PEU 455 Assignment 1

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1 8.2.3

Chebyshev:

$$(1-x^2)y'' - xy' + n^2y = 0$$

Operator:

$$\mathcal{L}(x) = P_0(x)\frac{d^2}{dx^2} + P_1(x)\frac{d}{dx} + P_2(x)$$

Self-adjoint Condition:

$$P_0'(x) = P_1(x)$$

Weighting Function:

$$w(x) = (1 - x^2)^{-\frac{1}{2}}$$

Chebyshev $* \, w(x)$:

$$(1 - x^2)^{\frac{1}{2}} y'' - x(1 - x^2)^{-\frac{1}{2}} y' + n^2 (1 - x^2)^{-\frac{1}{2}} y = 0$$

$$P_0(x) = (1 - x^2)^{\frac{1}{2}}$$

$$P_0'(x) = -x(1 - x^2)^{-\frac{1}{2}}$$

$$P_1(x) = -x(1 - x^2)^{-\frac{1}{2}}$$

We can since that Self-adjoint Condition holds.

2 8.2.5

Given:

$$\mathcal{L}u_1(x) = \lambda_1 u_1(x)$$

$$\mathcal{L}u_2(x) = \lambda_2 u_2(x)$$

$$\lambda_1 \neq \lambda_2 \tag{1}$$

To prove:

$$u_1(x) \neq \alpha u_2(x)$$

Prove by contradiction:

$$Let \quad u_1(x) = \alpha u_2(x) \tag{2}$$

$$\therefore \mathcal{L}u_1(x) = \alpha \mathcal{L}u_2(x) = \alpha \lambda_2 u_2(x) = \lambda_1 u_1(x)$$

$$\therefore u_1(x) = \frac{\lambda_2}{\lambda_1} \alpha u_2(x)$$

In order to satisfy (2) $\lambda_1 \stackrel{!}{=} \lambda_2$ which contradicts (1).

3 8.2.7

$$\int_{-1}^{1} T_0^*(x)V_1(x)w(x)\,dx$$

$$\int_{-1}^{1} (1-x^2)^{\frac{1}{2}}(1-x^2)^{-\frac{1}{2}}\,dx = \int_{-1}^{1} dx = 2$$

Since the result of the integral is not zero, therefore T_0 and V_1 are not orthogonal on the range $(-1, 1)$ with the weighting function $(1-x^2)^{-\frac{1}{2}}$.

4 8.2.8

Given:

$$\frac{d}{dx}\left[P(x)\frac{d}{dx}u_n(x)\right] + \lambda_n w(x)u_n(x) = 0$$

$$\mathcal{L}u_m(x) = \lambda_m u_m(x)$$

$$\mathcal{L}u_n(x) = \lambda_n u_n(x)$$

$$\lambda_1 \neq \lambda_2$$

$$\int_a^b w(x)u_m^*(x)u_n(x)\,dx = 0$$

To Prove:

$$\int_{a'}^{b'} {u'_m}^*(x)u'_n(x)P(x)\,dx = 0$$

Solution:

$$\frac{d}{dx}\left[P(x)u'_n(x)\right] + \lambda_n w(x)u_n(x) = 0$$

$$\int_a^b u_m^*(x)\frac{d}{dx}\left[P(x)u'_n(x)\right]\,dx + \lambda_n \int_a^b w(x)u_m^*(x)u_n(x)\,dx = 0$$

$$\int_a^b u_m^*(x)\frac{d}{dx}\left[P(x)u'_n(x)\right]\,dx = 0$$

$$\int_a^b P(x)u'_n(x)\frac{d}{dx}\left[u_m^*(x)\right]\,dx = P(x)u'_n(x)u_m^*(x)$$

$$\int_a^b P(x) u_n'(x) {u_m'}^*(x)\, dx = [P(x) u_n'(x) u_m^*(x)]_a^b$$

Under this boundary condition:

$$[P(x) u_n'(x) u_m^*(x)]_a^b = 0$$

Dot product of $u_n'(x)$ and u_m' lead to orthogonality under the weighting function $P(x)$

5 8.3.4

Given:

$$xL_n^{''}(x) + (1-x)L_n^{'}(x) + nL_n(x) = 0$$

Let:

$$L_n(x) = \sum_{j=0}^{\infty} a_j^{(n)} x^{s+j}$$

Therefore:

$$L_n^{'}(x) = \sum_{j=0}^{\infty} (s+j) a_j^{(n)} x^{s+j-1}$$

$$L_n^{''}(x) = \sum_{j=0}^{\infty} (s+j)(s+j-1) a_j^{(n)} x^{s+j-2}$$

$$x\sum_{j=0}^{\infty}(s+j)(s+j-1)a_j^{(n)}x^{s+j-2}+(1-x)\sum_{j=0}^{\infty}(s+j)a_j^{(n)}x^{s+j-1}+n\sum_{j=0}^{\infty}a_j^{(n)}x^{s+j}=0$$

$$\sum_{j=0}^{\infty}(s+j)(s+j-1)a_j^{(n)}x^{s+j-1}+\sum_{j=0}^{\infty}(s+j)a_j^{(n)}x^{s+j-1}-\sum_{j=0}^{\infty}(s+j)a_j^{(n)}x^{s+j}+\sum_{j=0}^{\infty}na_j^{(n)}x^{s+j}=0$$

$$\sum_{j=0}^{\infty} a_j^{(n)}(s+j)^2 x^{s+j-1} - \sum_{j=0}^{\infty} a_j^{(n)}(s+j-n)x^{s+j} = 0$$

$$\sum_{j=-1}^{\infty} a_{j+1}^{(n)}(s+j+1)^2 x^{s+j} - \sum_{j=0}^{\infty} a_j^{(n)}(s+j-n)x^{s+j} = 0$$

$$a_0^{(n)} s^2 x^{s-1} + \sum_{j=0}^{\infty} a_{j+1}^{(n)} (s+j+1)^2 x^{s+j} - \sum_{j=0}^{\infty} a_j^{(n)} (s+j-n) x^{s+j} = 0$$

$$a_0^{(n)} s^2 x^{s-1} + x^s \sum_{j=0}^{\infty} (a_{j+1}^{(n)} (s+j+1)^2 - a_j^{(n)} (s+j-n)) x^j = 0$$

$$a_0^{(n)} s^2 + \sum_{j=0}^{\infty} \left(a_{j+1}^{(n)} (s+j+1)^2 - a_j^{(n)} (s+j-n) \right) x^{j+1} = 0$$

We need to make the summation be always zero regardless of x so the constant term must be zero.

Therefore $a_0^{(n)}$ or s^2 must be zero.

$$\sum_{j=0}^{\infty} \left(a_{j+1}^{(n)} (s+j+1)^2 - a_j^{(n)} (s+j-n) \right) x^{j+1} = 0$$

For the same reason

$$a_{j+1}^{(n)} (s+j+1)^2 - a_j^{(n)} (s+j-n) = 0$$

$$a_{j+1}^{(n)} = \frac{s+j-n}{(s+j+1)^2} a_j^{(n)}$$

From this we can't set $a_0 = 0$ because all subsequent a_j must be zero as well making the solution a trivial one.

Therefore s must be zero.

$$a_{j+1}^{(n)} = \frac{j-n}{(j+1)^2} a_j^{(n)}$$

Where j starts from 0.

For the series to be finite the parameter n must be a positive integer.

and the polynomial solution will be of order n.

6 8.3.6

Given:

$$(1-x^2)U_n^{''}(x) - 3xU_n^{'}(x) + n(n+2)U_n(x) = 0$$

Let:

$$U_n(x) = \sum_{j=0}^{\infty} a_j^{(n)} x^{s+j}$$

Therefore:

$$U_n^{'}(x) = \sum_{j=0}^{\infty} (s+j) a_j^{(n)} x^{s+j-1}$$

$$U_n^{''}(x) = \sum_{j=0}^{\infty} (s+j)(s+j-1) a_j^{(n)} x^{s+j-2}$$

$$(1-x^2)\sum_{j=0}^{\infty}(s+j)(s+j-1)a_j^{(n)}x^{s+j-2} - 3x\sum_{j=0}^{\infty}(s+j)a_j^{(n)}x^{s+j-1} + n(n+2)\sum_{j=0}^{\infty}a_j^{(n)}x^{s+j} = 0$$

$$\sum_{j=0}^{\infty}(s+j)(s+j-1)a_j^{(n)}x^{s+j-2} - \sum_{j=0}^{\infty}(s+j)(s+j-1)a_j^{(n)}x^{s+j} - \sum_{j=0}^{\infty}3(s+j)a_j^{(n)}x^{s+j} + \sum_{j=0}^{\infty}n(n+2)a_j^{(n)}x^{s+j} = 0$$

$$\sum_{j=0}^{\infty}(s+j)(s+j-1)a_j^{(n)}x^{s+j-2} + \sum_{j=0}^{\infty}\left(n(n+2) - (s+j)(s+j+2)\right)a_j^{(n)}x^{s+j} = 0$$

$$\sum_{j=-2}^{\infty}(s+j+2)(s+j+1)a_{j+2}^{(n)}x^{s+j}+\sum_{j=0}^{\infty}\left(n(n+2)-(s+j)(s+j+2)\right)a_j^{(n)}x^{s+j}=0$$

$$s(s-1)a_0^{(n)}x^{s-2}+s(s+1)a_1^{(n)}x^{s-1}+\sum_{j=0}^{\infty}(s+j+2)(s+j+1)a_{j+2}^{(n)}x^{s+j}$$
$$+\sum_{j=0}^{\infty}\left(n(n+2)-(s+j)(s+j+2)\right)a_j^{(n)}x^{s+j}=0$$

$$s(s-1)a_0^{(n)}x^{s-2}+s(s+1)a_1^{(n)}x^{s-1}+$$
$$\sum_{j=0}^{\infty}\left((s+j+2)(s+j+1)a_{j+2}^{(n)}+\left(n(n+2)-(s+j)(s+j+2)\right)a_j^{(n)}\right)x^{s+j}=0$$

Similar to last question we figure that:

$$s(s-1)a_0^{(n)}=0$$

$$s(s+1)a_1^{(n)}=0$$

$$a_{j+2}^{(n)}=\frac{(s+j+2)(s+j)-n(n+2)}{(s+j+1)(s+j+2)}a_j^{(n)}$$

For the polynomial to be finite:

$$(s+j+2)(s+j)=n(n+2)$$

So, the truncation value of j is

$$j=n-s$$

Therefore $n - s$ is positive odd integer for odd solutions

Choosing odd solutions means that we could set $a_0 = 0$ so s could be 0 or -1

We will choose s = 0

$$a_{j+2}^{(n)} = \frac{j(j+2) - n(n+2)}{(j+1)(j+2)} a_j^{(n)}$$

Where n is an odd positive number and j starts from 1

References

[1] G.B. Arfken, H.J. Weber, and F.E. Harris. *Mathematical Methods for Physicists: A Comprehensive Guide*. Elsevier Science, 2013.

[2] M.H. El-Deeb. PEU-455 Assignments.