

PEU 323 Assignment 3

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1 Problem 1

$$\lambda = \frac{h}{p}$$

$$E_K = \frac{mv^2}{2} = \frac{p^2}{2m}$$

Where E_K is the kinetic energy.

$$p = \sqrt{2mE_K}$$

$$E_K = \frac{f}{2}K_B T$$

Where f is 3 for nitrogen atoms under room temperature ($T = 293.15K$) and $K_B = 1.380649 \times 10^{-23} J.K^{-1}$ is boltzmann constant.

$$E_K = \frac{3}{2}K_B T$$

$$p = \sqrt{3mK_B T} = \sqrt{3 \times 10^{-18} kg \times 1.380649 \times 10^{-23} J.K^{-1} \times 293.15 K}$$

$$= \sqrt{3 \times 10^{-18} \times 1.380649 \times 10^{-23} \times 293.15} \sqrt{kg.J} \approx 10^{-19} \frac{kg.m}{s}$$

$$\lambda = \frac{h}{\sqrt{3mK_B T}}$$

$$h = 6.62607015 \times 10^{-34} J.s$$

$$m = 10^{-18} kg$$

$$kgJ = \frac{kg^2m^2}{s^2}$$

$$\lambda = \frac{6.62607015 \times 10^{-34}}{\sqrt{3 \times 10^{-18} \times 1.380649 \times 10^{-23} \times 293.15}} \frac{J.s}{\sqrt{kgJ}}$$

$$\lambda \approx 6.0 \times 10^{-15}m$$

$$d = 10^{-7}m$$

$$\frac{h}{pd} = \frac{6.62607015 \times 10^{-34}}{10^{-7} \times 10^{-19}} \frac{J.s^2}{kg.m^2} = 6.6 \times 10^{-8}$$

Since the $\Delta p \Delta d \gg h$ We don't need to use quantum treatment.

2 Problem 2

$$\int_{-\infty}^{\infty} f(y) f(y)^* dy = c$$
$$F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$\begin{aligned} \int F(k) F(k)^* dk &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x) e^{-ikx} dx \int_{-\infty}^{\infty} f(y)^* e^{iky} dy \right] dk \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) f(y)^* e^{ik(y-x)} dx dy \right] dk \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) f(y)^* \int_{-\infty}^{\infty} e^{ik(y-x)} dk dx dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) f(y)^* \delta(y-x) dx dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) f(y)^* dy = \frac{c}{2\pi} \end{aligned}$$

We can see that the $F(k)$ is normalizable if c is finite.

3 Problem 3

$$\frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = i\hbar \frac{\partial \psi}{\partial t}$$

$$\Psi(x, t) = \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \omega(k)t)} dk$$

$$\frac{i\hbar}{2m} \frac{\partial^2 (\int_{-\infty}^{\infty} \phi(k) e^{i(kx - \omega(k)t)} dk)}{\partial x^2} = \frac{\partial (\int_{-\infty}^{\infty} \phi(k) e^{i(kx - \omega(k)t)} dk)}{\partial t}$$

$$\frac{\hbar}{2m} \int_{-\infty}^{\infty} k^2 \phi(k) e^{i(kx - \omega(k)t)} dk = \int_{-\infty}^{\infty} w(k) \phi(k) e^{i(kx - \omega(k)t)} dk$$

$$w(k) = \frac{k^2 \hbar}{2m}$$

We can clearly see that both sides of the equation are equal therefore Schrodinger equation is satisfied.

4 Problem 4

(a) So that in the integral of the probability density $f \cdot f^* = 1$ leaving us with the same normalization constant.

(b)

$$\begin{aligned} x' &= x - vt, & t' &= t \\ \frac{\partial x}{\partial x'} &= 1, & \frac{\partial t}{\partial x'} &= 0, & \frac{\partial x}{\partial t'} &= v, & \frac{\partial t}{\partial t'} &= 1 \\ \psi'(x', t') &= f(x, t)\psi(x, t) \\ \frac{-\hbar^2}{2m} \frac{\partial^2 \psi'}{\partial x'^2} &= i\hbar \frac{\partial \psi'}{\partial t'} \end{aligned}$$

$$\begin{aligned} \frac{\partial \psi'}{\partial x'} &= \frac{\partial(f(x, t)\psi(x, t))}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial(f(x, t)\psi(x, t))}{\partial t} \frac{\partial t}{\partial x'} \\ &= \frac{\partial(f(x, t)\psi(x, t))}{\partial x} \\ &= \frac{\partial f(x, t)}{\partial x} \psi(x, t) + f(x, t) \frac{\partial \psi(x, t)}{\partial x} \\ \frac{\partial^2 \psi'}{\partial x'^2} &= \partial_x^2[f(x, t)]\psi(x, t) + 2\partial_x[f(x, t)]\partial_x[\psi(x, t)] + f(x, t)\partial_x^2[\psi(x, t)] \end{aligned}$$

$$\begin{aligned} \frac{\partial \psi'}{\partial t'} &= \frac{\partial(f(x, t)\psi(x, t))}{\partial x} \frac{\partial x}{\partial t'} + \frac{\partial(f(x, t)\psi(x, t))}{\partial t} \frac{\partial t}{\partial t'} \\ &= v \frac{\partial(f(x, t)\psi(x, t))}{\partial x} + \frac{\partial(f(x, t)\psi(x, t))}{\partial t} \\ &= v \left(\frac{\partial f(x, t)}{\partial x} \psi(x, t) + f(x, t) \frac{\partial \psi(x, t)}{\partial x} \right) + \frac{\partial f(x, t)}{\partial t} \psi(x, t) + f(x, t) \frac{\partial \psi(x, t)}{\partial t} \end{aligned}$$

$$\begin{aligned} \frac{-\hbar^2}{2m} & \left(\partial_x^2[f(x, t)]\psi(x, t) + 2\partial_x[f(x, t)]\partial_x[\psi(x, t)] + f(x, t)\partial_x^2[\psi(x, t)] \right) \\ &= i\hbar \left(v (\partial_x[f(x, t)]\psi(x, t) + f(x, t)\partial_x[\psi(x, t)]) + \partial_t[f(x, t)]\psi(x, t) + f(x, t)\partial_t[\psi(x, t)] \right) \end{aligned}$$

Using

$$\frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = i\hbar \frac{\partial \psi}{\partial t}$$

$$\begin{aligned} & \frac{-\hbar^2}{2m} (\partial_x^2[f(x, t)]\psi(x, t) + 2\partial_x[f(x, t)]\partial_x[\psi(x, t)]) \\ & = i\hbar (v (\partial_x[f(x, t)]\psi(x, t) + f(x, t)\partial_x[\psi(x, t)]) + \partial_t[f(x, t)]\psi(x, t)) \end{aligned}$$

$$\begin{aligned} & \left(\frac{\hbar^2}{2m} \partial_x^2[f(x, t)] + i\hbar\partial_t[f(x, t)] + i\hbar v\partial_x[f(x, t)] \right) \psi(x, t) \\ & + \left(\frac{\hbar^2}{m} \partial_x[f(x, t)] + i\hbar v f(x, t) \right) \partial_x[\psi(x, t)] = 0 \end{aligned}$$

$$A = \frac{\hbar^2}{2m} \partial_x^2[f(x, t)] + i\hbar\partial_t[f(x, t)] + i\hbar v\partial_x[f(x, t)]$$

$$B = \frac{\hbar^2}{m} \partial_x[f(x, t)] + i\hbar v f(x, t)$$

(c) Since ψ is an arbitrary function A and B must be zero to satisfy the previous equation.

Using B

$$\frac{\hbar^2}{m} \partial_x[f(x, t)] = -i\hbar v f(x, t)$$

$$\frac{\partial_x[f(x, t)]}{f(x, t)} = -\frac{imv}{\hbar}$$

$$\int \frac{df(x, t)}{f(x, t)} = -\frac{imv}{\hbar} \int dx$$

$$\ln(f(x, t)) = -\frac{imv}{\hbar} x + c(t)$$

$$f(x, t) = c(t)e^{-\frac{imv}{\hbar} x}$$

Using A

$$\partial_x[f(x, t)] = -\frac{imv}{\hbar}c(t)e^{-\frac{imv}{\hbar}x}$$

$$\partial_x^2[f(x, t)] = -\left(\frac{mv}{\hbar}\right)^2 c(t)e^{-\frac{imv}{\hbar}x}$$

$$\frac{\hbar^2}{2m}\partial_x^2[f(x, t)] + i\hbar\partial_t[f(x, t)] + i\hbar v\partial_x[f(x, t)] = 0$$

$$i\hbar\partial_t[f(x, t)] = \frac{\hbar^2}{2m}\left(\frac{mv}{\hbar}\right)^2 c(t)e^{-\frac{imv}{\hbar}x} + i\hbar v\frac{imv}{\hbar}c(t)e^{-\frac{imv}{\hbar}x}$$

$$i\hbar\partial_t[f(x, t)] = \frac{mv^2}{2}c(t)e^{-\frac{imv}{\hbar}x} - mv^2c(t)e^{-\frac{imv}{\hbar}x}$$

$$= -\frac{mv^2}{2}c(t)e^{-\frac{imv}{\hbar}x}$$

$$\partial_t[f(x, t)] = \frac{imv^2}{2\hbar}c(t)e^{-\frac{imv}{\hbar}x}$$

$$\partial_t[f(x, t)] = \frac{imv^2}{2\hbar}f(x, t)$$

$$\partial_t[c(t)] = \frac{imv^2}{2\hbar}c(t)$$

$$c(t) = ge^{\frac{imv^2}{2\hbar}t}$$

$$f(x, t) = ge^{-\frac{imv}{\hbar}x}e^{\frac{imv^2}{2\hbar}t} = ge^{-\frac{imv}{\hbar}(x - \frac{1}{2}vt)}$$

$$f(x, t)f^*(x, t) = |g|^2 = 1$$

$$f(x, t) = e^{-\frac{imv}{\hbar}(x - \frac{1}{2}vt + \phi)}$$

$$f(x, t) = ge^{-\frac{i}{\hbar}\left(p_{S'}x - \frac{p_{S'}^2}{2m}t\right)}$$

(d)

$$\psi(x, t) = Ne^{\frac{i}{\hbar}\left(px - \frac{p^2}{2m}t\right)}$$

$$\psi'(x', t') = f(x, t)\psi(x, t)$$

$$= Nge^{\frac{i}{\hbar}\left((p-p_{S'})x - \frac{(p^2-p_{S'}^2)}{2m}t\right)}$$

$$x = x' + vt = x' + \frac{p_{S'}}{m}t', \quad t = t'$$

$$\psi'(x', t') = Nge^{\frac{i}{\hbar}\left((p-p_{S'})(x' + \frac{p_{S'}}{m}t') - \frac{(p^2-p_{S'}^2)}{2m}t'\right)}$$

$$= Nge^{\frac{i}{\hbar}\left((p-p_{S'})x' - \frac{(p^2-2pp_{S'}+p_{S'}^2)}{2m}t'\right)}$$

$$= Nge^{\frac{i}{\hbar}\left((p-p_{S'})x' - \frac{(p-p_{S'})^2}{2m}t'\right)}$$

The new momentum is $p - p_{S'}$ and the new energy is $\frac{(p-p_{S'})^2}{2m}$.

5 Problem 5

$$\frac{d}{dt}\langle Q \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{Q}] \rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle$$

$$\frac{d}{dt}\langle xp \rangle = 2\langle T \rangle - \left\langle x \frac{\partial V}{\partial x} \right\rangle$$

$$\frac{d}{dt}\langle xp \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{x}\hat{p}] \rangle + \left\langle \frac{\partial(xp)}{\partial t} \right\rangle$$

$$\left\langle \frac{\partial(xp)}{\partial t} \right\rangle = 0$$

$$\frac{d}{dt}\langle xp \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{x}\hat{p}] \rangle$$

$$\hat{p} = -i\hbar \frac{d}{dx}, \quad \hat{x} = x$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + V$$

$$\begin{aligned} [\hat{H}, \hat{x}\hat{p}]\psi &= \left[\frac{\hat{p}^2}{2m} + V, x\hat{p} \right]\psi \\ &= \left[\frac{\hat{p}^2}{2m}, x\hat{p} \right]\psi + [V, x\hat{p}]\psi \end{aligned}$$

$$[V, x\hat{p}]\psi = [Vx\hat{p} - x\hat{p}V]\psi = -xi\hbar[V\frac{d\psi}{dx} - \frac{d(V\psi)}{dx}]$$

$$= -xi\hbar[V\frac{d\psi}{dx} - \frac{dV}{dx}\psi - V\frac{d\psi}{dx}]$$

$$= xi\hbar \frac{dV}{dx} \psi$$

$$\begin{aligned} [\frac{\hat{p}^2}{2m}, x\hat{p}]\psi &= [\frac{\hat{p}^2}{2m}, x]\hat{p} + x[\frac{\hat{p}^2}{2m}, \hat{p}] \\ &= \frac{1}{2m}(\hat{p}[\hat{p}, x]\hat{p} + [\hat{p}, x]\hat{p}^2 + x[p^2, \hat{p}]) \\ &= \frac{1}{2m}(2\hat{p}^2[\hat{p}, x] + x[p^2, \hat{p}]) \\ &= \frac{1}{2m}(2\hat{p}^2(-i\hbar) + 0) \\ &= -i\hbar \frac{\hat{p}^2}{m} \end{aligned}$$

$$[\hat{H}, \hat{x}\hat{p}] = -i\hbar \frac{\hat{p}^2}{m} + i\hbar x \frac{dV}{dx}$$

$$\begin{aligned} \frac{d}{dt}\langle xp \rangle &= \frac{i}{\hbar} \langle [\hat{H}, \hat{x}\hat{p}] \rangle \\ &= \langle \frac{\hat{p}^2}{m} \rangle - \langle x \frac{dV}{dx} \rangle \\ &= 2\langle T \rangle - \langle x \frac{dV}{dx} \rangle \end{aligned}$$

The left hand side with vanish because p and x are not functions of time.

6 Problem 6

$$\hat{T} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}$$

$$\langle T \rangle = \int_{-\infty}^{\infty} \psi^*(x) \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \right) \psi(x) dx$$

$$\begin{aligned} \langle T \rangle &= -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \psi^*(x) \frac{d^2 \psi(x)}{dx^2} dx \\ &= \frac{\hbar^2}{2m} \left[\psi^*(x) \frac{d\psi(x)}{dx} \right]_{-\infty}^{\infty} + \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \left| \frac{d\psi(x)}{dx} \right|^2 dx \\ &= \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \left| \frac{d\psi(x)}{dx} \right|^2 dx \end{aligned}$$

The boundary conditions are zero as both ψ and its derivative are zero at the infinities.

$$\therefore \langle T \rangle \geq 0$$

7 Problem 7

(a)

$$\sigma_A^2 = \langle (\hat{A} - \langle A \rangle) \Psi | (\hat{A} - \langle A \rangle) \Psi \rangle = \langle f | f \rangle$$

$$\sigma_B^2 = \langle g | g \rangle$$

$$\sigma_A^2 \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2$$

$$|z|^2 = [\operatorname{Re}(z)]^2 + [\operatorname{Im}(z)]^2 = \left[\frac{1}{2}(z + z^*) \right]^2 + \left[\frac{1}{2i}(z - z^*) \right]^2$$

$$|\langle f | g \rangle|^2 = \left[\frac{1}{2}(\langle f | g \rangle + \langle g | f \rangle) \right]^2 + \left[\frac{1}{2i}(\langle f | g \rangle - \langle g | f \rangle) \right]^2$$

$$\sigma_A^2 \sigma_B^2 \geq \left[\frac{1}{2}(\langle f | g \rangle + \langle g | f \rangle) \right]^2 + \left[\frac{1}{2i}(\langle f | g \rangle - \langle g | f \rangle) \right]^2$$

$$\langle f | g \rangle = \langle \hat{A} \hat{B} \rangle - \langle A \rangle \langle B \rangle, \quad \langle g | f \rangle = \langle \hat{B} \hat{A} \rangle - \langle A \rangle \langle B \rangle$$

$$\begin{aligned} |\operatorname{Re}|^2 &= \left[\frac{1}{2}(\langle \hat{A} \hat{B} \rangle - \langle A \rangle \langle B \rangle + \langle \hat{B} \hat{A} \rangle - \langle A \rangle \langle B \rangle) \right]^2 \\ &= \left[\frac{1}{2}(\langle \hat{A} \hat{B} \rangle - 2\langle A \rangle \langle B \rangle + \langle \hat{B} \hat{A} \rangle) \right]^2 \\ &= \left[\frac{1}{2}(\langle \hat{A} \hat{B} - 2\langle A \rangle \langle B \rangle + \hat{B} \hat{A} \rangle) \right]^2 \\ &= \frac{1}{4} \langle \hat{D} \rangle^2 \end{aligned}$$

$$\begin{aligned}
|\text{Im}|^2 &= \left[\frac{1}{2i} (\langle \hat{A}\hat{B} \rangle - \langle A \rangle \langle B \rangle - \langle \hat{B}\hat{A} \rangle + \langle A \rangle \langle B \rangle) \right]^2 \\
&= \left[\frac{1}{2i} (\langle \hat{A}\hat{B} \rangle - \langle \hat{B}\hat{A} \rangle) \right]^2 \\
&= \left[\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right]^2 \\
&= \frac{1}{4} \langle \hat{C} \rangle^2
\end{aligned}$$

$$\sigma_A^2 \sigma_B^2 \geq \frac{1}{4} (\langle \hat{C} \rangle^2 + \langle \hat{D} \rangle^2)$$

(b)

$$\begin{aligned}
\langle \hat{C} \rangle^2 &= \left[\langle [\hat{A}, \hat{B}] \rangle \right]^2 \\
&= \left[\langle [\hat{A}, \hat{A}] \rangle \right]^2 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\langle \hat{D} \rangle^2 &= \left[\langle \hat{A}\hat{B} - 2\langle A \rangle \langle B \rangle + \hat{B}\hat{A} \rangle \right]^2 \\
&= \left[\langle 2\hat{A}^2 - 2\langle A \rangle^2 \rangle \right]^2 \\
&= 4 \left[\langle \hat{A}^2 \rangle - \langle A \rangle^2 \right]^2 \\
&= 4\sigma_A^4
\end{aligned}$$

$$\sigma_A^2 = \frac{1}{2} \langle \hat{D} \rangle$$

8 Problem 8

$$\langle h|\hat{Q}h\rangle = \langle \hat{Q}h|h\rangle$$

For $h = f + g$,

$$\langle (f + g)|\hat{Q}(f + g)\rangle = \langle \hat{Q}(f + g)|(f + g)\rangle$$

$$\langle f + g|\hat{Q}f + \hat{Q}g\rangle = \langle \hat{Q}f + \hat{Q}g|f + g\rangle$$

$$\langle f|\hat{Q}f\rangle + \langle f|\hat{Q}g\rangle + \langle g|\hat{Q}f\rangle + \langle g|\hat{Q}g\rangle = \langle \hat{Q}f|f\rangle + \langle \hat{Q}f|g\rangle + \langle \hat{Q}g|f\rangle + \langle \hat{Q}g|g\rangle$$

$$\because \langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle, \quad \langle g|\hat{Q}g\rangle = \langle \hat{Q}g|g\rangle$$

$$\therefore \langle f|\hat{Q}g\rangle + \langle g|\hat{Q}f\rangle = \langle \hat{Q}f|g\rangle + \langle \hat{Q}g|f\rangle \quad (1)$$

For $h = f + ig$,

$$\langle (f + ig)|\hat{Q}(f + ig)\rangle = \langle \hat{Q}(f + ig)|(f + ig)\rangle$$

$$\langle f + ig|\hat{Q}f + i\hat{Q}g\rangle = \langle \hat{Q}f + i\hat{Q}g|f + ig\rangle$$

$$\langle f|\hat{Q}f\rangle + i\langle f|\hat{Q}g\rangle - i\langle g|\hat{Q}f\rangle + \langle g|\hat{Q}g\rangle = \langle \hat{Q}f|f\rangle + i\langle \hat{Q}f|g\rangle - i\langle \hat{Q}g|f\rangle + \langle \hat{Q}g|g\rangle$$

$$\because \langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle, \quad \langle ig|i\hat{Q}g\rangle = \langle i\hat{Q}g|ig\rangle$$

$$\therefore \langle f|\hat{Q}g\rangle - \langle g|\hat{Q}f\rangle = \langle \hat{Q}f|g\rangle - \langle \hat{Q}g|f\rangle \quad (2)$$

Adding 1 and 2 we get,

$$\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle$$

References

- [1] M.H. El-Deeb. [PEU-323 Assignments](#).
- [2] D.J. Griffiths and D.F. Schroeter. *Introduction to Quantum Mechanics*. Cambridge University Press, 2018.