

# PEU 323 Assignment 2

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## 1 Problem 1

### 1.1 Problem

Consider a particle which is free to be anywhere, with equal probability, on a line segment of length  $L$ .

- (a) Find the normalized probability distribution for such a particle.
- (b) Calculate  $\langle x \rangle$  and  $\sigma_x$ .
- (c) Calculate the probability of finding the particle within  $\sigma_x$  of  $\langle x \rangle$ .
- (d) What are the dimensions of the probability density?

## 1.2 Solution

(a)  $\frac{1}{L}$

(b)

$$\langle x \rangle = \frac{1}{L} \int_0^L x \, dx = \frac{L}{2}$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\langle x^2 \rangle = \frac{1}{L} \int_0^L x^2 \, dx = \frac{L^2}{3}$$

$$\langle x \rangle^2 = \frac{L^2}{4}$$

$$\sigma_x = \sqrt{\frac{L^2}{3} - \frac{L^2}{4}} = \frac{L}{2\sqrt{3}}$$

(c)  $\frac{1}{L} \int_{\frac{L}{2}(1-\frac{1}{\sqrt{3}})}^{\frac{L}{2}(1+\frac{1}{\sqrt{3}})} dx = \frac{1}{\sqrt{3}}$

(d)  $length^{-1}$

## 2 Problem 2

### 2.1 Problem

Buffon's Needle: A needle of length  $l$  is dropped at random on a sheet of paper with parallel lines a distance  $l$  apart. What is the probability that it crosses a line?

### 2.2 Solution

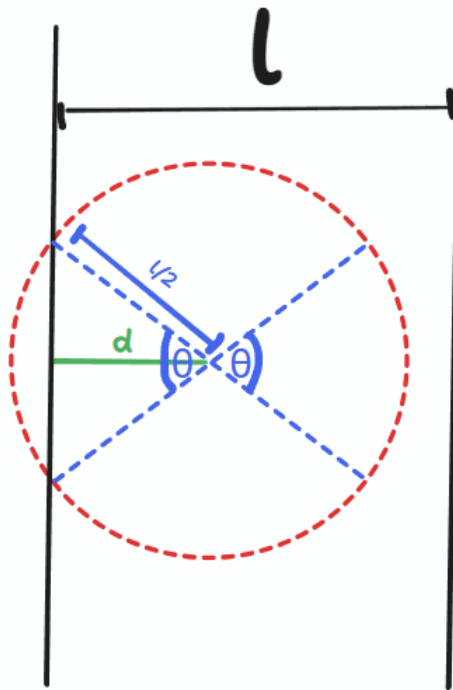


Figure 1: The black lines are the parallel lines of the paper, the red circle represents the possible area covered by the needle on a specific distance  $d$  from one of the lines.  $\theta$  is the angle in which the needle crosses one of the lines

$$\cos\left(\frac{\theta}{2}\right) = \frac{2d}{l}$$

$$\theta = 2 \cos^{-1}\left(\frac{2d}{l}\right)$$

$$P(d) = 4 \frac{\theta}{2\pi} = \frac{2}{\pi} \cos^{-1}\left(\frac{2d}{l}\right)$$

$$P = \int_0^{\frac{L}{2}} P(d) \, dd$$

$$= \frac{2}{\pi} \int_0^{\frac{L}{2}} \cos^{-1}\left(\frac{2d}{l}\right) \, dd$$

$$\because \int \cos^{-1}(x) \, dx = x \cos^{-1}(x) - \sqrt{1-x^2} + c$$

$$\therefore P = \left[ \frac{2d}{l} \cos^{-1}\left(\frac{2d}{l}\right) - \sqrt{1 - \left(\frac{2d}{l}\right)^2} \right]_0^{\frac{L}{2}} = \frac{2}{\pi}$$

### 3 Problem 3

#### 3.1 Problem

Find the probability distribution for the momentum of a harmonic oscillator with angular frequency  $\omega$ .

#### 3.2 Solution

$$y(t) = A \sin(\omega t)$$

$$p = my'(t) = mA\omega \cos(\omega t) = p_{max} \cos(\omega t)$$

$$t = \frac{\cos^{-1}(\frac{p}{p_{max}})}{\omega}$$

$$p' = -A\omega^2 \sin(\omega t) = -A\omega^2 \sin(\cos^{-1}(\frac{p}{p_{max}}))$$

$$p_f = \frac{p}{p_{max}}$$

We will absorb all constants and proportionality constants into  $\alpha$

$$\rho(p_f) \propto \frac{1}{p'} = \alpha \csc(\cos^{-1}(p_f))$$

$$\int_{-1}^1 \rho(p_f) dp_f = \alpha \int_{-1}^1 \csc(\cos^{-1}(p_f)) dp_f = \alpha\pi = 1$$

$$\alpha = \frac{1}{\pi}$$

$$\rho(p) = \frac{\csc(\cos^{-1}(\frac{p}{Am\omega}))}{\pi}$$

## 4 Problem 4

### 4.1 Problem

Consider the probability density for the location of the electron inside the Hydrogen atom:

$$\rho(r) = Ae^{-2r/a_0}, \quad (1)$$

where  $a_0$  is the Bohr radius.

- (a) Find  $A$  which normalizes this probability distribution.

*Hint:*

$$\int_0^\infty e^{-x} x^n dx = n!. \quad (2)$$

- (b) Calculate the probability for the electron to be found in a sphere, centered about the origin, of radius  $b_0$ , with  $b_0 \ll a_0$ .

*P.S.:* You can do this calculation exactly or approximately. The approximate one is much easier.

### 4.2 Solution

- (a)

$$\begin{aligned} \rho(r) &= Ae^{-2r/a_0} \\ \rho &= \int_0^\infty \rho(r) dr = 1 \\ \rho &= A \int_0^\infty e^{-\frac{2r}{a_0}} dr \\ \because \int e^{-\alpha x} dx &= -\frac{e^{-\alpha x}}{\alpha} + c \\ \therefore 1 &= -\frac{A}{\alpha} [e^{-\alpha r}]_0^\infty = \frac{A}{\alpha} \\ A &= \frac{2}{a_0} \end{aligned}$$

(b)

$$\begin{aligned}\rho(r) &= \frac{2}{a_0} e^{-2r/a_0} \\ \frac{2}{a_0} \int_0^{b_0} e^{-\alpha x} dx &= -\frac{2}{\alpha a_0} [e^{-\alpha x}]_0^{b_0} \\ &= 1 - e^{-2\frac{b_0}{a_0}} \approx 2\frac{b_0}{a_0}\end{aligned}$$

## 5 Problem 5

### 5.1 Problem

Consider the map

$$f : \mathbb{C} \rightarrow \mathbb{R}_{2 \times 2} \tag{3}$$

$$x + iy \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

from the complex numbers to the set of  $2 \times 2$  real matrices.

- (a) Show that this map is an isomorphism. That is, show that it is invertible and that for  $z_1, z_2 \in \mathbb{C}$ ,

$$f(z_1 z_2) = f(z_1) f(z_2) \tag{4}$$

and hence prove that, in two dimensions, rotations commute.

- (b) Prove De Moivre's formula for complex numbers:

$$(r(\cos(\theta) + i \sin(\theta)))^n = r^n (\cos(n\theta) + i \sin(n\theta)) \tag{5}$$

and hence prove that

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}^n = \begin{pmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{pmatrix} \tag{6}$$

## 5.2 Solution

(a)

$$\begin{aligned}
 f(z_1 z_2) &= f((x_1 + iy_1)(x_2 + iy_2)) = f((x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)) \\
 &= \begin{pmatrix} x_1 x_2 - y_1 y_2 & -x_1 y_2 - y_1 x_2 \\ x_1 y_2 + y_1 x_2 & x_1 x_2 - y_1 y_2 \end{pmatrix} \\
 f(z_1) f(z_2) &= \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix} \\
 &= \begin{pmatrix} x_1 x_2 - y_1 y_2 & -x_1 y_2 - y_1 x_2 \\ x_1 y_2 + y_1 x_2 & x_1 x_2 - y_1 y_2 \end{pmatrix}
 \end{aligned}$$

We can see that  $f(z_1 z_2)$  is just a multiplication of two complex numbers and that commutes.

(b)

$$\begin{aligned}
 e^{i\theta} &= \cos(\theta) + i \sin(\theta) \\
 (r(\cos(\theta) + i \sin(\theta)))^n &= r^n (\cos(\theta) + i \sin(\theta))^n \\
 &= r^n e^{in\theta} = r^n (\cos(n\theta) + i \sin(n\theta))
 \end{aligned} \tag{7}$$

Since the rotation transformation is isomorphic.

$$\begin{aligned}
 \therefore \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}^n &\mapsto (\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta) \\
 &\mapsto \begin{pmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{pmatrix}
 \end{aligned}$$



## References

- [1] M.H. El-Deeb. [PEU-323 Assignments](#).