# PEU 455 Assignment 4

Mohamed Hussien El-Deeb (201900052)

## Contents

## 11.4.7

$$\oint_C \frac{\sin^2 z - z^2}{(z-a)^3} dz$$

This function has a singularity at z = a within the contour.

We will assume that f(z) is a analytical within the contour.

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z-z_0)^{n+1}}$$

$$f^{(2)}(a) = \frac{1}{\pi i} \oint_C \frac{\sin^2 z - z^2 dz}{(z-a)^3}$$

$$\oint_C \frac{\sin^2 z - z^2 dz}{(z-a)^3} = \pi i \left[\sin^2 z - z^2\right]''_{z=a}$$

$$= \pi i[2\sin(z)\cos(z) - 2z]'_{z=a}$$

$$= 2\pi i\left[\cos^2(a) - \sin^2(a) - 1\right] = -4\pi i \sin^2(a)$$

## 11.4.8

$$I = \oint_C \frac{dz}{z(2z+1)}$$

We have two singularities ($z = 0, -\frac{1}{2}$) within the contour

$$\frac{1}{z(2z+1)} = \frac{A}{z} + \frac{B}{2z+1}$$

$$2A + B = 0$$
$$A = 1$$

$$\frac{1}{z(2z+1)} = \frac{1}{z} - \frac{1}{z+\frac{1}{2}}$$

$$I = \oint_C \frac{dz}{z} - \oint_C \frac{dz}{\left(z+\frac{1}{2}\right)}$$

$$\oint_C \frac{f(z)dz}{z - z_0} = \begin{cases} 2\pi i f(z_0), & z_0 \text{ within the contour} \\ 0, & z_0 \text{ exterior to the contour} \end{cases}$$

$$I = 2\pi i - 2\pi i = 0$$

## 11.4.9

$$I = \oint_C \frac{f(z)}{z(2z+1)^2} dz$$

$$\frac{1}{z(2z+1)^2} = \frac{A}{z} + \frac{Bz+C}{(2z+1)^2}$$

$$= \frac{4Az^2 + 4Az + A + Bz^2 + Cz}{z(2z+1)^2}$$

$$(4A+B)z^2 + (4A+C)z + A = 1$$

$$A = 1, B = C = -4$$

$$I = \oint_C f(z) \left( \frac{1}{z} - \frac{z+1}{\left(z+\frac{1}{2}\right)^2} \right) dz$$

$$= \oint_C \frac{f(z)}{z} dz - \oint_C \frac{f(z)(z+1)}{\left(z+\frac{1}{2}\right)^2} dz$$

$$= 2\pi i \Big( f(0) - [f(z)(z+1)]'_{z=-\frac{1}{2}} \Big)$$

$$= 2\pi i \left( f(0) - \frac{f'\left(-\frac{1}{2}\right)}{2} - f\left(-\frac{1}{2}\right) \right)$$

## 11.5.3

$\frac{f(z)}{z}$ is analytical for all points in unit circle as f(z) and z are analytical.

for z $=$ 0 $\lim_{z\to 0} \frac{f(z)}{z} = f'(0)$ using L'Hôpital's and f'(0) is defined because f(z) is analytical within the contour.

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z - z_0)^{n+1}}$$

$$\left(\frac{f(z_0)}{z_0}\right)^n = \frac{1}{2\pi i} \oint_C \frac{\left(\frac{f(z)}{z}\right)^n dz}{z - z_0}$$

Where n $> 0$ as we want the function to be always analytic over the contour.

$$\left(|\frac{f(z_0)}{z_0}|\right)^n = \frac{1}{2\pi} \,|\oint_C \frac{\left(\frac{f(z)}{z}\right)^n dz}{z - z_0}|$$

$$\because |\oint_C g(z)dz| \leq \oint_C |g(z)|\ |dz|$$

$$\left(|\frac{f(z_0)}{z_0}|\right)^n \leq \frac{1}{2\pi} \oint_C |\frac{\left(\frac{f(z)}{z}\right)^n}{z - z_0}|\ |dz|$$

Assuming that $z \neq z_0$

$$\left(|\frac{f(z_0)}{z_0}|\right)^n \leq \frac{1}{2\pi} \oint_C \frac{\left(|\frac{f(z)}{z}|\right)^n}{|z - z_0|}\ |dz| \leq \frac{1}{2\pi} \oint_C \frac{\left(|\frac{f(z)}{z}|\right)^n}{|z| - |z_0|}\ |dz|$$

$$(|g(z_0)|)^n \leq \frac{1}{2\pi} \oint_C \frac{(|g(z)|)^n}{|z| - |z_0|}\ |dz|$$

$$\left(|\frac{f(z_0)}{z_0}|\right)^n \leq \frac{1}{2\pi} \oint_{|z|=1} \frac{(|f(|z|=1)|)^n}{1 - |z_0|}\ |dz| \leq \frac{1}{2\pi} \frac{1}{1 - |z_0|} \oint_{|z|=1} |dz|$$

For a contour of unit circle

$$z = e^{i\theta}, \quad dz = ie^{i\theta} d\theta, \quad |dz| = d\theta$$

$$\left(|\frac{f(z_0)}{z_0}|\right)^n \leq \frac{1}{2\pi} \frac{1}{1 - |z_0|} \oint_{|z|=1} d\theta$$

$$\therefore \left(|\frac{f(z_0)}{z_0}|\right)^n \leq \frac{1}{1 - |z_0|}$$

$$|f(z_0)| \leq \frac{|z_0|}{(1-|z_0|)^{\frac{1}{n}}}$$

Since this is true for all n it must be true for the smallest value of $(1-|z_0|)^{-\frac{1}{n}}$ which is 1 for positive values of n

$$\therefore |f(z_0)| \leq |z_0|$$

## 11.5.4

$$f(z) = \sum_{n=-N}^{\infty} a_n z^n$$

Consider the function:

$$g(z) = z^N f(z)$$

$$\begin{aligned} g(z) &= z^N \sum_{n=-N}^{\infty} a_n z^n \\ &= \sum_{n=-N}^{\infty} a_n z^{n+N} \\ &= \sum_{n=0}^{\infty} a_{n-N} z^n \end{aligned}$$

Where in the last step $n \to n - N$

Note that $g(z)$ is analytic over all the space, so it is valid to apply it on the real axis $(z = x)$

$$\because g(x)^* = x^N f^*(x) = x^N f(x) = g(x)$$

$$\therefore \sum_{n=0}^{\infty} a_{n-N}^* x^n = \sum_{n=0}^{\infty} a_{n-N} x^n \therefore a_{n-N}^* = a_{n-N}$$

Therefore, all the coefficients $a_n$ in the Laurent expansion of $f(z)$ are real.

### 11.5.6

$$
\begin{aligned}
\frac{e^z}{z^2} &= \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{z^n}{n!} = \sum_{n=0}^{\infty}\frac{z^{n-2}}{n!} \\
&= \frac{1}{z^2} + \frac{1}{z} + \sum_{n=2}^{\infty}\frac{z^{n-2}}{n!} \\
&= \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty}\frac{1}{(n+2)!}z^n
\end{aligned}
$$

## 11.5.8

$$\begin{aligned}
(z-1)e^{\frac{1}{z}} &= ze^{\frac{1}{z}} - e^{\frac{1}{z}} \\
&= \sum_{n=0}^{\infty} \frac{z^{1-n}}{n!} - \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} \\
&= z + \sum_{n=1}^{\infty} \frac{z^{1-n}}{n!} - \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} \\
&= z + \sum_{n=0}^{\infty} \frac{z^{-n}}{(n+1)!} - \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} \\
&= z + \sum_{n=0}^{\infty} \left( \frac{1}{(n+1)!} - \frac{1}{n!} \right) z^{-n} \\
&= z - \sum_{n=0}^{\infty} \frac{n}{(n+1)!} z^{-n}
\end{aligned}$$

# References

[1] M. El-Deeb, "PEU-455 Assignments." [Online]. Available: https://github.com/mhdeeb/peu-455-latex