PEU 323 Assignment 4

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1 Problem 1

(a) Yes, this wave function is physically acceptable if the normalization constant A is chosen such that $\int_{-a}^{a} |\psi(x)|^2 \, dx = 1$. The wave function is continuous and single-valued within $[-a, a]$, meeting the requirements for physical acceptability.

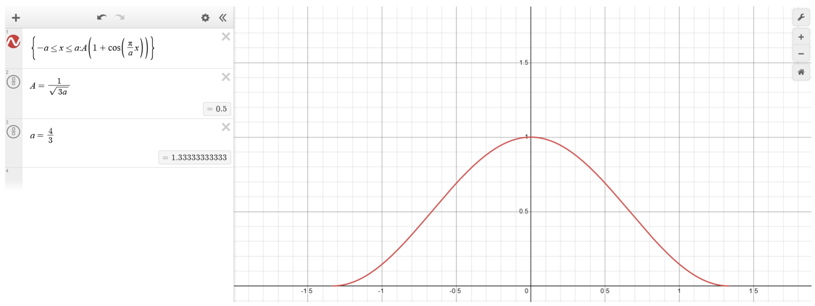

Figure 1: ψ

(b) The classically allowed region is:

$$[-a, a]$$

since the wave function $\psi(x) = 0$ outside this interval, meaning the classical particle cannot be found outside $[-a, a]$.

2 Problem 2

(a)

$$\psi(x,t) = \sin\left(\frac{n\pi}{a}x\right)e^{-i\omega t}$$

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$\begin{aligned}
V(x)\psi &= \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + i\hbar\frac{\partial\psi}{\partial t} \\
&= \frac{\hbar^2}{2m}\frac{\partial^2(\sin\left(\frac{n\pi}{a}x\right))}{\partial x^2}e^{-i\omega t} + i\hbar\sin\left(\frac{n\pi}{a}x\right)\frac{\partial(e^{-i\omega t})}{\partial t} \\
&= -\frac{\hbar^2 n^2\pi^2}{2ma^2}\sin\left(\frac{n\pi}{a}x\right)e^{-i\omega t} + \omega\hbar\sin\left(\frac{n\pi}{a}x\right)e^{-i\omega t} \\
&= (\omega\hbar - \frac{\hbar^2 n^2\pi^2}{2ma^2})\sin\left(\frac{n\pi}{a}x\right)e^{-i\omega t} \\
&= (\omega\hbar - \frac{\hbar^2 n^2\pi^2}{2ma^2})\psi \\
V &= \omega\hbar - \frac{\hbar^2 n^2\pi^2}{2ma^2}
\end{aligned}$$

(b) To find the expectation value $\langle x\rangle$, we observe that the term $\sin^2\left(\frac{n\pi}{a}x\right)$ is symmetric around $x = \frac{a}{2}$. By shifting the domain of $\psi(x)$ as $\psi(x) \to \psi\left(x - \frac{a}{2}\right)$, we redefine the domain as $-\frac{a}{2} \leq x \leq \frac{a}{2}$.

Under this transformation, the shifted wave function becomes an even function, while x (as a variable) is odd with respect to $x = 0$. As a result, the integrand for $\langle x'\rangle = \int_{-\frac{a}{2}}^{\frac{a}{2}} x'|\psi(x')|^2\,dx'$ is an odd function over a symmetric interval, leading to:

$$\langle x'\rangle = 0.$$

To obtain $\langle x\rangle$ in the original coordinates, we perform the inverse transformation $x' = x - \frac{a}{2}$, which implies $x = x' + \frac{a}{2}$. Thus:

$$\langle x\rangle = \langle x' + \frac{a}{2}\rangle = \langle x'\rangle + \frac{a}{2} = 0 + \frac{a}{2} = \frac{a}{2}.$$

Therefore, the average position of the particle is:

$$\langle x\rangle = \frac{a}{2}.$$

3 Problem 3

Fourier transform:

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-ikx}\, dx$$

Inverse Fourier transform:

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx}\, dk$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^*(x) \left(-i\hbar \frac{d}{dx} \right) \psi(x)\, dx$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi^*(k') e^{-ik'x}\, dk' \left(-i\hbar \frac{d}{dx} \right) \int_{-\infty}^{\infty} \phi(k) e^{ikx}\, dk\, dx$$

$$-i\hbar \frac{d}{dx} e^{ikx} = \hbar k e^{ikx}$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi^*(k') \phi(k) \hbar k \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x}\, dx \right) dk'\, dk$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x}\, dx = \delta(k - k')$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi^*(k') \phi(k) \hbar k \delta(k - k')\, dk'\, dk$$

$$\therefore \langle p \rangle = \int_{-\infty}^{\infty} \hbar k |\phi(k)|^2\, dk$$

where $|\phi(k)|^2$ represents the probability density in k-space and $\hbar k$ is the momentum associated with each k.

4 Problem 4

(a) If a system is in a pure energy eigenstate, its time evolution is given by:

$$\psi(x,t) = \psi(x,0)e^{-iEt/\hbar}$$

$$\psi(x,t+T) = \psi(x,t)$$

$$\psi(x,0)e^{-iE(t+T)/\hbar} = \psi(x,0)e^{-iEt/\hbar}$$

$$\therefore e^{-iET/\hbar} = 1$$

$$\therefore \frac{ET}{\hbar} = 2\pi n, \quad n \in \mathbb{N}$$

$$T = \frac{2\pi\hbar}{E}n$$

(b)

$$E = \frac{2\pi\hbar}{T}n$$

(c) For a particle in an infinite square well

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

$$T = \frac{2\pi\hbar}{E_n}n = \frac{4m}{n\pi\hbar}L^2$$

5 Problem 5

The wave function stays undisturbed (the same) after the sudden expansion

$$\psi_0 = \sqrt{\frac{2}{L}} \cos\left(\frac{\pi}{L}x\right), \quad -\frac{L}{2} \leq x \leq \frac{L}{2}$$

The new ground state:

$$\psi'_0 = \sqrt{\frac{1}{L}} \cos\left(\frac{\pi}{2L}x\right), \quad -L \leq x \leq L$$

$$\begin{aligned}
\langle \psi | \psi'_0 \rangle &= \frac{\sqrt{2}}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{\pi}{2L}x\right) \cos\left(\frac{\pi}{L}x\right) dx \\
&= \frac{\sqrt{2}}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{\pi}{2L}x\right)\left[1 - 2\sin^2\left(\frac{\pi}{2L}x\right)\right] dx \\
&= \frac{\sqrt{2}}{L} \left[\frac{2L}{\pi} \sin\left(\frac{\pi}{2L}x\right) - \frac{4L}{3\pi} \sin^3\left(\frac{\pi}{2L}x\right)\right]_{-\frac{L}{2}}^{\frac{L}{2}} \\
&= \frac{2\sqrt{2}}{\pi} \left[\sin\left(\frac{\pi}{2L}x\right) - \frac{2}{3} \sin^3\left(\frac{\pi}{2L}x\right)\right]_{-\frac{L}{2}}^{\frac{L}{2}} \\
&= \frac{8}{3\pi}
\end{aligned}$$

$$P = |\langle \psi | \psi'_0 \rangle|^2 = \left|\frac{8}{3\pi}\right|^2 = \frac{64}{9\pi^2}$$

6 Problem 6

References

[1] M.H. El-Deeb. PEU-323 Assignments.