# New resummation techniques of divergent series: the Painlevé equation $P_{II}$

UNDERGRADUATE RESEARCH THESIS

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# Abstract

A major goal in the study of ordinary differential equations (ODEs) is to be able to recover maximal information about the solution by resummation of divergent asymptotic series. Most differential equations have asymptotic series which are divergent; for example, in physics, most Hamiltonian perturbation expansions are divergent, only being accurate to a small number of terms. Resummation methods are used to sum divergent series and obtain actual solutions. We are interested in resummation methods that give maximum information about the associated function even when truncating the divergent asymptotic series.

One of the most prevalent resummation methods is Borel summation; however, Borel summation is very limited in its applicability to equations. In particular, it always fails along half-lines in which the terms of the divergent series have the same phase, known as Stokes lines. Écalle introduced a variation named Borel-Écalle resummation which removes this limitation. In the last few years, mathematicians have developed efficient improvements of Borel-Écalle resummation which employ the property of resurgence, a remarkable property of divergent series of natural origin which has been recently discovered. One can use this property to deal with incomplete information such as partial perturbation expansions.

We have developed such a method, applying it to the Painlevé equation $P_{II}$ (from now on, $P_{II}$) to test its efficiency and applicability for resumming divergent series. We provide herein a detailed analysis of the numerical accuracy of these different types of approximants for $P_{II}$. We have increased the accuracy greatly from the usual Borel-Padé approximants of a series used in physics by using asymptotics of continued fractions as well as conformal

Padé approximants, a combination of Padé with conformal mappings of the complex plane. Additionally, we observe that conformal Padé approximants converge much faster than the other approximants, making this new method much more efficient in resumming divergent series and dealing with incomplete information. Finally, we provide a preliminary calculation of the binary rational expansion for $P_{II}$, which is a re-expansion of the function in the Borel plane in attempts to more conveniently calculate the convergent form of the solution. We discuss the behavior of the binary rational expansion through two simple examples.

The study of $P_{II}$ is directly connected to random matrix theory in physics. We also hope to apply the method to other problems in mathematics and physics, such as obtaining higher precision in critical expansions at low and high temperatures for problems such as the 3-dimensional Ising model in statistical mechanics.

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# Chapter 1
# Introduction

The study of asymptotics is concerned with understanding the limiting behavior of functions, especially when approaching singular points [1]. An asymptotic expansion $\tilde{f}$ of a function $f$ at $t_0$ is a formal series of the form

$$\tilde{f}(t) = \sum_{k=0}^{\infty} f_k(t), \tag{1.1}$$

where the functions $f_k$ are simpler than $f$ and successively become smaller. This series usually depends on the direction along which $t_0$ is approached in the complex plane. For example, if $t_0$ is approached from the right, then the condition that the functions $f_k$ become successively smaller amounts to having

$$f_{k+1}(t) = o(f_k(t)) \text{ as } t \to t_0^+, \tag{1.2}$$

meaning that

$$\lim_{t \to t_0^+} \frac{f_{k+1}(t)}{f_k(t)} = 0. \tag{1.3}$$

This notation is also sometimes written as $f_{k+1}(t) \ll f_k(t)$. We say that a function $f$ is asymptotic to an asymptotic expansion $\tilde{f} = \sum_{k=0}^{\infty} f_k(t)$ if for each $N \in \mathbb{N}$,

$$f(t) - \sum_{k=0}^{N} f_k(t) = o(f_N(t)). \tag{1.4}$$

We can further define an asymptotic power series as a special type of asymptotic expansion where the functions $f_k(t)$ are powers of $t$. Notice that if a function $f$ is analytic at a point $t_0$ and hence admits a Taylor series there, the Taylor series $\tilde{f}$ is indeed an asymptotic power series expansion of the function $f$.

In the study of ordinary differential equations, it turns out that the asymptotic expansions of the solution are often divergent and hence can not be used to describe the solution as the error terms are unbounded. As such, the theory of resummation of divergent series has arisen, which deals with methods to obtain an expression of the convergent solution from the divergent asymptotic series. One of the most prevalent resummation methods is Borel-Écalle resummation. This essentially amounts to mapping the divergent series $\tilde{f}$ to a convergent series in the Borel plane by an inverse Laplace transform, summing the convergent series to a function $F$ in the Borel plane, and then mapping the function $F$ back to the physical plane through a Laplace transform to get the solution. We will introduce this in detail in Chapter 3. Over the last few years, mathematicians have developed efficient improvements of Borel-Écalle resummation which employ the property of resurgence, a property which we will briefly describe in Section 3.3.

When obtaining the asymptotic divergent series, one often deals with a truncated version of the series which gives incomplete information, such as a partial perturbation expansions, since one is unable to compute infinitely many terms of a series numerically. To analyze these problems properly, it is imperative that we are able to effectively deal with incomplete information. Thus, we are interested in resummation methods that give maximum information about the associated function when dealing with truncated asymptotic series. One of course could approach this problem in many different ways, but there are two main steps at which large improvements can be made in obtaining maximal information of the function. First, one can improve the description of the function in the Borel plane by finding better approximations to the function when given finitely many terms. We want to be able to "squeeze out" all of the information that we can from those finite number of terms. Second, one can improve efficiency in calculating the expression of the convergent solution by finding a more convenient form of the solution which requires fewer calculations or simply less expensive calculations. We have made progress in both areas.

## 1.1 Contributions

Our main contributions are:

- We discuss and analyze new approximations of the function in the Borel plane to improve accuracy and convergence. We use $\mathrm{P_{II}}$ as an example to provide numerical analysis of these new approximants in comparison to the standard Padé approximants used in physics.

- We do a preliminary calculation of the coefficients of the binary rational expansion for $\mathrm{P_{II}}$, which is a convenient re-expansion of the function $H$ in the Borel plane. We discuss its behavior and the next steps in calculating this re-expansion.

## 1.2 Outline

The rest of the thesis is as follows:

- In Chapter 2, we review the Laplace transform and its inverse. We discuss some of its basic properties and introduce an important example which is integral to the development of Borel-Écalle resummation.

- In Chapter 3, we introduce Borel-Écalle resummation, a method to obtain a convergent form of a function when dealing with its divergent asymptotic series. We discuss transseries, the Borel transform, Écalle critical time, the properties of Borel summation, Stokes phenomena, and Watson's lemma.

- In Chapter 4, we introduce the Painlevé equation $\mathrm{P_{II}}$. We find the asymptotic behavior of the solution we are exploring and discuss its Écalle critical time. Finally, we rewrite $\mathrm{P_{II}}$ in a normalized form, discuss obtaining the asymptotic series $\tilde{h}$, and list some important properties of the Borel transform of $\tilde{h}$.

- In Chapter 5, we discuss three different approximations of the function $H$ in the Borel plane: Padé approximants, finite continued fractions with terminants, and conformal Padé approximants. We then discuss the accuracy and convergence of each approximation. Finally, we touch on a relation between capacitors and Padé approximants discovered by Stahl (see Ref. [2]).

- In Chapter 6, we do a preliminary calculation of the binary rational expansion for $\mathrm{P_{II}}$. We analyze its behavior through two simple examples.

# Chapter 2
# The Laplace Transform

In this chapter, we introduce the Laplace transform, which is an integral part to Borel-Écalle resummation. We prove some of its basic properties, illustrate a few important examples, and discuss the inverse Laplace transform.

## 2.1 The Laplace transform

The Laplace transform of a function $F$, denoted $\mathcal{L}F$ [1], is given by

$$\mathcal{L}F(x) = \int_0^\infty e^{-xp} F(p)\, dp, \quad \operatorname{Re}(x) > \nu \geq 0. \tag{2.1}$$

Notice, this is not always well-defined as the function could simply not be integrable or the integral could diverge (for example, if $F(p) = e^{p^2}$). For this to be well-defined, it is assumed that $F$ is locally integrable in $[0, \infty)$, denoted $F \in L^1_{\text{loc}}([0, \infty))$, and that $F$ does not grow faster than exponentially. For example, we either take $F$ such that

$$\|F\|_{\infty,\nu} := \sup_{p \geq 0} |F(p)| e^{-\nu p} < \infty \tag{2.2}$$

or

$$\|F\|_{L^1,\nu} := \int_0^\infty |F(p)| e^{-\nu p}\, dp < \infty \tag{2.3}$$

for some $\nu \in [0, \infty)$, which both ensure the existence of $\mathcal{L}F(x)$ when $\mathrm{Re}(x) > \nu$. Indeed, if $\|F\|_{\infty,\nu} < \infty$, then for $\mathrm{Re}(x) > \nu$,

$$\int_0^\infty \left|e^{-xp}F(p)\right| \, dp = \int_0^\infty e^{-\mathrm{Re}(x)p}|F(p)| \leq \|F\|_{\infty,\nu} \int_0^\infty e^{-\mathrm{Re}(x)p}e^{\nu p} \, dp < \infty.$$

This ensures the existance of $\mathcal{L}F(x)$ when $\mathrm{Re}(x) > \nu$. This can be similarly shown for the condition in (2.3).

**Remark 2.1.** *Notice that if one satisfies the condition in either (2.2) or (2.3) for some $\nu \in [0, \infty)$, then one satisfies this condition for all $\mu \geq \nu$.*

**Remark 2.2.** *If $F \in L^1([0, \infty))$, then $F$ satisfies the condition given in either (2.2) or (2.3) for some $\nu \in [0, \infty)$.*

### 2.1.1 Properties of the Laplace transform

First, the Laplace transform extends the domain of analyticity of the function, as seen in the following two propositions [1].

**Proposition 2.3.** *If $F \in L^1(\mathbb{R}^+)$, then $\mathcal{L}F$ is analytic in $\mathbb{H}$ and continuous on the imaginary axis $\partial\mathbb{H}$.*

**Proposition 2.4.** *Let $F$ be analytic in the open sector $S_p := \{e^{i\phi}|p| : \phi \in (-\delta, \delta), |p| \in \mathbb{R}^+\}$ and such that $|F(|x|e^{i\phi})| \leq g(|x|) \in L^1([0, \infty))$. Then, $f = \mathcal{L}F$ is analytic in the sector $S_x = \{x : |\arg(x)| < \pi/2 + \delta\}$ and $f(x) \to 0$ as $|x| \to \infty$, $\arg(x) \in (-\pi/2 - \delta, \pi/2 + \delta)$.*

*Sketch of Proof.* The idea behind the proof is that the Laplace integration can be rotated by any angle $\phi$ in $(-\delta, \delta)$ without changing $\mathcal{L}F(x)$ since $F$ is analytic in $S_p$ and decays sufficiently for large $p$. One then utilizes the previous proposition. $\square$

Additionally, we note that the Laplace transform is injective [1].

**Proposition 2.5.** *Suppose that $F \in L^1([0,\infty))$. If $\mathcal{L}F(x) = 0$ for $x$ in a set $A$ with an accumulation point in $\mathbb{H}$, then $F = 0$ a.e. on $\mathbb{H}$.*

It is also useful for us to list some basic properties of the Laplace transform and compute a simple example.

**Proposition 2.6** (Basic Properties of the Laplace Transform)**.** *Suppose $F$ and $G$ satisfy the condition in either (2.2) or (2.3) for some $\nu_1, \nu_2 \in [0,\infty)$. Let $\nu = \max\{\nu_1, \nu_2\}$, so $F$ and $G$ both satisfy the condition for $\nu$.*

*(a) The Laplace transform is linear, that is for all $a \in \mathbb{C}$, $\mathcal{L}(aF + G) = a\mathcal{L}(F) + \mathcal{L}(G)$.*

*(b) $\mathcal{L}(pF)(x) = -\frac{d}{dx}\mathcal{L}F(x)$. Furthermore, for all $n \in \mathbb{N}$, $\mathcal{L}(p^nF)(x) = (-1)^n \frac{d^n}{dx^n}\mathcal{L}F(x)$.*

*(c) $\mathcal{L}(F * G) = \mathcal{L}(F) \cdot \mathcal{L}(G)$, where $(F * G)(p) \coloneqq \int_0^p F(s)G(p-s)\,ds$.*

*Proof.* (a) First, notice that $aF + G$ also satisfies the condition in either (2.2) or (2.3) for the given $\nu$ by triangle inequality. Then, for all $\mathrm{Re}(x) > \nu$,

$$\begin{aligned}\mathcal{L}(aF+G)(x) &= \int_0^\infty e^{-xp}(aF(p) + G(p))\,dp = a\int_0^\infty e^{-xp}F(p)\,dp + \int_0^\infty e^{-xp}G(p)\,dp \\ &= a\mathcal{L}F(x) + \mathcal{L}G(x).\end{aligned}$$

(b) Notice that for all $n \in \mathbb{N}$, $p^nF$ satisfies the condition in either (2.2) or (2.3) for all $\mu > \nu$ since $p^n e^{-(\mu-\nu)p}$ is bounded, so $\mathcal{L}(p^nF)(x)$ exists for all $\mathrm{Re}(x) > \nu$. Then, for all $\mathrm{Re}(x) > \nu$,

$$\begin{aligned}\mathcal{L}(pF)(x) &= \int_0^\infty e^{-xp}pF(p)\,dp = \int_0^\infty -\frac{d}{dx}(e^{-xp}F(p))\,dp = -\frac{d}{dx}\int_0^\infty e^{-xp}F(p)\,dp \\ &= -\frac{d}{dx}\mathcal{L}F(x),\end{aligned}$$

where passing the derivative through the integral is justified by dominated convergence theorem. The rest follows from induction on $n$.

(c) First, notice if $F$ and $G$ satisfy condition (2.2), then $F * G$ also satisfies condition (2.2) for all $\mu > \nu$. Indeed,

$$
\begin{aligned}
|(F*G)(p)|e^{-\mu p} \leq e^{-\mu p}\int_0^p |F(s)||G(p-s)|\, ds &\leq \|F\|_{\infty,\nu}\|G\|_{\infty,\nu}e^{-\mu p}\int_0^p e^{\nu s}e^{\nu(p-s)}\, ds \\
&= \|F\|_{\infty,\nu}\|G\|_{\infty,\nu}e^{-\mu p}e^{\nu p}\int_0^p ds = \|F\|_{\infty,\nu}\|G\|_{\infty,\nu}e^{-(\mu-\nu)p}p.
\end{aligned}
$$

Thus, $\sup_{p\geq 0}|(F*G)(p)|e^{-\mu p} < \infty$ since $e^{-(\mu-\nu)p}p$ is bounded. On the other hand, if $F$ and $G$ satisfy condition (2.3), then $F * G$ also satisfies condition (2.3) for $\nu$ by Fubini's theorem.

Now, for each $\mathrm{Re}(x) > \nu$, by Fubini's theorem,

$$
\begin{aligned}
\mathcal{L}(F*G)(x) &= \int_0^\infty e^{-xp}(F*G)(p)\, dp = \int_0^\infty e^{-xp}\int_0^p F(s)G(p-s)\, ds\, dp \\
&= \int_0^\infty \int_0^p e^{-xs}F(s)e^{-x(p-s)}G(p-s)\, ds\, dp \\
&= \int_0^\infty e^{-xs}F(s)\int_s^\infty e^{-x(p-s)}G(p-s)\, dp\, ds \\
&= \int_0^\infty e^{-xs}F(s)\int_0^\infty e^{-xt}G(t)\, dt\, ds = \mathcal{L}F(x)\cdot\mathcal{L}G(x).
\end{aligned}
$$

□

**Example 2.7.** *For all $n \in \mathbb{N}$, show that $\mathcal{L}(p^n)(x) = \dfrac{n!}{x^{n+1}}$.*

*Solution.* First, notice that

$$\mathcal{L}(1)(x) = \int_0^\infty e^{-xp}1\, dp = -\frac{1}{x}e^{-xp}\Big|_{p=0}^\infty = -\frac{1}{x}(0-1) = \frac{1}{x}.$$

Hence, by Proposition 2.6(b), for each $n \in \mathbb{N}$, $\mathcal{L}(p^n)(x) = (-1)^n\dfrac{d^n}{dx^n}(x^{-1}) = \dfrac{n!}{x^{n+1}}$. ■

This example will be the basis of defining the Borel transform, a formal inverse Laplace transform, as we will see in Section 3.2. The other basic properties we have proven will hold analogously for the Borel transform as well.

**Example 2.8.** *Let $a \in \mathbb{C}$. Then,* $\mathcal{L}(e^{ap})(x) = \dfrac{1}{x-a}$.

*Solution.* First, notice that $e^{ap}$ satisfies condition (2.2) for $\nu \geq \mathrm{Re}(a)$ and condition (2.3) for $\nu > \mathrm{Re}(a)$. Hence, for $\mathrm{Re}(x) > \mathrm{Re}(a)$,

$$\mathcal{L}(e^{ap})(x) = \int_0^\infty e^{-xp} e^{ap}\, dp = \int_0^\infty e^{(a-x)p}\, dp = \frac{1}{a-x} e^{(a-x)p}\Big|_{p=0}^{\infty} = \frac{1}{a-x}(0-1) = \frac{1}{x-a}.$$

$\blacksquare$

This example will be integral in the idea of the binary rational expansion, as we will see in Chapter 6.

### 2.1.2 Laplace inversion formula

We would now like to be able to invert the Laplace transform as this is a key component of the Borel-Écalle transformation. In the following, we denote the open right half plane by $\mathbb{H}$ and define the open sectors $\mathbb{H}_\delta := \{x : |\arg(x)| < \pi/2 + \delta\}$ for $\delta \geq 0$. Also, $\partial A$ will denote the boundary of a set $A$. We then have the following proposition which gives a formula for the inverse Laplace transform [1].

**Proposition 2.9.** *(a) Assume $f$ is analytic in an open sector $\mathbb{H}_\delta$ for some $\delta \geq 0$ and is continuous on $\partial\mathbb{H}_\delta$. Assume further that for some $K > 0$, we have for each $x \in \overline{\mathbb{H}_\delta}$ that*

$$|f(x)| \leq K(|x|^2 + 1)^{-1}. \tag{2.4}$$

*Then, $\mathcal{L}^{-1} f$ is well-defined by*

$$F = \mathcal{L}^{-1} f = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{pt} f(t)\, dt, \tag{2.5}$$

*and we further have that*

$$\mathcal{L}\left(\mathcal{L}^{-1} f\right) = \int_0^\infty e^{-px} F(p)\, dp = f(x). \tag{2.6}$$

*Finally, we have* $\|\mathcal{L}^{-1}f\|_\infty \leq K/2$ *and* $\mathcal{L}^{-1}f \to 0$ *as* $p \to \infty$.

(b) *If* $\delta > 0$, *then* $F = \mathcal{L}^{-1}f$ *is analytic in the sector* $S_\delta = \{p \neq 0 : |\arg(p)| < \delta\}$. *In addition,* $\sup_{S_\delta}|F| \leq K/2$ *and* $F(p) \to 0$ *as* $p \to \infty$ *along rays in* $S_\delta$.

**Proposition 2.10.** *The inverse Laplace transform is linear.*

*Proof.* It is clear that if $f$ and $g$ satisfy the conditions of Proposition 2.9(a), then $af + g$ does as well for any $a \in \mathbb{C}$. The rest follows from linearity of the integral. $\square$

The Borel transform is defined as a formal inverse Laplace transform according to Example 2.7, as seen in Section 3.2.

# Chapter 3
# Borel-Écalle Resummation

When finding formal asymptotic solutions to differential equations, one frequently encounters formal series in powers of $x^{-1}$ which are often divergent. These divergent series are of limited use in approximating the actual solutions, only being useful for large values of $x$ where they still intrinsically provide limited accuracy. Borel-Écalle resummation is a way to obtain a convergent form of the solution to a differential equation when dealing with a divergent asymptotic series. In this chapter, we briefly introduce transseries and then go on to introduce the Borel transform, Écalle critical time, Borel-Écalle resummation, Stokes phenomena, and a very important asymptotic tool, Watson's Lemma.

## 3.1 Transseries

Transseries are, by definition, the closure of formal asymptotic series under a particular set of operations, including algebraic operations, differentiation, integration, composition, and functional inversion [1]. Practically, they consist in all formally asymptotic expansions in powers, small exponentials, and logarithms, with coefficients that grow at most as powers of factorials. Transseries are extrememly useful in solving many asymptotic problems, and have been utilized to solve many ordinary differential equations.

We are particularly interested in meromorphic and nonresonant ordinary differential equations, which can be rewritten in the form

$$\boldsymbol{y}' = \boldsymbol{f}_0(x) - \hat{\boldsymbol{\Lambda}}\boldsymbol{y} - \frac{1}{x}\hat{\boldsymbol{B}}\boldsymbol{y} + \boldsymbol{g}(x, \boldsymbol{y}), \tag{3.1}$$

where $f_0$ is analytic at $\infty$ with $f_0(x) = \mathcal{O}(x^{-M-1})$ for some $M \in \mathbb{N}$, $\boldsymbol{g}(x, \boldsymbol{y}) = \mathcal{O}(\boldsymbol{y}^2, x^{-M-1}\boldsymbol{y})$, $\hat{\boldsymbol{\Lambda}} = \operatorname{diag}(\lambda_i)$, and $\hat{\boldsymbol{B}} = \operatorname{diag}(\beta_i)$. See sections 1.1 and 1.1.2 of Ref. [3] for details. These ODEs admit transseries solutions, and the transseries have the form

$$\tilde{\boldsymbol{y}} = \tilde{\boldsymbol{y}}_0 + \sum_{\boldsymbol{k} \geq 0; |\boldsymbol{k}| > 0} C_1^{k_1} \cdots C_n^{k_n} e^{-(\boldsymbol{k}\cdot\boldsymbol{\lambda})x} x^{-\boldsymbol{k}\cdot\boldsymbol{\beta}} \tilde{\boldsymbol{y}}_{\boldsymbol{k}}, \tag{3.2}$$

where $\boldsymbol{C} \in \mathbb{C}^n$ is an arbitrary vector of constants, $\boldsymbol{\lambda}, \boldsymbol{\beta} \in \mathbb{C}^n$ are the diagonals of $\hat{\boldsymbol{\Lambda}}$ and $\hat{\boldsymbol{B}}$ respectively, and $\tilde{\boldsymbol{y}}_{\boldsymbol{k}} = \sum_{l=0}^{\infty} \boldsymbol{a}_{\boldsymbol{k};l} x^{-l}$ are formal power series. Notice that the terms in the transseries become less significant as $|\boldsymbol{k}|$ increases since the exponential becomes smaller, and $\tilde{\boldsymbol{y}}_{\boldsymbol{0}}$, the only series not multiplied by an exponential, is the asymptotic series of the solution. One can think that the transseries gives the full form of the exponentially small corrections to the asymptotic power series $\tilde{\boldsymbol{y}}_{\boldsymbol{0}}$ of the solution.

**Remark 3.1.** *Notice that the exponent of the exponential is simply a constant multiplied by* $x$*. Additionally, the formal power series* $\tilde{\boldsymbol{y}}_{\boldsymbol{k}}$ *are integer power series in* $x$*. These are both a result of having the differential equation correctly normalized, and this variable* $x$ *(of the normalized differential equation) is called the Écalle critical time and is further discussed in Section 3.3.1. Since the exponent of the exponential is simply a constant multiplied by the Écalle critical time, this becomes useful in identifying the Écalle critical time in practice, as we will see for Painlevé equation* $P_{II}$ *in Section 4.2.1.*

## 3.2 The Borel transform

To define the Borel transform, it is first convenient to define the formal Laplace transform on series, $\tilde{\mathcal{L}} : \mathbb{C}[[p]] \to x^{-1}\mathbb{C}[[x^{-1}]]$, as expected from Example 2.7 [1]:

$$\tilde{\mathcal{L}}\left(\sum_{k=0}^{\infty} c_k p^k\right) = \sum_{k=0} c_k \tilde{\mathcal{L}}(p^k) = \sum_{k=0}^{\infty} c_k \frac{k!}{x^{k+1}}. \tag{3.3}$$

The Borel transform, $\mathcal{B} : x^{-1}\mathbb{C}[[x^{-1}]] \to \mathbb{C}[[p]]$, is the formal inverse of $\tilde{\mathcal{L}}$, defined by

$$\mathcal{B}\left(\sum_{k=0}^{\infty} \frac{c_k}{x^{k+1}}\right) = \sum_{k=0}^{\infty} c_k \mathcal{B}\left(\frac{1}{x^{k+1}}\right) = \sum_{k=0}^{\infty} \frac{c_k}{k!} p^k. \tag{3.4}$$

Both of these definitions can be extended to noninteger power series by defining $\tilde{\mathcal{L}}(p^s) = \Gamma(s+1)x^{-s-1}$ and analogously for $\mathcal{B}$, but this will not be relevant for us [1].

The Borel transform is a very interesting operator. First, notice that $\tilde{\mathcal{L}}\mathcal{B}$ is the identity operator on series in $x^{-1}\mathbb{C}[[x^{-1}]]$, and $\mathcal{L}\mathcal{B}$ is *formally* $\mathcal{L}\mathcal{L}^{-1}$, the identity operator [1]. Second, notice that for a series $\tilde{f}$, the coefficients of $\mathcal{B}(\tilde{f})$ are smaller by a factor of $k!$ than the corresponding coefficients of $\tilde{f}$. Hence, one could take a (factorially) divergent series $\tilde{f}$ (e.g., $\tilde{f} = \sum_{k=0}^{\infty} k! x^{-k-1}$), and after applying the Borel transform, the series $\mathcal{B}(\tilde{f})$ would be convergent and could be summed to some analytic function around 0 (in this example, $\mathcal{B}(\tilde{f}) = \sum_{k=0}^{\infty} p^k = (1-p)^{-1}$). Now, if $\mathcal{B}(\tilde{f})$ is convergent and converges to a function $f$ which can be Laplace transformed, the operator $\mathcal{L}\mathcal{B}$ which maps $\tilde{f} \mapsto \mathcal{L}f$ is effectively an identity operator from series to functions. This motivates the central role played by $\mathcal{L}\mathcal{B}$, the operator of Borel summation, in the resumming of factorially divergent asymptotic series.

## 3.3 Borel summation

Borel summation is simply a formulation of the resumming process described in the previous paragraph. First, Borel summation is relative to a particular direction. Here, we discuss Borel summation along $\mathbb{R}^+$, but see Section 3.3.3 for the formulation of Borel summation along other directions. Borel summation along $\mathbb{R}^+$ consists of three steps, assuming that all of the steps are possible [1]:

1. Borel transform the series, $\tilde{f} \mapsto \mathcal{B}(\tilde{f})$. We denote the plane that $\mathcal{B}(\tilde{f})$ takes values in as the Borel plane.

2. Sum the series $\mathcal{B}(\tilde{f})$ (assuming it is convergent) and analytically continue it along $\mathbb{R}^+$. Here, let $F$ denote the analytic continuation of $\mathcal{B}(\tilde{f})$ and $D$ denote an open set in $\mathbb{C}$ containing $[0, \infty)$ where $F$ is analytic.

3. Laplace transform back, $F \mapsto \mathcal{L}(F) =: \mathcal{LB}(\tilde{f})$, assuming $F$ satisfies some exponential bound, such as the condition given in (2.2) or (2.3). This is then defined in a half plane $\mathrm{Re}(x) > x_0$ for some $x_0 \geq 0$.

We call the space of series which are Borel summable $S_{\mathcal{B}}$. As it happens, Borel-Écalle summation has been shown to apply to generic systems of meromorphic and nonresonant linear or nonlinear ordinary differential equations of the form given in Equation (3.1) [3]. It also has been shown to apply to a wide class of partial differential equations (PDEs), including Navier-Stokes [1]. These systems generate solutions whose preimages in the Borel plane are resurgent functions, a large class of functions with very desirable properties.

Resurgent functions are analytic at the origin, admit analytic continuation along any path that avoids a discrete set of singularities in $\mathbb{C}$, and are exponentially bounded along these paths [3]. Additionally, the set of singularities is of the form $\cup\lambda_j\mathbb{N}$, and the singularities are of the form $(p-p_0)^\alpha A(p) + B(p)$ or $\log(p-p_0)(p-p_0)^{\alpha'} A(p) + B(p)$ where $A, B$ are locally analytic, $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, and $\alpha' \in \mathbb{Z}$ [3].

### 3.3.1 Écalle critical time

To ensure Borel summability, it is important to make a change of variables to a variable called the *Écalle critical time*. Essentially, one makes a change of variables to $t$ to ensure that the coefficient $c_k$ of $t^{-k-1}$ in the series $\tilde{f}$ goes like $k!$ (if it is already diverging factorially), as then the Borel transform of the series will be convergent. This is the same change of variables that makes the exponent of the exponentials in the transseries linear in $t$, as discussed in Remark 3.1. For example, suppose $\tilde{f} = \sum_{k=0}^{\infty}(2k)!x^{-k-1}$. Here, $c_k = (2k)!$, which grows much faster than $k!$, so we make the change of variables $t = \sqrt{x}$. Then $\tilde{f} = \sum_{k=0}^{\infty}(2k)!t^{-2k-2} = \sum_{k=0}^{\infty}(2k)!t^{-(2k+1)-1} = \sum_{j=0}^{\infty} b_j t^{-j-1}$ where $b_k = 0$ for $k$ even and $b_k = (k-1)!$ for $k$ odd, which goes like $k!$.

### 3.3.2 Properties of Borel summation

In this section, we list some important properties of Borel summation [1]:

**Proposition 3.2.** *(a) $S_{\mathcal{B}}$ is a differential field with respect to formal addition, multiplication, and differentiation of power series. Further, $\mathcal{LB} : S_{\mathcal{B}} \to \mathcal{LB}(S_{\mathcal{B}})$ is linear and commutes with multiplication, division (when it exists), and differentiation, that is, $\mathcal{LB}$ is a differential field isomorphism.*

*(b) If $S_C \subset S_{\mathcal{B}}$ denotes the differential algebra of convergent power series, then $\mathcal{LB}$ is the identity on $S_C$, where we identify a convergent power series with its sum.*

*(c) For $\tilde{f} \in S_{\mathcal{B}}$, $\mathcal{LB}(\tilde{f}) \sim \tilde{f}$ as $|x| \to \infty$, $\mathrm{Re}(x) > 0$.*

The previous proposition shows that $\mathcal{LB}$ has a lot of desirable properties, such as mapping a convergent power series back to itself. Most importantly, it states that the function obtained from the Borel summation behaves asymptotically like the series $\tilde{f}$ that one started with, which is a direct application of Watson's lemma (see Section 3.4). In fact, if one applies Borel-Écalle summation to the divergent asymptotic series for a differential equation, one recovers a function which is indeed a solution to the differential equation whose asymptotic series is the divergent series one started with.

### 3.3.3 Borel summation along other directions

The Borel sum of a series $\tilde{f}$ in a direction $\phi$, $(\mathcal{LB})_\phi(\tilde{f})$, is defined to the Laplace transform of $F$ along the ray $xp \in \mathbb{R}^+$ with $\arg(x) = \phi$, meaning that $\arg(p) = -\phi$ [1]:

$$(\mathcal{LB})_\phi(\tilde{f}) = \int_0^{\infty e^{-i\phi}} e^{-px} F(p)\, dp =: \mathcal{L}_{-\phi} F. \tag{3.5}$$

This, of course, requires that $\mathcal{B}(\tilde{f})$ can be analytically continued to a function $F$ in the direction $-\phi$ and that $F$ is exponentially bounded in this direction. We can similarly define the Borel sum of $\tilde{f}$ in a direction $\phi$ as the Borel summation along $\mathbb{R}^+$ of $\tilde{f}(xe^{i\phi})$, where $x \in \mathbb{R}^+$.

In general, if the analytic continuation $F$ of $\mathcal{B}(\tilde{f})$ is analytic in a sector $S_p := \{e^{i\phi}|p| : \phi \in (\delta, \gamma), |p| \in \mathbb{R}^+\}$ for some $\delta, \gamma$, one can do Borel summation of the series along any direction $\phi \in (\delta, \gamma)$.

### 3.3.4 Stokes phenomena

Here we will discuss an example of the Stokes phenomenon adapted from Section 4.4d of Ref. [1] in an attempt to illustrate it. Consider $\tilde{f} = \sum_{k=0}^\infty (-1)^k k! \cdot x^{-k-1}$. Then, $\mathcal{B}(\tilde{f}) = \sum_{k=0}^\infty (-1)^k p^k$, which converges to $(1+p)^{-1}$ in the unit disk. We can analytically continue this to $(1+p)^{-1}$ in $\mathbb{C} \setminus (-\infty, -1]$, as here the function has a singularity. Then, we Borel sum along $\mathbb{R}^+$ first, obtaining

$$f(x) := \mathcal{LB}(\tilde{f})(x) = \int_0^\infty \frac{e^{-px}}{1+p}\, dp. \tag{3.6}$$

Due to the rapid decay of the integrand for $|p| \to \infty$ in the right half plane, one can see that for $x \in \mathbb{R}^+$ we have

$$f(x) = \int_0^{\infty e^{-i\pi/4}} \frac{e^{-px}}{1+p}\, dp. \tag{3.7}$$

Since the two functions in (3.7) agree on $\mathbb{R}^+$, they agree everywhere they are analytic. One can easily see that $f$ is analytic for $\arg(x) \in (-\pi/2, \pi/2)$, while the right hand side of (3.7) is analytic for $\arg(x) \in (-\pi/4, 3\pi/4)$. Thus, the two expressions agree for $\arg(x) \in (-\pi/4, \pi/2)$, and by definition of analytic continuation, $f$ admits analytic continuation in the sector $\arg(x) \in (-\pi/2, 3\pi/4)$.

Now, we begin rotating $x$ counterclockwise, attempting to extend the analytic continuation. Taking $\arg(x) = \pi/4$, by a similar argument to before,

$$f(x) = \int_0^{\infty e^{-i\pi/2}} \frac{e^{-px}}{1+p}\, dp. \tag{3.8}$$

The integrals in (3.7) and (3.8) agree on $e^{i\pi/4}\mathbb{R}^+$ and the integral in (3.8) is analytic for $\arg(x) \in (0, \pi)$. Thus, we have that $f$ admits analytic continuation in the sector $\arg(x) \in (-\pi/2, \pi)$. We continue rotating as so until $\arg(x) = \pi - \epsilon$, where we get that

$$f(x) = \int_0^{\infty e^{-i(\pi-\epsilon)}} \frac{e^{-px}}{1+p}\, dp, \tag{3.9}$$

where the right hand side is analytic for $\arg(x) \in (\pi/2 - \epsilon, 3\pi/2 - \epsilon)$, so $f$ admits analytic continuation in the secton $\arg(x) \in (-\pi/2, 3\pi/2 - \epsilon)$.

To rotate beyond $\arg(x) = \pi$, we must collect the residue at the pole at $p = -1$:

$$\int_0^{\infty e^{-i(\pi+\epsilon)}} \frac{e^{-px}}{1+p}\, dp - \int_0^{\infty e^{-i(\pi-\epsilon)}} \frac{e^{-px}}{1+p}\, dp = 2\pi i e^{x}, \tag{3.10}$$

where the exponential is small as $\arg(x) = \pi$. Thus, we have

$$f(x) = \int_0^{\infty e^{-i(\pi+\epsilon)}} \frac{e^{-px}}{1+p}\, dp - 2\pi i e^{x}, \tag{3.11}$$

which itself is analytic for $\arg(x) \in (\pi/2+\epsilon, 3\pi/2+\epsilon)$. We can now continue with the analytic continuation as before until $\arg(x) = 2\pi$, finding that $f(xe^{2\pi i}) = f(x) - 2\pi i e^{x}$. Here, the exponential is now large since $\arg(x) = 2\pi$, and in fact, the exponential starts to become large at $\arg(x) = 3\pi/2$. Now, $f$ admits an asymptotic series, which actually happens to be $\tilde{f}$, in a full half-plane, which is justified by Watson's Lemma (see Section 3.4). However, $f$

ceases to admit an asymptotic series at $\arg(x) = \pi$ as seen in (3.11); the integral also has $\tilde{f}$ as its asymptotic series, but we have an exponential term that is left over. This exponential term begins to dominate the asymptotic series at $\arg(x) = 3\pi/2$.

We call the ray at which we collect the residue the Stokes ray [1]. On the other hand, the ray at which the exponential that has been collected becomes large is called the antistokes ray. In nonlinear problems, such as the Painlevé equation $\mathrm{P}_{\mathrm{II}}$, one often collects infinitely many exponentials at each Stokes ray. These exponentials all become large at the antistokes ray, which is usually a source of singularities in the represented function.

## 3.4 Watson's Lemma

Watson's lemma is a very important tool for understanding the asymptotic behavior of Laplace transforms, which is crucial in Borel summation. It provides the asymptotic series at infinity of $(\mathcal{L}F)(x)$ in terms of the asymptotic series of $F(p)$ at 0 (which is simply the Taylor series of $F$ at 0 if $F$ is analytic) [1].

**Proposition 3.3** (Watson's lemma)**.** *Let $F \in L^1(\mathbb{R}^+)$ and assume $F(p) \sim \sum_{k=0}^{\infty} c_k p^{k\beta_1+\beta_2-1}$ as $p \to 0^+$ for some constants $\beta_i$ with $\mathrm{Re}(\beta_i) > 0$, $i = 1, 2$. Then, for $a \leq \infty$,*

$$f(x) = \int_0^a e^{-xp} F(p)\, dp \sim \sum_{k=0}^{\infty} c_k \frac{\Gamma(k\beta_1 + \beta_2)}{x^{k\beta_1+\beta_2}} \tag{3.12}$$

*as $x \to \infty$ along any ray in $\mathbb{H}$.*

**Remark 3.4.** *Watson's lemma also holds for $F \in L^1_{loc}(\mathbb{R}^+)$ such that $F$ is exponentially bounded (as in condition (2.2) or (2.3)).*

*Proof.* Indeed, assume again that $F(p) \sim \sum_{k=0}^{\infty} c_k p^{k\beta_1+\beta_2-1}$ as $p \to 0^+$ for some constants $\beta_i$

with $\mathrm{Re}(\beta_i) > 0$, $i = 1, 2$. For $a \leq \infty$,

$$\begin{aligned} f(x) = \int_0^a e^{-xp} F(p)\, dp &= \int_0^a e^{-xp} F(p) \chi_{[0,1]}(p)\, dp + \int_0^a e^{-xp} F(p) \chi_{(1,\infty)}(p)\, dp \\ &= \int_0^a e^{-xp} F(p) \chi_{[0,1]}(p)\, dp + \int_1^a e^{-xp} F(p)\, dp. \end{aligned}$$

Now, define $G \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ by $G(s) = F(s+1)$. Then, notice that $G$ is also exponentially bounded and

$$\int_1^a e^{-xp} F(p)\, dp = \int_0^{a-1} e^{-x(s+1)} F(s+1)\, ds = e^{-x} \int_0^{a-1} e^{-xs} G(s)\, ds = C_a e^{-x}.$$

where $C_a \in \mathbb{R}$ is a constant and it is understood that for $a = \infty$, $a - 1 = \infty$. Now, notice that $F(p)\chi_{[0,1]}(p) \in L^1(\mathbb{R}^+)$ since $F \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$. Also, since $F(p) = F(p)\chi_{[0,1]}$ around 0, we have that $F(p)\chi_{[0,1]} \sim \sum_{k=0}^{\infty} c_k p^{k\beta_1 + \beta_2 - 1}$ as $p \to 0^+$. Thus, by Watson's lemma above, we have that

$$f(x) = \int_0^a e^{-xp} F(p) \chi_{[0,1]}(p)\, dp + \int_1^a e^{-xp} F(p)\, dp \sim \sum_{k=0}^{\infty} c_k \frac{\Gamma(k\beta_1 + \beta_2)}{x^{k\beta_1 + \beta_2}}$$

as $x \to \infty$ along any ray in $\mathbb{H}$ since exponentially small terms do not affect the asymptotic series. □

**Remark 3.5.** *We are mostly concerned with $F$ which are analytic around 0, meaning that we have a Taylor series expansion, $F(p) = \sum_{k=0}^{\infty} c_k p^k$, around 0. Thus, Watson's lemma applies with $\beta_1 = \beta_2 = 1$, and we get that*

$$f(x) = \int_0^a e^{-xp} F(p) dp \sim \sum_{k=0}^{\infty} c_k \frac{k!}{x^{k+1}}. \tag{3.13}$$

*Taking $a = \infty$, we see that*

$$\mathcal{L}F(x) \sim \sum_{k=0}^{\infty} c_k \frac{k!}{x^{k+1}}, \tag{3.14}$$

*as we desire for Borel summation.*

# Chapter 4
# Painlevé Equation $P_{II}$

In this chapter, we first introduce the history of the Painlevé equations. Then we examine the Painlevé equation $P_{II}$, finding the asymptotic behavior, justifying the use of the Borel-Écalle resummation, and discussing its Écalle critical time. We then rewrite $P_{II}$ in a more convenient normalized form, discussing how to obtain the asymptotic series. Finally, we list important properties of the Borel transform of the asymptotic series.

## 4.1 The Painlevé equations

There are six Painlevé equations, labelled $P_I$-$P_{VI}$, which were discovered around the beginning of the 1900s by Paul Painlevé and his colleagues [4]. They were interested in studying second-order ordinary differential equations of the following form:

$$y'' = F(x; y, y'), \tag{4.1}$$

where $F$ is a rational function of $x$, $y$, and $y'$ and is analytic in $x$. They wondered which of these differential equations have the property that the solutions have no movable branch points, meaning that the location of the branch point singularities only depends on the equation and not which particular solution is chosen. This property is now called the *Painlevé property*.

Painlevé showed that there were 50 canonical differential equations of this form with the

Painlevé property (up to a Möbius transformation) [4]. These included linear equations and other well-known equations that were integrable in terms of known special functions. However, there were also six new nonlinear differential equations, named the Painlevé equations, whose solutions are special functions that do not reduce to other known special functions.

Although their mathematical discovery is remarkable in its own right, the Painlevé equations are also incredibly important to many physical applications. They have arisen in the 2D Ising Model from statistical mechanics [5, 6], in random matrix theory [7], and as reductions of soliton equations which are solvable by inverse scattering [8, 9]. They also each correspond to a polynomial time-dependent Hamiltonian [5]. The presence of the Painlevé equations in such a variety of physical applications makes them incredibly interesting to study, garnering the attention of many mathematicians and physicists.

## 4.2 Painlevé equation $\mathrm{P_{II}}$

The Painlevé equation $\mathrm{P_{II}}$ is as follows:

$$y'' = 2y^3 + xy + \alpha, \tag{4.2}$$

where $\alpha$ is a complex constant. There are many different solutions to this equation, such as the Ablowitz-Segur and Hastings-McLeod solutions when $\alpha = 0$. However, we are interested in the solutions when $\alpha \neq 0$. The default value for $\alpha$ for all numerical calculations that follow is $\alpha = \frac{1}{2}$. It is convenient to fully normalize the equation as follows.

We first look at the asymptotic behavior of the solution which we find by balancing dominant terms in the differential equation. Here, we look at the solutions where the dominating terms in the differential equation are $xy$ and $\alpha$ (see Remark 4.1 to see a dominant balance which does not work in general). Asymptotically, we have that $xy + \alpha = 0$, so

$$y \sim -\frac{\alpha}{x} \tag{4.3}$$

as $x \to \infty$. Thus, the solution has the form

$$y(x) = -\frac{\alpha}{x}(1 + f(x)), \tag{4.4}$$

where $f$ is $o(1)$.

**Remark 4.1.** *Suppose that the dominating terms to Equation (4.2) are $y''$ and $2y^3$. Then, we have that $y \sim \sqrt{2}/x$. However, then $xy \sim \sqrt{2}$ and $\alpha$ are more dominant than $y^3$. Thus, this dominant balance doesn't make sense, and the actual underlying dominant balance in this case is the one considered above. This balance would only work if $\alpha = -\sqrt{2}$.*

**Remark 4.2.** *Borel-Écalle summation applies to the Painlevé equation $P_{II}$ as well as the other Painlevé equations since they can be rewritten in the form given in Equation (3.1). To see this normalization done for $P_I$, see Section 5.5 of Ref. [10]. The process for rewriting $P_{II}$ in this form is similar.*

### 4.2.1 Écalle critical time of $\mathbf{P_{II}}$

To correctly normalize the equation, we would also like to find the Écalle critical time of the solution $y(x)$ so that the asymptotic series diverges exactly factorially. To do this, we wish to find the exponent of the exponentials in the transseries solution, so we look at small exponential perturbations $\delta$ to the asymptotic power series solution $y_0$ to the Equation (4.2). Hence, we substitute $y = y_0 + \delta$ into Equation (4.2) to find the differential equation for $\delta$, noting that $y_0$ satisfies Equation (4.2) asymptotically. As a result, we have that

$$\delta'' = 6y_0^2\delta + 6y_0\delta^2 + 2\delta^3 + x\delta. \tag{4.5}$$

Noting that $y_0 \sim -\alpha/x$ from above and that $\delta \ll 1$ as it is exponentially small, we have that $\delta''$ and $x\delta$ are the dominant terms asymptotically. This gives us that $\delta'' = x\delta$, which can't be satisfied by any power. Hence, we try the substitution $\delta = e^w$, finding that

$$w'' + (w')^2 = 6y_0^2 + 6y_0 e^w + 2(e^w)^2 + x \tag{4.6}$$

after cancelling out factors of $e^w$. Since $\delta = e^w << 1$ and $y_0 \sim -\alpha/x$, the dominant terms here are $(w')^2$ and $x$, giving us that $w(x) \sim -\frac{2}{3}x^{3/2}$ and, thus, $\delta(x) \sim \exp\left(-\frac{2}{3}x^{3/2}\right)$. This tells us to make the change of variables to the Écalle critical time $t = \frac{2}{3}x^{3/2}$.

### 4.2.2 Normalized differential equation

We wish to make the change of variables to $t = \frac{2}{3}x^{3/2}$, so it is convenient to let $\alpha f(x) = th(t)$ in the form of the solution from Equation (4.4) (see Remark 4.3). Thus, we have

$$y(x) = x^{-1}(-\alpha + th(t)). \tag{4.7}$$

Making the change of variables $t = \frac{2}{3}x^{3/2}$ and the substitution from Equation (4.7) in Equation (4.2), we obtain a new differential equation for $h$. After simplifying, we have that

$$h'' + \frac{h'}{t} - \left(1 + \frac{24\alpha^2+1}{9t^2}\right)h - \frac{8}{9}h^3 + \frac{8\alpha}{3t}h^2 + \frac{8(\alpha^3-\alpha)}{9t^3} = 0. \tag{4.8}$$

**Remark 4.3.** *The reason we let $\alpha f(x) = th(t)$ instead of simply $\alpha f(x) = g(t)$ is because the differential equation for $h$ is slightly simpler algebraically. One sees that here the $h^3$ term is only multiplied by a constant, while in the other case the $g^3$ term would be additionally multiplied by a factor of $t^{-2}$.*

We now wish to find the asymptotic series for the solution $h$. To do this, we rearrange the equation so that $h$ is on one side as follows:

$$h = \left(1 + \frac{24\alpha^2+1}{9t^2}\right)^{-1}\left(h'' + \frac{h'}{t} - \frac{8}{9}h^3 + \frac{8\alpha}{3t}h^2 + \frac{8(\alpha^3-\alpha)}{9t^3}\right). \tag{4.9}$$

Since we know that $h$ is $o(1)$, we first guess that $h = 0$. Plugging this into the right hand side of Equation (4.9), we obtain a second guess that $h = \frac{8(\alpha^3-\alpha)}{9t^3}$. We continue recursively plugging these guesses into the right hand side of Equation (4.9) to obtain more and more terms in the asymptotic series of $h$, call it $\tilde{h}$. For numerical calculations, we computed 200 terms in the series $\tilde{h}$, giving us a series in $t^{-2}$ up to the power $t^{-401}$. This iteration procedure is justified by Watson's lemma: if one transforms to the Borel plane, $\mathcal{B}(\tilde{h})$ is convergent, so the iteration procedure would converge there. Then, by Watson's lemma, $\tilde{h}$ is given by the formal Laplace transform of $\mathcal{B}(\tilde{h})$.

### 4.2.3 Borel transform of $\tilde{h}$

We then Borel transform $\tilde{h}$, obtaining a convergent series $\tilde{H} := \mathcal{B}(\tilde{h})$ in $p$. For numerical calculations, we obtained 200 terms in the series $\tilde{H}$, a series in $p^2$ up to the power $p^{400}$. We denote by $H$ the analytic continuation of the sum of this convergent series. Notice that $\tilde{H}$ satisfies the Borel transform of the differential equation (4.8), which corresponds to the inverse Laplace transform of the equation. To find the inverse Laplace transform of Equation (4.8), we notice, from Proposition 2.6 and Example 2.7, that $\mathcal{L}^{-1}(h^{(n)}) = p^n\mathcal{L}^{-1}h = p^nH$, $\mathcal{L}^{-1}(fg) = \mathcal{L}^{-1}(f) * \mathcal{L}^{-1}(g)$, and $\mathcal{L}^{-1}\left(\frac{1}{t}\right) = 1$. Using these, we find that the Borel transform of Equation (4.8) is

$$p^2H + (pH) * 1 - H - \frac{24\alpha^2+1}{9}H * 1^{*2} - \frac{8}{9}H^{*3} + \frac{8\alpha}{3}H^{*2} * 1 + \frac{4(\alpha^3-\alpha)}{9}p^2 = 0, \quad (4.10)$$

where $f^{*n} = f * f * \cdots * f$ $n$-times. Solving for the terms with $H$, we get

$$H = \frac{1}{p^2-1}\left[-(pH) * 1 + \frac{24\alpha^2+1}{9}H * 1^{*2} + \frac{8}{9}H^{*3} - \frac{8\alpha}{3}H^{*2} * 1 - \frac{4(\alpha^3-\alpha)}{9}p^2\right]. \quad (4.11)$$

Now, we list some important properties of $H$ which will be used and discussed later:

(i) Since $\tilde{H}$ satisfies Equation (4.11), which is an even equation, and $\tilde{H}$ is analytic around 0, $\tilde{H}$ is even, so $H$ is an even function.

(ii) The set of singularities of $H$ is $\mathbb{Z}\setminus\{0\}$. In light of the discussion of resurgent functions in Section 3.3, we have that the set of singularities is $\cup\lambda_j\mathbb{N}$ with $\lambda_1 = 1$ and $\lambda_2 = -1$.

Thus, we have that the Stokes rays are $\mathbb{R}^+$ and $\mathbb{R}^-$, giving us that the antistokes rays are $i\mathbb{R}^+$ and $i\mathbb{R}^-$.

(iii) $H$ is single-valued and analytic on the simply connected domain $\mathbb{C}\backslash((-\infty,-1]\cup[1,\infty))$, denoted now by $\mathcal{D}$. However, $H$ is also analytic at the points on $(-\infty,-1]$ and $[1,\infty)$, excluding its singularities.

In practice, we only have finitely many terms of $\tilde{h}$, leading to only finitely many terms of $\mathcal{B}(\tilde{h})$. In the next chapter we will discuss different approximations to the analytic continuation $H$ using only finitely many terms of $\mathcal{B}(\tilde{h})$.

# Chapter 5
# Borel Plane Approximations

In this chapter, we first discuss approximations of the function $H$, the Borel transform of the asymptotic series $\tilde{h}$, in the Borel plane. We first introduce Padé approximations, continued fraction approximations, and conformal Padé approximations. We then compare the accuracy and convergence of the different approximations. Finally, we touch on a relation between capacitors and the placement of poles of the conformal Padé approximants.

## 5.1 Padé approximations

The $[m/n]$ Padé approximant of $F$ at $p = 0$ is a rational function $A_m/B_n$, where $A_m$ is a polynomial of degree at most $m$, $B_n$ is a polynomial of degree at most $n$, and

$$F(p) - \frac{A_m(p)}{B_n(p)} = \mathcal{O}(p^{m+n+1}) \tag{5.1}$$

as $p \to 0$ [11]. In order to have uniqueness, it is also required that $B_n(0) = 1$. The Padé approximants are guaranteed to converge except on a set of zero logarithmic capacity [2].

Notice, if $F$ admits a convergent Maclaurin series $\tilde{F}$, the Taylor series about 0, then the Taylor series of $A_m/B_n$ about 0 should match $\tilde{F}$ to the term $p^{m+n}$, so that the difference is of order $\mathcal{O}(p^{m+n+1})$. In fact, one can calculate the Padé approximant of $F$ from $\tilde{F}$ as $F = \tilde{F}$ about 0, so we can also say that $[m/n]$ is the Padé approximant of $\tilde{F}$.

A large advantage of Padé approximants over the standard Taylor series is that Padé approximants can mimic the behavior of singularities in a function, while Taylor series cannot. Taylor series are analytic at every point in their radius of convergence, and they can only converge in disks that avoid singularities of the function they are approximating. For example, the function $(1-p)^{-1}$ admits Taylor series expansion $\sum_0^\infty p^n$ around 0, but this only converges in the open unit disk, $\{p : |p| < 1\}$, and completely fails to approximate the function at, for example, $p = 2$. On the other hand, $(1-p)^{-1}$ can be exactly replicated by a Padé approximation of order $[0/1]$.

However, in the case of $\mathrm{P_{II}}$, and other similar problems, the Padé approximants have some clear limitations which come from an intimate connection to orthogonal polynomials [12]. The Padé approximants of our function $H$ diverge on the entire cuts $(-\infty, -1]$ and $[1, \infty)$, which include the sets of singularities of $H$; the Padé approximants place singularities densely behind the singularities at 1 and $-1$ along the cuts $[1, \infty)$ and $(-\infty, -1]$ respectively, as seen in Figure 5.1. This is problematic because (i) there are many points of analyticity of $H$ on the rays $(-\infty, -1]$ and $[1, \infty)$ which are inaccesible by the Padé approximants due to the divergence, as seen in Figure 5.5(a), and (ii) the singularities of $H$, other than 1 and $-1$, are undetectable by the Padé approximants. The divergence of these approximations along the ray of singularities makes it impossible to calculate the Borel summation of $\tilde{h}$ along $\mathbb{R}^+$. The direction $\mathbb{R}^+$ is particularly important since it is the direction along which resummation reconstructs a real-valued solution of $\mathrm{P_{II}}$ for real positive $x$.

## 5.2 Continued fraction approximations

Another type of approximation that can be useful is a generalized continued fraction approximation. In this section, we only care to understand continued fraction approximations to Maclaurin series $\tilde{F}$, as our function $H$ in the Borel plane is analytic about 0. The generalized

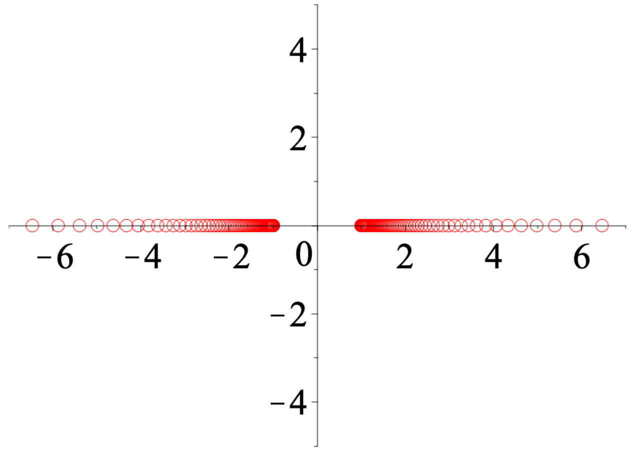



Figure 5.1: Poles produced by the [200/200] Padé approximant. Notice that the poles are densely placed behind 1 and −1.

continued fraction of $\tilde{F}$ is an expansion of the function in the form

$$\tilde{F}(p) = b_0 + \cfrac{\beta_1(p)}{1 + \cfrac{\beta_2(p)}{1 + \cdots}} =: b_0 + \frac{\beta_1}{1\,+}\,\frac{\beta_2}{1\,+}\,\cdots\,, \tag{5.2}$$

where, for all $i$, $\beta_i(p) = a_i p^{\alpha_i}$ with $a_i \neq 0$ and $\alpha_i \geq 1$ [11]. For such a continued fraction expansion, one defines the $n$-th convergents to the continued fraction as the finite continued fraction at each step $n$ in the expansion:

$$b_0 + \frac{\beta_1}{1\,+}\,\frac{\beta_2}{1\,+}\,\cdots\,\frac{\beta_n}{+\,1}. \tag{5.3}$$

We say that the continued fraction expansion in Equation (5.2) converges if the convergents in Equation (5.3) converge. It is not obvious that such an expansion in fact exists and converges. However, if one collapses the convergents, they are rational functions and actually correspond to Padé approximants of the Maclaurin series $\tilde{F}$ [11]. Hence, for the case of Maclaurin series, convergence of the infinite continued fraction is guaranteed in the same manner as it is guaranteed for Padé approximants.

Let us consider the following example of how to obtain such a continued fraction for a Maclaurin series, adapted from Ref. [11]. Suppose we are given a Maclaurin series

$$\tilde{F}(p) = \sum_{n=0}^{\infty} c_n p^n = c_0 + c_1 p + c_2 p^2 + \cdots \,, \tag{5.4}$$

where $c_i \neq 0$ for each $i$. One can find the continued fraction expansion as follows: We first calculate the reciprocal of the series

$$1 + \frac{c_2 p}{c_1} + \frac{c_3 p^2}{c_1} + \cdots = \left(1 + c_1^{(1)} p + c_2^{(1)} p^2 + \cdots\right)^{-1},$$

which exists since the left hand side converges about 0 (as $\tilde{F}$ converges about 0) and is nonzero. This allows the re-expansion

$$\tilde{F}(p) = c_0 + c_1 p + c_2 p^2 + \cdots = c_0 + \frac{c_1 p}{1 + c_1^{(1)} p + c_2^{(1)} p^2 + \cdots}. \tag{5.5}$$

One similarly continues, calculating the reciprocal of

$$1 + \frac{c_2^{(1)} p}{c_1^{(1)}} + \frac{c_3^{(1)} p^2}{c_1^{(1)}} + \cdots = \left(1 + c_1^{(2)} p + c_2^{(2)} p^2 + \cdots\right)^{-1},$$

allowing another re-expansion

$$\tilde{F}(p) = c_0 + \cfrac{c_1 p}{1 + \cfrac{c_1^{(1)} p}{1 + c_1^{(2)} p + c_2^{(2)} p^2 + \cdots}}. \tag{5.6}$$

Repeating this gives an iterative procedure to obtain a continued fraction expansion of the form given in Equation (5.2) [11]. Of course, the series $\tilde{H}$ for $\mathrm{P_{II}}$ is even, and so does not satisfy the premises of this calculation; however, by making a change of variables $z = p^2$, it then does. One can also adjust this calculation to more general Maclaurin series with no conditions on which coefficients are nonzero.

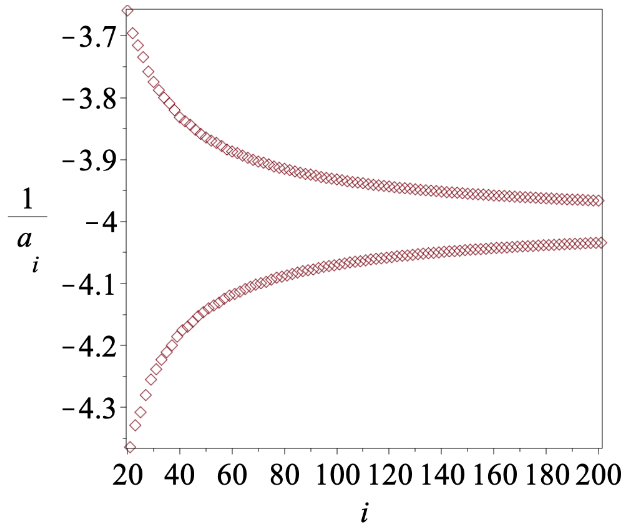



Figure 5.2: Plot of $a_i^{-1}$ versus $i$ for the continued fraction coefficients $\beta_i(p) = a_i p^2$ for $\tilde{H}$. We see that $a_i^{-1} \to -4$, so $a_i \to -\frac{1}{4}$.

### 5.2.1 Continued fractions with terminants

In practice, one clearly can not calculate the infinite continued fraction approximation to the function, especially when you have only calculated finitely many terms of the Maclaurin series. Thus, one approximates the function by the convergents of the continued fraction, which, as stated before, actually correspond to diagonal Padé approximants of the Maclaurin series $\tilde{F}$. To improve these approximations in comparison to Padé approximations, one can estimate the behavior of the infinite continued fraction by attaching a non-polynomial *terminant* $\phi(p)$ to the end of the finite continued fraction [13]:

$$b_0 + \frac{\beta_1}{1+} \frac{\beta_2}{1+} \cdots \frac{\beta_n}{1+} \frac{\phi}{1}. \tag{5.7}$$

Such a terminant can be found based on the asymptotic behaviors of the $\beta_i(p)$. For the continued fraction expansion of $\tilde{H}$ for $P_{II}$, we have that $\beta_i(p) = a_i p^2$, where $a_i \to -\frac{1}{4}$ as $i \to \infty$ [14], as seen in Figure 5.2. Thus, an appropriate approximation to the infinite continued fraction would be

$$b_0 + \frac{\beta_1}{1+}\frac{\beta_2}{1+}\cdots\frac{\beta_n}{+\ 1+}\frac{\tilde{\beta}}{1+}\frac{\tilde{\beta}}{1+}\cdots \tag{5.8}$$

where $\tilde{\beta}(p) = -\frac{p^2}{4}$. However, this can't be calculated numerically (as there are infinitely many steps), so we would like to write the tail of the continued fraction as one function, $\phi$. Then,

$$\phi(p) = \frac{\tilde{\beta}}{1+}\frac{\tilde{\beta}}{1+}\cdots = \cfrac{\tilde{\beta}(p)}{1+\cfrac{\tilde{\beta}(p)}{1+\cdots}} = \frac{-p^2/4}{1+\phi(p)}. \tag{5.9}$$

Solving this for $\phi$, you get

$$\phi(p) = -\frac{p^2}{2(1+\sqrt{1-p^2})}. \tag{5.10}$$

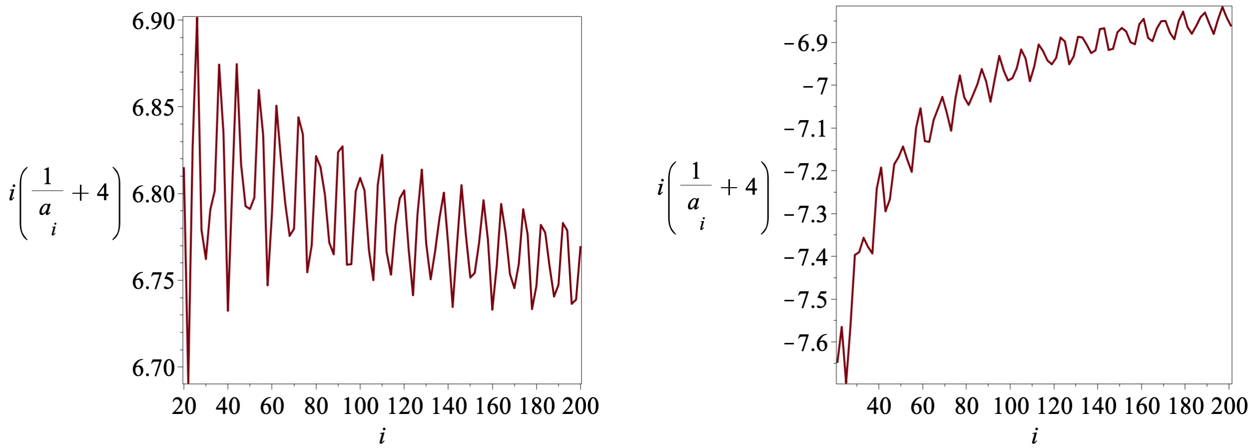
We refer to the approximant given in Equation (5.7) as a *CFwT approximant* with $n$ terms, which stands for continued fraction with terminant.

### 5.2.2 Asymptotic terminants

We can further improve the terminant described above by estimating the rate of convergence at which $a_i \to -\frac{1}{4}$. To do this, we looked at how fast $a_i^{-1} + 4 \to 0$ and found that $i(a_i^{-1} + 4) \sim 6.75(-1)^i$, as seen in Figure 5.3. Hence, $(a_i)^{-1} \sim -4 + (-1)^i \frac{6.75}{k}$. Using this rate of convergence for the coefficients of the continued fraction, we found a more accurate terminant to add onto the $n$-th convergent:

$$\phi_{\mathrm{asym}}(p) = -\frac{p^2}{2(1+\sqrt{1-p^2})} + (-1)^n \left(\frac{6.75}{16(n+1)}\right) p^2. \tag{5.11}$$

We refer to the approximant given in Equation (5.7), with $\phi$ replaced by $\phi_{\mathrm{asym}}$, as a *CFwAT approximant* with $n$ terms, which stands for continued fraction with asymptotic terminant. The $(n+1)^{-1}$ term in Equation (5.11) was obtained empirically, and we have no rigorous justification for it at this stage.




(a) Plot for even $i$.



(b) Plot for odd $i$.

Figure 5.3: Plots of $i(a_i^{-1}+4)$ versus $i$ for the continued fraction coefficients $\beta_i(p) = a_i p^2$ for $\tilde{H}$. We see that $i(a_i^{-1}+4) \sim 6.75(-1)^i$.

## 5.3 Conformal Padé

The conformal Padé approximant is another approximant that we have studied to increase the accuracy of the approximation along the line of singularities [13]. Here, we first apply a conformal map to the Taylor series to transform the problem into the unit disk and then use Padé approximants on this transformed Taylor series. This idea has been used in special models in physics, but it has not become widely used or known.

First, since our function $H$ is analytic in the simply connected domain $\mathcal{D} = \mathbb{C} \setminus ((-\infty,-1] \cup [1,\infty))$, we wish to find a map $\psi$ that maps $\mathcal{D}$ into the unit disk $\mathbb{D}$, which is given by

$$\psi(p) = \frac{1-\sqrt{1-p^2}}{p}. \tag{5.12}$$

Note that this map takes the cuts $(-\infty,-1]$ and $[1,\infty)$ to the boundary of the unit disk, $\partial\mathbb{D}$. This map has the inverse

$$\varphi(z) \coloneqq \psi^{-1}(z) = \frac{2z}{1+z^2}, \tag{5.13}$$

which takes the unit disk $\mathbb{D}$ into $\mathcal{D}$. Then, $G = H \circ \varphi$ is analytic in $\mathbb{D}$, and its Taylor series is given by $\tilde{G} = \tilde{H} \circ \varphi$ [13]. Notice, $G$ can be analytically continued outside of the unit disk due to the properties of resurgence. We denote the plane that $G$ lives in as the conformal plane.

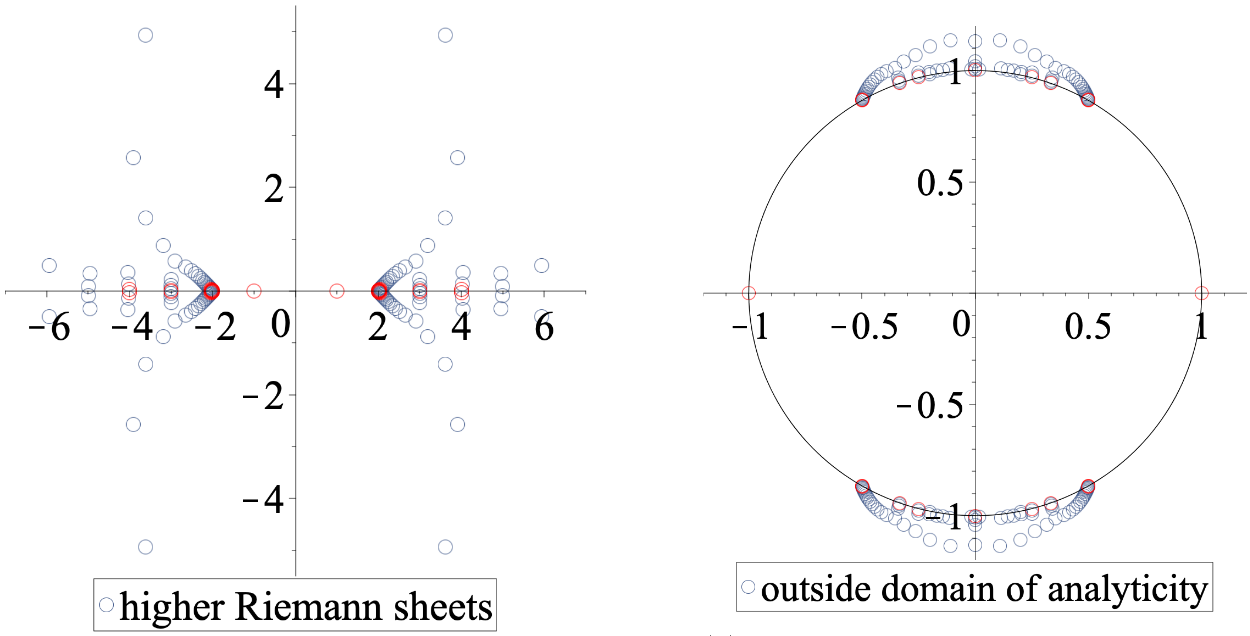
The set of singularities of $H$, $\mathbb{Z} \setminus \{0\}$, gets mapped under $\psi$ to a set of points of $\partial\mathbb{D}$, each at a distinct angle. Thus, we expect the singularities of the Padé approximants $\phi_{[m/n]}$ of $G$ to lie on rays originating at points of $\partial\mathbb{D}$ and going outwards into the conformal plane, as seen in Figure 5.4(b). When these Padé approximants are mapped back to $\mathcal{D}$ by $\phi_{[m/n]} \circ \psi$, the singularities outside of $\mathbb{D}$ lie on the second Riemann sheet, as seen in Figure 5.4(a), and therefore don't affect the accuracy of the function at those points. Compared to the singularities of the original Padé approximants, we see that the set of singularities on the first Riemann sheet obtained from the conformal Padé approximants are a subset of the set of actual singularities of $H$, so the conformal Padé approximants do not create unphysical singularities [15].

## 5.4 Accuracy of approximations

In this section, we compare the accuracy of the different approximation methods that we have mentioned. To see how these approximations are calculated, see Appendix A.

Our motivation in studying different approximation methods came from the divergence of the Padé approximations on the rays $(-\infty, -1]$ and $[1, \infty)$ due to the placement of unphysical poles. In Figure 5.5(a), we see that the unphysical poles produced by the Padé approximant create unwanted noise along the line of singularities, completely obscuring the behavior of the function. On the other hand, the other three approximants do not have unwanted noise as seen in Figure 5.5(b), only displaying visible jumps near singularities at integer points, as desired.

For the continued fraction approximations, attaching the terminants $\phi$ or $\phi_{\text{asym}}$ cause the




(a) Conformal Padé singularities in the Borel plane, with singularities on the first Riemann sheet in red.

(b) Conformal Padé singularities in the conformal plane, with singularities on the boundary of the disk in red.

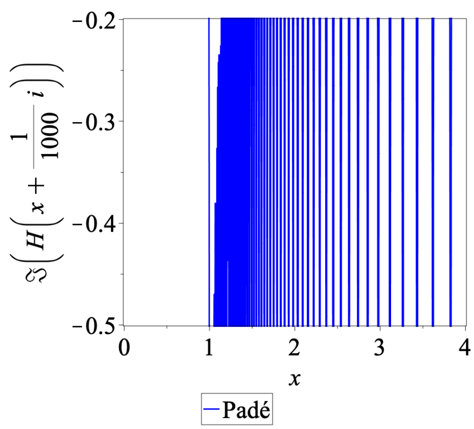
Figure 5.4: Singularities produced by the [200/200] conformal Padé approximant.

unphysical poles of the Padé approximant along the cuts $(-\infty, -1]$ and $[1, \infty)$ to disappear [13]. However, we see in Figure 5.5(b) that without the asymptotic correction to the terminant, the CFwT approximation still has frequent small oscillations. These small oscillations are smoothed out by attaching the asymptotic terminant in the CFwAT approximant, which agrees very closely with the conformal Padé approximant.

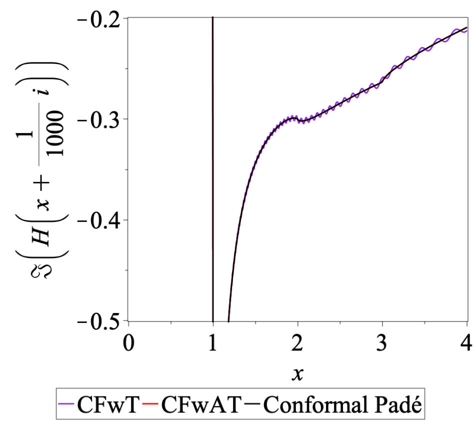
For the conformal Padé approximant, the only singularities produced on the first Riemann sheet are actual singularities of the function $H$, as seen in Figure 5.4(a), explaining why the conformal Padé approximant does not exhibit the noise seen from the Padé approximant.

### 5.4.1 Accuracy and convergence along the line of singularities

We now would like to study in depth the convergence and accuracy of the approximations along the line of singularities. To measure convergence, we computed the relative error of




(a) Plot of the imaginary part of the [200/200] Padé approximant along the line $p = x + 10^{-3}i$, $x \in \mathbb{R}$.



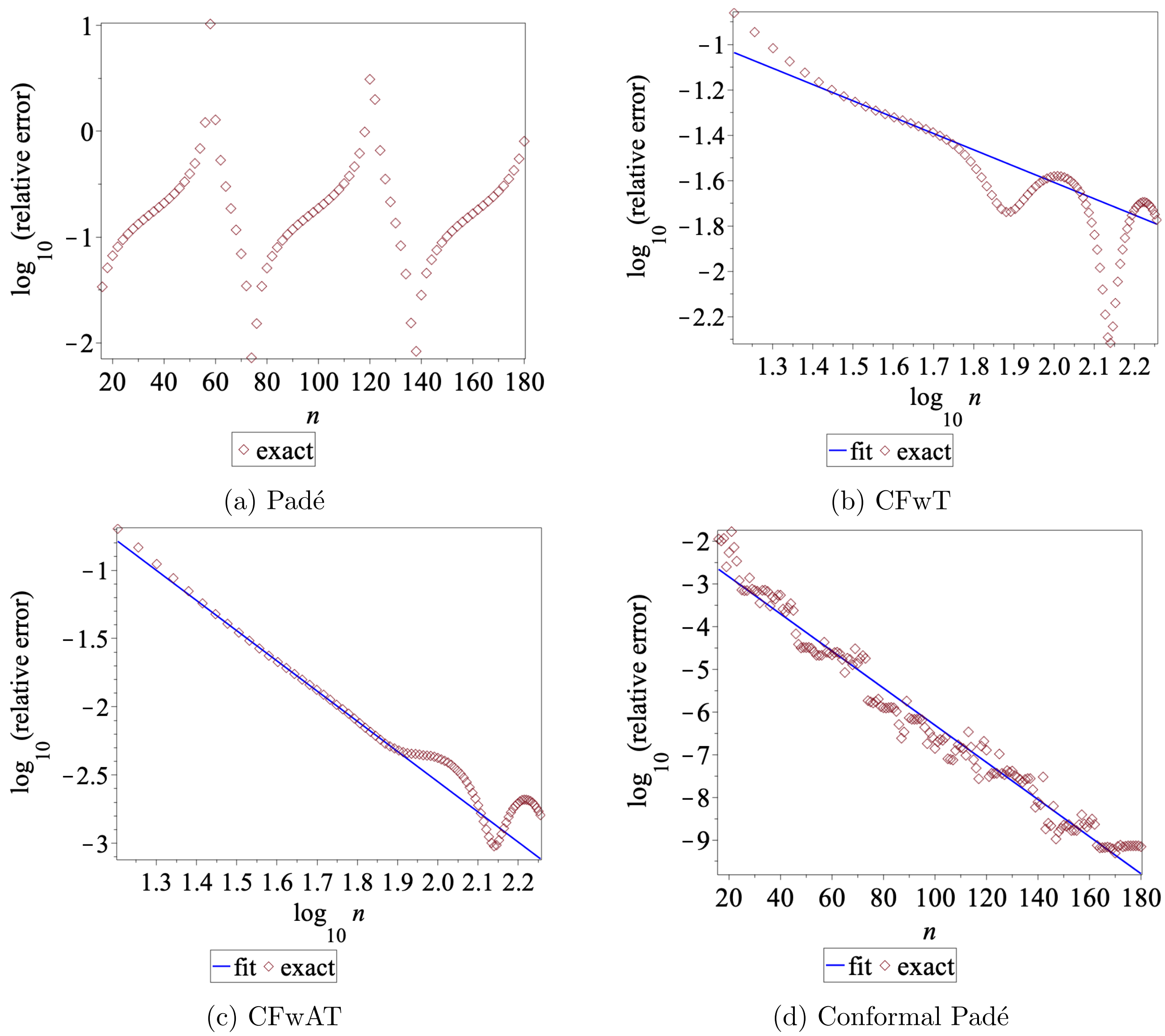

(b) Plot of the imaginary parts of the CFwT, CFwAT, and conformal Padé approximants, each at 200 terms, along the line $p = x + 10^{-3}i$, $x \in \mathbb{R}$. The red and black curves agree almost exactly.

Figure 5.5: Plots of the imaginary parts of various approximants along the line $p = x+10^{-3}i$.

each approximation at the point $p = 20 + 10^{-2}i$ as a function of the number of terms $n$ in the approximation, where the relative error is computed with respect to that approximant at 200 terms.

In Figure 5.6(a), we see that the relative error of the Padé approximants at $p = 20+10^{-2}i$ does not decrease as $n$ increases. In fact, the relative error oscillates, which is likely because the position of the singularities of the Padé approximants depends on the number of terms $n$ of the approximation. Hence, the behavior near the point $p = 20+10^{-2}i$ can vary immensely as $n$ increases. This numerically confirms that the Padé approximants diverge along the cuts $(-\infty, -1]$ and $[1, \infty)$.

On the other hand, the other approximants clearly converge at the point $p = 20 + 10^{-2}i$. In Figure 5.6(b), we see that relative error for the CFwT approximants decreases as $n$ increases. There is a small linear portion of the plot, which has slope $-0.719$. Since we plot $\log_{10}$(relative error) vs. $\log_{10} n$, this means that the relative error decreases like $n^{-0.719}$ along the line of singularities. However, the plot does not continue to follow the linear trend, instead oscillating around the linear fit as $n$ increases.




(a) Padé

(b) CFwT

(c) CFwAT

(d) Conformal Padé

Figure 5.6: Plots of relative error against number of terms $n$ for different approximants at $p = 20+10^{-2}i$ near the line of singularities, where the relative error is calculated with respect to the the same approximant at 200 terms. Note, the $x$-axis is $\log_{10}(n)$ for (b) and (c).

Similarly, we see that the CFwAT approximants converge at $p = 20 + 10^{-2}i$ as $n$ increases in Figure 5.6(c). In this plot, there is a much longer linear portion with slope $-2.212$, implying that the relative error decreases like $n^{-2.212}$ along the line of singularities. Notice that this is much faster convergence than that of the CFwT approximants. Again, the plot begins to deviate from the linear trend, oscillating around the fitted line as $n$ increases.

Finally, the conformal Padé approximants converge extremely fast at $p = 20 + 10^{-2}i$, as seen in Figure 5.6(d). The data follows a general linear trend with slope $-0.04348$. Since we plot $\log_{10}$(relative error) vs. $n$ in this figure, this implies that the relative error decreases like $10^{-0.04348n}$, which decreases much faster than the power decays of the two types of continued fraction approximants. Notice, the way in which the relative error decays is interesting; there seem to be periods in which the relative error is approximately constant as $n$ increases, while at other points there are jumps of an order of magnitude in the relative error.

To measure accuracy of the different approximations along the line of singularities, we computed the relative error of each approximation at 150 terms along the line $p = x + 10^{-3}i$, $x \in \mathbb{R}$, where the relative error is computed with respect to the conformal Padé approximant at 200 terms (which is empirically the most accurate). The results are seen in Figure 5.7. We see that there is a clear hierarchy of the approximations. The Padé approximant is extremely inaccurate, especially between $x = 1$ and $x = 4$ where most of the unphysical singularities lie. The CFwT approximants gives a large improvement along the rays of singularities as it gets rid of the unphysical singularities. The accuracy is then further improved by using the asymptotic terminant in the CFwAT approximant, as it approximates the infinite continued fraction more precisely than the regular terminant. Finally, the conformal Padé approximant is incredibly accurate compared to the other approximations, and is clearly the best approximation along the line of singularities.

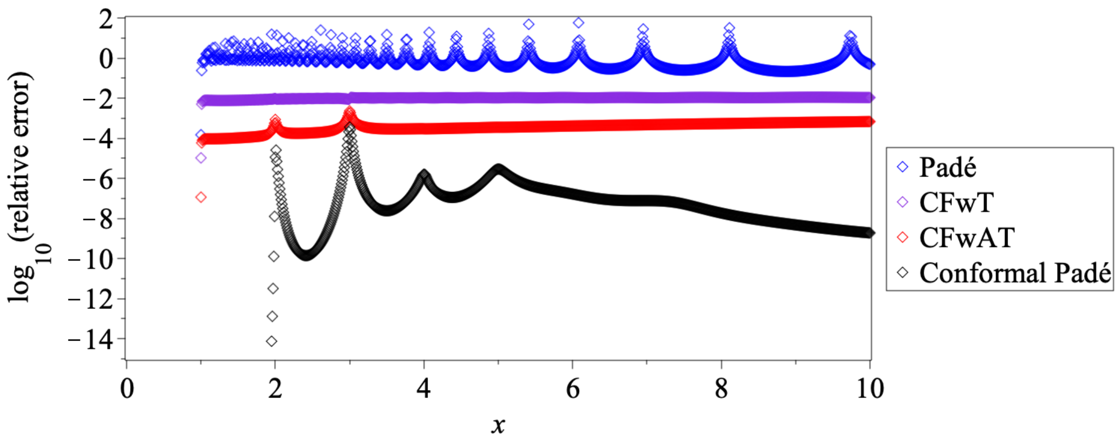



Figure 5.7: Comparison of relative error of different approximations, each at 150 terms, to conformal Padé at 200 terms along the line $p = x + 10^{-3}i$.

### 5.4.2 Accuracy and convergence along the softest line, $p = yi$

On the other hand, it is also important to study the behavior of the approximations away from the line of singularities. Here, we look at the softest line, $p = yi$ with $y \in \mathbb{R}$, which is between the two rays of singularities, $(-\infty, -1]$ and $[1, \infty)$. We call this the "softest" line because it is the line furthest from the effects of the singularities. To measure convergence, we computed the relative error of each approximation at the point $p = 20i$ as a function of the number of terms $n$ in the approximation, where the relative error is computed with respect to that approximant at 200 terms.

In Figure 5.8(a), we see that the relative error of the Padé approximants at $p = 20i$ decreases surprisingly quickly as $n$ increases, contrary to their behavior along the line of singularities. We see a linear relation between $\log_{10}$(relative error) and $n$ with slope -0.04337, which means that the relative error decreases like $10^{-0.04337}$ along the softest line. Notice that this is comparable to the convergence of the conformal Padé approximants along the line of singularities, which is the worst line for convergence.

In fact, for all the approximations, the relative error decreases exponentially along the

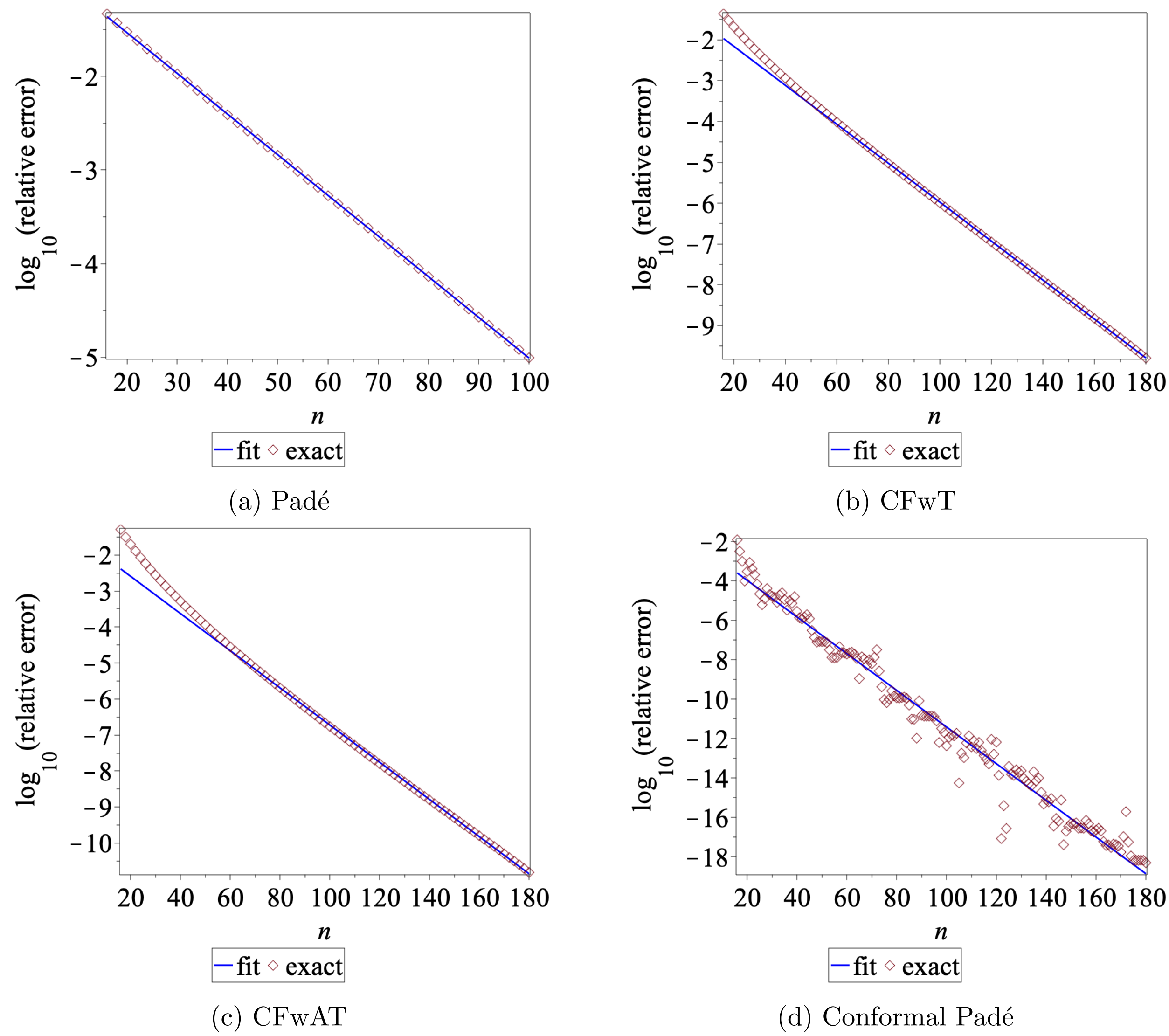



Figure 5.8: Plots of relative error against number of terms $n$ for different approximants at $p = 20i$ on the softest line, where the relative error is calculated with respect to the the same approximant at 200 terms.

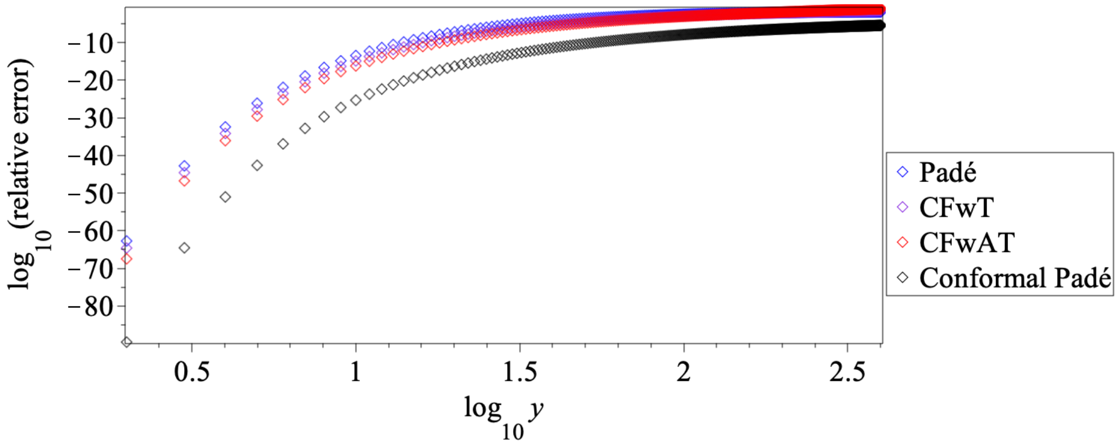



Figure 5.9: Comparison of relative error of different approximations, each at 150 terms, to conformal Padé at 200 terms along the softest line $p = yi$.

softest line, as seen in Figures 5.8(b), (c), and (d). For the CFwT approximants, the relative error decreases like $10^{-0.04770n}$, while for CFwAT approximants the relative error decreases like $10^{-0.05162n}$. Finally, the relative error for the conformal Padé approximants decreases like $10^{-0.09299n}$ along the softest line, which is still significantly faster than the convergence of all the other approximants.

To measure accuracy of the different approximations along the softest line, we computed the relative error of each approximation at 150 terms along the line $p = yi$, $y \in \mathbb{R}$, where the relative error is computed with respect to the conformal Padé approximant at 200 terms (which again is empirically the most accurate). The results are seen in Figure 5.9. We see that there is again the same hierarchy of the approximations, with the Padé approximant being the least accurate while the conformal Padé approximant is the most accurate. Notice also that along the softest line, the accuracy of the approximants is orders of magnitude better than along the line of singularities, as expected: the function should be easiest to approximate away from its singularities.

## 5.5 Conformal Padé and Capacitors

The placement of poles by the Padé approximants in the conformal plane can be predicted by the study of capacitors, as described by Stahl in Ref. [2]. We will shortly describe this relation.

We start with a function $G$ which is analytic in $\mathbb{D}$ which has analytic continuation outside the disk except for a set of zero capacity of poles and branch points, which exactly describes our function $H \circ \psi$ when mapped into the conformal plane. Then, we make the change of variables $z \to \frac{1}{z}$, placing the domain of analyticity of the function outside of the unit disk. Then, we make cuts $\mathcal{C} \subset \overline{\mathbb{D}}$ between the singularities of $G(1/z)$ on the unit circle, making $G(1/z)$ single-valued on $\mathbb{C} \setminus \mathcal{C}$. Now, we think of these cuts $\mathcal{C}$ as a bendable wire with charge 1 Coulomb, bending the wire so that the capacitance of the wire is minimal, where the capacitance of the wire is computed by the voltage with respect to infinity. Let us denote the cuts with minimum capacitance as $\mathcal{C}_0$. Then, the Padé approximants for $G(1/z)$ place poles on $\mathcal{C}_0$. See Figure 5.10 for an example.

Additionally, if we take the potential $V(z)$ created by the wire $\mathcal{C}_0$, and let $W(z) = e^{-V(z)}$, the asymptotic error in the Padé approximants $\tilde{P}_{[n/n]}$ for $G(1/z)$ (as $n$ becomes large) is

$$|\tilde{P}_{[n/n]}(z) - G(1/z)| \sim |W(z)|^{2n}. \tag{5.14}$$

Notice that $W$, and thus the error, depends only on the location of singularities of the Padé approximants. Translating back into the unit disk by the change of variables $z \to \frac{1}{z}$, we find that the asymptotic error in the Padé approximants $P_{[n/n]}$ for $G$ (as n becomes large) is

$$|P_{[n/n]}(z) - G(z)| \sim |W(1/z)|^{2n}. \tag{5.15}$$

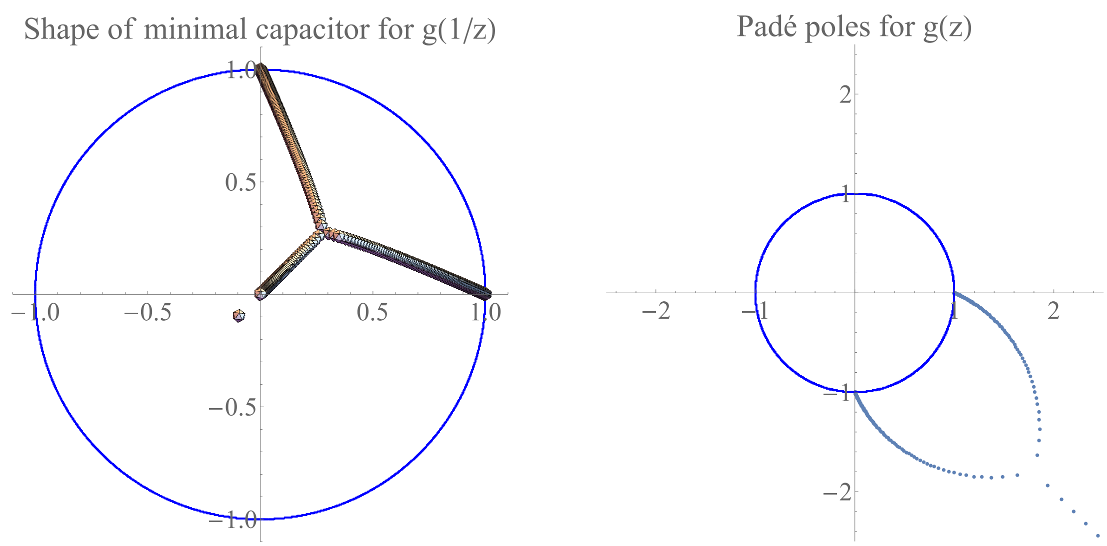



Figure 5.10: Plot of the wire of minimal capacitance for $g(1/z)$ and the locations of the Padé poles for $g(z)$, where $g(z) = [(1-z)(i+z)]^{1/3}$.

# Chapter 6
# Binary Rational Expansion

In this chapter we discuss the idea behind the binary rational expansion and see a *formal* derivation of the coefficients in the binary rational expansion for the solution to $\mathrm{P}_{\mathrm{II}}$.

## 6.1 Idea behind the binary rational expansion

The binary rational expansion is a rational expansion of the function $H$ to increase convenience in computing $\mathcal{L}(H)$ when doing the Borel summation of $\tilde{h}$. Although Borel summation gives an expression for the solution to the differential equation, it is an integral expression since $\mathcal{L}(H)$ is an integral. This means that we have to compute an integral for each value of the convergent solution, which is, numerically, a very expensive problem. Hence, we would like to rewrite $H$ in a rational expansion where each term has an exact Laplace transform, meaning that we get an exact form of the convergent solution in terms of only the coefficients of the expansion. This means we would only have to calculate the coefficients once and for all and would then be able to find any value of the convergent solution using those coefficients, which is much more convenient than the integral expression. We choose the expansion

$$H(p) = c_0 + \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} c_{m,k} \left(1 - e^{\beta i p / 2^k}\right)^m . \tag{6.1}$$

Using partial fractions and Example 2.8, one can find that

$$\mathcal{L}\left(\left(1-e^{\beta ip/2^k}\right)^m\right)(x)=\frac{(-1)^m(\beta i)^m m!}{x(2^kx-\beta i)(2^kx-2\beta i)\cdots(2^kx-m\beta i)},\tag{6.2}$$

so the Laplace transform of each term is indeed exact. See Appendix C for the full calculation. Thus, we get that the expansion of the function $h=\mathcal{L}^{-1}(H)$ in the physical plane would be

$$h(t)=\frac{c_0}{t}+\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^m(\beta i)^m m!c_{m,k}}{t(2^kt-\beta i)(2^kt-2\beta i)\cdots(2^kt-m\beta i)}.\tag{6.3}$$

In this expansion, $\beta\in\mathbb{C}$ is a parameter which can be chosen. If we choose $\beta\in\mathbb{R}^+$, this expansion for $h$ has singularities that are placed densely on the non-negative imaginary line. On the other hand, if we take $\beta\in\mathbb{R}^-$, this expansion for $h$ has singularities that are placed densely on the non-positive imaginary line.

To see why the singularities in the binary rational expansion are placed densely on a cut, let us discuss the domain of analyticity of the Borel summation of $\tilde{h}$. As mentioned in Section 4.2.3, for $\mathrm{P_{II}}$, the Stokes rays are $\mathbb{R}^+$ and $\mathbb{R}^-$. Taking the Laplace transform of $H$ along the Stokes ray is numerically difficult since the function has singularities, so we can Borel sum $\tilde{h}$ along a different direction as described in Section 3.3.3. We wish to Borel sum the series along $\arg(x)=\pi/2$, so that we have $\arg(p)=-\pi/2$, which is the softest line for $H$. Now, since $H$ is analytic in the lower half plane and the Laplace transform increases the domain of analyticity by Proposition 2.4, the Laplace transform of $H$ has a domain of analyticity that is $\mathbb{C}\setminus i\mathbb{R}^-$, which excludes one of the antistokes rays. This matches the ray of singularities of the binary rational expansion when $\beta\in\mathbb{R}^-$. If we instead Borel sum the series along $\arg(x)=-\pi/2$, we would see that the domain of analyticity would match the singularities of the binary rational expansion when $\beta\in\mathbb{R}^+$.

There is a mathematical theory for the binary rational expansions for the general class of resurgent functions that proves that these expansions exists and converge geometrically [16]. Other than the expansion's convenience, the expansion is also useful because its domain of convergence can include Stokes rays, which are asymptotically important. However, the general theory does not give efficient ways of computing the coefficients for more general types of functions. Thus, the question now becomes how to find the coefficients $c_{m,k}$ efficiently for more general types of functions, such as for the solutions to $\mathrm{P_{II}}$.

## 6.2 Basic identities

We first state some basic identities that are used throughout the calculation of the binary rational expansion.

### 6.2.1 Mittag Leffler identity

We begin with Mittag Leffler's Theorem and the pole expansion of $\cot(p)$:

$$\cot(p) = \sum_{n\in\mathbb{Z}} \frac{1}{p - n\pi}, \tag{6.4}$$

so we have that

$$\sum_{n\in\mathbb{Z}} \frac{1}{\pi(p-n)} = \cot(\pi p) = i\frac{e^{\pi ip} + e^{-\pi ip}}{e^{\pi ip} - e^{-\pi ip}} = i + i\frac{2e^{-\pi ip}}{e^{\pi ip} - e^{-\pi ip}} = i + i\frac{2}{e^{2\pi ip} - 1}.$$

Thus, we have that

$$\frac{1}{2\pi ip} + \sum_{n\neq 0} \frac{1}{2\pi i(p+n)} = \frac{1}{2} + \frac{1}{e^{2\pi ip} - 1}.$$

Making the change of variables $2\pi ip \to p$, we get

$$\frac{1}{e^p - 1} = \frac{1}{p} + \sum_{n\neq 0} \frac{1}{p + 2\pi in} - \frac{1}{2}. \tag{6.5}$$

We can also manipulate this to find that

$$\frac{1}{1 - e^{-p}} = \frac{1}{p} + \sum_{n\neq 0} \frac{1}{p + 2\pi in} + \frac{1}{2}. \tag{6.6}$$

### 6.2.2 $1/p$ identity

We will require an identity between $1/p$ and exponentials for the binary rational expansion [16]. The identity is as follows:

$$\frac{1}{p} = \frac{1}{1-e^{-p}} - \sum_{k=1}^{\infty} \frac{2^{-k}}{1+e^{-p/2^k}}. \tag{6.7}$$

This also gives us

$$\frac{1}{p} = \frac{1}{e^{p}-1} + \sum_{k=1}^{\infty} \frac{2^{-k}}{1+e^{p/2^k}}. \tag{6.8}$$

## 6.3 Binary rational expansion

In this section, we will do a *formal* derivation of the binary rational expansion. We first begin with Cauchy's integral formula

$$H(p) = \frac{1}{2\pi i} \oint \frac{H(s)}{s-p}\, ds. \tag{6.9}$$

By pushing the contour of integration towards infinity around the cuts $(-\infty,-1]$ and $[1,\infty)$ and using the fact that $H$ is even, we obtain that

$$H(p) = \frac{1}{2\pi i}\left(\int_1^{\infty} \frac{(H^+ - H^-)(s)}{s-p}\, ds + \int_1^{\infty} \frac{(H^+ - H^-)(s)}{s+p}\, ds\right), \tag{6.10}$$

where $H^+$ denotes the function $H$ right above the cut $[1,\infty)$ and $H^-$ denotes the function $H$ right below the cut. For shorthand, we will denote $F(s) \coloneqq H^+(s) - H^-(s)$, so we have that

$$H(p) = \frac{1}{2\pi i}\left(\int_1^{\infty} \frac{F(s)}{s-p}\, ds + \int_1^{\infty} \frac{F(s)}{s+p}\, ds\right). \tag{6.11}$$

Now, we include a parameter $\beta$ which we may adjust later to improve rates of convergence:

$$H(p) = \frac{\beta i}{2\pi i}\left(\int_1^{\infty} \frac{F(s)}{\beta i(s-p)}\, ds + \int_1^{\infty} \frac{F(s)}{\beta i(s+p)}\, ds\right). \tag{6.12}$$

Utilizing the identities in Equations (6.7) and (6.8), we have that

$$\begin{aligned} H(p) = \frac{\beta i}{2\pi i}\Bigg[ \int_1^\infty \left( \frac{F(s)}{1-e^{-\beta i(s-p)}} - \sum_{k=1}^\infty \frac{2^{-k}F(s)}{1+e^{-\beta i(s-p)/2^k}} \right) ds \\ + \int_1^\infty \left( \frac{F(s)}{e^{\beta i(s+p)}-1} + \sum_{k=1}^\infty \frac{2^{-k}F(s)}{1+e^{\beta i(s+p)/2^k}} \right) ds \Bigg]. \end{aligned} \tag{6.13}$$

We will now look at the $k = 0$ term in detail and then treat the $k \geq 1$ terms similarly.

### 6.3.1 $k = 0$ term

The $k = 0$ term in Equation (6.13) is

$$\varphi_0(p) = \frac{\beta i}{2\pi i}\left[ \int_1^\infty \frac{F(s)}{1-e^{-\beta i(s-p)}}\, ds + \int_1^\infty \frac{F(s)}{e^{\beta i(s+p)}-1}\, ds \right]. \tag{6.14}$$

We manipulate this expression in two different ways. First, we utilize the Mittag Leffler identities in Equations (6.5) and (6.6) to find that

$$\begin{aligned} \varphi_0(p) &= \frac{\beta i}{2\pi i}\left[ \int_1^\infty \left( \frac{F(s)}{\beta i(s-p)} + \sum_{n\neq 0} \frac{F(s)}{\beta i(s-p)+2\pi i n} + \frac{F(s)}{2} \right) ds \right. \\ &\qquad \left. + \int_1^\infty \left( \frac{F(s)}{\beta i(s+p)} + \sum_{n\neq 0} \frac{F(s)}{\beta i(s+p)+2\pi i n} - \frac{F(s)}{2} \right) ds \right] \\ &= H(p) + \sum_{n\neq 0} H\left(p + \frac{2\pi n}{\beta}\right) = \sum_{n\in\mathbb{Z}} H\left(p+\frac{2\pi n}{\beta}\right). \end{aligned} \tag{6.15}$$

On the other hand, we have that

$$\begin{aligned} \varphi_0(p) &= \frac{\beta i}{2\pi i}\left[ \int_1^\infty \frac{e^{\beta i s}F(s)}{e^{\beta i s}-e^{\beta i p}}\, ds + \int_1^\infty \frac{e^{-\beta i s}F(s)}{e^{\beta i p}-e^{-\beta i s}}\, ds \right] \\ &= \frac{\beta i}{2\pi i}\left[ \int_1^\infty \frac{e^{\beta i s}F(s)}{(e^{\beta i s}-1)\left(1+\frac{1-e^{\beta i p}}{e^{\beta i s}-1}\right)}\, ds - \int_1^\infty \frac{e^{-\beta i s}F(s)}{(e^{-\beta i s}-1)\left(1+\frac{1-e^{\beta i p}}{e^{-\beta i s}-1}\right)}\, ds \right] \\ &= \frac{\beta i}{2\pi i}\sum_{m=0}^\infty (-1)^m \left[ \int_1^\infty \frac{e^{\beta i s}F(s)}{(e^{\beta i s}-1)^{m+1}}\, ds - \int_1^\infty \frac{e^{-\beta i s}F(s)}{(e^{-\beta i s}-1)^{m+1}}\, ds \right] (1-e^{\beta i p})^m. \end{aligned} \tag{6.16}$$

Of course, this requires both $e^{\beta is}-1$ and $e^{-\beta is}-1$ to be large for the Taylor expansion to be valid. If we actually integrate along the ray $[1,\infty)$, both $e^{\beta is}-1$ and $e^{-\beta is}-1$ are periodically zero, or small, so we need to be careful which contour of integration we actually use for these integral coefficients.

Notice, the expansion of $\varphi_0$ in Equation (6.16) matches the form we are looking for from Equation (6.1). However, the coefficients are currently still in the form of integrals. We would like to be able to rewrite these coefficients in terms of the summation expansion of $\varphi_0$ from Equation (6.15) and its derivatives. Here, sums of derivatives are more efficient to calculate since the $H$ we will be using, for $\mathrm{P_{II}}$ for example, is a rational approximant whose derivative can be exactly calculated.

We start with

$$\varphi_0(p)=c_0^{(0)}+\sum_{m=1}^{\infty}c_{m,0}(1-e^{\beta ip})^m. \tag{6.17}$$

First, notice that

$$c_0^{(0)}=\varphi_0(0)=\sum_{n\in\mathbb{Z}}H\left(\frac{2\pi n}{\beta}\right), \tag{6.18}$$

which can't be accurately calculated for $H$ since the set of singularities of $H$ is $\mathbb{Z}\setminus\{0\}$; for any value of $\beta$, $(2\pi n/\beta)_{n\in\mathbb{Z}}$ will get arbitrarily close to the set of singularities of $H$. This could be solved by calculating the coefficients by their integral forms; however, for the sake of the calculation, let us continue.

By induction, we can show that for $n\geq 1$,

$$\left(e^{-\beta ip}\frac{d}{dp}\right)^n\varphi_0(p)=(-\beta i)^n\sum_{m=1}^{\infty}m(m-1)\cdots(m-(n-1))c_{m,0}(1-e^{\beta ip})^{m-n}. \tag{6.19}$$

Then,

$$\left[\left(e^{-\beta ip}\frac{d}{dp}\right)^n\varphi_0(p)\right]_{p=0}=(-\beta i)^n n!c_{n,0}. \tag{6.20}$$

For $n \geq 1$, we can show that

$$\left(e^{-\beta ip}\frac{d}{dp}\right)^n = \left(e^{-\beta ip}\right)^n \sum_{j=1}^{n} a_{n,j}(-\beta i)^{n-j}\frac{d^j}{dp^j}, \tag{6.21}$$

with $a_{n,j} = |s(n,j)|$ for $1 \leq j \leq n$ where $s(n,j)$ denote the signed Stirling numbers of the first kind. Using this and the summation expression of $\varphi_0$ given in Equation (6.15), we have that

$$\left[\left(e^{-\beta ip}\frac{d}{dp}\right)^n \varphi_0(p)\right]_{p=0} = \sum_{j=1}^{n} a_{n,j}(-\beta i)^{n-j}\sum_{l\in\mathbb{Z}} H^{(j)}\left(\frac{2\pi l}{\beta}\right). \tag{6.22}$$

Combining this with Equation (6.20), we have for $n \geq 1$ that

$$c_{n,0} = \sum_{j=1}^{n} \frac{a_{n,j}}{n!(-\beta i)^j}\sum_{l\in\mathbb{Z}} H^{(j)}\left(\frac{2\pi l}{\beta}\right). \tag{6.23}$$

Substituting in everything back into Equation (6.16), we obtain

$$\begin{aligned}\varphi_0(p) &= \sum_{l\in\mathbb{Z}} H\left(\frac{2\pi l}{\beta}\right) + \sum_{m=1}^{\infty}\left[\sum_{j=1}^{m}\frac{a_{m,j}}{(-\beta i)^j m!}\sum_{l\in\mathbb{Z}} H^{(j)}\left(\frac{2\pi l}{\beta}\right)\right](1-e^{\beta ip})^m && (6.24a)\\ &= \sum_{l\in\mathbb{Z}} H\left(\frac{2\pi l}{\beta}\right) + \sum_{j=1}^{\infty}\left[\sum_{m=j}^{\infty}\frac{a_{m,j}}{(-\beta i)^j m!}(1-e^{\beta ip})^m\right]\sum_{l\in\mathbb{Z}} H^{(j)}\left(\frac{2\pi l}{\beta}\right). && (6.24b)\end{aligned}$$

The expansion of $\varphi_0$ in Equation (6.24a) is the binary rational expansion form we would hope for, where all coefficients can be calculated by sums of derivatives of $H$, which would be much quicker and convenient. The expansion in Equation (6.24b) groups the derivatives together, since computing derivatives is expensive. This structure is helpful in writing more efficient code when doing the calculation of this expansion.

### 6.3.2 $k \geq 1$ terms

For $k \geq 1$, the corresponding term in Equation (6.13) is

$$\varphi_k(p) = \frac{\beta i}{2\pi i}\left[\int_1^\infty \frac{2^{-k}F(s)}{1+e^{\beta i(s+p)/2^k}}\,ds - \int_1^\infty \frac{2^{-k}F(s)}{1+e^{-\beta i(s-p)/2^k}}\,ds\right]. \tag{6.25}$$

Similarly to the $k = 0$ term, we use the identities in Equations (6.5) and (6.6) to find that

$$\varphi_k(p) = -\sum_{n\in\mathbb{Z}} H\left(p + \frac{2^k(2n+1)\pi}{\beta}\right). \tag{6.26}$$

On the other hand, we expand into the form of the binary rational expansion, obtaining

$$\varphi_k(p) = \frac{2^{-k}\beta i}{2\pi i}\sum_{m=0}^{\infty}\left[\int_1^{\infty}\frac{e^{-\beta is/2^k}F(s)}{\left(e^{-\beta is/2^k}+1\right)^{m+1}}\,ds - \int_1^{\infty}\frac{e^{\beta is/2^k}F(s)}{\left(e^{\beta is/2^k}+1\right)^{m+1}}\,ds\right]\left(1-e^{\beta ip/2^k}\right)^m. \tag{6.27}$$

Again, this requires both $e^{-\beta is/2^k}+1$ and $e^{\beta is/2^k}+1$ to be large for the Taylor expansion to be valid. If we actually integrate along the ray $[1,\infty)$, both $e^{-\beta is/2^k}+1$ and $e^{\beta is/2^k}+1$ are periodically zero, so we need to choose appropriate contours of integration for these integral coefficients.

We once again would like to rewrite the coefficients of the expansion in Equation (6.27) in terms of the summation expansion of $\varphi_k$ from Equation (6.26). Doing this, we obtain

$$\begin{aligned}\varphi_k(p) &= -\sum_{l\in\mathbb{Z}} H\left(\frac{2^k(2l+1)\pi}{\beta}\right)\\ &\qquad - \sum_{m=1}^{\infty}\left[\sum_{j=1}^{m}\frac{a_{m,j}}{(-2^{-k}\beta i)^j\, m!}\sum_{l\in\mathbb{Z}} H^{(j)}\left(\frac{2^k(2l+1)\pi}{\beta}\right)\right]\left(1-e^{\beta ip/2^k}\right)^m && \text{(6.28a)}\\ &= -\sum_{l\in\mathbb{Z}} H\left(\frac{2^k(2l+1)\pi}{\beta}\right)\\ &\qquad - \sum_{j=1}^{\infty}\left[\sum_{m=j}^{\infty}\frac{a_{m,j}}{(-2^{-k}\beta i)^j\, m!}\left(1-e^{\beta ip/2^k}\right)^m\right]\sum_{l\in\mathbb{Z}} H^{(j)}\left(\frac{2^k(2l+1)\pi}{\beta}\right). && \text{(6.28b)}\end{aligned}$$

The expansion of $\varphi_k$ in Equation (6.28a) is only in terms of sums of derivatives of $H$, as desired. Again, the expansion in Equation (6.28b) groups the derivatives together to aid in structuring the code for calculations efficiently.

### 6.3.3 Full binary rational expansion for $H$

To obtain the full binary rational expansion for $H$, we note that

$$H(p) = \sum_{k=0}^{\infty} \varphi_k(p). \tag{6.29}$$

Utilizing the summation forms of each $\varphi_k$ given in Equations (6.15) and (6.26), we obtain that

$$H(p) = \sum_{n\in\mathbb{Z}} H\left(p + \frac{2\pi n}{\beta}\right) - \sum_{k=1}^{\infty}\sum_{n\in\mathbb{Z}} H\left(p + \frac{2^k(2n+1)\pi}{\beta}\right). \tag{6.30}$$

First, we add the constant terms from the binary rational expansion for each $\varphi_k$ in Equations (6.24a) and (6.28a) to obtain the constant term in the binary rational expansion for $H$, getting

$$\sum_{l\in\mathbb{Z}} H\left(\frac{2\pi l}{\beta}\right) - \sum_{k=1}^{\infty}\sum_{l\in\mathbb{Z}} H\left(\frac{2^k(2l+1)\pi}{\beta}\right) = H(0), \tag{6.31}$$

where we utilize the identity in Equation (6.30). This is the expected constant term for the binary rational expansion of $H$, as all other terms in the expansion are zero at $p = 0$. Then, combining the expansion from Equations (6.24a) and (6.28a), we get

$$\begin{aligned}
H(p) = H(0) + \sum_{m=1}^{\infty}\Bigg(&\left[\sum_{j=1}^{m} \frac{a_{m,j}}{(-\beta i)^j m!} \sum_{l\in\mathbb{Z}} H^{(j)}\left(\frac{2\pi l}{\beta}\right)\right](1-e^{\beta i p})^m &\text{(6.32a)}\\
&- \sum_{k=1}^{\infty}\left[\sum_{j=1}^{m} \frac{a_{m,j}}{\left(-2^{-k}\beta i\right)^j m!} \sum_{l\in\mathbb{Z}} H^{(j)}\left(\frac{2^k(2l+1)\pi}{\beta}\right)\right]\left(1-e^{\beta i p/2^k}\right)^m\Bigg)\\
= H(0) + \sum_{j=1}^{\infty}\Bigg(&\left[\sum_{m=j}^{\infty} \frac{a_{m,j}}{(-\beta i)^j m!}(1-e^{\beta i p})^m\right] \sum_{l\in\mathbb{Z}} H^{(j)}\left(\frac{2\pi l}{\beta}\right) &\text{(6.32b)}\\
&- \sum_{k=1}^{\infty}\left[\sum_{m=j}^{\infty} \frac{a_{m,j}}{\left(-2^{-k}\beta i\right)^j m!}\left(1-e^{\beta i p/2^k}\right)^m\right] \sum_{l\in\mathbb{Z}} H^{(j)}\left(\frac{2^k(2l+1)\pi}{\beta}\right)\Bigg).
\end{aligned}$$

## 6.4 Two sample calculations

We do a sample calculation of the binary rational expansion for two even functions whose exact form we know, $H_1(p) = e^{-p^4}$ and $H_2(p) = (1-p^2)^{-2/3}$. We calculate the binary rational expansion up to $m = 120$ and $k = 30$. To see the code for the computation, see Appendix C.

Since we build the binary rational expansion of a function $H$ from the binary rational expansions of the functions $\varphi_k$, as described in Section 6.3, it is interesting to first examine the binary rational expansions to the functions $\varphi_k$. For $H_1(p) = e^{-p^4}$, we find that the binary rational expansion of $\varphi_0$ agrees with $\varphi_0$ up to 50 digits of precision at $p = i/10$, but only agrees with $\varphi_0$ up to 9 digits of precision at $p = i/2$. At $p = i$, the binary rational expansion of $\varphi_0$ completely fails to predict the correct value. Thus, it loses accuracy quickly as $p$ travels up the imaginary axis.

The binary rational expansion of $H_1$, which is the sum of the binary rational expansions for each $\varphi_k$, behaves similarly: at $p = i/10$, it agrees with $H_1$ up to 50 digits of precision. However, at $p = i/2$, it only agrees up to 9 digits of precision, and at $p = i$, the binary rational expansion completely fails to predict the value of $H_1$. On the other hand, the binary rational expansion of $H_1$ loses accuracy even quicker along the real axis. At $p = 1/10$, it agrees with $H_1$ to 50 digits of precision, but at $p = 1/2$, it already fails to predict the value of $H_1$, giving a value of order of magnitude $10^{20}$.

For $H_2(p) = (1 - p^2)^{-2/3}$, we find that the binary rational expansion of $\varphi_0$ fails to agree with $\varphi_0$ already at $p = i/10$. The binary rational expansions of the other $\varphi_k$ behave similarly, completely failing to approximate their respective functions on the positive imaginary line. On the other hand, the binary rational expansion for $H_2$ itself agrees with $H_2$ up to 7 digits of precision at $p = i/2$ and up to 6 digits of precision at $p = i$. It begins to fail to predict the value of $H_2$ at $p = 2i$. This is interesting since, unlike for $H_1$, the binary rational expansions for the $\varphi_k$ were completely inaccurate, but the binary rational expansion for $H_2$ was much more accurate than that of $H_1$. However, similarly to $H_1$, the binary rational expansion of $H_2$ loses accuracy extremely quickly along the real axis, failing to predict the value of $H_2$ at $p = 1/2$ as it gives a value of order of magnitude $10^{14}$.

**Remark 6.1.** *Since the Laplace transform is an integral with an exponentially small kernel, the expansion for $H$ is suppressed as $p$ grows larger. Thus, although the expansion for $H$ in the Borel plane seems to not converge very quickly at large values of $p$, it is possible that convergence in the physical plane is faster.*

## 6.5 Next steps for the binary rational expansion

The examples discussed in Section 6.4 were able to use the coefficients calculated by sums since they had very few singularities which were avoidable or no singularities at all. However, it still stands that the summation expressions for the coefficients would not work for $\mathrm{P_{II}}$ since they get arbitrarily close to the set of singularities of $H$. Instead, we must use the integral expressions of the coefficients although these are computationally more expensive. This requires a very careful choice of the contours of integration to ensure that the Taylor series expansion is valid and the coefficients are calculated properly. As we move forward, we would like to calculate the coefficients for $\mathrm{P_{II}}$, and perhaps simpler examples, from their integral expressions.

# Appendix A
# Calculating Approximants in the Borel Plane

## A.1 Calculating Padé approximants

We use the fact that the diagonal (i.e. $[n/n]$), or near diagonal, Padé approximants correspond to finite continued fractions (see Section 5.2). We use the `ContinuedFraction` method from the `NumberTheory` package of Maple to find the continued fraction coefficients. Once we have the list of coefficients, `coeff`, we calculate the Padé approximant with `n` terms at a certain value `p` with the following code:

```
pade := proc (n::integer, p::complex, coeff::list)::complex;
    local v; # the temporary value of the approximation
    local i; # the index variable for the loop
    local q; # a temporary value for p^2 to save computations
    q := p^2;
    v := evalf(coeff[n]*q); # the initial value of approximation
    # a loop to calculate the Pade approximation at the value p
    for i from 1 to n-1 do
        v := evalf(coeff[n-i]*q/(v+1));
    end do;
```

```
    return v;
end proc:
```

## A.2 Calculating continued fractions with the terminant

Once we have the list of continued fraction coefficients, `coeff`, obtained above, we calculate the `n`-th continued fraction with the terminant at a certain value `p` with the following code:

```
cf_terminant := proc (n::integer, p::complex, coeff::list)::complex;
    local v; # the temporary value of the approximation
    local i; # the index variable for the loop
    local q; # a temporary value for p^2 to save computations
    q := p^2;
    v := evalf(-q/(2+2*sqrt(1-q))); # the initial value of the terminant
    # a loop to calculate the approimation at the value p
    for i from 1 to n do
        v := evalf(coeff[n+1-i]*q/(v+1));
    end do;
    return v;
end proc:
```

## A.3 Calculating continued fractions with the asymptotic terminant

Once we have the list of continued fraction coefficients, `coeff`, obtained above, we calculate the `n`-th continued fraction with the asymptotic terminant at a certain value `p` with the following code:

```
cf_asympt := proc (n::integer, p::complex, coeff::list)::complex;
```

```
        local v; # the temporary value of the approximation
        local i; # the index variable for the loop
        local q; # a temporary value for p^2 to save computations
        q := p^2;
        # the initial value of the asymptotic terminant
        v := evalf(-q/(2*(1+sqrt(1-q)))+(-1)^n*(6.75/16)*q/(n+1));
        # a loop to calculate the approimation at the value p
        for i from 1 to n do
            v := evalf(coeff[n+1-i]*q/(v+1));
        end do;
        return v;
    end proc:
```

## A.4 Calculating conformal Padé approximants

We first transform our series to the conformal disk by $\tilde{G} = \tilde{H} \circ \phi$ (see 5.3). We then wish to calculate the Padé approximants to $\tilde{G}$. We use the fact that the diagonal (i.e. $[n/n]$), or near diagonal, Padé approximants correspond to finite continued fractions. We use the `ContinuedFraction` method from the `NumberTheory` package of Maple to find the continued fraction coefficients. Once we have the list of coefficients, `coeff`, we calculate the conformal Padé approximant with `n` terms at a certain value `p` with the following code:

```
    conf_pade := proc (n::integer, p::complex, coeff::list)::complex;
        local v; # the temporary value of the approximation
        local i; # the index variable for the loop
        local q; # a temporary value for the point we are evaluating at
        local g0; # the conformal map from the plane to the unit disk
        g0 := unapply((1-sqrt(-x^2+1))/x, x);
```

```
    # take the point from the plane into the disk and square it,
    # as the series is in p^2
    q := g0(p)^2;
    v := evalf(coeff[n]*q); # the initial value of the approximation
    for i from 1 to n-1 do
        v := evalf(coeff[n-i]*q/(v+1));
    end do;
    return v;
end proc:
```

The main idea here is that $H(p) = (H \circ \phi)(\psi(p))$, where $\psi = \phi^{-1}$ (in the code, $g0 = \psi$). Thus, we let the variable $q = \psi(p)^2$ and calculate the Padé approximation in the conformal plane at that point.

# Appendix B

# The Laplace Transform of the Binary Rational Expansion

In this appendix, we confirm the equality stated in Equation (6.2), which is rewritten here:

$$\mathcal{L}\left(\left(1 - e^{\beta i p/2^k}\right)^m\right)(x) = \frac{(-1)^m(\beta i)^m m!}{x(2^k x - \beta i)(2^k x - 2\beta i)\cdots(2^k x - m\beta i)}. \tag{B.1}$$

We calculate the inverse Laplace transform of the right hand side. For convenience, let $b = (-1)^m(\beta i)^m m!$. We decompose the right hand side by partial fractions:

$$\frac{b}{x(2^k x - \beta i)(2^k x - 2\beta i)\cdots(2^k x - m\beta i)} = \frac{a_0}{x} + \frac{a_1}{2^k x - \beta i} + \cdots + \frac{a_m}{2^k x - m\beta i}. \tag{B.2}$$

Multiplying through by the denominator, we get

$$b = a_0 \prod_{j=1}^{m}(2^k x - j\beta i) + a_1 x \prod_{j \neq 1}(2^k x - j\beta i) + a_2 x \prod_{j \neq 2}(2^k x - j\beta i) + \cdots + a_m x \prod_{j \neq m}(2^k x - j\beta i). \tag{B.3}$$

Solving for the coefficients $a_0, \ldots, a_m$ by plugging in values for $x$, we find

$$a_0 = \frac{b}{\prod_{j=1}^{m}(-j\beta i)} = \frac{b}{(-1)^m(\beta i)^m m!} = 1 \tag{B.4}$$

and

$$a_j = \frac{2^k b}{(-1)^{m-j}(\beta i)^m (m-j)! j!} = 2^k (-1)^j \binom{m}{j} \tag{B.5}$$

for $1 \leq j \leq m$. Now, using Example 2.8 and linearity of the inverse Laplace transform, we take the inverse Laplace transform of the right hand side, obtaining

$$\mathcal{L}^{-1}\left(\frac{a_0}{x}+\sum_{j=1}^{m}\frac{a_m}{2^k x - j\beta i}\right)(p) = \mathcal{L}^{-1}\left(\frac{a_0}{x}+\sum_{j=1}^{m}\frac{a_m/2^k}{x - j\beta i/2^k}\right)(p) = a_0 + \sum_{j=1}^{m}\frac{a_m}{2^k}e^{j\beta ip/2^k}. \tag{B.6}$$

By plugging in the values for $a_0$ and $a_m$ and using the binomial theorem, we have that the inverse Laplace transform of the right hand side becomes

$$1+\sum_{j=1}^{m}(-1)^j\binom{m}{j}e^{j\beta ip/2^k} = \sum_{j=0}\binom{m}{j}1^{m-j}\left(-e^{\beta ip/2^k}\right)^j = \left(1-e^{\beta ip/2^k}\right)^m. \tag{B.7}$$

This finishes the derivation.

# Appendix C
# Calculating the Binary Rational Expansion

Here we show the Mathematica code for calculating the binary rational expansion from Section 6.3. Here, `H` is the function we are doing the expansion of, `beta` is the parameter $\beta$, `mTerms` is the upper bound for $m$ in the expansion, and `kTerms` is the upper bound for $k$ in the expansion. We first initialize a coefficient array `c`, indexed both by $m$ and $k$.

```
c = {{H[0]}}; ck = {0};
Do[ (*make ck an array with kTerms+1 zeros*)
  AppendTo[ck, 0],
  {k, 1, kTerms}];
Do[ (*Append mTerms copies of ck to c*)
  AppendTo[c, ck],
  {m, 1, mTerms}];
```

Now, we calculate the coefficients:

```
dj = 0; dk = 0; Do[ (*dj, dk hold temporary values*)
  F[x_] = D[H[x], {x, j}]; (*F is the j-th derivative of H*)
  dj = F[0];
  (* First calculate the contribution to coefficients for k = 0*)
```

```
  Do[ (* Adjust length of summation by value of j *)
    dj += F[2*\[Pi]*l/beta] + F[-2*\[Pi]*l/beta],
    {l, 1, Floor[300/j] + 1}
  ];
  Do[
    c[[m + 1]][[1]] += N[Abs[StirlingS1[m, j]]/((-beta*I)^j*m!)*dj, 200],
    {m, j, mTerms}
  ];
  (* Now calculate the contribution to coefficients for k geq 1*)
  Do[
    dk = 0;
    Do[
      dk += F[2^k*(2 l + 1)*\[Pi]/beta] + F[-2^k*(2 l + 1)*\[Pi]/beta],
      {l, 0, Floor[300/j] + 1}
    ];
    Do[
      c[[m + 1]][[k + 1]] -=
      N[Abs[StirlingS1[m, j]]/((-2^(-k)*beta*I)^j*m!)*dk, 200],
      {m, j, mTerms}
    ],
  {k, 1, kTerms}
  ],
{j, 1, mTerms}
];
```

Now, the $c_{m,k}$ coefficient of the expansion is given by `c[[m+1]][[k+1]]`. Thus, we define the binary rational expansion of `H` as follows:

```
Binary[p_] := c[[1]][[1]] +
  Sum[Sum[c[[m + 1]][[ k + 1]]*(1 - Exp[beta*I*p/(2^k)])^m, {m, 1, mTerms}],
    {k, 0, kTerms}]
```

We can also similarly calculate the binary expansions of $\varphi_k$ for each $k$.

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