PEU 323: Fall 2024
Assignment 3

University of Science and Technology at Zewail City

1. Find the wavelength of a smoke particle of diameter $100nm$ and mass $1fg$ (fg is femto-gram) bouncing off of air molecules at room temperature. Let the particle's kinetic energy be equal to the average thermal energy of the air molecules.

   Does this warrant a quantum mechanical treatment?

2. Show that if a (wave)function is normalizable, then so is its Fourier Transform.

   Note: We will use this result later on to interpret the F.T. of a wave function.

3. A free particle is described by a plane wave of the form

$$\psi = e^{i(kx-\omega t)}. \tag{1}$$

   Consider a superposition of waves of definite momenta given by

$$\Psi = \int_{-\infty}^{\infty} \phi(k) e^{i(kx-\omega(k)t)}\, dk. \tag{2}$$

   Show that this superposition satisfies the Schrodinger equation.

   Hint: If eq. 1 describes a free particle, what function of k is $\omega(k)$?

4. Consider the wave function ψ under Galilean Transformation. For a frame S' that moves with velocity v w.r.t. frame S, we want to show that the free-particle Schrodinger equation

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\psi}{\partial t} \tag{3}$$

   is invariant under Galilean Transformations. In terms of the primed coordinates x' and t' given by

$$\begin{aligned} x' &= x - vt, \\ t' &= t, \end{aligned} \tag{4}$$

we want to find the function $f(x,t)$ which satisfies

$$\psi'(x',t') = f(x,t)\psi(x,t), \tag{5}$$

where demand that $|f(x,t)| = 1$.

(a) Why do we demand $|f(x,t)| = 1$?

(b) To find f, we demand that ψ' obeys the Schrodinger eq. in S'. In doing so, we can obtain an equation of the form

$$A\psi + B\frac{\partial\psi}{\partial x} = 0, \tag{6}$$

where A and B are expressions involving f, ψ and their derivatives. Find A and B.

(c) Use the previous result to find $f(x,t)$.

(d) Consider the plane wave

$$\psi(x,t) = Ne^{\frac{i}{\hbar}(px - \frac{p^2}{2m}t)}. \tag{7}$$

Compute ψ' in S' as a function of x' and t'. What are the energy and momentum of the particle in S'?

5. The expectation value of any operator $\hat{Q}$ evolves according to

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \left\langle\frac{\partial Q}{\partial t}\right\rangle \tag{8}$$

(a) Use this to show that

$$\frac{d}{dt}\langle xp\rangle = 2\langle T\rangle - \langle x\frac{\partial V}{\partial x}\rangle. \tag{9}$$

(b) The left hand side vanishes for stationary states. Why?

6. We define a "positive" operator as an operator whose expectation value is always positive. Show that the kinetic energy operator is positive.

7. The generalized uncertainty principle states that, for two operators $\hat{A}$ and $\hat{B}$, if $[\hat{A},\hat{B}] = i\hat{C}$, then

$$\sigma_A^2\sigma_B^2 \geqslant \frac{1}{4}\langle C\rangle^2. \tag{10}$$

(a) Show that it can be strengthened to read

$$\sigma_A^2\sigma_B^2 \geqslant \frac{1}{4}\Big(\langle C\rangle^2 + \langle D\rangle^2\Big), \tag{11}$$

where $\hat{D} = \hat{A}\hat{B} + \hat{B}\hat{A} - 2\langle A\rangle\langle B\rangle$

(b) Check eq. (11) for the case $\hat{B} = \hat{A}$.

8. Griffiths 3.3: Show that if $\langle h|\hat{Q}h\rangle = \langle\hat{Q}h|h\rangle$ for all functions h (in a Hilbert space), then

$$\langle f|\hat{Q}g\rangle = \langle\hat{Q}f|g\rangle \tag{12}$$

for all functions f and g. These are two equivalent definitions for a hermitian operator $\hat{Q}$. The goal of this problem is to show that they are equivalent.

Hint: First let $h = f + g$, then let $h = f + ig$. [1]

[1] The notation $\langle f|g\rangle$ is the Bracket(Dirac) notation for $\int_{allspace} f^* g dx$.