Condensed Formula Sheet for High-Energy Astrophysics and Radiative Transfer

1. Radiative Transfer Fundamentals

Specific Intensity I_ν:

$$dE = I_\nu \, dA \, dt \, d\Omega \, d\nu$$

Radiative Transfer Equation (RTE):

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

with optical depth $\tau_\nu = \int \alpha_\nu \, ds$, absorption coefficient α_ν, and source function $S_\nu = \dfrac{j_\nu}{\alpha_\nu}$.

General Solution to RTE:

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(t) e^{-(\tau_\nu - t)} \, dt$$

For constant S_ν:

$$I_\nu = I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu})$$

Limiting Cases:

Optically Thin ($\tau_\nu \ll 1$):

$$I_\nu \approx I_\nu(0)(1 - \tau_\nu) + S_\nu \tau_\nu$$

Optically Thick ($\tau_\nu \gg 1$):

$$I_\nu \approx S_\nu$$

2. Blackbody Radiation and Kirchhoff's Law

Planck Function:

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/(kT)} - 1}$$

Energy Density and Radiation Pressure:

$$u = aT^4, \quad P_{\text{rad}} = \frac{1}{3}u = \frac{a}{3}T^4$$

with $a = \dfrac{8\pi^5 k^4}{15h^3c^3} \approx 7.5657 \times 10^{-16}\,\mathrm{J\,m^{-3}\,K^{-4}}$.

Stefan-Boltzmann Law:

$$F = \sigma T^4$$

with $\sigma = \dfrac{2\pi^5 k^4}{15h^3c^2} \approx 5.6704 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$.

Average Photon Energy and Wien's Law:

$$\langle h\nu \rangle \approx 2.7\,kT, \quad \lambda_{\max} T = 2.8978 \times 10^{-3}\,\mathrm{m\,K}, \quad \nu_{\max} = 2.8214\,\frac{kT}{h}$$

Kirchhoff's Law:

$$\frac{j_\nu}{\alpha_\nu} = B_\nu(T), \quad a_\nu = 1 \quad \text{(for a blackbody)}$$

3. Bremsstrahlung (Free-Free Emission)

Emissivity per Unit Volume and Frequency j_ν:

$$j_\nu = \frac{16}{3}\left(\frac{2\pi}{3}\right)^{1/2} \frac{e^6}{m_e^2 c^3} Z^2 n_e n_i \frac{1}{\sqrt{kT}} e^{-h\nu/(kT)} \overline{g}_{ff}$$

where $\overline{g}_{ff}$ is the velocity-averaged Gaunt factor.

Total Emissivity Integrated over Frequency:

$$j(T) = C_1 Z^2 n_e n_i T^{1/2}$$

where C_1 is a constant dependent on fundamental constants and $\overline{g}_{ff}$.

Free-Free Absorption Coefficient α_ν^{ff}:

$$\alpha_\nu^{ff} = \frac{4}{3}\left(\frac{2\pi}{3}\right)^{1/2} \frac{e^6}{m_e^2 c} Z^2 n_e n_i \frac{1}{\nu^3\sqrt{kT}} \left(1 - e^{-h\nu/(kT)}\right) \overline{g}_{ff}$$

Optical Depth for Free-Free Absorption:

$$\tau_\nu = \alpha_\nu^{ff} L$$

4. Larmor's Formula and Kinetic Theory

Larmor's Formula:

$$P = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3}$$

Root-Mean-Square Speed and Kinetic Energy:

$$v_{\rm rms} = \sqrt{\frac{3kT}{m}}, \quad E_k = \frac{1}{2} m v_{\rm rms}^2 = \frac{3}{2} kT$$

Hydrostatic Equilibrium:

$$\frac{dP}{dr} = -\rho \frac{GM(r)}{r^2}$$

Virial Theorem for Spherical Systems:

$$kT = \frac{GMm_p}{3R}$$

5. Photon Energy and Spectral Approximations

Energy per Photon:

$$E = h\nu$$

Spectral Approximations:

$$I(\nu, T) \approx \begin{cases} \dfrac{2h\nu^3}{c^2} e^{-h\nu/(kT)} & (h\nu \gg kT) \\ \dfrac{2\nu^2 kT}{c^2} & (h\nu \ll kT) \end{cases}$$

Brightness Temperature T_b:

$$T_b = \frac{c^2 I_\nu}{2k\nu^2}$$

6. Emission and Absorption in Plasma

Integrated Volume Emissivity:

$$j(T) = C_1 Z^2 n_e n_i T^{1/2}$$

Optical Depth and Mean Free Path:

$$\tau_\nu = \alpha_\nu L, \quad \lambda_{\text{mfp}} = \frac{1}{n\sigma}$$

7. Flux and Luminosity

Flux from a Point Source:

$$F = \frac{L}{4\pi r^2}$$

Flux in Terms of Specific Intensity:

$$F = \int I_\nu \cos\theta \, d\Omega, \quad \text{For isotropic } I_\nu: \quad F = \pi I_\nu$$

Luminosity of a Spherical Blackbody:

$$L = 4\pi R^2 \sigma T^4$$

Luminosity of Optically Thin Bremsstrahlung Emission:

$$L = j(T)V = C_1 Z^2 n_e n_i T^{1/2} \times \frac{4}{3}\pi R^3$$

8. Radiative Transfer Solutions

Optically Thin Medium ($\tau_\nu \ll 1$):

$$I_\nu \approx I_\nu(0) + j_\nu L$$

Optically Thick Medium ($\tau_\nu \gg 1$):

$$I_\nu \approx S_\nu$$

General Solution with $I_\nu(0) = 0$ and Constant S_ν:

$$I_\nu = S_\nu(1 - e^{-\tau_\nu})$$

9. Thermal Bremsstrahlung Spectrum

Power Radiated in Frequency Interval $(\nu_1, f\nu_1)$:

$$P \propto n_e n_i Z^2 T^{1/2} \left(e^{-h\nu_1/(kT)} - e^{-hf\nu_1/(kT)} \right)$$

Frequency of Maximum Emission:

$$h\nu_{\max} \approx kT$$

10. Constants and Units

Physical Constants:

$h = 6.626 \times 10^{-34}\,\mathrm{J\,s}$	(Planck's constant)
$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$	(Speed of light)
$k = 1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$	(Boltzmann's constant)
$\varepsilon_0 = 8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$	(Vacuum permittivity)
$e = 1.602 \times 10^{-19}\,\mathrm{C}$	(Elementary charge)
$m_e = 9.109 \times 10^{-31}\,\mathrm{kg}$	(Electron mass)
$m_p = 1.673 \times 10^{-27}\,\mathrm{kg}$	(Proton mass)
$G = 6.674 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$	(Gravitational constant)
$\sigma = 5.6704 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$	(Stefan-Boltzmann constant)

Unit Conversions:

$$1\,\mathrm{eV} = 1.602 \times 10^{-19}\,\mathrm{J}, \quad 1\,\mathrm{erg} = 1 \times 10^{-7}\,\mathrm{J}, \quad 1\,\mathrm{ly} = 9.461 \times 10^{15}\,\mathrm{m}$$

11. Additional Relations

Density and Number Density:

$$\rho = \frac{M}{V}, \quad n = \frac{\rho}{\mu m_p}$$

where μ is the mean molecular weight.

Total Bremsstrahlung Luminosity:

$$L = C_1 Z^2 n_e n_i T^{1/2} V$$

Hydrostatic Equilibrium in Isothermal Gas Sphere:

$$\frac{dP}{dr} = -\rho \frac{GM(r)}{r^2}$$

Virial Theorem for Gravitational Systems:

$$2E_{\mathrm{kin}} + E_{\mathrm{grav}} = 0, \quad kT = \frac{GMm_p}{3R}$$