PEU 405 Assignment 1

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$$\text{Number of cycles per century} = \frac{1 \text{ century}}{90 \text{ min}} = \frac{100\frac{\text{year}}{\text{century}} \times 365.25\frac{\text{day}}{\text{year}} \times 24\frac{\text{hr}}{\text{day}} \times 60\frac{\text{min}}{\text{hr}}}{90 \text{ min}}$$
$$= 584400$$

For the one cycle

$$\Delta\phi_{\text{1-cycle}} = \frac{6\pi GM}{r_c c^2} = \frac{6\pi \text{ rad} \times 6.6743 \times 10^{-11}\frac{m^3}{s^2 \cdot \text{ kg}} \times 5.9722 \times 10^{24} \text{ kg}}{6500 \times 10^3 m \times \left(299792458\frac{m}{s}\right)^2} \cdot \left(3600 \times \frac{180}{\pi}\frac{\text{arcsec}}{\text{rad}}\right)$$
$$\approx 0.00265 \text{ arcsec}$$

Precession rate per century

$$\Delta\phi_{\text{century}} = \Delta\phi_{\text{1-cycle}} \times \text{Number of cycles per century}$$
$$= 1550.32 \text{ arcsec}$$

2 11.3

For the one cycle

$$\Delta\phi_{\text{1-cycle}} = \frac{6\pi GM}{r_c} = \frac{6\pi \text{ rad} \times 2.0 \text{ km}}{400 \text{ km}} \cdot \left(\frac{180}{\pi} \deg/\text{rad}\right)$$
$$\approx 5.4^{\circ}$$

Period of the orbit at infinity

$$T = 2\pi\sqrt{\frac{r_c^3}{GM}} = 2\pi\sqrt{\frac{(400 \text{ km})^3}{2.0 \text{ km}}} \approx 35543 \text{ km} \approx \frac{35543 \text{ km}}{3 \times 10^5 s/\text{km}}$$
$$\approx 0.1185$$

$$\therefore \Delta\phi_{\text{unit-time}} = \frac{\Delta\phi_{\text{1-cycle}}}{T} \approx 46^{\circ}/s$$

3 11.5

a.

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

For $\mu = t$,

$$\frac{d}{d\tau}\left(g_{tt}\frac{dt}{d\tau}\right) - \frac{1}{2}\partial_t g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$\frac{d}{d\tau}\left(g_{tt}\frac{dt}{d\tau}\right) = 0 \quad \Rightarrow \quad g_{tt}\frac{dt}{d\tau} = -e \quad \text{where } e \text{ is a constant}$$

$$\therefore \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} = e$$

For $\mu = \phi$,

$$\frac{d}{d\tau}\left(g_{\phi\phi}\frac{d\phi}{d\tau}\right) - \frac{1}{2}\partial_\phi g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$\frac{d}{d\tau}\left(g_{\phi\phi}\frac{d\phi}{d\tau}\right) = 0 \quad \Rightarrow \quad g_{\phi\phi}\frac{d\phi}{d\tau} = l \quad \text{where } l \text{ is a constant}$$

In the equatorial plane, $\theta = \frac{\pi}{2}$

$$\therefore r^2\frac{d\phi}{d\tau} = l \quad \Rightarrow \quad \frac{d\phi}{d\tau} = \frac{l}{r^2}$$

b. In the equatorial plane, $\theta = \frac{\pi}{2}$:

$$\sin(\theta) = 1, \quad \frac{d\theta}{d\tau} = 0$$

$$u \cdot u = g_{\mu\nu} u^\mu u^\nu = -1$$

$$g_{tt}\left(\frac{dt}{d\tau}\right)^2 + g_{rr}\left(\frac{dr}{d\tau}\right)^2 + g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 = -1$$

$$-\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2 = -1$$

$$\frac{dt}{d\tau} = e\left(1 - \frac{2GM}{r}\right)^{-1}, \quad \frac{d\phi}{d\tau} = \frac{l}{r^2}$$

$$-e^2\left(1 - \frac{2GM}{r}\right)^{-1} + \left(\frac{dr}{d\tau}\right)^2 + \frac{l^2}{r^2} = -1$$

$$e^2\left(1 - \frac{2GM}{r}\right)^{-1} - \left(\frac{dr}{d\tau}\right)^2 - \frac{l^2}{r^2} = 1$$

c.

$$\frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{dr}{d\phi}\frac{l}{r^2}$$

$$e^2\left(1 - \frac{2GM}{r}\right)^{-1} - \left(\frac{dr}{d\phi}\right)^2 \frac{l^2}{r^4} - \frac{l^2}{r^2} = 1$$

$$u = \frac{1}{r} \quad \Rightarrow \quad \frac{du}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi} \quad \Rightarrow \quad \frac{dr}{d\phi} = -r^2\frac{du}{d\phi} = -\frac{1}{u^2}\frac{du}{d\phi}$$

$$e^2(1 - (2GM)u)^{-1} - \left(\frac{du}{d\phi}\right)^2 l^2 - l^2u^2 = 1$$

Taking the ϕ derivative,

$$2GMe^2(1 - (2GM)u)^{-2}\frac{du}{d\phi} - 2\frac{du}{d\phi}\frac{d^2u}{d\phi^2}l^2 - 2l^2u\frac{du}{d\phi} = 0$$

$$2GMe^2(1 - (2GM)u)^{-2} - 2l^2\left(\frac{d^2u}{d\phi^2} + u\right) = 0$$

$$\frac{d^2u}{d\phi^2} + u = \frac{GMe^2}{l^2(1 - 2GMu)^2}$$

d. For $2GMu \ll 1$,

$$\frac{d^2u}{d\phi^2} + u \approx \frac{GMe^2}{l^2}[1 + 4GMu]$$

$$\therefore \frac{d^2u}{d\phi^2} + \left[1 - 4\left(\frac{GMe}{l}\right)^2\right]u = \frac{GMe^2}{l^2}$$

e. For $u = u_c$

$$\frac{d^2u}{d\phi^2} = 0$$

$$\left[1 - 4\left(\frac{GMe}{l}\right)^2\right] u_c = \frac{GMe^2}{l^2}$$

$$u_c = \frac{GMe^2}{l^2\left[1 - 4\left(\frac{GMe}{l}\right)^2\right]}$$

f.

$$u(\phi) = u_c + u_c w(\phi)$$

$$u_c \frac{d^2w}{d\phi^2} + \left[1 - 4\left(\frac{GMe}{l}\right)^2\right] u_c(1 + w(\phi)) = \frac{GMe^2}{l^2}$$

$$u_c \frac{d^2w}{d\phi^2} + \left[1 - 4\left(\frac{GMe}{l}\right)^2\right] u_c w(\phi) = 0$$

$$\frac{d^2w}{d\phi^2} + \left[1 - 4\left(\frac{GMe}{l}\right)^2\right] w(\phi) = 0$$

$$w(\varphi) = A\cos(\omega\phi + \phi_0), \quad \omega = \sqrt{1 - 4\left(\frac{GMe}{l}\right)^2}$$

$$\omega\Delta\phi = \sqrt{1 - 4\left(\frac{GMe}{l}\right)^2}\,\Delta\phi = 2\pi$$

$$\begin{aligned}\omega &= 2\pi\left[1 - 4\left(\frac{GMe}{l}\right)^2\right]^{-\frac{1}{2}} \\ &\approx 2\pi\left[1 + 2\left(\frac{GMe}{l}\right)^2\right] \\ &= 2\pi + 4\pi\left(\frac{GMe}{l}\right)^2\end{aligned}$$

References

[1] M. El-Deeb, “PEU-405 Assignments.” [Online]. Available: https://github.com/mhdeeb/peu-assignments/tree/main/peu-405