PEU 218 Assignment 4

Mohamed Hussien El-Deeb (201900052)

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1 Question 1

1.1 Problem

Paraboloidal coordinates u, v, φ are defined in terms of Cartesian coordinates

$$x = \alpha\beta\cos\varphi, \quad y = \alpha\beta\sin\varphi, \quad z = \frac{1}{2}\left(\alpha^2 - \beta^2\right)$$
$$\text{where } 0 \leq \alpha \leq \infty, \quad 0 \leq \beta \leq \infty, \quad 0 \leq \varphi \leq 2\pi$$

Prove that the α -component of $\vec{\nabla} \times \vec{A}$ is

$$\frac{1}{\left(\alpha^2 + \beta^2\right)^{1/2}} \left(\frac{A_\varphi}{\beta} + \frac{\partial\varphi}{\partial\beta}\right) - \frac{1}{\alpha\beta}\frac{\partial A_\beta}{\partial\varphi}$$

1.2 Solution

$$\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$$

$$(\vec{\nabla} \times \vec{A})_\alpha = \frac{1}{h_\beta h_\varphi}\left(\frac{\partial}{\partial\beta}\left(A_\varphi h_\varphi\right) - \frac{\partial}{\partial\varphi}\left(A_\beta h_\beta\right)\right)$$

$$= \frac{1}{h_\beta h_\varphi}\left(\frac{\partial A_\varphi}{\partial\beta} h_\varphi + \frac{\partial h_\varphi}{\partial\beta} A_\varphi - \frac{\partial A_\beta}{\partial\varphi} h_\beta - \frac{\partial h_\beta}{\partial\varphi} A_\beta\right)$$

$$h_\beta = \left|\frac{\partial\vec{r}}{\partial\beta}\right| = \sqrt{\left(\frac{\partial x}{\partial\beta}\right)^2 + \left(\frac{\partial y}{\partial\beta}\right)^2 + \left(\frac{\partial z}{\partial\beta}\right)^2}$$

$$= \sqrt{\left(\alpha\cos\varphi\right)^2 + \left(\alpha\sin\varphi\right)^2 + \left(\beta\right)^2} = \sqrt{\alpha^2 + \beta^2}$$

$$h_\varphi = \left|\frac{\partial\vec{r}}{\partial\varphi}\right| = \sqrt{\left(\frac{\partial x}{\partial\varphi}\right)^2 + \left(\frac{\partial y}{\partial\varphi}\right)^2 + \left(\frac{\partial z}{\partial\varphi}\right)^2}$$

$$= \sqrt{\left(\alpha\beta\sin\varphi\right)^2 + \left(\alpha\beta\cos\varphi\right)^2} = \alpha\beta$$

$$\frac{\partial h_\beta}{\partial \varphi} = \frac{\partial}{\partial \varphi}\left(\sqrt{\alpha^2+\beta^2}\right) = 0$$

$$\frac{\partial h_\varphi}{\partial \beta} = \frac{\partial}{\partial \beta}\left(\alpha\beta\right) = \alpha$$

$$(\vec{\nabla}\times\vec{A})_\alpha = \frac{1}{\alpha\beta\sqrt{\alpha^2+\beta^2}}\left(\frac{\partial A_\varphi}{\partial \beta}\alpha\beta + \alpha A_\varphi - \frac{\partial A_\beta}{\partial \varphi}\sqrt{\alpha^2+\beta^2}\right)$$

$$= \frac{1}{(\alpha^2+\beta^2)^{1/2}}\left(\frac{A_\varphi}{\beta} + \frac{\partial A_\varphi}{\partial \beta}\right) - \frac{1}{\alpha\beta}\frac{\partial A_\beta}{\partial \varphi}$$

2 Question 2

2.1 Problem

Using spherical coordinates, evaluate

$$\nabla^2 \left(\vec{\nabla} \cdot \frac{\vec{r}}{r^2} \right)$$

2.2 Solution

$$\vec{r} = \langle r, 0, 0 \rangle$$

$$\frac{\vec{r}}{r^2} = \vec{r} = \left\langle \frac{1}{r}, 0, 0 \right\rangle$$

$$\vec{\nabla} \cdot \vec{u} = \frac{1}{r^2} \frac{\partial \left(r^2 u_r \right)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial \left(\sin\theta u_\theta \right)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial \left(u_\varphi \right)}{\partial \varphi}$$

$$\vec{\nabla} \cdot \frac{\vec{r}}{r^2} = \frac{1}{r^2} \frac{\partial r}{\partial r} = \frac{1}{r^2}$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \varphi^2}$$

$$\nabla^2 \frac{1}{r^2} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \frac{1}{r^2}}{\partial r} \right) = -\frac{2}{r^2} \frac{\partial}{\partial r} \left(r^{-1} \right) = \frac{2}{r^4}$$

$$\nabla^2 \left(\vec{\nabla} \cdot \frac{\vec{r}}{r^2} \right) = \frac{2}{r^4}$$

3 Question 3

3.1 Problem

We introduce the so-called spheroidal coordinates (η, θ, φ) by the following equations expressed in rectangular coordinates

$$x = a \sinh \eta \sin \theta \cos \varphi,$$
$$y = a \sinh \eta \sin \theta \sin \varphi,$$
$$z = a \cosh \eta \cos \theta$$

where $0 \leq \eta \leq \infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi$

a) Show that this coordinate system is orthogonal.

b) Find the scale factors.

c) Show that the function $f(\eta, \theta, \varphi) = \ln \tanh \left(\frac{\eta}{2}\right)$ is a solution of Laplace's equation.

3.2 Solution

a)

$$\vec{r} = a \left\langle \sinh \eta \sin \theta \cos \varphi, \sinh \eta \sin \theta \sin \varphi, \cosh \eta \cos \theta \right\rangle$$

$$\vec{e}_\eta = \frac{\partial \vec{r}}{\partial \eta} = a \left\langle \cosh \eta \sin \theta \cos \varphi, \cosh \eta \sin \theta \sin \varphi, \sinh \eta \cos \theta \right\rangle$$

$$\vec{e}_\theta = \frac{\partial \vec{r}}{\partial \theta} = a \left\langle \sinh \eta \cos \theta \cos \varphi, \sinh \eta \cos \theta \sin \varphi, -\cosh \eta \sin \theta \right\rangle$$

$$\vec{e}_\varphi = \frac{\partial \vec{r}}{\partial \varphi} = a \left\langle -\sinh \eta \sin \theta \sin \varphi, \sinh \eta \sin \theta \cos \varphi, 0 \right\rangle$$

$$\frac{\partial \vec{r}}{\partial \eta} \cdot \frac{\partial \vec{r}}{\partial \theta} = a^2 \left(\sinh \eta \cosh \eta \sin \theta \cos \theta - \sinh \eta \cosh \eta \sin \theta \cos \theta \right) = 0$$

$$\frac{\partial \vec{r}}{\partial \eta} \cdot \frac{\partial \vec{r}}{\partial \varphi} =$$

$$a^2 \left(- \sinh \eta \cosh \eta \sin \varphi \cos \varphi \sin^2 \theta + \sinh \eta \cosh \eta \sin \varphi \cos \varphi \sin^2 \theta \right) = 0$$

$$\frac{\partial \vec{r}}{\partial \theta} \cdot \frac{\partial \vec{r}}{\partial \varphi} =$$

$$a^2 \left(- \sinh^2 \eta \sin \theta \cos \theta \sin \varphi \cos \varphi + \sinh^2 \eta \sin \theta \cos \theta \sin \varphi \cos \varphi \right) = 0$$

Since the dot product of different basis vectors is zero, the coordinate system is orthogonal.

b)

$$\cosh^2 \eta = 1 + \sinh^2 \eta$$

$$h_\eta = \left| \frac{\partial \vec{r}}{\partial \eta} \right| = a\sqrt{\cosh^2 \eta \sin^2 \theta + \sinh^2 \eta \cos^2 \theta}$$

$$= a\sqrt{(1 + \sinh^2 \eta) \sin^2 \theta + \sinh^2 \eta \cos^2 \theta} = a\sqrt{\sinh^2 \eta + \sin^2 \theta}$$

$$h_\theta = \left| \frac{\partial \vec{r}}{\partial \theta} \right| = a\sqrt{\sinh^2 \eta \cos^2 \theta + \cosh^2 \eta \sin^2 \theta} = h_\eta$$

$$h_\varphi = \left| \frac{\partial \vec{r}}{\partial \varphi} \right| = a \sinh \eta \sin \theta$$

c)

$$\nabla^2 f = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$$

In this coordinate system, $h_1 = h_2 = h_\eta, h_3 = h_\varphi$,

$$\nabla^2 f = \frac{1}{h_\eta^2 h_\varphi}\left[\frac{\partial}{\partial \eta}\left(h_\varphi\frac{\partial f}{\partial \eta}\right) + \frac{\partial}{\partial \theta}\left(h_\varphi\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{h_\eta^2}{h_\varphi}\frac{\partial f}{\partial \varphi}\right)\right]$$

Since we are solving laplace equation, we need only to focus on what is inside the square brackets.

$$\frac{\partial}{\partial \eta}\left(h_\varphi\frac{\partial f}{\partial \eta}\right) + \frac{\partial}{\partial \theta}\left(h_\varphi\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{h_\eta^2}{h_\varphi}\frac{\partial f}{\partial \varphi}\right)$$

Since f is a function of η only, we can take our solution to be a function of only η so second and third terms will be zero.

$$\frac{\partial}{\partial \eta}\left(h_\varphi\frac{\partial f}{\partial \eta}\right)$$

We could also induce that,

$$h_\varphi\frac{\partial f}{\partial \eta} = g(\theta, \varphi)$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial}{\partial \eta}\left(\ln\tanh\left(\frac{\eta}{2}\right)\right) = \frac{1}{2}\operatorname{sech}^2\frac{\eta}{2}\coth\left(\frac{\eta}{2}\right) = \frac{1}{2}\operatorname{csch}\frac{\eta}{2}\operatorname{sech}\frac{\eta}{2} = \operatorname{csch}\eta$$

$$h_\varphi\frac{\partial f}{\partial \eta} = a\sin\theta\sinh\eta\operatorname{csch}\eta = a\sin\theta$$

Since g is not a function of η, we could deduce that f is a solution of Laplace's equation.

References

[1] M.H. El-Deeb. PEU-218 Assignments.