$-\iint_S \nabla T \cdot d\vec{S}$$

$$\nabla T = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right\rangle$$

$$= 6 \langle x, 0, z \rangle$$

$$x = \sqrt{2}\cos(\theta), \quad y = y, \quad z = \sqrt{2}\sin(\theta)$$

$$d\vec{S} = \frac{\partial \vec{r}}{\partial y} \times \frac{\partial \vec{r}}{\partial \theta}$$

$$\vec{r} = \left\langle \sqrt{2}\cos(\theta), y, \sqrt{2}\sin(\theta) \right\rangle$$

$$\frac{\partial \vec{r}}{\partial y} = \langle 0, 1, 0 \rangle \, dy, \quad \frac{\partial \vec{r}}{\partial \theta} = \sqrt{2} \langle -\sin(\theta), 0, \cos(\theta) \rangle \, d\theta$$

$$d\vec{S} = \frac{\partial \vec{r}}{\partial y} \times \frac{\partial \vec{r}}{\partial \theta} = \sqrt{2} \langle \cos(\theta), 0, \sin(\theta) \rangle \, dy d\theta$$

$$\nabla T \cdot d\vec{S} = 12 \langle \cos(\theta), 0, \sin(\theta) \rangle \cdot \langle \cos(\theta), 0, \sin(\theta) \rangle \, dy d\theta = 12 dy d\theta$$

$$-\iint_S \nabla T \cdot d\vec{S} = -12 \int_0^{2\pi} \int_0^2 dy d\theta = -48\pi$$

The negative sign indicates that the heat is flowing into the surface. This is because heat moves from higher temperature in this case outside since temperature increases as we get away radially from x and z origin to lower temperature inside the surface.

6 Question 6

6.1 Problem

Evaluate $\iint_S \vec{F} \cdot dS$ where $\vec{F} = y\hat{\imath} + 2x\hat{\jmath} + (z-8)\hat{k}$ and S is the surface of the solid bounded by $4x + 2y + z = 8, z = 0, y = 0$ and $x = 0$ with the positive orientation. Note that all four surfaces of the solid are included in S.

6.2 Solution

$$\iint_S \vec{F} \cdot dS = \iint_{S_1} \vec{F} \cdot dS_1 + \iint_{S_2} \vec{F} \cdot dS_2 + \iint_{S_3} \vec{F} \cdot dS_3 + \iint_{S_4} \vec{F} \cdot dS_4$$

For $x = 0$,

$$S_1 : 4x + 2y + z = 8 \rightarrow z = 8 - 2y$$

$$\iint_{S_1} \vec{F} \cdot dS_1 = \int_0^4 \int_0^{8-2y} \left(\vec{F} \cdot \hat{\imath}\right) dzdy$$

$$= \int_0^4 \int_0^{8-2y} ydzdy = \int_0^4 y(8-2y)dy = \left[4y^2 - \frac{2}{3}y^3\right]_0^4 = \frac{64}{3}$$

For $y = 0$,

$$S_2 : 4x + 2y + z = 8 \rightarrow z = 8 - 4x$$

$$\iint_{S_2} \vec{F} \cdot dS_2 = \int_0^2 \int_0^{8-4x} \left(\vec{F} \cdot \hat{\jmath}\right) dzdx$$

$$= \int_0^2 \int_0^{8-4x} 2xdzdx = \int_0^2 2x(8-4x)dx = \left[8x^2 - \frac{4}{3}x^3\right]_0^2 = \frac{64}{3}$$

For $z = 0$,

$$S_3 : 4x + 2y + z = 8 \to y = 4 - 2x$$

$$\iint_{S_3} \vec{F} \cdot dS_3 = \int_0^2 \int_0^{4-x} \left(\vec{F} \cdot \hat{k} \right) dydx$$

$$= -8 \int_0^2 \int_0^{4-x} dydx = -8 \int_0^2 (4-x)dx = -8 \left[4x - \frac{1}{2}x^2 \right]_0^2 = -48$$

for $\hat{n} = \frac{1}{\sqrt{21}} \langle 4, 2, 1 \rangle$,

$$\iint_{S_4} \vec{F} \cdot dS_4 = \iint_{S_4} \left(\vec{F} \cdot \hat{n} \right) |\nabla f| \, dxdy$$

$$\iint_{S_4} \vec{F} \cdot dS_4 = \iint_{S_4} (2y) \, dxdy$$

$$= \int_0^{-4} \int_0^{4+y} 2ydxdy = \int_0^{-4} 2y(4+y)dy = \frac{32}{3}$$

$$\iint_S \vec{F} \cdot dS = \frac{64}{3} + \frac{64}{3} - 48 + \frac{32}{3} = \frac{16}{3}$$

References

[1] M.H. El-Deeb. PEU-218 Assignments.

[2] M.H. El-Deeb. PEU-218 Assignments (Python).