ARE 256B Winter 2020: Unit Root Testing to Detect Non-stationarity

Armando R. Colina

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The objective of this write-up is to build the intuition of how some unit-root tests work. This write-up is aimed for the Master's students at the ARE Program in UC Davis.

1 Dickey-Fuller Test

Suppose we have a stochastic, time-dependent process that looks like[1]:

$$y_t = \alpha + \rho y_{t-1} + \varepsilon_t \tag{1}$$

where $\varepsilon \sim i.i.d$ with mean of zero and variance of σ^2. We want to test if it is a stationary process. That is, we want to test:

- $H_0 : \rho = 1 \implies$ non-stationary time series
- $H_a : \rho < 1 \implies$ stationary time series

Then, why not just an do an OLS regression and do a Wald test on $\hat{\rho}$?

Challenge:

If the null-hypothesis is true, then y_t is a non-stationary process, which implies that the weak law of large numbers will not hold. Then $\hat{\rho}$ will not be consistent. ☹

Solution:

Substract y_{t-1} from both sides of Equation(1).

$$\begin{gathered} [y_t - y_{t-1}] = \alpha + [\rho - 1]\, y_{t-1} + \varepsilon_t \\ \Delta y_t = \alpha + \delta y_{t-1} + \varepsilon_t \end{gathered} \tag{2}$$

[1]Note that if $\alpha = 0$ the we have a random walk and if $\alpha \neq 0$ we have a random walk with drift

Note that if $\rho = 1 \implies \delta = 0 \implies \Delta y_t = \alpha + \varepsilon_t \implies \Delta y_t$ is a stationary process since ε_t is independent and identically distributed with mean equal to zero.

If $\delta < 0 \implies \rho < 1 \implies \Delta y_t$ is a stationary process. Then we can change our hypothesis to:

- $H_0 : \delta = 0$
- $H_a : \delta < 0$

This will help us with the estimation. However, the estimate of the parameter will not be exactly distributed as a t or a normal. We need to use the Dickey-Fuller distribution. However, if we use a statistcal package like Stata, it will take care of it.

The **decision rule** is: if test-statistic $< DF_{critical-value} \implies$ reject Null Hypothesis

2 Augmented Dickey-Fuller Test

What if our process takes a more complex form other than an AR(1) process? Suppose it is an AR(2) process of the form

$$y_t = +\beta_0 + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \varepsilon_t \tag{3}$$

Substract y_{t-1} from both sides of Equation(3) and add a special zero to the RHS of the equation: $0 = \beta_3 y_{t-1} - \beta_3 y_{t-1}$.

$$\begin{aligned} y_t - y_{t-1} &= \beta_0 + \beta_1 y_{t-1} - y_{y-1} + (\beta_2 y_{t-1} - \beta_2 y_{t-1}) + \beta_2 y_{t-2} + \varepsilon_t \\ &= \beta_0 + (\beta_1 + \beta_2 - 1) y_{t-1} - \beta_2 (y_{t-1} - y_{t-2}) + \varepsilon_t \\ \Delta y_t &= \beta_0 + \delta_0 y_{t-1} + \delta_1 \Delta y_{t-1} + \varepsilon_t \end{aligned} \tag{4}$$

Our hypothesis becomes

- $H_0 : \delta_0 = 0$ Note that if $\beta_1 + \beta_2 = 1 \implies \delta = 0$
- $H_a : \delta_0 < 0$ Note that if $\beta_1 + \beta_2 = 1 \implies \delta < 0$

More generally, if we are dealing with an AR(p) process, then:

$$y_t = +\beta_0 + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \cdots + \beta_p y_{t-p} + \varepsilon_t \tag{5}$$

The condition of stationarity of y_t is that $|\beta_1 + \beta_2 + \cdots + \beta_p| < 1$. So we can rewrite Equation(5) as

$$\Delta y_t = \beta_0 + \delta_0 y_{t-1} + \delta_1 \Delta y_{t-1} + \delta_2 \Delta y_{t-2} + \cdots + \delta_p \Delta y_{t-p} + \varepsilon_t \tag{6}$$

3 Implementing the Augmented Dickey-Fuller test in Stata

To run the Dickey-Fuller test in Stata is relatively easy. Suppose you want to know if the price of Apple's stock follows a stationary process. See Figure(1), it clearly doesn't.

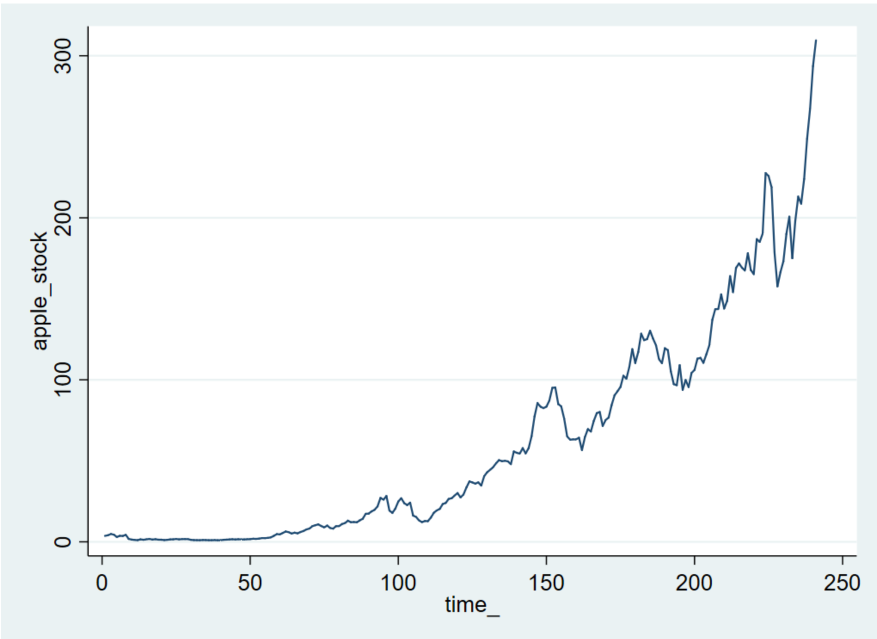

Figure 1: Apple's Stock Price in USD across time

In our database the price of Apple's stock is called *apple_stock*. You can run the following command:

```
dfuller apple_stock
```

The results of the test indicate that we cannot reject the null hypothesis since the test statistic

Table 1: Results from a Dickey-Fuller test

	Test Statistic	1% Critical Value	5% Critical Value	10% Critical Value
Z(t)	3.293	-3.463	-2.881	-2.571
		MacKinnon approximate p-value for Z(t) = 1.0000		

Table 2: Results from a Dickey-Fuller test assuming two lags and drift

	Test Statistic	1% Critical Value	5% Critical Value	10% Critical Value
Z(t)	2.785	-2.342	-1.651	-1.285
		MacKinnon approximate p-value for Z(t) = 0.9971		

is greater than the critical values. That is, we cannot reject that the price of Apple's stock follows a non-stationary process. See Table(1)

What if we want to test for stationarity assuming *apple_stock* follows an AR(2) process with drift?

```
dfuller apple_stock, lags(2) drift
```

Once again, we cannot reject the null. See Table(2)