PEU 323 Assignment 3

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1 Problem 1

$$\lambda = \frac{h}{p}$$

$$E_K = \frac{mv^2}{2} = \frac{p^2}{2m}$$

Where E_K is the kinetic energy.

$$p = \sqrt{2mE_K}$$

$$E_K = \frac{f}{2}K_B T$$

Where f is 3 for nitrogen atoms under room temperature (T = 293.15K) and $K_B = 1.380649 \times 10^{-23} J.K^{-1}$ is boltzmann constant.

$$E_K = \frac{3}{2}K_B T$$

$$p = \sqrt{3mK_B T} = \sqrt{3 \times 10^{-18} kg \times 1.380649 \times 10^{-23} J.K^{-1} \times 293.15K}$$

$$= \sqrt{3 \times 10^{-18} \times 1.380649 \times 10^{-23} \times 293.15}\sqrt{kg.J} \approx 10^{-19}\frac{kg.m}{s}$$

$$\lambda = \frac{h}{\sqrt{3mK_B T}}$$

$$h = 6.62607015 \times 10^{-34} J.s$$

$$m = 10^{-18} kg$$

$$kgJ = \frac{kg^2m^2}{s^2}$$

$$\lambda = \frac{6.62607015 \times 10^{-34}}{\sqrt{3 \times 10^{-18} \times 1.380649 \times 10^{-23} \times 293.15}} \frac{J.s}{\sqrt{kgJ}}$$

$$\lambda \approx 6.0 \times 10^{-15} m$$

$$d = 10^{-7} m$$

$$\frac{h}{pd} = \frac{6.62607015 \times 10^{-34}}{10^{-7} \times 10^{-19}} \frac{J.s^2}{kg.m^2} = 6.6 \times 10^{-8}$$

Since the $\Delta p \Delta d >> h$ We don't need to use quantum treatment.

2 Problem 2

$$\int_{-\infty}^{\infty} f(y)f(y)^*\,dy = c$$

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$

$$\begin{aligned}
\int F(k)F(k)^*\,dk &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx \int_{-\infty}^{\infty} f(y)^* e^{iky}\,dy\right] dk \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)f(y)^* e^{ik(y-x)}\,dx\,dy\right] dk \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)f(y)^* \int_{-\infty}^{\infty} e^{ik(y-x)}\,dk\,dx\,dy \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)f(y)^*\delta(y-x)\,dx\,dy \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)f(y)^*\,dy = \frac{c}{2\pi}
\end{aligned}$$

We can see that the $F(k)$ is normalizable if c is finite.

3 Problem 3

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\psi}{\partial t}$$

$$\Psi(x,t) = \int_{-\infty}^{\infty} \phi(k)e^{i(kx-\omega(k)t)}\,dk$$

$$\frac{i\hbar}{2m}\frac{\partial^2(\int_{-\infty}^{\infty} \phi(k)e^{i(kx-\omega(k)t)}\,dk)}{\partial x^2} = \frac{\partial(\int_{-\infty}^{\infty} \phi(k)e^{i(kx-\omega(k)t)}\,dk)}{\partial t}$$

$$\frac{\hbar}{2m}\int_{\infty}^{\infty} k^2\phi(k)e^{i(kx-\omega(k)t)}\,dk = \int_{\infty}^{\infty} w(k)\phi(k)e^{i(kx-\omega(k)t)}\,dk$$

$$w(k) = \frac{k^2\hbar}{2m}$$

We can clearly see that both sides of the equation are equal therefore Schrodinger equation is satisfied.

4 Problem 4

(a) So that in the integral of the probability density $f.f^* = 1$ leaving us with the same normalization constant.

(b)

$$x' = x - vt, \quad t' = t$$
$$\frac{\partial x}{\partial x'} = 1, \quad \frac{\partial t}{\partial x'} = 0, \quad \frac{\partial x}{\partial t'} = v, \quad \frac{\partial t}{\partial t'} = 1$$
$$\psi'(x', t') = f(x,t)\psi(x,t)$$
$$\frac{-\hbar^2}{2m}\frac{\partial^2 \psi'}{\partial x'^2} = i\hbar \frac{\partial \psi'}{\partial t'}$$

$$\begin{aligned}
\frac{\partial \psi'}{\partial x'} &= \frac{\partial(f(x,t)\psi(x,t))}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial(f(x,t)\psi(x,t))}{\partial t}\frac{\partial t}{\partial x'} \\
&= \frac{\partial(f(x,t)\psi(x,t))}{\partial x} \\
&= \frac{\partial f(x,t)}{\partial x}\psi(x,t) + f(x,t)\frac{\partial \psi(x,t)}{\partial x} \\
\frac{\partial^2 \psi'}{\partial x'^2} &= \partial_x^2[f(x,t)]\psi(x,t) + 2\partial_x[f(x,t)]\partial_x[\psi(x,t)] + f(x,t)\partial_x^2[\psi(x,t)]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \psi'}{\partial t'} &= \frac{\partial(f(x,t)\psi(x,t))}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial(f(x,t)\psi(x,t))}{\partial t}\frac{\partial t}{\partial t'} \\
&= v\frac{\partial(f(x,t)\psi(x,t))}{\partial x} + \frac{\partial(f(x,t)\psi(x,t))}{\partial t} \\
&= v\left(\frac{\partial f(x,t)}{\partial x}\psi(x,t) + f(x,t)\frac{\partial \psi(x,t)}{\partial x}\right) + \frac{\partial f(x,t)}{\partial t}\psi(x,t) + f(x,t)\frac{\partial \psi(x,t)}{\partial t}
\end{aligned}$$

$$\begin{aligned}
&\frac{-\hbar^2}{2m}\left(\partial_x^2[f(x,t)]\psi(x,t) + 2\partial_x[f(x,t)]\partial_x[\psi(x,t)] + f(x,t)\partial_x^2[\psi(x,t)]\right) \\
&= i\hbar\left(v\left(\partial_x[f(x,t)]\psi(x,t) + f(x,t)\partial_x[\psi(x,t)]\right) + \partial_t[f(x,t)]\psi(x,t) + f(x,t)\partial_t[\psi(x,t)]\right)
\end{aligned}$$

Using

$$\frac{-\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} = i\hbar\frac{\partial \psi}{\partial t}$$

$$\frac{-\hbar^2}{2m}\left(\partial_x^2[f(x,t)]\psi(x,t) + 2\partial_x[f(x,t)]\partial_x[\psi(x,t)]\right)$$
$$= i\hbar\left(v\left(\partial_x[f(x,t)]\psi(x,t) + f(x,t)\partial_x[\psi(x,t)]\right) + \partial_t[f(x,t)]\psi(x,t)\right)$$

$$\left(\frac{\hbar^2}{2m}\partial_x^2[f(x,t)] + i\hbar\partial_t[f(x,t)] + i\hbar v\partial_x[f(x,t)]\right)\psi(x,t)$$
$$+\left(\frac{\hbar^2}{m}\partial_x[f(x,t)] + i\hbar v f(x,t)\right)\partial_x[\psi(x,t)] = 0$$

$$A = \frac{\hbar^2}{2m}\partial_x^2[f(x,t)] + i\hbar\partial_t[f(x,t)] + i\hbar v\partial_x[f(x,t)]$$

$$B = \frac{\hbar^2}{m}\partial_x[f(x,t)] + i\hbar v f(x,t)$$

(c) Since ψ is an arbitrary function A and B must be zero to satisfy the previous equation.
Using B

$$\frac{\hbar^2}{m}\partial_x[f(x,t)] = -i\hbar v f(x,t)$$

$$\frac{\partial_x[f(x,t)]}{f(x,t)} = -\frac{imv}{\hbar}$$

$$\int\frac{df(x,t)}{f(x,t)} = -\frac{imv}{\hbar}\int dx$$

$$\ln(f(x,t)) = -\frac{imv}{\hbar}x + c(t)$$

$$f(x,t) = c(t)e^{-\frac{imv}{\hbar}x}$$

Using A

$$\partial_x[f(x,t)] = -\frac{imv}{\hbar}c(t)e^{-\frac{imv}{\hbar}x}$$

$$\partial_x^2[f(x,t)] = -(\frac{mv}{\hbar})^2 c(t)e^{-\frac{imv}{\hbar}x}$$

$$\frac{\hbar^2}{2m}\partial_x^2[f(x,t)] + i\hbar\partial_t[f(x,t)] + i\hbar v\partial_x[f(x,t)] = 0$$

$$i\hbar\partial_t[f(x,t)] = \frac{\hbar^2}{2m}(\frac{mv}{\hbar})^2 c(t)e^{-\frac{imv}{\hbar}x} + i\hbar v\frac{imv}{\hbar}c(t)e^{-\frac{imv}{\hbar}x}$$

$$i\hbar\partial_t[f(x,t)] = \frac{mv^2}{2}c(t)e^{-\frac{imv}{\hbar}x} - mv^2 c(t)e^{-\frac{imv}{\hbar}x}$$

$$= -\frac{mv^2}{2}c(t)e^{-\frac{imv}{\hbar}x}$$

$$\partial_t[f(x,t)] = \frac{imv^2}{2\hbar}c(t)e^{-\frac{imv}{\hbar}x}$$

$$\partial_t[f(x,t)] = \frac{imv^2}{2\hbar}f(x,t)$$

$$\partial_t[c(t)] = \frac{imv^2}{2\hbar}c(t)$$

$$c(t) = ge^{\frac{imv^2}{2\hbar}t}$$

$$f(x,t) = ge^{-\frac{imv}{\hbar}x}e^{\frac{imv^2}{2\hbar}t} = ge^{-\frac{imv}{\hbar}(x-\frac{1}{2}vt)}$$

$$f(x,t)f^*(x,t) = |g|^2 = 1$$

$$f(x,t) = e^{-\frac{imv}{\hbar}(x-\frac{1}{2}vt+\phi)}$$

$$f(x,t) = ge^{-\frac{i}{\hbar}\left(p_{S'}x-\frac{p_{S'}^2}{2m}t\right)}$$

(d)

$$\psi(x,t) = Ne^{\frac{i}{\hbar}\left(px-\frac{p^2}{2m}t\right)}$$

$$\begin{aligned}
\psi'(x',t') &= f(x,t)\psi(x,t) \\
&= Nge^{\frac{i}{\hbar}\left((p-p_{S'})x-\frac{(p^2-p_{S'}^2)}{2m}t\right)}
\end{aligned}$$

$$x = x' + vt = x' + \frac{p_{S'}}{m}t', \quad t = t'$$

$$\begin{aligned}
\psi'(x',t') &= Nge^{\frac{i}{\hbar}\left((p-p_{S'})(x'+\frac{p_{S'}}{m}t')-\frac{(p^2-p_{S'}^2)}{2m}t'\right)} \\
&= Nge^{\frac{i}{\hbar}\left((p-p_{S'})x'-\frac{(p^2-2pp_{S'}+p_{S'}^2)}{2m}t'\right)} \\
&= Nge^{\frac{i}{\hbar}\left((p-p_{S'})x'-\frac{(p-p_{S'})^2}{2m}t'\right)}
\end{aligned}$$

The new momentum is $p - p_{S'}$ and the new energy is $\frac{(p-p_{S'})^2}{2m}$.

5 Problem 5

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \left\langle \frac{\partial Q}{\partial t}\right\rangle$$

$$\frac{d}{dt}\langle xp\rangle = 2\langle T\rangle - \left\langle x\frac{\partial V}{\partial x}\right\rangle$$

$$\frac{d}{dt}\langle xp\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{x}\hat{p}]\rangle + \left\langle \frac{\partial (xp)}{\partial t}\right\rangle$$

$$\left\langle \frac{\partial (xp)}{\partial t}\right\rangle = 0$$

$$\frac{d}{dt}\langle xp\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{x}\hat{p}]\rangle$$

$$\hat{p} = -i\hbar\frac{d}{dx}, \quad \hat{x} = x$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + V$$

$$\begin{aligned}[\hat{H},\hat{x}\hat{p}]\psi &= [\frac{\hat{p}^2}{2m} + V, x\hat{p}]\psi \\ &= [\frac{\hat{p}^2}{2m}, x\hat{p}]\psi + [V, x\hat{p}]\psi\end{aligned}$$

$$[V, x\hat{p}]\psi = [Vx\hat{p} - x\hat{p}V]\psi = -xi\hbar[V\frac{d\psi}{dx} - \frac{d(V\psi)}{dx}]$$

$$= -xi\hbar[V\frac{d\psi}{dx} - \frac{dV}{dx}\psi - V\frac{d\psi}{dx}]$$

$$= xi\hbar\frac{dV}{dx}\psi$$

$$\begin{aligned}
[\frac{\hat{p}^2}{2m}, x\hat{p}]\psi &= [\frac{\hat{p}^2}{2m}, x]\hat{p} + x[\frac{\hat{p}^2}{2m}, \hat{p}] \\
&= \frac{1}{2m}(\hat{p}[\hat{p}, x]\hat{p} + [\hat{p}, x]\hat{p}^2 + x[p^2, \hat{p}]) \\
&= \frac{1}{2m}(2\hat{p}^2[\hat{p}, x] + x[p^2, \hat{p}]) \\
&= \frac{1}{2m}(2\hat{p}^2(-i\hbar) + 0) \\
&= -i\hbar\frac{\hat{p}^2}{m}
\end{aligned}$$

$$[\hat{H}, \hat{x}\hat{p}] = -i\hbar\frac{\hat{p}^2}{m} + i\hbar x\frac{dV}{dx}$$

$$\begin{aligned}
\frac{d}{dt}\langle xp\rangle &= \frac{i}{\hbar}\langle[\hat{H}, \hat{x}\hat{p}]\rangle \\
&= \langle\frac{\hat{p}^2}{m}\rangle - \langle x\frac{dV}{dx}\rangle \\
&= 2\langle T\rangle - \langle x\frac{dV}{dx}\rangle
\end{aligned}$$

The left hand side with vanish because p and x are not functions of time.

6 Problem 6

$$\hat{T} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

$$\langle T \rangle = \int_{-\infty}^{\infty} \psi^*(x) \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \right) \psi(x)\, dx$$

$$\begin{aligned}
\langle T \rangle &= -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \psi^*(x)\frac{d^2\psi(x)}{dx^2}\, dx \\
&= \frac{\hbar^2}{2m}\left[\psi^*(x)\frac{d\psi(x)}{dx}\right]_{-\infty}^{\infty} + \frac{\hbar^2}{2m}\int \left|\frac{d\psi(x)}{dx}\right|^2\, dx \\
&= \frac{\hbar^2}{2m}\int \left|\frac{d\psi(x)}{dx}\right|^2\, dx
\end{aligned}$$

The boundary conditions are zero as both ψ and its derivative are zero at the infinities.

$$\therefore \langle T \rangle \geq 0$$

7 Problem 7

(a)

$$\sigma_A^2 = \langle (\hat{A} - \langle A \rangle)\Psi | (\hat{A} - \langle A \rangle)\Psi \rangle = \langle f | f \rangle$$

$$\sigma_B^2 = \langle g | g \rangle$$

$$\sigma_A^2 \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2$$

$$|z|^2 = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2 = \left[\frac{1}{2}(z + z^*)\right]^2 + \left[\frac{1}{2i}(z - z^*)\right]^2$$

$$|\langle f | g \rangle|^2 = \left[\frac{1}{2}(\langle f | g \rangle + \langle g | f \rangle)\right]^2 + \left[\frac{1}{2i}(\langle f | g \rangle - \langle g | f \rangle)\right]^2$$

$$\sigma_A^2 \sigma_B^2 \geq \left[\frac{1}{2}(\langle f | g \rangle + \langle g | f \rangle)\right]^2 + \left[\frac{1}{2i}(\langle f | g \rangle - \langle g | f \rangle)\right]^2$$

$$\langle f | g \rangle = \langle \hat{A}\hat{B} \rangle - \langle A \rangle \langle B \rangle, \quad \langle g | f \rangle = \langle \hat{B}\hat{A} \rangle - \langle A \rangle \langle B \rangle$$

$$\begin{aligned}
|\mathrm{Re}|^2 &= \left[\frac{1}{2}(\langle \hat{A}\hat{B} \rangle - \langle A \rangle \langle B \rangle + \langle \hat{B}\hat{A} \rangle - \langle A \rangle \langle B \rangle)\right]^2 \\
&= \left[\frac{1}{2}(\langle \hat{A}\hat{B} \rangle - 2\langle A \rangle \langle B \rangle + \langle \hat{B}\hat{A} \rangle)\right]^2 \\
&= \left[\frac{1}{2}(\langle \hat{A}\hat{B} - 2\langle A \rangle \langle B \rangle + \hat{B}\hat{A} \rangle)\right]^2 \\
&= \frac{1}{4}\langle \hat{D} \rangle^2
\end{aligned}$$

$$
\begin{aligned}
|\mathrm{Im}|^2 &= \left[\frac{1}{2i}(\langle\hat{A}\hat{B}\rangle - \langle A\rangle\langle B\rangle - \langle\hat{B}\hat{A}\rangle + \langle A\rangle\langle B\rangle)\right]^2 \\
&= \left[\frac{1}{2i}(\langle\hat{A}\hat{B}\rangle - \langle\hat{B}\hat{A}\rangle\right]^2 \\
&= \left[\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right]^2 \\
&= \frac{1}{4}\langle\hat{C}\rangle^2
\end{aligned}
$$

$$
\sigma_A^2\sigma_B^2 \geq \frac{1}{4}(\langle\hat{C}\rangle^2 + \langle\hat{D}\rangle^2)
$$

(b)

$$
\begin{aligned}
\langle\hat{C}\rangle^2 &= \left[\langle[\hat{A},\hat{B}]\rangle\right]^2 \\
&= \left[\langle[\hat{A},\hat{A}]\rangle\right]^2 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\langle\hat{D}\rangle^2 &= \left[\langle\hat{A}\hat{B} - 2\langle A\rangle\langle B\rangle + \hat{B}\hat{A}\rangle\right]^2 \\
&= \left[\langle 2\hat{A}^2 - 2\langle A\rangle^2\rangle\right]^2 \\
&= 4\left[\langle\hat{A}^2\rangle - \langle A\rangle^2\right]^2 \\
&= 4\sigma_A^4
\end{aligned}
$$

$$
\sigma_A^2 = \frac{1}{2}\langle\hat{D}\rangle
$$

8 Problem 8

$$\langle h|\hat{Q}h\rangle = \langle \hat{Q}h|h\rangle$$

For $h = f + g$,

$$\langle (f+g)|\hat{Q}(f+g)\rangle = \langle \hat{Q}(f+g)|(f+g)\rangle$$

$$\langle f+g|\hat{Q}f+\hat{Q}g\rangle = \langle \hat{Q}f+\hat{Q}g|f+g\rangle$$

$$\langle f|\hat{Q}f\rangle + \langle f|\hat{Q}g\rangle + \langle g|\hat{Q}f\rangle + \langle g|\hat{Q}g\rangle = \langle \hat{Q}f|f\rangle + \langle \hat{Q}f|g\rangle + \langle \hat{Q}g|f\rangle + \langle \hat{Q}g|g\rangle$$

$$\because \langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle, \quad \langle g|\hat{Q}g\rangle = \langle \hat{Q}g|g\rangle$$

$$\therefore \langle f|\hat{Q}g\rangle + \langle g|\hat{Q}f\rangle = \langle \hat{Q}f|g\rangle + \langle \hat{Q}g|f\rangle \tag{1}$$

For $h = f + ig$,

$$\langle (f+ig)|\hat{Q}(f+ig)\rangle = \langle \hat{Q}(f+ig)|(f+ig)\rangle$$

$$\langle f+ig|\hat{Q}f+i\hat{Q}g\rangle = \langle \hat{Q}f+i\hat{Q}g|f+ig\rangle$$

$$\langle f|\hat{Q}f\rangle + i\langle f|\hat{Q}g\rangle - i\langle g|\hat{Q}f\rangle + \langle g|\hat{Q}g\rangle = \langle \hat{Q}f|f\rangle + i\langle \hat{Q}f|g\rangle - i\langle \hat{Q}g|f\rangle + \langle \hat{Q}g|g\rangle$$

$$\because \langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle, \quad \langle ig|i\hat{Q}g\rangle = \langle i\hat{Q}g|ig\rangle$$

$$\therefore \langle f|\hat{Q}g\rangle - \langle g|\hat{Q}f\rangle = \langle \hat{Q}f|g\rangle - \langle \hat{Q}g|f\rangle \tag{2}$$

Adding 1 and 2 we get,

$$\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle$$

References

[1] M.H. El-Deeb. PEU-323 Assignments.

[2] D.J. Griffiths and D.F. Schroeter. *Introduction to Quantum Mechanics*. Cambridge University Press, 2018.