PEU 323: Fall 2024
Probability and Complex Numbers

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1. Consider a particle which is free to be anywhere, with equal probability, on a line segment of length L.

   (a) Find the normalized probability distribution for such a particle.

   (b) Calculate $< x >$ and σ_x.

   (c) Calculate the probability of finding the particle within σ_x of $< x >$.

   (d) What are the dimensions of the probability density?

2. Buffon's Needle: A needle of length l is dropped at random on a sheet of paper with parallel lines a distance l apart. What is the probability that it crosses a line?

3. Find the probability distribution for the momentum of a harmonic oscillator with angular frequency ω.

4. Consider the probability density for the location of the electron inside the Hydrogen atom:

$$\rho(r) = Ae^{-2r/a_0}, \tag{1}$$

   where a_0 is the Bohr radius.

   (a) Find A which normalizes this probability distribution.

   Hint:

$$\int_0^\infty e^{-x}x^n dx = n!. \tag{2}$$

   (b) Calculate the probability for the electron to be found in a sphere, centered about the origin, of radius b_0, with $b_0 << a_0$.

   P.S.: You can do this calculation exactly or approximately. The approximate one is much easier.

5. Consider the map

$$\begin{aligned} f : \mathbb{C} &\to \mathbb{R}_{2x2} \\ x + iy &\mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \end{aligned} \tag{3}$$

from the complex numbers to the set of 2×2 real matrices.

(a) Show that this map is an isomorphism. That is, show that it is invertible and that for $z_1, z_2 \in \mathbb{C}$,

$$f(z_1 z_2) = f(z_1) f(z_2), \tag{4}$$

and hence prove that, in two dimensions, rotations commute.

(b) Prove De Moivre's formula for complex numbers:

$$\Big(r(cos(\theta) + isin(\theta))\Big)^n = r^n(cos(n\theta) + isin(n\theta)) \tag{5}$$

and hence prove that

$$\begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}^n = \begin{pmatrix} cos(n\theta) & -sin(n\theta) \\ sin(n\theta) & cos(n\theta) \end{pmatrix} \tag{6}$$