

# PEU 455 Assignment 1

Mohamed Hussien El-Deeb (201900052)

## Contents

<b>1</b>	<b>8.2.3</b>	<b>1</b>
<b>2</b>	<b>8.2.5</b>	<b>3</b>
<b>3</b>	<b>8.2.7</b>	<b>4</b>
<b>4</b>	<b>8.2.8</b>	<b>5</b>
<b>5</b>	<b>8.3.4</b>	<b>7</b>
<b>6</b>	<b>8.3.6</b>	<b>9</b>

## **1 8.2.3**

Chebyshev:

$$(1 - x^2)y'' - xy' + n^2y = 0$$

Operator:

$$\mathcal{L}(x) = P_0(x)\frac{d^2}{dx^2} + P_1(x)\frac{d}{dx} + P_2(x)$$

Self-adjoint Condition:

$$P'_0(x) = P_1(x)$$

Weighting Function:

$$w(x) = (1 - x^2)^{-\frac{1}{2}}$$

Chebyshev  $\ast w(x)$ :

$$(1-x^2)^{\frac{1}{2}}y'' - x(1-x^2)^{-\frac{1}{2}}y' + n^2(1-x^2)^{-\frac{1}{2}}y = 0$$

$$P_0(x) = (1-x^2)^{\frac{1}{2}}$$

$$P'_0(x) = -x(1-x^2)^{-\frac{1}{2}}$$

$$P_1(x) = -x(1-x^2)^{-\frac{1}{2}}$$

We can since that Self-adjoint Condition holds.

## 2 8.2.5

Given:

$$\mathcal{L}u_1(x) = \lambda_1 u_1(x)$$

$$\mathcal{L}u_2(x) = \lambda_2 u_2(x)$$

$$\lambda_1 \neq \lambda_2 \tag{1}$$

To prove:

$$u_1(x) \neq \alpha u_2(x)$$

Prove by contradiction:

$$\text{Let } u_1(x) = \alpha u_2(x) \tag{2}$$

$$\therefore \mathcal{L}u_1(x) = \alpha \mathcal{L}u_2(x) = \alpha \lambda_2 u_2(x) = \lambda_1 u_1(x)$$

$$\therefore u_1(x) = \frac{\lambda_2}{\lambda_1} \alpha u_2(x)$$

In order to satisfy (2)  $\lambda_1 \stackrel{!}{=} \lambda_2$  which contradicts (1).

### 3 8.2.7

$$\int_{-1}^1 T_0^*(x) V_1(x) w(x) dx$$

$$\int_{-1}^1 (1-x^2)^{\frac{1}{2}} (1-x^2)^{-\frac{1}{2}} dx = \int_{-1}^1 dx = 2$$

Since the result of the integral is not zero, therefore  $T_0$  and  $V_1$  are not orthogonal on the range  $(-1, 1)$  with the weighting function  $(1-x^2)^{-\frac{1}{2}}$ .

## 4 8.2.8

Given:

$$\frac{d}{dx} \left[ P(x) \frac{d}{dx} u_n(x) \right] + \lambda_n w(x) u_n(x) = 0$$

$$\mathcal{L}u_m(x) = \lambda_m u_m(x)$$

$$\mathcal{L}u_n(x) = \lambda_n u_n(x)$$

$$\lambda_1 \neq \lambda_2$$

$$\int_a^b w(x) u_m^*(x) u_n(x) dx = 0$$

To Prove:

$$\int_{a'}^{b'} u_m'^*(x) u_n'(x) P(x) dx = 0$$

Solution:

$$\frac{d}{dx} [P(x) u_n'(x)] + \lambda_n w(x) u_n(x) = 0$$

$$\int_a^b u_m^*(x) \frac{d}{dx} [P(x) u_n'(x)] dx + \lambda_n \int_a^b w(x) u_m^*(x) u_n(x) dx = 0$$

$$\int_a^b u_m^*(x) \frac{d}{dx} [P(x) u_n'(x)] dx = 0$$

$$\int_a^b P(x) u_n'(x) \frac{d}{dx} [u_m^*(x)] dx = P(x) u_n'(x) u_m^*(x)$$

$$\int_a^b P(x) u'_n(x) u'^{*}_m(x) dx = [P(x) u'_n(x) u^*_m(x)]_a^b$$

Under this boundary condition:

$$[P(x) u'_n(x) u^*_m(x)]_a^b = 0$$

Dot product of  $u'_n(x)$  and  $u'_m$  lead to orthogonality under the weighting function  $P(x)$

## 5 8.3.4

Given:

$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0$$

Let:

$$L_n(x) = \sum_{j=0}^{\infty} a_j^{(n)} x^{s+j}$$

Therefore:

$$L_n'(x) = \sum_{j=0}^{\infty} (s+j) a_j^{(n)} x^{s+j-1}$$

$$L_n''(x) = \sum_{j=0}^{\infty} (s+j)(s+j-1) a_j^{(n)} x^{s+j-2}$$

$$x \sum_{j=0}^{\infty} (s+j)(s+j-1) a_j^{(n)} x^{s+j-2} + (1-x) \sum_{j=0}^{\infty} (s+j) a_j^{(n)} x^{s+j-1} + n \sum_{j=0}^{\infty} a_j^{(n)} x^{s+j} = 0$$

$$\sum_{j=0}^{\infty} (s+j)(s+j-1) a_j^{(n)} x^{s+j-1} + \sum_{j=0}^{\infty} (s+j) a_j^{(n)} x^{s+j-1} - \sum_{j=0}^{\infty} (s+j) a_j^{(n)} x^{s+j} + \sum_{j=0}^{\infty} n a_j^{(n)} x^{s+j} = 0$$

$$\sum_{j=0}^{\infty} a_j^{(n)} (s+j)^2 x^{s+j-1} - \sum_{j=0}^{\infty} a_j^{(n)} (s+j-n) x^{s+j} = 0$$

$$\sum_{j=-1}^{\infty} a_{j+1}^{(n)} (s+j+1)^2 x^{s+j} - \sum_{j=0}^{\infty} a_j^{(n)} (s+j-n) x^{s+j} = 0$$

$$a_0^{(n)} s^2 x^{s-1} + \sum_{j=0}^{\infty} a_{j+1}^{(n)} (s+j+1)^2 x^{s+j} - \sum_{j=0}^{\infty} a_j^{(n)} (s+j-n) x^{s+j} = 0$$

$$a_0^{(n)} s^2 x^{s-1} + x^s \sum_{j=0}^{\infty} (a_{j+1}^{(n)} (s+j+1)^2 - a_j^{(n)} (s+j-n)) x^j = 0$$

$$a_0^{(n)} s^2 + \sum_{j=0}^{\infty} \left( a_{j+1}^{(n)} (s+j+1)^2 - a_j^{(n)} (s+j-n) \right) x^{j+1} = 0$$

We need to make the summation be always zero regardless of x so the constant term must be zero.

Therefore  $a_0^{(n)}$  or  $s^2$  must be zero.

$$\sum_{j=0}^{\infty} \left( a_{j+1}^{(n)} (s+j+1)^2 - a_j^{(n)} (s+j-n) \right) x^{j+1} = 0$$

For the same reason

$$a_{j+1}^{(n)} (s+j+1)^2 - a_j^{(n)} (s+j-n) = 0$$

$$a_{j+1}^{(n)} = \frac{s+j-n}{(s+j+1)^2} a_j^{(n)}$$

From this we can't set  $a_0 = 0$  because all subsequent  $a_j$  must be zero as well making the solution a trivial one.

Therefore s must be zero.

$$a_{j+1}^{(n)} = \frac{j-n}{(j+1)^2} a_j^{(n)}$$

Where j starts from 0.

For the series to be finite the parameter n must be a positive integer.

and the polynomial solution will be of order n.



## 6 8.3.6

Given:

$$(1 - x^2)U_n''(x) - 3xU_n'(x) + n(n+2)U_n(x) = 0$$

Let:

$$U_n(x) = \sum_{j=0}^{\infty} a_j^{(n)} x^{s+j}$$

Therefore:

$$U_n'(x) = \sum_{j=0}^{\infty} (s+j)a_j^{(n)} x^{s+j-1}$$

$$U_n''(x) = \sum_{j=0}^{\infty} (s+j)(s+j-1)a_j^{(n)} x^{s+j-2}$$

$$(1-x^2) \sum_{j=0}^{\infty} (s+j)(s+j-1)a_j^{(n)} x^{s+j-2} - 3x \sum_{j=0}^{\infty} (s+j)a_j^{(n)} x^{s+j-1} + n(n+2) \sum_{j=0}^{\infty} a_j^{(n)} x^{s+j} = 0$$

$$\begin{aligned} & \sum_{j=0}^{\infty} (s+j)(s+j-1)a_j^{(n)} x^{s+j-2} - \sum_{j=0}^{\infty} (s+j)(s+j-1)a_j^{(n)} x^{s+j} \\ & - \sum_{j=0}^{\infty} 3(s+j)a_j^{(n)} x^{s+j} + \sum_{j=0}^{\infty} n(n+2)a_j^{(n)} x^{s+j} = 0 \end{aligned}$$

$$\sum_{j=0}^{\infty} (s+j)(s+j-1)a_j^{(n)} x^{s+j-2} + \sum_{j=0}^{\infty} (n(n+2) - (s+j)(s+j+2)) a_j^{(n)} x^{s+j} = 0$$

$$\sum_{j=-2}^{\infty} (s+j+2)(s+j+1)a_{j+2}^{(n)}x^{s+j} + \sum_{j=0}^{\infty} (n(n+2) - (s+j)(s+j+2))a_j^{(n)}x^{s+j} = 0$$

$$\begin{aligned} s(s-1)a_0^{(n)}x^{s-2} + s(s+1)a_1^{(n)}x^{s-1} + \sum_{j=0}^{\infty} (s+j+2)(s+j+1)a_{j+2}^{(n)}x^{s+j} \\ + \sum_{j=0}^{\infty} (n(n+2) - (s+j)(s+j+2))a_j^{(n)}x^{s+j} = 0 \end{aligned}$$

$$\begin{aligned} s(s-1)a_0^{(n)}x^{s-2} + s(s+1)a_1^{(n)}x^{s-1} + \\ \sum_{j=0}^{\infty} \left( (s+j+2)(s+j+1)a_{j+2}^{(n)} + (n(n+2) - (s+j)(s+j+2))a_j^{(n)} \right) x^{s+j} = 0 \end{aligned}$$

Similar to last question we figure that:

$$s(s-1)a_0^{(n)} = 0$$

$$s(s+1)a_1^{(n)} = 0$$

$$a_{j+2}^{(n)} = \frac{(s+j+2)(s+j) - n(n+2)}{(s+j+1)(s+j+2)}a_j^{(n)}$$

For the polynomial to be finite:

$$(s+j+2)(s+j) = n(n+2)$$

So, the truncation value of j is

$$j = n - s$$

Therefore  $n - s$  is positive odd integer for odd solutions

Choosing odd solutions means that we could set  $a_0 = 0$  so  $s$  could be 0 or -1

We will choose  $s = 0$

$$a_{j+2}^{(n)} = \frac{j(j+2) - n(n+2)}{(j+1)(j+2)} a_j^{(n)}$$

Where  $n$  is an odd positive number and  $j$  starts from 1

## References

- [1] G.B. Arfken, H.J. Weber, and F.E. Harris. *Mathematical Methods for Physicists: A Comprehensive Guide*. Elsevier Science, 2013.
- [2] M.H. El-Deeb. [PEU-455 Assignments](#).