PEU 323 Assignment 9

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$$A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

B is the identity matrix, which any unitary transformation can diagonalize

$$U^\dagger B U = U^\dagger U B = B$$

Let's find the unitary matrix that diagonalizes the matrix A

$$A\psi_i = \alpha_i \psi_i$$
$$(A - \alpha_i I)\psi_i = 0$$
$$\det(A - \alpha_i I) = 0$$
$$\begin{vmatrix} -\alpha_i & -i \\ i & -\alpha_i \end{vmatrix} = 0$$
$$(\alpha_i)^2 - 1 = 0$$
$$\alpha_i = \pm 1$$

$$(A \mp I)\psi_i = 0$$
$$\begin{pmatrix} \mp 1 & -i \\ i & \mp 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$
$$\mp a - ib = 0$$
$$a = \mp ib$$
$$\therefore \psi = b \begin{pmatrix} \mp i \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mp i \\ 1 \end{pmatrix}$$
$$\therefore U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$$

For verification:

$$U^T = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$$
$$U^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}$$
$$U^\dagger A U = \frac{1}{2} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \begin{pmatrix} -i & -i \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

For a hermitian matrix,

$$H^\dagger = H$$
$$H^\dagger = \begin{pmatrix} H_{11}^* & H_{21}^* \\ H_{12}^* & H_{22}^* \end{pmatrix}$$
$$\begin{pmatrix} H_{11}^* & H_{21}^* \\ H_{12}^* & H_{22}^* \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$
$$H_{11}^* = H_{11}, \quad H_{22}^* = H_{22}, \quad H_{12}^* = H_{21}$$

a)
$$A\psi_i = E_i\psi_i$$
$$(A - E_i I)\psi_i = 0$$
$$\det(A - E_i I) = 0$$
$$\begin{vmatrix} H_{11} - E_i & H_{12} \\ H_{21} & H_{22} - E_i \end{vmatrix} = 0$$
$$(H_{11} - E_i)(H_{22} - E_i) - H_{12}H_{21} = 0$$
$$E_i^2 - (H_{11} + H_{22})E_i + H_{11}H_{22} - H_{12}H_{21} = 0$$

$$E_i = \frac{-(H_{11} + H_{22}) \pm \sqrt{(H_{11} + H_{22})^2 - 4(H_{11}H_{22} - H_{12}H_{21})}}{2}$$
$$= \frac{-H_{11} - H_{22} \pm \sqrt{(H_{11} - H_{22})^2 + 4H_{12}H_{21}}}{2}$$
$$= \frac{-H_{11} - H_{22} \pm \sqrt{(H_{11} - H_{22})^2 + |2H_{12}|^2}}{2}$$
$$= -\frac{H_{11} + H_{22}}{2} \pm \sqrt{\left(\frac{H_{11} - H_{22}}{2}\right)^2 + |H_{12}|^2}$$

b)

$$(A - E_i I)\psi_i = 0$$

$$\begin{pmatrix} H_{11} - E_i & H_{12} \\ H_{21} & H_{22} - E_i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$(H_{11} - E_i)a + H_{12}b = 0 \Rightarrow a = \frac{H_{12}}{H_{11} - E_i} b$$

$$|a|^2 + |b|^2 = 1$$

$$|\frac{H_{12}}{H_{11} - E_i}|^2 \ |b|^2 + |b|^2 = 1$$

$$\left(|\frac{H_{12}}{H_{11} - E_i}|^2 + 1 \right) |b|^2 = 1$$

$$|b|^2 = \frac{1}{|\frac{H_{12}}{H_{11}-E_i}|^2 + 1}$$

$$\psi_i = \begin{pmatrix} a \\ b \end{pmatrix} = b \begin{pmatrix} \frac{H_{12}}{H_{11}-E_i} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{|\frac{H_{12}}{H_{11}-E_i}|^2 + 1}} \begin{pmatrix} \frac{H_{12}}{H_{11}-E_i} \\ 1 \end{pmatrix}$$

where $E_i = -\frac{H_{11}+H_{22}}{2} \pm \sqrt{\left(\frac{H_{11}-H_{22}}{2}\right)^2 + |H_{12}|^2}$

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$$\hat{L}_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{L}_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{L}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

a) Since $\hat{L}_z$ is diagonal, we can read off the eigenvalues directly. $E_i = -1, 0, 1$

b) Since the eigenvectors of diagonal matrices are the columns of the identity matrix,

For $E = 1$, $\psi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{aligned}
\langle L_x \rangle &= \langle \psi | L_x | \psi \rangle \\
&= \frac{1}{\sqrt{2}}(1\ \ 0\ \ 0)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}(1\ \ 0\ \ 0)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0
\end{aligned}$$

$$\begin{aligned}
\langle L_x^2 \rangle &= \langle \psi | L_x^2 | \psi \rangle \\
&= \frac{1}{2}(1\ \ 0\ \ 0)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \frac{1}{2}(1\ \ 0\ \ 0)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{2}(1\ \ 0\ \ 0)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}
\end{aligned}$$

$$\sigma_{L_x} = \sqrt{\langle L_x^2 \rangle - \langle L_x \rangle^2} = \frac{1}{\sqrt{2}}$$

c)

$$\hat{L}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det\left(\hat{L}_x - \lambda_i I\right) = 0$$

$$\begin{vmatrix} -\lambda_i & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda_i & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda_i \end{vmatrix} = 0$$

$$-\lambda_i\left(\lambda_i^2 - \frac{1}{2}\right) + \frac{1}{2}\lambda_i = 0$$

$$-\lambda_i^3 + \lambda_i = 0$$

$$\lambda_i(-\lambda_i^2 + 1) = 0$$

$$\therefore \lambda_i = -1, 0, 1$$

$$\left(\hat{L}_x - \lambda_i I\right)\psi_i = 0$$

$$\begin{pmatrix} -\lambda_i & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda_i & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda_i \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$-\lambda_i a + \frac{1}{\sqrt{2}} b = 0, \quad \frac{1}{\sqrt{2}} b - \lambda_i c = 0, \quad \frac{1}{\sqrt{2}}(a + c) - \lambda_i b = 0$$

$$\therefore b = \sqrt{2}\lambda_i c, \quad a = \sqrt{2}\lambda_i b - c = (2\lambda_i^2 - 1)c$$

$$|a|^2 + |b|^2 + |c|^2 = 1$$

$$|2\lambda_i^2 - 1|^2 \ |c|^2 + 2|\lambda_i|^2 \ |c|^2 + |c|^2 = 1$$

$$|4\lambda_i^4 - 4\lambda_i^2 + 1\| c|^2 + 2|\lambda_i|^2 \ |c|^2 + |c|^2 = 1$$

For that case, $|4\lambda_i^4 - 2\lambda_i^2 + 1| = (4\lambda_i^4 - 2\lambda_i^2 + 1)$

$$2(2\lambda_i^4 - \lambda_i^2 + 1)|c|^2 = 1$$
$$(2 + 2\lambda_i^2)\ |c|^2 = 1$$
$$|c|^2 = \frac{1}{2(2\lambda_i^4 - \lambda_i^2 + 1)}$$
$$\therefore \psi_i = c\begin{pmatrix}(2\lambda_i^2 - 1)\\ \sqrt{2}\lambda_i\\ 1\end{pmatrix} = \frac{1}{\sqrt{2(2\lambda_i^4 - \lambda_i^2 + 1)}}\begin{pmatrix}(2\lambda_i^2 - 1)\\ \sqrt{2}\lambda_i\\ 1\end{pmatrix}$$
$$\psi_{-1} = \frac{1}{2}\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}, \quad \psi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}, \quad \psi_1 = \frac{1}{2}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}$$

For $\hat{L}_z$, let the basis be

$$v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad v_0 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad v_{-1} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\therefore \psi_{-1} = \frac{1}{2}(v_1 + v_{-1}) - \sqrt{2}v_0$$
$$\psi_0 = \frac{1}{\sqrt{2}}(-v_1 + v_{-1})$$
$$\psi_1 = \frac{1}{2}(v_1 + v_{-1}) + \sqrt{2}v_0$$

d) For $\hat{L}_z$ measured to be -1, the eigenstate is v_{-1}

$$\begin{aligned}|v_{-1}\rangle &= \left(\sum_i |\psi_i\rangle\langle\psi_i|\right)|v_{-1}\rangle \\ &= \langle\psi_{-1}|v_{-1}\rangle\ |\psi_{-1}\rangle + \langle\psi_0|v_{-1}\rangle\ |\psi_0\rangle + \langle\psi_1|v_{-1}\rangle\ |\psi_1\rangle \\ &= \frac{1}{2}\ |\psi_{-1}\rangle - \frac{1}{\sqrt{2}}\ |\psi_0\rangle + \frac{1}{2}\ |\psi_1\rangle\end{aligned}$$

Therefore the possible outcomes are:
- $\lambda = -1$, with $p = \frac{1}{4}$
- $\lambda = 0$, with $p = \frac{1}{2}$
- $\lambda = 1$, with $p = \frac{1}{4}$

e)

$$|\psi\rangle = \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$\hat{L}_z^2\psi_i = \lambda_i^2\psi_i$$

For $\hat{L}_z^2$ measured to be 1, the possible eigenstates are v_{-1} and v_1.

probability of getting this result:

$$\begin{aligned} p_{+1} &= |\langle v_1|\psi\rangle|^2 + |\langle v_{-1}|\psi\rangle|^2 \\ &= \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \end{aligned}$$

Post-Measurement State:

$$P_{+1} = |v_1\rangle\langle v_1| + |v_{-1}\rangle\langle v_{-1}|$$

$$\begin{aligned} |\psi_{\text{post}}\rangle &= \frac{P_{+1}\ |\psi\rangle}{\sqrt{p_{+1}}} \\ &= \frac{\frac{1}{2}\ |v_1\rangle + \frac{1}{\sqrt{2}}\ |v_{-1}\rangle}{\sqrt{\frac{3}{4}}} \\ &= \frac{1}{\sqrt{3}}\ |v_1\rangle + \sqrt{\frac{2}{3}}\ |v_{-1}\rangle \end{aligned}$$

After an immediate measurement of $\hat{L}_z$, the outcomes would be:

- $\lambda = 1$, with $p = \frac{1}{3}$
- $\lambda = -1$, with $p = \frac{2}{3}$

f)
$$|\psi\rangle = \sum_i c_i|v_i\rangle, \quad \text{where } |c_i|^2 = p(v_i) \text{ or } c_i = \sqrt{p(v_i)}e^{i\varphi_i}$$

$$\therefore |\psi\rangle = \frac{1}{\sqrt{2}}e^{i\varphi_{-1}}|v_{-1}\rangle + \frac{1}{2}e^{i\varphi_0}\ |v_0\rangle + \frac{1}{\sqrt{2}}e^{i\varphi_1}|v_1\rangle$$

where φ_{-1}, φ_0 and φ_1 are arbitrary phases g)

$$\begin{aligned}
p\left(\hat{L}_x = 0\right) &= |\langle \psi_0 |\ \psi \rangle|^2 \\
&= |\langle \frac{1}{\sqrt{2}}(-v_1 + v_{-1})|\ \psi\rangle|^2 \\
&= |\frac{1}{\sqrt{2}}(-\langle v_1|\ \psi\rangle + \langle v_{-1}|\ \psi\rangle)|^2 \\
&= |\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}e^{-i\varphi_{-1}} + \frac{1}{\sqrt{2}}e^{i\varphi_1}\right)|^2 \\
&= |\frac{1}{2}(-e^{-i\varphi_{-1}} + e^{i\varphi_1})|^2 \\
&= |\frac{1}{2}e^{i\varphi_{-1}}\big(e^{i(\varphi_1-\varphi_{-1})} - 1\big)|^2 \\
&= \frac{1}{4}\big(e^{i(\varphi_1-\varphi_{-1})} - 1\big)\big(e^{-i(\varphi_1-\varphi_{-1})} - 1\big) \\
&= \frac{1}{4}\big(2 - \big(e^{i(\varphi_1-\varphi_{-1})} + e^{-i(\varphi_1-\varphi_{-1})}\big)\big) \\
&= \frac{1}{2}(1 - \cos(\varphi_1 - \varphi_{-1}))
\end{aligned}$$

Therefore a global phase like $e^{i\varphi_{-1}}$ multiplied in all result in an equivalent state, however, relative phases between basis states result in a different behavior when computing the angular momenta, or any observables in general.

4

a) We solve the two-dimensional time-independent Schrödinger equation:

For $0 \leq x \leq L_x, 0 \leq y \leq L_y$:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y) = E\psi(x,y)$$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi(x,y)}{\partial x^2} + \frac{\partial^2\psi(x,y)}{\partial y^2}\right) = E\psi(x,y)$$

Assume the wavefunction is separable:

$$\psi(x,y) \equiv X(x)Y(y)$$

Substituting into the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\left(Y(y)\frac{\partial^2 X(x)}{\partial x^2} + X(x)\frac{\partial^2 Y(y)}{\partial y^2}\right) = EX(x)Y(y)$$

$$-\frac{\hbar^2}{2m}Y(y)\frac{\partial^2 X(x)}{\partial x^2} = E_x X(x)Y(y)$$

$$-\frac{\hbar^2}{2m}X(x)\frac{\partial^2 Y(y)}{\partial y^2} = E_y X(x)Y(y)$$

where $E_x + E_y = E$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 X(x)}{\partial x^2} = E_x X(x)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 Y(y)}{\partial y^2} = E_y Y(y)$$

By solving both equations and applying boundary conditions $X(0) = X(L_x) = 0$ and $Y(0) = Y\left(L_y\right) = 0$,

$$X(x) = \sqrt{\frac{2}{L_x}}\sin\left(\frac{n_x\pi x}{L_x}\right), \quad n_x \in N$$

$$Y(y) = \sqrt{\frac{2}{L_y}}\sin\left(\frac{n_y\pi y}{L_y}\right), \quad n_y \in N$$

The normalized wavefunction is:

$$\psi(x,y) = \frac{2}{\sqrt{L_x L_y}}\sin\left(\frac{n_x\pi x}{L_x}\right)\sin\left(\frac{n_y\pi y}{L_y}\right)$$

b) Energies:

$$E_x = \frac{\hbar^2\pi^2 n_x^2}{2mL_x^2}, \quad E_y = \frac{\hbar^2\pi^2 n_y^2}{2mL_y^2}$$

The total energy is:

$$E = E_x + E_y = \frac{\hbar^2\pi^2}{2m}\left(\left(\frac{n_x}{L_x}\right)^2 + \left(\frac{n_y}{L_y}\right)^2\right)$$

c) For $L_x = L_y = L$,

$$E(n_x, n_y) = \frac{\hbar^2\pi^2}{2mL^2}(n_x^2 + n_y^2)$$

For degenerate states where $E(n'_x, n,y) = E(n_x, n_y)$,

$$(n_x'^2 + n_y'^2) = (n_x'^2 + n_y'^2)$$

d) i. For $\frac{L_x}{L_y}$ is rational,

$$E = \frac{\hbar^2\pi^2}{2mL_x^2}\left(n_x^2 + n_y^2\left(\frac{L_x}{L_y}\right)^2\right)$$

For degenerate states to exist:

$$n_x'^2 + n_y'^2\left(\frac{L_x}{L_y}\right)^2 = n_x^2 + n_y^2\left(\frac{L_x}{L_y}\right)^2$$

$$\frac{L_x}{L_y} = \frac{p}{q}$$

$$n_x'^2 + n_y'^2\left(\frac{p}{q}\right)^2 = n_x^2 + n_y^2\left(\frac{p}{q}\right)^2$$

$$q^2 n_x'^2 + p^2 n_y'^2 = q^2 n_x^2 + p^2 n_y^2$$

Because p and q are integers, there may exist integer solutions (n'_x, n'_y) and (n_x, n_y) that satisfy the degeneracy condition, in other words, **degenerate states are allowable to exist** in this case,

ii. For $\frac{L_x}{L_y}$ is irrational, then the relation

$$n_x'^2 + n_y'^2\left(\frac{L_x}{L_y}\right)^2 = n_x^2 + n_y^2\left(\frac{L_x}{L_y}\right)^2$$

can't be satisfied for distinct $\left(n_x, n_y\right)$ because an irrational ratio ensures that no linear combination of n_x^2 and n_y^2 can satisfy the equality. Therefore there will be **no degeneracy** in that case.

References

[1] M. El-Deeb, "PEU-323 Assignments." [Online]. Available: https://github.com/mhdeeb/peu-323-latex