PEU 323 Assignment 2

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1 Problem 1

1.1 Problem

Consider a particle which is free to be anywhere, with equal probability, on a line segment of length L.

(a) Find the normalized probability distribution for such a particle.

(b) Calculate $\langle x \rangle$ and σ_x.

(c) Calculate the probability of finding the particle within σ_x of $\langle x \rangle$.

(d) What are the dimensions of the probability density?

1.2 Solution

(a) $\frac{1}{L}$

(b)

$$\begin{aligned}
\langle x \rangle &= \frac{1}{L}\int_0^L x\,dx = \frac{L}{2} \\
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
\langle x^2 \rangle &= \frac{1}{L}\int_0^L x^2\,dx = \frac{L^2}{3} \\
\langle x \rangle^2 &= \frac{L^2}{4} \\
\sigma_x &= \sqrt{\frac{L^2}{3} - \frac{L^2}{4}} = \frac{L}{2\sqrt{3}}
\end{aligned}$$

(c) $\frac{1}{L}\int_{\frac{L}{2}(1-\frac{1}{\sqrt{3}})}^{\frac{L}{2}(1+\frac{1}{\sqrt{3}})} dx = \frac{1}{\sqrt{3}}$

(d) $length^{-1}$

2 Problem 2

2.1 Problem

Buffon's Needle: A needle of length l is dropped at random on a sheet of paper with parallel lines a distance l apart. What is the probability that it crosses a line?

2.2 Solution

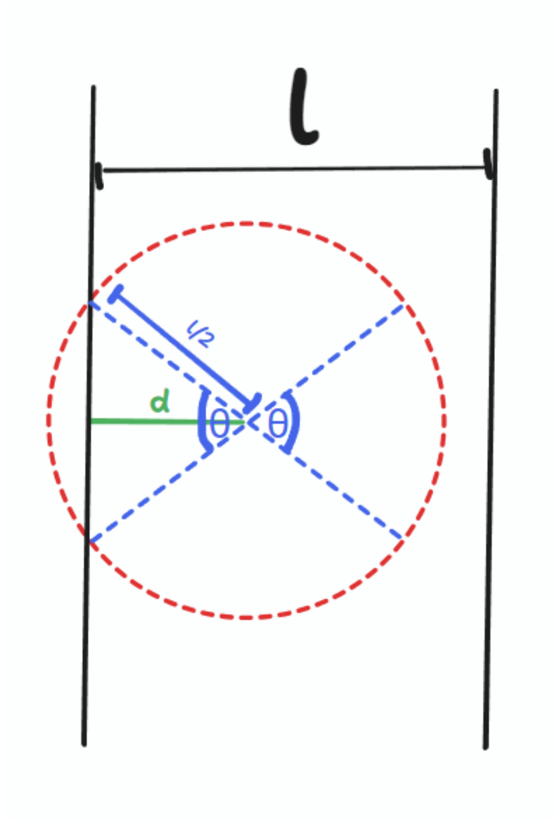

Figure 1: The black lines are the parallel lines of the paper, the red circle represents the possible area covered by the needle on a specific distance d from one of the lines. θ is the angle in which the needle crosses one of the lines

$$\cos\left(\frac{\theta}{2}\right) = \frac{2d}{l}$$

$$\theta = 2\cos^{-1}\left(\frac{2d}{l}\right)$$

$$P(d) = 4\frac{\theta}{2\pi} = \frac{2}{\pi}\cos^{-1}\left(\frac{2d}{l}\right)$$

$$P = \int_0^{\frac{L}{2}} P(d)\,dd$$

$$= \frac{2}{\pi}\int_0^{\frac{L}{2}} \cos^{-1}\left(\frac{2d}{l}\right) dd$$

$$\because \int \cos^{-1}(x)\,dx = x\cos^{-1}(x) - \sqrt{1-x^2} + c$$

$$\therefore P = \left[\frac{2d}{l}\cos^{-1}(\frac{2d}{l}) - \sqrt{1-\left(\frac{2d}{l}\right)^2}\right]_0^{\frac{L}{2}} = \frac{2}{\pi}$$

3 Problem 3

3.1 Problem

Find the probability distribution for the momentum of a harmonic oscillator with angular frequency ω.

3.2 Solution

$$y(t) = A\sin(\omega t)$$

$$p = y'(t) = A\omega\cos(\omega t)$$

$$t = \frac{\cos^{-1}(\frac{p}{A\omega})}{\omega}$$

$$p' = -A\omega^2\sin(\omega t) = -A\omega^2\sin(\cos^{-1}(\frac{p}{A\omega}))$$

$$p_f = \frac{p}{Aw}$$

We will absorb all constants and proportionality constants into α

$$\rho(p_f) \propto \frac{1}{p'} = \alpha\csc(\cos^{-1}(p_f))$$

$$\int_{-1}^{1}\rho(p_f)\,dp_f = \alpha\int_{-1}^{1}\csc(\cos^{-1}(p_f))\,dp_f = \frac{\pi}{2}$$

4 Problem 4

4.1 Problem

Consider the probability density for the location of the electron inside the Hydrogen atom:

$$\rho(r) = Ae^{-2r/a_0}, \tag{1}$$

where a_0 is the Bohr radius.

(a) Find A which normalizes this probability distribution.

Hint:

$$\int_0^\infty e^{-x} x^n dx = n!. \tag{2}$$

(b) Calculate the probability for the electron to be found in a sphere, centered about the origin, of radius b_0, with $b_0 << a_0$.

P.S.: You can do this calculation exactly or approximately. The approximate one is much easier.

4.2 Solution

(a)

$$\begin{aligned}
\rho(r) &= Ae^{-2r/a_0} \\
\rho &= \int_0^\infty \rho(r)\, dr = 1 \\
\rho &= A \int_0^\infty e^{-\frac{2r}{a_0}}\, dr \\
\because \int e^{-\alpha x}\, dx &= -\frac{e^{-\alpha x}}{\alpha} + c \\
\therefore 1 &= -\frac{A}{\alpha}\left[e^{-\alpha r}\right]_0^\infty = \frac{A}{\alpha} \\
A &= \frac{2}{a_0}
\end{aligned}$$

(b)

$$\rho(r) = \frac{2}{a_0} e^{-2r/a_0}$$

$$\frac{2}{a_0} \int_0^{b_0} e^{-\alpha x}\, dx = -\frac{2}{\alpha a_0} \left[e^{-\alpha x}\right]_0^{b_0}$$

$$= 1 - e^{-2\frac{b_0}{a_0}} \approx 2\frac{b_0}{a_0}$$

5 Problem 5

5.1 Problem

Consider the map

$$f : \mathbb{C} \to \mathbb{R}_{2\times 2} \tag{3}$$

$$x + iy \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

from the complex numbers to the set of 2×2 real matrices.

(a) Show that this map is an isomorphism. That is, show that it is invertible and that for $z_1, z_2 \in \mathbb{C}$,

$$f(z_1 z_2) = f(z_1)\, f(z_2) \tag{4}$$

and hence prove that, in two dimensions, rotations commute.

(b) Prove De Moivre's formula for complex numbers:

$$(r(\cos(\theta) + i\sin(\theta)))^n = r^n(\cos(n\theta) + i\sin(n\theta)) \tag{5}$$

and hence prove that

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}^n = \begin{pmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{pmatrix} \tag{6}$$

5.2 Solution

(a)

$$f(z_1z_2) = f((x_1+iy_1)(x_2+iy_2)) = f((x_1x_2-y_1y_2)+i(x_1y_2+y_1x_2))$$

$$= \begin{pmatrix} x_1x_2-y_1y_2 & -x_1y_2-y_1x_2 \\ x_1y_2+y_1x_2 & x_1x_2-y_1y_2 \end{pmatrix}$$

$$f(z_1)\,f(z_2) = \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1x_2-y_1y_2 & -x_1y_2-y_1x_2 \\ x_1y_2+y_1x_2 & x_1x_2-y_1y_2 \end{pmatrix}$$

We can that $f(z_1z_2)$ is just a multiplication of two complex number and that commutes.

(b)

$$\begin{aligned} e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\ (r(\cos(\theta)+i\sin(\theta)))^n &= r^n(\cos(\theta)+i\sin(\theta))^n \\ &= r^n e^{in\theta} = r^n(\cos(n\theta)+i\sin(n\theta)) \end{aligned} \tag{7}$$

Since the rotation transformation is isomorphic.

$$\therefore \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}^n \mapsto (\cos(\theta)+i\sin(\theta))^n = \cos(n\theta)+i\sin(n\theta)$$

$$\mapsto \begin{pmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{pmatrix}$$

References

[1] M.H. El-Deeb. PEU-323 Assignments.