Formula Sheet for High-Energy Astrophysics and Radiative Transfer

1. Radiative Transfer Basics

Specific Intensity I_ν:

- Describes the amount of energy passing through a unit area, in a unit time, within a unit solid angle, and per unit frequency interval.
$$dE = I_\nu \, dA \, dt \, d\Omega \, d\nu$$

Radiative Transfer Equation (RTE):

- Governs the change in specific intensity as radiation travels through a medium.
$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$
  - τ_ν: Optical depth, measures how opaque the medium is.
$$\tau_\nu = \int \alpha_\nu \, ds$$
  - α_ν: Absorption coefficient.
  - S_ν: Source function, represents emission per unit absorption.
$$S_\nu = \frac{j_\nu}{\alpha_\nu}$$
  - j_ν: Emissivity, the energy emitted per unit volume, time, solid angle, and frequency.

General Solution to RTE (with constant S_ν):

- Describes how I_ν changes with optical depth.
$$I_\nu = I_\nu(0) e^{-\tau_\nu} + S_\nu \left(1 - e^{-\tau_\nu}\right)$$

Limiting Cases:

- **Optically Thin ($\tau_\nu \ll 1$):**
$$I_\nu \approx I_\nu(0) \left(1 - \tau_\nu\right) + S_\nu \tau_\nu$$
- **Optically Thick ($\tau_\nu \gg 1$):**
$$I_\nu \approx S_\nu$$

2. Blackbody Radiation

Planck's Law (Blackbody Spectrum):

- Describes the specific intensity of blackbody radiation at frequency ν and temperature T.
$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/(kT)} - 1}$$
  - h: Planck's constant.
  - c: Speed of light.
  - k: Boltzmann's constant.

Stefan-Boltzmann Law:

- Total energy flux emitted by a blackbody per unit area.

$$F = \sigma T^4$$

  - σ: Stefan-Boltzmann constant ($\approx 5.6704 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$).

Wien's Displacement Law:

- Relates the temperature of a blackbody to the wavelength at which it emits most intensely.

$$\lambda_{\max} T = 2.898 \times 10^{-3}\,\mathrm{m\,K}$$

  - $\lambda_{\max}$: Wavelength at peak emission.

3. Kirchhoff's Law

- States that, at thermal equilibrium, the emissivity (j_ν) and absorptivity (α_ν) are related by the blackbody function.

$$\frac{j_\nu}{\alpha_\nu} = B_\nu(T)$$

4. Bremsstrahlung (Free-Free Emission)

Emissivity j_ν:

- Emission from electrons decelerating in the electric fields of ions.

$$j_\nu \propto n_e n_i Z^2 T^{-1/2} e^{-h\nu/(kT)}$$

  - n_e: Electron density.
  - n_i: Ion density.
  - Z: Charge number of ions.
  - The exponential term shows that higher frequencies are suppressed.

Free-Free Absorption Coefficient α_ν^{ff}:

- Represents how free electrons absorb photons in the presence of ions.

$$\alpha_\nu^{ff} \propto n_e n_i Z^2 T^{-1/2} \nu^{-3} \left(1 - e^{-h\nu/(kT)}\right)$$

5. Larmor's Formula

- Calculates the power radiated by an accelerating charged particle.

$$P = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3}$$

  - q: Charge of the particle.
  - a: Acceleration.
  - ε_0: Vacuum permittivity.

6. Kinetic Theory and Thermal Velocities

Root-Mean-Square Speed v_{rms}:

- Average speed of particles in a gas at temperature T.

$$v_{\mathrm{rms}} = \sqrt{\frac{3kT}{m}}$$

  - m: Mass of a gas particle.

Kinetic Energy per Particle:

- Energy associated with the motion of particles.

$$E_k = \frac{1}{2}mv_{\mathrm{rms}}^2 = \frac{3}{2}kT$$

7. Hydrostatic Equilibrium

- Describes the balance between gravity and pressure in a star or gas cloud.

$$\frac{dP}{dr} = -\rho\frac{GM(r)}{r^2}$$

  - P: Pressure.
  - ρ: Density.
  - G: Gravitational constant.
  - $M(r)$: Mass enclosed within radius r.

8. Virial Theorem

- Relates kinetic and potential energy in gravitational systems.

$$2E_{\mathrm{kin}} + E_{\mathrm{grav}} = 0$$

- For an ideal gas in a spherical distribution:

$$kT = \frac{GMm_p}{3R}$$

  - m_p: Proton mass.
  - R: Radius of the system.

9. Photon Energy and Frequency Relations

Energy of a Photon:

- The energy carried by a single photon.

$$E = h\nu$$

Brightness Temperature T_b:

- Temperature corresponding to the observed brightness at radio frequencies.

$$T_b = \frac{c^2 I_\nu}{2k\nu^2}$$

  - I_ν: Specific intensity.

10. Flux and Luminosity

Flux F:

- Energy per unit area per unit time received from a source.

$$F = \frac{L}{4\pi r^2}$$

  - L: Luminosity of the source.
  - r: Distance to the source.

Luminosity of a Blackbody Sphere:

- Total power emitted by a spherical blackbody.

$$L = 4\pi R^2 \sigma T^4$$

  - R: Radius of the sphere.

11. Units and Constants

Fundamental Constants:

$$\begin{aligned}
h &= 6.626 \times 10^{-34}\,\mathrm{J\,s} && \text{(Planck's constant)} \\
c &= 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}} && \text{(Speed of light)} \\
k &= 1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}} && \text{(Boltzmann's constant)} \\
G &= 6.674 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}} && \text{(Gravitational constant)} \\
\sigma &= 5.6704 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}} && \text{(Stefan-Boltzmann constant)} \\
e &= 1.602 \times 10^{-19}\,\mathrm{C} && \text{(Elementary charge)} \\
m_e &= 9.109 \times 10^{-31}\,\mathrm{kg} && \text{(Electron mass)} \\
m_p &= 1.673 \times 10^{-27}\,\mathrm{kg} && \text{(Proton mass)} \\
\varepsilon_0 &= 8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}} && \text{(Vacuum permittivity)}
\end{aligned}$$

Useful Conversions:

- $1\,\mathrm{eV} = 1.602 \times 10^{-19}\,\mathrm{J}$
- $1\,\mathrm{erg} = 1 \times 10^{-7}\,\mathrm{J}$
- $1\,\mathrm{ly} = 9.461 \times 10^{15}\,\mathrm{m}$ (Light-year)

12. Additional Relations

Density ρ and Number Density n:

- Mass density and particle number density.

$$\rho = nm, \quad n = \frac{\rho}{m}$$

  - m: Mass of a single particle.

Mean Free Path λ_{mfp}:

- Average distance a particle travels before interacting.

$$\lambda_{\mathrm{mfp}} = \frac{1}{n\sigma}$$

  - σ: Cross-sectional area for interaction.