PEU 438 Assignment 1

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Contents

1 Problem 1

1.1 Blackbody Radiation Energy Inside the Eye

Radius of the Eye (r):

$$r = 1.5\,\text{cm} = 0.015\,\text{m}$$

Volume of the Eye (V_{eye}):

$$V_{\text{eye}} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(0.015\,\text{m})^3 = 1.4137 \times 10^{-5}\,\text{m}^3$$

Energy Density of Blackbody Radiation (u):

$$u = aT^4$$

Where $a = 7.5657 \times 10^{-16}\,\text{J} \cdot \text{m}^{-3} \cdot \text{K}^{-4}$ is the radiation constant, and $T = 37°\text{C} = 310.15\,\text{K}$.

$$T^4 = (310.15\,\text{K})^4 = 9.254 \times 10^9\,\text{K}^4$$

Then,

$$u = (7.5657 \times 10^{-16}\,\text{J} \cdot \text{m}^{-3} \cdot \text{K}^{-4})(9.254 \times 10^9\,\text{K}^4) = 7.0013 \times 10^{-6}\,\text{J/m}^3$$

Total Energy Inside the Eye (E_{eye}):

$$E_{\text{eye}} = u \times V_{\text{eye}} = (7.0013 \times 10^{-6}\,\text{J/m}^3)(1.4137 \times 10^{-5}\,\text{m}^3) = 9.9 \times 10^{-11}\,\text{J}$$

1.2 Energy Entering the Eye from Light Bulb

Intensity at 1 Meter (I):

$$I = \frac{\text{Power}}{4\pi r^2} = \frac{100\,\text{W}}{4\pi(1\,\text{m})^2} = 7.9577\,\text{W/m}^2$$

Area of the Pupil (A_{pupil}):

$$A_{\text{pupil}} = 0.1\,\text{cm}^2 = 1 \times 10^{-5}\,\text{m}^2$$

Power Entering the Eye (P_{eye}):

$$P_{\text{eye}} = I \times A_{\text{pupil}} = (7.9577\,\text{W/m}^2)(1 \times 10^{-5}\,\text{m}^2) = 7.9577 \times 10^{-5}\,\text{W}$$

Energy Entering the Eye in 1 Second (E_{in}):

$$E_{\text{in}} = P_{\text{eye}} \times t = (7.9577 \times 10^{-5}\,\text{W})(1\,\text{s}) = 7.9577 \times 10^{-5}\,\text{J}$$

1.3 Comparison

$$\frac{E_{\text{in}}}{E_{\text{eye}}} = \frac{7.9577 \times 10^{-5}\,\text{J}}{9.9 \times 10^{-11}\,\text{J}} \approx 8 \times 10^5$$

1.4 Answer

It is dark when we close our eyes because the energy of the blackbody photons inside the eye is below the threshold needed to stimulate the photoreceptors.

2 Problem 2

2.1 General Solution

$$I_\nu = I_\nu(\tau_\nu = 0)e^{-\tau_{\lambda,0}} + \int_0^{\tau_{\lambda,0}} S_\nu(\tau_\nu)e^{-(\tau_{\lambda,0}-\tau_\nu)}\,d\tau_\nu$$

$$I_\nu(\tau_\nu = 0) = 0, \quad S_\nu(\tau_\nu) = S_\nu$$

$$I_\nu = S_\nu e^{-\tau_{\lambda,0}} \int_0^{\tau_{\lambda,0}} e^{\tau_\nu}\,d\tau_\nu$$

$$I_\nu = S_\nu(1 - e^{-\tau_{\lambda,0}})$$

2.2 Small Optical Depth

$$e^{-\tau_{\lambda,0}} \approx 1 - \tau_{\lambda,0}$$

$$I_\nu = S_\nu \tau_{\lambda,0}$$

$$S_\nu = \frac{j_\nu}{\alpha_\nu}, \quad \tau_{\lambda,0} = \alpha_\nu L$$

$$I_\nu = j_\nu L$$

Where j_ν is the emission function of a specific wavelength.

2.3 Large Optical Depth

$$I_\nu = S_\nu$$

$$I_\nu = B_\lambda(T) = \frac{2hc^2}{\lambda^5} \cdot \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1}$$

3 Problem 3

3.1 General Solution

$$I_\nu = I_\nu(\tau_\nu = 0)e^{-\tau_{\lambda,0}} + \int_0^{\tau_{\lambda,0}} S_\nu(\tau_\nu)e^{-(\tau_{\lambda,0}-\tau_\nu)}\, d\tau_\nu$$

$$I_\nu(\tau_\nu = 0) = I_{\lambda,0}, \quad S_\nu(\tau_\nu) = S_\nu$$

$$I_\nu = I_{\lambda,0}e^{-\tau_{\lambda,0}} + S_\nu e^{-\tau_{\lambda,0}} \int_0^{\tau_{\lambda,0}} e^{\tau_\nu}\, d\tau_\nu$$

$$I_\nu = I_{\lambda,0}e^{-\tau_{\lambda,0}} + S_\nu(1 - e^{-\tau_{\lambda,0}})$$

3.2 Small Optical Depth

$$e^{-\tau_{\lambda,0}} \approx 1 - \tau_{\lambda,0}$$

$$I_\nu = (1 - \tau_{\lambda,0})I_{\lambda,0} + \tau_{\lambda,0}S_\nu = I_{\lambda,0} + (S_\nu - I_{\lambda,0})\tau_{\lambda,0}$$

$$S_\nu = \frac{j_\nu}{\alpha_\nu}, \quad \tau_{\lambda,0} = \alpha_\nu L$$

$$I_\nu = I_{\lambda,0}(1 - \alpha_\nu L) + j_\nu L = I_{\lambda,0} + (j_\nu - \alpha_\nu I_{\lambda,0})L$$

Where j_ν is the emission function of a specific wavelength and a_ν is the absorption coefficient for a specific wavelength.

3.3 Large Optical Depth

$$I_\nu = S_\nu$$

$$I_\nu = B_\lambda(T) = \frac{2hc^2}{\lambda^5} \cdot \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1}$$

4 Problem 4

4.1 Brightness Temperature and Energy Regime

4.1.1 Brightness Temperature

Convert Angular Diameter to Radians

$$\theta = 4.3' \times \left(\frac{1\,\text{deg}}{60'}\right) \times \left(\frac{\pi \text{ rad}}{180\,\text{deg}}\right) = 4.3 \times \frac{\pi}{10800} \approx 1.2507 \times 10^{-3} \text{ rad}$$

Calculate the Solid Angle Ω

$$\Omega = \pi\left(\frac{\theta}{2}\right)^2 = \pi\left(\frac{1.2507 \times 10^{-3} \text{ rad}}{2}\right)^2 = \pi\big(6.2535 \times 10^{-4} \text{ rad}\big)^2 \approx 1.2277 \times 10^{-6} \text{ sr}$$

Convert Flux Density to SI Units

$$F_{100} = \left(1.6 \times 10^{-19} \frac{\text{erg}}{\text{cm}^2 \text{ s Hz}}\right) \times \left(\frac{1 \times 10^{-7}\text{J}}{1 \text{ erg}}\right) \times \left(\frac{1}{(1 \times 10^{-2} \text{ m})^2}\right) = 1.6 \times 10^{-19} \times 1 \times 10^{-7} \times 1 \times 10^{4} \frac{\text{W}}{\text{m}^2 \text{ Hz}}$$

$$F_{100} = 1.6 \times 10^{-22} \frac{\text{W}}{\text{m}^2 \text{ Hz}}$$

Find the Wavelength λ

$$\lambda = \frac{c}{\nu} = \frac{3.0 \times 10^8 \text{m s}^{-1}}{1.0 \times 10^8 \text{Hz}} = 3.0\text{m}$$

Brightness Temperature

$$T_b = \frac{F_\nu \lambda^2}{2k\Omega}$$

Where:

- F_ν is the flux density at frequency ν
- λ is the wavelength

- k is Boltzmann's constant (1.38×10^{-23}J K^{-1})

Substitute the values:

$$T_b = \frac{(1.6 \times 10^{-22}\text{W m}^{-2}\text{Hz}^{-1}) \times (3.0\text{m})^2}{2 \times (1.38 \times 10^{-23}\text{J K}^{-1}) \times (1.2277 \times 10^{-6}\text{sr})} = 4.247 \times 10^7\text{K}$$

4.1.2 Energy Regime

Compare $h\nu$ and kT_b:

$$h\nu = (6.626 \times 10^{-34}\text{J s})(1.0 \times 10^8\text{Hz}) = 6.626 \times 10^{-26}\text{J}$$

$$kT_b = (1.38 \times 10^{-23}\text{J K}^{-1})(4.247 \times 10^7\text{K}) = 5.861 \times 10^{-16}\text{J}$$

Since $h\nu \ll kT_b$, the emission is in the **Rayleigh-Jeans** regime (long-wavelength limit of the blackbody curve).

4.2 Effect of Compact Emitting Region on Brightness Temperature

If the actual emitting region is more compact, the angular diameter θ is smaller, leading to a smaller solid angle Ω.

Since:

$$T_b \propto \frac{1}{\Omega}$$

A smaller Ω results in a higher brightness temperature T_b.

4.3 Frequency at Which the Radiation Peaks (Wien's Law)

Use Wien's Law for frequency:

$$\nu_{\max} = \frac{kT}{h} \times x$$

Where:

- k is Boltzmann's constant
- h is Planck's constant
- x is a constant approximately equal to 2.8214

Calculate $\nu_{\max}$:

$$\nu_{\max} = \frac{(1.38 \times 10^{-23}\text{J K}^{-1})(4.247 \times 10^{7}\text{K})}{6.626 \times 10^{-34}\text{J s}} \times 2.8214 = 2.497 \times 10^{18}\text{Hz}$$

5 Problem 5

a. Note that $j_\nu = P_\nu/4\pi$ and that, effectively, $\alpha_\nu = 0$, since the cloud is optically thin. Then, using Eq. (1.24),

$$I_\nu(b) = \int j_\nu(z)\, dz = \frac{P_\nu}{2\pi}\sqrt{R^2 - b^2}.$$

b. The total power emitted by the cloud is $L = (4/3)\pi R^3 P_\nu$, where $P = \int P_\nu\, d\nu$. Then

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4,$$

by definition of T_{eff}, so that

$$T_{\text{eff}} = \left(\frac{PR}{3\sigma}\right)^{1/4}.$$

c. Let d be the distance from the spherical cloud to the earth. Energy conservation gives a relation between F_ν, the flux at the earth, and P_ν:

$$4\pi d^2 F_\nu = \frac{4}{3}\pi R^3 P_\nu,$$

$$F_\nu = \frac{P_\nu R^3}{3d^2}.$$

d. From Eq. (1.30), with $S_\nu = B_\nu(T)$, $I_\nu(0) = 0$, $\tau_\nu \ll 1$,

$$I_\nu = B_\nu(T)(1 - e^{-\tau_\nu}) \approx \tau_\nu B_\nu(T) \ll B_\nu(T).$$

With the definition of T_b from Eq. (1.59),

$$B_\nu(T_b) \ll B_\nu(T),$$

and the monotonicity of $B_\nu(T)$ with T, we have $T_b \ll T$.

e. For the optically thick case the results are:

a'. From Eq. (1.30) with $\tau_\nu \gg 1$ and with $S_\nu = B_\nu(T)$ we have $I_\nu = B_\nu(T)$ independent of b.

b'. Since $I_\nu = B_\nu$, the flux at the surface is the blackbody flux, so $T_{\text{eff}} = T$.

c'. The monochromatic flux at the surface is $\pi B_\nu(T)$ [cf. Eq. (1.14)], so using the inverse square law gives

$$F_\nu(d) = \pi \left(\frac{R}{d}\right)^2 B_\nu(T).$$

d'. From (a') and Eq. (1.59) we have $B_\nu(T_b) = B_\nu(T)$, which implies $T_b = T$.

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A certain gas emits thermally at the rate $P(\nu)$ (power per unit volume and frequency range). A spherical cloud of this gas has radius R, temperature T, and is a distance d from Earth ($d \gg R$).

(a) Optically Thin Cloud: Brightness as a Function of b

For an optically thin cloud, the observed brightness $I_\nu(b)$ at Earth is the integrated emission along the line of sight at a distance b from the cloud center. The emission coefficient j_ν (power emitted per unit volume per unit solid angle per unit frequency) relates to $P(\nu)$ as:

$$j_\nu = \frac{P(\nu)}{4\pi}$$

The path length through the cloud at impact parameter b is:

$$L(b) = 2\sqrt{R^2 - b^2} \quad \text{for } b \leq R$$

Thus, the brightness $I_\nu(b)$ is:

$$I_\nu(b) = \int_{-\infty}^{+\infty} j_\nu(s)\, ds = \frac{P(\nu)}{4\pi} \times 2\sqrt{R^2 - b^2} = \frac{P(\nu)}{2\pi}\sqrt{R^2 - b^2}$$

(b) Optically Thin Cloud: Effective Temperature

$L = (4/3)\pi R^3 P_\nu$, where $P = \int P_\nu \, d\nu$.

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4,$$

$$T_{\text{eff}} = \left(\frac{PR}{3\sigma} \right)^{1/4}.$$

(c) Optically Thin Cloud: Flux F_ν Measured at Earth

The total luminosity L_ν emitted by the cloud at frequency ν is:

$$L_\nu = P(\nu) \times V = P(\nu) \times \frac{4}{3}\pi R^3$$

The flux F_ν measured at Earth is:

$$F_\nu = \frac{L_\nu}{4\pi d^2} = \frac{P(\nu) \times \frac{4}{3}\pi R^3}{4\pi d^2} = \frac{P(\nu) R^3}{3d^2}$$

(d) Optically Thin Cloud: Comparison of Brightness Temperatures

Since $T_B(b) \propto \sqrt{R^2 - b^2}$ and $P(\nu)$ is finite, the measured brightness temperatures $T_B(b)$ are much less than the actual cloud temperature T:

$$T_B(b) \ll T$$

This is because the cloud is optically thin and does not emit as a blackbody along the line of sight.

(e) Optically Thick Cloud

(a) Brightness as a Function of b

For an optically thick cloud, the specific intensity $I_\nu(b)$ at any impact parameter $b \leq R$ is equal to the blackbody intensity:

$$I_\nu(b) = B_\nu(T)$$

since the cloud is opaque and emits as a blackbody at temperature T.

(b) Effective Temperature

In this case, the brightness temperature is equal to the actual temperature:

$$T_B(b) = T$$

(c) Flux F_ν Measured at Earth

The flux measured at Earth is calculated from the projected area of the cloud and the blackbody intensity:

$$F_\nu = \frac{\text{Total Power Emitted Towards Earth}}{\text{Area at Distance } d} = \frac{I_\nu(b) \times \text{Projected Area}}{d^2} = \frac{B_\nu(T) \times \pi R^2}{d^2}$$

(d) Comparison of Brightness Temperatures

For an optically thick cloud, the measured brightness temperatures are equal to the cloud's temperature:

$$T_B(b) = T$$

This is because the cloud emits as a blackbody along every line of sight within its angular size.

References

[1] M.H. El-Deeb. PEU-438 Assignments.