PEU 455 Assignment 3

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1 11.2.3

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(a)

$$u(x, y) = x^3 - 3xy^2$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}$$

$$v = \int 3x^2 - 3y^2 \, dy = 3yx^2 - y^3 + f(x) \tag{1}$$

$$\frac{\partial u}{\partial y} = -6xy = -\frac{\partial v}{\partial x}$$

$$v = 3x^2y + g(y) \tag{2}$$

from eq.1 and eq.2

$$v = 3x^2y - y^3$$

(b)

$$v(x, y) = e^{-y} \sin x$$

$$\frac{\partial v}{\partial y} = -e^{-y} \sin x = \frac{\partial u}{\partial x}$$

$$u = e^{-y} \cos x + f(y) \tag{3}$$

$$\frac{\partial v}{\partial x} = e^{-y} \cos x = -\frac{\partial u}{\partial y}$$

$$u = e^{-y} \cos x + g(x) \tag{4}$$

from eq.3 and eq.4

$$u = e^{-y} \cos x$$

2 11.2.4

From the fact w is analytical,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

From the fact w^* is analytical,

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

We can see that,

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = -\frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial y} = 0,$$

$$\therefore u = c_1, \quad v = c_2$$

3 11.2.7

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$u = R\cos\Theta, \quad v = R\sin\Theta$$

$$\frac{\partial u}{\partial R} = \cos\Theta, \quad \frac{\partial u}{\partial \Theta} = -R\sin\Theta, \quad \frac{\partial v}{\partial R} = \sin\Theta, \quad \frac{\partial v}{\partial \Theta} = R\cos\Theta$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial R}\frac{\partial R}{\partial x} + \frac{\partial u}{\partial \Theta}\frac{\partial \Theta}{\partial x}$$
$$\frac{\partial u}{\partial x} = \cos\Theta\frac{\partial R}{\partial x} - R\sin\Theta\frac{\partial \Theta}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial R}\frac{\partial R}{\partial y} + \frac{\partial u}{\partial \Theta}\frac{\partial \Theta}{\partial y}$$
$$\frac{\partial u}{\partial y} = \cos\Theta\frac{\partial R}{\partial y} - R\sin\Theta\frac{\partial \Theta}{\partial y}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial R}\frac{\partial R}{\partial x} + \frac{\partial v}{\partial \Theta}\frac{\partial \Theta}{\partial x}$$
$$\frac{\partial v}{\partial x} = \sin\Theta\frac{\partial R}{\partial x} + R\cos\Theta\frac{\partial \Theta}{\partial x}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial R}\frac{\partial R}{\partial y} + \frac{\partial v}{\partial \Theta}\frac{\partial \Theta}{\partial y}$$
$$\frac{\partial v}{\partial y} = \sin\Theta\frac{\partial R}{\partial y} + R\cos\Theta\frac{\partial \Theta}{\partial y}$$

From first relation,

$$\cos\Theta\frac{\partial R}{\partial x} - R\sin\Theta\frac{\partial \Theta}{\partial x} = \sin\Theta\frac{\partial R}{\partial y} + R\cos\Theta\frac{\partial \Theta}{\partial y}$$

$$\cos\Theta(\frac{\partial R}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial R}{\partial \theta}\frac{\partial \theta}{\partial x}) - R\sin\Theta(\frac{\partial \Theta}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \Theta}{\partial \theta}\frac{\partial \theta}{\partial x})$$
$$= \sin\Theta(\frac{\partial R}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial R}{\partial \theta}\frac{\partial \theta}{\partial y}) + R\cos\Theta(\frac{\partial \Theta}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \Theta}{\partial \theta}\frac{\partial \theta}{\partial y})$$

$$\cos\Theta(\frac{\partial R}{\partial r}\cos(\theta)-\frac{\partial R}{\partial \theta}\frac{\sin\theta}{r})-R\sin\Theta(\frac{\partial \Theta}{\partial r}\cos(\theta)-\frac{\partial \Theta}{\partial \theta}\frac{\sin\theta}{r})$$
$$=\sin\Theta(\frac{\partial R}{\partial r}\sin(\theta)+\frac{\partial R}{\partial \theta}\frac{\cos\theta}{r})+R\cos\Theta(\frac{\partial \Theta}{\partial r}\sin(\theta)+\frac{\partial \Theta}{\partial \theta}\frac{\cos\theta}{r})$$

$$(\cos(\Theta)\cos(\theta)-\sin(\Theta)\sin(\theta))\frac{\partial R}{\partial r}-\frac{1}{r}(\sin(\Theta)\cos(\theta)+\sin(\theta)\cos(\Theta))\frac{\partial R}{\partial \theta}$$
$$=\frac{R}{r}(\cos(\Theta)\cos(\theta)-\sin(\Theta)\sin(\theta))\frac{\partial \Theta}{\partial \theta}+R(\sin(\Theta)\cos(\theta)+\sin(\theta)\cos(\Theta))\frac{\partial \Theta}{\partial r}$$

$$(\cos(\Theta)\cos(\theta)-\sin(\Theta)\sin(\theta))(\frac{\partial R}{\partial r}-\frac{R}{r}\frac{\partial \Theta}{\partial \theta})-(\sin(\Theta)\cos(\theta)+\sin(\theta)\cos(\Theta))(\frac{1}{r}\frac{\partial R}{\partial \theta}+R\frac{\partial \Theta}{\partial r})=0 \tag{5}$$

From second relation,

$$\cos\Theta(\frac{\partial R}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial R}{\partial \theta}\frac{\partial \theta}{\partial y})-R\sin\Theta(\frac{\partial \Theta}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial \Theta}{\partial \theta}\frac{\partial \theta}{\partial y})$$
$$=-\sin\Theta(\frac{\partial R}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial R}{\partial \theta}\frac{\partial \theta}{\partial x})-R\cos\Theta(\frac{\partial \Theta}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial \Theta}{\partial \theta}\frac{\partial \theta}{\partial x})$$

$$\cos\Theta(\frac{\partial R}{\partial r}\sin(\theta)+\frac{\partial R}{\partial \theta}\frac{\cos\theta}{r})-R\sin\Theta(\frac{\partial \Theta}{\partial r}\sin(\theta)+\frac{\partial \Theta}{\partial \theta}\frac{\cos\theta}{r})$$
$$=-\sin\Theta(\frac{\partial R}{\partial r}\cos(\theta)-\frac{\partial R}{\partial \theta}\frac{\sin\theta}{r})-R\cos\Theta(\frac{\partial \Theta}{\partial r}\cos(\theta)-\frac{\partial \Theta}{\partial \theta}\frac{\sin\theta}{r})$$

$$(\sin(\theta)\cos(\Theta)+\sin(\Theta)\cos(\theta))(\frac{\partial R}{\partial r}-\frac{R}{r}\frac{\partial \Theta}{\partial \theta})+(\cos(\Theta)\cos(\theta)-\sin(\theta)\sin(\Theta))(\frac{1}{r}\frac{\partial R}{\partial \theta}+R\frac{\partial \Theta}{\partial r})=0 \tag{6}$$

(a)

$$(\cos(\Theta)\cos(\theta)-\sin(\theta)\sin(\Theta))\text{eq.}5+(\sin(\Theta)\cos(\theta)+\sin(\theta)\cos(\Theta))\text{eq.}6$$

$$((\sin(\theta)\cos(\Theta)+\sin(\Theta)\cos(\theta))^2+(\cos(\Theta)\cos(\theta)-\sin(\Theta)\sin(\theta))^2)(\frac{\partial R}{\partial r}-\frac{R}{r}\frac{\partial \Theta}{\partial \theta})=0$$

Since,

$$\sin(\theta)\cos(\Theta) + \sin(\Theta)\cos(\theta) = \sin(\theta + \Theta)$$

$$\cos(\Theta)\cos(\theta) - \sin(\Theta)\sin(\theta) = \cos(\theta + \Theta)$$

$$(\sin(\theta)\cos(\Theta) + \sin(\Theta)\cos(\theta))^2 + (\cos(\Theta)\cos(\theta) - \sin(\Theta)\sin(\theta))^2 = 1$$

$$\frac{\partial R}{\partial r} = \frac{R}{r}\frac{\partial \Theta}{\partial \theta} \tag{7}$$

(b) Substituting angle addition identity and eq.7 in eq.6 we get,

$$-\cos(\Theta + \theta)(\frac{1}{r}\frac{\partial R}{\partial \theta} + R\frac{\partial \Theta}{\partial r}) = 0$$

Since this equation holds for all θ and θ then,

$$\frac{1}{r}\frac{\partial R}{\partial \theta} \stackrel{!}{=} -R\frac{\partial \Theta}{\partial r}$$

4 11.2.8

$$\frac{\partial R}{\partial r} = \frac{R}{r}\frac{\partial \Theta}{\partial \theta}, \quad \frac{1}{r}\frac{\partial R}{\partial \theta} = -R\frac{\partial \Theta}{\partial r}$$

$$\frac{\partial \Theta}{\partial \theta} = \frac{r}{R}\frac{\partial R}{\partial r}$$

$$\begin{aligned}\frac{1}{r^2}\frac{\partial^2 \Theta}{\partial \theta^2} &= \frac{1}{r}\frac{\partial}{\partial \theta}(\frac{1}{R}\frac{\partial R}{\partial r}) \\ &= \frac{1}{Rr}(\frac{\partial^2 R}{\partial \theta \partial r} - \frac{1}{R}\frac{\partial R}{\partial \theta}\frac{\partial R}{\partial r})\end{aligned} \tag{1}$$

$$\begin{aligned}\frac{\partial^2 \Theta}{\partial r^2} &= \frac{\partial}{\partial r}(\frac{\partial \Theta}{\partial r}) \\ &= -\frac{\partial}{\partial r}(\frac{1}{Rr}\frac{\partial R}{\partial \theta}) \\ &= -\frac{\partial}{\partial r}(\frac{1}{Rr})\frac{\partial R}{\partial \theta} - \frac{1}{Rr}\frac{\partial}{\partial r}(\frac{\partial R}{\partial \theta}) \\ &= \frac{1}{Rr}(\frac{1}{R}\frac{\partial R}{\partial r}\frac{\partial R}{\partial \theta} + \frac{1}{r}\frac{\partial R}{\partial \theta} - \frac{\partial^2 R}{\partial r \partial \theta})\end{aligned} \tag{2}$$

$$\frac{1}{r}\frac{\partial \Theta}{\partial r} = -\frac{1}{Rr^2}\frac{\partial R}{\partial \theta} \tag{3}$$

$$\frac{\partial^2 \Theta}{\partial r^2} + \frac{1}{r}\frac{\partial \Theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Theta}{\partial \theta^2} = 0$$

5 11.2.11

(a)

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = V_x - i\frac{\partial u}{\partial y} = V_x - iV_y$$

(b) Since the Laplacian of an analytic function of z is zero

$$\nabla.V = \nabla.\nabla u = \nabla^2 u = 0$$

(c) From the curl of gradient identity

$$\nabla \times \nabla u = 0$$

6 11.3.6

$$z^* = x - iy$$

For path 1,
Segment 1: $dz = dx, \quad x_1 = 0, \quad x_2 = 1, \quad y = 0$
Segment 2: $dz = idy, \quad y_1 = 0, \quad y_2 = 1, \quad x = 1$

$$\int_0^{1+i} z^* \, dz = \int_0^1 x dx + i \int_0^1 (1 - iy) dy = 1 + i$$

For path 2,
Segment 1: $dz = idy, \quad y_1 = 0, \quad y_2 = 1, \quad x = 0$
Segment 2: $dz = dx, \quad x_1 = 0, \quad x_2 = 1, \quad y = 1$

$$\int_0^{1+i} z^* \, dz = \int_0^1 y dy + \int_0^1 (x - i) dx = 1 - i$$

7 11.3.7

$$\int_C \frac{1}{z(z+1)} dz$$

$$\int_C \frac{1}{z} dz - \int_C \frac{1}{z+1} dz = 2\pi i - 2\pi i = 0$$

8 11.4.1

$$\frac{1}{2\pi i}\oint z^{m-n-1}\,dz$$

for m = n,

$$\frac{1}{2\pi i}\oint z^{-1}\,dz = \frac{1}{2\pi i}2\pi i = 1$$

for $m \neq n$ the exponent won't be -1 so,

$$\oint z^{m-n-1}\,dz = 0$$

$$\frac{1}{2\pi i}\oint z^{m-n-1}\,dz = \delta_{mn}$$

9 11.4.6

The contours encapsulate the singularity point of $z = 0$

$$f(z) = e^{iz}$$

$$n = 3$$

$$\oint \frac{f(z)}{z^n}\, dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(0)$$

$$\oint \frac{e^{iz}}{z^3}\, dz = -\pi i$$

References

[1] G.B. Arfken, H.J. Weber, and F.E. Harris. *Mathematical Methods for Physicists: A Comprehensive Guide*. Elsevier Science, 2013.

[2] M.H. El-Deeb. PEU-455 Assignments.