

# PEU 323 Assignment 9

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# 1

$$A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

B is the identity matrix, which any unitary transformation can diagonalize

$$U^\dagger B U = U^\dagger U B = B$$

Let's find the unitary matrix that diagonalizes the matrix A

$$A\psi_i = \alpha_i\psi_i$$

$$(A - \alpha_i I)\psi_i = 0$$

$$\det(A - \alpha_i I) = 0$$

$$\begin{vmatrix} -\alpha_i & -i \\ i & -\alpha_i \end{vmatrix} = 0$$

$$(\alpha_i)^2 - 1 = 0$$

$$\alpha_i = \pm 1$$

$$(A \mp I)\psi_i = 0$$

$$\begin{pmatrix} \mp 1 & -i \\ i & \mp 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\mp a - ib = 0$$

$$a = \mp ib$$

$$\therefore \psi = b \begin{pmatrix} \mp i \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mp i \\ 1 \end{pmatrix}$$

$$\therefore U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$$

For verification:

$$U^T = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$$

$$U^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}$$

$$U^\dagger A U = \frac{1}{2} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \begin{pmatrix} -i & -i \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

## 2

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

For a hermitian matrix,

$$H^\dagger = H$$

$$H^\dagger = \begin{pmatrix} H_{11}^* & H_{21}^* \\ H_{12}^* & H_{22}^* \end{pmatrix}$$

$$\begin{pmatrix} H_{11}^* & H_{21}^* \\ H_{12}^* & H_{22}^* \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

$$H_{11}^* = H_{11}, \quad H_{22}^* = H_{22}, \quad H_{12}^* = H_{21}$$

a)  $A\psi_i = E_i\psi_i$

$$(A - E_i I)\psi_i = 0$$

$$\det(A - E_i I) = 0$$

$$\begin{vmatrix} H_{11} - E_i & H_{12} \\ H_{21} & H_{22} - E_i \end{vmatrix} = 0$$

$$(H_{11} - E_i)(H_{22} - E_i) - H_{12}H_{21} = 0$$

$$E_i^2 - (H_{11} + H_{22})E_i + H_{11}H_{22} - H_{12}H_{21} = 0$$

$$E_i = \frac{-(H_{11} + H_{22}) \pm \sqrt{(H_{11} + H_{22})^2 - 4(H_{11}H_{22} - H_{12}H_{21})}}{2}$$

$$= \frac{-H_{11} - H_{22} \pm \sqrt{(H_{11} - H_{22})^2 + 4H_{12}H_{21}}}{2}$$

$$= \frac{-H_{11} - H_{22} \pm \sqrt{(H_{11} - H_{22})^2 + |2H_{12}|^2}}{2}$$

$$= -\frac{H_{11} + H_{22}}{2} \pm \sqrt{\left(\frac{H_{11} - H_{22}}{2}\right)^2 + |H_{12}|^2}$$

b)

$$(A - E_i I)\psi_i = 0$$

$$\begin{pmatrix} H_{11} - E_i & H_{12} \\ H_{21} & H_{22} - E_i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$(H_{11} - E_i)a + H_{12}b = 0 \Rightarrow a = \frac{H_{12}}{H_{11} - E_i}b$$

$$|a|^2 + |b|^2 = 1$$

$$\left| \frac{H_{12}}{H_{11} - E_i} \right|^2 |b|^2 + |b|^2 = 1$$

$$\left( \left| \frac{H_{12}}{H_{11} - E_i} \right|^2 + 1 \right) |b|^2 = 1$$

$$|b|^2 = \frac{1}{\left| \frac{H_{12}}{H_{11} - E_i} \right|^2 + 1}$$

$$\psi_i = \begin{pmatrix} a \\ b \end{pmatrix} = b \begin{pmatrix} \frac{H_{12}}{H_{11} - E_i} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{\left| \frac{H_{12}}{H_{11} - E_i} \right|^2 + 1}} \begin{pmatrix} \frac{H_{12}}{H_{11} - E_i} \\ 1 \end{pmatrix}$$

$$\text{where } E_i = -\frac{H_{11} + H_{22}}{2} \pm \sqrt{\left( \frac{H_{11} - H_{22}}{2} \right)^2 + |H_{12}|^2}$$

### 3

$$\hat{L}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{L}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{L}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- a) Since  $\hat{L}_z$  is diagonal, we can read off the eigenvalues directly.  $E_i = -1, 0, 1$   
b) Since the eigenvectors of diagonal matrices are the columns of the identity matrix,

For  $E = 1$ ,  $\psi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{aligned} \langle L_x \rangle &= \langle \psi | L_x | \psi \rangle \\ &= \frac{1}{\sqrt{2}} (1 \ 0 \ 0) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} (1 \ 0 \ 0) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \end{aligned}$$

$$\begin{aligned} \langle L_x^2 \rangle &= \langle \psi | L_x^2 | \psi \rangle \\ &= \frac{1}{2} (1 \ 0 \ 0) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{1}{2} (1 \ 0 \ 0) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{2} (1 \ 0 \ 0) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2} \end{aligned}$$

$$\sigma_{L_x} = \sqrt{\langle L_x^2 \rangle - \langle L_x \rangle^2} = \frac{1}{\sqrt{2}}$$

c)

$$\hat{L}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det(\hat{L}_x - \lambda_i I) = 0$$

$$\begin{vmatrix} -\lambda_i & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda_i & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda_i \end{vmatrix} = 0$$

$$-\lambda_i \left( \lambda_i^2 - \frac{1}{2} \right) + \frac{1}{2} \lambda_i = 0$$

$$-\lambda_i^3 + \lambda_i = 0$$

$$\lambda_i(-\lambda_i^2 + 1) = 0$$

$$\therefore \lambda_i = -1, 0, 1$$

$$(\hat{L}_x - \lambda_i I) \psi_i = 0$$

$$\begin{pmatrix} -\lambda_i & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda_i & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda_i \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$-\lambda_i a + \frac{1}{\sqrt{2}} b = 0, \quad \frac{1}{\sqrt{2}} b - \lambda_i c = 0, \quad \frac{1}{\sqrt{2}}(a + c) - \lambda_i b = 0$$

$$\therefore b = \sqrt{2} \lambda_i c, \quad a = \sqrt{2} \lambda_i b - c = (2\lambda_i^2 - 1)c$$

$$|a|^2 + |b|^2 + |c|^2 = 1$$

$$|2\lambda_i^2 - 1|^2 |c|^2 + 2|\lambda_i|^2 |c|^2 + |c|^2 = 1$$

$$|4\lambda_i^4 - 4\lambda_i^2 + 1| |c|^2 + 2|\lambda_i|^2 |c|^2 + |c|^2 = 1$$

For that case,  $|4\lambda_i^4 - 4\lambda_i^2 + 1| = (4\lambda_i^4 - 4\lambda_i^2 + 1)$

$$\begin{aligned}
2(2\lambda_i^4 - \lambda_i^2 + 1)|c|^2 &= 1 \\
(2 + 2\lambda_i^2) |c|^2 &= 1 \\
|c|^2 &= \frac{1}{2(2\lambda_i^4 - \lambda_i^2 + 1)} \\
\therefore \psi_i &= c \begin{pmatrix} (2\lambda_i^2 - 1) \\ \sqrt{2}\lambda_i \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2(2\lambda_i^4 - \lambda_i^2 + 1)}} \begin{pmatrix} (2\lambda_i^2 - 1) \\ \sqrt{2}\lambda_i \\ 1 \end{pmatrix} \\
\psi_{-1} &= \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, \quad \psi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \psi_1 = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}
\end{aligned}$$

For  $\hat{L}_z$ , let the basis be

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_{-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore \psi_{-1} = \frac{1}{2}(v_1 + v_{-1}) - \sqrt{2}v_0$$

$$\psi_0 = \frac{1}{\sqrt{2}}(-v_1 + v_{-1})$$

$$\psi_1 = \frac{1}{2}(v_1 + v_{-1}) + \sqrt{2}v_0$$

d) For  $\hat{L}_z$  measured to be  $-1$ , the eigenstate is  $v_{-1}$

$$\begin{aligned}
|v_{-1}\rangle &= \left( \sum_i |\psi_i\rangle \langle \psi_i| \right) |v_{-1}\rangle \\
&= \langle \psi_{-1} | v_{-1} \rangle |\psi_{-1}\rangle + \langle \psi_0 | v_{-1} \rangle |\psi_0\rangle + \langle \psi_1 | v_{-1} \rangle |\psi_1\rangle \\
&= \frac{1}{2} |\psi_{-1}\rangle - \frac{1}{\sqrt{2}} |\psi_0\rangle + \frac{1}{2} |\psi_1\rangle
\end{aligned}$$

Therefore the possible outcomes are:

- $\lambda = -1$ , with  $p = \frac{1}{4}$
- $\lambda = 0$ , with  $p = \frac{1}{2}$
- $\lambda = 1$ , with  $p = \frac{1}{4}$

e)

$$|\psi\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\hat{L}_z^2 \psi_i = \lambda_i^2 \psi_i$$

For  $\hat{L}_z^2$  measured to be 1, the possible eigenstates are  $v_{-1}$  and  $v_1$ .  
probability of getting this result:

$$\begin{aligned} p_{+1} &= |\langle v_1 | \psi \rangle|^2 + |\langle v_{-1} | \psi \rangle|^2 \\ &= \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \end{aligned}$$

Post-Measurement State:

$$\begin{aligned} P_{+1} &= |v_1\rangle\langle v_1| + |v_{-1}\rangle\langle v_{-1}| \\ |\psi_{\text{post}}\rangle &= \frac{P_{+1} |\psi\rangle}{\sqrt{p_{+1}}} \\ &= \frac{\frac{1}{2} |v_1\rangle + \frac{1}{\sqrt{2}} |v_{-1}\rangle}{\sqrt{\frac{3}{4}}} \\ &= \frac{1}{\sqrt{3}} |v_1\rangle + \sqrt{\frac{2}{3}} |v_{-1}\rangle \end{aligned}$$

After an immediate measurement of  $\hat{L}_z$ , the outcomes would be:

- $\lambda = 1$ , with  $p = \frac{1}{3}$
- $\lambda = -1$ , with  $p = \frac{2}{3}$

f) 
$$|\psi\rangle = \sum_i c_i |v_i\rangle, \quad \text{where } |c_i|^2 = p(v_i) \text{ or } c_i = \sqrt{p(v_i)} e^{i\varphi_i}$$

$$\therefore |\psi\rangle = \frac{1}{\sqrt{2}} e^{i\varphi_{-1}} |v_{-1}\rangle + \frac{1}{2} e^{i\varphi_0} |v_0\rangle + \frac{1}{\sqrt{2}} e^{i\varphi_1} |v_1\rangle$$

where  $\varphi_{-1}, \varphi_0$  and  $\varphi_1$  are arbitrary phases



g)

$$\begin{aligned}
p(\hat{L}_x = 0) &= |\langle \psi_0 | \psi \rangle|^2 \\
&= \left| \langle \frac{1}{\sqrt{2}}(-v_1 + v_{-1}) | \psi \rangle \right|^2 \\
&= \left| \frac{1}{\sqrt{2}}(-\langle v_1 | \psi \rangle + \langle v_{-1} | \psi \rangle) \right|^2 \\
&= \left| \frac{1}{\sqrt{2}} \left( -\frac{1}{\sqrt{2}}e^{-i\varphi_{-1}} + \frac{1}{\sqrt{2}}e^{i\varphi_1} \right) \right|^2 \\
&= \left| \frac{1}{2}(-e^{-i\varphi_{-1}} + e^{i\varphi_1}) \right|^2 \\
&= \left| \frac{1}{2}e^{i\varphi_{-1}}(e^{i(\varphi_1 - \varphi_{-1})} - 1) \right|^2 \\
&= \frac{1}{4}(e^{i(\varphi_1 - \varphi_{-1})} - 1)(e^{-i(\varphi_1 - \varphi_{-1})} - 1) \\
&= \frac{1}{4}(2 - (e^{i(\varphi_1 - \varphi_{-1})} + e^{-i(\varphi_1 - \varphi_{-1})})) \\
&= \frac{1}{2}(1 - \cos(\varphi_1 - \varphi_{-1}))
\end{aligned}$$

Therefore a global phase like  $e^{i\varphi_{-1}}$  multiplied in all result in an equivalent state, however, relative phases between basis states result in a different behavior when computing the angular momenta, or any observables in general.

## 4

a) We solve the two-dimensional time-independent Schrödinger equation:

For  $0 \leq x \leq L_x, 0 \leq y \leq L_y$ :

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(x, y) = E \psi(x, y)$$

$$-\frac{\hbar^2}{2m} \left( \frac{\partial^2 \psi(x, y)}{\partial x^2} + \frac{\partial^2 \psi(x, y)}{\partial y^2} \right) = E \psi(x, y)$$

Assume the wavefunction is separable:

$$\psi(x, y) \equiv X(x)Y(y)$$

Substituting into the Schrödinger equation:

$$-\frac{\hbar^2}{2m} \left( Y(y) \frac{\partial^2 X(x)}{\partial x^2} + X(x) \frac{\partial^2 Y(y)}{\partial y^2} \right) = EX(x)Y(y)$$

$$-\frac{\hbar^2}{2m} Y(y) \frac{\partial^2 X(x)}{\partial x^2} = E_x X(x)Y(y)$$

$$-\frac{\hbar^2}{2m} X(x) \frac{\partial^2 Y(y)}{\partial y^2} = E_y X(x)Y(y)$$

where  $E_x + E_y = E$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 X(x)}{\partial x^2} = E_x X(x)$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 Y(y)}{\partial y^2} = E_y Y(y)$$

By solving both equations and applying boundary conditions  $X(0) = X(L_x) = 0$  and  $Y(0) = Y(L_y) = 0$ ,

$$X(x) = \sqrt{\frac{2}{L_x}} \sin\left(\frac{n_x \pi x}{L_x}\right), \quad n_x \in N$$

$$Y(y) = \sqrt{\frac{2}{L_y}} \sin\left(\frac{n_y \pi y}{L_y}\right), \quad n_y \in N$$

with energy:

$$E_x = -\frac{\hbar^2 \pi^2 n_x^2}{2m L_x^2}, \quad E_y = -\frac{\hbar^2 \pi^2 n_y^2}{2m L_y^2}$$

The total energy is:

$$E = E_x + E_y = -\frac{\hbar^2 \pi^2}{2m} \left( \left( \frac{n_x}{L_x} \right)^2 + \left( \frac{n_y}{L_y} \right)^2 \right)$$

## References

- [1] M. El-Deeb, “PEU-453 Assignments.” [Online]. Available: <https://github.com/mhdeeb/peu-453-latex>