UNIT - 1

1 Matrices

1.1 Introduction

The theory of matrices was first introduced by the well known mathematician, Arthur Cayley in the second half of the nineteenth century. For more than 65 years after its introduction, the application of it was never felt by any of the mathematicians. It's validity was first felt by mathematicians when Heisenberg used the theory of matrices in quantum mechanics. Its applications are plenty in the fields of almost all branches of science and engineering. One can notice the use of matrices in Astronomy, Mechanics, Nuclear physics, Electrical circuits, Differential equations, Computer graphics, Graph theory, Optimization techniques and so on.

Many research articles are noticed on the applications of the inverse of the matrices and eigen value properties on the study of equilibrium analysis of industrial organizations. A recent research paper in the year 2010 in one of the most popular research journals reveals a special class of M_k matrices which arise from strategic behaviour in industrial organizations and by suggesting a new inverse, lead directly to closed form expressions of strategic equilibria for arbitrary coalition structures in linear oligopolies, which include four classes of oligopoly equilibria, Bertrand equilibria for an arbitrary coalition structure, Cournot equilibria for an arbitrary coalition structure, Bertrand and Cournot equilibria with multi product firms which are previously unknown and now it will be useful to scholars in industrial organization.

Many applications of matrices can be cited on recent studies, for example in environmental analysis, Thermosetting composite fibres and Polyester via matrices. As matrices are widely used by engineers in industry, it is essential for every engineering graduate to have a thorough knowledge about matrices. As you are exposed already to the basic concepts of operations on matrices like addition, subtraction, scalar multiplication, multiplication of the matrices, let us now concentrate on further analysis on matrices. In the lower classes you are aware of the concepts of finding the inverse of a square matrix, solution of simultaneous equations using matrices, rank of a matrix. Let us concentrate on the evaluation of eigen values and its applications and move further to know more on matrix theory.

1.2 Eigen values and Eigen vectors of a Real Matrix

Definition

Let A be a square matrix of order n. A number λ is called an eigen value of A if there exists a nonzero column matrix X such that $AX = \lambda X$. Then X is called an eigen vector of A corresponding to λ.

Note

If λ is an eigen value and X is an eigen vector corresponding to λ, then $AX = \lambda X \Rightarrow AX - \lambda X = 0 \Rightarrow (A - \lambda I)X = 0$.

This is a homogeneous system of equations. It will have a nontrivial solution if $|A - \lambda I| = 0$. This equation is called the characteristic equation.
$|A - \lambda I|$ is called the characteristic polynomial of A. It is an n^{th} degree polynomial in λ. The roots of the characteristic equation are called the eigen values of A.
(Eigen in German means characteristic)
The following are simple ways to find the characteristic equation of a given square matrix.

Results

1. If A is a 2×2 matrix, then the characteristic equation takes the form $\lambda^2 - s_1\lambda + s_2 = 0$ where s_1 = sum of the diagonal elements of A = trace of A ($tr(A)$) and $s_2 = |A|$.

2. If A is a square matrix of order 3, then the characteristic equation $|A - \lambda I| = 0$ takes the form $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$ where
   s_1 = sum of the main diagonal elements of A.
   s_2 = sum of the minors of the elements of the main diagonal.
   $s_3 = |A|$.

3. Sum of the eigen values of a matrix A is equal to the trace of the matrix.

4. Product of the eigen values $= |A|$.

Worked Examples

Example 1.1. Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$.

Solution. Let $A = \begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 2×2 matrix, the characteristic equation takes the form

$$\lambda^2 - s_1\lambda + s_2 = 0$$

where s_1 = sum of the diagonal elements of $A = 1 + 4 = 5$.

$$s_2 = |A| = \begin{vmatrix} 1 & -2 \\ -5 & 4 \end{vmatrix} = 4 - 10 = -6.$$

$\therefore$ (1) becomes

$$\lambda^2 - 5\lambda - 6 = 0 \Rightarrow (\lambda - 6)(\lambda + 1) = 0 \Rightarrow \lambda = -1, \lambda = 6.$$

$\therefore$ The eigen values are -1 and 6.

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Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector of A corresponding to λ.

$$\therefore \ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0. \tag{1}$$

when $\lambda = -1$, (1) becomes

$$\begin{pmatrix} 2 & -2 \\ -5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{2}$$

$$2x_1 - 2x_2 = 0 \Rightarrow x_1 - x_2 = 0.$$

$$-5x_1 + 5x_2 = 0 \Rightarrow x_1 - x_2 = 0.$$

Both the equations are same. They are reduced to one single equation namely $x_1 = x_2 \Longrightarrow \frac{x_1}{1} = \frac{x_2}{1} \Longrightarrow x_1 = 1, x_2 = 1.$

$\therefore$ The eigen vector X_1 corresponding to $\lambda = -1$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

When $\lambda = 6$, (1) becomes $\begin{pmatrix} -5 & -2 \\ -5 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$

$$-5x_1 - 2x_2 = 0 \Rightarrow 5x_1 + 2x_2 = 0$$

$$-5x_1 - 2x_2 = 0 \Rightarrow 5x_1 + 2x_2 = 0.$$

We have the same situation as above. Both the equations are reduced to $5x_1 = -2x_2$

$$\Longrightarrow \frac{x_1}{2} = \frac{x_2}{-5}$$

$$\Longrightarrow x_1 = 2, x_2 = -5.$$

Hence, the eigen vector X_2 corresponding to the eigen value $\lambda = 6$ is $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

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Example 1.2. Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$.

[Jan 1997]

Solution. Let $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 2×2 matrix, the characteristic equation takes the form

$$\lambda^2 - s_1\lambda + s_2 = 0$$

where, s_1 = sum of the main diagonal elements of $A = 4 + 2 = 6$.

$$s_2 = |A| = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 8 - 3 = 5.$$

The characteristic equation is $\lambda^2 - 6\lambda + 5 = 0 \Rightarrow (\lambda - 1)(\lambda - 5) = 0 \Rightarrow \lambda = 1, \lambda = 5$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector of A corresponding to λ.

$$\therefore (A - \lambda I)X = 0$$

$$\Longrightarrow \begin{pmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{1}$$

When $\lambda = 1$, the equation (1) becomes $\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$$i.e., 3x_1 + x_2 = 0 \Rightarrow x_2 = -3x_1 \Rightarrow \frac{x_2}{-3} = \frac{x_1}{1}$$

$$\Longrightarrow x_1 = 1, x_2 = -3.$$

$\therefore$ The eigen vector corresponding to $\lambda = 1$ is $X_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

When $\lambda = 5$,(1) becomes $\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

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$$i.e., -x_1 + x_2 = 0$$

$$\Rightarrow x_1 = x_2 = 1$$

Hence, the eigen vector X_2 corresponding to the eigen value $\lambda = 5$ is $X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Note. In the above example in case (i), if we take $x_1 = k$, we get $x_2 = -3k$.
The corresponding eigen vector is $\begin{pmatrix} k \\ -3k \end{pmatrix} = k\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.
i.e., we get infinite number of eigen vectors corresponding to an eigen value.
i.e., the eigen vector corresponding to an eigen value is not unique.

Type-I: Evaluation of eigen vectors when all the eigen values are different.

Example 1.3. Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{pmatrix}$.

Solution. Let $A = \begin{pmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

$s_1 = tr(A) = 3 - 2 + 3 = 4.$

s_2 = sum of the minors of the main diagonal elements of A.

$= \begin{vmatrix} -2 & 4 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix}$

$s_2 = -6 + 4 + 9 - 4 - 6 + 4 = -2 + 5 - 2 = 1.$

$$s_3 = |A| = \begin{vmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{vmatrix} = 3(-2) + 4(-1) + 4 = -6 - 4 + 4 = -6.$$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$.

i.e., $\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$.

$\lambda = -1$ is a root.

By synthetic division,

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 1 & 6 \\ & 0 & -1 & 5 & -6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$\lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda = 2, \lambda = 3.$$

The eigen values are $-1, 2, 3$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to λ.

$$\therefore \qquad (A - \lambda I)X = 0$$

$$\begin{pmatrix} 3-\lambda & -4 & 4 \\ 1 & -2-\lambda & 4 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = -1$, (1) becomes

$$\begin{pmatrix} 4 & -4 & 4 \\ 1 & -1 & 4 \\ 1 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

The equations are

$$4x_1 - 4x_2 + 4x_3 = 0$$

$$x_1 - x_2 + 4x_3 = 0$$

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$$x_1 - x_2 + 4x_3 = 0.$$

Taking the first two equations and on solving by the method of cross multiplication, we obtain

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -4 & \times & 4 & \times & 4 & \times & -4 \\ -1 & & 4 & & 1 & & -1 \end{array}$$

$$\frac{x_1}{-12} = \frac{x_2}{-12} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0}$$

$$\Rightarrow x_1 = 1, x_2 = 1, x_3 = 0.$$

$$\Rightarrow X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

When $\lambda = 2$, (1) becomes

$$\begin{pmatrix} 1 & -4 & 4 \\ 1 & -4 & 4 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

The equations are

$$x_1 - 4x_2 + 4x_3 = 0$$

$$x_1 - 4x_2 + 4x_3 = 0$$

$$x_1 - x_2 + x_3 = 0.$$

Taking the last two equations and on solving by the method of cross multiplication, we obtain

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -4 & \times & 4 & \times & 1 & \times & -4 \\ -1 & & 1 & & 1 & & -1 \end{array}$$

$$\frac{x_1}{0} = \frac{x_2}{3} = \frac{x_3}{3}$$
$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$
$$\Rightarrow x_1 = 0, x_2 = 1, x_3 = 1.$$
$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

When $\lambda = 3$, (1) becomes

$$\begin{pmatrix} 0 & -4 & 4 \\ 1 & -5 & 4 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$
$$-4x_2 + 4x_3 = 0 \Rightarrow x_2 = x_3$$
$$x_1 - 5x_2 + 4x_3 = 0 \Rightarrow x_1 - x_2 = 0$$
$$x_1 = x_2 \quad \Rightarrow x_1 = x_2 = x_3 = 1.$$
$$\therefore X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore$ The eigen vectors are $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Example 1.4. Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$.

[Jan 2013]

Solution. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$.

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Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

where s_1 = sum of the main diagonal elements of A

$s_1 = 1 - 1 - 1 = -1.$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} -1 & 0 \\ -2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 6 & -1 \end{vmatrix}$$

$$= 1 - 0 + (-1) + 1 + (-1 - 12)$$

$$= 1 + 0 - 13$$

$$= -12.$$

$$s_3 = |A| = \begin{vmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{vmatrix}$$

$$= 1(1 - 0) - 2(-6 - 0) + 1(-12 - 1)$$

$$= 1 + 12 - 13$$

$$= 0.$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - 12\lambda = 0$$

$$\lambda(\lambda^2 + \lambda - 12) = 0$$

$$\lambda(\lambda + 4)(\lambda - 3) = 0$$

$$\lambda = 0, -4, 3.$$

The eigen values are $0, -4, 3$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value λ.

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$$\therefore (A - \lambda I)X = 0$$

$$i.e., \begin{pmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \quad (1)$$

When $\lambda = 0$, (1) becomes

$$\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\text{i.e., } x_1 + 2x_2 + x_3 = 0 \quad (2)$$

$$6x_1 - x_2 = 0 \quad (3)$$

$$-x_1 - 2x_2 - x_3 = 0. \quad (4)$$

(3) can be written as

$$6x_1 = x_2$$

$$x_1 = \frac{x_2}{6}$$

$$\Rightarrow x_1 = 1, x_2 = 6.$$

Substituting in (2) we obtain, $1 + 12 + x_3 = 0$

$$\Rightarrow x_3 = -13.$$

$\therefore$ The eigen vector is $X_1 = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}$.

When $\lambda = -4$, (1) becomes

$$\begin{pmatrix} 5 & 2 & 1 \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

—

$$\text{i.e., } 5x_1 + 2x_2 + x_3 = 0 \quad (5)$$

$$6x_1 + 3x_2 = 0 \quad (6)$$

$$-x_1 - 2x_2 + 3x_3 = 0. \quad (7)$$

(6) can be written as

$$6x_1 = -3x_2$$

$$\frac{x_1}{-3} = \frac{x_2}{6}$$

$$\frac{x_1}{1} = \frac{x_2}{-2}$$

$$\therefore x_1 = 1, x_2 = -2.$$

Substituting in (5) we obtain

$$5 - 4 + x_3 = 0$$

$$\Rightarrow x_3 = -1.$$

$\therefore$ The eigen vector is $X_2 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$.

When $\lambda = 3$, (1) becomes

$$\begin{pmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

$$\text{i.e., } -2x_1 + 2x_2 + x_3 = 0 \quad (8)$$

$$6x_1 - 4x_2 = 0 \quad (9)$$

$$-x_1 - 2x_2 - 4x_3 = 0. \quad (10)$$

(9) can be written as

$$6x_1 = 4x_2$$

$$\Rightarrow 3x_1 = 2x_2$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{3}$$

$$\therefore x_1 = 2, x_2 = 3.$$

Substituting in (8) we obtain

$$-4 + 6 + x_3 = 0$$

$$2 + x_3 = 0$$

$$\Rightarrow x_3 = -2.$$

$\therefore$ The eigen vector is $X_3 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$.

$\therefore$ The eigen vectors are $\begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$.

Type-II: Evaluation of Eigen vectors when two of the eigen values are equal.

Example 1.5. Find the eigen values and eigen vectors of $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$.[Jan 2009]

Solution. Let $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

where $s_1 = tr(A) = -2 + 1 + 0 = -1$.

s_2 = sum of the minors of the main diagonal elements of A.

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$

$s_2 = 0 - 12 + 0 - 3 - 2 - 4 = -21.$

$$s_3 = |A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

$$= -2(0-12) - 2(0-6) - 3(-4+1) = 24 + 12 + 9 = 45.$$

The characteristic equation is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0.$

$\lambda = -3$ is a root. By synthetic division,

-3	1	1	-21	-45
	0	-3	6	45
	1	-2	-15	0

$$\Longrightarrow (\lambda+3)(\lambda^2 - 2\lambda - 15) = 0$$

$$(\lambda+3)(\lambda-5)(\lambda+3) = 0 \Rightarrow \lambda = -3, \lambda = -3, \lambda = 5.$$

The eigen values are $-3,\ -3,\ 5.$

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to λ.

$$\therefore \qquad (A - \lambda I)X = 0.$$

$$\begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = -3$, (1) becomes $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The above three equations are reduced to

$$x_1 + 2x_2 - 3x_3 = 0. \tag{2}$$

Since two of the eigen values are equal, from (2) we have to get two eigen vectors by assigning arbitrary values for two variables. First, choosing $x_3 = 0$, we obtain $x_1 + 2x_2 = 0$ which implies $x_1 = -2x_2 \Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} \Rightarrow x_1 = 2, x_2 = -1$

$$\therefore X_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

Also in (2) assign $x_2 = 0$, we obtain $x_1 - 3x_3 = 0$,
which implies $x_1 = 3x_3 \Rightarrow \frac{x_1}{3} = \frac{x_3}{1} \Rightarrow x_1 = 3, x_3 = 1$.

$$X_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$$

When $\lambda = 5$, (1)$\Longrightarrow \begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The equations are

$$\left.\begin{aligned} -7x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 - 4x_2 - 6x_3 &= 0 \\ -x_1 - 2x_2 - 5x_3 &= 0. \end{aligned}\right\} \tag{A}$$

Taking the first two equations and on solving by the method of cross multiplication, we obtain

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 2 & \times & -3 & \times & -7 & \times & 2 \\ -4 & & -6 & & 2 & & -4 \end{array}$$

$$\frac{x_1}{-12-12} = \frac{x_2}{-6-42} = \frac{x_3}{28-4}$$

$$\frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}.$$

$$\implies x_1 = 1, x_2 = 2, x_3 = -1.$$

The values of x_1, x_2, x_3 satisfy the last equation in (A).

$$\therefore\ X_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

$\therefore$ The eigen vectors are $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

Example 1.6. Find the eigen values and eigen vectors of $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$.

[Jun 2010, Jan 2007]

Solution. Let $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements of A

$= 2 + 3 + 2 = 7.$

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s_2 = sum of the minors of the main diagonal elements of A.

$$= \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= (6-2) + (4-1) + (6-2)$$

$$= 4 + 3 + 4$$

$$= 11.$$

$$s_3 = |A| = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= 2(6-2) - 2(2-1) + 1(2-3)$$

$$= 8 - 2 - 1$$

$$= 5.$$

The characterstic equation is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$.

$\lambda = 1$ is a root.

By synthetic division we have

$$\begin{array}{c|cccc} 1 & 1 & -7 & 11 & -5 \\ & 0 & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & \boxed{0} \end{array}$$

$$\lambda^3 - 6\lambda + 5 = 0$$

$$(\lambda - 1)(\lambda - 5) = 0$$

$$\lambda = 1, 5.$$

The given eigen values are 1, 1, 5.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

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$$\therefore (A - \lambda I)X = 0$$

$$\begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \tag{1}$$

When $\lambda = 1$, (1) becomes

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

All the three equations are reduced to one single equation

$$x_1 + 2x_2 + x_3 = 0 \tag{2}$$

Since two of the eigen values are equal, from(2), we have to get two eigen vectors by assigning arbitrary values for two variables.

First, choosing $x_3 = 0$, we obtain $\quad x_1 + 2x_2 = 0$

$$x_1 = -2x_2$$

$$\frac{x_1}{-2} = \frac{x_2}{1}$$

$$\Rightarrow x_1 = -2, x_2 = 1$$

$\therefore$ One eigen vector is $X_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$.

Assign $x_2 = 0$, (2) becomes

$$x_1 + x_3 = 0$$

$$x_1 = -x_3$$

$$\frac{x_1}{1} = \frac{x_3}{-1}$$

$$\Rightarrow x_1 = 1, x_3 = -1.$$

$\therefore$ The second eigen vector is $X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

When $\lambda = 5$, (1) becomes

$$\begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$\left.\begin{aligned} -3x_1 + 2x_2 + x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 + 2x_2 - 3x_3 &= 0. \end{aligned}\right\} \qquad \text{(A)}$$

All the three equations are different.

Taking the first two equations and applying the rule of cross multiplication we obtain

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 2 & \times & 1 & \times & -3 & \times & 2 \\ -2 & & 1 & & 1 & & -2 \end{array}$$

$$\frac{x_1}{2+2} = \frac{x_2}{1+3} = \frac{x_3}{6-2}$$

$$\frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

$$\Rightarrow x_1 = 1, x_2 = 1, x_3 = 1.$$

The eigen vector is $X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$\therefore$ The eigen vectors are $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

—

Example 1.7. Find the eigen values and eigen vectors of $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$.

[May 2000]

Solution. Given $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

Now, S_1 = sum of the main diagonal elements

$= 3 - 3 + 7 = 7.$

S_2 = sum of the minors of the main diagonal elements of A.

$$= \begin{vmatrix} -3 & -4 \\ 5 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 3 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 10 \\ -2 & -3 \end{vmatrix}$$

$= -21 + 20 + 21 - 15 + (-9 + 20)$

$= -1 + 6 + 11$

$= 16.$

$S_3 = |A|$

$$= \begin{vmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{vmatrix}$$

$= 3(-21 + 20) - 10(-14 + 12) + 5(-10 + 9)$

$= -3 + 20 + (-5)$

$= 20 - 8$

$= 12.$

The characteristic equation is $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$.

$$\lambda = 2 \text{ is a root.}$$

By synthetic division, we have

$$\begin{array}{c|cccc} 2 & 1 & -7 & 16 & -12 \\ & 0 & 2 & -10 & 12 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 2, 3.$$

The eigen values are 2, 2, 3.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \tag{1}$$

When $\lambda = 2$, (1) becomes

$$\begin{pmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations becomes

$$x_1 + 10x_2 + 5x_3 = 0$$

$$2x_1 + 5x_2 + 4x_3 = 0$$

$$3x_1 + 5x_2 + 5x_3 = 0.$$

—

Since all the three equations are different, we will get only one eigen vector. Since we need two eigen vectors corresponding to the twice repeated eigen value 2, the two eigen vectors are the same.

Taking the first two equations and using the rule of cross multiplication we obtain

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 10 & \times & 5 & \times & 1 & \times & 10 \\ 5 & & 4 & & 2 & & 5 \end{array}$$

$$\frac{x_1}{40-25} = \frac{x_2}{10-4} = \frac{x_3}{5-20}$$

$$\frac{x_1}{15} = \frac{x_2}{6} = \frac{x_3}{-15}$$

$$\frac{x_1}{5} = \frac{x_2}{2} = \frac{x_3}{-5}$$

$$\therefore x_1 = 5, \quad x_2 = 2, \quad x_3 = -5$$

The two eigen vectors are $X_1 = \begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix}$.

When $\lambda = 3$, (1) becomes

$$\begin{pmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations are

$$10x_2 + 5x_3 = 0$$

$$2x_1 + 6x_2 + 4x_3 = 0$$

$$3x_1 + 5x_2 + 4x_3 = 0.$$

From $10x_2 + 5x_3 = 0$ we obtain

$$10x_2 = -5x_3$$

$$2x_2 = -x_3$$

—

$$\frac{x_2}{1} = \frac{x_3}{-2}$$

$$\therefore x_2 = 1, x_3 = -2.$$

Substituting this in the second equation we obtain

$$2x_1 + 6 - 8 = 0$$

$$2x_1 - 2 = 0$$

$$2x_1 = 2$$

$$x_1 = 1.$$

$\therefore$ The eigen vector is $X_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

The eigen vectors are $\begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

Type-III: Evaluation of eigen vectors when all the eigen values are equal.

Example 1.8. Find the eigen values and eigen vectors of $A = \begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}$.

[Dec 1998]

Solution. Given $A = \begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$= 6 - 13 + 4$

$= -3.$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} -13 & 10 \\ -6 & 4 \end{vmatrix} + \begin{vmatrix} 6 & 5 \\ 7 & 4 \end{vmatrix} + \begin{vmatrix} 6 & -6 \\ 14 & -13 \end{vmatrix}$$

$= -52 + 60 + 24 - 35 + (-78 + 84)$

$= 8 - 11 + 6$

$= 3.$

$$s_3 = |A| = \begin{vmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{vmatrix}$$

$= 6(-52 + 60) + 6(56 - 70) + 5(-84 + 91)$

$= 6(8) + 6(-14) + 5(7) = 48 - 84 + 35$

$= 83 - 84$

$= -1.$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0.$$

$$(\lambda + 1)^3 = 0$$

$$\therefore \lambda = -1, -1, -1.$$

The eigen values are $-1, -1, -1$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0$$

—

$$\begin{pmatrix} 6-\lambda & -6 & 5 \\ 14 & -13-\lambda & 10 \\ 7 & -6 & 4-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{1}$$

when $\lambda = -1$, (1) becomes

$$\begin{pmatrix} 7 & -6 & 5 \\ 14 & -12 & 10 \\ 7 & -6 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above three equations are reduced to one single equation

$$7x_1 - 6x_2 + 5x_3 = 0.$$

With this single equation, we must get three different eigen vectors, which can be achieved by assigning arbitrary values to x_1, x_2 & x_3.

$$x_1 = 0 \;\Rightarrow\; -6x_2 + 5x_3 = 0 \;\Rightarrow\; \frac{x_2}{5} = \frac{x_3}{6} \;\therefore\; X_1 = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}.$$

$$x_2 = 0 \;\Rightarrow\; 7x_1 + 5x_3 = 0 \;\Rightarrow\; \frac{x_1}{-5} = \frac{x_3}{7} \;\therefore\; X_2 = \begin{pmatrix} 5 \\ 0 \\ -7 \end{pmatrix}.$$

$$x_3 = 0 \;\Rightarrow\; 7x_1 - 6x_2 = 0 \;\Rightarrow\; \frac{x_1}{6} = \frac{x_2}{7} \;\therefore\; X_3 = \begin{pmatrix} 6 \\ 7 \\ 0 \end{pmatrix}.$$

Example 1.9. Find the eigen values and eigen vectors of $A = \begin{pmatrix} -5 & -5 & -9 \\ 8 & 9 & 18 \\ -2 & -3 & -7 \end{pmatrix}$.

Solution. Given $A = \begin{pmatrix} -5 & -5 & -9 \\ 8 & 9 & 18 \\ -2 & -3 & -7 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements.

$$= -5 + 9 - 7$$

$$= 9 - 12 = -3.$$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 9 & 18 \\ -3 & -7 \end{vmatrix} + \begin{vmatrix} -5 & -9 \\ -2 & -7 \end{vmatrix} + \begin{vmatrix} -5 & -5 \\ 8 & 9 \end{vmatrix}$$

$$= -63 + 54 + 35 - 18 + (-45 + 40)$$

$$= -9 + 17 - 5 = 3.$$

$$s_3 = |A| = \begin{vmatrix} -5 & -5 & -9 \\ 8 & 9 & 18 \\ -2 & -3 & -7 \end{vmatrix}$$

$$= -5(-63 + 54) + 5(-56 + 36) - 9(-24 + 18)$$

$$= -5(-9) + 5(-20) - 9(-6)$$

$$= 45 - 100 + 54 = -1.$$

The characteristic equation is

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

$$(\lambda + 1)^3 = 0$$

$$\lambda = -1, -1, -1.$$

The eigen values are $-1, -1, -1$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

—

$$\therefore (A - \lambda I)X = 0.$$

$$\begin{pmatrix} -5-\lambda & -5 & -9 \\ 8 & 9-\lambda & 18 \\ -2 & -3 & -7-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \tag{1}$$

When $\lambda = -1$, (1) becomes

$$\begin{pmatrix} -4 & -5 & -9 \\ 8 & 10 & 18 \\ -2 & -3 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

The equations are

$$-4x_1 - 5x_2 - 9x_3 = 0$$

$$8x_1 + 10x_2 + 18x_3 = 0$$

$$-2x_1 - 3x_2 - 6x_3 = 0.$$

The above three equations are reduced to two equations

$$4x_1 + 5x_2 + 9x_3 = 0$$

$$2x_1 + 3x_2 + 6x_3 = 0.$$

Using the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 5 & \times & 9 & \times & 4 & \times & 5 \\ 3 & & 6 & & 2 & & 3 \end{array}$$

$$\frac{x_1}{30-27} = \frac{x_2}{18-24} = \frac{x_3}{12-10}$$

$$\frac{x_1}{3} = \frac{x_2}{-6} = \frac{x_3}{2}$$

The only eigen vector is $X = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$.

In this case $X_1 = X_2 = X_3 = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$.

Exercise I(A)

I. Find the characteristic equation, eigen values and eigen vectors of the following matrices.

1. $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$.

2. $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$.

II. Find the eigen values and eigen vectors of the following matrices.

1. $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$

3. $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

4. $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

5. $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

6. $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -1 & 2 & -1 \end{pmatrix}$

7. $\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -2 \\ 1 & -1 & 2 \end{pmatrix}$

8. $\begin{pmatrix} 3 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 5 \end{pmatrix}$

9. $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$

10. $\begin{pmatrix} 2 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$

11. $\begin{pmatrix} 4 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 5 & 4 \end{pmatrix}$.

—

1.3 Properties of Eigen values and Eigen vectors

Property 1.1. A square matrix A and its transpose A^T have the same eigen values.

Proof. The eigen values of the matrix A are the roots of its characteristic equation $|A - \lambda I| = 0$.

Now, $(A - \lambda I)^T = A^T - (\lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$.

Also, $|(A - \lambda I)^T| = |A - \lambda I| \Rightarrow |A^T - \lambda I| = |A - \lambda I|$.

This implies that the characteristic polynomial of A and A^T are equal.

$\Rightarrow$ The characteristic equations of A and A^T are equal.

$\Rightarrow$ A and A^T have the same eigen values. □

Worked Examples

Example 1.10. 1, $\sqrt{5}$ and $-\sqrt{5}$ are the eigen values of the matrix $\begin{pmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$.

Write down the eigen values of the matrix $\begin{pmatrix} -1 & 1 & -1 \\ 2 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix}$.

Solution. Let $A = \begin{pmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 & -1 \\ 2 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix}$.

B is the transpose of A.

By the above property, A & B have the same eigen values.

$\therefore$ The eigen values of B are 1, $\sqrt{5}$ and $-\sqrt{5}$.

Example 1.11. If 1, 3 and -4 are the eigen values of the matrix $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$, what are the eigen values of the matrix $\begin{pmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{pmatrix}$?

Solution. Let $A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{pmatrix}$.

B is the transpose of A.

By the above property, A and B have the same eigen values.

$\therefore$ The eigen values of B are 1, 3 and -4.

Property 1.2. Sum of the eigen values of a square matrix A is equal to the sum of the elements on its main diagonal.

Proof. Let A be a square matrix of order n.

The characteristic equation is $|A - \lambda I| = 0$.

It is of the form $\lambda^n - s_1\lambda^{n-1} + s_2\lambda^{n-2} + \cdots + (-1)^n s_n = 0$ (1)

where, s_1 = sum of the elements of the leading diagonal.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of (1).

From the theory of equations,

Sum of the roots $= -\dfrac{\text{coeff. of } \lambda^{n-1}}{\text{coeff. of } \lambda^n}$.

i.e., $\lambda_1 + \lambda_2 + \cdots + \lambda_n = -(-s_1) = s_1 = tr(A)$.

i.e., Sum of the eigen values = sum of the main diagonal elements of A. □

Property 1.3. Product of the eigen values of a square matrix A is equal to $|A|$.

Proof. For the square matrix A , the characteristic equation is $|A - \lambda I| = 0$.

It is of the form $\lambda^n - s_1\lambda^{n-1} + s_2\lambda^{n-2} + \cdots + (-1)^n s_n = 0,$ (1)

where $s_n = |A|$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of (1).

From the theory of equations, we have

Product of the roots $= (-1)^n \dfrac{\text{constant term}}{\text{coeff. of } \lambda^n}$.

i.e., $\lambda_1, \lambda_2, \ldots, \lambda_n = (-1)^n(-1)^n s_n = (-1)^{2n}|A|$.

i.e., Product of the eigen values = $|A|$. $\square$

Worked Examples

Example 1.12. Find the sum and product of the eigen values of the matrix $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$. [Dec 1999]

Solution. Sum of the eigen values = $tr(A) = -2 + 1 + 0 = -1$.

$$\text{Product of the eigen values} = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} = -2(-12) - 2(-6) - 3(-3) = 45.$$

Example 1.13. Find the sum and product of the eigen values of the matrix $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$. [March 1996]

Solution. Sum of the eigen values = Sum of the elements along the main diagonal

$$= -1 - 1 - 1 = -3.$$

$$\text{Product of the eigen values} = |A| = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1 - 1) - 1(-1 - 1) + 1(1 + 1)$$

$$= -1 \times 0 + 2 + 2 = 4.$$

Example 1.14. The product of the two eigen values of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16, find the third eigen value. [Jan 2003,2012]

Solution. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values.

Given $\lambda_1\lambda_2 = 16$.

$$\text{But, product of eigen values} = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$i.e., \lambda_1\lambda_2\lambda_3 = 6(8) + 2(-4) + 2(-4)$$

$$16\lambda_3 = 48 - 8 - 8 = 32$$

$$\therefore \lambda_3 = 2.$$

Example 1.15. If 2 and 3 are the eigen values of $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ x & 0 & 2 \end{pmatrix}$, find the value of x.

Solution. Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ x & 0 & 2 \end{pmatrix}$.

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

We know that, $\lambda_1 + \lambda_2 + \lambda_3 = tr(A) = 6$.

$$2 + 3 + \lambda_3 = 6$$

$$\lambda_3 = 1.$$

Also, product of the eigen values $= |A|$.

$$\lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ x & 0 & 2 \end{vmatrix}$$

$$2.3.1 = 2(4) - 0(0) + 1(0 - 2x)$$

$$6 = 8 - 2x$$

$$2x = 2$$

$$x = 1.$$

—

Example 1.16. Find the values of a and b, such that the matrix $\begin{pmatrix} a & 4 \\ 1 & b \end{pmatrix}$ has 3 and -2 its eigen values. [May 2011]

Solution. Sum of the eigen values $= tr(A) = a + b$

$$3 - 2 = a + b$$

$$\Rightarrow a + b = 1 \tag{1}$$

Product of the eigen values $= |A|$

$$\text{i.e., } 3 \times (-2) = \begin{vmatrix} a & 4 \\ 1 & b \end{vmatrix} = ab - 4$$

$$-6 = ab - 4$$

$$ab = -6 + 4 = -2$$

$$b = -\frac{2}{a}.$$

Substituting in (1) we get

$$a - \frac{2}{a} = 1$$

$$a^2 - 2 = a$$

$$a^2 - a - 2 = 0$$

$$(a - 2)(a + 1) = 0$$

$$a = 2, -1.$$

When $a = 2, b = -1$.

When $a = -1, b = 2$.

Example 1.17. If the sum of two eigen values and trace of a 3×3 matrix A are equal, find the value of $|A|$.

Solution. Let the eigen values be $\lambda_1, \lambda_2, \lambda_3$.

We know that $\lambda_1 + \lambda_2 + \lambda_3 = tr(A)$.

But, given that $\lambda_1 + \lambda_2 = tr(A)$.

—

$\therefore\ tr(A) + \lambda_3 = tr(A)$.

$\Longrightarrow \lambda_3 = 0$.

Now, $|A|$ = product of the eigen values $= \lambda_1 \times \lambda_2 \times 0 = 0$.

Property 1.4. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the non zero eigen values of a square matrix A of order n, then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ are the eigen values of A^{-1}.

Proof. Let λ be a non zero eigen value of A. Then, there exists a non zero column matrix X such that

$$AX = \lambda X. \tag{1}$$

Since all the eigen values are non zero, A is non-singular. Hence, A^{-1} exists.
Premultiplying both sides of (1) by A^{-1} we get,

$$A^{-1}(AX) = A^{-1}\lambda X$$

$$(A^{-1}A)X = \lambda(A^{-1}X)$$

$$IX = \lambda A^{-1}X$$

$$\frac{1}{\lambda}X = A^{-1}X.$$

$\Rightarrow \frac{1}{\lambda}$ is an eigen value of A^{-1}.
This is true for all eigen values of A.
$\therefore\ \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ are the eigen values of A^{-1}. □

Property 1.5. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of A, then
(i) $c\lambda_1, c\lambda_2, \ldots, c\lambda_n$ are the eigen values of cA where $c \neq 0$.
(ii) $\lambda_1^m, \lambda_2^m, \lambda_3^m, \ldots, \lambda_n^m$ are the eigen values of A^m where m is a positive integer.

Proof. Let λ be an eigen value of A.
Then, there exists a column matrix X such that

$$AX = \lambda X. \tag{1}$$

(i). Let $c \neq 0$.

Multiply (1) by c we get, $c(AX) = c(\lambda X)$

$$(cA)X = (c\lambda)X.$$

$\Rightarrow c\lambda$ is an eigen value of cA.

This is true for all eigen values of A.

$\therefore c\lambda_1, c\lambda_2, \ldots, c\lambda_n$ are the eigen values of cA.

(ii) We have

$$AX = \lambda X$$

$$\Longrightarrow A(AX) = A(\lambda X)$$

$$\Longrightarrow (AA)X = \lambda(AX)$$

$$A^2X = \lambda(\lambda X)$$

$$\Longrightarrow A^2X = \lambda^2 X.$$

$\Longrightarrow \lambda^2$ is an eigen value of A^2.

$$A(A^2X) = A(\lambda^2 X)$$

$$\Longrightarrow (AA^2)X = \lambda^2(AX)$$

$$\Longrightarrow A^3X = \lambda^2\lambda X = \lambda^3 X.$$

$\Longrightarrow \lambda^3$ is an eigen value of A^3. Proceeding like this, we arrive at $A^mX = \lambda^m X$.

$\Rightarrow \lambda^m$ is an eigen value of A^m.

This is true for all eigen values of A.

$\therefore \lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ are the eigen values of A^m. □

Worked Examples

Example 1.18. Given $\begin{pmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix}$. Find the eigen values of A^2. [Jan 2010]

Solution. The given matrix is triangular.

Hence, the eigen values are the elements along the leading diagonal.

The Eigen values of A are $-1, -3, 2$.

$\therefore$ The Eigen values of A^2 are $(-1)^2, (-3)^2, 2^2$.

i.e., $1, 9, 4$.

Example 1.19. If 2 and 3 are the eigen values of $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$, find the eigen values of A^{-1} and A^3. [Dec 2000]

Solution. Let λ be the third eigen value.

we know that, sum of the eigen values = $tr(A)$.

$$2 + 3 + \lambda = 3 - 3 + 7$$

$$5 + \lambda = 7 \Rightarrow \lambda = 2.$$

The Eigen values of A are $2, 2, 3$.

The Eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{2}, \frac{1}{3}$.

The Eigen values of A^3 are 2^3, 2^3, 3^3 (i.e) 8, 8, 27.

Example 1.20. Two of the eigen values of the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ are equal to 1 each. Find the eigen values of A^{-1}. [Dec.2002]

Solution. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

Given $\lambda_1 = \lambda_2 = 1$.

$\therefore \lambda_1 + \lambda_2 + \lambda_3$ = sum of the elements along the main diagonal

—

$$1 + 1 + \lambda_3 = 2 + 3 + 2$$
$$2 + \lambda_3 = 7$$
$$\lambda_3 = 5.$$

$\therefore$ The eigen values of A are 1,1,5

$\therefore$ Eigen values of A^{-1} are 1, 1, $\frac{1}{5}$.

Example 1.21. Two of the eigen values of the matrix $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ are 3 and 6. Find the eigen values of A^{-1}. [May 2002]

Solution. Let λ_1, λ_2 and λ_3 be the eigen values of the given matrix A.

$\therefore \lambda_1 + \lambda_2 + \lambda_3 =$ sum of the elements along the main diagonal

$$3 + 6 + \lambda_3 = 3 + 5 + 3$$
$$9 + \lambda_3 = 11$$
$$\lambda_3 = 2.$$

The eigen values of A are 3, 6, 2.

$\therefore$ The eigen values of A^{-1} are $\frac{1}{3}, \frac{1}{6}, \frac{1}{2}$.

Example 1.22. Find the eigen values of the matrix $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$. Hence, find the matrix whose eigen values are $\frac{1}{6}$ and -1. [Jan 2001]

Solution. Let $A = \begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$.

The characteristic equation is $|A - \lambda I| = 0$.

This is of the form $\lambda^2 - s_1\lambda + s_2 = 0$.

$$s_1 = tr(A) = 1 + 4 = 5$$

$$s_2 = \begin{vmatrix} 1 & -2 \\ -5 & 4 \end{vmatrix} = 4 - 10 = -6.$$

—

The characteristic equation is $\lambda^2 - 5\lambda - 6 = 0 \Rightarrow (\lambda - 6)(\lambda + 1) = 0 \Rightarrow \lambda = -1, 6$.

We know that, the matrix whose eigen values are $\frac{1}{-1}$ and $\frac{1}{6}$ is A^{-1}.

$$\therefore A^{-1} = \frac{1}{|A|} A_c^T = \frac{-1}{6}\begin{pmatrix} 4 & 2 \\ 5 & 1 \end{pmatrix}.$$

Example 1.23. Find the eigen values of $adj(A)$ if $A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

Solution. Given $A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

Since A is triangular, the eigen values are the elements along the main diagonal.

$\therefore$ The eigen values are $3, 4, 1$.

We have $A^{-1} = \frac{1}{|A|} adj(A)$

$\Rightarrow adj(A) = |A|\, A^{-1}$

The eigen values of A^{-1} are $\frac{1}{3}, \frac{1}{4}, 1$

$\therefore$ The eigen values of $adj(A)$ are $|A| \times \frac{1}{3}, |A| \times \frac{1}{4}, |A| \times 1$

Now, $|A| = \begin{vmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 3 \times 4 \times 1 = 12.$

$\therefore$ Eigen values of $adj(A)$ are $12 \times \frac{1}{3}, 12 \times \frac{1}{4}, 12 \times 1$

i.e., $4, 3, 12$.

Example 1.24. Find the eigen values of $adj(A)$ if $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

Step 1. To find the eigen values of A

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

—

Now, s_1 = sum of the main diagonal elements

$= 2 + 2 + 2 = 6.$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$$

$= 4 - 0 + 4 - 1 + 4 - 0$

$= 4 + 3 + 4 = 11.$

$s_3 = |A|$

$$= \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$= 2(4 - 0) - 0 + 1(0 - 2) = 8 - 2 = 6.$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

$$\lambda = 1 \text{ is a root.}$$

By synthetic division, we have

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 3)(\lambda - 2) = 0$$

$$\lambda = 2, 3.$$

The eigen values of A are $1, 2, 3$.

We know that $adj(A) = |A|\, A^{-1}$.

—

$\therefore$ The eigen values of $adj(A)$ are $|A| \times \frac{1}{1}, |A| \times \frac{1}{2}, |A| \times \frac{1}{3}$

$$\text{i.e.,} 6 \times 1, 6 \times \frac{1}{2}, 6 \times \frac{1}{3}$$
$$\text{i.e.,} 6, 3, 2$$

Property 1.6. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of A, then

(i) $\lambda_1 - k, \lambda_2 - k, \ldots, \lambda_n - k$ are the eigen values of $A - kI$.

(ii) $a_0\lambda_1^2 + a_1\lambda_1 + a_2, a_0\lambda_2^2 + a_1\lambda_2 + a_2, \ldots, a_0\lambda_n^2 + a_1\lambda_n + a_2$ are the eigen values of $a_0A^2 + a_1A + a_2I$.

Proof. (i) let λ be an eigen value of A.

Then, there exists a column matrix X such that $AX = \lambda X$.

Since $X \neq 0$ is a column matrix, we have

$$AX - kX = \lambda X - kX$$
$$\Rightarrow (A - kI)X = (\lambda - k)X.$$

$\Rightarrow \lambda - k$ is an eigen value of $A - kI$.

This is true for all eigen values of A.

$\therefore$ $\lambda_1 - k, \lambda_2 - k, \ldots, \lambda_n - k$ are the eigen values of $A - kI$.

(ii) We have

$$AX = \lambda X \Rightarrow A^2X = \lambda^2X \Rightarrow a_0(A^2X) = a_0(\lambda^2X)$$
$$\Longrightarrow (a_0A^2)X = (a_0\lambda^2)X.$$
$$a_1(AX) = a_1(\lambda X) \Rightarrow (a_1A)X = (a_1\lambda)X.$$
$$(a_0A^2)X + (a_1A)X = (a_0\lambda^2)X + (a_1\lambda)X.$$

Add a_2X both sides we get,

$$(a_0A^2 + a_1A + a_2I)X = (a_0\lambda^2 + a_1\lambda + a_2)X$$

$\Rightarrow a_0\lambda^2 + a_1\lambda + a_2$ is an eigen value of $a_0A^2 + a_1A + a_2I$.

This is true for all eigen values of A.

$\therefore a_0\lambda_1^2 + a_1\lambda_1 + a_2, a_0\lambda_2^2 + a_1\lambda_2 + a_2, \ldots$ are the eigen values of $a_0A^2 + a_1A + a_2I$. □

—

Worked Examples

Example 1.25. If $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$, find the eigen values of $A^2 - 2A + I$.

Solution. Since the given matrix is triangular, the eigen values are $3, 2, 5$.

Therefore, eigen values of $A^2 - 2A + I$ are $3^2 - 2(3) + 1, 2^2 - 2(2) + 1$ and $5^2 - 2(5) + 1$.

i.e., $4, 1, 16$.

Example 1.26. Form the matrix whose eigen values are $\alpha - 5, \beta - 5$ and $\gamma - 5$ where α, β and γ are the eigen values of $A = \begin{pmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{pmatrix}$. [Jan 2001]

Solution. Using (i) of Property 1.6, we obtain the matrix whose eigen values are $\alpha - 5, \beta - 5$ and $\gamma - 5$ is $A - 5I$.

$\therefore$ The required matrix is

$$A - 5I = \begin{pmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{pmatrix} - 5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -6 & -2 & -3 \\ 4 & 0 & -6 \\ 7 & -8 & 4 \end{pmatrix}.$$

Example 1.27. What is the sum of the squares of the eigen values of $\begin{pmatrix} 1 & 7 & 5 \\ 0 & 2 & 9 \\ 0 & 0 & 5 \end{pmatrix}$. [Jan 2001]

Solution. The given matrix is triangular.

$\therefore$ Eigen values are $1, 2, 5$.

Sum of squares of eigen values $= 1^2 + 2^2 + 5^2 = 30$.

Example 1.28. Find the eigen values of the matrix. $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$. Find also

—

the sum of the cubes of the eigen values.

Solution. Since A is triangular, the eigen values are the elements along the leading diagonal.

$\therefore$ The eigen values are $1, -1$ and 2.

$$\text{Sum of the cubes of the eigen values } = 1^3 + (-1)^3 + 2^3 = 1 - 1 + 8 = 8.$$

Example 1.29. Find the sum of the fourth powers of the eigen values of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 3 & -2 \end{pmatrix}$.

Solution. Since the matrix A is triangular, the eigen values are the elements along the leading diagonal.

$\therefore$ The eigen values are $1, 2$ and -2.

$$\text{Sum of the fourth powers of the eigen values} = 1^4 + 2^4 + (-2)^4 = 1 + 16 + 16 = 33.$$

Example 1.30. If $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is an eigen vector of the matrix $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$, find the corresponding eigen value.

Solution. Let $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ and $X = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Let λ be the corresponding eigen value.

$$\text{Now, } (A - \lambda I)X = 0.$$

$$\text{i.e., } \begin{pmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = 0$$

$$4(5-\lambda) + 4 = 0 \Rightarrow 5 - \lambda = -1 \Rightarrow \lambda = 6.$$

$$\text{Also, } 4 + 2 - \lambda = 0 \Rightarrow \lambda = 6.$$

Hence, the eigen value corresponding to the given eigen vector is 6.

—

Example 1.31. If $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are the eigen vectors of the matrix $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$, find the corresponding eigen values.

Solution. Let λ_1 be the eigen value corresponding to the eigen vector $X_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

$$\therefore (A - \lambda_1 I)X_1 = 0.$$

$$\begin{pmatrix} 1-\lambda_1 & 1 \\ 3 & -1-\lambda_1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = 0.$$

$$i.e., \;\; 1 - \lambda_1 - 3 = 0 \Rightarrow \lambda_1 = -2.$$

Also the second equation is

$$3 - 3(-1 - \lambda_1) = 0$$
$$3 + 3 + 3\lambda_1 = 0$$
$$3\lambda_1 = -6$$
$$\lambda_1 = -2.$$

Let λ_2 be the eigen value corresponding to the eigen vector $X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$\therefore (A - \lambda_2 I)X_2 = 0.$$

$$\begin{pmatrix} 1-\lambda_2 & 1 \\ 3 & -1-\lambda_2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow 1 - \lambda_2 + 1 = 0$$

$$\lambda_2 = 2.$$

Also $3 - 1 - \lambda_2 = 0 \Rightarrow \lambda_2 = 2$

$\therefore$ The eigen values of A are -2 and 2.

—

Definition [Linearly dependent and independent vectors]

(i) The vectors $X_1, X_2, \ldots, X_n$ are said to be linearly dependent if there exist real numbers $c_1, c_2, \ldots, c_n$ not all zero such that $c_1X_1 + c_2X_2 + \cdots + c_nX_n = 0$.

(ii) The vectors $X_1, X_2, \ldots, X_n$ are said to be linearly independent if they are not linearly dependent.

Results

(i) If the vectors $X_1, X_2, \ldots, X_n$ are linearly independent and satisfying the relation $c_1X_1 + c_2X_2 + \cdots + c_nX_n = 0$, then $c_1 = c_2 = \cdots = c_n = 0$.

(ii) If the vectors $X_1, X_2, \ldots, X_n$ are linearly dependent, then any one vector can be written as the linear combination of all the other vectors.

Note

(i) The eigen values of the unit matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are $1, 1, 1$, and the corresponding eigen vectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. These vectors are linearly independent.

(ii) The eigen values of the triangular matrix $\begin{pmatrix} \lambda_1 & a & b \\ 0 & \lambda_2 & c \\ 0 & 0 & \lambda_3 \end{pmatrix}$ are $\lambda_1, \lambda_2, \lambda_3$.

i.e., In a triangular matrix, the eigen values are the elements in the main diagonal.

(iii) The eigen values of $A, A^2, \ldots, A^m$ are $\lambda, \lambda^2, \ldots, \lambda^m$. which are all different. But they all have the same eigen vector X.

(iv) λ and $a_0\lambda^2 + a_1\lambda + a_2$ are the eigen values of A and $a_0A^2 + a_1A + a_2I$ respectively. But they have the same eigen vector X.

—

Exercise I(B)

1. Find the sum and product of the eigen values of the matrix $\begin{pmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & -3 \end{pmatrix}$.

2. If α and β are the eigen values of $\begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}$, form the matrix whose eigen values are α^3 and β^3.

3. Prove that $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $-3A^{-1}$ have the same eigen values.

4. Two of the eigen values of the matrix $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ are 3 and 15. Find the third eigen value.

5. Two eigen values of the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ are equal to unity each. Find the eigen values of A^{-1}.

6. If the product of two of the eigen values of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ is 2, find the third eigen value.

7. Find the constants a and b such that the matrix $\begin{pmatrix} a & 4 \\ 1 & b \end{pmatrix}$ has 3 and -2 as eigen values.

8. Two eigen values of $A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{pmatrix}$ are equal and are double the third.

—

Find the eigen values of A^2.

9. If the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ has an eigen vector $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, find the corresponding eigen value of A.

10. If the eigen values of the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ are $-2, 3, 6$, find the eigen values of A^T.

1.4 Cayley Hamilton Theorem

Statement. Every square matrix satisfies its own characteristic equation.

Example. If the characteristic equation of a square matrix A is $\lambda^3 - 6\lambda^2 + 5\lambda + 3 = 0$, then we have, $A^3 - 6A^2 + 5A + 3I = 0$.

Results. Cayley Hamilton theorem is useful for

(i) finding the inverse of a non-singular matrix,and

(ii) finding the higher powers of A.

The above results are demonstrated in the following examples.

Worked Examples

Example 1.32. If the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2+i & -1 & 0 & 0 \\ -3 & 2i & i & 0 \\ 4 & -i & 1 & -i \end{pmatrix}$ where $i = \sqrt{-1}$, then using Cayley Hamilton theorem, prove that $A^4 = I$. [Jun 2011]

Solution. A is a triangular matrix.

The eigen values of A are $1, -1, i, -i$.

The characteristic equation is

$$(\lambda-1)(\lambda+1)(\lambda-i)(\lambda+i)=0$$
$$(\lambda^2-1)(\lambda^2-i^2)=0$$
$$(\lambda^2-1)(\lambda^2+1)=0$$
$$\lambda^4-1=0$$

By Cayley Hamilton theorem, $A^4-I=0 \quad \Rightarrow A^4=I$.

Example 1.33. If $A=\begin{pmatrix}1 & 0\\ 4 & 5\end{pmatrix}$, express A^3 interms of A and I, using Cayley Hamilton theorem. [Jan 2001]

Solution. Since A is a 2×2 matrix, the characteristic equation is of the form

$$\lambda^2-s_1\lambda+s_2=0.$$

Now, s_1 = sum of the main diagonal elements = $1+5=6$.

$$s_2=|A|=\begin{vmatrix}1 & 0\\ 4 & 5\end{vmatrix}=5-0=5.$$

The characteristic equation is $\lambda^2-6\lambda+5=0$.

By Cayley Hamilton theorem we get

$$A^2-6A+5I=0. \qquad (1)$$

Premultiplying by A, we get

$$A^3-6A^2+5A=0$$
$$\begin{aligned}A^3&=6A^2-5A\\ &=6(6A-5I)-5A \qquad [\text{ by } (1)]\\ &=36A-30I-5A\\ &=31A-30I.\end{aligned}$$

Example 1.34. Find the value of A^4 if $A=\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}$, using Cayley Hamilton theorem. [Jan 2003]

Solution. Since A is a 2×2 matrix, the characteristic equation is of the form

$$\lambda^2-s_1\lambda+s_2=0.$$

Now, s_1 = sum of the main diagonal elements

$= 1 + (-1) = 0.$

$$s_2 = |A| = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5.$$

The characteristic equation is

$$\lambda^2 - 5 = 0.$$

By Cayley Hamilton theorem, $A^2 - 5I = 0$.

$$\text{i.e., } A^2 = 5I$$

$$\text{Now, } A^4 = A^2.A^2$$

$$= 5I.5I = 25I = 25\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}.$$

Example 1.35. Using Cayley Hamilton theorem, find the inverse of $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

[Jan 2003]

Solution. Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

Since A is a 2×2 matrix, the characteristic equation is of the form

$$\lambda^2 - s_1\lambda + s_2 = 0.$$

Now, s_1 = sum of the main diagonal elements $= 1 + 3 = 4.$

$s_2 = |A| = 3 - 8 = -5.$

The characteristic equation is $\lambda^2 - 4\lambda - 5 = 0$.

By Cayley Hamilton theorem, we have

$$A^2 - 4A - 5I = 0.$$

Multiplying by A^{-1} we get,

$$A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I = 0$$

$$A - 4I - 5A^{-1} = 0$$

$$5A^{-1} = A - 4I$$

$$A^{-1} = \frac{1}{5}[A - 4I]$$
$$= \frac{1}{5}\left(\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right). = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}.$$

Example 1.36. Verify Cayley Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and find its inverse. Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A. [Jun 2009]

Step 1. To find the characteristic equation

Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

Since A is a 2×2 matrix, the characteristic equation is of the form

$$\lambda^2 - s_1\lambda + s_2 = 0.$$

Now, s_1 = sum of the main diagonal elements = $1 + 3 = 4$.

$s_2 = |A| = 3 - 8 = -5$.

The characteristic equation is

$$\lambda^2 - 4\lambda - 5 = 0.$$

Step 2. Verification of Cayley Hamilton theorem

By cayley Hamilton theorem, we have

$$A^2 - 4A - 5I = 0.$$

Now, $A^2 = A.A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix}$.

$$A^2 - 4A - 5I = \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix} - \begin{pmatrix} 4 & 16 \\ 8 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

$\therefore$ Cayley Hamilton theorem is verified.

—

Step 3. To find A^{-1}

we have

$$A^2 - 4A - 5I = 0.$$

Multiplying by A^{-1} we get,

$$\begin{aligned} A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I &= 0 \\ A - 4I - 5A^{-1} &= 0 \\ 5A^{-1} &= A - 4I \\ A^{-1} = \frac{1}{5}[A - 4I] &= \frac{1}{5}\left(\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}. \end{aligned}$$

Step 4. To find $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

Consider the polynomial $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10$.

Divide by $\lambda^2 - 4\lambda - 5$ we get,

$\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 = (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3) + \lambda + 5.$

Replacing λ by A we get,

$$\begin{aligned} A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I &= (A^2 - 4A - 5I)(A^3 - 2A + 3I) + A + 5I \\ &= 0 + A + 5I \quad [\text{ Using Cayley Hamilton theorem}] \\ &= A + 5I. \end{aligned}$$

Example 1.37. Verify Cayley Hamilton theorem for the matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

Find A^{-1} and A^4. [Jan 2009]

Solution.

Step 1. To find the characteristic equation

Since A is a 3×3 matrix, the characteristic equation is of the form

—

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0. \tag{1}$$

Now, s_1 = sum of the main diagonal elements

$= 2 + 2 + 2 = 6.$

s_2 = sum of the minors of the elements of the main diagonal

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$= 4 - 1 + 4 - 1 + 4 - 1$

$= 3 + 3 + 3$

$= 9.$

$$s_3 = |A| \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 2(3) + 1(-1) + 1(-1) = 6 - 1 - 1 = 4.$$

The characteristic equation is $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$.

Step 2. Verification of Cayley Hamilton theorem.

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$A^2 = A \cdot A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{pmatrix} = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

$$A^3 = A^2 \times A = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{pmatrix} = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 6\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + 9\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

$\therefore$ Cayley Hamilton theorem is verified.

Step 3. To find A^{-1}

By Cayley Hamilton theorem, we have $A^3 - 6A^2 + 9A - 4I = 0$.

Multiplying by A^{-1} we get,

$$A^2 - 6A + 9I - 4A^{-1} = 0.$$

$$\Rightarrow 4A^{-1} = A^2 - 6A + 9I.$$

$$\Rightarrow A^{-1} = \frac{1}{4}[A^2 - 6A + 9I].$$

$$A^2 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{4}\left(\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - \begin{pmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{pmatrix} + \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}\right).A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}.$$

Step 4. To find A^4

Consider λ^4. Dividing λ^4 by $\lambda^3 - 6\lambda^2 + 9\lambda - 4$ we get,

$\lambda^4 = (\lambda+6)(\lambda^3 - 6\lambda^2 + 9\lambda - 4) + 27\lambda^2 - 50\lambda + 24.$

Replacing λ by A we get,

$$A^4 = (A+6I)(A^3 - 6A^2 + 9A - 4I) + 27A^2 - 50A + 24I.$$
$$= 27A^2 - 50A + 24I \quad \text{[By Cayley Hamilton theorem].}$$
$$= \begin{pmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{pmatrix}.$$

Example 1.38. If $A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$, verify Cayley Hamilton theorem and hence find A^{-1}. [Jan 2005]

Solution.

Step 1. To find the characteristic equation

Since A is 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$$= 1 + 3 + 1 = 5$$

s_2 = sum of the minors of the elements of the main diagonal

$$= \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$
$$= 3 - 0 + 1 - 0 + 3 + 2$$
$$= 3 + 1 + 5 = 9$$

$$s_3 = |A| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$
$$= 1(3-0) - 2(-1-0) - 2(2-0) = 3 + 2 - 4 = 1.$$

The characteristic equation is $\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$.

By Cayley Hamilton theorem, we have $A^3 - 5A^2 + 9A - I = 0$.

Step2. Verification of Cayley Hamilton theorem

$$A^2 = A.A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix}$$

$$A^3 = A^2.A = \begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{pmatrix}$$

Now $A^3 - 5A^2 + 9A - I = \begin{pmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{pmatrix} - 5\begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix} + 9\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Hence, Cayley Hamilton theorem is verified.

Step 3. To find A^{-1}

By Cayley Hamilton theorem, we have $A^3 - 5A^2 + 9A - I = 0$.

Multiplying by A^{-1} we get

$$A^2 - 5A + 9I - A^{-1} = 0$$

$$\therefore\ A^{-1} = A^2 - 5A + 9I = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}.$$

Example 1.39. Find the matrix $A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$ if $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$, using Cayley Hamilton theorem. [Jun 2009]

—

Solution.

Step 1. To find the characteristic equation

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

$$\begin{aligned}
\text{Now, } s_1 &= \text{sum of the main diagonal elements} \\
&= 2 + 1 + 2 = 5 \\
s_2 &= \text{sum of the minors of the main diagonal elements} \\
&= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2 + 3 + 2 = 7. \\
s_3 = |A| &= \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 2(2-0) - 1(0-0) + 1(0-1) = 4 - 1 = 3.
\end{aligned}$$

The characteristic equation is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$.

By Cayley Hamilton theorem we get $A^3 - 5A^2 + 7A - 3I = 0$.

Step 2. To find the value of given matrix expression

Consider the polynomial $\lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + 8\lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1$.

Divide this polynomial by $\lambda^3 - 5\lambda^2 + 7\lambda - 3$ we get

$$\lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + 8\lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1 = (\lambda^5 + 8\lambda + 35)(\lambda^3 - 5\lambda^2 + 7\lambda - 3) + 127\lambda^2 - 223\lambda + 106.$$

Replacing λ by A we get

$$\begin{aligned}
&A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I \\
&\qquad = (A^5 + 8A + 35I)(A^3 - 5A^2 + 7A - 3I) + 127A^2 - 223A + 106I \\
&\qquad = 127A^2 - 223A + 106I \quad \text{[By Cayley Hamilton theorem].} \\
&\qquad = \begin{pmatrix} 295 & 285 & 285 \\ 0 & 10 & 0 \\ 285 & 285 & 295 \end{pmatrix}.
\end{aligned}$$

—

Example 1.40. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ then show that $A^n = A^{n-2} + A^2 - I$ for $n \geq 3$. Hence find A^{50}. [Jan 2006]

Solution. Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$= 1 + 0 + 0 = 1$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$$

$= 0 - 1 - 0 + 0 = -1.$

$$s_3 = |A| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$= 1(0 - 1) = -1.$

$\therefore$ The characteristic equation is $\lambda^3 - \lambda^2 - \lambda - 1 = 0$.

By Cayley Hamilton theorem we have,

$$A^3 - A^2 - A + I = 0$$
$$A^3 - A^2 = A - I$$
$$A^4 - A^3 = A^2 - A$$
$$A^5 - A^4 = A^3 - A^2$$
$$\cdots\cdots\cdots\cdots$$
$$A^n - A^{n-1} = A^{n-2} - A^{n-3}$$

Adding all these equations we get, $A^n - A^2 = A^{n-2} - I$.

$$\therefore \; A^n = A^{n-2} + A^2 - I$$

—

$$A^{n-2} = A^{n-4} + A^2 - I.$$

$$\therefore\ A^n = A^{n-4} + 2(A^2 - I).$$

$$A^{n-4} = A^{n-8} + 2(A^2 - I).$$

$$\therefore\ A^n = A^{n-8} + 4(A^2 - I).$$

$$A^{n-8} = A^{n-16} + 4(A^2 - I).$$

$$A^n = A^{n-16} + 8(A^2 - I).$$

$$\text{If } n \text{ is even, } A^n = A^{n-(n-2)} + \frac{n-2}{2}(A^2 - I)$$

$$= A^2 + \frac{n-2}{2}(A^2 - I)$$

$$= A^2 + \frac{n-2}{2}A^2 - \frac{n-2}{2}I$$

$$A^n = \frac{n}{2}A^2 - \frac{n-2}{2}I.$$

When $n = 50$, we have

$$A^{50} = 25A^2 - 24I.$$

Now, $A^2 = A \cdot A$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A^{50} = 25\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} - 24\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{pmatrix} - \begin{pmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{pmatrix}.$$

Example 1.41. If n is a positive integer, find A^n for the matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ using Cayley Hamilton theorem. [Jan 2005]

Solution. Since A is a 2×2 matrix, the characteristic equation is of the form

$$\lambda^2 - s_1\lambda + s_2 = 0.$$

Now, s_1 = sum of the main diagonal elements

$$= 1 + 3 = 4.$$

$$s_2 = |A| = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}$$

$$= 3 - 8 = -5.$$

The characteristic equation is

$$\lambda^2 - 4\lambda - 5 = 0.$$

$$(\lambda - 5)(\lambda + 1) = 0.$$

The eigen values are $-1, 5$.

By Cayley Hamilton theorem we get, $A^2 - 4A - 5I = 0$.

Dividing λ^n by $\lambda^2 - 4\lambda - 5$, let $Q(\lambda)$ be the quotient and $R(\lambda)$ be the remainder.

Since we are dividing by $\lambda^2 - 4\lambda - 5$, $R(\lambda)$ is of degree atmost 1.

$\therefore$ Let $R(\lambda) = a\lambda + b$.

Hence, we can write λ^n as

$$\lambda^n = (\lambda^2 - 4\lambda - 5)Q(\lambda) + a\lambda + b. \qquad \text{(A)}$$

When $\lambda = -1$, we get

$$-a + b = (-1)^n \qquad (1)$$

When $\lambda = 5$, we get

$$5a + b = 5^n \qquad (2)$$

$$(2) - (1) \;\Rightarrow\; 6a = 5^n - (-1)^n. \;\Rightarrow\; a = \frac{1}{6}[5^n - (-1)^n].$$

Substituting in (1) we get

$$b = a + (-1)^n = \frac{1}{6}[5^n - (-1)^n] + (-1)^n = \frac{1}{6}[5^n + 5(-1)^n].$$

—

Replacing λ by A in (A) we get

$$\begin{aligned} A^n &= (A^2 - 4A - 5I)Q(A) + aA + bI \\ &= aA + bI \text{ [by C.H Theorem]} \\ &= \frac{1}{6}[5^n - (-1)^n]A + \frac{1}{6}[5^n + 5(-1)^n]I. \end{aligned}$$

Example 1.42. If $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ find A^n in terms of A and I. [Jun 2003]

Solution. Since A is a 2×2 matrix, the characteristic equation is of the form $\lambda^2 - s_1\lambda + s_2 = 0$.

Now, s_1 = sum of the main diagonal elements

$$= 1 - 1 = 0.$$

$$s_2 = |A| = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$$= -1 - 4 = -5.$$

The characteristic equation is $\lambda^2 - 5 = 0$

$$\text{i.e., } \lambda^2 - 5 = 0$$

$$\Longrightarrow \lambda = \pm\sqrt{5}.$$

By Cayley Hamilton theorem we get $A^2 - 5I = 0$.

Dividing λ^n by $\lambda^2 - 5$, let $Q(\lambda)$ be the quotient and $R(\lambda)$ be the remainder.

Since we are dividing λ^n by $\lambda^2 - 5$, $R(\lambda)$ is of degree atmost 1.

$$\therefore R(\lambda) = a\lambda + b.$$

$$\therefore \lambda^n = (\lambda^2 - 5)Q(\lambda) + a\lambda + b. \tag{1}$$

$$\text{When } \lambda = \sqrt{5}, \text{we get} (\sqrt{5})^n = a\sqrt{5} + b. \tag{2}$$

$$\text{When } \lambda = -\sqrt{5}, \text{we get } (-\sqrt{5})^n = -\sqrt{5}a + b \tag{3}$$

—

$(2)-(3) \Rightarrow 2\sqrt{5}a = (\sqrt{5})^n - (-\sqrt{5})^n$
$= (\sqrt{5})^n[1-(-1)^n]$

$a = (\sqrt{5})^{n-1}\dfrac{[1-(-1)^n]}{2}.$

substituting in (2) we get

$$\begin{aligned} b &= (\sqrt{5})^n - a(\sqrt{5}) \\ &= (\sqrt{5})^n - (\sqrt{5})^n\frac{[1-(-1)^n]}{2} \\ &= (\sqrt{5})^n\frac{[2-1-+(-1)]^n}{2} \\ b &= (\sqrt{5})^n\frac{[1+(-1)^n]}{2}. \end{aligned}$$

$\therefore \lambda^n = (\lambda^2-5)\phi(\lambda) + \dfrac{\sqrt{5}^{n-1}}{2}[1-(-1)^n]\lambda + \dfrac{(\sqrt{5})^n + (-\sqrt{5})^n}{2}.$

Replacing λ by A we get

$$A^n = \frac{(\sqrt{5})^{n-1}}{2}[1-(-1)^n]A + (\sqrt{5})^n\frac{[1+(-1)^n]}{2}.$$

Exercise I(C)

1. Verify Cayley Hamilton theorem find the inverse of $A = \begin{pmatrix} 13 & -3 & 5 \\ 0 & 4 & 0 \\ -15 & 9 & -7 \end{pmatrix}$.

2. Verify Cayley Hamilton theorem for the matrix $\begin{pmatrix} 2 & 2 & 0 \\ -2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$. Hence find its inverse.

3. Using Cayley Hamilton theorem find the inverse of $A = \begin{pmatrix} 2 & 1 \\ 1 & -5 \end{pmatrix}$.

4. Using Cayley Hamilton theorem, find the inverse of the following matrices.

   (i) $\begin{pmatrix} 5 & 4 \\ 1 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 2 \\ 1 & -1 & 2 \end{pmatrix}$

—

5. Show that the matrix $\begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$ satisfies Cayley Hamilton theorem.

6. Verify Cayley Hamilton theorem for the matrix $A = \begin{pmatrix} -1 & 0 & 3 \\ 8 & 1 & -7 \\ -3 & 0 & 8 \end{pmatrix}$.

7. Using Cayley Hamilton theorem find A^4 for the matrix $A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

8. Find the characteristic equation of the matrix A and hence Compute $2A^8 - 3A^5 + A^4 - 4I$ where A is $\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

9. If $A = \begin{pmatrix} 7 & 3 \\ 2 & 6 \end{pmatrix}$, find A^n interms of A and I using Cayley Hamilton theorem. Hence find the value of A^3.

10. If $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$, find A^n interms of A and I.

11. If $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$, prove that $A^n = 7^{n-1}\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$.

12. If $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$, prove that $A^n = \begin{pmatrix} 1+2n & -4n \\ n & 1-2n \end{pmatrix}$.

13. If $A = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix}$, using Cayley Hamilton theorem obtain the value of $A^6 - 25A^2 + 122A$.

—

1.5 Diagonalization of Matrices - Similarity Transformation

Similar matrices. Let A and B be square matrices of order n. A is similar to B if there exists a nonsingular matrix P such that $A = P^{-1}BP$.

The transformation which transforms B into A is called a similarity transformation. The matrix P is called a similarity matrix.

Results.

1. If A and B are similar then $|A| = |B|$.
2. If A and B are similar, then they have the same eigen values.

Diagonalisation of a square matrix

A square matrix A is said to be diagonalizable, if there exists a nonsingular matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix. The matrix P is called the modal matrix of A. D is also called as the spectral matrix of A.

Working rule to diagonalize a matrix

Step 1. Find the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Step 2. Find the linearly independent eigen vectors $X_1, X_2, \ldots, X_n$, corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Step 3. Form the modal matrix $P = [X_1, X_2, \ldots, X_n]$.

Step 4. Find P^{-1}.

Step 5. Find $P^{-1}AP = D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$.

Note. P is possible only when A has n linearly independent eigen vectors.

—

Computation of powers of a square matrix using similarity transformation

$$\begin{aligned}
\text{we have } D &= P^{-1}AP \\
PDP^{-1} &= P(P^{-1}AP)P^{-1} \\
&= (PP^{-1})A(PP^{-1}) \\
&= IAI = A \\
\therefore\ A &= PDP^{-1} \\
A^2 &= A \cdot A \\
&= (PDP^{-1})(PDP^{-1}) \\
&= PD(P^{-1}P)DP^{-1} \\
&= PDIDP^{-1} \\
A^2 &= PD^2P^{-1}
\end{aligned}$$

In the same way we get $A^3 = PD^3P^{-1}$.

In general $A^n = PD^nP^{-1}$, where $D = \begin{pmatrix} \lambda_1^n & 0 & \ldots & 0 \\ 0 & \lambda_2^n & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n^n \end{pmatrix}$

Properties of eigen values of similar matrices

Property 1.7. If A and B are similar matrices, they have the same characteristic equation.

Proof. Since A and B are similar, there exists a matrix P such that

$$\begin{aligned}
B &= P^{-1}AP \\
\therefore\ B - \lambda I &= P^{-1}AP - \lambda I \\
&= P^{-1}AP - P^{-1}\lambda IP \\
&= P^{-1}(AP - \lambda IP) \\
&= P^{-1}(A - \lambda I)P
\end{aligned}$$

—

Hence, $|B-\lambda I| = |P^{-1}(A-\lambda I)P|$

$$= |P^{-1}||A-\lambda I||P|$$

$$= |P^{-1}||P||A-\lambda I|$$

$$= |P^{-1}P||A-\lambda I|$$

$$= |I||A-\lambda I|$$

$$= |A-\lambda I|.$$

The characteristic equations of A and B are respectively $|A-\lambda I| = 0$ and $|B-\lambda I| = 0$. Hence, they are equal. □

Corollary. Two similar matrices have the same eigen values.

Property 1.8. If A and B are $n\times n$ matrices and B is a non singular matrix, then A and $B^{-1}AB$ have the same eigen values.

Proof. The characteristic polynomial of $B^{-1}AB$ is

$$|B^{-1}AB-\lambda I| = |B^{-1}AB - B^{-1}(\lambda I)B|$$

$$= |B^{-1}(A-\lambda I)B|$$

$$= |B^{-1}||A-\lambda I||B|$$

$$= |B^{-1}||B||A-\lambda I|$$

$$= |B^{-1}B||A-\lambda I|$$

$$= |I||A-\lambda I|$$

$$= |A-\lambda I|.$$

= Characteristic polynomial of A.

$\Rightarrow$ $B^{-1}AB$ and A have the same characteristic equation.

$\Rightarrow$ They have the same eigen values. □

Result

If we are asked to find $f(A)$, then $f(A)$ can be evaluated by $f(A) = Pf(D)P^{-1}$

Worked Examples

Example 1.43. Reduce the matrix $A = \begin{pmatrix} -19 & 7 \\ -42 & 16 \end{pmatrix}$ to the diagonal form.

Solution. Let the matrix be $A = \begin{pmatrix} -19 & 7 \\ -42 & 16 \end{pmatrix}$.

Step 1. To find the eigen values

$$s_1 = tr(A) = -19 + 16 = -3.$$

$$s_2 = |A| = \begin{vmatrix} -19 & 7 \\ -42 & 16 \end{vmatrix}$$

$$= -304 + 294 = -10.$$

The characteristic equation is

$$\lambda^2 - s_1\lambda + s_2 = 0$$

$$i.e., \lambda^2 + 3\lambda - 10 = 0$$

$$(\lambda + 5)(\lambda - 2) = 0$$

$$\lambda = 2, -5.$$

The eigen values are $\lambda_1 = 2$, $\lambda_2 = -5$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0$$

$$i.e. \begin{pmatrix} -19 - \lambda & 7 \\ -42 & 16 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\left.\begin{aligned} (-19 - \lambda)x_1 + 7x_2 &= 0 \\ -42x_1 + (16 - \lambda)x_2 &= 0. \end{aligned}\right\} \qquad \text{(A)}$$

—

case(i) When $\lambda = 2$, (A) become

$$-21x_1 + 7x_2 = 0$$

$$-42x_1 + 14x_2 = 0.$$

Both the equations are reduced to one single equation

$$3x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{3}.$$

$\therefore$ The eigen vector corresponding to $\lambda = 2$ is $X_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

case(ii) When $\lambda = -5$, (A) reduces to

$$-14x_1 + 7x_2 = 0$$

$$-42x_1 + 21x_2 = 0.$$

Both the equations are reduced to one single equation

$$-2x_1 = -x_2$$

$$\frac{x_1}{1} = \frac{x_2}{2}$$

$\therefore$ The eigen vector corresponding to $\lambda = -5$ is $X_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Step 3. To form the modal matrix P

The modal matrix P is obtained with X_1, X_2 as columns.

$$\therefore P = \begin{pmatrix} X_1 & X_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}.$$

Step 4. To find P^{-1}

We know that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

—

Hence $P^{-1} = \frac{1}{-1}\begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$.

Step 5. To find $P^{-1}AP$

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} -19 & 7 \\ -42 & 16 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} -19+21 & -19+14 \\ -42+48 & -42+32 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ 6 & -10 \end{pmatrix} \\ &= \begin{pmatrix} -4+6 & 10-10 \\ 6-6 & -15+10 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \end{aligned}$$

which is the required diagonal form of A.

Example 1.44. Reduce the matrix $A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$ to the diagonal form. Hence find A^4.

Solution.

Step 1. To find the eigen values

$$s_1 = tr(A) = \text{sum of the elements along the main diagonal} = 1+2+0 = 3.$$

$$\begin{aligned} s_2 &= \text{sum of the minors of the elements of the main diagonal} \\ &= \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0+1+0-2+2-2 = -1. \end{aligned}$$

$$s_3 = |A| = \begin{vmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{vmatrix} = 1(0+1) - 2(0+1) - 2(-1+2) = 1-2-2 = -3.$$

The characteristic equation is

$$\lambda^3 - s_1\lambda^2 + s_2\lambda + s_3 = 0$$

$$i.e., \lambda^3 - 3\lambda^2 - \lambda + 3 = 0$$

$$\lambda = 1 \text{ is a root.}$$

By synthetic division we have

$$\begin{array}{c|cccc} 1 & 1 & -3 & -1 & 3 \\ & 0 & 1 & -2 & -3 \\ \hline & 1 & -2 & -3 & \boxed{0} \end{array}$$

$\therefore$ The characteristic equation becomes

$$(\lambda - 1)(\lambda^2 - 2\lambda - 3) = 0$$

$$(\lambda - 1)(\lambda - 3)(\lambda + 1) = 0$$

$$\lambda = -1, 1, 3.$$

The eigen values are $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 3$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0$$

$$i.e. \begin{pmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

$$\left.\begin{aligned} (1-\lambda)x_1 + 2x_2 - 2x_3 &= 0 \\ x_1 + (2-\lambda)x_2 + x_3 &= 0 \\ -x_1 - x_2 - \lambda x_3 &= 0 \end{aligned}\right\} \quad \text{(A)}$$

case(i) When $\lambda = -1$, (A) become

$$2x_1 + 2x_2 - 2x_3 = 0. \quad (1)$$

—

$$x_1 + 3x_2 + x_3 = 0. \tag{2}$$

$$-x_1 - x_2 + x_3 = 0. \tag{3}$$

The above three equations are reduced to the following two equations.

$$x_1 + x_2 - x_3 = 0$$

$$x_1 + 3x_2 + x_3 = 0.$$

By the rule of cross multiplication we have,

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & \times & -1 & \times & 1 & \times & 1 \\ 3 & & 1 & & 1 & & 3 \end{array}$$

$$\frac{x_1}{1+3} = \frac{x_2}{-1-1} = \frac{x_3}{3-1}$$
$$\frac{x_1}{4} = \frac{x_2}{-2} = \frac{x_3}{2}$$
$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}.$$

$$\therefore X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

case(ii) When $\lambda = 1$, (A) become

$$2x_2 - 2x_3 = 0 \tag{4}$$

$$x_1 + x_2 + x_3 = 0 \tag{5}$$

$$-x_1 - x_2 - x_3 = 0. \tag{6}$$

The above three equations are reduced to

$$x_2 = x_3$$

$$\text{and } x_1 + x_2 + x_3 = 0.$$

$$\text{let } x_3 = 1 \Rightarrow x_2 = 1.$$

$$\text{Hence } \; x_1 = -2$$

$$\therefore X_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

case(iii) When $\lambda = 3$, (A) become

$$-2x_1 + 2x_2 - 2x_3 = 0 \tag{7}$$

$$x_1 - x_2 + x_3 = 0 \tag{8}$$

$$-x_1 - x_2 - 3x_3 = 0. \tag{9}$$

The above three equations are reduced to the following two equations.

$$x_1 - x_2 + x_3 = 0$$

$$x_1 + x_2 + 3x_3 = 0.$$

By the rule of cross multiplication we have

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -1 & \times & 1 & \times & 1 & \times & -1 \\ 1 & & 3 & & 1 & & 1 \end{array}$$

$$\frac{x_1}{-3-1} = \frac{x_2}{1-3} = \frac{x_3}{1+1}$$
$$\frac{x_1}{-4} = \frac{x_2}{-2} = \frac{x_3}{2}$$
$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-1}.$$

$$\therefore X_3 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

Step 3. To form the modal matrix P

The modal matrix P is formed with the eigen vectors X_1, X_2, X_3 as columns.

$$i.e, \quad P = \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Step 4. To find P^{-1}

$$|P| = \begin{vmatrix} 2 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 2(-1-1) + 2(1-1) + 2(-1-1)$$

$$= 2(-2) + 2(0) + 2(-2) = -4 + 0 - 4 = -8.$$

$$\text{cofactor of } 2 = +\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2.$$

$$\text{cofactor of } -2 = -\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = -(1 - 1) = 0.$$

$$\text{cofactor of } 2 = +\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1 - 1 = -2.$$

$$\text{cofactor of } -1 = -\begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix} = -(2 - 2) = 0.$$

$$\text{cofactor of } 1 = +\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} = -2 - 2 = -4.$$

$$\text{cofactor of } 1 = -\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} = -(2 + 2) = -4.$$

$$\text{cofactor of } 1 = +\begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} = -2 - 2 = -4.$$

$$\text{cofactor of } 1 = -\begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} = -(2 + 2) = -4.$$

cofactor of $-1 = +\begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} = 2 - 2 = 0.$

$$\therefore P_c = \begin{pmatrix} -2 & 0 & -2 \\ 0 & -4 & -4 \\ -4 & -4 & 0 \end{pmatrix}.$$

$$P^{-1} = \frac{1}{|P|} P_c^T = \frac{-1}{8} \begin{pmatrix} -2 & 0 & -4 \\ 0 & -4 & -4 \\ -2 & -4 & 0 \end{pmatrix}.$$

Step 5. To find $P^{-1}AP$

$$P^{-1}AP = \frac{-1}{8}\begin{pmatrix} -2 & 0 & -4 \\ 0 & -4 & -4 \\ -2 & -4 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \frac{-1}{8}\begin{pmatrix} -2+0+4 & -4+0+4 & 4+0+0 \\ 0-4+4 & 0-8+4 & 0-4+0 \\ -2-4+0 & -4-8+0 & 4-4+0 \end{pmatrix}\begin{pmatrix} 2 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \frac{-1}{8}\begin{pmatrix} 2 & 0 & 4 \\ 0 & -4 & -4 \\ -6 & -12 & 0 \end{pmatrix}\begin{pmatrix} 2 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \frac{-1}{8}\begin{pmatrix} 4+0+4 & -4+0+4 & 4+0-4 \\ 0+4-4 & 0-4-4 & 0-4+4 \\ -12+12+0 & 12-12+0 & -12-12+0 \end{pmatrix}$$

$$= \frac{-1}{8}\begin{pmatrix} 8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -24 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

which is the required diagonal form.

—

Step 6. To find A^4

By similarity transformation, $D = P^{-1}AP$.

$$\therefore D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Hence

$$D^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 81 \end{pmatrix}$$

Now $\quad A^4 = PD^4P^{-1}$

$$= \frac{-1}{8}\begin{pmatrix} 2 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 81 \end{pmatrix}\begin{pmatrix} -2 & 0 & -4 \\ 0 & -4 & -4 \\ -2 & -4 & 0 \end{pmatrix}$$

$$= \frac{-1}{8}\begin{pmatrix} 2+0+0 & 0-2+0 & 0+0+162 \\ -1+0+0 & 0+1+0 & 0+0+81 \\ 1+0+0 & 0+1+0 & 0+0-81 \end{pmatrix}\begin{pmatrix} -2 & 0 & -4 \\ 0 & -4 & -4 \\ -2 & -4 & 0 \end{pmatrix}$$

$$= \frac{-1}{8}\begin{pmatrix} 2 & -2 & 162 \\ -1 & 1 & 81 \\ 1 & 1 & -81 \end{pmatrix}\begin{pmatrix} -2 & 0 & -4 \\ 0 & -4 & -4 \\ -2 & -4 & 0 \end{pmatrix}$$

$$= \frac{-1}{8}\begin{pmatrix} -4+0-324 & 0+8-648 & -8+8+0 \\ 2+0-162 & 0-4-324 & 4-4+0 \\ -2+0+162 & 0-4+324 & -4-4+0 \end{pmatrix}$$

$$= \frac{-1}{8}\begin{pmatrix} -328 & -640 & 0 \\ -160 & -328 & 0 \\ 160 & 320 & -8 \end{pmatrix} = \begin{pmatrix} 41 & 80 & 0 \\ 20 & 41 & 0 \\ -20 & -40 & 1 \end{pmatrix}$$

Result. A simple way to find the inverse of a 3×3 matrix.

—

Let $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

Step1. Rearrange the elements in the following way

$$\begin{matrix} b_2 & b_3 & b_1 \\ c_2 & c_3 & c_1 \\ a_2 & a_3 & a_1 \end{matrix}$$

Step2. Attach the first column as the 4^{th} column

$$\begin{matrix} b_2 & b_3 & b_1 & b_2 \\ c_2 & c_3 & c_1 & c_2 \\ a_2 & a_3 & a_1 & a_2 \end{matrix}$$

Step3. Attach the first row as the 4^{th} row

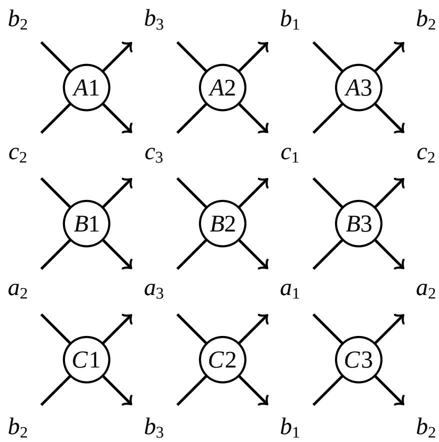
$$\begin{matrix} b_2 & b_3 & b_1 & b_2 \\ c_2 & c_3 & c_1 & c_2 \\ a_2 & a_3 & a_1 & a_2 \\ b_2 & b_3 & b_1 & b_2 \end{matrix}$$

Step4. The matrix A_c formed by the Cofactors of the elements of A is given by

$$A_c = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}.$$

A_i^s, B_i^s and C_i^s are calculated as follows, similar to the rule of Cross multiplication.

—



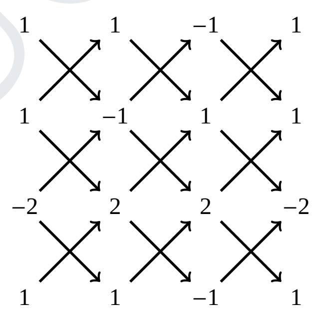
$$A_c = \begin{pmatrix} b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \\ c_2a_3 - c_3a_2 & c_3a_1 - c_1a_3 & c_1a_2 - c_2a_1 \\ a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \end{pmatrix}$$

Now $A^{-1} = \dfrac{1}{|A|} A_c^T$.

Example. Consider the previous example.

To find P^{-1} we have

$$P = \begin{pmatrix} 2 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$



$$P_c = \begin{pmatrix} -2 & 0 & -2 \\ 0 & -4 & -4 \\ -4 & -4 & 0 \end{pmatrix}$$

$$P^{-1} = \frac{1}{|P|} P_c^T.$$

$$= -\frac{1}{8}\begin{pmatrix} 2 & 0 & -4 \\ 0 & -4 & -4 \\ -2 & -4 & 0 \end{pmatrix}$$

Note. We shall apply this procedure to find the inverse of any 3×3 matrix here after.

Example 1.45. Diagonalize the matrix $A = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$

Solution. Given $A = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$= 2 + 1 - 1 = 2.$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}$$

$= -4 - 5 + 4 = -5.$

$$s_3 = |A| = \begin{vmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$$

$= 2(-1 - 3) + 2(-1 - 1) + 3(3 - 1)$

$= -8 - 4 + 6$

$= -6.$

The characteristic equation is

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0.$$

$$\lambda = 1 \text{ is a root.}$$

By synthetic division we have

$$\begin{array}{c|cccc} 1 & 1 & -2 & -5 & 6 \\ & 0 & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$$\lambda^2 - \lambda - 6 = 0$$

$$(\lambda - 3)(\lambda + 2) = 0$$

$$\lambda = -2, 3.$$

The eigen values are $1, -2, 3$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \tag{1}$$

When $\lambda = 1$, (1) becomes

$$\begin{pmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$x_1 - 2x_2 + 3x_3 = 0 \tag{2}$$

—

$$x_1 + x_3 = 0 \tag{3}$$

$$x_1 + 3x_2 - 2x_3 = 0. \tag{4}$$

(3) $\Rightarrow \; x_1 = -x_3$

$\dfrac{x_1}{-1} = \dfrac{x_3}{1}$

$\Rightarrow x_1 = -1, x_3 = 1.$

Substituting in (2) we get

$-1 - 2x_2 + 3 = 0$

$2 = 2x_2$

$x_2 = 1.$

$\therefore X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$

When $\lambda = -2$ (1) becomes

$$\begin{pmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations are reduced to

$4x_1 - 2x_2 + 3x_3 = 0$

$x_1 + 3x_2 + x_3 = 0.$

By the rule of cross multiplication, we obtain

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -2 & \times & 3 & \times & 4 & \times & -2 \\ 3 & & 1 & & 1 & & 3 \end{array}$$

$$\frac{x_1}{-2-9} = \frac{x_2}{3-4} = \frac{x_3}{12+2}$$

$$\frac{x_1}{-11} = \frac{x_2}{-1} = \frac{x_3}{14}$$

$$\therefore x_1 = 11, x_2 = 1, x_3 = -14$$

—

$$\therefore X_2 = \begin{pmatrix} 11 \\ 1 \\ -14 \end{pmatrix}.$$

When $\lambda = 3$, (1) becomes

$$\begin{pmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations are reduced to

$$-x_1 - 2x_2 + 3x_3 = 0 \tag{5}$$

$$x_1 - 2x_2 + x_3 = 0 \tag{6}$$

$$x_1 + 3x_2 - 4x_3 = 0. \tag{7}$$

All the three equations are different.

Taking (5) & (6) and by the rule of cross multiplication we obtain

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -2 & \times & 3 & \times & -1 & \times & -2 \\ -2 & & 1 & & 1 & & -2 \end{array}$$

$$\frac{x_1}{-2+6} = \frac{x_2}{3+1} = \frac{x_3}{2+2}$$

$$\frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

$$\therefore x_1 = x_2 = x_3 = 1$$

$$X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

—

Step 3. To form the modal matrix P

The modal matrix P is formed with the eigen vectors X_1, X_2, X_3 as columns.

$$\text{i.e., } P = \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix} = \begin{pmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{pmatrix}.$$

Step 4. To find P^{-1}

$$|P| = \begin{vmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{vmatrix}$$

$$= -1(1+14) - 11(1-1) + 1(-14-1)$$

$$= -15 + 0 - 15$$

$$= -30.$$

To find P_c

$$\begin{matrix} 1 & 1 & 1 & 1 \\ -14 & 1 & 1 & -14 \\ 11 & 1 & -1 & 11 \\ 1 & 1 & 1 & 1 \end{matrix}$$

$$P_c = \begin{pmatrix} 15 & 0 & -15 \\ -25 & -2 & -3 \\ 10 & 2 & -12 \end{pmatrix}$$

$$P^{-1} = \frac{1}{|P|} P_c^T$$

$$= -\frac{1}{30} \begin{pmatrix} 15 & -25 & 10 \\ 0 & -2 & 2 \\ -15 & -3 & -12 \end{pmatrix}$$

$$= \frac{1}{30} \begin{pmatrix} -15 & 25 & -10 \\ 0 & 2 & -2 \\ 15 & 3 & 12 \end{pmatrix}$$

Step 5. To find $P^{-1}AP$

$$P^{-1}AP = \frac{1}{30}\begin{pmatrix} -15 & 25 & -10 \\ 0 & 2 & -2 \\ 15 & 3 & 12 \end{pmatrix}\begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}\begin{pmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{pmatrix}$$

$$= \frac{1}{30}\begin{pmatrix} -15 & 25 & -10 \\ 0 & -4 & 4 \\ 45 & 9 & 36 \end{pmatrix}\begin{pmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{pmatrix}$$

$$= \frac{1}{30}\begin{pmatrix} 30 & 0 & 0 \\ 0 & -60 & 0 \\ 0 & 0 & 90 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

which is the required diagonal form.

Example 1.46. Diagonalize the matrix $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$.

Solution. Let $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$.

Step 1. To find the eigen values

$$s_1 = tr(A) = 6 + 3 + 3 = 12.$$

s_2 = sum of the minors of the main diogonal elements.

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= 9 - 1 + 18 - 4 + 18 - 4$$

$$= 8 + 14 + 14 = 36.$$

$$s_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(8) + 2(-4) + 2(-4)$$

$$= 48 - 8 - 8 = 32.$$

The characteristic equation is

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

$$i.e., \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0.$$

$$\lambda = 2 \text{ is a root.}$$

By synthetic division we have

$$\begin{array}{r|rrr|r} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

Hence, the characteristic equation becomes

$$(\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$$

$$(\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$$

$$\lambda = 2, 2, 8.$$

$\therefore$ The eigen values are $\lambda_1 = 2$, $\lambda_2 = 2$, $\lambda_3 = 8$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0.$$

—

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

$$\left.\begin{aligned} (6-\lambda)x_1 - 2x_2 + 2x_3 = 0 \\ -2x_1 + (3-\lambda)x_2 - x_3 = 0 \\ 2x_1 - x_2 + (3-\lambda)x_3 = 0 \end{aligned}\right\}. \tag{A}$$

case(i) <u>when $\lambda = 2$</u>, (A) reduces to

$$4x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 + x_2 - x_3 = 0$$

$$2x_1 - x_2 + x_3 = 0.$$

The above three equations are reduced to the single equation

$$2x_1 - x_2 + x_3 = 0.$$

Since $\lambda = 2$ is a repeated root, we have to find two eigen vectors by assigning a particular value to two variables, one at a time.

Assigning $x_3 = 0$, we obtain

$$2x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{2}$$

$$\therefore X_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

Assigning $x_2 = 0$, we obtain

$$2x_1 = -x_3$$

$$\frac{x_1}{-1} = \frac{x_3}{2}.$$

—

$$\therefore X_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

case(ii) when $\lambda = 8$, (A) reduces to

$$-2x_1 - 2x_2 + 2x_3 = 0 \tag{1}$$

$$-2x_1 - 5x_2 - x_3 = 0 \tag{2}$$

$$2x_1 - x_2 - 5x_3 = 0. \tag{3}$$

Since all the three equations are different, we can consider the the first two equations and solve for $x_1, x_2, \&, x_3$ by the method of cross multiplication

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -2 & & 2 & & -2 & & -2 \\ -5 & & -1 & & -2 & & -5 \end{array}$$

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$
$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$
$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$
$$\therefore x_1 = 2, x_2 = -1, x_3 = 1.$$

x_1, x_2, x_3 satisfy (3).

$$\therefore X_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Step 3. To form the modal matrix P

P can be obtained by considering X_1, X_2, X_3 as columns.

$$\therefore P = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

Step 4. To find P^{-1}

$$|P| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= 1(0+2) + 1(2-0) + 2(4-0) = 2 + 2 + 8 = 12.$$

To find P_c

$$\begin{matrix} 0 & -1 & 2 & 0 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 1 & -1 \\ 0 & -1 & 2 & 0 \end{matrix}$$

$$P_c = \begin{pmatrix} 2 & -2 & 4 \\ 5 & 1 & -2 \\ 1 & 5 & 2 \end{pmatrix}$$

$$P^{-1} = \frac{1}{|P|} P_c^T.$$

$$= \frac{1}{12} \begin{pmatrix} 2 & 5 & 1 \\ -2 & 1 & 5 \\ 4 & -2 & 2 \end{pmatrix}.$$

Step 5. To find $P^{-1}AP$

$$P^{-1}AP = \frac{1}{12} \begin{pmatrix} 2 & 5 & 1 \\ -2 & 1 & 5 \\ 4 & -2 & 2 \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$= \frac{1}{12} \begin{pmatrix} 12-10+2 & -4+15-1 & 4-5+3 \\ -12-2+10 & 4+3-5 & -4-1+15 \\ 24+4+4 & -8-6-2 & 8+2+6 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

—

$$= \frac{1}{12}\begin{pmatrix} 4 & 10 & 2 \\ -4 & 2 & 10 \\ 32 & -16 & 16 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix} 4+20+0 & -4+0+4 & 8-10+2 \\ -4+4+0 & 4+0+20 & -8-2+10 \\ 32-32+0 & -32+0+32 & 64+16+16 \end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 96 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

which is the required diagonal form.

Example 1.47. Find e^A and 4^A if $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

Solution.

Step 1. To find the eigen values

$$s_1 = \text{sum of the main diagonal elements.}$$
$$= 3 + 3 = 6.$$
$$s_2 = |A| = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 9 - 1 = 8.$$

The characteristic equation is

$$\lambda^2 - s_1\lambda + s_2 = 0$$
$$\lambda^2 - 6\lambda + 8 = 0$$
$$(\lambda - 2)(\lambda - 4) = 0$$
$$\lambda = 2, 4.$$

The eigen values are $\lambda_1 = 2$, $\lambda_2 = 4$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0$$

$$i.e. \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

$$\left.\begin{aligned} (3-\lambda)x_1 + x_2 &= 0 \\ x_1 + (3-\lambda)x_2 &= 0. \end{aligned}\right\} \quad \text{(A)}$$

case(i) when $\lambda = 2$, (A) reduces to one single equation

$$x_1 + x_2 = 0$$

$$i.e. x_1 = -x_2$$

$$\frac{x_1}{-1} = \frac{x_2}{1}.$$

$$\therefore X_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

case(ii) when $\lambda = 4$, (A) reduces to

$$-x_1 + x_2 = 0$$

$$x_1 - x_2 = 0$$

Both the equations are reduced to one single equation

$$x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{1}.$$

$$\therefore X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

—

Step 3. To form the modal matrix P

P is obtained by taking X_1, X_2 as columns.

$$\therefore P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Step 4. To find P^{-1}

$$|P| = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1 - 1 = -2.$$

$$P^{-1} = \frac{1}{|P|} P_c^T = \frac{1}{-2} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Step 5. To find $P^{-1}AP$

$$\begin{aligned} P^{-1}AP &= \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -3+1 & -1+3 \\ 3+1 & 1+3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 2+2 & -2+2 \\ -4+4 & 4+4 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}. \end{aligned}$$

Step 6. To find e^A and 4^A

By similarity transformation

$$D = P^{-1}AP = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}.$$

—

Let $f(A) = e^A$, then $f(D) = e^D = \begin{pmatrix} e^2 & 0 \\ 0 & e^4 \end{pmatrix}$.

$$\begin{aligned}
\text{Now, } e^A &= Pf(D)P^{-1} \\
&= \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^2 & 0 \\ 0 & e^4 \end{pmatrix}\frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} -e^2 & e^4 \\ e^2 & e^4 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\
e^A &= \frac{1}{2}\begin{pmatrix} e^2 + e^4 & -e^2 + e^4 \\ -e^2 + e^4 & e^2 + e^4 \end{pmatrix}.
\end{aligned}$$

Replacing e by 4, we get

$$\begin{aligned}
4^A &= \frac{1}{2}\begin{pmatrix} 4^2 + 4^4 & -4^2 + 4^4 \\ -4^2 + 4^4 & 4^2 + 4^4 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 16 + 256 & -16 + 256 \\ -16 + 256 & 16 + 256 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 272 & 240 \\ 240 & 272 \end{pmatrix} = \begin{pmatrix} 136 & 120 \\ 120 & 136 \end{pmatrix}.
\end{aligned}$$

Exercise I(D)

1. Find a nonsingular matrix P such that $P^{-1}AP$ is in a diagonal form where A is given by

i. $\begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$ ii. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ iii. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

iv. $\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$ v. $\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}$ vi. $\begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -2 & -4 & -3 \end{pmatrix}$

—

2. Reduce the following matrices to the diagonal form using similarity transformation.

i. $\begin{pmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$ ii. $\begin{pmatrix} -1 & -\sqrt{3} & 0 \\ -\sqrt{3} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ iii. $\begin{pmatrix} 7 & -2 & -2 \\ -2 & 1 & 4 \\ -2 & 4 & 1 \end{pmatrix}$

iv. $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$ v. $\begin{pmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{pmatrix}$ vi. $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$ vii. $\begin{pmatrix} 2 & -1 & 0 \\ 9 & 4 & 6 \\ -8 & 0 & -3 \end{pmatrix}$.

3. Reduce the matrix $A = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}$ to the diagonal form and hence evaluate A^n and A^4.

4. Find A^4 if $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

5. Reduce the matrix $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$ to the diagonal form by similarity transformation. Hence find A^3.

1.6 Diagonlization by orthogonal transformation

A real square matrix A is said to be orthogonal if $AA^T = A^T A = I$.

Result. A real square matrix A is orthogonal if $A^T = A^{-1}$.

Normalized eigen vector

Let $X = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be an eigen vector corresponding to an eigen value λ. Then the normalized eigen vector of X is defined as $\dfrac{X}{\sqrt{a^2+b^2+c^2}}$.

Example

If $X = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ is an eigen vector, then its normalised eigen vector is $\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}$.

Diagonalization by orthogonal transformation

A real square matrix A is said to be orthogonally diagonalizable, if there exists an orthogonal matrix N such that $N^{-1}AN = D \Rightarrow N^TAN = D$. This transformation which transforms A into D is called an orthogonal transformation or orthogonal reduction.

Symmetric Matrices

A real square matrix A is said to be symmetric if $A^T = A$.

Working rule for orthogonal reduction

Step 1. Find the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Step 2. Find eigen vectors $X_1, X_2, \ldots, X_n$ which are pairwise orthogonal.

Step 3. Form the normalised modal matrix N with the normalised eigen vectors as columns.

Step 4. Find N^T and $N^TAN = D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$.

Definition 1.1. If $X_i, X_j, i, j = 1, 2, \ldots, n$ are orthogonal, then $X_i^T X_j = 0, i \neq j$.

Properties of eigen values of orthogonal matrices

Property 1.9. The eigen values of a real symmetric matrix are real.

—

Proof. Let λ (real or complex) be an eigen value of A.

Let X be the eigen vector corresponding to the eigen value λ.

$$\therefore AX = \lambda X. \tag{1}$$

Let $\overline{X}$ be the Complex conjugate of X.

Premultiply (1) by $\overline{X}^T$ we get

$$\overline{X}^T AX = \lambda \overline{X}^T X$$

$$\overline{\overline{X}^T AX} = \overline{\lambda \overline{X}^T X}$$

i.e., $X^T \overline{A}\overline{X} = \overline{\lambda} X^T \overline{X}$.

Since A is a real symmetric matrix, $\overline{A} = A$ & $A^T = A$.

$$\therefore \ X^T A\overline{X} = \overline{\lambda} X^T \overline{X}.$$

Taking transpose on both sides we get

$$\left(X^T A\overline{X}\right)^T = \left(\overline{\lambda} X^T \overline{X}\right)^T$$

$$\overline{X}^T A^T X = \overline{X}^T X \overline{\lambda}^T = \overline{\lambda}\overline{X}^T X$$

$$\overline{X}^T AX = \overline{\lambda}\overline{X}^T X$$

$$\overline{X}^T \lambda X = \overline{\lambda}\overline{X}^T X$$

$$\lambda \overline{X}^T X = \overline{\lambda}\overline{X}^T X.$$

Now X is an $n \times 1$ matrix and $\overline{X}^T$ is a $1 \times n$ matrix.

Hence, $\overline{X}^T X$ is a 1×1 matrix, which is a positive real.

$$\Rightarrow \ \lambda = \overline{\lambda}.$$

$\Rightarrow$ λ is a real number. □

Property 1.10. The eigen vectors corresponding to distinct eigen values of a real symmetric matrix are orthogonal. [Dec.2002]

Proof. By property 1.9, we know that, if A is a real symmetric matrix, then its eigen values are real.

Let λ_1 and λ_2 be two real eigen values such that $\lambda_1 \neq \lambda_2$.

—

Let X_1 and X_2 be the eigen vectors corresponding to the eigen values λ_1 and λ_2 respectively.

$$\therefore \ AX_1 = \lambda_1 X_1 \tag{1}$$

$$AX_2 = \lambda_2 X_2. \tag{2}$$

Premultiplying (1) by X_2^T we get

$$X_2^T AX_1 = X_2^T \lambda_1 X_1 = \lambda_1 X_2^T X_1.$$

Taking transpose both sides we get

$$\left(X_2^T AX_1\right)^T = \left(\lambda_1 X_2^T X_1\right)^T$$

$$\Rightarrow \ X_1^T A^T X_2 = \lambda_1 X_1^T X_2$$

$$\text{i.e., } X_1^T AX_2 = \lambda_1 X_1^T X_2 \quad [\text{since, } A \text{ is symmetric, } A^T = A]. \tag{3}$$

Premultiplying (2) by X_1^T we get

$$X_1^T AX_2 = X_1^T \lambda_2 X_2 = \lambda_2 X_1^T X_2. \tag{4}$$

From (3) and (4) we get

$$\lambda_1 X_1^T X_2 = \lambda_2 X_1^T X_2$$

$$\text{i.e., } \lambda_1 X_1^T X_2 - \lambda_2 X_1^T X_2 = 0$$

$$(\lambda_1 - \lambda_2) X_1^T X_2 = 0.$$

$$\text{Since } \lambda_1 \neq \lambda_2, \ \lambda_1 - \lambda_2 \neq 0.$$

$$\therefore \ X_1^T X_2 = 0$$

$$\Rightarrow X_1 \text{ and } X_2 \text{ are orthogonal.}$$

□

Property 1.11. If λ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also its eigen value.

—

Proof. Let A be an orthogonal matrix.

$$\Rightarrow A^T = A^{-1}. \tag{1}$$

We know that A and A^T have the same eigen values.

Also we know that if λ is an eigen value of A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

From (1), we get $\frac{1}{\lambda}$ is an eigen value of A^T.

$\Rightarrow \frac{1}{\lambda}$ is an eigen value of A. □

Results

1. Diagonalization of orthogonal transformation is possible only for a real symmetric martix.
2. If A is a real symmetric matrix, then the eigen vectors of A will be not only linearly independent but also pairwise orthogonal.
3. If A is orthogonal, then A^T is orthogonal.
4. If A is an orthogonal matrix, then $|A| = \pm 1$.
5. The eigen values of an orthogonal matrix are of magnitude 1.

Worked Examples

Example 1.48. Prove that $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is orthogonal. [Jan 2009]

Solution. $|A| = \cos^2\theta + \sin^2\theta = 1$, which is nonsingular.

$$AA^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$\therefore A$ is orthogonal.

Example 1.49. Show that $A = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}$ is orthogonal.

Solution. Given $A = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}$

$$A^T = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}$$

$$AA^T = \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$\therefore$ A is orthogonal.

Example 1.50. If A is an orthogonal matrix, prove that A^{-1} is also orthogonal.

Solution. Since A is orthogonal, $A^T = A^{-1}$.

Let $A^T = A^{-1} = B$.

Now $B^T = (A^{-1})^T = (A^T)^{-1} = B^{-1}$.

$\therefore$ B is orthogonal.

i.e., A^{-1} is orthogonal.

Example 1.51. Prove that if A and B are orthogonal matrices, then AB is orthogonal.

Solution. Given A & B are orthogonal.

$$\therefore\ A^T = A^{-1} \text{ and } B^T = B^{-1}.$$

Now, $(AB)^T = B^T A^T = B^{-1}A^{-1} = (AB)^{-1}$ $\Rightarrow AB$ is orthogonal.

—

Example 1.52. Diagonalise the symmetric matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$ by an orthogonal transformation. [Jan 2004]

Solution. A is a real symmetric matrix.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$.

Now s_1 = sum of the main diagonal elements

$$= 2 + 1 + 1 = 4.$$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 1 - 4 + 2 - 1 + 2 - 1$$

$$= -3 + 1 + 1 = -1.$$

$$s_3 = |A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{vmatrix}$$

$$= 2(-3) - 1(-1) - 1(-1)$$

$$= -6 + 1 + 1 = -4.$$

The characteristic equation is $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$.

$$\lambda = 1 \text{ is a root.}$$

By synthetic division we get,

$$\begin{array}{c|cccc} 1 & 1 & -4 & -1 & 4 \\ & 0 & 1 & -3 & -4 \\ \hline & 1 & -3 & -4 & \boxed{0} \end{array}$$

$$\lambda^2 - 3\lambda - 4 = 0.$$

$$(\lambda+1)(\lambda-4)=0.$$

$\therefore$ The characteristic equation becomes $(\lambda-1)(\lambda+1)(\lambda-4)=0$.

$\therefore$ The eigen values are $-1, 1, 4$.

Step 2. To find the eigen vectors

Let $X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to λ. Then,

$$(A-\lambda I)X=0.$$

$$\begin{pmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1}$$

When $\lambda=-1$, (1) becomes

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0.$$

The equations are reduced to two equations

$3x_1+x_2-x_3=0.$

$x_1+2x_2-2x_3=0.$

By the rule of cross multiplication we get,

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & \times & -1 & \times & 3 & \times & 1 \\ 2 & & -2 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{0}=\frac{x_2}{5}=\frac{x_3}{5}\implies x_1=0, x_2=1, x_3=1.$$

$\therefore X_1=\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$

When $\lambda = 1$ (1) become

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & -2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

$$\text{i.e., } \ x_1 + x_2 - x_3 = 0.$$

$$x_1 - 2x_3 = 0 \Rightarrow x_1 = 2x_3 \Rightarrow \frac{x_1}{2} = \frac{x_3}{1} \Longrightarrow x_1 = 2, x_3 = 1.$$

$$-x_1 - 2x_2 = 0 \Rightarrow x_1 = -2x_2 \Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} \Longrightarrow x_1 = 2, x_2 = -1.$$

$$\therefore \ x_1 = 2, x_2 = -1, x_3 = 1$$

$$\therefore X_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

When $\lambda = 4$, (1) become

$$\begin{pmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

i.e., $-2x_1 + x_2 - x_3 = 0 \Rightarrow 2x_1 - x_2 + x_3 = 0.$

$x_1 - 3x_2 - 2x_3 = 0.$

$-x_1 - 2x_2 - 3x_3 = 0 \Rightarrow x_1 + 2x_2 + 3x_3 = 0.$

All the three equations are different.

Taking the first two equations and solving by the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -1 & \times & 1 & \times & 2 & \times & -1 \\ -3 & & -2 & & 1 & & -3 \end{array}$$

$$\text{i.e., } \frac{x_1}{5} = \frac{x_2}{5} = \frac{x_3}{-5}.$$

$$\Longrightarrow x_1 = 1, x_2 = 1, x_3 = -1.$$

$\therefore X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

Since A is symmetric and all the eigen values are different X_1, X_2, X_3 are pairwise orthogonal.

Step 3. To form the modal matrix N

The normalized eigen vectors are $\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}$.

The modal matrix N is formed with the normalized eigen vectors as columns.

i.e., $N = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$.

Step 4. To find $N^T AN$

$$N^T AN = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} = D.$$

Example 1.53. Diagonalise the matrix $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ by an orthogonal reduction.

Solution. Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

A is a real symmetric matrix.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements of A.

$= 1 + 5 + 1 = 7.$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}$$

$= 5 - 1 + 1 - 9 + 5 - 1$

$= 4 - 8 + 4 = 0.$

$$s_3 = |A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$

$= 1(5 - 1) - 1(1 - 3) + 3(1 - 15)$

$= 4 + 2 - 42$

$= -36.$

The characteristic equation is $\lambda^3 - 7\lambda^2 + 36 = 0.$

$\lambda = -2$ is a root.

By synthetic division we have

$$\begin{array}{r|rrrr} -2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$\lambda^2 - 9\lambda + 18 = 0$

$(\lambda - 3)(\lambda - 6) = 0$

$\lambda = 3, \lambda = 6.$

The eigen values are $-2, 3, 6.$

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0$$

$$\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \tag{1}$$

When $\lambda = -2$, (1) becomes

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations are reduced to

$$3x_1 + x_2 + 3x_3 = 0$$

$$x_1 + 7x_2 + x_3 = 0.$$

By the rule of cross multiplication we obtain

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & \times & 3 & \times & 3 & \times & 1 \\ 7 & & 1 & & 1 & & 7 \end{array}$$

$$\frac{x_1}{1-21} = \frac{x_2}{3-3} = \frac{x_3}{21-1}$$

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$$

$$x_1 = 1, x_2 = 0, x_3 = -1.$$

$$\therefore \; X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

When $\lambda = 3$, (1) becomes

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations become

$$-2x_1 + x_2 + 3x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$
$$3x_1 + x_2 - 2x_3 = 0.$$

Taking the first two equations and applying the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & \times & 3 & \times & -2 & \times & 1 \\ 2 & & 1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{1-6} = \frac{x_2}{3+2} = \frac{x_3}{-4-1}$$
$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5}$$
$$\therefore \ x_1 = 1, x_2 = -1, x_3 = 1.$$
$$\therefore \ X_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

When $\lambda = 6$, (1) becomes

$$\begin{pmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations become

$$-5x_1 + x_2 + 3x_3 = 0$$
$$x_1 - x_2 + x_3 = 0$$
$$3x_1 + x_2 - 5x_3 = 0.$$

Taking the first two equations and applying rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & \times & 3 & \times & -5 & \times & 1 \\ -1 & & 1 & & 1 & & -1 \end{array}$$

$$\frac{x_1}{1+3} = \frac{x_2}{3+5} = \frac{x_3}{5-1}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

$$\therefore\ x_1 = 1, x_2 = 2, x_3 = 1.$$

$$\therefore\ X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Since A is symmetric and all the eigen values are different x_1, x_2, x_3 are pairwise orthogonal.

Step 3. To form the modal matrix N

The normalized eigen vectors are $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}.$

The modal matrix N is formed with the normalized eigen vectors as columns.

$$\therefore\ N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

Step 4. To find $N^T AN$

$$N^T AN = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} -\sqrt{2} & 0 & \sqrt{2} \\ \sqrt{3} & -\sqrt{3} & \sqrt{3} \\ \sqrt{6} & 2\sqrt{6} & \sqrt{6} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

—

Example 1.54. Diagonalise the matrix $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ by means of an orthogonal transformation. [Jan 2004]

Solution. A is a symmetric matrix.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now s_1 = sum of the main diagonal elements

$= 3 + 3 + 3 = 9.$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$

$= 9 - 1 + 9 - 1 + 9 - 1 = 24.$

$$s_3 = |A| = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$= 3(9 - 1) - 1(3 + 1) + 1(-1 - 3)$

$= 24 - 4 - 4 = 16.$

$\therefore$ The characteristic equation is $\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0.$

$$\lambda = 1 \text{ is a root.}$$

By synthetic division we have

$$\begin{array}{c|cccc} 1 & 1 & -9 & 24 & -16 \\ & 0 & 1 & -8 & 16 \\ \hline & 1 & -8 & 16 & \boxed{0} \end{array}$$

$$\Longrightarrow \lambda^2 - 8\lambda + 16 = 0.$$

$$(\lambda - 4)(\lambda - 4) = 0.$$

The characteristic equation becomes $(\lambda - 1)(\lambda - 4)(\lambda - 4) = 0$.

$\therefore$ The eigen values are $1, 4, 4$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to λ. Therefore,

$$(A - \lambda I)X = 0.$$

$$\begin{pmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \tag{1}$$

When $\lambda = 1$, (1) becomes

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations become

$$2x_1 + x_2 + x_3 = 0.$$

$$x_1 + 2x_2 - x_3 = 0.$$

$$x_1 - x_2 + 2x_3 = 0.$$

Taking the first two equations and solving for x_1, x_2, x_3 by the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & \times & 1 & \times & 2 & \times & 1 \\ 2 & & -1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{-1-2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\implies \frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3}.$$

$$\implies x_1 = -1, x_2 = 1, x_3 = 1.$$

$\therefore X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

When $\lambda = 4$, (1) becomes

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations are reduced to one single equation

$$x_1 - x_2 - x_3 = 0. \tag{1}$$

Put $x_3 = 0 \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2 \implies x_1 = 1, x_2 = 1.$

$\therefore X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be orthogonal to X_2.

By the condition of orthogonality, $X_3^T X_2 = 0$.

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\implies a + b = 0 \Rightarrow a = -b.$$

Also X_3 should satisfy (1).

$$\therefore a - b - c = 0 \Rightarrow -2b - c = 0 \Rightarrow 2b = -c$$

$$\Rightarrow \frac{b}{-1} = \frac{c}{2} \Rightarrow b = -1, c = 2, a = 1.$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Step 3. To find the modal matrix N

The normalised eigen vectors are

$$\begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}.$$

The normalised modal matrix is formed by the normalised eigen vectors as columns.

$$\therefore \; N = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}, N^T = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix}$$

Step 4. To find $N^T AN$

$$N^T AN = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{4}{\sqrt{2}} & \frac{4}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{2}} & \frac{-4}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{8}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

which is the required diagonal form.

Computation of powers of a square matrix

Let A be the given square matrix. Diagonalising A, we obtain,

$$D = N^T AN = N^{-1}AN.$$

$$D^2 = D.D = (N^{-1}AN)(N^{-1}AN) = N^{-1}A(NN^{-1})AN$$

—

$$= N^{-1}AAN = N^{-1}A^2N$$

$$D^3 = N^{-1}A^3N$$

$$\text{In general, } D^r = N^{-1}A^rN$$

Where $D^r = \begin{pmatrix} \lambda_1^r & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \lambda_n^r \end{pmatrix}$.

Premultiplying by N and postmultiplying by N^{-1} we get,

$$A^r = ND^rN^{-1}.$$

Example 1.55. Diagonalise the matrix $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Hence find A^3. [Jan 2006]

Solution. A is a symmetric matrix.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now s_1 = sum of the main diagonal elements

$= 2 + 3 + 2 = 7.$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}$$

$$= 6 - 0 + 4 - 1 + 6 - 0$$

$$= 6 + 3 + 6 = 15.$$

$$s_3 = |A| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= 2(6-0) + 1(0-3)$$

$$= 12 - 3 = 9.$$

The characteristic equation is $\lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$.

$$\lambda = 1 \text{ is a root.}$$

By synthetic division, we get

$$\begin{array}{c|cccc} 1 & 1 & -7 & 15 & -9 \\ & 0 & 1 & -6 & 9 \\ \hline & 1 & -6 & 9 & \multicolumn{1}{|c}{0} \end{array}$$

$$\lambda^2 - 6\lambda + 9 = 0.$$

$$(\lambda - 3)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 3, 3.$$

The eigen values are $1, 3, 3$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0$$

$$\text{i.e., } \begin{pmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \quad (1)$$

When $\lambda = 1$, (1) becomes

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations become

$$x_1 + x_3 = 0$$
$$2x_2 = 0.$$

From the last equation we get $x_2 = 0$.

Also $x_1 = -x_3$

$$\frac{x_1}{1} = \frac{x_3}{-1}$$
$$\therefore\ x_1 = 1, x_3 = -1.$$
$$\therefore\ X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

When $\lambda = 3$, (1) becomes

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations are reduced to one single equation

$$x_1 - x_3 = 0 \tag{2}$$
$$\Rightarrow\ x_1 = x_3$$

i.e., $x_1 = 1, x_3 = 1$ and x_2 = any value = 0 (say)

$$\therefore\ X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be orthogonal to X_2.

$$\therefore\ X_3^T X_2 = 0.$$

$$\text{i.e., } \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0 \qquad \Longrightarrow a + c = 0. \tag{3}$$

—

i.e., Also X_3 will satisfy by (2).

Solving (2) & (3) we get $a = 0, c = 0$. Choose $b = 1$.

We get the corresponding eigen vector $X_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Step 3. To find the modal matrix N

The normalised eigen vectors are $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

The modal matrix N is formed with the normalised eigen vectors as columns.

$$\therefore \; N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

Step 4. To find $N^T AN$

$$N^T AN = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & 0 & \frac{3}{\sqrt{2}} \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = D.$$

Step 5. To find A^3

$$D^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{pmatrix}$$

$$\text{Now, } A^3 = ND^3N^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 14 & 0 & 13 \\ 0 & 27 & 0 \\ 13 & 0 & 14 \end{pmatrix}.$$

—

Example 1.56. Diagonalise the matrix $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$. Hence find the value of A^4. [Jan 2013]

Solution. A is a symmetric matrix.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements.

$= 8 + 7 + 3 = 18.$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$= 21 - 16 + 24 - 4 + 56 - 36$$

$$= 5 + 20 + 20 = 45.$$

$$s_3 = |A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8(21 - 16) + 6(-18 + 8) + 2(24 - 14)$$

$$= 40 - 60 + 20 = 0$$

$\therefore$ The characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda(\lambda - 3)(\lambda - 15) = 0$$

$$\lambda = 0, 3, 15.$$

The eigen values are $0, 3, 15$

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore \ (A - \lambda I)X = 0$$

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = 0$, (1) becomes

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations become

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 3x_3 = 0.$$

Taking the first two equations and applying the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -6 & & 2 & & 8 & & -6 \\ 7 & & -4 & & -6 & & 7 \end{array}$$

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$x_1 = 1, x_2 = 2, x_3 = 2.$$

$$\therefore \ X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

When $\lambda = 3$, (1) becomes

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$5x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 4x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 = 0$$

i.e., $x_1 = 2x_2$

$$\frac{x_1}{2} = \frac{x_2}{1}$$

$\therefore\ x_1 = 2, x_2 = 1.$

Substituting in the first equation we get

$$10 - 6 + 2x_3 = 0$$
$$4 + 2x_3 = 0$$
$$2x_3 = -4$$
$$x_3 = -2.$$

$$\therefore\ X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$

When $\lambda = 15$, (1) becomes

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations become

$$-7x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 - 8x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 - 12x_3 = 0.$$

Taking the first two equations and applying the rule of cross multiplication, we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -6 & \times & 2 & \times & -7 & \times & -6 \\ 8 & & 4 & & 6 & & 8 \end{array}$$

$$\frac{x_1}{-24-16} = \frac{x_2}{12+28} = \frac{x_3}{-56+36}$$
$$\frac{x_1}{-40} = \frac{x_2}{40} = \frac{x_3}{-20}$$
$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$
$$\therefore \; x_1 = 2, x_2 = -2, x_3 = 1.$$

$\therefore \; X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$

Since A is a symmetric matrix and all the eigen values are different, $X_1, \; X_2$ & X_3 are pairwise orthogonal.

Step 3. To find the modal matrix N

The normalized eigen vectors are $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{pmatrix}$

The modal matrix N is formed with the normalized eigen vectors as columns.

$$\therefore \; N = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

Step 4. To find N^TAN

$$N^TAN = \frac{1}{9}\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{pmatrix}\begin{pmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix}0 & 0 & 0\\ 6 & 3 & -6\\ 30 & -30 & 15\end{pmatrix}\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix}0 & 0 & 0\\ 0 & 27 & 0\\ 0 & 0 & 135\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 15\end{pmatrix} = D$$

Step 5. To find A^4

$$D = \begin{pmatrix}0 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 15\end{pmatrix}.$$

$$D^4 = \begin{pmatrix}0 & 0 & 0\\ 0 & 3^4 & 0\\ 0 & 0 & 15^4\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 81 & 0\\ 0 & 0 & 50625\end{pmatrix}.$$

Now $A^4 = ND^4N^T$

$$= \frac{1}{9}\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 81 & 0\\ 0 & 0 & 50625\end{pmatrix}\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix}0 & 162 & 101250\\ 0 & 81 & -101250\\ 0 & -162 & 50625\end{pmatrix}\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix}202824 & -202338 & 100926\\ -202338 & 202581 & -101412\\ 100926 & -101412 & 50949\end{pmatrix} = \begin{pmatrix}22536 & -22482 & 11214\\ -22482 & 22509 & -11268\\ 11214 & -11268 & 5661\end{pmatrix}.$$

—

Example 1.57. The eigen vectors of a 3×3 real symmetric matrix A corresponding to the eigen values $2, 3, 6$ are $[1\ 0\ -1]^T, [1\ 1\ 1]^T, [-1\ 2\ -1]^T$ respectively. Find A.

[Jan 2012]

Solution. Given, A is symmetric and the eigen values are different.

$\therefore$ The eigen vectors are pairwise disjoint.

$$\text{Now, } N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \end{pmatrix}.$$

$\therefore$ By the orthogonal transformation, we obtain

$$N^TAN = D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

$$\text{Now, } A = NDN^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}.$$

Exercise I(E)

1. Prove that the matrix $B = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is orthogonal.

2. Show that the matrix $\frac{1}{7}\begin{pmatrix} 6 & -3 & 2 \\ -3 & -2 & 6 \\ 2 & 6 & 3 \end{pmatrix}$ is orthogonal.

—

3. Prove that the following matrices are orthogonal.

(i) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (ii) $\frac{1}{9}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$.

4. Diagonalise the matrix $\begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}$.

5. Reduce $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ to the diagonal form by an orthogonal reduction.

6. Reduce the matrix $A = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}$ to the diagonal form.

7. By an orthogonal transformation, diagonalise the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$.

8. Reduce the matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ to diagonal form by an orthogonal reduction. Hence find A^3.

9. The eigen vectors corresponding to the eigen values $1, 2, 3$ of the symmetric matrix A are $[1\ -1\ 0]^T, [0\ 0\ 1]^T$ and $[1\ 1\ 0]^T$ respectively. Find A.

10. $[1\ \ -1]^T$ and $[1\ \ 1]^T$ are the eigen vectors of the symmetric matrix A corresponding to the eigen values 0 and 2 respectively. Find A.

—

1.7 Quadratic Form

Definition. A homogeneous polynomial of second degree in any number of variables is called a quadratic form.

Example.

(1) $x^2 + 2xy + 2y^2$ is a quadratic form in 2 variables.

(2) $ax^2 + 2hxy + by^2 + cz^2 + 2gyz + 2fzx$ is a quadratic form in 3 variables.

(3) $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gyz + 2fzx + 2\ell xw + 2myw + 2nzw$ is a quadratic form in 4 variables.

Definition. The general quadratic form in n variables $x_1, x_2, \ldots, x_n$ is $\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j$ where a_{ij}'s are real numbers such that $a_{ij} = a_{ji}$ for all $i, j = 1, 2, 3, \ldots, n$.

Notation. Usually the quadratic form is denoted by Q.

$$\therefore\ Q = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j.$$

Matrix form of Q

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$ where $a_{ij} = a_{ji}$. Then, A is a symmetric matrix.

Now the quadratic form $Q = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j$ can be written as $Q = X^TAX$. A is called the matrix of the quadratic form.

Example. Consider the quadratic form $x^2 + 4xy + y^2$.

i.e., $x^2 + 2xy + 2yx + y^2$.

The quadratic form is $[x\ y]\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Hence $X = \begin{pmatrix} x \\ y \end{pmatrix}, A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

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1.8 Canonical form of Q

A quadratic form of Q which contains only the square terms of the variables is said to be in canonical form.

Example. $x^2 + 2y^2, x^2 - y^2, x^2 + y^2 + z^2, x_1^2 + x_2^2 + x_3^2 + 4x_4^2$ are in canonical forms.

Reduction of Q to canonical form by orthogonal transformation

Let $Q = X^T AX$ be a quadratic form in n variables $x_1, x_2, \ldots, x_n$ and $A = [a_{ij}]$ be the symmetric matrix of order n of the quadratic form. We shall reduce A to the diagonal form by an orthogonal transformation. Let $X = NY$ where N is the normalized modal matrix of A so that $N^T AN = D$, where $D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of A.

$$\text{We have } Q = X^T AX = (NY)^T A(NY) = (y^T N^Y)A(NY)$$
$$= Y^T (N^T AN)Y = Y^T DY.$$

If $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ then $Q = [y_1\ y_2\ \ldots\ y_n] \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

which is the required canonical form.

Note

(i) In the above canonical form, some λ_i's may be positive or negative or zero.

(ii) In the canonical form, the coefficients are the eigen values of the matrix A.

(iii) A quadratic form is said to be real if the elements of the symmetric matrix are real.

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Definition

If A is the matrix of the quadratic form Q in the variables $x_1, x_2, \ldots, x_n$, then the rank of Q is equal to the rank of A. The rank of A is denoted by $\rho(A)$. If rank of $A < n$, where n is the number of variables or order of A, then $|A| = 0$ and Q is called a singular form.

Definition

Let $Q = X^T AX$ be a quadratic form in n variables $x_1, x_2, \ldots, x_n$, where $X = [x_1\ x_2\ \ldots\ x_n]^T$ and A is the matrix of the quadratic form.

(i) The number of positive and negative eigen values of A is called the *rank* of the quadratic form. It is denoted by r.

(ii) The number of positive eigen values of A is defined as the *index* of the quadratic form. It is also denoted by p. It is equal to the number of positive terms in the canonical form.

(iii) The difference between the number of positive and negative eigen values of A is called the *signature* of the quadratic form. It is denoted by s. It is also equal to the difference between the number of positive and negative terms in the canonical form. i.e., $s = p - (r - p) = 2p - r$ where $\rho(A) = r$.

(iv) Q is said to be *positive definite* if all the n eigen values of A are positive. i.e., $r = n, p = r$.

Ex. $y_1^2 + y_2^2 + \cdots + y_n^2$ is positive definite.

(v) Q is said to be *negative definite* if all the n eigen values of A are negative. i.e.,$r = n, p = 0$.

Ex. $-y_1^2 - y_2^2 - \cdots - y_n^2$ is negative definite.

(vi) Q is said to be *positive semi definite* if all the n eigen values are ≥ 0 with atleast one eigen value $= 0$.

i.e., if $r < n$ and $p = r$.

Ex. $y_1^2 + y_2^2 + y_4^2 + \cdots + y_r^2$ is positive semi definite.

(vii) Q is said to be *negative semi definite* if all the n eigen values of A are ≤ 0 with atleast one value $= 0$.

i.e., if $r < n$ and $p = 0$.

Ex. $-y_1^2 - y_2^2 - y_5^2 - \cdots - y_r^2, (r < n)$ is negative semi definite.

(viii) Q is said to be *indefinite* if A has positive and negative eigen values.

Ex. $y_1^2 - y_2^2 + y_3^2 - y_4^2 - y_5^2 + \cdots + y_n^2$ is indefinite.

Result. The nature of the quadratic form can also be found without finding the eigen values of A or without reducing to canonical form but by using the principal minors of A.

Definition. Let $Q = X^T AX$ be a quadratic form in n variables $x_1, x_2, \ldots, x_n$ and let the matrix of the form A be

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}.$$

Let $D_1 = |a_{11}|, D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ etc.

Finally $D_n = |A|$. The determinants $D_1, D_2, \ldots, D_n$ are called the principal minors of A. The quadratic form Q is said to be

(i) Positive definite if $D_i > 0$ for all $i = 1, 2, \ldots, n$.

(ii) Negative definite if $(-1)^i D_i > 0$ for all $i = 1, 2, \ldots, n$. i.e.,$D_1, D_3, D_5, \ldots$ are negative and $D_2, D_4, D_6, \ldots$ are positive.

(iii) Positive semidefinite if $D_i \geq 0$ for all $i = 1, 2, 3, \ldots, n$ and atleast one $D_i = 0$.

(iv) Negative semidefinite if $(-1)^i D_i \geq 0$ for all $i = 1, 2, 3, \ldots, n$ and atleast one $D_i = 0$.

(v) Indefinite in all other cases.

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Law of inertia of a quadratic form

The index of a real quadratic form is invariant under a real non singular transformation. This property is called the law of inertia of the quadratic form.

Worked Examples

Example 1.58. Write down the matrix of the quadratic form $2x^2 + 3y^2 + 6xy$.

[Jan 2001]

Solution. The given quadratic form is in two variables.

Hence, the matrix of the quadratic form is a 2×2 symmetric matrix. Let A be the required matrix. Then

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

a_{11} = coefficient of $x^2 = 2$.

$a_{12} = a_{21} = \frac{1}{2}$ coefficient of $xy = \frac{1}{2} \times 6 = 3$.

a_{22} = coefficient of $y^2 = 3$.

$\therefore\ A = \begin{pmatrix} 2 & 3 \\ 3 & 3 \end{pmatrix}$

Example 1.59. Write down the matrix of the quadratic form

$2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$. [Jan 2002]

Solution. The given quadratic form contains 3 variables. Hence, the matrix A of the quadratic form is a 3×3 symmetric matrix. The matrix of the quadratic form is $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

a_{11} = Coeff. of $x_1^2 = 2$.

a_{22} = Coeff. of $x_2^2 = -2$.

a_{33} = Coeff. of $x_3^2 = 4$.

$a_{12} = a_{21} = \frac{1}{2}$ Coeff. of $x_1x_2 = \frac{1}{2}2 = 1$.

$a_{13} = a_{31} = \frac{1}{2}$ Coeff. of $x_1x_3 = \frac{1}{2}(-6) = -3$.

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$a_{23} = a_{32} = \frac{1}{2}$ Coeff. of $x_2x_3 = \frac{1}{2}6 = 3$.

$$\therefore A = \begin{pmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{pmatrix}.$$

Example 1.60. Write down the matrix of the quadratic form $2x^2+8z^2+4xy+10xz-2yz$ [Jun 2013, Dec 2001].

Solution. The given quadratic form contain 3 variables. Hence the matrix A of the quadratic form is a 3×3 matrix.

$$\text{Hence } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\begin{aligned} \text{Now, } a_{11} &= \text{Coeff.of } x^2 = 2 \\ a_{12} = a_{21} &= \frac{1}{2}\times \text{Coeff.of } xy = \frac{1}{2}\times 4 = 2. \\ a_{13} = a_{31} &= \frac{1}{2}\times \text{Coeff.of } xz = \frac{1}{2}\times 10 = 5. \\ a_{22} &= \text{Coeff.of } y^2 = 0 \\ a_{23} = a_{32} &= \frac{1}{2}\times \text{Coeff.of } yz = \frac{1}{2}\times(-2) = -1. \\ a_{33} &= \text{Coeff.of } z^2 = 8. \end{aligned}$$

$$\therefore\ A = \begin{pmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{pmatrix}.$$

Example 1.61. Write down the quadratic form of the matrix $\begin{pmatrix} 2 & 4 & 5 \\ 4 & 3 & 1 \\ 5 & 1 & 1 \end{pmatrix}$.[Jan 2003]

Solution. The given matrix is a 3×3 symmetric matrix. Hence, the quadratic form is in 3 variables. The required quadratic form must be of the form

$$Q = a_{11}x^2 + 2a_{12}xy + 2a_{13}xz + a_{22}y^2 + 2a_{23}yz + a_3z^2.$$

Comparing the given matrix with $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ we get,

$a_{11} = 2, a_{12} = 4, a_{13} = 5, a_{22} = 3, a_{23} = 1, a_{33} = 1.$

$\therefore Q = 2x^2 + 8xy + 10xz + 3y^2 + 2yz + z^2.$

$= 2x^2 + 3y^2 + z^2 + 8xy + 10xz + 2yz.$

Example 1.62. Write down the quadratic form corresponding to the matrix $\begin{pmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{pmatrix}$.

Solution. Since the given matrix is a 3×3 symmetric matrix, the quadratic form must be in 3 variables x, y, z. Let Q be of the required quadratic form

$\therefore \; Q = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz.$

Comparing the given matrix with $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ we have

$a_{11} = 2, a_{22} = 0, a_{33} = 8, a_{12} = 2, a_{13} = 5, a_{23} = -1.$

$\therefore \; Q = 2x^2 + 8z^2 + 4xy - 2yz + 10xz.$

Example 1.63. Write down the quadratic form corresponding to the matrix $\begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & -2 \end{pmatrix}$. [Jan 2013]

Solution. Since the given matrix is a 3×3 symmetric matrix, the quadratic form must be in 3 variables x, y, z. Let Q be the required quadratic form.

$\therefore \; Q = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz.$

Comparing the given matrix with $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ we obtain

$$a_{11} = 2, a_{12} = 0, a_{13} = -2, a_{22} = 2, a_{23} = 1, a_{33} = -2.$$

$$\therefore \; Q = 2x^2 + 2y^2 - 2z^2 - 4xz + 2yz.$$

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Example 1.64. Determine the nature of the quadratic form $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2$.
[Jan 2003]

Solution. Since it contains only square terms it is of the canonical form. The canonical form contains 3 variables but the RHS expression contains only two terms. The coefficients of the canonical form are the eigen values of the matrix of the quadratic form. Hence, $\lambda_1 = 1, \lambda_2 = 2$ and $\lambda_3 = 0$. Hence, the given quadratic form is positive semi-definite.

Example 1.65. Find the nature of the quadratic form $2x^2 + 2xy + 3y^2$.

Solution. Let us form the matrix A of the given quadratic form. Since the Q.F contains only two variables, A is a 2×2 matrix.

$\therefore\ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. From the given Q.E, we have

$$a_{11} = \text{Coeff.of } x^2 = 2.$$
$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } xy = \frac{1}{2} \times 2 = 1$$
$$a_{22} = \text{Coeff.of } y^2 = 3.$$
$$\therefore\ A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

Let us analyse the nature of the Q.F with the help of the principal minors.

$$D_1 = |2| = 2.$$

$$D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5.$$

Since all $D_i > 0$, the quadratic form is positive definite.

Example 1.66. Find the nature of the quadratic form $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx$.
[Jan 2010]

Solution. Since the given Q.F is in 3 variables, the matrix A of the Q.E is a 3×3 matrix.

$$\therefore \; A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

From the given Q.F we have

$$a_{11} = \text{Coeff.of } x^2 = 6.$$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } xy = \frac{1}{2} \times (-4) = -2$$

$$a_{13} = a_{31} = \frac{1}{2} \times \text{Coeff.of } xz = \frac{1}{2} \times 4 = 2$$

$$a_{22} = \text{Coeff.of } y^2 = 3.$$

$$a_{23} = a_{32} = \frac{1}{2} \times \text{Coeff.of } yz = \frac{1}{2} \times (-2) = -1.$$

$$a_{33} = \text{Coeff.of } z^2 = 3$$

$$\therefore \; A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}.$$

The principal minors are $D_1 = 6 > 0$.

$$D_2 = \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} = 18 - 4 = 14 > 0.$$

$D_3 = |A| = 6(8) + 2(-4) + 2(-4) = 48 - 8 - 8 = 32 > 0.$

Since D_1, D_2, D_3 are positive, the Q.F is positive definite.

Example 1.67. Determine λ so that the quadratic form $\lambda(x^2+y^2+z^2)+2xy-2yz+2zx$ is positive definite.

Solution. If A is the matrix of the given quadratic form, then $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\text{Now, } a_{11} = \text{Coeff.of } x^2 = \lambda$$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } xy = \frac{1}{2} \times 2 = 1$$

$$a_{13} = a_{31} = \frac{1}{2} \times \text{Coeff.of } xz = \frac{1}{2} \times 2 = 1$$

$$a_{22} = \text{Coeff.of } y^2 = \lambda$$
$$a_{23} = a_{32} = \frac{1}{2} \times \text{Coeff.of } yz = \frac{1}{2} \times (-2) = -1$$
$$a_{33} = \text{Coeff.of } z^2 = \lambda$$
$$\therefore\ A = \begin{pmatrix} \lambda & 1 & 1 \\ 1 & \lambda & -1 \\ 1 & -1 & \lambda \end{pmatrix}.$$

Let us find out the principal minors D_1, D_2 and D_3.

$$D_1 = \lambda.$$
$$D_2 = \lambda^2 - 1 = (\lambda + 1)(\lambda - 1).$$
$$D_3 = \lambda(\lambda^2 - 1) - 1(\lambda + 1) + 1(-1 - \lambda)$$
$$= \lambda(\lambda - 1)(\lambda + 1) - (\lambda + 1) - (\lambda + 1) = (\lambda + 1)(\lambda^2 - \lambda - 1 - 1)$$
$$= (\lambda + 1)(\lambda^2 - \lambda - 2) = (\lambda + 1)(\lambda - 2)(\lambda + 1) = (\lambda + 1)^2(\lambda - 2).$$

Since the Q.F is positive definite, we have $D_1 > 0, D_2 > 0$ and $D_3 > 0$.

$$D_1 > 0 \Longrightarrow \lambda > 0$$

$$D_2 > 0 \Longrightarrow (\lambda + 1)(\lambda - 1) > 0.$$

$$\Longrightarrow \lambda + 1 > 0 \text{ and } \lambda - 1 > 0.$$

$$\Longrightarrow \lambda > -1 \text{ and } \lambda > 1.$$

OR

$$\lambda + 1 < 0 \text{ and } \lambda - 1 < 0$$

$$\Longrightarrow \lambda < -1 \text{ and } \lambda < 1.$$

$$D_3 > 0 \Longrightarrow (\lambda + 1)^2(\lambda - 2) > 0$$

$$\Longrightarrow \lambda > 2 \text{ [since } (\lambda + 1)^2 \text{ is always } > 0].$$

—

We have to consider the following cases.

case(i). $\lambda > 0, \lambda > -1, \lambda > 1, \lambda > 2$.

or

case(ii). $\lambda > 0, \lambda < -1, \lambda < 1, \lambda > 2$. In the above cases the value of λ that satisfies the above conditions is $\lambda > 2$.

Example 1.68. Show that the Q.F $ax_1^2 - 2bx_1x_2 + cx_2^2$ is positive definite if $a > 0$ and $ac - b^2 > 0$.

Solution. The given Q.F. is in 2 variables. Hence, the matrix A of the Q.F is a 2×2 matrix.

Let A be $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

$$\text{Now, } a_{11} = \text{Coeff.of } x_1^2 = a.$$
$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } x_1x_2 = \frac{1}{2} \times (-2b) = -b.$$
$$a_{22} = \text{Coeff.of } x_2^2 = c.$$

$$\therefore\ A = \begin{pmatrix} a & -b \\ -b & c \end{pmatrix}.$$

We have $D_1 = a, D_2 = ac - b^2$.

Given $a > 0$ and $ac - b^2 > 0$.

$\implies D_1 > 0$ and $D_2 > 0$.

$\therefore$ The Q.F is positive definite.

Example 1.69. Find the index, signature, rank and nature of the quadratic form in 3 variables $x^2 + 2y^2 - 3z^2$. [May 2011]

Solution. Since the given Q.F contains only square terms, it is in the canonical form in 3 variables.

The eigen values are the coefficients of x^2, y^2 and z^2.

$$\therefore\ \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -3$$

—

No. of positive eigen values = 2.

$\therefore$ Index = 2.

No. of negative eigen values = 1.

Signature = Difference of the number of positive and negative eigen values

$= 2 - 1 = 1.$

Rank = 3 = Number of positive and negative eigen values.

Nature of the Q.F is indefinite.

Example 1.70. If the quadratic form $ax^2+2bxy+cy^2$ is positive definite (or negative definite), then prove that the quadratic equation $ax^2 + 2bx + c = 0$ has imaginary roots.

Solution. Since the given Q.F contains two variables, the matrix of the Q.F. A is a 2×2 matrix.

Let A be $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

$$\text{Now, } a_{11} = \text{Coeff.of } x^2 = a.$$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } xy = \frac{1}{2} \times 2b = b.$$

$$a_{22} = \text{Coeff.of } y^2 = c.$$

$\therefore \; A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

The principal minors are $D_1 = a, D_2 = ac - b^2$.

If the quadratic form is positive definite then $D_1 > 0$ and $D_2 > 0$.

$$\implies a > 0 \text{ and } ac - b^2 > 0.$$

$$\implies a > 0 \text{ and } b^2 - ac < 0. \qquad (1)$$

If the quadratic form is negative definite, then $(-1)^i D_i > 0$ for all i.

$$\implies -D_1 > 0 \text{ and } D_2 > 0$$

$$\implies -a > 0 \text{ and } ac - b^2 > 0.$$

—

$$\Longrightarrow a < 0 \text{ and } b^2 - ac < 0. \tag{2}$$

Consider the quadratic equation

$$ax^2 + 2bx + c = 0. \tag{3}$$

The discriminant of the quadratic equation is

$$\Delta = 4b^2 - 4ac = 4(b^2 - ac) < 0.$$

$\therefore$ The roots of (3) are imaginary.

Example 1.71. Reduce the quadratic form $x_1^2 + 5x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 6x_3x_1$ to canonical form through an orthogonal transformation. [Jan 2006]

Solution.

Step 1. To form the matrix of the Q.F

Let A be the matrix of the Q.F.

Since the given Q.F is in 3 variables, A must be a 3×3 matrix.

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

$$\begin{aligned}
\text{Now, } a_{11} &= \text{Coeff.of } x_1^2 = 1 \\
a_{12} = a_{21} &= \frac{1}{2} \times \text{Coeff.of } x_1x_2 = \frac{1}{2} \times 2 = 1 \\
a_{13} = a_{31} &= \frac{1}{2} \times \text{Coeff.of } x_1x_3 = \frac{1}{2} \times 6 = 3 \\
a_{22} &= \text{Coeff.of } x_2^2 = 5. \\
a_{23} = a_{32} &= \frac{1}{2} \times \text{Coeff.of } x_2x_3 = \frac{1}{2} \times 2 = 1. \\
a_{33} &= \text{Coeff.of } x_3^2 = 1.
\end{aligned}$$

$\therefore \; A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

—

Step 2. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$= 1 + 5 + 1 = 7.$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}$$

$= 5 - 1 + 1 - 9 + 5 - 1$

$= 4 - 8 + 4 = 0.$

$$s_3 = |A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$

$= 1(5 - 1) - 1(1 - 3) + 3(1 - 15)$

$= 4 + 2 - 42$

$= -36.$

$\therefore$ The characteristic equation is $\lambda^3 - 7\lambda^2 + 36 = 0.$

$\lambda = -2$ is a root.

By synthetic division, we get

$$\begin{array}{r|rrrr} -2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$$\lambda^2 - 9\lambda + 18 = 0.$$

$$(\lambda - 3)(\lambda - 6) = 0.$$

The characteristic equation becomes $(\lambda + 2)(\lambda - 3)(\lambda - 6) = 0.$

The eigen values are $-2, 3, 6.$

—

Step 3.To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to the eigen value λ.

$$\therefore\ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \tag{1}$$

When $\lambda = -2$, (1) becomes

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above two equations are reduced to two equations

$$3x_1 + x_2 + 3x_3 = 0$$

$$x_1 + 7x_2 + x_3 = 0.$$

Solving the two equations by the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & \times & 3 & \times & 3 & \times & 1 \\ 7 & & 1 & & 1 & & 7 \end{array}$$

$$\frac{x_1}{1-21} = \frac{x_2}{3-3} = \frac{x_3}{21-1}$$

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$$

$$\implies x_1 = 1, x_2 = 0, x_3 = -1.$$

$$\therefore\ X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

When $\lambda = 3$, (1) become $\begin{pmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The above equations are reduced to

$$-2x_1 + x_2 + 3x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

$$3x_1 + x_2 - 2x_3 = 0.$$

From the first two equations by the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & \times & 3 & \times & -2 & \times & 1 \\ 2 & & 1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{1-6} = \frac{x_2}{3+2} = \frac{x_3}{-4-1}$$
$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Longrightarrow x_1 = 1, x_2 = -1, x_3 = 1.$$

$$\therefore\ X_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

When $\lambda = 6$, (1) becomes $\begin{pmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The above equations are reduced to

$$-5x_1 + x_2 + 3x_3 = 0$$

$$x_1 - x_2 + x_3 = 0$$

$$3x_1 + x_2 - 5x_3 = 0.$$

By the rule of cross multiplication, from the first two equations we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & \times & 3 & \times & -5 & \times & 1 \\ -1 & & 1 & & 1 & & -1 \end{array}$$

$$\frac{x_1}{1+3} = \frac{x_2}{3+5} = \frac{x_3}{5-1}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \Longrightarrow x_1 = 1, x_2 = 2, x_3 = 1.$$

$$X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Step 4. To form the modal matrix

Since the eigen values are different, the eigen vectors are mutually orthogonal.

The normalized eigen vectors are $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}.$

The modal matrix N is formed with columns as the normalized eigen vectors.

$$\therefore\ N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Step 5. To find $N^T AN$

$$N^T AN = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} -\sqrt{2} & 0 & \sqrt{2} \\ \sqrt{3} & -\sqrt{3} & \sqrt{3} \\ \sqrt{6} & 2\sqrt{6} & \sqrt{6} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} = D.$$

Step 6. Reduction to Canonical form

Let $X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

The given Q.F is $Q = X^T AX$.

$$= (NY)^T A(NY)$$
$$= Y^T N^T ANY$$
$$= Y^T DY$$
$$= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$
$$= -2y_1^2 + 3y_2^2 + 6y_3^2$$

which is canonical.

Example 1.72. Reduce the quadratic form $8x^2 + 7y^2 + 3z^2 - 12xy + 4xz - 8yz$ to the canonical form by an orthogonal transformation. Find one set of values of x, y, z (not all zero) which will make the Quadratic form zero. [Jan 2012]

Solution.

Step 1. To find the matrix of the Q.F

Since the given Q.F contains 3 variables, the matrix of the Q.F A is a 3×3 matrix.

—

$$\therefore\ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Now, $a_{11} = \text{Coeff.of } x_1^2 = 8$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } xy = \frac{1}{2} \times (-12) = -6$$

$$a_{13} = a_{31} = \frac{1}{2} \times \text{Coeff.of } xz = \frac{1}{2} \times 4 = 2$$

$$a_{22} = \text{Coeff.of } y^2 = 7.$$

$$a_{23} = a_{32} = \frac{1}{2} \times \text{Coeff.of } yz = \frac{1}{2} \times (-8) = -4.$$

$$a_{33} = \text{Coeff.of } z^2 = 3.$$

$$\therefore\ A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}.$$

Step 2. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$= 8 + 7 + 3 = 18.$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$= 21 - 16 + 24 - 4 + 56 - 36$$

$$= 5 + 20 + 20 = 45.$$

$$s_3 = |A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8(21 - 16) + 6(-18 + 8) + 2(24 - 14)$$

$$= 40 - 60 + 20 = 0.$$

The characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda(\lambda - 3)(\lambda - 15) = 0$$

$$\lambda = 0, 3, 15.$$

Step 3. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value λ.

$$\therefore \ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \quad (1)$$

When $\lambda = 0$, (1) becomes

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 3x_3 = 0.$$

Taking the first two equations and applying the rule of cross multiplication, we obtain

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -6 & \times & 2 & \times & 8 & \times & -6 \\ 7 & & -4 & & -6 & & 7 \end{array}$$

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$
$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$
$$\implies x_1 = 1, x_2 = 2, x_3 = 2.$$
$$\therefore \; X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

When $\lambda = 3$, (1) become $\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The above equations become

$$5x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 4x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 = 0.$$

From the last equation we get

$$2x_1 = 4x_2$$
$$\frac{x_1}{4} = \frac{x_2}{2}$$

$\Rightarrow x_1 = 2, x_2 = 1.$

Substituting in the first equation we get

$$10 - 6 + 2x_3 = 0$$
$$4 + 2x_3 = 0$$
$$2x_3 = -4$$
$$x_3 = -2.$$

—

$$\therefore\ X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$

When $\lambda = 15$, (1) becomes $\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The above equations become

$$-7x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 - 8x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 - 12x_3 = 0.$$

Taking the first two equations and applying the rule of cross multiplication, we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -6 & \times & 2 & \times & -7 & \times & -6 \\ -8 & & -4 & & -6 & & -8 \end{array}$$

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20} \Longrightarrow x_1 = 2, x_2 = -2, x_3 = 1.$$

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

Step 4. To form the modal matrix

Since all the eigen values are different, X_1, X_2, X_3 are pairwise orthogonal.

The modal matrix N is formed with the normalized eigen vectors are columns.

The normalized eigen vectors are $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{pmatrix}$.

$$\therefore \ N = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}.$$

Step 5. To find $N^T AN$

$$N^T AN = \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 0 & 0 & 0 \\ 6 & 3 & -6 \\ 30 & -30 & 15 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 135 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix} = D.$$

Step 6. Reduction to Canonical form

Let $X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

$$\begin{aligned} \text{The given Q.F is } Q &= X^T AX. \\ &= (NY)^T A(NY) \\ &= Y^T N^T ANY \\ &= Y^T DY \\ &= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \end{aligned}$$

$$= 3y_2^2 + 15y_3^2.$$

which is canonical.

we have $X = NY$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$x = \frac{1}{3}(y_1 + 2y_2 + 2y_3)$$

$$y = \frac{1}{3}(2y_1 + y_2 - 2y_3)$$

$$z = \frac{1}{3}(2y_1 - 2y_2 + y_3).$$

Equating the quadratic form to zero we get

$$3y_2^2 + 15y_3^2 = 0.$$

$$\Rightarrow \; y_2 = 0, y_3 = 0, \text{ and assume } y_1 = 3.$$

Substituting these values in x, y, z we obtain $x = 1, y = 2, z = 2$.

These set of values will make the quadratic form zero.

Example 1.73. Reduce the quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ to the canonical form through an orthogonal transformation and hence show that it is positive semidefinite. Also give a nonzero set of values (x_1, x_2, x_3) which makes the quadratic form to zero. [Jan 2009]

Solution.

Step 1. To find the matrix of the Q.F

Since the given Q.F contains 3 variables, the matrix of the Q.F A is a 3×3 matrix.

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

—

$$\text{Now, } a_{11} = \text{Coeff.of } x_1^2 = 1$$
$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } x_1x_2 = \frac{1}{2} \times (-2) = -1$$
$$a_{13} = a_{31} = \frac{1}{2} \times \text{Coeff.of } x_1x_3 = \frac{1}{2} \times 0 = 0$$
$$a_{22} = \text{Coeff.of } x_2^2 = 2.$$
$$a_{23} = a_{32} = \frac{1}{2} \times \text{Coeff.of } x_2x_3 = \frac{1}{2} \times 2 = 1$$
$$a_{33} = \text{Coeff.of } x_3^2 = 1.$$

$$\therefore\ A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Step 2. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

$$\text{Now, } s_1 = \text{sum of the main diagonal elements}$$
$$= 1 + 2 + 1 = 4.$$
$$s_2 = \text{sum of the minors of the main diagonal elements.}$$
$$= \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$
$$= 2 - 1 + 1 - 0 + 2 - 1$$
$$= 3.$$
$$s_3 = |A| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$
$$= 1(2 - 1) + 1(-1 - 0) + 0$$
$$= 1 - 1 = 0.$$

—

$\therefore$ The characteristic equation is $\lambda^3 - 4\lambda^2 + 3\lambda = 0$

$$\lambda(\lambda^2 - 4\lambda + 3) = 0$$

$$\lambda(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda = 0, 1, 3.$$

The eigen values are $0, 1, 3$.

Step 3. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value λ.

$$\therefore \ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \tag{1}$$

When $\lambda = 0$, (1) becomes

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations are

$$x_1 - x_2 = 0$$

$$-x_1 + 2x_2 + x_3 = 0$$

$$x_2 + x_3 = 0.$$

From the first equation we get

$$x_1 = x_2 \implies x_1 = 1, \ x_2 = 1.$$

From the last equation we get

$$x_2 = -x_3$$
$$\Rightarrow \; x_3 = -1.$$

$$\therefore \; X_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

When $\lambda = 1$, (1) becomes

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations are

$$x_2 = 0$$
$$-x_1 + x_2 + x_3 = 0$$

Substituting $x_2 = 0$, the second equation becomes

$$x_1 = x_3 \;\Rightarrow\; x_1 = 1, \; x_3 = 1.$$

$$\therefore \; X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

When $\lambda = 3$, (1) become $\begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The equations are

$$2x_1 + x_2 = 0$$
$$x_1 + x_2 - x_3 = 0$$
$$x_2 - 2x_3 = 0.$$

From the first equation we have

$$2x_1 = -x_2$$
$$\frac{x_1}{-1} = \frac{x_2}{2}$$

—

$\Rightarrow x_1 = -1, x_2 = 2.$

From the last equation we have

$$x_2 = 2x_3$$
$$\frac{x_2}{2} = \frac{x_3}{1}$$
$$\Longrightarrow x_2 = 2, x_3 = 1.$$

$$\therefore\ X_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Step 4. To find the modal matrix

Since all the eigen values are different, X_1, X_2, X_3 are pairwise orthogonal.

The normalized eigen vectors are $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}.$

The modal matrix N is formed with the normalized eigen vectors as columns.

$$\therefore\ N = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}.$$

Step 5. To find N^TAN

$$N^TAN = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-3}{\sqrt{6}} & \frac{6}{\sqrt{6}} & \frac{3}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} = D.$$

—

Step 6. Reduction to Canonical form

Let $X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

The given Q.F is $Q = X^T AX$.

$$\begin{aligned}
&= (NY)^T A(NY) \\
&= Y^T N^T ANY \\
&= Y^T DY \\
&= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\
&= y_2^2 + 3y_3^2.
\end{aligned}$$

which is canonical.

Step 7. To find the nature of the Q.F

Since two eigen values are positive and one eigen value is 0, the given quadratic form is positive semidefinite.

Step 8. To find the value of x_1, x_2, x_3 for which the $Q.F = 0$

Taking $y_2^2 + 3y_3^2 = 0 \Longrightarrow y_2 = 0, y_3 = 0$.

Now $X = NY$ is reduced to the equations.

$$x_1 = \frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 - \frac{1}{\sqrt{6}} y_3$$

$$x_2 = \frac{1}{\sqrt{3}} y_1 + \frac{2}{\sqrt{6}} y_3$$

$$x_3 = \frac{-1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 + \frac{1}{\sqrt{6}} y_3.$$

Taking $y_1 = \sqrt{3}$, we get $x_1 = 1, x_2 = 1, x_3 = -1$.

$\therefore x_1 = 1, x_2 = 1, x_3 = -1$ make the given quadratic form zero.

—

Example 1.74. Reduce the Q.F $2x_1^2 + 6x_2^2 + 2x_3^2 + 8x_1x_3$ to canonical form. State the type of the canonical form. [Jan 2008]

Solution.

Step 1. To find the matrix of the Q.F

Let A be the matrix of the Q.F.

Since the given Q.F contains 3 variables, A must be a 3×3 matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Now, $a_{11} = \text{Coeff.of } x_1^2 = 2.$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } x_1x_2 = 0.$$

$$a_{13} = a_{31} = \frac{1}{2} \times \text{Coeff.of } x_1x_3 = \frac{1}{2} \times 8 = 4$$

$a_{22} = \text{Coeff.of } x_2^2 = 6.$

$$a_{23} = a_{32} = \frac{1}{2} \times \text{Coeff.of } x_2x_3 = \frac{1}{2} \times 0 = 0.$$

$a_{33} = \text{Coeff.of } x_3^2 = 2.$

$$\therefore\ A = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix}.$$

Step 2. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$= 2 + 6 + 2 = 10.$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 6 \end{vmatrix}$$

$$= 12 - 0 + 4 - 16 + 12 - 0$$

$$= 12 - 12 + 12 = 12.$$

$$s_3 = |A| = \begin{vmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{vmatrix}$$

$$= 2(12 - 0) - 0 + 4(0 - 24)$$

$$= 24 - 96$$

$$= -72.$$

$\therefore$ The characteristic equation is $\lambda^3 - 10\lambda^2 + 12\lambda + 72 = 0$

$$\lambda = -2 \text{ is a root.}$$

By synthetic division we get

$$\begin{array}{r|rrrr} -2 & 1 & -10 & 12 & 72 \\ & 0 & -2 & 24 & -72 \\ \hline & 1 & -12 & 36 & \boxed{0} \end{array}$$

$$\lambda^2 - 12\lambda + 36 = 0$$

$$(\lambda - 6)(\lambda - 6) = 0$$

$$\lambda = 6, 6.$$

The eigen values are $-2, 6, 6$.

Step 3. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value λ.

$$\therefore \ (A - \lambda I)X = 0.$$

—

$$\begin{pmatrix} 2-\lambda & 0 & 4 \\ 0 & 6-\lambda & 0 \\ 4 & 0 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \tag{1}$$

When $\lambda = -2$, (1) becomes

$$\begin{pmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations are reduced to

$$4x_1 + 4x_3 = 0$$

$$8x_2 = 0$$

$$\therefore\ x_2 = 0.$$

$$\text{and } 4x_1 = -4x_3$$

$$\frac{x_1}{4} = \frac{x_3}{-4}$$

$$\Rightarrow\ x_1 = 1, x_3 = -1.$$

$$\therefore\ X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

When $\lambda = 6$, (1) becomes

$$\begin{pmatrix} -4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations are reduced to one single equation

$$-4x_1 + 4x_3 = 0$$

$$\text{i.e., } 4x_1 = 4x_3$$

$$x_1 = x_3.$$

—

$$\Rightarrow \; x_1 = 1, x_3 = 1, x_2 = \text{any value. Take } x_2 = 0.$$

$X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ orthogonal to X_2.

$\therefore \; X_3^T X_2 = 0$

$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0$

$a + c = 0$ (2)

X_3 also satisfy

$-4a + 4c = 0$

i.e., $a - c = 0$ (3)

Solving (2) and (3) we get

$a = 0, c = 0$ and $b =$any value.

Take $b = 1$.

$\therefore \; X_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$

Step 4. To find the modal matrix

The normalized eigen vectors are $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$

The modal matrix is formed with the normalised eigen vectors as columns.

$\therefore \; N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$

—

Step 5. To find N^TAN

$$N^TAN = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\sqrt{2} & 0 & \sqrt{2} \\ 3\sqrt{2} & 0 & 3\sqrt{2} \\ 0 & 6 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = D.$$

Step 6. Reduction to Canonical form

Let $X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

The given Q.F is $Q = X^TAX$.

$$= (NY)^TA(NY)$$

$$= Y^TN^TANY$$

$$= Y^TDY$$

$$= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= -2y_1^2 + 6y_2^2 + 6y_3^2$$

which is canonical.

Step 7. To find the nature

Since two of the eigen values are positive and one eigen value is negative, the given Q.F is indefinite.

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Example 1.75. Reduce $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$ into a canonical form by an orthogonal reduction and find the rank, signature, index and nature of the quadratic form. [Jan 2015, Jan 2005]

Solution.

Step 1. To find the matrix of the Q.F

Since the given Q.F contains 3 variables, the matrix of the Q.F A is a 3×3 matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

$$\text{Now, } a_{11} = \text{Coeff.of } x^2 = 6$$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } xy = \frac{1}{2} \times (-4) = -2.$$

$$a_{13} = a_{31} = \frac{1}{2} \times \text{Coeff.of } xz = \frac{1}{2} \times 4 = 2$$

$$a_{22} = \text{Coeff.of } y^2 = 3.$$

$$a_{23} = a_{32} = \frac{1}{2} \times \text{Coeff.of } yz = \frac{1}{2} \times (-2) = -1.$$

$$a_{33} = \text{Coeff.of } z^2 = 3.$$

$$\therefore\ A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}.$$

Step 2. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

$$\text{Now, } s_1 = \text{sum of the main diagonal elements}$$

$$= 6 + 3 + 3 = 12.$$

$$s_2 = \text{sum of the minors of the main diagonal elements.}$$

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= 9 - 1 + 18 - 4 + 18 - 4$$

$$= 8 + 14 + 14 = 36.$$

$$s_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9 - 1) + 2(-6 + 2) + 2(2 - 6)$$

$$= 48 - 8 - 8$$

$$= 32.$$

$\therefore$ The characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$.

$\lambda = 2$ is a root. By synthetic division we get

$$\begin{array}{c|cccc} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda - 2)(\lambda - 8) = 0.$$

The characteristic equation becomes $(\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$.

The eigen values are $2, 2, 8$.

Step 3. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to the eigen value λ.

$$\therefore \ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

$$\left.\begin{aligned} (6-\lambda)x_1 - 2x_2 + 2x_3 &= 0 \\ -2x_1 + (3-\lambda)x_2 - x_3 &= 0 \\ 2x_1 - x_2 + (3-\lambda)x_3 &= 0. \end{aligned}\right\} \quad (1)$$

When $\lambda = 8$ (1) $\Rightarrow -2x_1 - 2x_2 + 2x_3 = 0 \Rightarrow x_1 + x_2 - x_3 = 0.$

$$-2x_1 - 5x_2 - x_3 = 0 \Rightarrow 2x_1 + 5x_2 + x_3 = 0.$$

$$2x_1 - x_2 - 5x_3 = 0 \Rightarrow 2x_1 - x_2 - 5x_3 = 0.$$

From the first two equations using the rule of cross multiplication we get

$$\frac{x_1}{6} = \frac{x_2}{-3} = \frac{x_3}{3} \Rightarrow x_1 = 2, x_2 = -1, x_3 = 1.$$

$$\therefore\ X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

When $\lambda = 2$, (1) is reduced to a single equation, $2x_1 - x_2 + x_3 = 0$.
Choose $x_3 = 0 \Rightarrow 2x_1 - x_2 = 0 \Rightarrow 2x_1 = x_2 \Rightarrow \dfrac{x_1}{1} = \dfrac{x_2}{2} \Rightarrow x_1 = 1, x_2 = 2.$

$$\therefore\ X_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be orthogonal to X_2.

$$\Longrightarrow\ X_2^T X_3 = 0 \Rightarrow a + 2b = 0. \quad (2)$$

$$X_3 \text{ also satisfy } 2a - b + c = 0. \quad (3)$$

(2) $\Rightarrow a = -2b \Longrightarrow \dfrac{a}{-2} = \dfrac{b}{1}.$
(3) $\Rightarrow -4b - b + c = 0 \Rightarrow 5b = c \Rightarrow \dfrac{b}{1} = \dfrac{c}{5}.$

—

Combining the above two equations we obtain $\frac{a}{-2} = \frac{b}{1} = \frac{c}{5}$.

Hence, $b = 1, c = 5, a = -2$.

$$\therefore \; X_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}.$$

Step 4. To find the modal matrix

The normalised eigen vectors are
$\left[\frac{2}{\sqrt{6}} \; \frac{-1}{\sqrt{6}} \; \frac{1}{\sqrt{6}}\right]^T, \left[\frac{1}{\sqrt{5}} \; \frac{2}{\sqrt{5}} \; 0\right]^T, \left[\frac{-2}{\sqrt{30}} \; \frac{1}{\sqrt{30}} \; \frac{5}{\sqrt{30}}\right]^T$

$$\therefore \; N = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{pmatrix}.$$

Step5. To find N^TAN

$$N^TAN = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{-2}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{16}{\sqrt{6}} & \frac{-8}{\sqrt{6}} & \frac{8}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-8}{\sqrt{5}} & 0 \\ \frac{-4}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{10}{\sqrt{30}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D.$$

Step 6. Reduction to canonical form

Let $X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

—

The given Q.F. is $Q = X^T AX$

$$= (NY)^T ANY$$

$$= Y^T N^T ANY$$

$$= Y^T DY$$

$$Y^T DY = [y_1\ y_2\ y_3]\begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= [y_1\ y_2\ y_3]\begin{pmatrix} 8y_1 \\ 2y_2 \\ 2y_3 \end{pmatrix}$$

$$= 8y_1^2 + 2y_2^2 + 2y_3^2$$

which is the canonical form.

Rank of the Q.F = 3.

Index = 3.

Signature = 3.

Since all the eigen values are positive, the Q.F is positive definite.

Example 1.76. Reduce the quadratic form $2x_1x_2 + 2x_2x_3 + 2x_3x_1$ in to canonical form. Discuss also its nature. [Jan 2013].

Solution.

Step 1. To find the matrix of the quadratic form

Since given Q.F. contains 3 variables, the matrix of the Q.F. A is a 3×3 matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

—

2 ANNA UNIVERSITY SOLVED QUESTION PAPERS

2.1 Unit I MATRICES

2.1.1 APRIL/ MAY 2015 (R 2013)

Part A

1. Find the sum and product of all the eigen values of $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$.

Solution. Sum of the eigen values $=$ trace of A

$$= 8 + 7 + 3 = 18.$$

Product of the eigen values $= |A|$

$$= \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8(21 - 16) + 6(-18 + 8) + 2(24 - 14)$$

$$= 40 - 60 + 20 = 0.$$

2. Give the nature of the quadratic form whose matrix is $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

Solution. The given matrix is a diagonal matrix whose elements along the leading diagonal are the eigen values. We notice that all the eigen values are negative. Hence, the given quadratic form is negative definite.

Part B

11. (a) (i) Find the eigen values and eigen vectors of $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$.

Solution. Let $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$.

Step 1. To find the eigen values

$$s_1 = tr(A) = 6 + 3 + 3 = 12.$$

s_2 = sum of the minors of the main diogonal elements.

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= 9 - 1 + 18 - 4 + 18 - 4$$

$$= 8 + 14 + 14 = 36.$$

$$s_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(8) + 2(-4) + 2(-4)$$

$$= 48 - 8 - 8 = 32.$$

The characteristic equation is

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

$$i.e., \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0.$$

$$\lambda = 2 \text{ is a root.}$$

By synthetic division we have

$$\begin{array}{c|cccc} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

Hence, the characteristic equation becomes

$$(\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$$

$$(\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$$

$$\lambda = 2, 2, 8.$$

$\therefore$ The eigen values are $\lambda_1 = 2$, $\lambda_2 = 2$, $\lambda_3 = 8$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

$$\left.\begin{aligned} (6-\lambda)x_1 - 2x_2 + 2x_3 &= 0 \\ -2x_1 + (3-\lambda)x_2 - x_3 &= 0 \\ 2x_1 - x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\}. \qquad \text{(A)}$$

case(i) when $\lambda = 2$, (A) reduces to

$$4x_1 - 2x_2 + 2x_3 = 0$$
$$-2x_1 + x_2 - x_3 = 0$$
$$2x_1 - x_2 + x_3 = 0.$$

The above three equations are reduced to the single equation

$$2x_1 - x_2 + x_3 = 0.$$

Since $\lambda = 2$ is a repeated root, we have to find two eigen vectors by assigning a particular value to two variables, one at a time.

Assigning $x_3 = 0$, we obtain

$$2x_1 = x_2$$
$$\frac{x_1}{1} = \frac{x_2}{2}$$
$$\therefore X_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

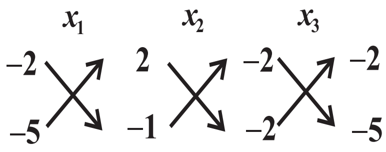
Assigning $x_2 = 0$, we obtain

$$2x_1 = -x_3$$
$$\frac{x_1}{-1} = \frac{x_3}{2}.$$
$$\therefore X_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

case(ii) when $\lambda = 8$, (A) reduces to

$$-2x_1 - 2x_2 + 2x_3 = 0 \qquad (1)$$
$$-2x_1 - 5x_2 - x_3 = 0 \qquad (2)$$
$$2x_1 - x_2 - 5x_3 = 0. \qquad (3)$$

Since all the three equations are different, we can consider the the first two equations and solve for $x_1, x_2, \&, x_3$ by the method of cross multiplication



$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$
$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$
$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$
$$\therefore x_1 = 2, x_2 = -1, x_3 = 1.$$

x_1, x_2, x_3 satisfy (3).

$$\therefore X_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

11. (a) (ii) If $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$, verify Cayley-Hamilton theorem and hence find A^{-1}.

Solution.

Step 1. To find the characteristic equation

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0. \qquad (1)$$

Now, s_1 = sum of the diagonal elements

$= 3 + 5 + 3 = 11.$

s_2 = sum of the minors of the elements of the main diagonal

$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$$= 15 - 1 + 9 - 1 + 15 - 1 = 14 + 8 + 14 = 36.$$

$$s_3 = |A| \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 3(15-1) + 1(-3+1) + 1(1-5) = 3 \times 14 - 2 - 4$$

$$= 42 - 6 = 36.$$

The characteristic equation is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$.

Step 2. Verification of Cayley Hamilton theorem.

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

$$A^2 = A \cdot A$$

$$= \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 9+1+1 & -3-5-1 & 3+1+3 \\ -3-5-1 & 1+25+1 & -1-5-3 \\ 3+1+3 & -1-5-3 & 1+1+9 \end{pmatrix} = \begin{pmatrix} 11 & -9 & 7 \\ -9 & 27 & -9 \\ 7 & -9 & 11 \end{pmatrix}$$

$$A^3 = A^2 \times A$$

$$= \begin{pmatrix} 11 & -9 & 7 \\ -9 & 27 & -9 \\ 7 & -9 & 11 \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 33+9+7 & -11-45-7 & 11+9+21 \\ -27-27-9 & 9+135+9 & -9-27-27 \\ 21+9+11 & -7-45-11 & 7+9+33 \end{pmatrix} = \begin{pmatrix} 49 & -63 & 41 \\ -63 & 153 & -63 \\ 41 & -63 & 49 \end{pmatrix}$$

$$A^3 - 11A^2 + 36A - 36I$$

$$= \begin{pmatrix} 49 & -63 & 41 \\ -63 & 153 & -63 \\ 41 & -63 & 49 \end{pmatrix} - \begin{pmatrix} 121 & -99 & 77 \\ -99 & 297 & -99 \\ 77 & -99 & 121 \end{pmatrix} + \begin{pmatrix} 108 & -36 & 36 \\ -36 & 180 & -36 \\ 36 & -36 & 108 \end{pmatrix}$$

$$- \begin{pmatrix} 36 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 36 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

$\therefore$ Cayley Hamilton theorem is verified.

Step 3. To find A^{-1}

By Cayley Hamilton theorem, we have $A^3 - 11A^2 + 36A - 36I = 0$.

Multiplying by A^{-1} we get,

$$A^2 - 11A + 36I - 36A^{-1} = 0.$$

$$\Rightarrow 36A^{-1} = A^2 - 11A + 36I.$$

$$\Rightarrow A^{-1} = \frac{1}{36}[A^2 - 11A + 36I].$$

$$\therefore A^{-1} = \frac{1}{36}\left(\begin{pmatrix} 11 & -9 & 7 \\ -9 & 27 & -9 \\ 7 & -9 & 11 \end{pmatrix} - \begin{pmatrix} 33 & -11 & 11 \\ -11 & 55 & -11 \\ 11 & -11 & 33 \end{pmatrix} + \begin{pmatrix} 36 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 36 \end{pmatrix}\right).$$

$$= \frac{1}{36}\begin{pmatrix} 11-33+36 & -9+11+0 & 7-11+0 \\ -9+11+0 & 27-55+36 & -9+11+0 \\ 7-11+0 & -9+11+0 & 11-33+36 \end{pmatrix}$$

$$A^{-1} = \frac{1}{36}\begin{pmatrix} 14 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 14 \end{pmatrix}.$$

11. (b) Reduce the quadratic form $x^2 + 5y^2 + z^2 + 2xy + 2yz + 6zx$ into canonical form and hence find its rank.

Solution.

Step 1. To form the matrix of the Q.F

Let A be the matrix of the Q.F.

Since the given Q.F is in 3 variables, A must be a 3×3 matrix.

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Now, $a_{11} = \text{Coeff.of } x^2 = 1$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } xy = \frac{1}{2} \times 2 = 1$$

$$a_{13} = a_{31} = \frac{1}{2} \times \text{Coeff.of } xz = \frac{1}{2} \times 6 = 3$$

$$a_{22} = \text{Coeff.of } y^2 = 5.$$

$$a_{23} = a_{32} = \frac{1}{2} \times \text{Coeff.of } yz = \frac{1}{2} \times 2 = 1.$$

$$a_{33} = \text{Coeff.of } z^2 = 1.$$

$$\therefore\ A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}.$$

Step 2. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$= 1 + 5 + 1 = 7.$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}$$

$$= 5 - 1 + 1 - 9 + 5 - 1 = 4 - 8 + 4 = 0.$$

$$s_3 = |A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$

$$= 1(5-1) - 1(1-3) + 3(1-15) = 4 + 2 - 42 = -36.$$

$\therefore$ The characteristic equation is $\lambda^3 - 7\lambda^2 + 36 = 0$.

$\lambda = -2$ is a root.

By synthetic division, we get

$$\begin{array}{r|rrrr} -2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$$\lambda^2 - 9\lambda + 18 = 0.$$

$$(\lambda - 3)(\lambda - 6) = 0.$$

The characteristic equation becomes $(\lambda + 2)(\lambda - 3)(\lambda - 6) = 0$.

The eigen values are $-2, 3, 6$.

Step 3.To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to the eigen value λ.

$$\therefore\ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = -2$, (1) becomes

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above two equations are reduced to two equations

$$3x_1 + x_2 + 3x_3 = 0$$

$$x_1 + 7x_2 + x_3 = 0.$$

Solving the two equations by the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 3 & & 3 & & 1 \\ 7 & & 1 & & 1 & & 7 \end{array}$$

$$\frac{x_1}{1-21} = \frac{x_2}{3-3} = \frac{x_3}{21-1}$$

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$$

$$\implies x_1 = 1, x_2 = 0, x_3 = -1.$$

$$\therefore\ X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

When $\lambda = 3$, (1) become $\begin{pmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The above equations are reduced to

$$-2x_1 + x_2 + 3x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

$$3x_1 + x_2 - 2x_3 = 0.$$

From the first two equations by the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 3 & & -2 & & 1 \\ 2 & & 1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{1-6} = \frac{x_2}{3+2} = \frac{x_3}{-4-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \implies x_1 = 1, x_2 = -1, x_3 = 1.$$

$$\therefore\ X_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

When $\lambda = 6$, (1) becomes $\begin{pmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The above equations are reduced to

$$-5x_1 + x_2 + 3x_3 = 0$$

$$x_1 - x_2 + x_3 = 0$$

$$3x_1 + x_2 - 5x_3 = 0.$$

By the rule of cross multiplication, from the first two equations we get

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 1 & & 3 & & -5 & & 1 \\ -1 & \times & 1 & \times & 1 & \times & -1 \end{array}$$

$$\frac{x_1}{1+3} = \frac{x_2}{3+5} = \frac{x_3}{5-1}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \implies x_1 = 1, x_2 = 2, x_3 = 1.$$

$$X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Step 4. To form the modal matrix

Since the eigen values are different, the eigen vectors are mutually orthogonal.

The normalized eigen vectors are $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$.

The modal matrix N is formed with columns as the normalized eigen vectors.

$$\therefore \quad N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Step 5. To find N^TAN

$$N^TAN = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} -\sqrt{2} & 0 & \sqrt{2} \\ \sqrt{3} & -\sqrt{3} & \sqrt{3} \\ \sqrt{6} & 2\sqrt{6} & \sqrt{6} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} = D.$$

Step 6. Reduction to Canonical form

Let $X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

The given Q.F is $Q = X^TAX$.

$$= (NY)^TA(NY)$$

$$= Y^TN^TANY$$

$$= Y^T DY$$

$$= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= -2y_1^2 + 3y_2^2 + 6y_3^2$$

which is canonical.

Since two of the eigen values are positive and one eigen value is negative, the rank = 3.

2.1.2 NOV/DEC 2014 (R 2013)

Part A

1.If 2, 3 are the eigen values of $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ b & 0 & 2 \end{pmatrix}$, then find the value of b.

Solution. Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ b & 0 & 2 \end{pmatrix}$.

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

We know that, $\lambda_1 + \lambda_2 + \lambda_3 = tr(A) = 6$.

$$2 + 3 + \lambda_3 = 6$$

$$\lambda_3 = 1.$$

Also, product of the eigen values = $|A|$.

$$\lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ b & 0 & 2 \end{vmatrix}$$

$$2.3.1 = 2(4) - 0(0) + 1(0 - 2b)$$

$$6 = 8 - 2b$$
$$2b = 2$$
$$b = 1.$$

2. State Cayley-Hamilton theorem.

Solution. Every square matrix satisfies its own characteristic equation.

Part B

11. (a) (i) Using Cayley-Hamilton theorem find A^4 for the matrix $A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

Solution.

Step 1. To find the characteristic equation

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0. \qquad (1)$$

Now, s_1 = sum of the main diagonal elements

$= 2 + 2 + 2 = 6.$

s_2 = sum of the minors of the elements of the main diagonal

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$= 4 - 1 + 4 - 1 + 4 - 1$

$= 3 + 3 + 3 = 9.$

$$s_3 = |A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$= 2(3) + 1(-1) + 1(-1) = 6 - 1 - 1 = 4.$

The characteristic equation is $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$.

Step 2. Verification of Cayley Hamilton theorem.

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$A^2 = A \cdot A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{pmatrix} = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

$$A^3 = A^2 \times A$$

$$= \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{pmatrix} = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix}$$

$$A^3 - 6A^2 + 9A - 4I$$

$$= \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 6\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + 9\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$-4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

$\therefore$ Cayley Hamilton theorem is verified.

Step 3. To find A^4

Consider λ^4. Dividing λ^4 by $\lambda^3 - 6\lambda^2 + 9\lambda - 4$ we get,

$\lambda^4 = (\lambda + 6)(\lambda^3 - 6\lambda^2 + 9\lambda - 4) + 27\lambda^2 - 50\lambda + 24.$

Replacing λ by A we get,

$$A^4 = (A + 6I)(A^3 - 6A^2 + 9A - 4I) + 27A^2 - 50A + 24I.$$

$$= 27A^2 - 50A + 24I \quad \text{[By Cayley Hamilton theorem].}$$

$$= \begin{pmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{pmatrix}.$$

11. (a) (ii) Find the eigenvalues and eigen vectors of $\begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}$.

Solution. Given $A = \begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1=sum of the main diagonal elements

$$= 7 + 6 + 5 = 18$$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ -2 & 6 \end{vmatrix}$$

$$= (30 - 4) + (35 - 0) + (42 - 4) = 99.$$

$$s_3 = |A| = \begin{vmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{vmatrix}$$

$$= 7(30 - 4) + 2(-10 - 0) + 0 = 182 - 20 = 162.$$

$\therefore$ The characteristic equation is $\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$.

$\lambda = 3$ is a root.

By synthetic division we get

$$\begin{array}{r|rrrr} 3 & 1 & -18 & 99 & -162 \\ & 0 & 3 & -45 & 162 \\ \hline & 1 & -15 & 54 & \boxed{0} \end{array}$$

The remaining factor is

$\lambda^2 - 15\lambda + 54 = 0$

$(\lambda - 6)(\lambda - 9) = 0$

$\lambda = 6, 9.$

$\therefore$ The eigen values are $3, 6, 9$.

Step 3. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to the eigen value λ.

$$\therefore \ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \tag{1}$$

When $\lambda = 3$ (1) becomes

$$\begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$4x_1 - 2x_2 = 0 \Rightarrow 2x_1 - x_2 = 0 \tag{2}$$

$$-2x_1 + 3x_2 - 2x_3 = 0 \tag{3}$$

$$-2x_2 + 2x_3 = 0 \Rightarrow -x_2 + x_3 = 0. \tag{4}$$

From (2) and (4) we get $2x_1 = x_2 = x_3$.

If we set $x_1 = 1$, then $x_2 = 2$, $x_3 = 2$.

$\therefore$ One eigen vector is $X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

When $\lambda = 6$ (1) becomes

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$x_1 - 2x_2 = 0 \Rightarrow x_1 = 2x_2 \tag{5}$$

$$-2x_1 - 2x_3 = 0 \Rightarrow x_1 = x_3 \tag{6}$$

$$-2x_1 - 2x_3 = 0 \Rightarrow x_3 = -2x_2. \tag{7}$$

From (5), (6) and (7) we obtain $x_1 = 2x_2 = -x_3$.

When $x_2 = 1$, $x_1 = 2$, $x_3 = -2$

$\therefore$ The second eigen vector is $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$.

When $\lambda = 9$ (1) becomes

$$\begin{pmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$-2x_1 - 2x_2 = 0 \Rightarrow x_1 = -x_2 \tag{8}$$

$$-2x_1 - 3x_2 - 2x_3 = 0 \tag{9}$$

$$-2x_2 - 4x_3 = 0 \Rightarrow x_2 = -2x_3. \tag{10}$$

From (8), (9) and (10) we obtain $x_1 = -x_2 = 2x_3$.

Let $x_3 = 1$. Then $x_1 = 2$, $x_2 = -2$

$\therefore$ The third eigen vector is $X_2 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

11. (b) (i) Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$ to the canonical form through orthogonal transformation.

Solution.

Step 1. To find the matrix of the Q.F

Since the given Q.F contains 3 variables, the matrix A of the Q.F is a 3×3 matrix.

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

$$\text{Now, } a_{11} = \text{Coeff.of } x^2 = 3.$$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } xy = \frac{1}{2} \times (-2) = -1.$$

$$a_{13} = a_{31} = \frac{1}{2} \times \text{Coeff.of } xz = \frac{1}{2} \times 2 = 1.$$

$$a_{22} = \text{Coeff.of } y^2 = 5.$$

$$a_{23} = a_{32} = \frac{1}{2} \times \text{Coeff.of } yz = \frac{1}{2} \times (-2) = -1.$$

$$a_{33} = \text{Coeff.of } z^2 = 3.$$

$$\therefore\ A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}.$$

Step 2. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$, where

$$s_1 = \text{sum of the main diagonal elements} = 3 + 5 + 3 = 11.$$

$$s_2 = \text{sum of the minors of the main diagonal elements.}$$

$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$$= 14 + 8 + 14 = 36.$$

$$s_3 = |A| = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 3(15-1) + 1(-3+1) + 1(1-5)$$

$$= 3 \times 14 - 2 - 4 = 42 - 6 = 36.$$

$\therefore$ The characteristic equation is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$.

$\lambda = 2$ is a root.

By synthetic division we get

$$\begin{array}{c|cccc} 2 & 1 & -11 & 36 & -36 \\ & 0 & 2 & -18 & 36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$$\lambda^2 - 9\lambda + 18 = 0$$

$$(\lambda - 3)(\lambda - 6) = 0$$

$$\lambda = 3, 6.$$

$\therefore$ The eigen values are $2, 3, 6$.

Step 3. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to the eigen value λ.

$$\therefore \ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = 2$ (1) becomes

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$x_1 - x_2 + x_3 = 0$$
$$-x_1 + 3x_2 - x_3 = 0$$
$$x_1 - x_2 + x_3 = 0.$$

From the last two equations, using the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 3 & & -1 & & -1 & & 3 \\ & \times & & \times & & \times & \\ -1 & & 1 & & 1 & & -1 \end{array}$$

$$\frac{x_1}{3-1} = \frac{x_2}{-1+1} = \frac{x_3}{1-3}$$
$$\frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{-2} \Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$
$$\Rightarrow x_1 = 1, x_2 = 0, x_3 = -1.$$

$$\therefore\ X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

When $\lambda = 3$ (1) becomes

$$\begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$-x_2 + x_3 = 0$$
$$-x_1 + 2x_2 - x_3 = 0$$
$$x_1 - x_2 = 0.$$

From the first and the last equations we get

$x_1 = x_2 = x_3 = 1$(say)

$$\therefore \; X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

When $\lambda = 6$ (1) becomes

$$\begin{pmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$-3x_1 - x_2 + x_3 = 0$$

$$-x_1 - x_2 - x_3 = 0$$

$$x_1 - x_2 - 3x_3 = 0.$$

From the last two equations using the rule of cross multiplication, we get

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ -1 & & -1 & & -1 & & -1 \\ -1 & & -3 & & 1 & & -1 \end{array}$$

$$\frac{x_1}{3-1} = \frac{x_2}{-1-3} = \frac{x_3}{1+1}$$

$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2} \Rightarrow\Rightarrow \frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$\Rightarrow x_1 = 1, x_2 = -2, x_3 = 1.$$

$$\therefore \; X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

$\therefore$ The eigen vectors are $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$

Step 4. To find the modal matrix

The normalised eigen vectors are
$[\frac{1}{\sqrt{2}}\ 0\ \frac{-1}{\sqrt{2}}]^T, [\frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}]^T, [\frac{1}{\sqrt{6}}\ \frac{-2}{\sqrt{6}}\ \frac{1}{\sqrt{6}}]^T.$

$$\therefore\ N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}.$$

Step5. To find N^TAN

$$N^TAN = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} = D.$$

Step 6. Reduction to canonical form

Let $X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

The given Q.F. is $Q = X^TAX = (NY)^TANY = Y^TN^TANY = Y^TDY$

$$Y^TDY = [y_1\ y_2\ y_3] \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = [y_1\ y_2\ y_3] \begin{pmatrix} 2y_1 \\ 3y_2 \\ 6y_3 \end{pmatrix} = 2y_1^2 + 3y_2^2 + 6y_3^2,$$

which is in the canonical form.

11. (b) (ii) If β is a eigenvalue of a matrix, then prove that $\frac{1}{\beta}$ is the eigen value of A^{-1}.

Solution. Given β is a non zero eigen value of A. Then, there exists a non zero column matrix X such that

$$AX = \beta X. \tag{1}$$

Since all the eigen values are non zero, A is non-singular. Hence, A^{-1} exists.

Premultiplying both sides of (1) by A^{-1} we get,

$$A^{-1}(AX) = A^{-1}\beta X$$

$$(A^{-1}A)X = \beta(A^{-1}X)$$

$$IX = \beta A^{-1}X$$

$$\frac{1}{\beta}X = A^{-1}X.$$

$\Rightarrow \frac{1}{\beta}$ is an eigen value of A^{-1}.

2.1.3 MAY/JUNE 2014 (R 2013)

Part A

1. Find the Eigen values of the inverse of the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix}$.

Solution. A is a triangular matrix.

The eigen values of A are $2, 3, 4$.

$\therefore$ Eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

2. If $2, -1, -3$ are the Eigen values of the matrix A, then find the Eigen values of the matrix $A^2 - 2I$.

Solution. If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A, then the eigen values of

$A^2 - 2I$ are $\lambda_1^2 - 2, \lambda_2^2 - 2, \lambda_3^2 - 2$.

Here $\lambda_1 = 2, \lambda_2 = -1, \lambda_3 = -3$.

$\therefore$ The required eigen values are $2^2 - 2, (-1)^2 - 2, (-3)^2 - 2$.

i.e., $2, -1, 7$.

Part B

11. (a) (i) Verify Cayley Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$, hence find A^{-1}. (8)

Solution.

Step 1. To find the characteristic equation

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements.

$= 1 + 1 + 1 = 3.$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$

$$= 1 - 1 + 1 - 3 + 1 - 0 = -1.$$

$$s_3 = |A| = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= 1(1 - 1) + 3(-2 - 1) = -9.$$

The characteristic equation is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$.

Step 2. Verification of Cayley Hamilton theorem

By Cayley Hamilton theorem, $A^3 - 3A^2 - A + 9I = 0$.

$$\text{Now, } A^2 = A \times A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{pmatrix} = \begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix}$$

$$A^3 = A^2 \times A = \begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4-6+6 & 0-3-6 & 12+3+6 \\ 3+4+4 & 0+2-4 & 9-2+4 \\ 0-4+5 & 0-2-5 & 0+2+5 \end{pmatrix} = \begin{pmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{pmatrix}$$

$$A^3 - 3A^2 - A + 9I$$

$$= \begin{pmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{pmatrix} - 3\begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4-12-1+9 & -9+9-0+0 & 21-18-3+0 \\ 11-9-2+0 & -2-6-1+9 & 11-12+1+0 \\ 1+0-1+0 & -7+6+1+0 & 7-15-1+9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Hence, Cayley Hamilton theorem is verified.

Step 3. To find A^{-1}

By Cayley Hamilton theorem we have $A^3 - 3A^2 - A + 9I = 0$.

Premultiply by A^{-1} we get, $A^2 - 3A - I + 9A^{-1} = 0$.

$$9A^{-1} = I + 3A - A^2$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1+3-4 & 0+0+3 & 0+9-6 \\ 0+6-3 & 1+3-2 & 0-3-4 \\ 0+3-0 & 0-3+2 & 1+3-5 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{9}\begin{pmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{pmatrix}$$

11. (a) (ii) Find the Eigen values and Eigen vectors of $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$.

Solution. Let $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

where $s_1 = tr(A) = -2 + 1 + 0 = -1$.

$s_2 =$ sum of the minors of the main diagonal elements of A.

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$

$s_2 = 0 - 12 + 0 - 3 - 2 - 4 = -21.$

$$s_3 = |A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

$$= -2(0 - 12) - 2(0 - 6) - 3(-4 + 1) = 24 + 12 + 9 = 45.$$

The characteristic equation is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$.

$\lambda = -3$ is a root. By synthetic division,

$$\begin{array}{r|rrr|r} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

$$\Longrightarrow (\lambda + 3)(\lambda^2 - 2\lambda - 15) = 0$$

$$(\lambda + 3)(\lambda - 5)(\lambda + 3) = 0 \Rightarrow \lambda = -3, \lambda = -3, \lambda = 5.$$

The eigen values are $-3,\ -3,\ 5$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to λ.

$$\therefore \qquad (A - \lambda I)X = 0.$$

$$\begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = -3$, (1) becomes $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The above three equations are reduced to

$$x_1 + 2x_2 - 3x_3 = 0. \qquad (2)$$

Since two of the eigen values are equal, from (2) we have to get two eigen vectors by assigning arbitrary values for two variables. First, choosing $x_3 = 0$, we obtain $x_1 + 2x_2 = 0$ which implies $x_1 = -2x_2 \Rightarrow \dfrac{x_1}{2} = \dfrac{x_2}{-1} \Rightarrow x_1 = 2, x_2 = -1$

$$\therefore X_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

Also in (2) assign $x_2 = 0$, we obtain $x_1 - 3x_3 = 0$, which implies $x_1 = 3x_3 \Rightarrow \dfrac{x_1}{3} = \dfrac{x_3}{1} \Rightarrow x_1 = 3, x_3 = 1$.

$$X_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$$

When $\lambda = 5$, (1)$\Longrightarrow \begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The equations are

$$\left.\begin{aligned} -7x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 - 4x_2 - 6x_3 &= 0 \\ -x_1 - 2x_2 - 5x_3 &= 0. \end{aligned}\right\} \qquad \text{(A)}$$

Taking the first two equations and on solving by the method of cross multiplication, we obtain

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 2 & & -3 & & -7 & & 2 \\ -4 & & -6 & & 2 & & -4 \end{array}$$

$$\frac{x_1}{-12-12} = \frac{x_2}{-6-42} = \frac{x_3}{28-4}$$
$$\frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24}$$
$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}.$$
$$\Longrightarrow x_1 = 1, x_2 = 2, x_3 = -1.$$

The values of x_1, x_2, x_3 satisfy the last equation in (A).

$$\therefore \; X_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

$\therefore$ The eigen vectors are $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

11. (b) (i) Reduce the quadratic form $x_1^2 + 5x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 6x_3x_1$ to the canonical form through orthogonal transformation and find

its nature.

Solution.

Step 1. To form the matrix of the Q.F

Let A be the matrix of the Q.F.

Since the given Q.F is in 3 variables, A must be a 3×3 matrix.

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

$$\text{Now, } a_{11} = \text{Coeff.of } x_1^2 = 1$$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } x_1x_2 = \frac{1}{2} \times 2 = 1$$

$$a_{13} = a_{31} = \frac{1}{2} \times \text{Coeff.of } x_1x_3 = \frac{1}{2} \times 6 = 3$$

$$a_{22} = \text{Coeff.of } x_2^2 = 5.$$

$$a_{23} = a_{32} = \frac{1}{2} \times \text{Coeff.of } x_2x_3 = \frac{1}{2} \times 2 = 1.$$

$$a_{33} = \text{Coeff.of } x_3^2 = 1.$$

$$\therefore\ A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}.$$

Step 2. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$= 1 + 5 + 1 = 7.$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}$$

$$= 5 - 1 + 1 - 9 + 5 - 1 = 4 - 8 + 4 = 0.$$

$$s_3 = |A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$

$$= 1(5 - 1) - 1(1 - 3) + 3(1 - 15)$$

$$= 4 + 2 - 42 = -36.$$

$\therefore$ The characteristic equation is $\lambda^3 - 7\lambda^2 + 36 = 0$.

$\lambda = -2$ is a root.

By synthetic division, we get

$$\begin{array}{r|rrrr} -2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & \multicolumn{1}{|r}{0} \end{array}$$

$$\lambda^2 - 9\lambda + 18 = 0.$$

$$(\lambda - 3)(\lambda - 6) = 0.$$

The characteristic equation becomes $(\lambda + 2)(\lambda - 3)(\lambda - 6) = 0$.

The eigen values are $-2, 3, 6$.

Step 3.To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to the eigen value λ.

$$\therefore \ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = -2$, (1) becomes

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above two equations are reduced to two equations

$$3x_1 + x_2 + 3x_3 = 0$$

$$x_1 + 7x_2 + x_3 = 0.$$

Solving the two equations by the rule of cross multiplication we get

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 1 & & 3 & & 3 & & 1 \\ 7 & & 1 & & 1 & & 7 \end{array}$$

$$\frac{x_1}{1-21} = \frac{x_2}{3-3} = \frac{x_3}{21-1}$$
$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$$
$$\Longrightarrow \quad x_1 = 1, x_2 = 0, x_3 = -1.$$

$$\therefore \; X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

When $\lambda = 3$, (1) become $\begin{pmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The above equations are reduced to

$$-2x_1 + x_2 + 3x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

$$3x_1 + x_2 - 2x_3 = 0.$$

From the first two equations by the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 3 & & -2 & & 1 \\ 2 & & 1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{1-6} = \frac{x_2}{3+2} = \frac{x_3}{-4-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \implies x_1 = 1, x_2 = -1, x_3 = 1.$$

$$\therefore \; X_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

When $\lambda = 6$, (1) becomes $\begin{pmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The above equations are reduced to

$$-5x_1 + x_2 + 3x_3 = 0$$

$$x_1 - x_2 + x_3 = 0$$

$$3x_1 + x_2 - 5x_3 = 0.$$

By the rule of cross multiplication, from the first two equations we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 3 & & -5 & & 1 \\ -1 & & 1 & & 1 & & -1 \end{array}$$

$$\frac{x_1}{1+3} = \frac{x_2}{3+5} = \frac{x_3}{5-1}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \implies x_1 = 1, x_2 = 2, x_3 = 1.$$

$$X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Step 4. To form the modal matrix

Since the eigen values are different, the eigen vectors are mutually orthogonal.

The normalized eigen vectors are $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$.

The modal matrix N is formed with columns as the normalized eigen vectors.

$$\therefore\ N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Step 5. To find $N^T AN$

$$N^T AN = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} -\sqrt{2} & 0 & \sqrt{2} \\ \sqrt{3} & -\sqrt{3} & \sqrt{3} \\ \sqrt{6} & 2\sqrt{6} & \sqrt{6} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} = D.$$

Step 6. Reduction to Canonical form

Let $X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

The given Q.F is $Q = X^T AX$.

$$= (NY)^T A(NY)$$
$$= Y^T N^T ANY$$
$$= Y^T DY$$
$$= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$
$$= -2y_1^2 + 3y_2^2 + 6y_3^2$$

which is canonical.

Since two of the eigen values are positive and one eigen value is negative, the given Q.F is indefinite.

11. (b) (ii) Prove that the Eigen values of a real symmetric matrix are real.

Solution. Let λ (real or complex) be an eigen value of A.

Let X be the eigen vector corresponding to the eigen value λ.

$$\therefore AX = \lambda X. \qquad (1)$$

Let $\bar{X}$ be the Complex conjugate of X.

Premultiply (1) by $\bar{X}^T$ we get

$$\bar{X}^T AX = \lambda \bar{X}^T X$$
$$\overline{\bar{X}^T AX} = \overline{\lambda \bar{X}^T X}$$
$$\text{i.e., } X^T \bar{A}\bar{X} = \bar{\lambda} X^T \bar{X}.$$

Since A is a real symmetric matrix, $\bar{A} = A$ & $A^T = A$.

$$\therefore \; X^T A\bar{X} = \bar{\lambda} X^T \bar{X}.$$

Taking transpose on both sides we get

$$\left(X^T A\bar{X}\right)^T = \left(\bar{\lambda} X^T \bar{X}\right)^T$$
$$\bar{X}^T A^T X = \bar{X}^T X \bar{\lambda}^T = \bar{\lambda}\bar{X}^T X$$
$$\bar{X}^T AX = \bar{\lambda}\bar{X}^T X$$

$$\overline{X}^T \lambda X = \overline{\lambda}\overline{X}^T X$$

$$\lambda \overline{X}^T X = \overline{\lambda}\overline{X}^T X.$$

Now X is an $n \times 1$ matrix and $\overline{X}^T$ is a $1 \times n$ matrix.

Hence, $\overline{X}^T X$ is a 1×1 matrix, which is a positive real.

$$\Rightarrow \lambda = \overline{\lambda}.$$

$\Rightarrow$ λ is a real number.

2.1.4 JANUARY 2014 (R 2013)

Part A

1. If the eigen values of the matrix A of order 3×3 are $2, 3, 1$ then find the eigen values of adjoint of A.

Solution. Eigen values of A are $2, 3, 1$.

$|A|$ = Product of eigen values

$= 2 \times 3 \times 1 = 6.$

We know that $adj(A) = |A|\ A^{-1}$.

Eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, 1$.

$\therefore$ The eigen values of $adj(A)$ are $6 \times \frac{1}{2}, 6 \times \frac{1}{3}, 6 \times 1$

i.e., $3, 2, 6$.

2. If λ is the eigen value of the matrix A ,then prove that λ^2 is the eigen value of A^2.

Solution. Since λ is an eigen value of A, there exists a column matrix X such that

$$AX = \lambda X$$

$$A(AX) = A(\lambda X)$$

$$(AA)X = \lambda(AX)$$

$$A^2X = \lambda(\lambda X)$$

$$A^2X = \lambda^2 X.$$

$\Rightarrow$ λ^2 is an eigen value of A^2.

Part B

11. (a) (i) Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

Solution. Let $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$, where

s_1 = sum of the main diagonal elements of A

$= 2 + 3 + 2 = 7.$

s_2 = sum of the minors of the main diagonal elements of A.

$$= \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$$

$= (6 - 2) + (4 - 1) + (6 - 2) = 4 + 3 + 4 = 11.$

$$s_3 = |A| = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 2(6 - 2) - 2(2 - 1) + 1(2 - 3) = 8 - 2 - 1 = 5.$$

The characteristic equation is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$.

$\lambda = 1$ is a root.

By synthetic division we have

$$\begin{array}{c|cccc} 1 & 1 & -7 & 11 & -5 \\ & 0 & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & \boxed{0} \end{array}$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$(\lambda - 1)(\lambda - 5) = 0 \quad \Rightarrow \lambda = 1, 5.$$

$\therefore$ The eigen values are 1, 1, 5.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$\therefore (A - \lambda I)X = 0$

$$\begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = 1$, (1) becomes

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

All the three equations are reduced to one single equation

$$x_1 + 2x_2 + x_3 = 0. \qquad (2)$$

Since two of the eigen values are equal, from(2), we have to get two eigen vectors by assigning arbitrary values for two variables.

First, choosing $x_3 = 0$, we obtain $x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$

$$i.e., \frac{x_1}{-2} = \frac{x_2}{1} \quad \Rightarrow \quad x_1 = -2, x_2 = 1.$$

$\therefore$ One eigen vector is $X_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$.

Assign $x_2 = 0$, (2) becomes

$$x_1 + x_3 = 0$$

$$x_1 = -x_3$$

$$\frac{x_1}{1} = \frac{x_3}{-1}$$

$$\Rightarrow x_1 = 1, x_3 = -1.$$

$\therefore$ The second eigen vector is $X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

When $\lambda = 5$, (1) becomes

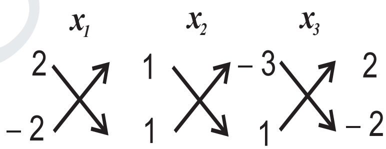
$$\begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$\left.\begin{aligned} -3x_1 + 2x_2 + x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 + 2x_2 - 3x_3 &= 0. \end{aligned}\right\} \quad \text{(A)}$$

All the three equations are different.

Taking the first two equations and applying the rule of cross multiplication we obtain



$$\frac{x_1}{2+2} = \frac{x_2}{1+3} = \frac{x_3}{6-2}$$

$$\frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

$$\Rightarrow x_1 = 1, x_2 = 1, x_3 = 1.$$

The third eigen vector is $X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$\therefore$ The eigen vectors are $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

11. (a) (ii) Using Cayley-Hamilton theorem find A^{-1} and A^4, if $A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$.

Solution.

Step 1. To find the characteristic equation

Since A is 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$= 1 + 3 + 1 = 5.$

s_2 = sum of the minors of the elements of the main diagonal

$$= \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$

$= 3 - 0 + 1 - 0 + 3 + 2$

$= 3 + 1 + 5$

$= 9.$

$$s_3 = |A| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

$= 1(3 - 0) - 2(-1 - 0) - 2(2 - 0)$

$= 3 + 2 - 4 = 1.$

The characteristic equation is $\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$.

By Cayley Hamilton theorem, we have

$$A^3 - 5A^2 + 9A - I = 0.$$

Step2. Verification of Cayley Hamilton theorem

$$A^2 = A.A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix}$$

$$A^3 = A^2.A = \begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{pmatrix}$$

$\therefore\ A^3 - 5A^2 + 9A - I$

$$= \begin{pmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{pmatrix} - 5\begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix} + 9\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Hence, Cayley Hamilton theorem is verified.

Step 3. To find A^{-1}

By Cayley Hamilton theorem, we have $A^3 - 5A^2 + 9A - I = 0$.

Multiplying by A^{-1} we get

$$A^2 - 5A + 9I - A^{-1} = 0$$

$$\therefore\ A^{-1} = A^2 - 5A + 9I = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}.$$

Step 4. To find A^4

By Cayley Hamilton theorem, we have

$$A^3 - 5A^2 + 9A - I = 0.$$
$$A^3 = 5A^2 - 9A + I.$$

Multiplying by A we get

$$A^4 = 5A^3 - 9A^2 + A$$
$$= 5\begin{pmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{pmatrix} - 9\begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -65 & 210 & -10 \\ -55 & 45 & 50 \\ 50 & -110 & -15 \end{pmatrix} + \begin{pmatrix} 9 & -108 & 36 \\ 36 & -63 & -18 \\ -18 & 72 & -9 \end{pmatrix} + \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -65+9+1 & 210-108+2 & -10+36-2 \\ -55+36-1 & 45-63+3 & 50-18+0 \\ 50-18+0 & -110+72-2 & -15-9+1 \end{pmatrix}$$
$$= \begin{pmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -40 & -23 \end{pmatrix}.$$

11. (b) Reduce the quadratic form $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$ into a canonical form by an orthogonal reduction.Hence find its rank and nature.

Solution.

Step 1. To find the matrix of the Q.F

Since the given Q.F contains 3 variables, the matrix of the Q.F A is a 3×3 matrix.

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

$$\begin{aligned}
\text{Now, } a_{11} &= \text{Coeff.of } x^2 = 6 \\
a_{12} = a_{21} &= \frac{1}{2} \times \text{Coeff.of } xy = \frac{1}{2} \times (-4) = -2. \\
a_{13} = a_{31} &= \frac{1}{2} \times \text{Coeff.of } xz = \frac{1}{2} \times 4 = 2 \\
a_{22} &= \text{Coeff.of } y^2 = 3. \\
a_{23} = a_{32} &= \frac{1}{2} \times \text{Coeff.of } yz = \frac{1}{2} \times (-2) = -1. \\
a_{33} &= \text{Coeff.of } z^2 = 3.
\end{aligned}$$

$$\therefore\ A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}.$$

Step 2. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$.

$$\begin{aligned}
s_1 &= \text{sum of the main diagonal elements} \\
&= 6 + 3 + 3 = 12. \\
s_2 &= \text{sum of the minors of the main diagonal elements.} \\
&= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} \\
&= 9 - 1 + 18 - 4 + 18 - 4 = 8 + 14 + 14 = 36. \\
s_3 = |A| &= \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} \\
&= 6(9 - 1) + 2(-6 + 2) + 2(2 - 6) = 48 - 8 - 8 = 32.
\end{aligned}$$

$\therefore$ The characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$.

$\lambda = 2$ is a root. By synthetic division, we get

$$\begin{array}{r|rrrr} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda - 2)(\lambda - 8) = 0.$$

The characteristic equation becomes $(\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$.

The eigen values are $2, 2, 8$.

Step 3. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to the eigen value λ.

$$\therefore \ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = 8$ (1) becomes

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations are $-2x_1 - 2x_2 + 2x_3 = 0 \Rightarrow x_1 + x_2 - x_3 = 0$.

$$-2x_1 - 5x_2 - x_3 = 0 \Rightarrow 2x_1 + 5x_2 + x_3 = 0.$$

$$2x_1 - x_2 - 5x_3 = 0 \Rightarrow 2x_1 - x_2 - 5x_3 = 0.$$

From the first two equations, using the rule of crossmultiplication we get,

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & -1 & & 1 & & 1 \\ 5 & & 1 & & 2 & & 5 \end{array}$$

$$\frac{x_1}{1+5} = \frac{x_2}{-2-1} = \frac{x_3}{5-2}$$

$$\frac{x_1}{6} = \frac{x_2}{-3} = \frac{x_3}{3} \Rightarrow x_1 = 2, x_2 = -1, x_3 = 1.$$

$$\therefore \; X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

When $\lambda = 2$ (1) becomes

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

$$\begin{aligned} i.e., 4x_1 - 2x_2 + 2x_3 = 0 &\Rightarrow 2x_1 - x_2 + x_3 = 0 \\ -2x_1 + x_2 - x_3 = 0 &\Rightarrow 2x_1 - x_2 + x_3 = 0 \\ 2x_1 - x_2 + x_3 &= 0. \end{aligned}$$

The above three equations are reduced to one single equation $2x_1 - x_2 + x_3 = 0$. Choose $x_3 = 0. \Rightarrow 2x_1 - x_2 = 0 \Rightarrow 2x_1 = x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{2}$ $\Rightarrow x_1 = 1, x_2 = 2.$

$$\therefore \; X_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be orthogonal to X_2.

$$\Rightarrow X_2^T X_3 = 0 \Rightarrow \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow a + 2b = 0. \qquad (2)$$

X_3 also satisfy $2a - b + c = 0$. (3)

$(2) \Rightarrow a = -2b \Longrightarrow \frac{a}{-2} = \frac{b}{1}$.

$(3) \Rightarrow -4b - b + c = 0 \Rightarrow 5b = c \Rightarrow \frac{b}{1} = \frac{c}{5}$.

Combining the above two equations we obtain, $\frac{a}{-2} = \frac{b}{1} = \frac{c}{5}$.

$$\text{Hence, } a = -2, b = 1, c = 5.$$

$$\therefore \; X_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}.$$

Step 4. To find the modal matrix

The normalised eigen vectors are

$\left[\frac{2}{\sqrt{6}} \; \frac{-1}{\sqrt{6}} \; \frac{1}{\sqrt{6}}\right]^T, \left[\frac{1}{\sqrt{5}} \; \frac{2}{\sqrt{5}} \; 0\right]^T, \left[\frac{-2}{\sqrt{30}} \; \frac{1}{\sqrt{30}} \; \frac{5}{\sqrt{30}}\right]^T$

$$\therefore \; N = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{pmatrix}.$$

Step5. To find $N^T AN$

$$N^T AN = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{-2}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{16}{\sqrt{6}} & \frac{-8}{\sqrt{6}} & \frac{8}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-8}{\sqrt{5}} & 0 \\ \frac{-4}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{10}{\sqrt{30}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D.$$

Step 6. Reduction to canonical form

Let $X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

The given Q.F. is

$Q = X^T AX = (NY)^T ANY = Y^T N^T ANY = Y^T DY$

$$Y^T DY = [y_1\ y_2\ y_3]\begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = [y_1\ y_2\ y_3]\begin{pmatrix} 8y_1 \\ 2y_2 \\ 2y_3 \end{pmatrix} = 8y_1^2 + 2y_2^2 + 2y_3^2,$$

which is in the canonical form.

Rank of the Q.F = 3.

Index = 3.

Signature = 3.

Since all the eigen values are positive, the Q.F is positive definite.

2.1.5 JANUARY 2014 (R 2008)

Part A

1. The product of two eigen values of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16. Find the third eigenvalue.

Solution. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

Given $\lambda_1\lambda_2 = 16$.

$$|A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1) + 2(-6+2) + 2(2-6) = 48 - 8 - 8 = 32.$$

We know that $\lambda_1\lambda_2\lambda_3 = 32$

$$16\lambda_3 = 32$$

$$\lambda_3 = 2.$$

$\therefore$ The third eigen value is 2.

2. Discuss the nature of the quadratic form $2x^2 + 3y^2 + 2z^2 + 2xy$.

Solution. The matrix of the quadratic form is $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

The principal minors are

$$D_1 = 2 > 0.$$

$$D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5 > 0.$$

$$D_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2(6-0) - 1(2-0) = 12 - 2 = 10 > 0.$$

Since all D_i's are positive, the given quadratic form is positive definite.

Part B

11. (a) (i) Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$.

Solution.

Step 1. To find the characteristic equation

Since A is 3×3 matrix, the characteristic equation is of the form $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$, where

$$s_1 = \text{sum of the main diagonal elements of A}$$

$$= 8 + 7 + 3 = 18.$$

s_2 = sum of the minors of the main diagonal elements of A

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$= (21 - 16) + (24 - 4) + (56 - 36)$$

$$= 5 + 20 + 20 = 45.$$

$$s_3 = |A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8(21 - 16) + 6(-18 + 8) + 2(24 - 14)$$

$$= 40 - 60 + 20 = 0.$$

The characteristic equation is

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda(\lambda - 3)(\lambda - 15) = 0$$

$$\lambda = 0, 3, 15.$$

Step 3. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value λ.

$$\therefore\ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = 0$ (1) becomes

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 3x_3 = 0.$$

Taking the first two equations and applying the rule of cross multiplication, we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -6 & & 2 & & 8 & & -6 \\ 7 & & -4 & & -6 & & 7 \end{array}$$

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\therefore\ X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

When $\lambda = 3$ (1) becomes

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$5x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 4x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 0x_3 = 0.$$

Taking the last two equations and applying the rule of cross multiplication we get,

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 4 & & -4 & & -6 & & 4 \\ -4 & & 0 & & 2 & & -4 \end{array}$$

$$\frac{x_1}{0-16} = \frac{x_2}{-8} = \frac{x_3}{24-8}$$

$$\frac{x_1}{-16} = \frac{x_2}{-8} = \frac{x_3}{16}$$

$$\therefore\ X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$

When $\lambda = 15$ (1) becomes

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$-7x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 - 8x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 - 12x_3 = 0.$$

Taking the first two equations and applying the rule of cross multiplication we get,

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -6 & & 2 & & -7 & & -6 \\ -8 & & -4 & & -6 & & -8 \end{array}$$

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

$$\therefore\ X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

$\therefore$ The eigen vectors are $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

11. (a) (ii) Using Cayley-Hamilton theorem find A^{-1} for the matrix $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$.

Solution. $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$, where

s_1 = sum of the main diagonal elements

$= 1 + 1 + 1 = 3.$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$

$= (1 - 1) + (1 - 3) + (1 - 0) = -1.$

$$s_3 = |A| = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = 1(1 - 1) - 0(2 + 1) + 3(-2 - 1) = 0 - 0 - 9 = -9.$$

The characteristic equation is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$.

By Cayley-Hamilton theorem $A^3 - 3A^2 - A + 9I = 0$.

Premultiply by A^{-1}, we have $A^2 - 3A - I + 9A^{-1} = 0$.

$$-9A^{-1} = I + 3A - A^2.$$

$$A^2 = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix}$$

$$-9A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{pmatrix} - \begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix}$$

$$-9A^{-1} = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{pmatrix} \Rightarrow \therefore A^{-1} = -\frac{1}{9}\begin{pmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{pmatrix}$$

11. (b) Reduce the quadratic form $Q = 3x^2 - 3y^2 - 5z^2 - 2xy - 6yz - 6xz$ to its canonical form using orthogonal transformation. Also find its rank, index and signature.

Solution.

Step 1. To find the matrix of the Q.F

Since the given Q.F contains 3 variables, the matrix of the Q.F A is a 3×3 matrix.

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

$$\begin{aligned}
\text{Now, } a_{11} &= \text{Coeff.of } x^2 = 3 \\
a_{12} = a_{21} &= \frac{1}{2} \times \text{Coeff.of } xy = \frac{1}{2} \times (-2) = -1. \\
a_{13} = a_{31} &= \frac{1}{2} \times \text{Coeff.of } xz = \frac{1}{2} \times (-6) = -3. \\
a_{22} &= \text{Coeff.of } y^2 = -3. \\
a_{23} = a_{32} &= \frac{1}{2} \times \text{Coeff.of } yz = \frac{1}{2} \times (-6) = -3. \\
a_{33} &= \text{Coeff.of } z^2 = -5.
\end{aligned}$$

$$\therefore A = \begin{pmatrix} 3 & -1 & -3 \\ -1 & -3 & -3 \\ -3 & -3 & -5 \end{pmatrix}.$$

Step 2. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$, where

s_1 = sum of the main diagonal elements

$$= 3 - 3 - 5 = -5.$$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} -3 & -3 \\ -3 & -5 \end{vmatrix} + \begin{vmatrix} 3 & -3 \\ -3 & -5 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & -3 \end{vmatrix}$$

$$= (15 - 9) + (-15 - 9) + (-9 - 1) = 6 - 10 - 24 = -28.$$

$$s_3 = |A| = \begin{vmatrix} 3 & -1 & -3 \\ -1 & -3 & -3 \\ -3 & -3 & -5 \end{vmatrix}$$

$$= 3(15 - 9) + 1(5 - 9) - 3(3 - 9) = 18 - 4 + 18 = 32.$$

$\therefore$ The characteristic equation is $\lambda^3 + 5\lambda^2 - 28\lambda - 32 = 0$.

$\lambda = -1$ is a root. By synthetic division we get

-1	1	5	-28	-32
	0	-1	-4	32
	1	4	-32	0

$$\lambda^2 + 4\lambda - 32 = 0$$

$$(\lambda - 4)(\lambda + 8) = 0.$$

The characteristic equation becomes $(\lambda - 4)(\lambda + 1)(\lambda + 8) = 0$.

The eigen values are $4, -1, -8$.

Step 3. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to the eigen value λ.

$$\therefore \ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 3-\lambda & -1 & -3 \\ -1 & -3-\lambda & -3 \\ -3 & -3 & -5-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = 4$ (1) becomes

$$\begin{pmatrix} -1 & -1 & -3 \\ -1 & -7 & -3 \\ -3 & -3 & -9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations are $-x_1 - x_2 - 3x_3 = 0$

$$-x_1 - 7x_2 - 3x_3 = 0$$

$$-3x_1 - 3x_2 - 9x_3 = 0 \Rightarrow -x_1 - x_2 - 3x_3 = 0.$$

From the first two equations, using the rule of crossmultiplication we get,

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -1 & & -3 & & -1 & & -1 \\ -7 & & -3 & & -1 & & -7 \end{array}$$

$$\frac{x_1}{3-21} = \frac{x_2}{3-3} = \frac{x_3}{7-1}$$

$$\frac{x_1}{-18} = \frac{x_2}{0} = \frac{x_3}{6}$$

$$\frac{x_1}{-3} = \frac{x_2}{0} = \frac{x_3}{1}.$$

$$\therefore\ X_1 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

When $\lambda = -1$ (1) becomes

$$\begin{pmatrix} 4 & -1 & -3 \\ -1 & -2 & -3 \\ -3 & -3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations are $4x_1 - x_2 - 3x_3 = 0$

$$-x_1 - 2x_2 - 3x_3 = 0$$

$$-3x_1 - 3x_2 - 4x_3 = 0.$$

From the first two equations, using the rule of crossmultiplication we get,

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -1 & & -3 & & 4 & & -1 \\ -2 & & -3 & & -1 & & -2 \end{array}$$

$$\frac{x_1}{3-6} = \frac{x_2}{3+12} = \frac{x_3}{-8-1}$$

$$\frac{x_1}{-3} = \frac{x_2}{15} = \frac{x_3}{-9}$$

$$\frac{x_1}{1} = \frac{x_2}{-5} = \frac{x_3}{3}.$$

$$\therefore \; X_2 = \begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}.$$

When $\lambda = -8$ (1) becomes

$$\begin{pmatrix} 11 & -1 & -3 \\ -1 & 5 & -3 \\ -3 & -3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations are $11x_1 - x_2 - 3x_3 = 0$

$$-x_1 + 5x_2 - 3x_3 = 0$$

$$-3x_1 - 3x_2 + 3x_3 = 0.$$

From the last two equations, using the rule of crossmultiplication, we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 5 & & -3 & & -1 & & 5 \\ -3 & & 3 & & -3 & & -3 \end{array}$$

$$\frac{x_1}{15-9} = \frac{x_2}{9+3} = \frac{x_3}{3+15}$$

$$\frac{x_1}{6} = \frac{x_2}{12} = \frac{x_3}{18}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}$$

$$\therefore \; X_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Since all the eigen values are different, the eigen vectors are pairwise orthogonal.

Step 4. To find the modal matrix

The normalised eigen vectors are

$$[\frac{-3}{\sqrt{10}} \; 0 \; \frac{1}{\sqrt{10}}]^T, [\frac{1}{\sqrt{35}} \; \frac{-5}{\sqrt{35}} \; \frac{3}{\sqrt{35}}]^T, [\frac{1}{\sqrt{14}} \; \frac{2}{\sqrt{14}} \; \frac{3}{\sqrt{14}}]^T$$

$$\therefore \; N = \begin{pmatrix} \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{35}} & \frac{1}{\sqrt{14}} \\ 0 & \frac{-5}{\sqrt{35}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{35}} & \frac{3}{\sqrt{14}} \end{pmatrix}.$$

Step5. To find $N^T AN$

$$N^T AN = \begin{pmatrix} \frac{-3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{35}} & \frac{-5}{\sqrt{35}} & \frac{3}{\sqrt{35}} \\ \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} 3 & -1 & -3 \\ -1 & -3 & -3 \\ -3 & -3 & -5 \end{pmatrix} \begin{pmatrix} \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{35}} & \frac{1}{\sqrt{14}} \\ 0 & \frac{-5}{\sqrt{35}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{35}} & \frac{3}{\sqrt{14}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-12}{\sqrt{10}} & 0 & \frac{4}{\sqrt{10}} \\ \frac{-1}{\sqrt{35}} & \frac{5}{\sqrt{35}} & \frac{-3}{\sqrt{35}} \\ \frac{-8}{\sqrt{14}} & \frac{-16}{\sqrt{14}} & \frac{-24}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{35}} & \frac{1}{\sqrt{14}} \\ 0 & \frac{-5}{\sqrt{35}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{35}} & \frac{3}{\sqrt{14}} \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -8 \end{pmatrix} = D.$$

Step 6. Reduction to canonical form

Let $X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

The given Q.F. is $Q = X^T AX = (NY)^T ANY = Y^T N^T ANY = Y^T DY$

$$Y^T DY = [y_1 \; y_2 \; y_3] \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -8 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = [y_1 \; y_2 \; y_3] \begin{pmatrix} 4y_1 \\ -y_2 \\ -8y_3 \end{pmatrix} = 4y_1^2 - y_2^2 - 8y_3^2,$$

which is the canonical form.

Rank of the Q.F = 3, Index = 1, Signature = −1.

2.1.6 JANUARY 2013 (R 2008)

Part A

1. Find the symmetric matrix A whose eigenvalues are 1 and 3 with corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Solution.

Let the required symmetric matrix be $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

The eigen values are $\lambda_1 = 1$ and $\lambda_2 = 3$, with corresponding eigen vectors are $X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

By definition, $AX_1 = \lambda_1 X_1$

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

which yields the equations

$$a - b = 1. \tag{1}$$

$$b - c = -1. \tag{2}$$

Also, $AX_2 = \lambda_2 X_2$

i.e., $\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ which yields the equations

$$a + b = 3 \tag{3}$$

$$b + c = 3 \tag{4}$$

$(1) + (3) \Longrightarrow 2a = 4 \Longrightarrow a = 2.$

$(2) + (4) \Longrightarrow 2b = 2 \Longrightarrow b = 1.$

$(4) \Longrightarrow 1 + c = 3 \Longrightarrow c = 2.$

$$\therefore \text{The required matrix is } A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

2. Write down the quadratic form corresponding to the matrix $\begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & -2 \end{pmatrix}$.

Solution.

From the matrix of the quadratic form we observe that $a_{11} = 2, a_{12} = 0, a_{13} = -2, a_{22} = 2, a_{23} = 1, a_{33} = -2.$

The required quadratic form is

$$\begin{aligned} & a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz \\ &= 2x^2 + 2y^2 - 2z^2 - 4xz + 2yz. \end{aligned}$$

Part B

11. (a) (i) Find the eigenvalues and eigenvectors of $\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$.

Solution. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$, where

s_1 = sum of the main diagonal elements of A.

$= 1 - 1 - 1 = -1.$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} -1 & 0 \\ -2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 6 & -1 \end{vmatrix}$$

$$= 1 - 0 + (-1) + 1 + (-1 - 12) = 1 + 0 - 13 = -12.$$

$$s_3 = |A| = \begin{vmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{vmatrix}$$

$$= 1(1-0) - 2(-6-0) + 1(-12-1) = 1 + 12 - 13 = 0.$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - 12\lambda = 0$$

$$\lambda(\lambda^2 + \lambda - 12) = 0$$

$$\lambda(\lambda + 4)(\lambda - 3) = 0$$

$$\lambda = 0, -4, 3.$$

$\therefore$ The eigen values are $0, -4, 3$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value λ.

$\therefore (A - \lambda I)X = 0$

$$\text{i.e., } \begin{pmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = 0$, (1) becomes

$$\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\text{i.e., } x_1 + 2x_2 + x_3 = 0 \qquad (2)$$

$$6x_1 - x_2 = 0 \qquad (3)$$

$$-x_1 - 2x_2 - x_3 = 0. \qquad (4)$$

(3) can be written as

$$6x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{6}$$

$$\Rightarrow x_1 = 1, x_2 = 6.$$

Substituting in (2) we obtain, $1 + 12 + x_3 = 0$

$$\Rightarrow x_3 = -13.$$

$\therefore$ The eigen vector is $X_1 = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}$.

When $\lambda = -4$, (1) becomes

$$\begin{pmatrix} 5 & 2 & 1 \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\text{i.e., } 5x_1 + 2x_2 + x_3 = 0 \quad (5)$$

$$6x_1 + 3x_2 = 0 \quad (6)$$

$$-x_1 - 2x_2 + 3x_3 = 0. \quad (7)$$

(6) can be written as

$$6x_1 = -3x_2$$

$$\frac{x_1}{-3} = \frac{x_2}{6}$$

$$\frac{x_1}{1} = \frac{x_2}{-2}$$

$$\therefore x_1 = 1, x_2 = -2.$$

Substituting in (5) we obtain

$$5 - 4 + x_3 = 0 \Rightarrow x_3 = -1.$$

$\therefore$ The second eigen vector is $X_2 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$.

When $\lambda = 3$, (1) becomes

$$\begin{pmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

$$\text{i.e.,} \quad -2x_1 + 2x_2 + x_3 = 0 \qquad (8)$$

$$6x_1 - 4x_2 = 0 \qquad (9)$$

$$-x_1 - 2x_2 - 4x_3 = 0. \qquad (10)$$

(9) can be written as

$$6x_1 = 4x_2$$

$$\Rightarrow 3x_1 = 2x_2$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{3}$$

$$\therefore x_1 = 2, x_2 = 3.$$

Substituting in (8) we obtain

$$-4 + 6 + x_3 = 0$$

$$2 + x_3 = 0$$

$$\Rightarrow x_3 = -2.$$

$\therefore$ The third eigen vector is $X_3 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$.

$\therefore$ The eigen vectors are $\begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$.

11. (a) (ii) If the eigenvalues of $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ are $0, 3, 15$ find the eigenvectors of A and diagonalize the matrix A.

Solution.

The given eigen values are 0,3,15.

Step 1. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value λ.

$$\therefore\ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = 0$ (1) becomes

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

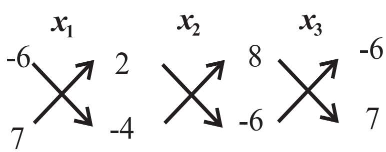
The equations become

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 3x_3 = 0.$$

Taking the first two equations and applying the rule of cross multiplication we get,



$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36} \Rightarrow \frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\therefore \; X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

When $\lambda = 3$ (1) becomes

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$5x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 4x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 0x_3 = 0.$$

Taking the last two equations and applying the rule of cross multiplication we get,

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 4 & & -4 & & -6 & & 4 \\ -4 & & 0 & & 2 & & -4 \end{array}$$

$$\frac{x_1}{0-16} = \frac{x_2}{-8} = \frac{x_3}{24-8}$$

$$\frac{x_1}{-16} = \frac{x_2}{-8} = \frac{x_3}{16}$$

$$\therefore \; X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$

When $\lambda = 15$ (1) becomes

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$-7x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 - 8x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 - 12x_3 = 0.$$

Taking the first two equations and applying the rule of cross multiplication we get,

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -6 & & 2 & & -7 & & -6 \\ -8 & & -4 & & -6 & & -8 \end{array}$$

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

$$\therefore \; X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

$\therefore$ The eigen vectors are $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

Step 2. To find the modal matrix

The normalised eigen vectors are

$\left[\frac{1}{3} \; \frac{2}{3} \; \frac{2}{3}\right]^T, \left[\frac{2}{3} \; \frac{1}{3} \; \frac{-2}{3}\right]^T, \left[\frac{2}{3} \; \frac{-2}{3} \; \frac{1}{3}\right]^T$

$$\therefore \; N = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix}.$$

Step 3. To find $N^T AN$

$$N^T AN = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & -2 \\ 10 & -10 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix} = D.$$

11. (b) (i) Reduce the quadratic form $2x_1x_2 + 2x_2x_3 + 2x_3x_1$ into canonical form.

Solution.

Step 1. To find the matrix of the quadratic form

Since given Q.F. contains 3 variables, the matrix of the Q.F. A is a 3×3 matrix.

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Now, $a_{11} = \text{Coeff.of } x_1^2 = 0$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } x_1x_2 = \frac{1}{2} \times 2 = 1.$$

$$a_{13} = a_{31} = \frac{1}{2} \times \text{Coeff.of } x_1x_3 = \frac{1}{2} \times 2 = 1$$

$$a_{22} = \text{Coeff.of } x_2^2 = 0.$$

$$a_{23} = a_{32} = \frac{1}{2} \times \text{Coeff.of } x_2x_3 = \frac{1}{2} \times 2 = 1.$$

$$a_{33} = \text{Coeff.of } x_3^2 = 0.$$

$$\therefore A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Step 2. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$, where

$$s_1 = \text{sum of the main diagonal elements}$$
$$= 0 + 0 + 0 = 0.$$
$$s_2 = \text{sum of the minors of the main diagonal elements.}$$
$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 + 0 - 1 + 0 - 1 = -3.$$
$$s_3 = |A| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0 - 1(0-1) + 1(1-0) = 1 + 1 = 2.$$

$\therefore$ The characteristic equation is $\lambda^3 - 3\lambda - 2 = 0$.

$\lambda = -1$ is a root. By synthetic division we get

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & 0 & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$$\lambda^2 - \lambda - 2 = 0$$
$$(\lambda - 2)(\lambda + 1) = 0.$$
$$\lambda = 2, -1.$$

The eigen values are $-1, -1, 2$..

Step 3. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value λ.

$$\therefore \ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \tag{1}$$

When $\lambda = 2$ (1) becomes

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$-2x_1 + x_2 + x_3 = 0$$

$$x_1 - 2x_2 + x_3 = 0$$

$$x_1 + x_2 - 2x_3 = 0.$$

Taking the first two equations and applying the rule of cross multiplication we get

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 1 & \times & 1 & \times & -2 & \times \quad 1 \\ -2 & & 1 & & 1 & \quad -2 \end{array}$$

$$\frac{x_1}{1+2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$\implies x_1 = x_2 = x_3 = 1.$$

$$\therefore \; X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

When $\lambda = -1$, (1) becomes $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The above three equations are equivalent to one equation

$$x_1 + x_2 + x_3 = 0$$

$$\text{Let } x_3 = 0.$$

$$\therefore \ x_1 = -x_2$$

$$\frac{x_1}{1} = \frac{x_2}{-1}$$

$$\Rightarrow \ x_1 = 1, x_2 = -1$$

$$\therefore \ X_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ orthogonal to X_2

$$\therefore \quad X_3^T X_2 = 0 \Rightarrow \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow \ a - b = 0 \qquad (2)$$

$$X_3 \text{ also satisfy } a + b + c = 0 \qquad (3)$$

From (2) we have $a = b \Rightarrow \ a = b = 1$.

Substituting in (3) we get $1 + 1 + c = 0 \quad \Rightarrow \quad c = -2$.

$$\therefore \ X_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Step 4. To find the modal matrix

The normalised eigen vectors are $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \end{pmatrix}$.

The modal matrix N is formed with the normalised eigen vectors as columns

$$\therefore\ N = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{pmatrix}.$$

Step 5. To find N^TAN

$$N^TAN = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = D.$$

Step 6. Reduction to canonical form

Let $X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

The given Q.F. is

$Q = X^TAX = (NY)^TANY = Y^TN^TANY = Y^TDY$

$$Y^TDY = [y_1\ y_2\ y_3] \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = [y_1\ y_2\ y_3] \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 2y_1^2 - y_2^2 - y_3^2.$$

which is the required canonical form.

Step 7. To find the nature of the Q.F

Since the Q.F contains both positive and negative eigen values, it is indefinite.

11. (b) (ii) Show that the matrix $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix}$ satisfies its own

characteristic equation. Find also its inverse.

Solution.

Step 1. To find the characteristic equation

Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix}$

Since A is a 3×3 matrix, the characteristic equation is of the form $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$ where

$$s_1 = \text{sum of the main diagonal elemnets.}$$
$$= 1 + 1 + 3 = 5.$$
$$s_2 = \text{sum of the minors of the main diagonal elments.}$$
$$= \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$$
$$= (3-0) + (3-2) + (1-0) = 3 + 1 + 1 = 5.$$
$$s_3 = |A| = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = 1(3-0) + 1(0-0) + 1(0-2) = 3 - 2 = 1.$$

Therefore the characteristic equation of A is $\lambda^3 - 5\lambda^2 + 5\lambda - 1 = 0$.

By Cayley-Hamilton theorem we get $A^3 - 5A^2 + 5A - I = 0$.

Step 2. To verify Cayley Hamilton theorem

$$A^2 = A \times A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 4 \\ 0 & 1 & 0 \\ 8 & -2 & 11 \end{pmatrix}$$

$$A^3 = A \times A^2 = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix}\begin{pmatrix} 3 & -2 & 4 \\ 0 & 1 & 0 \\ 8 & -2 & 11 \end{pmatrix} = \begin{pmatrix} 11 & -5 & 15 \\ 0 & 1 & 0 \\ 30 & -10 & 41 \end{pmatrix}$$

Now, $A^3 - 5A^2 + 5A - I$

$$= \begin{pmatrix} 11 & -5 & 15 \\ 0 & 1 & 0 \\ 30 & -10 & 41 \end{pmatrix} - 5\begin{pmatrix} 3 & -2 & 4 \\ 0 & 1 & 0 \\ 8 & -2 & 11 \end{pmatrix} + 5\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -5 & 15 \\ 0 & 1 & 0 \\ 30 & -10 & 41 \end{pmatrix} + \begin{pmatrix} -15 & 10 & -20 \\ 0 & -5 & 0 \\ -40 & 10 & -55 \end{pmatrix} + \begin{pmatrix} 5 & -5 & 5 \\ 0 & 5 & 0 \\ 10 & 0 & 15 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Hence Cayley Hamilton theorem verified.

Step 3. To find A^{-1}

By Cayley Hamilton theorem, we have $A^3 - 5A^2 + 5A - I = 0$

Multiply by A^{-1}, we get $A^2 - 5A + 5I - A^{-1} = 0$

$$\therefore A^{-1} = A^2 - 5A + 5I$$

$$= \begin{pmatrix} 3 & -2 & 4 \\ 0 & 1 & 0 \\ 8 & -2 & 11 \end{pmatrix} - 5\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 & 4 \\ 0 & 1 & 0 \\ 8 & -2 & 11 \end{pmatrix} + \begin{pmatrix} -5 & 5 & -5 \\ 0 & -5 & 0 \\ -10 & 0 & -15 \end{pmatrix} + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 3 & -1 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{pmatrix}$$

2.1.7 JUNE 2013 (R 2008)

Part A

1. Find the eigenvalues of A^{-1} where $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$.

Solution. Since A is triangular, the eigenvalues of A are the elements in the principal diagonal.

Hence, the eigen values of A are $3, 2, 5$.

$\therefore$ The eigenvalues of A^{-1} are $\frac{1}{3}, \frac{1}{2}, \frac{1}{5}$.

2. Write down matrix of the quadratic form

$$2x^2 + 8z^2 + 4xy + 10xz - 2yz.$$

Solution. If A is the matrix of the quadratic form, then $A = \begin{pmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{pmatrix}$.

Part B

11. (a) (i) Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

Solution.

Step 1. To find the eigenvalues

Since A is a 3×3 matrix, the characteristic equation is of the form $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$, where

$$s_1 = \text{sum of the main diagonal elements} = 2 + 2 + 2 = 6.$$

$$s_2 = \text{sum of the minors of the main diagonal elements}$$

$$= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 + (4 - 1) + 4 = 11.$$

$$s_3 = |A| = 2(4 - 0) + 1(0 - 2) = 8 - 2 = 6.$$

The characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\lambda = 1$ is a root. By synthetic division we have

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$\lambda^2 - 5\lambda + 6 = 0$$
$$(\lambda - 2)(\lambda - 3) = 0$$
$$\Rightarrow \lambda = 2, 3.$$

$\therefore$ The eigen values are $\lambda = 1, 2, 3.$

Step 2. To find the eigenvectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore [A - \lambda I]\, X = 0$$

$$\begin{pmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1}$$

When $\lambda = 1$ equation (1) becomes

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3.$$
$$x_2 = 0.$$
$$x_1 + x_3 = 0$$

If $x_1 = 1$ then $x_3 = -1$.

$$\therefore \quad X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

When $\lambda = 3$ equation (1) becomes

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$-x_1 + x_3 = 0.$$
$$-x_2 = 0.$$

$$x_1 - x_3 = 0.$$

If $x_1 = 1$ then $x_3 = 1$.

$$\therefore X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

When $\lambda = 2$ equation (1) becomes

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$x_3 = 0.$$

$$x_1 = 0.$$

$$x_2 = \text{any value say} = 1.$$

$$\therefore \ X_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

$\therefore$ The eigen vectors are $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

11. (a) (ii) Show that the matrix $A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ satisfies its own characteristic equation. Find also its inverse.

Solution.

Step 1. To find the characteristics equation

Since A is a 3×3 matrix, the characteristic equation is of the form $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$, where

$$s_1 = \text{sum of its leading diagonal elements}$$

$$= 2 + 2 + 2 = 6.$$

$$s_2 = \text{sum of the minors of its leading diagonal elements}$$

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4-1) + (4-2) + (4-1) = 3 + 2 + 3 = 8.$$

$$s_3 = |A| = \begin{vmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= 2(4-1) + 1(-2+1) + 2(1-2)$$

$$= 2(3) + 1(-1) + 2(-1) = 6 - 1 - 2 = 3.$$

$\therefore$ The characteristic equation of A is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$.

By Cayley Hamilton theorem, we have $A^3 - 6A^2 + 8A - 3I = 0$.

Step 2. Verification of Cayley Hamilton theorem.

$$A^2 = A \times A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix}$$

$$A^3 = A \times A^2 = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} = \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix}$$

Now, $A^3 - 6A^2 + 8A - 3I$

$$= \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix} - 6\begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} + 8\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix} - \begin{pmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{pmatrix} + \begin{pmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$\therefore$ Cayley Hamilton theorem is verified.

Step 3. To find A^{-1}

By Cayley Hamilton theorem we have $A^3 - 6A^2 + 8A - 3I = 0$.

Multiply by A^{-1} we get

$$A^2 - 6A + 8I - 3A^{-1} = 0$$

$$3A^{-1} = A^2 - 6A + 8I$$

$$3A^{-1} = \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 8\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} + \begin{pmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$3A^{-1} = \begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}.$$

11. (b) Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2zx$ into canonical form.

Solution. Refer question no. 11. (b) (i) of Nov/Dec. 2014 [2.1.2]

2.2 Unit II Vector Calculus

2.2.1 May/June 2016 (R 2013)

Part A

1. Evaluate $\nabla^2 \log r$.

   Solution. We have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

   $$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}.$$

   $$\therefore r^2 = x^2 + y^2 + z^2.$$

UNIT 2

BY EASYENGINEERING.NET

2 Vector Calculus

2.1 Introduction

Scalar point function

Let P be a point with position vector $\vec{r}$ in a region R. To each point $P(\vec{r})$ of the region R in space, if there is a unique scalar or real number denoted by $\phi(\vec{r})$, then ϕ is called a scalar point function in R. The region R is called a scalar field.

Vector point function

To each point $P(\vec{r})$ of a region R in space, if there exists a unique vector denoted by $\vec{F}(\vec{r})$, then $\vec{F}$ is called a vector point function in R. The region R is called a vector field.

Derivative of a vector function

The derivative of a vector function $\vec{f}(t)$ at t denoted by $\frac{d\vec{f}}{dt}$ is defined as $\frac{d\vec{f}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{f}(t+\Delta t) - \vec{f}(t)}{\Delta t}$ if the limit exists. It is also denoted by $\vec{f}'(t)$.

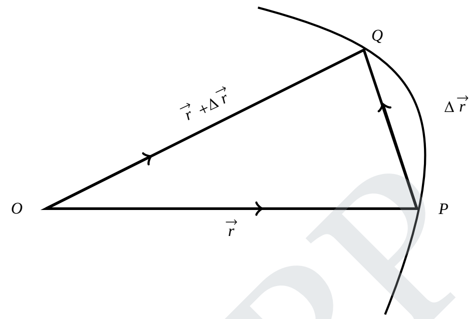
Results

(i) If $\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$, then $\vec{f}'(t) = f_1'(t)\vec{i} + f_2'(t)\vec{j} + f_3'(t)\vec{k}$.

(ii) The higher derivatives of $\vec{f}$ are $\frac{d^2\vec{f}}{dt^2}, \frac{d^3\vec{f}}{dt^3}, \ldots$

(iii) If $\vec{c}$ is a constant vector, then $\frac{d\vec{c}}{dt} = \vec{0}$.

Geometrical meaning of the derivative



Let P be a point on a curve with position vector $\vec{r}$. Let Q be an adjacent point with position vector $\vec{r}+\Delta\vec{r}$. Then, $\frac{d\vec{r}}{dt} = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t}$, which is along the tangent at P.

$\therefore \frac{d\vec{r}}{dt}$ denotes the tangent vector at P.

The unit tangent vector is $\frac{\frac{d\vec{r}}{dt}}{|\frac{d\vec{r}}{dt}|}$.

Differentiation formulae

Let $\vec{f}$ and $\vec{g}$ be two differentiable vector functions. Then the following are easy to derive.

(i) $\frac{d}{dt}(\vec{f}\pm\vec{g}) = \frac{d\vec{f}}{dt}\pm\frac{d\vec{g}}{dt}$.

(ii) If ϕ is a scalar point function, then $\frac{d}{dt}(\phi\vec{f}) = \phi\frac{d\vec{f}}{dt}+\frac{d\phi}{dt}\vec{f}$.

(iii) $\frac{d}{dt}(\vec{f}\cdot\vec{g}) = \vec{f}\cdot\frac{d\vec{g}}{dt}+\frac{d\vec{f}}{dt}\cdot\vec{g}$.

(iv) $\frac{d}{dt}(\vec{f}\times\vec{g}) = \vec{f}\times\frac{d\vec{g}}{dt}+\frac{d\vec{f}}{dt}\times\vec{g}$.

(v) $\frac{d}{dt}[\vec{f}\,\vec{g}\,\vec{h}] = \frac{d}{dt}[\vec{f}\cdot(\vec{g}\times\vec{h})] = [\frac{d\vec{f}}{dt}\,\vec{g}\,\vec{h}]+[\vec{f}\,\frac{d\vec{g}}{dt}\,\vec{h}]+[\vec{f}\,\vec{g}\,\frac{d\vec{h}}{dt}]$.

(vi) If $\vec{f}$ is a function of the scalar s and s is a function of t then $\frac{d\vec{f}}{dt} = \frac{d\vec{f}}{ds}\frac{ds}{dt}$.

Level Surfaces

Let ϕ be a continuous scalar point function defined in the region R. The set of all points satisfying the equation $\phi(x,y,z) = c$ defines a surface called the level surface.

2.2 Gradient of a scalar point function

The definition of ∇

The Hamiltonian operator or the vector differential operator denoted by ∇ is defined as $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$.

If $\phi(x, y, z)$ is a scalar point function, continuously differentiable in a given region R, then the gradient of ϕ is defined as

$$grad\ \phi = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}.$$

Geometrical meaning of $\nabla\phi$

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ be the position vector of any point P on the level surface $\phi(x, y, z) = c$. Now,

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k},$$

which is a tangent to the surface at P. Also,

$$\nabla\phi \cdot d\vec{r} = \left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right) \cdot \left(dx\vec{i} + dy\vec{j} + dz\vec{k}\right)$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= d\phi = 0 \qquad [\text{Since } \phi = c].$$

$\therefore$ $\nabla\phi$ is perpendicular to $d\vec{r}$.

But $d\vec{r}$ is the tangent vector at P. Hence $\nabla\phi$ is normal at P.

$\therefore$ $\frac{\nabla\phi}{|\nabla\phi|}$ is the unit normal vector at P.

2.3 Directional derivative

The directional derivative of a scalar point function ϕ in a given direction $\vec{a}$ is the rate of change of ϕ in the direction of $\vec{a}$. It is the component of $\nabla\phi$ in the direction of $\vec{d}$, which is given by $\dfrac{\nabla\phi\cdot\vec{d}}{|\vec{d}|}$.

$$\begin{aligned}\frac{\nabla\phi\cdot\vec{d}}{|\vec{d}|} &= \nabla\phi\cdot\frac{\vec{d}}{|\vec{d}|}\\ &= |\nabla\phi|\times\left|\frac{\vec{d}}{|\vec{d}|}\right|\times\cos\theta,\ \text{ where } \theta \text{ is the angle between } \nabla\phi \text{ and } \vec{d}.\\ &= |\nabla\phi|\cos\theta.\end{aligned}$$

Note

1. The directional derivative at a given point is maximum when $\cos\theta$ is maximum. The maximum value of $\cos\theta$ is 1. i.e. when $\theta = 0$.
   Therefore, Maximum directional derivative = $|\nabla\phi|$.
2. The directional derivative is minimum when $\cos\theta = -1$. i.e. when $\theta = \pi$.
   Therefore, Minimum directional derivative = $-|\nabla\phi|$.

Equation of the tangent plane

Let $\phi(x, y, z) = c$ be a level surface and A be a given point with position vector $\vec{r}_0$.
Let P be any point on the tangent plane at A with position vector $\vec{r}$. Let

$$\begin{aligned}\vec{r} &= x\vec{i} + y\vec{j} + z\vec{k}.\\ \vec{r}_0 &= x_0\vec{i} + y_0\vec{j} + z_0\vec{k}.\\ \text{Now, } \vec{AP} = \vec{r} - \vec{r}_0 &= (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}.\end{aligned}$$

Since $\nabla\phi$ is normal at A, $\vec{AP} = \vec{r} - \vec{r}_0$ lies in the tangent plane at A.
Therefore, $(\vec{r} - \vec{r}_0)\cdot\nabla\phi = 0$.
i.e., $(x - x_0)\dfrac{\partial\phi}{\partial x} + (y - y_0)\dfrac{\partial\phi}{\partial y} + (z - z_0)\dfrac{\partial\phi}{\partial z} = 0$, which is the equation of the tangent plane at $A(\vec{r}_0)$. The partial derivatives are evaluated at (x_0, y_0, z_0).

Equation of the normal

If $P(\vec{r})$ is any point on the normal at A, then $\nabla\phi$ and $\vec{r} - \vec{r}_0$ are parallel.
Therefore, the equation of the normal at A is $(\vec{r} - \vec{r}_0)\times\nabla\phi = \vec{0}$.

—

It's cartesian equation is $\frac{x-x_0}{\frac{\partial\phi}{\partial x}} = \frac{y-y_0}{\frac{\partial\phi}{\partial y}} = \frac{z-z_0}{\frac{\partial\phi}{\partial z}}$, the partial derivatives are to be evaluated at $A(x_0, y_0, z_0)$.

Angle between two surfaces

Let $f(x,y,z) = c_1$ & $g(x,y,z) = c_2$ be two surfaces. Let P be a common point. The angle between the two surfaces is equal to the angle between the normals at P. If θ is the angle between the normals, then

$$\cos\theta = \frac{\nabla f \cdot \nabla g}{|\nabla f||\nabla g|}.$$

If the surfaces are perpendicular or orthogonal, then $\nabla f \cdot \nabla g = 0$.

The surfaces touch each other when $\theta = 0$.

Properties of $\nabla\phi$

Let f and g be two scalar point functions and c is a constant. Then

(i) $\nabla c = \vec{0}$.

(ii) $\nabla(cf) = c\nabla f$.

(iii) $\nabla(f \pm g) = \nabla f \pm \nabla g$.

(iv) $\nabla(fg) = f\nabla g + g\nabla f$.

(v) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}, g \neq 0$.

Proof.

(i) $\nabla c = \vec{i}\frac{\partial c}{\partial x} + \vec{j}\frac{\partial c}{\partial y} + \vec{k}\frac{\partial c}{\partial z} = \vec{0}$. $\left[\because c \text{ is a constant, } \frac{\partial c}{\partial x} = \frac{\partial c}{\partial y} = \frac{\partial c}{\partial z} = 0\right]$.

(ii) $\nabla cf = \vec{i}\frac{\partial(cf)}{\partial x} + \vec{j}\frac{\partial(cf)}{\partial y} + \vec{k}\frac{\partial(cf)}{\partial z} = c\vec{i}\frac{\partial f}{\partial x} + c\vec{j}\frac{\partial f}{\partial y} + c\vec{k}\frac{\partial f}{\partial z}$

$= c\left(\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}\right) = c\nabla f.$

$(iii)\nabla(f \pm g) = \vec{i}\frac{\partial}{\partial x}(f \pm g) + \vec{j}\frac{\partial}{\partial y}(f \pm g) + \vec{k}\frac{\partial}{\partial z}(f \pm g)$

$= \vec{i}\left[\frac{\partial f}{\partial x} \pm \frac{\partial g}{\partial x}\right] + \vec{j}\left[\frac{\partial f}{\partial y} \pm \frac{\partial g}{\partial y}\right] + \vec{k}\left[\frac{\partial f}{\partial z} \pm \frac{\partial g}{\partial z}\right]$

$= \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z} \pm \left(\vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial g}{\partial z}\right)$

$= \nabla f \pm \nabla g.$

$$(iv)\nabla(fg) = \vec{i}\frac{\partial}{\partial x}(fg) + \vec{j}\frac{\partial}{\partial y}(fg) + \vec{k}\frac{\partial}{\partial z}(fg)$$
$$= \vec{i}\left[f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right] + \vec{j}\left[f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right] + \vec{k}\left[f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right]$$
$$= f\left(\vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial g}{\partial z}\right) + g\left(\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}\right)$$
$$= f\nabla g + g\nabla f.$$
$$(v)\nabla\left(\frac{f}{g}\right) = \vec{i}\frac{\partial}{\partial x}\left(\frac{f}{g}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{f}{g}\right) + \vec{k}\frac{\partial}{\partial z}\left(\frac{f}{g}\right)$$
$$= \vec{i}\left[\frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\right] + \vec{j}\left[\frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}\right] + \vec{k}\left[\frac{g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}}{g^2}\right]$$
$$= \frac{g\left[\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}\right]}{g^2} - \frac{f\left[\vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial g}{\partial z}\right]}{g^2}$$
$$= \frac{g\nabla f}{g^2} - \frac{f\nabla g}{g^2}$$
$$= \frac{g\nabla f - f\nabla g}{g^2}.$$

Worked Examples

Example 2.1. Find $|\nabla\phi|$ if $\phi = 2xz^4 - x^2y$ at $(2, -2, 1)$. [Jan 2009]

Solution. Given $\phi = 2xz^4 - x^2y$.

$$\frac{\partial\phi}{\partial x} = 2z^4 - 2xy.$$
$$\frac{\partial\phi}{\partial y} = -x^2.$$
$$\frac{\partial\phi}{\partial z} = 8xz^3.$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = \vec{i}(2z^4 - 2xy) + \vec{j}(-x^2) + \vec{k}(8xz^3).$$

$$(\nabla\phi)_{(2,-2,1)} = \vec{i}(2(1) - 2(2)(-2)) - 4\vec{j} + \vec{k}8(2)1 = 10\vec{i} - 4\vec{j} + 16\vec{k}$$

$$|\nabla\phi| = \sqrt{100 + 16 + 256} = \sqrt{372}.$$

Example 2.2. Find grad ϕ at $(1, -2, 1)$ if $\phi = 3x^2y - y^3z^2$. [Jan 2006]

Solution. Given $\phi = 3x^2y - y^3z^2$.

$$\frac{\partial \phi}{\partial x} = 6xy.$$
$$\frac{\partial \phi}{\partial y} = 3x^2 - 3y^2z^2.$$
$$\frac{\partial \phi}{\partial z} = -2y^3z.$$

$$\text{grad } \phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z} = \vec{i}(6xy) + \vec{j}(3x^2 - 3y^2z^2) + \vec{k}(-2y^3z).$$

$$(\text{grad } \phi)_{(1,-2,1)} = -12\vec{i} + \vec{j}(-9) + 16\vec{k} = -12\vec{i} - 9\vec{j} + 16\vec{k}.$$

Example 2.3. If $\phi = x^2 + y^2 + z^2$, find $\nabla\phi$ at (1,1,-1). [Jan 2005]

Solution. Given $\phi = x^2 + y^2 + z^2$.

We have $\nabla\phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$

$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}.$

$(\nabla\phi)_{(1,1,-1)} = 2\vec{i} + 2\vec{j} - 2\vec{k}.$

Example 2.4. Find the directional derivative of $\phi = xyz$ at $(1, 1, 1)$ in the direction of $\vec{i} + \vec{j} + \vec{k}$. [Jun 2013]

Solution. Given $\phi = xyz$.

$$\nabla\phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$$
$$= yz\vec{i} + xz\vec{j} + xy\vec{k}.$$
$$(\nabla\phi)_{(1,1,1)} = \vec{i} + \vec{j} + \vec{k}.$$

Let $\vec{a} = \vec{i} + \vec{j} + \vec{k}$.

$|\vec{a}| = \sqrt{1 + 1 + 1} = \sqrt{3}.$

Directional derivative of ϕ in the direction of $\vec{a} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$

$$= (\vec{i} + \vec{j} + \vec{k}) \cdot \frac{(\vec{i} + \vec{j} + \vec{k})}{\sqrt{3}}$$
$$= \frac{1 + 1 + 1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

—

Example 2.5. Find the directional derivative of $\phi = xy + yz + zx$ in the direction of the vector $\vec{i} + 2\vec{j} + 2\vec{k}$ at $(1, 2, 0)$. [Jun 2009]

Solution. Given $\phi = xy + yz + zx$.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$
$$= (y + z)\vec{i} + (x + z)\vec{j} + (y + x)\vec{k}.$$
$$(\nabla\phi)_{(1,2,0)} = 2\vec{i} + \vec{j} + 3\vec{k}.$$

Let $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$.

$$|\vec{a}| = \sqrt{1 + 4 + 4} = \sqrt{9} = 3.$$

The directional derivative of ϕ in the direction of $\vec{a} = \nabla\phi.\dfrac{\vec{a}}{|\vec{a}|}$

$$= (2\vec{i} + \vec{j} + 3\vec{k}).\frac{(\vec{i} + 2\vec{j} + 2\vec{k})}{3} = \frac{2 + 2 + 6}{3} = \frac{10}{3}.$$

Example 2.6. Find the directional derivative of $\phi = xy + yz + zx$ at the point (1,2,3) in the direction $3\vec{i} + 4\vec{j} + 5\vec{k}$. [Apr 2003]

Solution. Given $\phi = xy + yz + zx$.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$
$$= (y + z)\vec{i} + (x + z)\vec{j} + (x + y)\vec{k}.$$
$$(\nabla\phi)_{(1,2,3)} = (2 + 3)\vec{i} + (1 + 3)\vec{j} + (1 + 2)\vec{k}$$
$$= 5\vec{i} + 4\vec{j} + 3\vec{k}.$$
$$\vec{a} = 3\vec{i} + 4\vec{j} + 5\vec{k}.$$
$$|\vec{a}| = \sqrt{9 + 16 + 25} = \sqrt{50} = 5\sqrt{2}.$$

Directional derivative in the direction of $\vec{a} = \nabla\phi.\dfrac{\vec{a}}{|\vec{a}|}$

$$= (5\vec{i} + 4\vec{j} + 3\vec{k}).\frac{(3\vec{i} + 4\vec{j} + 5\vec{k})}{5\sqrt{2}}$$
$$= \frac{15 + 16 + 15}{5\sqrt{2}}$$
$$= \frac{46}{5\sqrt{2}}.$$

Example 2.7. Find the directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ at the point $(1, -2, -1)$ in the direction of the vector $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution. Given $\phi(x, y, z) = x^2yz + 4xz^2$.

$$\begin{aligned}
\nabla\phi &= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \\
&= (2xyz + 4z^2)\vec{i} + x^2z\vec{j} + (x^2y + 8xz)\vec{k}. \\
(\nabla\phi)_{(1,-2,-1)} &= (2(1)(-2)(-1) + 4)\vec{i} - \vec{j} + (-2 - 8)\vec{k} \\
&= 8\vec{i} - \vec{j} - 10\vec{k}.
\end{aligned}$$

$$\begin{aligned}
\text{Let } \vec{a} &= 2\vec{i} - \vec{j} - 2\vec{k}. \\
|\vec{a}| &= \sqrt{4 + 1 + 4} = 3. \\
\frac{\vec{a}}{|\vec{a}|} &= \frac{1}{3}(2\vec{i} - \vec{j} - 2\vec{k}).
\end{aligned}$$

The directional derivative of ϕ in the direction of $\vec{a} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{1}{3}(16 + 1 + 20) = \frac{37}{3}$.

Example 2.8. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $P(1, -2, -1)$ in the direction of PQ where Q is $(3, -3, -3)$. [Jun 2008]

Solution. Given $\phi = x^2yz + 4xz^2$.

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= 2xyz + 4z^2. \\
\left(\frac{\partial\phi}{\partial x}\right)_{(1,-2,-1)} &= 2 \cdot 1 \cdot (-2)(-1) + 4(-1)^2 = 4 + 4 = 8. \\
\frac{\partial\phi}{\partial y} &= x^2z \\
\left(\frac{\partial\phi}{\partial y}\right)_{(1,-2,-1)} &= 1(-1) = -1. \\
\frac{\partial\phi}{\partial z} &= x^2y + 8xz \\
\left(\frac{\partial\phi}{\partial z}\right)_{(1,-2,-1)} &= 1(-2) + 8 \cdot 1(-1) = -10. \\
(\nabla\phi)_{(1,-2,-1)} &= \left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)_{(1,-2,-1)} \\
&= 8\vec{i} - \vec{j} - 10\vec{k}.
\end{aligned}$$

$$\vec{a} = \overrightarrow{PQ} = 3\vec{i} - 3\vec{j} - 3\vec{k} - (\vec{i} - 2\vec{j} - \vec{k})$$
$$= 2\vec{i} - \vec{j} - 2\vec{k}.$$
$$|\vec{a}| = \sqrt{4+1+4} = 3.$$
$$\frac{\vec{a}}{|\vec{a}|} = \frac{1}{3}(2\vec{i} - \vec{j} - 2\vec{k}).$$
$$\text{Directional derivative} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{1}{3}(16+1+20) = \frac{37}{3}.$$

Example 2.9. Find the maximum value of the directional derivative of $\phi = x^3yz$ at $(1, 4, 1)$.

Solution. Given $\phi = x^3yz$.

$$\frac{\partial\phi}{\partial x} = 3x^2yz.$$
$$\left(\frac{\partial\phi}{\partial x}\right)_{(1,4,1)} = 3\cdot 1\cdot 4\cdot 1 = 12.$$
$$\frac{\partial\phi}{\partial y} = x^3z.$$
$$\left(\frac{\partial\phi}{\partial y}\right)_{(1,4,1)} = 1\cdot 1 = 1.$$
$$\frac{\partial\phi}{\partial z} = x^3y.$$
$$\left(\frac{\partial\phi}{\partial z}\right)_{(1,4,1)} = 1\cdot 4 = 4.$$
$$(\nabla\phi)_{(1,4,1)} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 12\vec{i} + \vec{j} + 4\vec{k}.$$

Maximum value of directional derivative $= |\nabla\phi| = \sqrt{144+1+16} = \sqrt{161}.$

Example 2.10. In what direction from the point $(1, 1, -2)$ is the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ maximum. Also find the maximum directional derivative.

Solution. Given $\phi = x^2 - 2y^2 + 4z^2$.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 2x\vec{i} - 4y\vec{j} + 8z\vec{k}.$$
$$(\nabla\phi)_{(1,1,-2)} = 2\vec{i} - 4\vec{j} - 16\vec{k}.$$

—

The directional derivative is maximum in the direction of $\nabla\phi = 2\vec{i} - 4\vec{j} - 16\vec{k}$.

$$\begin{aligned}\text{Maximum value of the directional derivative} &= |\nabla\phi| \\ &= \sqrt{4+16+256} \\ &= \sqrt{276} \\ &= \sqrt{4\times 69} \\ &= 2\sqrt{69}.\end{aligned}$$

Example 2.11. In what direction from the point (1,-1,-2) is the directional derivative of $\phi = x^3y^3z^3$ a maximum? What is the magnitude of this maximum.

[May 2005]

Solution. Given $\phi = x^3y^3z^3$.

$$\begin{aligned}\nabla\phi &= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \\ &= 3x^2y^3z^3\vec{i} + 3x^3y^2z^3\vec{j} + 3x^3y^3z^2\vec{k}. \\ (\nabla\phi)_{(1,-1,-2)} &= 3\times 1\times(-1)\times(-8)\vec{i} + 3\times 1\times 1\times(-8)\vec{j} + 3\times 1\times(-1)\times 4\vec{k} \\ &= 24\vec{i} - 24\vec{j} - 12\vec{k}. \\ |\nabla\phi| &= \sqrt{(24)^2 + (-24)^2 + (-12)^2} \\ &= \sqrt{576+576+144} \\ &= \sqrt{1296} \\ &= 36.\end{aligned}$$

The directional derivative is maximum in the direction of $24\vec{i} - 24\vec{j} - 12\vec{k}$ and its maximum value is 36.

Example 2.12. Find a unit normal vector to the surface $x^3+y^3+3xyz = 3$ at $(1, 2, -1)$.

Solution. Given $\phi = x^3 + y^3 + 3xyz - 3$.

$$\frac{\partial\phi}{\partial x} = 3x^2 + 3yz.$$

$$\left(\frac{\partial\phi}{\partial x}\right)_{(1,2,-1)} = 3 - 6 = -3.$$

$$\frac{\partial\phi}{\partial y} = 3xz + 3y^2.$$

$$\left(\frac{\partial\phi}{\partial y}\right)_{(1,2,-1)} = -3 + 12 = 9.$$

$$\frac{\partial\phi}{\partial z} = 3xy.$$

$$\left(\frac{\partial\phi}{\partial z}\right)_{(1,2,-1)} = 6.$$

$$(\nabla\phi) = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= -3\vec{i} + 9\vec{j} + 6\vec{k}.$$

$$|\nabla\phi| = \sqrt{9 + 81 + 36}$$

$$= \sqrt{126} = \sqrt{9 \times 14} = 3\sqrt{14}.$$

$$\text{Unit normal vector} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3(-\vec{i} + 3\vec{j} + 2\vec{k})}{3\sqrt{14}}$$

$$= \frac{-\vec{i} + 3\vec{j} + 2\vec{k}}{\sqrt{14}}.$$

Example 2.13. Find a unit normal vector to the surface $x^2y + 2xz^2 = 8$ at $(1, 0, 2)$.

Solution. Given $\phi = x^2y + 2xz^2 - 8$.

$$(\nabla\phi) = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}.$$

$$= (2xy + 2z^2)\vec{i} + x^2\vec{j} + 4xz\vec{k}$$

$$(\nabla\phi)_{(1,0,2)} = 8\vec{i} + \vec{j} + 8\vec{k}.$$

$$|\nabla\phi| = \sqrt{64 + 1 + 64} = \sqrt{129}.$$

$$\text{Unit normal vector} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{8\vec{i} + \vec{j} + 8\vec{k}}{\sqrt{129}}.$$

Example 2.14. Find the angle between the surfaces $x^2+y^2+z^2 = 9$ and $x^2+y^2-z = 3$ at the point $(2, -1, 2)$.

Solution. Let the given surfaces be f and g.

$\therefore\ f = x^2 + y^2 + z^2 - 9$ and $g = x^2 + y^2 - z - 3$.

$$\nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z} = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}.$$

$$(\nabla f)_{(2,-1,2)} = 4\vec{i} - 2\vec{j} + 4\vec{k}.$$

$$|\nabla f| = \sqrt{16+4+16} = \sqrt{36} = 6.$$

$$\nabla g = \vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial g}{\partial z} = 2x\vec{i} + 2y\vec{j} - \vec{k}.$$

$$(\nabla g)_{(2,-1,2)} = 4\vec{i} - 2\vec{j} - \vec{k}.$$

$$|\nabla g| = \sqrt{16+4+1} = \sqrt{21}.$$

Let θ be the angle between the surfaces. Then

$$\cos\theta = \frac{\nabla f \cdot \nabla g}{|\nabla f||\nabla g|} = \frac{16+4-4}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}.$$

$$\therefore\quad \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

Example 2.15. Find the angle between the surfaces $x\log z = y^2 - 1$ and $x^2 y = 2 - z$ at the point (1,1,1). [May 2001]

Solution. Let ϕ_1 and ϕ_2 be the given two surfaces.

Given $\phi_1 = x\log z - y^2 + 1$.

$\phi_2 = x^2 y - 2 + z$.

$$\nabla\phi_1 = \vec{i}\frac{\partial\phi_1}{\partial x} + \vec{j}\frac{\partial\phi_1}{\partial y} + \vec{k}\frac{\partial\phi_1}{\partial z}$$

$$= \log z\vec{i} - 2y\vec{j} + \frac{x}{z}\vec{k}.$$

$$(\nabla\phi_1)_{(1,1,1)} = 0\times\vec{i} - 2\vec{j} + \vec{k}.$$

$$= -2\vec{j} + \vec{k}.$$

$$\nabla\phi_2 = \vec{i}\frac{\partial\phi_2}{\partial x} + \vec{j}\frac{\partial\phi_2}{\partial y} + \vec{k}\frac{\partial\phi_2}{\partial z}$$

$$= 2xy\vec{i} + x^2\vec{j} + \vec{k}$$

$$(\nabla\phi_2)_{(1,1,1)} = 2\vec{i} + \vec{j} + \vec{k}.$$

—

Let θ be the acute angle between the two surfaces.

$$\therefore \cos\theta = \left|\frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}\right|$$
$$= \left|\frac{-2+1}{\sqrt{4+1}\sqrt{4+1+1}}\right|$$
$$= \frac{1}{\sqrt{5}\sqrt{6}} = \frac{1}{\sqrt{30}}$$
$$\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{30}}\right).$$

Example 2.16. Show that the surfaces $5x^2 - 2yz - 9x = 0$ and $4x^2y + z^3 - 4 = 0$ are orthogonal at $(1, -1, 2)$.

Solution. Let the given surfaces be f and g.

$$\therefore f = 5x^2 - 2yz - 9x \text{ and } g = 4x^2y + z^3 - 4.$$

$$\nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z} = (10x - 9)\vec{i} + (-2z)\vec{j} - 2y\vec{k}.$$
$$(\nabla f)_{(1,-1,2)} = \vec{i} - 4\vec{j} + 2\vec{k}.$$
$$\nabla g = \vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial g}{\partial z} = 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}.$$
$$(\nabla g)_{(1,-1,2)} = -8\vec{i} + 4\vec{j} + 12\vec{k}.$$
$$\nabla f.\nabla g = (\vec{i} - 4\vec{j} + 2\vec{k}).(-8\vec{i} + 4\vec{j} + 12\vec{k})$$
$$= -8 - 16 + 24 = 0.$$

$\therefore$ The normals to the surfaces are orthogonal.

$\Longrightarrow$ The given two surfaces are orthogonal.

Example 2.17. Find the angle between the normals to the surface $xy = z^2$ at the points (1,4,2) and (-3,-3,3). [May 2011, Jan 2004]

Solution. Given $\phi = xy - z^2$.

Let $\vec{n}_1$ and $\vec{n}_2$ be the normals to the surface ϕ at (1,4,2) and (-3,-3,3) respectively.

—

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = y\vec{i} + x\vec{j} - 2z\vec{k}.$$

$$\vec{n}_1 = (\nabla\phi)_{(1,4,2)} = 4\vec{i} + \vec{j} - 4\vec{k}.$$

$$\vec{n}_2 = (\nabla\phi)_{(-3,-3,3)} = -3\vec{i} - 3\vec{j} - 6\vec{k}.$$

Let θ be the angle between the two normals.

$$\therefore \cos\theta = \frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}.$$

$$= \frac{-12-3+24}{\sqrt{16+1+16}\,\sqrt{9+9+36}} = \frac{9}{\sqrt{33}\,\sqrt{54}} = \frac{9}{\sqrt{33\times 54}}$$

$$= \frac{9}{\sqrt{3\times 11\times 6\times 9}} = \frac{9}{\sqrt{81\times 22}} = \frac{1}{\sqrt{22}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{22}}\right).$$

Example 2.18. Find a and b if the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1,-1,2)$.

Solution. Let the given surfaces be f and g.

$$\therefore f = ax^2 - byz - (a+2)x \text{ and } g = 4x^2y + z^3 - 4.$$

$$\nabla f = (2ax - (a+2))\vec{i} + \vec{j}(-bz) - by\vec{k}.$$

$$(\nabla f)_{(1,-1,2)} = (2a - a - 2)\vec{i} - 2b\vec{j} + b\vec{k}.$$

$$= (a-2)\vec{i} - 2b\vec{j} + b\vec{k}.$$

$$\nabla g = \vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial g}{\partial z}.$$

$$= 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}.$$

$$(\nabla g)_{(1,-1,2)} = -8\vec{i} + 4\vec{j} + 12\vec{k}.$$

Given that the surfaces are orthogonal.

$$\therefore \nabla f \cdot \nabla g = 0$$

$$-8(a-2) - 8b + 12b = 0$$

$$-8a + 16 + 4b = 0$$

$$2a - b = 4. \quad (1)$$

—

$(1,-1,2)$ is a point on the surface $f=0$.

$$\therefore a+2b = a+2$$

$$2b = 2 \Rightarrow b = 1.$$

subtitute $b=1$ in (1) we get

$$a = \frac{5}{2}.$$

Example 2.19. Find the values of a and b so that the surfaces $ax^3-by^2z=(a+3)x^2$ and $4x^2y-z^3=11$ may cut orthogonally at (2,-1,-3). [Dec 2013, June 2006]

Solution. Let ϕ_1 and ϕ_2 be the given surfaces.

$$\therefore \phi_1 = ax^3 - by^2z - (a+3)x^2.$$

$$\phi_2 = 4x^2y - z^3 - 11.$$

$$\nabla\phi_1 = \vec{i}\frac{\partial\phi_1}{\partial x} + \vec{j}\frac{\partial\phi_1}{\partial y} + \vec{k}\frac{\partial\phi_1}{\partial z}$$

$$= [3ax^2 - 2(a+3)x]\vec{i} - 2byz\vec{j} - by^2\vec{k}$$

$$(\nabla\phi_1)_{(2,-1,-3)} = (12a - 4(a+3))\vec{i} - 6b\vec{j} - b\vec{k}$$

$$= (8a-12)\vec{i} - 6b\vec{j} - b\vec{k}.$$

$$\nabla\phi_2 = \vec{i}\frac{\partial\phi_2}{\partial x} + \vec{j}\frac{\partial\phi_2}{\partial y} + \vec{k}\frac{\partial\phi_2}{\partial z}$$

$$\nabla\phi_2 = 8xy\vec{i} + 4x^2\vec{j} - 3z^2\vec{k}.$$

$$(\nabla\phi_2)_{(2,-1,-3)} = -16\vec{i} + 16\vec{j} - 27\vec{k}.$$

Given that the surfaces are orthogonal.

$$\therefore \nabla\phi_1.\nabla\phi_2 = 0$$

$$-16(8a-12) + 16(-6b) + 27b = 0$$

$$-128a + 192 - 96b + 27b = 0$$

$$-128a - 69b + 192 = 0$$

$$128a + 69b = 192. \qquad (1)$$

$(2,-1,-3)$ lies on ϕ_1.

$$\therefore\ 8a + 3b - 4(a+3) = 0$$

$$8a + 3b - 4a - 12 = 0$$

$$4a + 3b = 12. \quad (2)$$

$$(2) \times 23 \Rightarrow 92a + 69b = 276. \quad (3)$$

$$(1) - (3) \Rightarrow 36a = -84$$

$$a = \frac{-84}{36} = \frac{-7}{3}.$$

Substituting in (2) we get

$$4\left(-\frac{7}{3}\right) + 3b = 12$$

$$3b = 12 + \frac{28}{3}$$

$$3b = \frac{36+28}{3} = \frac{64}{3}$$

$$b = \frac{64}{9}.$$

Example 2.20. Find the directional derivative of the function $\phi = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of the normal to the surface $x\log z - y^2 - 4 = 0$ at the point $(-1,2,1)$. [Jan 2008]

Solution. Let $\phi = xy^2 + yz^3$.

$$\nabla\phi = \vec{i}y^2 + (2xy + z^3)\vec{j} + 3yz^2\vec{k}.$$

$$(\nabla\phi)_{(2,-1,1)} = \vec{i} - 3\vec{j} - 3\vec{k}.$$

$$\text{Let } f = x\log z - y^2 - 4.$$

$$\nabla f = \vec{i}\log z + \vec{j}(-2y) + \vec{k}\frac{x}{z}.$$

$$(\nabla f)_{(-1,2,1)} = -4\vec{j} - \vec{k} = \vec{a}.$$

$$\text{Directional Derivative} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{0+12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}.$$

Example 2.21. Find the directional derivative of $\phi = x^2y^2z^2$ at the point (1,1,1) along the normal to the surface $x^2 + 2xy - z^2$ at the point (1,1,1). [May 2002]

Solution. Given $\phi = x^2y^2z^2$.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$
$$= 2xy^2z^2\vec{i} + 2x^2yz^2\vec{j} + 2x^2y^2z\vec{k}.$$
$$(\nabla\phi)_{(1,1,1)} = 2\vec{i} + 2\vec{j} + 2\vec{k}.$$

Let $\phi_1 = x^2 + 2xy - z^2$.

$$\nabla\phi_1 = \vec{i}\frac{\partial\phi_1}{\partial x} + \vec{j}\frac{\partial\phi_1}{\partial y} + \vec{k}\frac{\partial\phi_1}{\partial z}$$
$$= (2x+2y)\vec{i} + 2x\vec{j} + (-2z)\vec{k}.$$
$$(\nabla\phi_1)_{(1,1,1)} = 4\vec{i} + 2\vec{j} - 2\vec{k}.$$

Let $\vec{n}$ be the unit normal vector to the surface ϕ_1.

$$\therefore \vec{n} = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{4\vec{i}+2\vec{j}-2\vec{k}}{\sqrt{16+4+4}}$$
$$= \frac{4\vec{i}+2\vec{j}-2\vec{k}}{\sqrt{24}} = \frac{4\vec{i}+2\vec{j}-2\vec{k}}{2\sqrt{6}}$$
$$= \frac{2\vec{i}+\vec{j}-\vec{k}}{\sqrt{6}}$$

Directional derivative of ϕ along $\vec{n} = \nabla\phi.\vec{n}$

$$= (2\vec{i}+2\vec{j}+2\vec{k}).\frac{(2\vec{i}+\vec{j}-\vec{k})}{\sqrt{6}}$$
$$= \frac{4+2-2}{\sqrt{6}}.$$
$$= \frac{4}{\sqrt{6}}.$$

Example 2.22. Find ϕ if $\nabla\phi = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$.

Solution. We have $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$.

But it is given that $\nabla\phi = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$

Comparing the two values of $\nabla\phi$ we obtain

$$\frac{\partial\phi}{\partial x} = 6xy + z^3. \qquad (1)$$
$$\frac{\partial\phi}{\partial y} = 3x^2 - z. \qquad (2)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 - y. \tag{3}$$

Integrating (1), (2) and (3) partially w.r.to. x, y and z respectively we get,

$$\phi = 6y\frac{x^2}{2} + xz^3 + f_1(y,z)$$

$$\text{i.e., } \phi = 3x^2y + xz^3 + f_1(y,z). \tag{4}$$

$$\phi = 3x^2y - zy + f_2(x,z). \tag{5}$$

$$\text{and } \phi = 3x\frac{z^3}{3} - yz + f_3(x,y)$$

$$\text{i.e., } \phi = xz^3 - yz + f_3(x,y). \tag{6}$$

From (4), (5) and (6) we obtain, $\phi = 3x^2y + xz^3 - yz + c$.

Example 2.23. If $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$, find ϕ if $\phi(1,-2,2) = 4$.

Solution. We have $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$.

But it is given that $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$

Comparing the two values of $\nabla\phi$ we obtain

$$\frac{\partial\phi}{\partial x} = 2xyz^3. \tag{1}$$

$$\frac{\partial\phi}{\partial y} = x^2z^3. \tag{2}$$

$$\frac{\partial\phi}{\partial z} = 3x^2yz^2. \tag{3}$$

Integrating (1), (2) and (3) partially w.r.to. x, y and z respectively we get,

$$\phi = 2yz^3\int x\,dx + f_1(y,z)$$

$$= 2yz^3\frac{x^2}{2} + f_1(y,z)$$

$$= x^2yz^3 + f_1(y,z). \tag{4}$$

$$\text{Also, } \phi = x^2z^3\int dy + f_2(x,z)$$

$$\phi = x^2yz^3 + f_2(x,z). \tag{5}$$

—

Again, $\phi = 3x^2y\int z^2dz + f_3(x,y)$

$$\phi = 3x^2y\frac{z^3}{3} + f_3(x,y)$$

$$\phi = x^2yz^3 + f_3(x,y) \tag{6}$$

From (4), (5) and (6) we obtain,

$$\phi = x^2yz^3 + c.$$

Given that $\phi(1,-2,2) = 4 \quad \Rightarrow 1(-2)8 + c = 4 \quad \Rightarrow -16 + c = 4 \quad \Rightarrow c = 20.$

$$\therefore \phi = x^2yz^3 + 20.$$

Example 2.24. Find the equation of the tangent plane and normal to the surface $x^2 + y^2 + z^2 = 25$ at $(4,0,3)$.

Solution. Let the given point be (x_0, y_0, z_0)

$\therefore x_0 = 4, y_0 = 0, z_0 = 3.$

Now, $\phi = x^2 + y^2 + z^2 - 25.$

$$\frac{\partial\phi}{\partial x} = 2x \Rightarrow \left(\frac{\partial\phi}{\partial x}\right)_{(4,0,3)} = 8.$$
$$\frac{\partial\phi}{\partial y} = 2y \Rightarrow \left(\frac{\partial\phi}{\partial y}\right)_{(4,0,3)} = 0.$$
$$\frac{\partial\phi}{\partial z} = 2z \Rightarrow \left(\frac{\partial\phi}{\partial z}\right)_{(4,0,3)} = 6.$$

Equation of the tangent plane is

$$(x - x_0)\frac{\partial\phi}{\partial x} + (y - y_0)\frac{\partial\phi}{\partial y} + (z - z_0)\frac{\partial\phi}{\partial z} = 0.$$

$$8(x-4) + 0 + (z-3)6 = 0.$$

Dividing by 2 and expanding we get

$$4x - 16 + 3z - 9 = 0 \Rightarrow 4x + 3z = 25.$$

—

Equation of the normal to the surface at $(4,0,3)$ is

$$\frac{x-x_0}{\frac{\partial\phi}{\partial x}} = \frac{y-y_0}{\frac{\partial\phi}{\partial y}} = \frac{z-z_0}{\frac{\partial\phi}{\partial z}}$$
$$\frac{x-4}{8} = \frac{y-0}{0} = \frac{z-3}{6}$$
$$\frac{x-4}{8} = \frac{y}{0} = \frac{z-3}{6}.$$

Example 2.25. Find the equation of the tangent plane and normal to the surface $xz^2 + x^2y - z + 1 = 0$ at the point $(1,-3,2)$.

Solution. Let the given point be (x_0, y_0, z_0).

$\therefore x_0 = 1, y_0 = -3, z_0 = 2.$

$$\text{Now, } \phi = xz^2 + x^2y - z + 1.$$
$$\frac{\partial\phi}{\partial x} = z^2 + 2xy \Rightarrow \left(\frac{\partial\phi}{\partial x}\right)_{(1,-3,2)} = 4 + 2(1)(-3) = 4 - 6 = -2.$$
$$\frac{\partial\phi}{\partial y} = x^2 \Rightarrow \left(\frac{\partial\phi}{\partial y}\right)_{(1,-3,2)} = 1.$$
$$\frac{\partial\phi}{\partial z} = 2xz - 1 \Rightarrow \left(\frac{\partial\phi}{\partial z}\right)_{(1,-3,2)} = 2(1)2 - 1 = 3.$$

Equation of the tangent plane is

$$(x-x_0)\frac{\partial\phi}{\partial x} + (y-y_0)\frac{\partial\phi}{\partial y} + (z-z_0)\frac{\partial\phi}{\partial z} = 0.$$
$$-2(x-1) + 1(y+3) + 3(z-2) = 0.$$
$$-2x + 2 + y + 3 + 3z - 6 = 0$$
$$-2x + y + 3z - 1 = 0$$
$$2x - y - 3z + 1 = 0.$$

Equation of the normal is

$$\frac{x-x_0}{\frac{\partial\phi}{\partial x}} = \frac{y-y_0}{\frac{\partial\phi}{\partial y}} = \frac{z-z_0}{\frac{\partial\phi}{\partial z}}$$
$$\frac{x-1}{-2} = \frac{y+3}{1} = \frac{z-2}{3}.$$

—

Example 2.26. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$ prove that

(i) $\nabla r = \dfrac{\vec{r}}{r}$.

(ii) $\nabla r^n = nr^{n-2}\vec{r}$.

(iii) $\nabla\left(\dfrac{1}{r}\right) = -\dfrac{\vec{r}}{r^3}$.

(iv) $\nabla(\log r) = \dfrac{\vec{r}}{r^2}$. [Jun 2006]

Proof. Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}.$$

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}.$$

$$\text{Similarly, } \frac{\partial r}{\partial y} = \frac{y}{r}, \ \frac{\partial r}{\partial z} = \frac{z}{r}.$$

(i)
$$\begin{aligned}\nabla r &= \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}\\ &= \vec{i}\left(\frac{x}{r}\right) + \vec{j}\left(\frac{y}{r}\right) + \vec{k}\left(\frac{z}{r}\right)\\ &= \frac{1}{r}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{\vec{r}}{r}.\end{aligned}$$

(ii)
$$\begin{aligned}\nabla r^n &= \vec{i}\frac{\partial(r^n)}{\partial x} + \vec{j}\frac{\partial(r^n)}{\partial y} + \vec{k}\frac{\partial(r^n)}{\partial z}\\ &= \vec{i}.n.r^{n-1}\frac{\partial r}{\partial x} + \vec{j}.n.r^{n-1}\frac{\partial r}{\partial y} + \vec{k}.n.r^{n-1}\frac{\partial r}{\partial z}\\ &= nr^{n-1}(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r})\\ &= nr^{n-2}\vec{r}.\end{aligned}$$

(iii) $\nabla\left(\dfrac{1}{r}\right) = (-1)r^{-3}\vec{r} = -\dfrac{\vec{r}}{r^3}$. [put n=-1 in (ii)]

(iv)
$$\begin{aligned}\nabla(\log r) &= \vec{i}\frac{\partial}{\partial x}(\log r) + \vec{j}\frac{\partial}{\partial y}(\log r) + \vec{k}\frac{\partial}{\partial z}(\log r)\\ &= \vec{i}\frac{1}{r}.\frac{\partial r}{\partial x} + \vec{j}\frac{1}{r}.\frac{\partial r}{\partial y} + \vec{k}\frac{1}{r}.\frac{\partial r}{\partial z}\\ &= \frac{1}{r}\left[\frac{x\vec{i}}{r} + \frac{y\vec{j}}{r} + \frac{z\vec{k}}{r}\right] = \frac{\vec{r}}{r^2}.\end{aligned}$$

—

Exercise 2 A

1. Find $\nabla\phi$ at the point $(1,1,1)$ if $\phi = x^2y + y^2x + z^2$.

2. If $\phi = \log(x^2 + y^2 + z^2)$, find $\nabla\phi$. [Dec 2010]

3. Prove that $\nabla f(r) = \frac{f'(r)}{r}\vec{r}$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

4. Find $grad\phi$ at $(1,2,1)$ if $\phi = \log(x^2 + y^2 + z^2)$.

5. If $\phi = x + xy^2 + yz^2$, find $grad\phi$.

6. Find the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ at the point $(1,1,-1)$ in the direction $2\vec{i} - \vec{j} - \vec{k}$.

7. Find the directional derivative of the function $\phi = xy + yz + zx$ in the direction of the vector $2\vec{i} + 3\vec{j} + 6\vec{k}$ at the point $(3,1,2)$.

8. Find the directional derivative of the function $2yz + z^2$ in the direction of the vector $\vec{i} + 2\vec{j} + 2\vec{k}$ at the point $(1,-1,3)$.

9. Find the directional derivative of $x^3 + y^3 + z^3$ at the point $(1,-1,2)$ in the direction of $\vec{i} + 2\vec{j} + \vec{k}$.

10. In what direction from $(3,1,-2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum? Find also the magnitude of this maximum. [May 2015]

11. If the directional derivative of $\phi(x,y,z) = a(x+y)+b(y+z)+c(z+x)$ has maximum value 12 at $(1,2,1)$ in the direction parallel to the line $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-1}{3}$, find the values of a, b and c.

12. Find the directional derivative of $\phi = 3x^2 + 2y - 3z$ at $(1,1,1)$ in the direction of $2\vec{i} + 2\vec{j} - \vec{k}$. [Jun 2010]

—

13. In what direction from the point $(1, 1, -1)$ is the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ maximum? What is the magnitude of this maximum directional derivative?

14. Find a unit normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$.

15. Find a unit normal vector to the surface $x^2 + y^2 - z^2 = 1$ at $(1, 1, 1)$.

16. Find a unit normal to the surface $xy^2z^3 = 1$ at $(1, 1, 1)$.

17. Find the angle between the normals to the surface $xy = z^2$ at the points $(1, 4, 2)$ and $(-3, -3, 3)$.

18. Find the angle between the spheres $x^2 + y^2 + z^2 = 29$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z - 47 = 0$ at $(4, -3, 2)$.

19. Find the angle between the surfaces $2yz + z^2 = 3$ and $x \log z = y^2 - 1$ at $(1, 1, 1)$.

20. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.

21. If $\nabla\phi = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k}$, find ϕ.

22. If $\nabla\phi = \left(y^2 - 2xyz^3\right)\vec{i} + \left(3 + 2xy - x^2z^3\right)\vec{j} + \left(6z^3 - 3x^2yz^2\right)\vec{k}$, find ϕ.

23. Find the equation of the tangent plane and the equation of the normal to the surface $x^2 - 4y^2 + 3z^2 + 4 = 0$ at the point $(3, 2, 1)$.

24. Find the equation of the tangent plane and normal to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1, -1, 2)$.

25. Find the directional derivative of $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ at the point $P(1, 1, 1)$ in the direction of the line $\dfrac{x-1}{2} = \dfrac{y-3}{-2} = z$.

—

2.3.1 Divergence and curl of a vector point function

Let $\vec{F}(x, y, z)$ be a vector point function continuously differentiable in the region R. Then the divergence of $\vec{F}$ is defined as

$div\vec{F} = \nabla \cdot \vec{F} = \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} + \vec{j} \cdot \frac{\partial \vec{F}}{\partial y} + \vec{k} \cdot \frac{\partial \vec{F}}{\partial z}$.

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then $div\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$.

Result

(1). If $\vec{F}$ is a constant vector, then $\nabla \cdot \vec{F} = 0$.

(2). $\vec{F} \cdot \nabla \neq \nabla \cdot \vec{F}$.

Solenoidal vector. If $div\vec{F} = 0$ everywhere in a region R, then $\vec{F}$ is called a solenoidal vector in R. [Dec 2013]

Curl of a vector. If $\vec{F}$ is a vector point function continuously differentiable in a region R, then $curl\vec{F}$ is defined by

$curl\vec{F} = \nabla \times \vec{F} = \vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \times \frac{\partial \vec{F}}{\partial z}$.

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$.

Note. If $\vec{F}$ is a constant vector, then $curl\vec{F} = \vec{0}$.

Irrotational vector. Let $\vec{F}(x, y, z)$ be a vector point function. If $curl\vec{F} = \vec{0}$ at all points in a region R, then $\vec{F}$ is said to be an irrotational vector in R.

Conservative vector field. A vector field $\vec{F}$ is said to be conservative, if there exists a scalar point function ϕ such that $\vec{F} = \nabla\phi$.

Note

(1) In a conservative vector field $\vec{F} = \nabla\phi$.

$\therefore \nabla \times \vec{F} = \nabla \times \nabla\phi = \vec{0}$.

$\Rightarrow \vec{F}$ is irrotational.

—

(2) In the conservative vector field, the scalar function ϕ is called the scalar potential of $\vec{F}$.

Worked Examples

Example 2.27. Find the divergence and curl of the vector $\vec{F} = xyz\vec{i}+3x^2y\vec{j}+(xz^2-y^2z)\vec{k}$ at the point $(2,-1,1)$. [Jun 1996]

Solution. Given, $\vec{F} = xyz\vec{i} + 3x^2y\vec{j} + (xz^2 - y^2z)\vec{k}$

$$\Longrightarrow F_1 = xyz, F_2 = 3x^2y, F_3 = xz^2 - y^2z.$$
$$\frac{\partial F_1}{\partial x} = yz, \frac{\partial F_2}{\partial y} = 3x^2, \frac{\partial F_3}{\partial z} = 2xz - y^2.$$

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = yz + 3x^2 + 2xz - y^2.$$

$$(\nabla \cdot \vec{F})_{(2,-1,1)} = -1 + 12 + 4 - 1 = 14.$$

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$
$$= \vec{i}\Big(\frac{\partial(xz^2 - y^2z)}{\partial y} - \frac{\partial(3x^2y)}{\partial z}\Big) - \vec{j}\Big(\frac{\partial(xz^2 - y^2z)}{\partial x} - \frac{\partial(xyz)}{\partial z}\Big)$$
$$+ \vec{k}\Big(\frac{\partial(3x^2y)}{\partial x} - \frac{\partial(xyz)}{\partial y}\Big).$$
$$= \vec{i}(-2yz) - \vec{j}(z^2 - xy) + \vec{k}(6xy - xz).$$
$$(curl\vec{F})_{(2,-1,1)} = 2\vec{i} - 3\vec{j} - 14\vec{k}.$$

Example 2.28. Find $\nabla.\vec{F}$, and *curl* $\vec{F}$ of the vector point function $\vec{F} = xz^3\vec{i} - 2x^2yz\vec{j} + 2yz^4\vec{k}$ at the point $(1,-1,1)$. [Apr 2008]

Solution. Given $\vec{F} = xz^3\vec{i} - 2x^2yz\vec{j} + 2yz^4\vec{k}$.

$$\Longrightarrow F_1 = xz^3, F_2 = -2x^2yz, F_3 = 2yz^4.$$
$$\frac{\partial F_1}{\partial x} = z^3, \quad \frac{\partial F_2}{\partial y} = -2x^2z, \quad \frac{\partial F_3}{\partial z} = 8yz^3$$

$$\nabla.\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = z^3 - 2x^2z + 8yz^3.$$

—

$$(\nabla.\vec{F})_{(1,-1,1)} = 1 - 2.1.1 + 8(-1).1 = 1 - 2 - 8 = -9.$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right) - \vec{j}\left(\frac{\partial}{\partial x}(2yz^4) - \frac{\partial}{\partial z}(xz^3)\right) + \vec{k}\left(\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right)$$

$$= \vec{i}(2z^4 + 2x^2y) - \vec{j}(0 - 3xz^2) + \vec{k}(-4xyz - 0)$$

$$= (2z^4 + 2x^2y)\vec{i} + 3xz^2\vec{j} - 4xyz\vec{k}$$

$$(\nabla \times \vec{F})_{(1,-1,1)} = 0\vec{i} + 3\vec{j} + 4\vec{k} = 3\vec{j} + 4\vec{k}.$$

Example 2.29. Prove that $\nabla \times \nabla\phi = \vec{0}$ where ϕ is a scalar point function.

Solution. We have $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$.

$$\nabla \times \nabla\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}.$$

$$= \vec{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - \vec{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) + \vec{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)$$

$$= 0 \qquad \left[\text{assuming } \frac{\partial^2\phi}{\partial y\partial z} = \frac{\partial^2\phi}{\partial z\partial y} \text{ etc}\right]$$

$$\nabla \times \nabla\phi = 0 \quad \text{always.}$$

Example 2.30. Find divergence $\vec{F}$ and *curl* $\vec{F}$ where $\vec{F} = grad\,(x^3 + y^3 + z^3 - 3xyz)$.

[May 2001]

Solution. Given, $\vec{F} = grad(\phi)$, where $\phi = x^3 + y^3 + z^3 - 3xyz$.

$$\frac{\partial\phi}{\partial x} = 3x^2 - 3yz,\ \frac{\partial\phi}{\partial y} = 3y^2 - 3xz,\ \frac{\partial\phi}{\partial z} = 3z^2 - 3xy.$$

$$\text{Now, } \vec{F} = grad\phi$$

$$\text{i.e., } \vec{F} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}.$$

$$\vec{F} = (3x^2 - 3yz)\vec{i} + (3y^2 - 3xz)\vec{j} + (3z^2 - 3xy)\vec{k}$$

$$F_1 = 3x^2 - 3yz, F_2 = 3y^2 - 3xz, F_3 = 3z^2 - 3xy$$

$$div\ \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 6x + 6y + 6z = 6(x + y + z).$$

$$curl\ F = \nabla \times \vec{F} = \nabla \times \nabla\phi = \vec{0}. \text{ [By the previous example.]}$$

Example 2.31. Prove that $div\vec{r} = 3$, $curl\vec{r} = \vec{0}$, where $\vec{r}$ is the position vector of a point (x, y, z) in space. [May 2005]

Solution. Given, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$div\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3.$$

$$curl\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \vec{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \vec{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= \vec{0}.$$

Example 2.32. Prove that the vector $\vec{F} = (x + 3y)\vec{i} + (y - 3z)\vec{j} + (x - 2z)\vec{k}$ is solenoidal.

Solution. We have, $F_1 = x + 3y, F_2 = y - 3z, F_3 = x - 2z$.

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 3z) + \frac{\partial}{\partial z}(x - 2z)$$

$$= 1 + 1 - 2$$

$$= 0.$$

$\therefore \vec{F}$ is solenoidal.

Example 2.33. Prove that $\vec{F} = (2x^2y + yz)\vec{i} + (xy^2 - xz^2)\vec{j} - (6xyz + 2x^2y^2)\vec{k}$ is solenoidal.

—

Solution. We have, $F_1 = 2x^2y + yz, F_2 = xy^2 - xz^2, F_3 = -6xyz - 2x^2y^2$.

$$\frac{\partial F_1}{\partial x} = 4xy, \frac{\partial F_2}{\partial y} = 2xy, \frac{\partial F_3}{\partial z} = -6xy.$$

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\\ &= 4xy + 2xy - 6xy\\ &= 0.\end{aligned}$$

$\therefore \vec{F}$ is solenoidal.

Example 2.34. Find the value of a if the vector $\vec{F} = (2x^2y + yz)\vec{i} + x(y^2 - z^2)\vec{j}$ $+(axyz - 2x^2y^2)\vec{k}$ is solenoidal.

Solution. Given, $\vec{F}$ is solenoidal.

$$\therefore \nabla \cdot \vec{F} = 0.$$

$$\Rightarrow \frac{\partial}{\partial x}(2x^2y + yz) + \frac{\partial}{\partial y}(xy^2 - xz^2) + \frac{\partial}{\partial z}(axyz - 2x^2y^2) = 0.$$

$$4xy + 2xy + axy = 0$$

$$6xy + axy = 0$$

$$xy(6 + a) = 0 \Rightarrow a = -6.$$

Example 2.35. Find a such that $(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal. [May 2015, Jun 2012, Jun 2004]

Solution. Let $\vec{F} = (3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$.

Given that $\vec{F}$ is solenoidal.

$\therefore \nabla.\vec{F} = 0$.

$$\Rightarrow \frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + ay - z) + \frac{\partial}{\partial z}(x - y + 2z) = 0.$$

i.e., $3 + a + 2 = 0$

$a + 5 = 0 \quad \Rightarrow \quad a = -5.$

—

Example 2.36. If $\vec{V} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+2\lambda z)\vec{k}$ is solenoidal, find the value of λ.

[Dec 2013, May 2011]

Solution. Given $\vec{V} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+2\lambda z)\vec{k}$.

Given $\vec{V}$ is solenoidal.

$$\therefore \ \nabla \cdot \vec{V} = 0$$

$$i.e., \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+2\lambda z) = 0$$

$$1 + 1 + 2\lambda = 0$$

$$2\lambda = -2$$

$$\lambda = -1.$$

Example 2.37. Show that the vector $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational.

Solution.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= \vec{i}(-1+1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x - 6x) = \vec{0}$$

$\therefore \vec{F}$ is irrotational.

Example 2.38. Prove that $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is irrotational. [Dec 2012, Dec 2011]

Solution.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right) - \vec{j}\left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right) + \vec{k}\left(\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right)$$

—

$$= \vec{i}(x - x) - \vec{j}(y - y) + \vec{k}(z - z)$$
$$= \vec{0}.$$

$\therefore$ $\vec{F}$ is irrotational.

Example 2.39. Find the constants a, b, c so that
$\vec{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ is irrotational.

Solution. Given that $\vec{F}$ is irrotational.

$$\Rightarrow \nabla \times \vec{F} = \vec{0}.$$

i.e. $$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = 0$$

$$\vec{i}(c + 1) - \vec{j}(4 - a) + \vec{k}(b - 2) = 0.$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$ both sides, we get

$a = 4, b = 2, c = -1$

Example 2.40. Show that $\vec{F} = (y^2 + 2xz^2)\vec{i} + (2xy - z)\vec{j} + (2x^2z - y + 2z)\vec{k}$ is irrotational and hence find its scalar potential. [Dec 2014, Jun 2012]

Solution. Given that $\vec{F} = (y^2 + 2xz^2)\vec{i} + (2xy - z)\vec{j} + (2x^2z - y + 2z)\vec{k}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + 2xz^2 & 2xy - z & 2x^2z - y + 2z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(2x^2z - y + 2z) - \frac{\partial}{\partial z}(2xy - z)\right) - \vec{j}\left(\frac{\partial}{\partial x}(2x^2z - y + 2z) - \frac{\partial}{\partial z}(y^2 + 2xz^2)\right)$$
$$+ \vec{k}\left(\frac{\partial}{\partial x}(2xy - z) - \frac{\partial}{\partial y}(y^2 + 2xz^2)\right)$$
$$= \vec{i}(-1 + 1) - \vec{j}(4xz - 4xz) + \vec{k}(2y - 2y)$$
$$= \vec{i}.0 - \vec{j}.0 + \vec{k}.0$$
$$= \vec{0}.$$

—

$\therefore \vec{F}$ is irrotational.

Since $\nabla \times \vec{F} = 0$, then $\vec{F} = \nabla\phi$ where ϕ is the scalar potential.

$$\therefore \vec{F} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}.$$

Comparing the components of $\vec{F}$ we obtain

$$\frac{\partial\phi}{\partial x} = y^2 + 2xz^2 \tag{1}$$

$$\frac{\partial\phi}{\partial y} = 2xy - z \tag{2}$$

$$\frac{\partial\phi}{\partial z} = 2x^2z - y + 2z. \tag{3}$$

Integrating partially (1), (2), (3) w.r.t. x, y and z respectively we get

$$\begin{aligned}\phi &= \int (y^2 + 2xz^2)dx + f_1(y,z)\\ &= y^2x + 2z^2\frac{x^2}{2} + f_1(y,z)\\ &= xy^2 + x^2z^2 + f_1(y,z).\end{aligned} \tag{4}$$

From (2) we get

$$\begin{aligned}\phi &= \int (2xy - z)dy + f_2(x,z)\\ &= 2x\frac{y^2}{2} - zy + f_2(x,z)\\ &= xy^2 - yz + f_2(x,z)\end{aligned} \tag{5}$$

From (3) we obtain

$$\begin{aligned}\phi &= \int (2x^2z - y + 2z)dz + f_3(x,y)\\ &= 2x^2\frac{z^2}{2} - yz + 2\frac{z^2}{2} + f_3(x,y)\\ &= x^2z^2 - yz + z^2 + f_3(x,y).\end{aligned} \tag{6}$$

From (4), (5) and (6) we get

$$\phi = xy^2 + x^2z^2 - yz + z^2 + c.$$

—

Example 2.41. Prove that $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$ is irrotational and find its scalar potential. [Dec.2005]

Solution. Given $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$.

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(3z^2 - 3z^2) + \vec{k}(2y\cos x - 2y\cos x)$$

$$= \vec{0}.$$

$\therefore \vec{F}$ is irrotational.

Since $\nabla\times\vec{F} = \vec{0}, \vec{F} = \nabla\phi$, where ϕ is a scalar potential.

$\therefore \vec{F} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}.$

Comparing the components of $\vec{F}$, we get

$$\frac{\partial\phi}{\partial x} = y^2\cos x + z^3 \tag{1}$$

$$\frac{\partial\phi}{\partial y} = 2y\sin x - 4 \tag{2}$$

$$\frac{\partial\phi}{\partial z} = 3xz^2. \tag{3}$$

Integrating (1), (2), and (3) partially w.r.t. x, y and z respectively we get,

$$\phi = \int\left(y^2\cos x + z^3\right)dx + f_1(y,z)$$

$$= y^2\sin x + xz^3 + f_1(y,z). \tag{4}$$

$$\phi = \int(2y\sin x - 4)dy + f_2(x,z).$$

$$= 2\sin x\frac{y^2}{2} - 4y + f_2(x,z)$$

$$= y^2 sinx - 4y + f_2(x,z). \tag{5}$$

$$\phi = \int 3xz^2 dz + f_3(x,y).$$

$$= 3x\frac{z^3}{3} + f_3(x,y)$$

$$= xz^3 + f_3(x,y). \tag{6}$$

From (4), (5) and (6) we get

$$\phi = y^2\sin x + xz^3 - 4y + c, \text{ where } c \text{ is a constant.}$$

—

Example 2.42. Show that the vector field $\vec{F} = (x^2 + xy^2)\vec{i} + (y^2 + x^2y)\vec{j}$ is irrotational. Find its scalar potential. [Dec 2013]

Solution. Given $\vec{F} = (x^2 + xy^2)\vec{i} + (y^2 + x^2y)\vec{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(2xy - 2xy)$$

$$= \vec{0}.$$

$\therefore$ $\vec{F}$ is irrotational.

Since $\nabla \times \vec{F} = \vec{0}, \vec{F} = \nabla\phi$, where ϕ is a scalar potential.

$$\therefore \vec{F} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}.$$

Comparing the components of $\vec{F}$, we get

$$\frac{\partial\phi}{\partial x} = x^2 + xy^2 \tag{1}$$

$$\frac{\partial\phi}{\partial y} = y^2 + x^2y \tag{2}$$

$$\frac{\partial\phi}{\partial z} = 0.$$

Integrating (1), (2) and (3) partially w.r.t. x, y and z respectively we get

$$\phi = \int (x^2 + xy^2)dx + f_1(y,z)$$

$$\phi = \frac{x^3}{3} + \frac{x^2y^2}{2} + f_1(y,z) \tag{4}$$

$$\phi = \int (y^2 + x^2y) + f_2(x,z)$$

$$\phi = \frac{y^3}{3} + \frac{x^2y^2}{2} + f_2(x,z) \tag{5}$$

$$\phi = f_3(x,y) \tag{6}$$

From (4), (5) and (6) we get

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + c, \text{ where } c \text{ is a constant.}$$

—

Example 2.43. Show that $\vec{F} = (2xy - z^2)\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 - 2zx)\vec{k}$ is irrotational and hence find its scalar potential. [Dec 2012]

Solution. Given $\vec{F} = (2xy - z^2)\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 - 2zx)\vec{k}$.

$$\begin{aligned}
\nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - z^2 & x^2 + 2yz & y^2 - 2zx \end{vmatrix} \\
&= \vec{i}(2y - 2y) - \vec{j}(-2z + 2z) + \vec{k}(2x - 2x) \\
&= \vec{0}.
\end{aligned}$$

$\therefore$ $\vec{F}$ is irrotational.

Since $\nabla \times \vec{F} = \vec{0}, \vec{F} = \nabla\phi$, where ϕ is a scalar potential.

$$\therefore \vec{F} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}.$$

Comparing the components of $\vec{F}$ we get

$$\frac{\partial\phi}{\partial x} = 2xy - z^2. \qquad (1)$$

$$\frac{\partial\phi}{\partial y} = x^2 + 2yz. \qquad (2)$$

$$\frac{\partial\phi}{\partial z} = y^2 - 2zx. \qquad (3)$$

Integrating (1), (2) and (3) partially w.r.t. x, y and z respectively we get

$$\begin{aligned}
\phi &= \int (2xy - z^2)dx + f_1(y,z) \\
&= 2y \cdot \frac{x^2}{2} - xz^2 + f_1(y,z) \\
&= x^2y - xz^2 + f_1(y,z). \qquad (4)
\end{aligned}$$

$$\begin{aligned}
\phi &= \int (x^2 + 2yz)dy + f_2(x,z) \\
&= x^2y + 2z \cdot \frac{y^2}{2} + f_2(x,z) \\
&= x^2y + y^2z + f_2(x,z). \qquad (5)
\end{aligned}$$

$$\begin{aligned}\phi &= \int (y^2 - 2zx)dz + f_3(x,y)\\ &= zy^2 - 2x \cdot \frac{z^2}{2} + f_3(x,y)\\ &= zy^2 - xz^2 + f_3(x,y). \qquad (6)\end{aligned}$$

From (4), (5) and (6) we get

$\phi = x^2y - xz^2 + y^2z + c$, where c is a constant.

Example 2.44. Prove that $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is an irrotational vector and find the scalar potential such that $\vec{F} = \nabla\phi$. [Jun 2010]

Solution. Given $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$

$$\begin{aligned}\nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}\\ &= \vec{i}(-1 + 1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x - 6x)\\ &= \vec{0}.\end{aligned}$$

$\therefore$ $\vec{F}$ is irrotational.

Since $\nabla \times \vec{F} = \vec{0}, \vec{F} = \nabla\phi$, where ϕ is a scalar potential.

$$\therefore \vec{F} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}.$$

Comparing the components of $\vec{F}$ we get

$$\frac{\partial\phi}{\partial x} = 6xy + z^3. \qquad (1)$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z. \qquad (2)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 - y. \qquad (3)$$

Integrating (1), (2) and (3) partially w.r.t. x, y and z respectively we get

$$\begin{aligned}\phi &= \int (6xy + z^3)dx + f_1(y,z)\\ &= 6y \cdot \frac{x^2}{2} + xz^3 + f_1(y,z)\\ &= 3x^2y + xz^3 + f_1(y,z). \qquad (4)\\ \phi &= \int (3x^2 - z)dy + f_2(x,z)\\ &= 3x^2y - yz + f_2(x,z). \qquad (5)\\ \phi &= \int (3xz^2 - y)dz + f_3(x,y)\\ &= 3x \cdot \frac{z^3}{3} - yz + f_3(x,y)\\ &= xz^3 - yz + f_3(x,y). \qquad (6)\end{aligned}$$

From (4), (5) and (6) we get

$$\phi = 3x^2y + xz^3 - yz + c, \text{ where c is constant.}$$

Example 2.45. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$ prove that $r^n\vec{r}$ is solenoidal if $n = -3$ and irrotational for all values of n. [Dec 2007]

Solution. Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$r^2 = x^2 + y^2 + z^2. \quad \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

Let $\vec{F} = r^n\vec{r} = r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}$.

$$F_1 = r^n x, F_2 = r^n y, F_3 = r^n z$$

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial x} + \frac{\partial F_3}{\partial x}.\\ &= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial x}(r^n y) + \frac{\partial}{\partial x}(r^n z).\\ &= r^n + x \times n \times r^{(n-1)}\frac{\partial r}{\partial x} + r^n + y \times n \times r^{(n-1)}\frac{\partial r}{\partial y} + r^n + z \times n \times r^{(n-1)}\frac{\partial r}{\partial z}\\ &= 3r^n + nr^{(n-1)}\left(x \times \frac{x}{r} + y \times \frac{y}{r} + z \times \frac{z}{r}\right)\end{aligned}$$

—

$$= 3r^n + nr^{(n-2)}(x^2 + y^2 + z^2)$$

$$= 3r^n + nr^{(n-2)} \times r^2$$

$$= 3r^n + nr^n = (n+3)r^n.$$

If $r^n\vec{r}$ is solenoidal then $\nabla \cdot \vec{F} = 0$.

i.e., $(n+3)r^n = 0$.

$\Rightarrow n + 3 = 0$.

$\Rightarrow n = -3$.

$$\text{Now, } curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \vec{i}\Big(\frac{\partial(r^n z)}{\partial y} - \frac{\partial(r^n y)}{\partial z}\Big) - \vec{j}\Big(\frac{\partial(r^n z)}{\partial x} - \frac{\partial(r^n x)}{\partial z}\Big) + \vec{k}\Big(\frac{\partial(r^n y)}{\partial x} - \frac{\partial(r^n x)}{\partial y}\Big).$$

$$= \vec{i}\Big(z \times n \times r^{(n-1)} \times \frac{y}{r} - y \times n \times r^{n-1}\frac{z}{r}\Big) - \vec{j}(0) + \vec{k}(0) = \vec{0}$$

$\therefore$ $r^n\vec{r}$ is irrotational for all values of n.

Example 2.46. If $\vec{r}$ is the position vector of a point (x, y, z) in space and $\vec{A}$ is a constant vector, prove that $\vec{A} \times \vec{r}$ is solenoidal.

Solution. We have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Let $A = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, a_1, a_2, a_3 are constants.

$$\vec{A} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{i}(a_2 z - a_3 y) - \vec{j}(a_1 z - a_3 x) + \vec{k}(a_1 y - a_2 x).$$

$$\nabla.(\vec{A} \times \vec{r}) = \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(-(a_1 z - a_3 x)) + \frac{\partial}{\partial z}(a_1 y - a_2 x)$$

$$= 0 + 0 + 0 = 0.$$

$\Longrightarrow \vec{A} \times \vec{r}$ is solenoidal.

—

Example 2.47. Determine $f(r)$ so that $f(r)\vec{r}$ is both solenoidal and irrotational.

[Jan 2008]

Solution. Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r^2 = x^2 + y^2 + z^2$$
$$f(r)\vec{r} = f(r)x\vec{i} + f(r)y\vec{j} + f(r)z\vec{k}.$$

Given, $f(r)\vec{r}$ is solenoidal.

$$\Longrightarrow \nabla \cdot f(r)\vec{r} = 0$$
$$\frac{\partial}{\partial x}(f(r)x) + \frac{\partial}{\partial y}(f(r)y) + \frac{\partial}{\partial z}(f(r)z) = 0$$
$$f(r) + xf'(r)\frac{\partial r}{\partial x} + f(r) + yf'(r)\frac{\partial r}{\partial y} + f(r) + zf'(r)\frac{\partial r}{\partial z} = 0$$
$$3f(r) + f'(r)\Big(\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\Big) = 0$$
$$3f(r) + f'(r)\frac{r^2}{r} = 0$$
$$3f(r) + f'(r)r = 0$$
$$f'(r)r = -3f(r)$$
$$\frac{f'(r)}{f(r)} = -\frac{3}{r}.$$

Integrating w.r.to. r. we get,

$$\log f(r) = -3\log r + \log c$$
$$= \log r^{-3} + \log c$$
$$= \log \frac{c}{r^3}$$
$$f(r) = \frac{c}{r^3} \text{where c is the constant of integration.}$$

Given $f(r)\vec{r}$ is irrotational.
$\nabla \times f(r)\vec{r} = \vec{0}$.

—

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(r)x & f(r)y & f(r)z \end{vmatrix} = 0$$

$$\vec{i}\Big(\frac{\partial}{\partial y}(f(r)z) - \frac{\partial}{\partial z}(f(r)y)\Big) - \vec{j}\Big(\frac{\partial}{\partial x}(f(r)z) - \frac{\partial}{\partial z}(f(r)x)\Big) + \vec{k}\Big(\frac{\partial}{\partial x}(f(r)y) - \frac{\partial}{\partial y}(f(r)x)\Big) = 0$$

$$\vec{i}\Big[zf'(r)\frac{\partial r}{\partial y} - yf'(r)\frac{\partial r}{\partial z}\Big] - \vec{j}\Big[zf'(r)\frac{\partial r}{\partial x} - xf'(r)\frac{\partial r}{\partial z}\Big] + \vec{k}\Big[yf'(r)\frac{\partial r}{\partial x} - xf'(r)\frac{\partial r}{\partial y}\Big] = 0$$

$$\vec{i}f'(r)\Big\{\frac{yz}{r} - \frac{yz}{r}\Big\} - \vec{j}f'(r)\Big\{\frac{xz}{r} - \frac{xz}{r}\Big\} + \vec{k}f'(r)\Big\{\frac{xy}{r} - \frac{xy}{r}\Big\} = 0$$

$$\vec{i}\cdot 0 - \vec{j}\cdot 0 + \vec{k}\cdot 0 = 0$$

i.e., $0 = 0$

$\therefore$ $f(r)\vec{r}$ is irrotational for all $f(r)$ and it is solenoidal for $f(r) = \frac{c}{r^3}$ where c is an arbitrary constant.

Example 2.48. If $\vec{v} = \vec{w}\times\vec{r}$, prove that $\vec{w} = \frac{1}{2}curl\ \vec{v}$ where $\vec{w}$ is a constant vector and $\vec{r}$ is the position vector of the point (x, y, z).

Solution. Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Let $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$.

Also given that, $\vec{v} = \vec{w}\times\vec{r}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{i}(w_2z - w_3y) - \vec{j}(w_1z - w_3x) + \vec{k}(w_1y - w_2x).$$

$$\text{Now, } curl\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_2z - w_3y & -w_1z + w_3x & w_1y - w_2x \end{vmatrix}$$

$$= \vec{i}\Big(\frac{\partial(w_1y - w_2x)}{\partial y} - \frac{\partial(-w_1z + w_3x)}{\partial z}\Big) - \vec{j}\Big(\frac{\partial(w_1y - w_2x)}{\partial x} - \frac{\partial(w_2z - w_3y)}{\partial z}\Big)$$

$$+\vec{k}\Big(\frac{\partial(-w_1 z + w_3 x)}{\partial x} - \frac{\partial(w_2 z - w_3 y)}{\partial y}\Big).$$

$$= \vec{i}(w_1 + w_1) - \vec{j}(-w_2 - w_2) + \vec{k}(w_3 + w_3).$$

$$= 2(\vec{i}w_1 + \vec{j}w_2 + \vec{k}w_3) = 2\vec{w}.$$

$$\vec{w} = \frac{1}{2} curl \vec{v}.$$

2.3.2 Vector identities

Identity I. $\nabla \cdot (\vec{F} + \vec{G}) = \nabla \cdot \vec{F} + \nabla \cdot \vec{G}$ (or) $div(\vec{F} + \vec{G}) = div\vec{F} + div\vec{G}$.

Proof.

$$\nabla \cdot (\vec{F} + \vec{G}) = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}) \cdot (\vec{F} + \vec{G})$$

$$= \vec{i} \cdot \frac{\partial(\vec{F} + \vec{G})}{\partial x} + \vec{j} \cdot \frac{\partial(\vec{F} + \vec{G})}{\partial y} + \vec{k} \cdot \frac{\partial(\vec{F} + \vec{G})}{\partial z}$$

$$= \vec{i} \cdot \Big(\frac{\partial \vec{F}}{\partial x} + \frac{\partial \vec{G}}{\partial x}\Big) + \vec{j} \cdot \Big(\frac{\partial \vec{F}}{\partial y} + \frac{\partial \vec{G}}{\partial y}\Big) + \vec{k} \cdot \Big(\frac{\partial \vec{F}}{\partial z} + \frac{\partial \vec{G}}{\partial z}\Big)$$

$$= \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} + \vec{j} \cdot \frac{\partial \vec{F}}{\partial y} + \vec{k} \cdot \frac{\partial \vec{F}}{\partial z} + \vec{i} \cdot \frac{\partial \vec{G}}{\partial x} + \vec{j} \cdot \frac{\partial \vec{G}}{\partial y} + \vec{k} \cdot \frac{\partial \vec{g}}{\partial z}$$

$$= \nabla \cdot \vec{F} + \nabla \cdot \vec{G}.$$

$$= div\vec{F} + div\vec{G}.$$

Identity II. If f is a scalar point function and $\vec{G}$ is a vector point function, then $\nabla \cdot (f\vec{G}) = \nabla f \cdot \vec{G} + f(\nabla \cdot \vec{G})$.

Proof.

$$\nabla \cdot (f\vec{G}) = \vec{i} \cdot \frac{\partial(f\vec{G})}{\partial x} + \vec{j} \cdot \frac{\partial(f\vec{G})}{\partial y} + \vec{k} \cdot \frac{\partial(f\vec{G})}{\partial z}$$

$$= \vec{i} \cdot \Big\{f\frac{\partial \vec{G}}{\partial x} + \frac{\partial f}{\partial x}\vec{G}\Big\} + \vec{j} \cdot \Big\{f\frac{\partial \vec{G}}{\partial y} + \frac{\partial f}{\partial y}\vec{G}\Big\} + \vec{k} \cdot \Big\{f\frac{\partial \vec{G}}{\partial z} + \frac{\partial f}{\partial z}\vec{G}\Big\}$$

$$= f\Big(\vec{i} \cdot \frac{\partial \vec{G}}{\partial x} + \vec{j} \cdot \frac{\partial \vec{G}}{\partial y} + \vec{k} \cdot \frac{\partial \vec{G}}{\partial z}\Big) + \Big(\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}\Big) \cdot \vec{G}$$

$$= f(\nabla \cdot \vec{G}) + \nabla f \cdot \vec{G}.$$

—

Identity III. If f is a scalar point function and $\vec{G}$ is a vector point function then $\nabla \times (f\vec{G}) = \nabla f \times \vec{G} + f(\nabla \times \vec{G})$.

Proof.

$$\begin{aligned}
\nabla \times f\vec{G} &= \vec{i} \times \frac{\partial}{\partial x}(f\vec{G}) + \vec{j} \times \frac{\partial}{\partial y}(f\vec{G}) + \vec{k} \times \frac{\partial}{\partial z}(f\vec{G}) \\
&= \vec{i} \times \left(\frac{\partial f}{\partial x}\vec{G} + f\frac{\partial \vec{G}}{\partial x}\right) + \vec{j} \times \left(\frac{\partial f}{\partial y}\vec{G} + f\frac{\partial \vec{G}}{\partial y}\right) + \vec{k} \times \left(\frac{\partial f}{\partial z}\vec{G} + f\frac{\partial \vec{G}}{\partial z}\right) \\
&= \vec{i} \times \frac{\partial f}{\partial x}\vec{G} + \vec{j} \times \frac{\partial f}{\partial y}\vec{G} + \vec{k} \times \frac{\partial f}{\partial z}\vec{G} + \vec{i} \times f\frac{\partial \vec{G}}{\partial x} + \vec{j} \times f\frac{\partial \vec{G}}{\partial y} + \vec{k} \times f\frac{\partial \vec{G}}{\partial z} \\
&= \vec{i} \times \frac{\partial f}{\partial x}\vec{G} + \vec{j} \times \frac{\partial f}{\partial y}\vec{G} + \vec{k} \times \frac{\partial f}{\partial z}\vec{G} + f\left(\vec{i} \times \frac{\partial \vec{G}}{\partial x} + \vec{j} \times \frac{\partial \vec{G}}{\partial y} + \vec{k} \times \frac{\partial \vec{G}}{\partial z}\right) \\
&= \left(\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}\right) \times \vec{G} + f\left(\nabla \times \vec{G}\right) \\
&= \nabla f \times \vec{G} + f(\nabla \times \vec{G}).
\end{aligned}$$

Identity IV. If $\vec{F}$ and $\vec{G}$ are vector point functions, then

$$\nabla(\vec{F} \cdot \vec{G}) = (\vec{F} \cdot \nabla)\vec{G} + (\vec{G} \cdot \nabla)\vec{F} + \vec{F} \times (\nabla \times \vec{G}) + \vec{G} \times (\nabla \times \vec{F}).$$

Proof. We have $\nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z} = \sum \vec{i}\frac{\partial f}{\partial x}$.

$$\begin{aligned}
\nabla(\vec{F} \cdot \vec{G}) &= \sum \vec{i}\frac{\partial}{\partial x}(\vec{F} \cdot \vec{G}) = \sum \vec{i}\left\{\frac{\partial \vec{F}}{\partial x} \cdot \vec{G} + \vec{F} \cdot \frac{\partial \vec{G}}{\partial x}\right\} \\
&= \sum \left(\vec{F} \cdot \frac{\partial \vec{G}}{\partial x}\right)\vec{i} + \sum \left(\frac{\partial \vec{F}}{\partial x} \cdot \vec{G}\right)\vec{i} \qquad (1)
\end{aligned}$$

We know that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$$\begin{aligned}
(\vec{a} \cdot \vec{b})\vec{c} &= (\vec{a} \cdot \vec{c})\vec{b} - \vec{a} \times (\vec{b} \times \vec{c}) \\
(\vec{F} \cdot \frac{\partial \vec{G}}{\partial x})\vec{i} &= (\vec{F} \cdot \vec{i})\frac{\partial \vec{G}}{\partial x} - \vec{F} \times (\frac{\partial \vec{G}}{\partial x} \times \vec{i}) \\
&= (\vec{F} \cdot \vec{i})\frac{\partial \vec{G}}{\partial x} + \vec{F} \times (\vec{i} \times \frac{\partial \vec{G}}{\partial x}) \\
\sum(\vec{F} \cdot \frac{\partial \vec{G}}{\partial x})\vec{i} &= (\vec{F} \cdot \sum \vec{i}\frac{\partial}{\partial x})\vec{G} + \vec{F} \times (\sum \vec{i} \times \frac{\partial \vec{G}}{\partial x}) \\
&= (\vec{F} \cdot \nabla)\vec{G} + \vec{F} \times (\nabla \times \vec{G}). \qquad (2)
\end{aligned}$$

Interchanging $\vec{F}$ and $\vec{G}$, we get

$$\sum(\vec{G}\cdot\frac{\partial F}{\partial x})\vec{i} = (G\cdot\nabla)\vec{F} + \vec{G}\times(\nabla\times\vec{F}) \tag{3}$$

Substituting (2) and (3) in (1), we get

$\nabla(\vec{F}\cdot\vec{G}) = (\vec{F}\cdot\nabla)\vec{G} + (\vec{G}\cdot\nabla)\vec{F} + \vec{F}\times(\nabla\times\vec{G}) + \vec{G}\times(\nabla\times\vec{F})$.

Identity V. If $\vec{F}$ and $\vec{G}$ are vector point functions, then

$$\nabla\cdot(\vec{F}\times\vec{G}) = \vec{G}\cdot(\nabla\times\vec{F}) - \vec{F}\cdot(\nabla\times\vec{G})$$

i.e., $div(\vec{F}\times\vec{G}) = \vec{G}\cdot curl\vec{F} - \vec{F}\cdot curl\vec{G}$.

Proof. We have

$$\begin{aligned}
\nabla\cdot(\vec{F}\times\vec{G}) &= \sum\vec{i}\cdot\frac{\partial}{\partial x}(\vec{F}\times\vec{G})\\
&= \sum\vec{i}\cdot(\frac{\partial\vec{F}}{\partial x}\times\vec{G} + \vec{F}\times\frac{\partial\vec{G}}{\partial x})\\
&= \sum\vec{i}\cdot(\frac{\partial\vec{F}}{\partial x}\times\vec{G}) + \sum\vec{i}\cdot(\vec{F}\times\frac{\partial\vec{G}}{\partial x})\\
&= \sum\vec{i}\cdot(\frac{\partial\vec{F}}{\partial x}\times\vec{G}) - \sum\vec{i}\cdot(\frac{\partial\vec{G}}{\partial x}\times\vec{F})\\
&= \sum\left(\vec{i}\times\frac{\partial\vec{F}}{\partial x}\right)\cdot\vec{G} - \sum\left(\vec{i}\times\frac{\partial\vec{G}}{\partial x}\right)\cdot\vec{F}\\
&= (\nabla\times\vec{F})\cdot\vec{G} - (\nabla\times\vec{G})\cdot\vec{F}\\
\nabla\cdot(\vec{F}\times\vec{G}) &= \vec{G}\cdot(\nabla\times\vec{F}) - \vec{F}\cdot(\nabla\times\vec{G}).
\end{aligned}$$

Identity VI. If $\vec{F}$ and $\vec{G}$ are vector point functions, then

$$\nabla\times(\vec{F}\times\vec{G}) = \vec{F}(\nabla\cdot\vec{G}) - \vec{G}(\nabla\cdot\vec{F}) + (\vec{G}\cdot\nabla)\vec{F} - (\vec{F}\cdot\nabla)\vec{G}.$$

Proof. We have

$$\begin{aligned}
\nabla\times(\vec{F}\times\vec{G}) &= \sum\vec{i}\times\frac{\partial}{\partial x}(\vec{F}\times\vec{G})\\
&= \sum\vec{i}\times\left\{\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right) + \left(\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right)\right\}\\
&= \sum\vec{i}\times\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right) + \sum\vec{i}\times\left(\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right)
\end{aligned}$$

—

We have $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$$\begin{aligned}
\nabla \times (\vec{F} \times \vec{G}) &= \sum \left\{(\vec{i} \cdot \vec{G})\frac{\partial \vec{F}}{\partial x} - (\vec{i} \cdot \frac{\partial \vec{F}}{\partial x})\vec{G}\right\} + \sum \left\{(\vec{i} \cdot \frac{\partial \vec{G}}{\partial x})\vec{F} - (\vec{i} \cdot \vec{F})\frac{\partial \vec{G}}{\partial x}\right\} \\
&= \sum (\vec{G} \cdot \vec{i})\frac{\partial \vec{F}}{\partial x} - \sum (\vec{i} \cdot \frac{\partial \vec{F}}{\partial x})\vec{G} + \sum (\vec{i} \cdot \frac{\partial \vec{G}}{\partial x})\vec{F} - \sum (\vec{F} \cdot \vec{i})\frac{\partial \vec{G}}{\partial x} \\
&= \left(\vec{G} \cdot \sum \vec{i}\frac{\partial}{\partial x}\right)\vec{F} + \left(\sum \vec{i} \cdot \frac{\partial \vec{F}}{\partial x}\right)\vec{G} + \left(\sum \vec{i} \cdot \frac{\partial \vec{G}}{\partial x}\right)\vec{F} - \left(\vec{F} \cdot \sum \vec{i}\frac{\partial}{\partial x}\right)\vec{G} \\
&= (\vec{G} \cdot \nabla)\vec{F} - (\nabla \cdot \vec{F})\vec{G} + (\nabla \cdot \vec{G})\vec{F} - (\vec{F} \cdot \nabla)\vec{G} \\
\nabla \times (\vec{F} \times \vec{G}) &= \vec{F}(\nabla \cdot \vec{G}) - \vec{G}(\nabla \cdot \vec{F}) + (\vec{G} \cdot \nabla)\vec{F} - (\vec{F} \cdot \nabla)\vec{G} \\
curl(\vec{F} \times \vec{G}) &= \vec{F}(div \cdot \vec{G}) - \vec{G}(div \cdot \vec{F}) + (\vec{G} \cdot \nabla)\vec{F} - (\vec{F} \cdot \nabla)\vec{G}.
\end{aligned}$$

Laplacian Operator. If f is a scalar point function, then $div(grad\, f) = \nabla.\nabla f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

Solution. We have $grad\, f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$.

$$\begin{aligned}
div(grad\, f) &= \nabla.\nabla f \\
&= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}\right) \\
\nabla^2 f &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) \\
\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\
\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.
\end{aligned}$$

∇^2 is called the Laplacian operator.

Identity VII. If $\vec{F}$ is a vector point function then, $div\; curl\vec{F} = 0$.

Solution. Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$curl\; \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big) - \vec{j}\Big(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\Big) + \vec{k}\Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big).$$

$$div\ curl\ \vec{F} = \nabla \cdot (\nabla \times \vec{F})$$

$$= \frac{\partial}{\partial x}\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big) - \frac{\partial}{\partial y}\Big(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\Big) + \frac{\partial}{\partial z}\Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0.$$

Identity VIII. If $\vec{F}$ is a vector point function, then

$$curl(curl\ \vec{F}) = \nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}.$$

Solution. Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big) - \vec{j}\Big(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\Big) + \vec{k}\Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big).$$

$$curl(curl\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix}$$

$$= \vec{i}\bigg(\frac{\partial}{\partial y}\Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big) - \frac{\partial}{\partial z}\Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\bigg)$$

$$- \vec{j}\bigg(\frac{\partial}{\partial x}\Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big) - \frac{\partial}{\partial z}\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\bigg)$$

$$+ \vec{k}\bigg(\frac{\partial}{\partial x}\Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big) - \frac{\partial}{\partial y}\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\bigg)$$

$$= \vec{i}\bigg(\frac{\partial^2 F_2}{\partial y \partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 F_3}{\partial z \partial x}\bigg)$$

$$+ \vec{j}\bigg(\frac{\partial^2 F_1}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_3}{\partial z \partial y} - \frac{\partial^2 F_2}{\partial z^2}\bigg)$$

$$+ \vec{k}\bigg(\frac{\partial^2 F_1}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial x^2} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y \partial z}\bigg)$$

$$= \vec{i}\bigg(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} - \bigg(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\bigg)\bigg)$$

$$+\vec{j}\left(\frac{\partial^2 F_2}{\partial y^2}+\frac{\partial^2 F_1}{\partial y\partial x}+\frac{\partial^2 F_3}{\partial y\partial z}-\left(\frac{\partial^2 F_2}{\partial x^2}+\frac{\partial^2 F_2}{\partial y^2}+\frac{\partial^2 F_2}{\partial z^2}\right)\right)$$

$$+\vec{k}\left(\frac{\partial^2 F_3}{\partial z^2}+\frac{\partial^2 F_1}{\partial z\partial x}+\frac{\partial^2 F_2}{\partial z\partial y}-\left(\frac{\partial^2 F_3}{\partial x^2}+\frac{\partial^2 F_3}{\partial y^2}+\frac{\partial^2 F_3}{\partial z^2}\right)\right)$$

$$=\vec{i}\left(\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)-\nabla^2 F_1\right)+\vec{j}\left(\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)-\nabla^2 F_2\right)$$

$$+\vec{k}\left(\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)-\nabla^2 F_3\right)$$

$$=\vec{i}\left(\frac{\partial}{\partial x}(\nabla.\vec{F})-\nabla^2 F_1\right)+\vec{j}\left(\frac{\partial}{\partial y}(\nabla.\vec{F})-\nabla^2 F_2\right)+\vec{k}\left(\frac{\partial}{\partial z}(\nabla.\vec{F})-\nabla^2 F_3\right)$$

$$=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)(\nabla.\vec{F})-\nabla^2(F_1\vec{i}+F_2\vec{j}+F_3\vec{k})$$

$$\nabla\times(\nabla\times\vec{F})=\nabla(\nabla\cdot\vec{F})-\nabla^2\vec{F}.$$

Identity IX. Prove that $\nabla\left(\frac{1}{r^n}\right)=-\frac{n}{r^{n+2}}\vec{r}$.

Solution. Let $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$.

$$r^2=x^2+y^2+z^2.$$

We have, $\nabla\left(\frac{1}{r^n}\right)=\vec{i}\frac{\partial}{\partial x}\left(\frac{1}{r^n}\right)+\vec{j}\frac{\partial}{\partial y}\left(\frac{1}{r^n}\right)+\vec{k}\frac{\partial}{\partial z}\left(\frac{1}{r^n}\right)$

$$=\vec{i}(-n)r^{(-n-1)}\frac{\partial r}{\partial x}+\vec{j}(-n)r^{(-n-1)}\frac{\partial r}{\partial y}+\vec{k}(-n)r^{(-n-1)}\frac{\partial r}{\partial z}$$

$$=\frac{-n}{r^{n+1}}\left(\vec{i}\frac{x}{r}+\vec{j}\frac{y}{r}+\vec{k}\frac{z}{r}\right)$$

$$=\frac{-n}{r^{n+2}}(x\vec{i}+y\vec{j}+z\vec{k})=-\frac{n\vec{r}}{r^{n+2}}.$$

Identity X. Prove that $\nabla^2\left(\frac{1}{r}\right)=0$. [May 2015]

Solution. We have $\nabla^2\left(\frac{1}{r}\right)=\nabla.\nabla\left(\frac{1}{r}\right)=\nabla.\nabla(r^{-1})=\nabla.\left(\frac{(-1)}{r^3}\vec{r}\right)$ [$n=-1$ in the previous identity.]

$$=-\nabla.\left(\frac{\vec{r}}{r^3}\right)=-\left[\nabla\left(\frac{1}{r^3}\right).\vec{r}+\frac{1}{r^3}\nabla.\vec{r}\right]$$

$$=-\left[\frac{-3}{r^5}\vec{r}.\vec{r}+\frac{1}{r^3}3\right]\ [\because\ \nabla.\vec{r}=3]$$

$$=-\left[\frac{-3r^2}{r^5}+\frac{3}{r^3}\right]$$

$$= -\left[\frac{-3}{r^3} + \frac{3}{r^3}\right] = 0.$$

Identity XI. Prove that $\nabla^2(r^n) = n(n+1)r^{n-2}$. [May 2015, Dec 2010]

Solution. We have $\nabla r^n = nr^{n-2}\vec{r}$

$$\begin{aligned}
\nabla^2 r^n &= \nabla \cdot \nabla r^n = \nabla \cdot (nr^{n-2}\vec{r}) \\
&= n[\nabla r^{n-2} \cdot \vec{r} + r^{n-2}\nabla \cdot \vec{r}] \\
&= n[(n-2)r^{n-4}\vec{r} \cdot \vec{r} + r^{n-2} \cdot 3] \\
&= n[(n-2)r^{n-2} + 3r^{n-2}] = nr^{n-2}[n-2+3] = n(n+1)r^{n-2}.
\end{aligned}$$

Worked Examples

Example 2.49. Prove that $\nabla^2 u = 0$ if $u = x^2 - y^2$.

Solution. $u = x^2 - y^2$.

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y, \frac{\partial u}{\partial z} = 0.$$

$$\frac{\partial^2 u}{\partial x^2} = 2, \frac{\partial^2 u}{\partial y^2} = -2, \frac{\partial^2 u}{\partial z^2} = 0.$$

$$\text{Now, } \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= 2 - 2 + 0 = 0.$$

Example 2.50. If ϕ is a scalar point function, prove that $\nabla\phi$ is solenoidal and irrotational if ϕ is a solution of the Laplace equation. [Jun 2009]

Solution. Given that ϕ satisfies Laplace equation.

i.e, $\nabla^2\phi = 0$

$$\Longrightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

Now $div(\nabla\phi) = \nabla.\nabla\phi = \nabla^2\phi = 0$.

$\therefore$ $\nabla\phi$ is solenoidal.

$curl(\nabla\phi) = \nabla \times \nabla\phi = 0$ (always).

$\Longrightarrow$ $\nabla\phi$ is irrotational.

Example 2.51. If $\vec{A}$ and $\vec{B}$ are irrotational, prove that $\vec{A}\times\vec{B}$ is solenoidal. [Jun.2013]

Solution. Given that $\vec{A}$ and $\vec{B}$ are irrotational.

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Therefore, $\nabla \times \vec{A} = 0$ and $\nabla \times \vec{B} = 0$.

Now we have, $\nabla.(\vec{A} \times \vec{B}) = \vec{A}.(\nabla \times \vec{B}) - \vec{B}.(\nabla \times \vec{A})$

$$= \vec{A}.\vec{0} - B.\vec{0} = 0.$$

$\Longrightarrow \vec{A} \times \vec{B}$ is solenoidal.

Example 2.52. Prove that $\nabla \times (\nabla r^n) = \vec{0}$.

Solution. We have $\nabla r^n = nr^{n-2}\vec{r}$.

$$\begin{aligned}
\text{Now, } \nabla \times (\nabla r^n) &= \nabla \times (nr^{n-2}\vec{r}) = n[\nabla \times r^{n-2}\vec{r}] \\
&= n[\nabla(r^{n-2}) \times \vec{r} + r^{n-2}\nabla \times \vec{r}] \\
&= n[(n-2)r^{n-4}\vec{r} \times \vec{r} + 0] = n \cdot \vec{0} = \vec{0}.
\end{aligned}$$

Example 2.53. If ϕ and ψ satisfies Laplace equation, prove that $(\phi\nabla\psi - \psi\nabla\phi)$ is solenoidal.

Solution. Given ϕ and ψ satisfies Laplace equation.

Therefore, $\nabla^2\phi = 0$ and $\nabla^2\psi = 0$.

We have to prove that $\phi\nabla\psi - \psi\nabla\phi$ is solenoidal.

i.e., to prove $div(\phi\nabla\psi - \psi\nabla\phi) = 0$.

$$\begin{aligned}
\text{Now, } div(\phi\nabla\psi - \psi\nabla\phi) &= \nabla.(\phi\nabla\psi - \psi\nabla\phi) \\
&= \nabla.(\phi\nabla\psi) - \nabla.\psi(\nabla\phi) \\
&= \nabla\phi.\nabla\psi + \phi\nabla.\nabla\psi - [\nabla\psi.\nabla\phi + \psi\nabla.\nabla\phi] \\
&= \nabla\phi.\nabla\psi + \phi\nabla^2\psi - \nabla\psi.\nabla\phi - \psi\nabla^2\phi \\
&= \phi.0 - \psi.0 = 0.
\end{aligned}$$

$\Longrightarrow \phi\nabla\psi - \psi\nabla\phi$ is solenoidal.

Example 2.54. Show that $\nabla^2 f(r) = \dfrac{d^2 f}{dr^2} + \dfrac{2}{r}.\dfrac{df}{dr}$.

Solution. We have $\nabla f(r) = f'(r)\dfrac{\vec{r}}{r}$.

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Hence, $\nabla^2 f(r) = \nabla.\nabla f(r)$

$$= \nabla.\left(f'(r)\frac{\vec{r}}{r}\right)$$

$$= \left(\nabla\frac{f'(r)}{r}\right).\vec{r} + \frac{f'(r)}{r}(\nabla.\vec{r})$$

$$= \left(\nabla\frac{f'(r)}{r}\right).\vec{r} + \frac{3f'(r)}{r}$$

$$= \left(\frac{r\nabla f'(r) - f'(r)\nabla r}{r^2}\right).\vec{r} + \frac{3f'(r)}{r}[\text{using } \nabla\left(\frac{f}{g}\right)]$$

$$= \left(\frac{rf''(r)\frac{\vec{r}}{r} - f'(r)\frac{\vec{r}}{r}}{r^2}\right).\vec{r} + \frac{3f'(r)}{r}$$

$$= \frac{rf''(r) - f'(r)}{r^3}\vec{r}.\vec{r} + \frac{3f'(r)}{r}$$

$$= \frac{rf''(r) - f'(r)}{r^3}r^2 + \frac{3f'(r)}{r}$$

$$= \frac{rf''(r) - f'(r)}{r} + \frac{3f'(r)}{r}$$

$$= f''(r) - \frac{f'(r)}{r} + \frac{3f'(r)}{r}$$

$$= \frac{d^2 f(r)}{dr^2} + \frac{2}{r}\frac{df}{dr}.$$

Example 2.55. Prove that $\nabla^2(r^n\vec{r}) = n(n+3)r^{n-2}\vec{r}$.

Solution. We have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$.

$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$.

We know that

$$\nabla^2(r^n\vec{r}) = \sum\frac{\partial^2}{\partial x^2}(r^n\vec{r})$$

$$= \sum\frac{\partial}{\partial x}\left(\frac{\partial(r^n\vec{r})}{\partial x}\right)$$

$$= \sum\frac{\partial}{\partial x}\left(r^n\frac{\partial\vec{r}}{\partial x} + \vec{r}nr^{n-1}\frac{\partial r}{\partial x}\right)$$

$$= \sum\frac{\partial}{\partial x}\left(r^n\frac{\partial\vec{r}}{\partial x} + nr^{n-1}\frac{x}{r}\vec{r}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(r^n\vec{i} + nr^{n-2}x\vec{r}\right)$$

$$= \sum \left[nr^{n-1}\frac{\partial r}{\partial x}\vec{i} + n\left\{r^{n-2}x\frac{\partial \vec{r}}{\partial x} + r^{n-2}\vec{r}.1 + (n-2)r^{n-3}\frac{\partial r}{\partial x}x\vec{r}\right\}\right]$$

$$= \sum \left[nr^{n-1}\frac{x}{r}\vec{i} + nr^{n-2}x\vec{i} + nr^{n-2}\vec{r} + n(n-2)r^{n-3}\frac{x}{r}x\vec{r}\right]$$

$$= nr^{n-2}[x\vec{i} + y\vec{j} + z\vec{k}] + nr^{n-2}[x\vec{i} + y\vec{j} + z\vec{k}] + 3nr^{n-2}\vec{r}$$

$$+ n(n-2)r^{n-4}\vec{r}(x^2 + y^2 + z^2)$$

$$= nr^{n-2}.\vec{r} + nr^{n-2}.\vec{r} + 3nr^{n-2}\vec{r} + n(n-2)r^{n-4}\vec{r}.r^2$$

$$= 2nr^{n-2}\vec{r} + 3nr^{n-2}\vec{r} + n(n-2)r^{n-2}\vec{r}$$

$$= (2n + 3n + n^2 - 2n)r^{n-2}\vec{r}$$

$$= (n^2 + 3n)r^{n-2}\vec{r}$$

$$= n(n+3)r^{n-2}\vec{r}.$$

Example 2.56. If ϕ and ψ are scalar point functions, prove that $\phi\nabla\phi$ is irrotational and $\nabla\phi \times \nabla\psi$ is solenoidal.

Solution. We have $curl(\phi\nabla\phi) = \nabla \times (\phi\nabla\phi)$

$$= \phi(\nabla \times \nabla\phi) + \nabla\phi \times \nabla\phi = \phi.0 + 0 = 0.$$

$\therefore\ \phi\nabla\phi$ is irrotational.

Also, $div(\nabla\phi \times \nabla\psi) = \nabla.(\nabla\phi \times \nabla\psi)$

$$= \nabla\psi.(\nabla \times \nabla\phi) - \nabla\phi.\nabla \times (\nabla\psi)$$

$$= \nabla\psi.0 - \nabla\phi.0 = 0.$$

$\Longrightarrow \nabla\phi \times \nabla\psi$ is solenoidal.

Exercise 2 B

1. Prove that $div\left(\frac{\vec{r}}{r}\right) = \frac{2}{r}$. [Jun 2003]

2. If $\vec{A}$ is a constant vector, prove that $div(\vec{A}) = 0$ and $curl\vec{A} = \vec{0}$.

3. If $\vec{a}$ is a constant vector, prove that $\nabla.(\vec{a} \times \vec{r}) = 0$.

4. If $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$, find $\nabla.\vec{F}$ and $\nabla \times \vec{F}$.

5. Find the divergence and curl of the vector point function $xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$.

6. If $\vec{F} = 3xyz^2\vec{i} + 2xy^3\vec{j} - x^2yz\vec{k}$ and $f = 3x^2 - yz$, find (i) $\vec{F}.\nabla f$ and (ii) $\nabla.\nabla f$ at $(1, -1, 1)$.

7. Prove that the vector $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is solenoidal.

8. Show that $\vec{V} = xyz^2\vec{u}$ is solenoidal where $\vec{u} = (2x^2 + 8xy^2z)\vec{i} + (3x^3 - 3xy)\vec{j}$ $-(4y^2z^2 + 2x^3z)\vec{k}$.

9. Show that the vector $\vec{F} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$ is solenoidal.

10. Find the value of a so that the vector $\vec{F} = (x + 3y)\vec{i} + (y - 2z)\vec{j} + (x + az)\vec{k}$ is solenoidal.

11. Find a such that $(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal. [Jun 2004]

12. Show that $\vec{F} = (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k}$ is irrotational and solenoidal.

13. Show that the vector $2xy\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 + 1)\vec{k}$ is irrotational.

14. Show that $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is irrotational.

15. If $\vec{a}$ is a constant vector and $\vec{r}$ is the position vector of any point, then prove that $\nabla(\vec{a}.\vec{r}) = \vec{a}$ and $curl(\vec{a} \times \vec{r}) = 2\vec{a}$.

16. Find a, b, c if $(x + y + az)\vec{i} + (bx + 2y - z)\vec{j} + (-x + cy + 2z)\vec{k}$ is irrotational.

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17. Find the value of the arbitrary constant a so that the vector $\vec{F} = (axy - z^3)\vec{i} + (a-2)x^2\vec{j} + (1-a)xz^2\vec{k}$ is irrotational.

18. Find the constants a, b and c so that the vector $\vec{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ is irrotational.

19. Show that the vector $(3x^2 + 2y^2 + 1)\vec{i} + (4xy - 3y^2z - 3)\vec{j} + (2 - y^3)\vec{k}$ is irrotational and find its scalar potential.

20. Show that $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational and find its scalar potential. [May 2005]

21. Prove that $\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y \sin x - 4)\vec{j} + 3xz^2\vec{k}$ s irrotational and find its scalar potential.

22. If $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ show that $\nabla^2\vec{F} = 0$.

23. If $\vec{F} = x^2y\vec{i} + y^2z\vec{j} + z^2x\vec{k}$, find $curl\ curl\vec{F}$.

24. If $\vec{F} = (x + y + 1)\vec{i} + \vec{j} - (x + y)\vec{k}$, prove that $\vec{F}.curl\vec{F} = 0$.

25. If u and v are scalar point functions, prove that $\nabla(u\nabla v - v\nabla u) = u\nabla^2 v - v\nabla^2 u$.

26. If $\vec{F}$ is solenoidal, prove that $\nabla \times \nabla \times \nabla \times \vec{F} = \nabla^4\vec{F}$.

27. Prove that $\nabla.\left(\vec{r}\nabla\left(\frac{1}{r^3}\right)\right) = \frac{3}{r^4}$.

2.4 Integration of vector functions

Let $\vec{f}(t)$ be a vector function of the scalar variable t.

If $\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$, then

$$\int \vec{f}(t)dt = \vec{i}\int f_1(t)dt + \vec{j}\int f_2(t)dt + \vec{k}\int f_3(t)dt$$

Line Integral

A line integral of a vector point function $\vec{F}(\vec{r})$ over a curve C where $\vec{r}$ is the

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position vector of any point on C is defined by $\int_C \{\vec{F}\cdot d\vec{r}\}$.

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$.

Now, $\int_C \{\vec{F}\cdot d\vec{r}\} = \int_C (F_1dx + F_2dy + F_3dz)$.

Note

1. Since $\frac{d\vec{r}}{dt}$ is a tangent vector, to the curve C, the line integral $\int_C \vec{F}\cdot d\vec{r}$ is also called the tangential line integral of $\vec{F}$ over C.

2. This line integral is a scalar . The other types of line integrals are $\int_C \phi d\vec{r}$ and $\int_C \vec{F}\times d\vec{r}$ and these integrals are vectors.

Conservative Vector field. If the line integral depends only on the end points but not on the path C, then $\vec{F}$ is called a conservative vector field.

Result. If $\vec{F}$ is conservative, then $curl\vec{F} = \vec{0}$. Hence, if $\vec{F}$ is conservative, then $\vec{F} = \nabla\phi$.

Physical interpretation of $\int_c \vec{F}.d\vec{r}$.

Work done by a force [Physical interpretation of $\int_A^B \vec{F}\cdot d\vec{r}$] [May 2011]

If $\vec{F}(x,y,z)$ is a force acting on a particle which is moving along arc AB, then $\int_A^B \vec{F}\cdot d\vec{r}$ gives the total work done by the force $\vec{F}$ in displacing the particle from A to B.

Worked Examples

Example 2.57. If $\vec{F} = x^2\vec{i} + xy^2\vec{j}$, evaluate $\int_c \vec{F}\cdot d\vec{r}$ from $(0,0)$ to $(1,1)$ along the path $y = x$. [Jan 2006, Jun 2005]

Solution.

$$\text{Given } \vec{F} = x^2\vec{i} + xy^2\vec{j}.$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}.$$

$$\vec{F}\cdot d\vec{r} = x^2dx + xy^2dy.$$

Since the path is given by $y = x$, we have $dy = dx$ and x varies from 0 to 1.

$$\begin{aligned}\therefore \vec{F}\cdot d\vec{r} &= x^2dx + xx^2dx.\\ &= x^2dx + x^3dx\\ &= (x^2+x^3)dx.\end{aligned}$$

$$\begin{aligned}\int_c \vec{F}\cdot d\vec{r} &= \int_0^1 (x^2+x^3)dx\\ &= \left(\frac{x^3}{3}+\frac{x^4}{4}\right)_0^1 = \frac{1}{3}+\frac{1}{4} = \frac{7}{12}.\end{aligned}$$

Example 2.58. If $\vec{F} = 3xy\vec{i} - y^2\vec{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the arc of the parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$. [Jan 2001]

Solution. $\vec{F} = 3xy\vec{i} - y^2\vec{j}$.

$\vec{r} = x\vec{i} + y\vec{j}$.

$\vec{F}\cdot d\vec{r} = 3xydx - y^2dy$.

Given that the path is along the parabola $y = 2x^2 \Longrightarrow dy = 4xdx$.

Now, $\vec{F}\cdot d\vec{r} = 3x\cdot 2x^2dx - 4x^4\cdot 4xdx$

$= 6x^3dx - 16x^5dx$

Along $y = 2x^2$, x varies from 0 to 1.

$$\begin{aligned}\therefore \int_C \vec{F}\cdot d\vec{r} &= \int_0^1 (6x^3 - 16x^5)dx = \left(6\cdot\frac{x^4}{4} - 16\cdot\frac{x^6}{6}\right)_0^1\\ &= \frac{6}{4} - \frac{16}{6} = \frac{9-16}{6} = -\frac{7}{6}.\end{aligned}$$

Example 2.59. If $\vec{F} = 5xy\vec{i} + 2y\vec{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the part of the curve $y = x^2$ between $x = 1$ and $x = 2$. [Jun 2003]

Solution. We have $\vec{F} = 5xy\vec{i} + 2y\vec{j}$.

$$\begin{aligned}\vec{r} &= x\vec{i} + y\vec{j}.\\ d\vec{r} &= dx\vec{i} + dy\vec{j}.\\ \vec{F}\cdot d\vec{r} &= 5xydx + 2ydy.\end{aligned}$$

$$= 5x \cdot x^2 dx + 2 \cdot x^2 \cdot 2x dx \qquad [\text{since } y = x^2, dy = 2xdx]$$
$$= 5x^3 dx + 4x^3 dx = 9x^3 dx$$

Along $y = x^2$, x varies from 1 to 2.

$$\int_C \vec{F} \cdot d\vec{r} = \int_1^2 9x^3 dx = 9\left(\frac{x^4}{4}\right)_1^2 = \frac{9}{4}(16-1) = \frac{9}{4} \times 15 = \frac{135}{4}.$$

Example 2.60. If $\vec{F} = x^2\vec{i} + xy\vec{j}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$ from $(0,0)$ to $(1,1)$ along the line C given by $y = x$. [May 2010]

Solution. Given $\vec{F} = x^2\vec{i} + xy\vec{j}$

$$\vec{r} = x\vec{i} + y\vec{j}.$$
$$d\vec{r} = dx\vec{i} + dy\vec{j}.$$
$$\vec{F} \cdot d\vec{r} = x^2 dx + xydy.$$

The curve C is given by $y = x$.

$\therefore dy = dx$ and x varies from 0 to 1.

$$\vec{F} \cdot d\vec{r} = x^2 dx + xxdx$$
$$= 2x^2 dx.$$

$$\text{Now, } \int_C \vec{F} \cdot d\vec{r} = \int_0^1 2x^2 dx = 2\left(\frac{x^3}{3}\right)_0^1 = 2\left(\frac{1}{3}\right) = \frac{2}{3}.$$

Example 2.61. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ and C is the straight line from $A(0,0,0)$ to $B(2,1,3)$. [May 2008]

Solution. We have $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$.

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$
$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}.$$
$$\vec{F} \cdot d\vec{r} = 3x^2 dx + (2xz - y)dy + zdz.$$

The equation of the line joining the points $(0,0,0)$ and $(2,1,3)$ is

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t \text{ (say)}.$$

$$\text{i.e., } \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t.$$

Hence, $x = 2t, y = t, z = 3t$.

$dx = 2dt, dy = dt, dz = 3dt$.

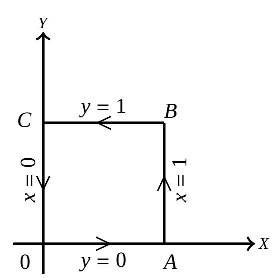
$$\vec{F} \cdot d\vec{r} = 3.4t^2.2dt + (2.2t.3t - t)dt + 3t.3dt$$
$$= (24t^2 + 12t^2 - t + 9t)dt$$
$$= (36t^2 + 8t)dt.$$

As x varies from 0 to 2, we have t varies from 0 to 1.

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (36t^2 + 8t)dt$$
$$= 36\left(\frac{t^3}{3}\right)_0^1 + 8\left(\frac{t^2}{2}\right)_0^1 = 36\left(\frac{1}{3}\right) + 4(1) = 12 + 4 = 16.$$

Example 2.62. Evaluate $\int_C (x^2 + xy)dx + (x^2 + y^2)dy$ where C is the square bounded by the lines $x = 0, x = 1, y = 0$ and $y = 1$. [Dec 2011]

Solution. Let $F_1 = x^2 + xy$ and $F_2 = x^2 + y^2$.



The given integral can be written as $\int_C (F_1dx + F_2dy)$.

C is the rectangle $OABC$. C contains four paths namely, the lines OA, AB, BC and CO.

$$\int_c (F_1dx + F_2dy) = \int_{OA} (F_1dx + F_2dy) + \int_{AB} (F_1dx + F_2dy) + \int_{BC} (F_1dx + F_2dy) + \int_{CO} (F_1dx + F_2dy).$$

Along OA, $y = 0 \Rightarrow dy = 0$ and x varies from 0 to 1.

$$F_1dx + F_2dy = x^2dx.$$

$$\int_{OA} (F_1dx + F_2dy) = \int_0^1 x^2dx = \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3}.$$

Along AB, $x = 1 \Rightarrow dx = 0$ and y varies from 0 to 1.

$$F_1dx + F_2dy = (y^2+1)dy.$$

$$\int_{AB}(F_1dx+F_2dy) = \int_0^1 (y^2+1)dy = \left(\frac{y^3}{3}\right)_0^1 + (y)_0^1 = \frac{1}{3}+1 = \frac{4}{3}.$$

Along BC, $y = 1 \Rightarrow dy = 0$ and x varies from 1 to 0.

$$F_1dx + F_2dy = (x^2+x)dx$$

$$\int_{BC}(F_1dx+F_2dy) = \int_1^0 (x^2+x)dx = \left(\frac{x^3}{3}\right)_1^0 + \left(\frac{x^2}{2}\right)_1^0 = \left(0-\frac{1}{3}\right)+\left(0-\frac{1}{2}\right)$$

$$= -\frac{1}{3}-\frac{1}{2} = \frac{-2-3}{6} = -\frac{5}{6}.$$

Along CO, $x = 0 \Rightarrow dx = 0$ and y varies from 1 to 0.

$$F_1dx + F_2dy = y^2dy.$$

$$\int_{CO}(F_1dx+F_2dy) = \int_1^0 y^2dy = \left(\frac{y^3}{3}\right)_1^0 = 0-\frac{1}{3} = -\frac{1}{3}.$$

$$\therefore \int_c (F_1dx+F_2dy) = \frac{1}{3}+\frac{4}{3}-\frac{5}{6}-\frac{1}{3} = \frac{4}{3}-\frac{5}{6} = \frac{8-5}{6} = \frac{3}{6} = \frac{1}{2}.$$

Example 2.63. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ and C is the rectangle in the XY plane bounded by $x = 0, x = a, y = b$ and $y = 0$.

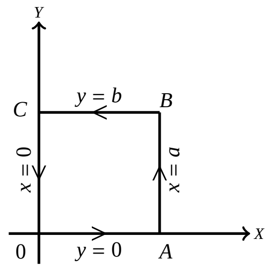
Solution. We have $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$.

$$\vec{r} = x\vec{i} + y\vec{j}.$$

$$d\vec{r} = dx\vec{i} + dy\vec{j}.$$

$$\vec{F}\cdot d\vec{r} = (x^2+y^2)dx - 2xydy.$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C [(x^2+y^2)dx - 2xydy]$$



Now, C is the rectangle $OABC$.

Now C contains four paths namely, the lines OA, AB, BC and CO.

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA}\vec{F}\cdot d\vec{r} + \int_{AB}\vec{F}\cdot d\vec{r} + \int_{BC}\vec{F}\cdot d\vec{r} + \int_{CO}\vec{F}\cdot d\vec{r}.$$

Along $OA, y = 0 \implies dy = 0$ and x varies from 0 to a.

$$\int_{OA} \vec{F} \cdot d\vec{r} = \int_0^a x^2 dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}.$$

Along $AB, x = a \implies dx = 0$ and y varies from 0 to b.

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^b (-2ay)dy = -2a\left(\frac{y^2}{2}\right)_0^b = -ab^2.$$

Along $BC, y = b \implies dy = 0$ and x varies from a to 0.

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_a^0 (x^2 + b^2)dx = \left(\frac{x^3}{3}\right)_a^0 + b^2(x)_a^0 = 0 - \frac{a^3}{3} + b^2(0 - a) = -\frac{a^3}{3} - ab^2.$$

Along $CO, x = 0 \implies dx = 0$ and y varies from b to 0.

$$\int_{CO} \vec{F} \cdot d\vec{r} = \int_b^0 (0)dy = 0.$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = -2ab^2.$$

Example 2.64. Show that $\vec{F} = (e^x z - 2xy)\vec{i} - (x^2 - 1)\vec{j} + (e^x + z)\vec{k}$ is a conservative field. Hence evaluate $\int_C \vec{F} \cdot d\vec{r}$ where the end points of C are $(0, 1, -1)$ and $(2, 3, 0)$.

Solution. If $\vec{F}$ is conservative, we must have $\nabla \times \vec{F} = \vec{0}$.

$$\therefore \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x z - 2xy & 1 - x^2 & e^x + z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(e^x + z) - \frac{\partial}{\partial z}(1 - x^2)\right) - \vec{j}\left(\frac{\partial}{\partial x}(e^x + z) - \frac{\partial}{\partial z}(e^x z - 2xy)\right)$$
$$+ \vec{k}\left(\frac{\partial}{\partial x}(1 - x^2) - \frac{\partial}{\partial y}(e^x z - 2xy)\right)$$
$$= (0)\vec{i} - (e^x - e^x)\vec{j} + (-2x + 2x)\vec{k} = \vec{0}.$$

$\therefore \vec{F}$ is conservative.

Hence $\vec{F} = \nabla\vec{\phi}$.
i.e., $\vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z} = (e^x z - 2xy)\vec{i} - (x^2-1)\vec{j} + (e^x+z)\vec{k}$.
Equating the coefficients of $\vec{i}$, $\vec{j}$ and $\vec{k}$ we get

$$\frac{\partial\phi}{\partial x} = e^x z - 2xy \tag{1}$$
$$\frac{\partial\phi}{\partial y} = 1 - x^2 \tag{2}$$
$$\frac{\partial\phi}{\partial z} = e^x + z. \tag{3}$$

On integration we obtain

$$\phi = \int (e^x z - 2xy)dx + f_1(y,z)$$
$$= ze^x - 2y\frac{x^2}{2} + f_1(y,z)$$
$$\phi = ze^x - x^2y + f_1(y,z) \tag{4}$$
$$\text{Also, } \phi = \int (1-x^2)dy + f_2(x,z)$$
$$= y - x^2y + f_2(x,z) \tag{5}$$
$$\text{Again, } \phi = \int (e^x+z)dz + f_3(x,y)$$
$$= e^x z + \frac{z^2}{2} + f_3(x,y) \tag{6}$$

From (4), (5) and (6) we get $\phi = e^x z - x^2y + y + \frac{z^2}{2} + C$, where C is a constant.

$$\begin{aligned}\text{Now, } \int_C \vec{F}\cdot d\vec{r} &= \int_C \nabla\phi\cdot d\vec{r}\\ &= \int_C \left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right).(\vec{i}dx + \vec{j}dy + \vec{k}dz)\\ &= \int_C \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)\\ &= \int_C d\phi\\ &= [\phi]_{(0,1,-1)}^{(2,3,0)}\end{aligned}$$

—

$$= \left(e^x z + \frac{z^2}{2} - x^2 y + y\right)_{(0,1,-1)}^{(2,3,0)}$$
$$= [0 + 0 - 12 + 3] - [-1 + \frac{1}{2} - 0 + 1]$$
$$= -9 - \frac{1}{2} = -\frac{19}{2}.$$

Example 2.65. Show that $\vec{F} = (2xy+z^3)\vec{i}+x^2\vec{j}+3xz^2\vec{k}$ is a conservative field. Find the scalar potential and the work done in moving an object in this field from $(1, -2, 1)$ to $(3, 1, 4)$.

Solution. To prove $\vec{F}$ is conservative, we have to prove that $\nabla \times \vec{F} = 0$.

$$\text{Now, } \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$
$$= \vec{i}\left(\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(x^2)\right) - \vec{j}\left(\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(2xy + z^3)\right)$$
$$+ \vec{k}\left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy + z^3)\right)$$
$$= (0)\vec{i} - (3z^2 - 3z^2)\vec{j} + (2x - 2x)\vec{k}$$
$$= \vec{0}.$$

$\therefore \vec{F}$ is conservative.

Hence $\vec{F} = \nabla\vec{\phi}$.

i.e., $\vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}.$

Comparing the coefficients of $\vec{i}, \vec{j}$ and $\vec{k}$ we get

$$\frac{\partial \phi}{\partial x} = 2xy + z^3. \tag{1}$$
$$\frac{\partial \phi}{\partial y} = x^2. \tag{2}$$
$$\frac{\partial \phi}{\partial z} = 3xz^2. \tag{3}$$

On integration we obtain

$$\phi = \int (2xy + z^3)dx + f_1(y,z)$$
$$= 2y\frac{x^2}{2} + z^3x + f_1(y,z)$$
$$\phi = x^2y + xz^3 + f_1(y,z). \tag{4}$$
$$\text{Also, } \phi = \int (x^2)dy + f_2(x,z)$$
$$= x^2y + f_2(x,z). \tag{5}$$
$$\text{Again, } \phi = \int 3xz^2dz + f_3(x,y)$$
$$= 3x\frac{z^3}{3} + f_3(x,y)$$
$$= xz^3 + f_3(x,y). \tag{6}$$

From (4), (5) and (6) we get $\phi = x^2y + xz^3 + C$, where C is a constant.

$$\text{Now, work done} = \int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\phi\cdot d\vec{r}$$
$$= [\phi]_{(1,-2,1)}^{(3,1,4)}$$
$$= \left(xz^3 + x^2y\right)_{(1,-2,1)}^{(3,1,4)}$$
$$= [3(4^3) + 9.1] - [1.1 + 1(-2)]$$
$$= 192 + 9 + 1 = 202.$$

Example 2.66. Find the work done when the force $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ moves a particle from the origin to the point $(1,1)$ along $y^2 = x$. [Jan 1996]

Solution.

$$\text{Work done} = \int_C \vec{F}\cdot d\vec{r}$$
$$= \int_C \left((x^2 - y^2 + x)dx - (2xy + y)dy\right)$$
$$\text{Along } C, y^2 = x \quad \Longrightarrow dx = 2ydy.$$

—

y varies from 0 to 1.

$$\therefore \text{Work done } = \int_0^1 \left((y^4 - y^2 + y^2)2ydy - (2y^3 + y)dy\right)$$
$$= \int_0^1 (2y^5 - 2y^3 - y)dy$$
$$= \left(2\frac{y^6}{6} - 2\frac{y^4}{4} - \frac{y^2}{2}\right)_0^1$$
$$= \frac{1}{3} - \frac{1}{2} - \frac{1}{2}$$
$$= -\frac{2}{3}.$$

Example 2.67. Find the work done in moving a particle in the vector field $\vec{A} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$.

[Jun 2012, Dec 2004]

Solution. Given $\vec{A} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$.

The equation of the straight line joining $(0,0,0)$ and $(2,1,3)$ is

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t \, (say)$$
$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t.$$
$$\therefore x = 2t, \; y = t, \; z = 3t.$$
$$dx = 2dt, \; dy = dt, \; dz = 3dt.$$

Now $\vec{A} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$.

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}.$$
$$\vec{A}.d\vec{r} = 3x^2dx + (2xz - y)dy + zdz$$
$$= 3.4t^2.2dt + (2.2t.3t - t)dt + 3t.3dt$$
$$= 24t^2dt + (12t^2 - t)dt + 9tdt$$
$$= (36t^2 + 8t)dt.$$

To find the limits for t.

when $x = 0, t = 0$.

when $x = 2, t = 1$.

$$\therefore \text{Now, work done } = \int_c \vec{A}.d\vec{r}$$

$$= \int_0^1 (36t^2 + 8t)dt$$

$$= 36.\left(\frac{t^3}{3}\right)_0^1 + 8.\left(\frac{t^2}{2}\right)_0^1$$

$$= 36.\frac{1}{3} + 8.\frac{1}{2}$$

$$= 12 + 4$$

$$= 16.$$

Exercise 2 C

1. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$, from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve C given by $x = t, y = t^2, z = t^3$.

2. Evaluate the line integral $\int_C (y^2dx - x^2dy)$ around the triangle whose vertices are $(1, 0), (0, 1), (-1, 0)$ in the positive sense.

3. If $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is the straight line joining $(0, 0, 0)$ and $(1, 1, 1)$.

4. Show that the vector field $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ is conservative vector field. [Jun 2001]

5. Evaluate $\int (xdy - ydx)$ around the circle $x^2 + y^2 = 1$.

6. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = xy\vec{i} + (x^2 + y^2)\vec{j}$, and C is the arc of the parabola $y = x^2 - 4$ from $(2, 0)$ to $(4, 12)$ in the xy plane.

7. If $\vec{F} = 3xy\vec{i} - y^3\vec{j}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is the curve $y = 2x^2$ in the xy plane from $(0, 0)$ to $(1, 2)$.

—

8. Given the vector field $\vec{F} = xz\vec{i} + yz\vec{j} + z^2\vec{k}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$ from the point $(0,0,0)$ to $(1,1,1)$ where C is the curve (i) $x = t, y = t^2, z = t^3$ and (ii) the straight line from $(0,0,0)$ to $(1,1,1)$.

9. Find the total work done in moving a particle in a force field $\vec{F} = (2x - y + z)\vec{i} + (x + y - z)\vec{j} + (3x - 2y - 5z)\vec{k}$ along a circle C in the xy plane $x^2 + y^2 = 9, z = 0$.
[Jan 2010]

10. If $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\vec{k}$, check whether the integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of the path C. [May 2000]

11. Show that the vector field $\vec{F} = (y + y^2 + z^2)\vec{i} + (x + z + 2xy)\vec{j} + (y + 2xz)\vec{k}$ is conservative and find its scalar potential.

12. Find the work done in moving a particle once round the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ if the vector field is $\vec{F} = (3x - 4y + 2z)\vec{i} + (4x + 2y - 3z^2)\vec{j} + (2xz - 4y^2 + z^3)\vec{k}$.

2.5 Green's theorem in a plane

In the previous semester, we have learned the methodology of evaluating double integral over a region R. Green's theorem demonstrates an easy way of evaluating line integrals. It provides a relation between a double integral over a region R in the xy plane and the line integral over a closed curve C which encloses the given region R. [Dec 2014, Dec 2013]

Statement. If $P(x,y)$ and $Q(x,y)$ are continuous functions with continuous partial derivatives in a region R in the xy plane and on its boundary C which is a simple closed curve, then $\int_C (Pdx + Qdy) = \iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$ where C is described in the anticlockwise sense.

Corollary. Area of the region bounded by C is $\iint_R dxdy = \frac{1}{2}\int_C (xdy - ydx)$.
[Dec 2011]

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Proof. Assume $P = -y, Q = x$. $\dfrac{\partial P}{\partial y} = -1, \dfrac{\partial Q}{\partial x} = 1$.

By Green's theorem we have

$$\therefore \int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$$\text{i.e.,} \int_C (-ydx + xdy) = \iint_R (1+1)dxdy$$

$$\int_C (xdy - ydx) = 2\iint_R dxdy$$

$$\frac{1}{2}\int_C (xdy - ydx) = \iint_R dxdy.$$

But $\iint_R dxdy$ = Area of the region R bounded by C.

$$\therefore \text{ Area } = \frac{1}{2}\int_C (xdy - ydx).$$

Worked Examples

Example 2.68. Using Green's theorem, evaluate $\int_C [(2x - y)dx + (x + y)dy]$ where C is the circle $x^2 + y^2 = 4$ in the xy plane.

Solution. Comparing the given integral with Green's theorem we have

$P = 2x - y$ and $Q = x + y$.

$\dfrac{\partial P}{\partial y} = -1$ $\dfrac{\partial Q}{\partial x} = 1$.

By Green's theorem,

$$\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$$\begin{aligned} \text{i.e.,} \int_C \{(2x - y)dx + (x + y)dy\} &= \iint_R (1+1)dxdy \\ &= 2\iint_R dxdy \\ &= 2 \times \text{Area of the region} \\ &= 2 \times \text{Area of the circle with radius } 2 \\ &= 2 \times \pi \times 2^2 = 8\pi. \end{aligned}$$

—

Example 2.69. Using Green's theorem, evaluate $\int_C [(2x^2-y^2)dx+(x^2+y^2)dy]$ where C is the boundary of the square enclosed by the lines $x=0, y=0, x=2$ and $y=2$.

Solution. Comparing the given integral with Green's theorem we have

$P = 2x^2 - y^2$ and $Q = x^2 + y^2$.

$\dfrac{\partial P}{\partial y} = -2y \; \dfrac{\partial Q}{\partial x} = 2x.$

By Green's theorem,

$$\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

$$\text{i.e.,} \int_C \{(2x^2-y^2)dx + (x^2+y^2)dy\} = \iint_R (2x+2y)dxdy = 2\iint_R (x+y)dxdy$$

$$= 2\int_0^2\int_0^2 (x+y)dxdy = 2\int_0^2 \left(\frac{x^2}{2} + xy\right)_0^2 dy$$

$$= 2\int_0^2 \left(2+2y\right)dy = 2\left(2y + 2\frac{y^2}{2}\right)_0^2 dy$$

$$= 2[4+4] = 16.$$

Example 2.70. Using Green's theorem, evaluate $\int_C [(x^2-y^2)dx + 2xydy]$ where C is the closed curve of the region $y^2 = x$ and $x^2 = y$.

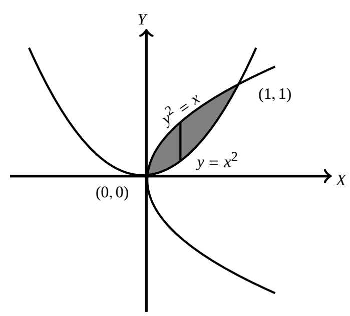
Solution.

Comparing the given integral with Green's theorem we have

$P = x^2 - y^2, Q = 2xy.$

$\dfrac{\partial P}{\partial y} = -2y \; \dfrac{\partial Q}{\partial x} = 2y.$

$\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 2y + 2y = 4y.$



By Green's theorem $\int_C (Pdx + Qdy) = \iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy.$

$$\text{i.e., } \int_C \{(x^2 - y^2)dx + 2xydy\} = \iint_R 4ydxdy = \iint_R 4ydydx$$

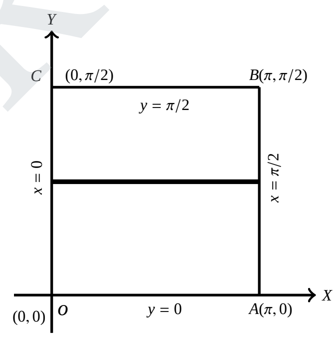
$$= 4\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} ydydx = 4\int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

$$= 2\int_0^1 (x - x^4)dx = 2\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 2\left[\frac{1}{2} - \frac{1}{5}\right] = 2\left[\frac{5-2}{10}\right] = \frac{3}{5}.$$

Example 2.71. Evaluate by Green's theorem $\int_C e^{-x}(\sin ydx + \cos ydy)$ where C being the rectangle with vertices $(0,0), (\pi,0), (\pi,\frac{\pi}{2}), (0,\frac{\pi}{2})$. [Jun 1995]

Solution.



Comparing the given integral with Green's theorem we have

$P = e^{-x}\sin y Q = e^{-x}\cos y.$

$\frac{\partial P}{\partial y} = e^{-x}\cos y \frac{\partial Q}{\partial x} = -e^{-x}\cos y.$

By Green's theorem

$$\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

$$= \iint_R (-e^{-x}\cos y - e^{-x}\cos y)dxdy$$

$$= -2\iint_R e^{-x}\cos ydxdy$$

$$= -2\int_{y=0}^{\frac{\pi}{2}}\int_{x=0}^{\pi} e^{-x}\cos ydxdy$$

$$= -2\int_{y=0}^{\frac{\pi}{2}} \cos y(-e^{-x})_{x=0}^{\pi} dy$$

$$= 2\int_0^{\frac{\pi}{2}} \cos y(e^{-\pi} - 1)dy$$

—

$$= 2(e^{-\pi} - 1)(\sin y)_0^{\frac{\pi}{2}}$$
$$= 2(e^{-\pi} - 1)(\sin\frac{\pi}{2} - \sin 0)$$
$$= 2(e^{-\pi} - 1).$$

Example 2.72. Evaluate by Green's theorem $\int_C \{(\sin x - y)dx - \cos x dy\}$ where C is the triangle with vertices $(0,0), \left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, 1\right)$. [Jun 2003]

Solution. Comparing the given integral with Green's theorem we have

$P = \sin x - yQ = -\cos x.$

$\frac{\partial P}{\partial y} = -1\frac{\partial Q}{\partial x} = \sin x.$

By Green's theorem we have

$$\int_C (Pdx + Qdy) = \iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy.$$

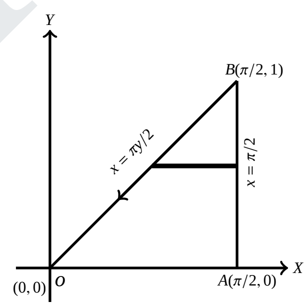
$$\text{i.e., } \int_C \{(\sin x - y)dx - \cos x dy\} = \iint_R (\sin x + 1)dxdy \tag{1}$$

Equation of OB is $\frac{y-0}{1-0} = \frac{x-0}{\frac{\pi}{2}-0}$

$$\frac{y}{1} = \frac{x}{\frac{\pi}{2}}$$
$$y = \frac{2}{\pi}x.$$

Equation to AB is $\frac{y-0}{1-0} = \frac{x-\frac{\pi}{2}}{\frac{\pi}{2}-\frac{\pi}{2}}$

$$\frac{y}{1} = \frac{x-\frac{\pi}{2}}{0} \quad \Rightarrow x = \frac{\pi}{2}.$$



Now (1) becomes

$$\int_C \{(\sin x - y)dx - \cos x dy\} = \int_0^1 \int_{x=\frac{\pi y}{2}}^{\frac{\pi}{2}} (\sin x + 1)dxdy$$
$$= \int_0^1 \left(-\cos x + x\right)_{\frac{\pi y}{2}}^{\frac{\pi}{2}} dy$$
$$= \int_0^1 \left\{-\cos\frac{\pi}{2} + \frac{\pi}{2} - (-\cos\frac{\pi y}{2} + \frac{\pi y}{2})\right\}dy$$

—

$$= \int_0^1 \left(\frac{\pi}{2} + \cos\frac{\pi y}{2} - \frac{\pi y}{2}\right) dy$$
$$= \frac{\pi}{2}(y)_0^1 + \left(\frac{\sin\frac{\pi y}{2}}{\frac{\pi}{2}}\right)_0^1 - \frac{\pi}{2}\left(\frac{y^2}{2}\right)_0^1$$
$$= \frac{\pi}{2} + \frac{2}{\pi}\left(\sin\frac{\pi}{2} - \sin 0\right) - \frac{\pi}{4}(1-0)$$
$$= \frac{\pi}{2} + \frac{2}{\pi}\cdot 1 - \frac{\pi}{4}$$
$$= \frac{\pi}{4} + \frac{2}{\pi}.$$

Example 2.73. Find the area of the circle of radius a using Green's theorem.

Solution.

We know that, the area enclosed by a simple closed curve C is $\frac{1}{2}\int_C (xdy - ydx)$.

Here, the curve is a circle of radius a given by $x^2 + y^2 = a^2$.

Consider the parametric coordinates $x = a\cos\theta, y = a\sin\theta, 0 \le \theta \le 2\pi$

$$dx = -a\sin\theta d\theta,\ dy = a\cos\theta d\theta$$

$$\text{Area of the circle} = \frac{1}{2}\int_0^{2\pi} (xdy - ydx)$$
$$= \frac{1}{2}\int_0^{2\pi} a\cos\theta a\cos\theta d\theta - a\sin\theta(-a\sin\theta)d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} (a^2\cos^2\theta + a^2\sin^2\theta)d\theta$$
$$= \frac{1}{2}a^2\int_0^{2\pi} d\theta = \frac{1}{2}a^2(\theta)_0^{2\pi}$$
$$= \frac{1}{2}a^2\cdot 2\pi = \pi a^2.$$

Example 2.74. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by using Green's theorem. [May 2004]

Solution. By Green's theorem,

Area of the ellipse $= \frac{1}{2}\int_c (xdy - ydx)$, where C is the ellipse.

The parametric coordinates of the ellipse are

$x = a\cos\theta, y = b\sin\theta, 0 \le \theta \le 2\pi$

—

$dx = -a\sin\theta d\theta, dy = b\cos\theta d\theta.$

$$\therefore \text{ Area of the ellipse } = \frac{1}{2}\int_0^{2\pi}(a\cos\theta b\cos\theta d\theta - b\sin\theta(-a\sin\theta d\theta))$$

$$= \frac{1}{2}\int_0^{2\pi}(ab\cos^2\theta + ab\sin^2\theta)d\theta$$

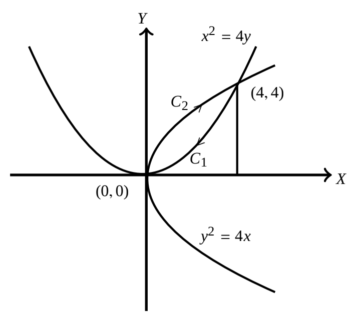
$$= \frac{1}{2}ab\int_0^{2\pi}d\theta$$

$$= \frac{1}{2}ab[\theta]_0^{2\pi}$$

$$= \frac{1}{2}ab2\pi = \pi ab.$$

Example 2.75. Using Green's theorem, find the area bounded between the parabolas $y^2 = 4x$ and $x^2 = 4y$.

Solution.



The given curves are $y^2 = 4x$ and $x^2 = 4y$.

Squaring the second equation we get

$$x^4 = 16y^2 = 16.4x = 64x$$

$$x^4 - 64x = 0$$

$$x(x^3 - 64) = 0$$

$$x = 0, x^3 = 64.$$

$$x = 4.$$

When $x = 0, y = 0,$

When $x = 4, y = 4.$

Hence, the points of intersection are $(0, 0)$ and $(4, 4)$.

Let A be $(4, 4)$.

Now C is the curve OAO.

Divide C into two curves C_1 and C_2 where C_1 represents the portion between O and A along $x^2 = 4y$ and C_2 represents the portion between A and O along $y^2 = 4x$.

By Green's theorem,

$$\text{Area} = \frac{1}{2}\int_C (xdy - ydx)$$

$$= \frac{1}{2}\left[\int_{C_1}(xdy - ydx) + \int_{C_2}(xdy - ydx)\right]$$

$$= \frac{1}{2}[I_1 + I_2] \text{ where}$$

$$I_1 = \int_{C_1}(xdy - ydx)$$

$$I_2 = \int_{C_2}(xdy - ydx).$$

$$\text{Consider } I_1 = \int_{C_1}(xdy - ydx)$$

Along $C_1, x^2 = 4y$.

$$\text{i.e., } y = \frac{x^2}{4}$$

$$dy = \frac{1}{4}2xdx = \frac{x}{2}dx.$$

x varies from 0 to 4.

$$\therefore I_1 = \int_0^4 \left(x\frac{x}{2}dx - \frac{x^2}{4}dx\right)$$

$$= \int_0^4 \left(\frac{x^2}{2}dx - \frac{x^2}{4}dx\right)$$

$$= \int_0^4 \frac{x^2}{4}dx = \frac{1}{4}\left(\frac{x^3}{3}\right)_0^4$$

$$= \frac{1}{12}[4^3 - 0] = \frac{1}{12}(64) = \frac{16}{3}.$$

$$\text{Consider } I_2 = \int_{C_2}(xdy - ydx)$$

Along $C_2, y^2 = 4x$

—

i.e., $x = \frac{y^2}{4}$

$$dx = \frac{1}{4}2ydy = \frac{y}{2}dy.$$

y varies from 4 to 0.

$$\therefore I_2 = \int_4^0 \left(\frac{y^2}{4}dy - y\frac{y}{2}dy\right)$$

$$= \int_4^0 \left(\frac{y^2}{4} - \frac{y^2}{2}\right)dy = \int_4^0 \left(-\frac{y^2}{4}\right)dy$$

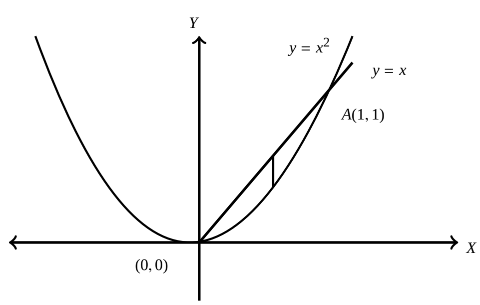
$$= \int_0^4 \left(\frac{y^2}{4}\right)dy = \frac{1}{4}\left(\frac{y^3}{3}\right)_0^4 = \frac{1}{12} \times 64 = \frac{16}{3}.$$

$$\therefore \text{Area } = \frac{1}{2}[I_1 + I_2]$$

$$= \frac{1}{2}\left[\frac{16}{3} + \frac{16}{3}\right] = \frac{16}{3}.$$

Example 2.76. Apply Green's theorem in a plane to evaluate $\int_C ((xy + y^2)dx + x^2dy)$ where C is the boundary of the area between $y = x^2$ and $y = x$. [Nov 2002]

Solution.



On solving the two equations we get the points of intersection as $(0, 0)$ and $(1, 1)$.

By Green's theorem

$$\int_C \{(Pdx + Qdy)\} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy,$$

Now, $P = xy + y^2$ and $Q = x^2$.

$$\frac{\partial P}{\partial y} = x + 2y, \frac{\partial Q}{\partial x} = 2x.$$

—

$$\text{Now, } \iint_R (xy+y^2)dx + x^2dy = \iint_R (2x - x - 2y)dxdy$$
$$= \iint_R (x-2y)dxdy$$
$$= \int_0^1 \int_{y=x^2}^{x} (x-2y)dydx$$
$$= \int_0^1 \left(xy - 2\frac{y^2}{2}\right)_{y=x^2}^{x} dx$$
$$= \int_0^1 [(x^2-x^2)-(x^3-x^4)]dx$$
$$= \int_0^1 (x^4-x^3)dx = \left(\frac{x^5}{5}-\frac{x^4}{4}\right)_0^1$$
$$= \frac{1}{5}-\frac{1}{4} = -\frac{1}{20}.$$

Example 2.77. Verify Green's theorem in a plane for the integral $\int_C ((x-2y)dx + xdy)$ taken around the circle $x^2+y^2=4$. [May 2000]

Solution. By Green's theorem, $\int_C \{(Pdx+Qdy)\} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$, C is the circle $x^2+y^2=4$.

Now, $P = x-2y \qquad Q = x.$

$$\frac{\partial P}{\partial y} = -2 \qquad \frac{\partial Q}{\partial x} = 1$$

$$\text{Now, } \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \iint_R (1+2)dxdy$$
$$= 3\iint_R dxdy$$
$$= 3\cdot \text{Area of the circle} = 3\cdot\pi\cdot 2^2 = 12\pi.$$

Let us evaluate $\int_C \{(Pdx+Qdy)\}$.

The parametric equations are

—

$x = 2\cos\theta \qquad y = 2\sin\theta$

$dx = -2\sin\theta d\theta \qquad dy = 2\cos\theta d\theta.$

Since C is the entire circle, θ varies from 0 to 2π.

$$\text{Now, } \int_C \{(Pdx + Qdy)\} = \int_0^{2\pi} (2\cos\theta - 2\cdot 2\sin\theta)(-2\sin\theta d\theta) + (2\cos\theta 2\cos\theta d\theta)$$
$$= \int_0^{2\pi} (-4\sin\theta cos\theta + 8sin^2\theta + 4\cos^2\theta)d\theta$$
$$= \int_0^{2\pi} (-4\sin\theta cos\theta + 8sin^2\theta + 4(1-\sin^2\theta)d\theta$$
$$= \int_0^{2\pi} (-2\sin 2\theta + 4 + 4\sin^2\theta)d\theta$$
$$= \int_0^{2\pi} (-2\sin 2\theta + 4 + 4\cdot\frac{1-\cos 2\theta}{2})d\theta$$
$$= \int_0^{2\pi} (-2\sin 2\theta + 4 + 2 - 2\cos 2\theta)d\theta$$
$$= 6(\theta)_0^{2\pi} - 2\left(\frac{-\cos 2\theta}{2}\right)_0^{2\pi} - \left(2\frac{\sin 2\theta}{2}\right)_0^{2\pi}$$
$$= 6\cdot 2\pi + [\cos 4\pi - \cos 0 - (\sin 4\pi - \sin 0)]$$
$$= 12\pi + (1-1) - (0-0) = 12\pi = LHS.$$

Hence, Green's theorem is verified.

Example 2.78. Verify Green's theorem in the plane for $\int_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ where C is the boundary of the region bounded by $x = 0, y = 0, x + y = 1.$ [Jun 2012, May 2011, Dec 2010, Dec 2007]

Solution. By Green's theorem,

$$\int_C (Pdx + Qdy) = \iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy.$$

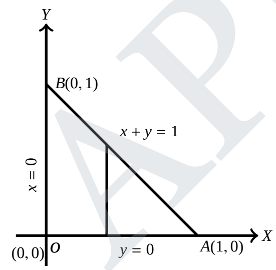
$P = 3x^2 - 8y^2 \qquad Q = 4y - 6xy.$

$\frac{\partial P}{\partial y} = -16y \qquad \frac{\partial Q}{\partial x} = -6y.$

$$\text{Now, } \iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy = \iint_R (-6y + 16y)dxdy = \iint_R 10ydxdy$$

$$= 10 \iint_R y\,dy\,dx = 10 \int_{x=0}^{1} \int_{y=0}^{1-x} y\,dy\,dx$$
$$= 10 \int_{x=0}^{1} \left(\frac{y^2}{2}\right)_0^{1-x} dx = \frac{10}{2} \int_0^1 (1-x)^2 dx$$
$$= 5\left[\frac{(1-x)^3}{-3}\right]_0^1 = -\frac{5}{3}[0-1] = \frac{5}{3}.$$

Now, $\int_C (Pdx + Qdy) = \int_{OA}(Pdx + Qdy) + \int_{AB}(Pdx + Qdy) + \int_{BO}(Pdx + Qdy)$.



Along OA, $y = 0 \Rightarrow dy = 0$ and x varies from 0 to 1.

$\therefore \int_{OA} (Pdx + Qdy) = \int_0^1 3x^2 dx = 3\left(\frac{x^3}{3}\right)_0^1 = 1.$

Along AB, $y = 1 - x$, $dy = -dx$, x varies from 1 to 0

$$\therefore \int_{AB} (Pdx + Qdy) = \int_1^0 (3x^2 - 8(1-x)^2)dx + (4(1-x) - 6x(1-x))(-dx)$$
$$= \int_1^0 \{3x^2 - 8 - 8x^2 + 16x - (4 - 4x - 6x + 6x^2)\}dx$$
$$= \int_1^0 (-5x^2 + 16x - 8 - 4 + 10x - 6x^2)dx$$
$$= \int_1^0 (-11x^2 + 26x - 12)dx$$
$$= -11\left(\frac{x^3}{3}\right)_1^0 + 26\left(\frac{x^2}{2}\right)_1^0 - 12\left(x\right)_1^0$$
$$= -\frac{11}{3}(0-1) + 13(0-1) - 12(0-1) = \frac{11}{3} - 13 + 12$$
$$= \frac{11}{3} - 1 = \frac{8}{3}.$$

Along BO, $x = 0 \Rightarrow dx = 0$, y varies from 1 to 0.

$$\int_{BO} (Pdx + Qdy) = \int_1^0 4y\,dy = \left[4\frac{y^2}{2}\right]_1^0 = 2[0-1] = -2.$$

$$\therefore \int_C (Pdx + Qdy) = 1 + \frac{8}{3} - 2 = \frac{8}{3} - 1 = \frac{5}{3} = RHS.$$

—

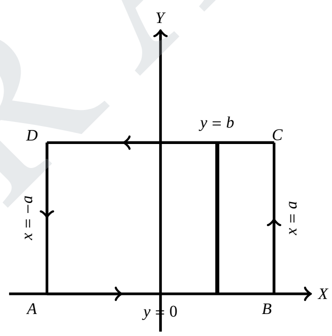
Example 2.79. Verify Green's theorem for $\vec{V} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$ taken around the rectangle bounded by the lines $x = \pm a, y = 0, y = b$. [Nov 2012]

Solution. Consider the line integral $\int_C \vec{V} \cdot d\vec{r}$ where C is the given rectangle.

$$\therefore \int_C \vec{V} \cdot d\vec{r} = \int_C [(x^2 + y^2)dx - 2xydy]$$
$$= \int_C (Pdx + Qdy) \quad \text{where } P = x^2 + y^2, Q = -2xy.$$

By Green's theorem $\int_C (Pdx + Qdy) = \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$

$$\frac{\partial P}{\partial y} = 2y, \frac{\partial Q}{\partial x} = -2y.$$



Now, $$\iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint\limits_R (-2y - 2y)dxdy$$
$$= -\iint\limits_R 4ydxdy$$
$$= -\int\limits_{x=-a}^{a} \int\limits_{y=0}^{b} 4ydydx$$
$$= -4 \int\limits_{x=-a}^{a} \left[\frac{y^2}{2}\right]_0^b dx$$
$$= -4 \int\limits_{-a}^{a} \frac{b^2}{2} dx$$
$$= -2b^2[x]_{-a}^{a} = -2b^2[a + a] = -4ab^2. \quad (1)$$

Now let us evaluate $\int_C (Pdx + Qdy)$.

$$\int_C (Pdx + Qdy) = \int\limits_{AB} (Pdx + Qdy) + \int\limits_{BC} (Pdx + Qdy) + \int\limits_{CD} (Pdx + Qdy) + \int\limits_{DA} (Pdx + Qdy).$$

—

Along $AB, y = 0, dy = 0$ and x varies from $-a$ to a.

$$\int_{AB}(Pdx + Qdy) = \int_{-a}^{a} x^2 dx = \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{1}{3}(a^3 + a^3) = \frac{2a^3}{3}.$$

Along $BC, x = a, dx = 0$ and y varies from 0 to b.

$$\int_{BC}(Pdx + Qdy) = \int_{0}^{b} -2aydy = -2a\left[\frac{y^2}{2}\right]_{0}^{b} = -ab^2.$$

Along $CD, y = b, dy = 0$ and x varies from a to $-a$.

$$\begin{aligned}\therefore \int_{CD}(Pdx + Qdy) &= \int_{a}^{-a}(x^2 + b^2)dx = \left[\frac{x^3}{3} + b^2x\right]_{a}^{-a}\\ &= -\frac{a^3}{3} - ab^2 - \left(\frac{a^3}{3} + ab^2\right)\\ &= -\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2\\ &= -\frac{2a^3}{3} - 2ab^2.\end{aligned}$$

Along $DA, x = -a, dx = 0$ and y varies from b to 0.

$$\therefore \int_{DA}(Pdx + Qdy) = \int_{b}^{0} 2aydy = 2a\left[\frac{y^2}{2}\right]_{b}^{0} = a(0 - b^2) = -ab^2.$$

$$\therefore \int_C(Pdx + Qdy) = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 = -4ab^2. \quad (2)$$

From (1) and (2) we have

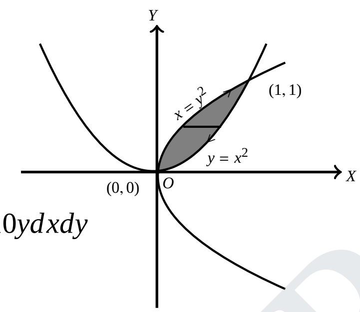
$$\int_C(Pdx + Qdy) = \iint_R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy.$$

$\therefore$ Green's theorem is verified.

Example 2.80. Verify Green's theorem in the plane for $\int_C(3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region bounded by the curves $x = y^2, y = x^2$.

[Jun 2010]

Solution. By Green's theorem,



$$\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

Now, $P = 3x^2 - 8y^2$. $\quad Q = 4y - 6xy$.

$\dfrac{\partial P}{\partial y} = -16y$. $\quad \dfrac{\partial Q}{\partial x} = -6y$.

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint_R (-6y + 16y) dxdy = \iint_R 10ydxdy$$

$$= \int_{y=0}^{1} \int_{x=y^2}^{\sqrt{y}} 10ydxdy = 10 \int_{y=0}^{1} y[x]_{y^2}^{\sqrt{y}} dy$$

$$= 10 \int_0^1 y(\sqrt{y} - y^2) dy = 10 \int_0^1 \left(y^{3/2} - y^3\right) dy$$

$$= 10 \left[\frac{y^{5/2}}{\frac{5}{2}} - \frac{y^4}{4}\right]_0^1 = 10\left(\frac{2}{5} - \frac{1}{4}\right)$$

$$= 10\left(\frac{8-5}{20}\right) = \frac{3}{2}. \qquad (1)$$

Now let us evaluate $\int_C (Pdx + Qdy)$.

We have $\int_C (Pdx + Qdy) = \int_{OA} (Pdx + Qdy) + \int_{AO} (Pdx + Qdy)$.

Along $OA, y = x^2, dy = 2xdx$ and x varies from 0 to 1.

$$\therefore \int_{OA} (Pdx + Qdy) = \int_0^1 \left\{(3x^2 - 8x^4)dx + (4x^2 - 6x \cdot x^2)2xdx\right\}$$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4) dx.$$

$$= \left[3 \cdot \frac{x^3}{3} - 8\frac{x^5}{5} + 8\frac{x^4}{4} - 12\frac{x^5}{5}\right]_0^1$$

$$= 1 - \frac{8}{5} + 2 - \frac{12}{5}$$

$$= 3 - \frac{20}{5} = 3 - 4 = -1.$$

Along AO, $x = y^2$, $dx = 2ydy$ and y varies from 1 to 0.

$$\therefore \int_{AO} (Pdx + Qdy) = \int_1^0 \left\{(3y^4 - 8y^2)2ydy + (4y - 6y^2 \cdot y)dy\right\}$$

$$= \int_1^0 (6y^5 - 16y^3 + 4y - 6y^3)dy$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)dy$$

$$= \left[6 \cdot \frac{y^6}{6} - 22\frac{y^4}{4} + 4\frac{y^2}{2}\right]_1^0$$

$$= \left[y^6 - \frac{11}{2}y^4 + 2y^2\right]_1^0$$

$$= 0 - \left(1 - \frac{11}{2} + 2\right) = -\left(3 - \frac{11}{2}\right) = \frac{5}{2}.$$

$$\therefore \int_C (Pdx + Qdy) = -1 + \frac{5}{2} = \frac{3}{2}. \quad (2)$$

From (1) and (2)

$$\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

$\therefore$ Green's theorem is verified.

Example 2.81. If C is a simple closed curve in the xy plane not enclosing the origin, show that $\int_C \vec{F} \cdot d\vec{r} = 0$ where $\vec{F} = \dfrac{y\vec{i} - x\vec{j}}{x^2 + y^2}$.

Solution. $\vec{F} \cdot d\vec{r} = \dfrac{y\vec{i} - x\vec{j}}{x^2 + y^2} \cdot \left(dx\vec{i} + dy\vec{j}\right)$

$$= \frac{ydx - xdy}{x^2 + y^2} = \frac{y}{x^2 + y^2}dx - \frac{x}{x^2 + y^2}dy$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \frac{y}{x^2 + y^2}dx - \frac{x}{x^2 + y^2}dy = \int_C Pdx + Qdy$$

where $P = \frac{y}{x^2+y^2}$ $\qquad Q = -\frac{x}{x^2+y^2}$

$$\frac{\partial P}{\partial y} = \frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

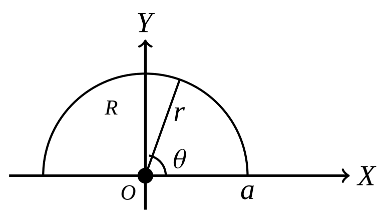
$$\frac{\partial Q}{\partial x} = -\frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

By Green's theorem,

$$\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

i.e., $$\int_C \frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy = \iint_R \left(\frac{x^2-y^2}{(x^2+y^2)^2} - \frac{x^2-y^2}{(x^2+y^2)^2}\right)dxdy = 0.$$

Example 2.82. Apply Green's theorem to evaluate $\int_C [(2x^2-y^2)dx + (x^2+y^2)dy]$ where C is the boundary of the area enclosed by the x–axis and the upper half of the circle $x^2+y^2=a^2$.



Solution. Comparing the given integral with Green's theorem we have

$P = 2x^2 - y^2$ and $Q = x^2 + y^2$.

$\frac{\partial P}{\partial y} = -2y, \frac{\partial Q}{\partial x} = 2x.$

By Green's theorem,

$$\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

i.e., $$\int_C \{(2x^2-y^2)dx + (x^2+y^2)dy\} = \iint_R (2x+2y)dxdy$$

$$= 2\iint_R (x+y)dxdy$$

By changing into polar coordinates we have $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = rdrd\theta$. Along R, r varies from 0 to a and θ varies from 0 to π.

$$\int_C \{(2x^2 - y^2)dx + (x^2 + y^2)dy\} = 2\int_0^a \int_0^\pi (r\cos\theta + r\sin\theta) r dr d\theta$$

$$= 2\int_0^a r^2 dr \int_0^\pi (\cos\theta + \sin\theta)d\theta$$

$$= 2\int_0^a r^2 dr \Big(\sin\theta - \cos\theta\Big)_0^\pi$$

$$= 2\int_0^a r^2 dr\,(\sin\pi - \sin 0 - \cos\pi + \cos 0)$$

$$= 2\int_0^a r^2 dr \cdot 2 = 4\left(\frac{r^3}{3}\right)_0^a = \frac{4}{3}\Big(a^3 - 0\Big) = \frac{4a^3}{3}.$$

Example 2.83. Using Green's theorem evaluate the integral $\int_C [(2x - y)dx + (x + 3y)dy]$ where C is the boundary of the ellipse $x^2 + 4y^2 = 4$.

Solution. Comparing the given integral with Green's theorem we have

$P = 2x - y$ and $Q = x + 3y$.

$$\frac{\partial P}{\partial y} = -1, \qquad \frac{\partial Q}{\partial x} = 1.$$

By Green's theorem,

$$\int_C (Pdx + Qdy) = \iint_R \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dxdy$$

i.e., $$\int_C (2x - y)dx + (x + 3y)dy = \iint_R (1 + 1)dxdy$$

$$= 2\iint_R dxdy$$

$$= 2 \times \text{Area of the ellipse} \qquad \left[\because \frac{x^2}{4} + \frac{y^2}{1} = 1 \Rightarrow a = 2, b = 1\right]$$

$$= 2 \times \pi ab = 2 \times \pi \times 2 \times 1 = 4\pi.$$

—

Exercise 2 D

1. Verify Green's theorem in the plane for $\int_C [x^2(1+y)dx + (x^3+y^3)dy]$ where C is the square bounded by $x = \pm a, y = \pm a$. [Dec 2006]

2. Verify Green's theorem in the plane for $\int_C [(x^2-y^2)dx + 2xydy]$ where C is the boundary of the rectangle in the xy plane bounded by the lines $x = 0, x = a$, $y = 0, y = b$. [Jan 2004]

3. Verify Green's theorem in the plane for $\int_C [(x-2y)dx + xdy]$ where C is the unit circle $x^2+y^2 = 1$. [Jun 2007]

4. Verify Green's theorem in the plane for $\int_C [(x^2-xy^3)dx + (y^2-2xy)dy]$ where C is the square with vertices $(0,0), (2,0), (2,2), (0,2)$. [Apr 2002]

5. Verify Green's theorem for $\int_C [(x^2-2xy)dx + (x^2y+3)dy]$ where C is the boundary of the region bounded by $y^2 = 6x$ and $x = 2$. [Jan 2005]

6. Apply Green's theorem to evaluate $\int_C [(2xy-x^2)dx + (x^2+y^2)dy]$ where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$.

7. Apply Green's theorem to evaluate $\int_C [(x^2+1+y)dx + (x^3+y^3)dy]$ where C is the square formed by the lines $x = \pm 1$ and $y = \pm 1$.

8. Using Green's theorem find the area between the curves $y^2 = 4ax$ and $x^2 = 4ay$.

2.6 Surface Integral

Consider a domain D in the cartesian coordinate system. Let a surface S be bounded by a simple closed curve C. Because of the space curve C, the surface S can be visualized as two sided, one arbitrarily chosen side can be regarded as positive and the other as negative. If S is closed, then the outer side of S can be taken as positive and the inner side can be considered as the negative side.

At any point P on the outer side of the surface, the unit normal $\vec{n}$ drawn at P is considered as the outward drawn normal and is along the positive direction. The inward drawn normal will be along the negative direction.

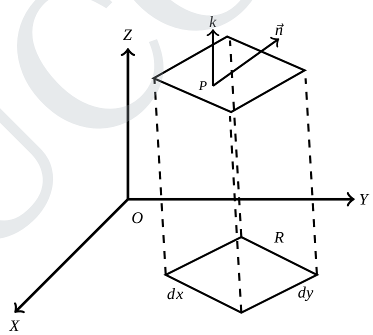
An integral which is evaluated over a surface is called a surface integral.

Definition. Let S be a surface of finite area which is smooth or piecewise smooth [Eg: Sphere is a smooth surface, cube is a piecewise smooth surface]. Let $\vec{F}(x, y, z)$ be a vector point function defined at each point of S. Let P be any point on the surface and $\vec{n}$ be the outward unit normal at P. Then the surface integral of $\vec{F}$ over S is defined as $\iint_S \vec{F} \cdot \vec{n} ds$.

Note. If we associate $d\vec{s}$ with the differential of the surface area ds such that $|d\vec{s}| = ds$ and direction of ds is $\vec{n}$ then $d\vec{s} = \vec{n} \cdot ds$
$\iint_S \vec{F} \cdot \vec{n} ds = \iint_S \vec{F} \cdot d\vec{s}$.

Note. $\iint_S \vec{F} \cdot d\vec{s}$ is called the normal flux of $\vec{F}$ through the surface S.

2.6.1 Evaluation of surface integral



Let R be the orthonormal projection of S on the xy plane. Then, the element surface ds is projected to an element area $dxdy$ in the xy plane. Now $dxdy = dscos\theta$ where θ is the angle between the planes of ds and the xy plane.

Let $\vec{n}$ be the unit normal to ds and $\vec{k}$ be the unit normal to the xy plane. Angle

—

between $\vec{n}$ and $\vec{k}$ is θ.

$$\therefore cos\theta = \frac{\vec{n} \cdot \vec{k}}{|\vec{n}||\vec{k}|} = \vec{n} \cdot \vec{k}$$

$$\therefore dxdy = ds|(\vec{n} \cdot \vec{k})| \qquad \text{(Taking the acute angle)}$$

$$\therefore ds = \frac{dxdy}{|\vec{n} \cdot \vec{k}|}$$

$$\therefore \iint_S \vec{F} \cdot \vec{n} ds = \iint_R \left\{\vec{F} \cdot \vec{n} \frac{dxdy}{|\vec{n} \cdot \vec{k}|}\right\}$$

Similarly, the projection on the yz plane and the zx planes give

$$\iint_S \{\vec{F} \cdot \vec{n} ds\} = \iint_R \vec{F} \cdot \vec{n} \frac{dydz}{|\vec{n} \cdot \vec{i}|}$$

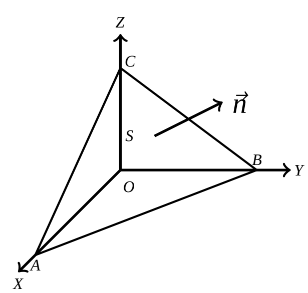
and $\displaystyle\iint_S \vec{F} \cdot \vec{n} ds = \iint_R \vec{F} \cdot \vec{n} \frac{dxdz}{|\vec{n} \cdot \vec{j}|}$.

Corollary. **Surface area** $= \displaystyle\iint_S ds = \iint_R \frac{dxdy}{|\vec{n} \cdot \vec{k}|} = \iint_R \frac{dxdz}{|\vec{n} \cdot \vec{j}|} = \iint_R \frac{dydz}{|\vec{n} \cdot \vec{i}|}$.

Worked Examples

Example 2.84. Evaluate $\displaystyle\iint_S \{\vec{F} \cdot \vec{n} ds\}$ if $\vec{F} = 4y\vec{i} + 18z\vec{j} - x\vec{k}$ and S is the surface of the portion of the plane $3x + 2y + 6z = 6$ contained in the first octant. [Jan 2005]

Solution.



$\vec{F} = 4y\vec{i} + 18z\vec{j} - x\vec{k}$.

The surface is $\phi = 3x + 2y + 6z - 6$.

Let R be the projection of S in the xy plane.

$\therefore R$ is the ΔAOB

$$\therefore \iint_S \vec{F} \cdot \vec{n} ds = \iint_R \vec{F} \cdot \vec{n} \frac{dxdy}{|\vec{n} \cdot \vec{k}|}$$

where $\vec{n}$ is the unit normal to S and $\vec{k}$ is the unit normal to the xy plane. Normal to the surface is $\nabla\phi$.

$$\therefore\ \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 3\vec{i} + 2\vec{j} + 6\vec{k}.$$

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3\vec{i} + 2\vec{j} + 6\vec{k}}{\sqrt{9 + 4 + 36}} = \frac{1}{7}(3\vec{i} + 2\vec{j} + 6\vec{k}).$$

$$\vec{F}\cdot\vec{n} = \frac{1}{7}(12y + 36z - 6x) = \frac{6}{7}(2y + 6z - x)$$

$$\vec{n}\cdot\vec{k} = \frac{6}{7}.$$

$$\therefore\ \iint_S \vec{F}\cdot\vec{n}ds = \iint_R \frac{\frac{6}{7}(2y + 6z - x)}{\frac{6}{7}}dxdy = \iint_R (2y + 6z - x)dxdy.$$

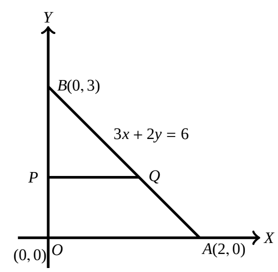
We have, $3x + 2y + 6z = 6$

$6z = 6 - 3x - 2y$

$2y + 6z - x = 6 - 3x - x = 6 - 4x$

$\therefore\ \iint_R \vec{F}\cdot\vec{n}ds = \iint_R (6 - 4x)dxdy.$

The plane $3x + 2y + 6z = 6$ meets the xy plane in line AB.



Equation of AB is $3x + 2y = 6$ [put $z = 0$]. The point A is $(2,0)$ and B is $(0,3)$. Divide the region OAB into strips parallel to the x–axis. PQ is one such strip. Along AB, x varies from 0 to $\frac{6-2y}{3}$ and finally as this strip traverses the surface OAB, y varies from 0 to 3.

$$\therefore\ \iint_S \vec{F}\cdot\vec{n}ds = \int_{y=0}^{3}\int_{x=0}^{\frac{6-2y}{3}} (6 - 4x)dxdy$$

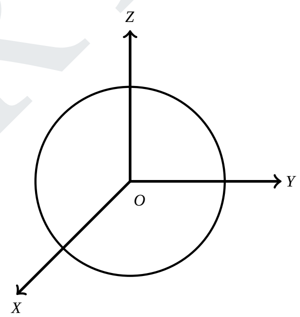
$$= \int_{y=0}^{3} \left(6x - 4\frac{x^2}{2}\right)_0^{\frac{6-2y}{3}} dy$$

$$= \int_{y=0}^{3} \left\{6\left(\frac{6-2y}{3}\right) - 2\left(\frac{6-2y}{3}\right)^2\right\}dy$$

$$= \int_{y=0}^{3} \left\{2(6-2y) - \frac{2}{9}(36+4y^2-24y)\right\} dy$$
$$= \int_{y=0}^{3} (12 - 4y - \frac{72+8y^2-48y}{9}) dy$$
$$= \frac{1}{9}\int_{0}^{3} (108 - 36y - 72 - 8y^2 + 48y) dy$$
$$= \frac{1}{9}\int_{0}^{3} (36 + 12y - 8y^2) dy$$
$$= \frac{1}{9}\left(36\left(y\right)_0^3 + 12\left(\frac{y^2}{2}\right)_0^3 - 8\left(\frac{y^3}{3}\right)_0^3\right) = \frac{1}{9}(36\cdot 3 + 6\cdot 9 - \frac{8}{3}\cdot 27)$$
$$= \frac{1}{9}(108 + 54 - 72) = \frac{1}{9}(162-72) = \frac{1}{9}\cdot 90 = 10.$$

Example 2.85. Evaluate $\iint_S \{\vec{F}\cdot\vec{n}\} ds$ where $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ and S is the part of the surface of the sphere $x^2+y^2+z^2 = 1$ which lies in the first octant. [Nov 2008]

Solution.



Given $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$.

Let $\phi = x^2 + y^2 + z^2 - 1$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}.$$

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{2\sqrt{x^2+y^2+z^2}} = x\vec{i} + y\vec{j} + z\vec{k}[\because x^2+y^2+z^2 = 1]$$

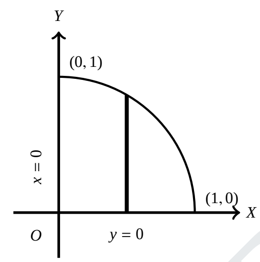
$$\vec{F}.\vec{n} = xyz + xyz + xyz = 3xyz.$$

Let R be the projection of S in the xy plane. Then R is bounded by the lines $x = 0, y = 0$ and the circle $x^2 + y^2 = 1, z = 0$. [Circle in the first quadrant of the xy plane]

Now $\iint\limits_S \vec{F}.\vec{n} ds = \iint\limits_R \vec{F}.\vec{n}\frac{dxdy}{|\vec{n}.\vec{k}|}$

$|\vec{n}.\vec{k}| = z$

$$\iint\limits_S \vec{F}.\vec{n}ds = \iint\limits_R 3xyz\frac{dxdy}{z} = 3\iint\limits_R xydxdy$$



$$= 3\int_0^1\int_{y=0}^{\sqrt{1-x^2}} xydxdy = 3\int_0^1 x\left(\frac{y^2}{2}\right)_0^{\sqrt{1-x^2}} dx$$

$$= \frac{3}{2}\int_0^1 x(1-x^2)dx = \frac{3}{2}\int_0^1 (x-x^3)dx = \frac{3}{2}\left(\frac{x^2}{2}-\frac{x^4}{4}\right)_0^1 = \frac{3}{2}\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{3}{8}.$$

Example 2.86. Evaluate $\iint_S \{\vec{F}\cdot\vec{n}\}ds$ where $\vec{F} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$ and S is that part of the plane $2x+3y+6z = 12$ which lies in the first octant. [Dec 2011]

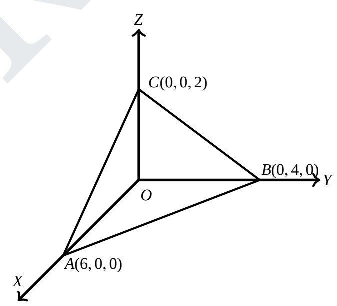
Solution.

Given $\vec{F} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$.

Let $\phi = 2x+3y+6z-12$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= 2\vec{i} + 3\vec{j} + 6\vec{k}.$$



$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i}+3\vec{j}+6\vec{k}}{\sqrt{4+9+36}} = \frac{2\vec{i}+3\vec{j}+6\vec{k}}{7}$$

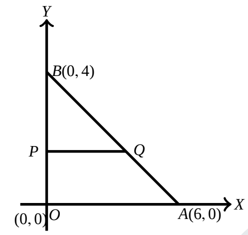
$$\vec{F}.\vec{n} = \frac{36z-36+18y}{7} = \frac{36\left(\frac{12-2x-3y}{6}\right)-36+18y}{7}.\left(\begin{array}{c}\text{Substitute } z \text{ from the}\\ \text{equation of the plane}\end{array}\right)$$

$$= \frac{1}{7}[72-12x-18y-36+18y] = \frac{1}{7}[36-12x].$$

Let R be the projection of S in the xy plane and hence R is the triangle OAB. AB is the line given by $2x+3y = 12$. Divide R into strips parallel to the x−axis. PQ is one such strip. Along this strip x varies from 0 to $\frac{12-3y}{2}$ and as this strip traverses the entire R, y varies from 0 to 4.

—

Now $\displaystyle\iint_S \vec{F}.\vec{n}ds = \iint_R \vec{F}.\vec{n}\frac{dxdy}{|\vec{n}.\vec{k}|} = \iint_R \frac{1}{7}(36-12x)\frac{dxdy}{\frac{6}{7}}$



$$= \int_0^4 \int_{x=0}^{\frac{12-3y}{2}} \frac{12}{6}(3-x)dxdy$$

$$= 2\int_0^4 \left(3x - \frac{x^2}{2}\right)_{x=0}^{\frac{12-3y}{2}} dy = 2\int_0^4 \left(3\Big(\frac{12-3y}{2}\Big) - \frac{1}{2}\Big(\frac{12-3y}{2}\Big)^2\right) dy$$

$$= 2\int_0^4 \left(18 - \frac{9}{2}y - \frac{1}{8}(144+9y^2-72y)\right) dy = 2\int_0^4 \left(18 - \frac{9}{2}y - 18 - \frac{9}{8}y^2 + 9y)\right) dy$$

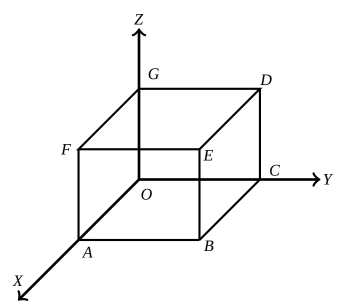
$$= 2\int_0^4 \left(\frac{9}{2}y - \frac{9}{8}y^2\right) dy = 2\left(\frac{9}{2}\Big(\frac{y^2}{2}\Big)_0^4 - \frac{9}{8}\Big(\frac{y^3}{3}\Big)_0^4\right) = 2\left(\frac{9}{4}16 - \frac{9}{24}64\right) = 2[36-24] = 24.$$

Example 2.87. Evaluate $\iint_S \{\vec{F}\cdot\vec{n}\}ds$ where $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ and S is the surface of the cube bounded by the planes $x=0, x=1, y=0, y=1, z=0, z=1$.

[Dec 2008]

Solution. $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$.

The given surface is the cube $OABCDEFG$. S is piecewise smooth which consists of six smooth surfaces namely $ABEF, OCDG, BCDE, OAFG, OABC$ and $DEFG$.



$$\therefore \iint_S \{\vec{F}\cdot\vec{n}\}ds = \iint_{ABEF}\{\vec{F}\cdot\vec{n}\}ds + \iint_{OCDG}\{\vec{F}\cdot\vec{n}\}ds + \iint_{OABC}\{\vec{F}\cdot\vec{n}\}ds$$
$$+ \iint_{DEFG}\{\vec{F}\cdot\vec{n}\}ds + \iint_{BCDE}\{\vec{F}\cdot\vec{n}\}ds + \iint_{OAFG}\{\vec{F}\cdot\vec{n}\}ds.$$

The following table gives the necessary quantities for the computation of the surface integral on each surface.

Face	Equation	Outward normal	$\vec{F}\cdot\vec{n}$	ds
ABEF	$x=1$	$\vec{i}$	$4z$	$dydz$
OCDG	$x=0$	$-\vec{i}$	$-4xz=0$	$dydz$
BCDE	$y=1$	$\vec{j}$	$-y^2=-1$	$dxdz$
OAFG	$y=0$	$-\vec{j}$	$-y^2=0$	$dxdz$
OABC	$z=0$	$-\vec{k}$	$-yz=0$	$dxdy$
DEFG	$z=1$	$\vec{k}$	$yz=y$	$dxdy$

$\iint_{ABEF}\vec{F}\cdot\vec{n}ds=\int_0^1\int_0^1 4zdydz=4\int_0^1 z(y)_0^1dz{=}4\int_0^1 zdz=4\left(\frac{z^2}{2}\right)_0^1=2.$
$\iint_{BCDE}\vec{F}\cdot\vec{n}ds=\int_0^1\int_0^1(-1)dxdz=-\int_0^1(x)_0^1dz{=}-\int_0^1 dz=-(z)_0^1=-1.$
$\iint_{DEFG}\vec{F}\cdot\vec{n}ds=\int_0^1\int_0^1 ydxdy=\int_{x=0}^1\left(\frac{y^2}{2}\right)_0^1dx{=}\frac{1}{2}(x)_0^1=\frac{1}{2}.$
All the other integrals will be zero.
$\therefore\iint_S\vec{F}\cdot\vec{n}ds=2-1+\frac{1}{2}=\frac{3}{2}.$

2.6.2 Volume Integral

If V is the volume bounded by a surface S then $\iiint\limits_V\phi(x,y,z)dv$ and $\iiint\limits_V\vec{F}dv$ are called volume integrals.

If we divide V into rectangular blocks by drawing planes parallel to the coordinate planes, then $dv=dxdydz$
$\therefore\iiint\phi dv=\iiint\limits_V\phi(x,y,z)dxdydz.$
If $\vec{F}=F_1\vec{i}+F_2\vec{j}+F_3\vec{k}$ then

$$\iiint\limits_V\vec{F}dv=\vec{i}\iiint\limits_V F_1dxdydz+\vec{j}\iiint\limits_V F_2dxdydz+\vec{k}\iiint\limits_V F_3dxdydz.$$

—

Worked Examples

Example 2.88. Evaluate $\iiint_V \nabla\cdot\vec{F}dv$ if $\vec{F} = x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$ and V is the volume of the region enclosed by the cube $x=0, x=1, y=0, y=1, z=0, z=1$. [Apr 2004]

Solution. $\vec{F} = x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^2)+\frac{\partial}{\partial y}(y^2)+\frac{\partial}{\partial z}(z^2) = 2x+2y+2z = 2(x+y+z).$$

$$\iiint_V \nabla\cdot\vec{F}dv = \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} 2(x+y+z)dxdydz$$

$$= 2\int_0^1\int_0^1 \left[(\frac{x^2}{2})_0^1 + y(x)_0^1 + z(x)_0^1\right]dydz$$

$$= 2\int_0^1\int_0^1 (\frac{1}{2}+y+z)dydz = 2\int_0^1 [\frac{1}{2}(y)_0^1 + (\frac{y^2}{2})_0^1 + z(y)_0^1]dz$$

$$= 2\int_0^1 (\frac{1}{2}+\frac{1}{2}+z)dz = 2\int_0^1 (1+z)dz = 2\left(z+\frac{z^2}{2}\right)_0^1 = 2[1+\frac{1}{2}] = 2\cdot\frac{3}{2} = 3.$$

Example 2.89. If $\vec{F} = (2x^2-3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$, evaluate $\iiint_V \nabla\times\vec{F}dV$ where V is the region bounded by $x=0, y=0, z=0$ and $2x+2y+z=4$.

Solution.

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x^2-3z) & -2xy & -4x \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(-4x) - \frac{\partial}{\partial z}(-2xy)\right] - \vec{j}\left[\frac{\partial}{\partial x}(-4x) - \frac{\partial}{\partial z}(2x^2-3z)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(2x^2-3z)\right]$$

$$= \vec{i}.0 - \vec{j}(-4+3) + \vec{k}(-2y+0) = \vec{j} - 2y\vec{k}.$$

$$\iiint_V \nabla\times\vec{F}dV = \iiint_V (\vec{j}-2y\vec{k})dxdydz = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} (\vec{j}-2y\vec{k})dzdydx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} (\vec{j}-2y\vec{k})(z)_0^{4-2x-2y}dydx$$

—

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} (4-2x-2y)\vec{j} - (8y-4xy-4y^2)\vec{k}dydx$$

$$= \int_{x=0}^{2} ((4y-2xy-y^2)\vec{j} - (4y^2-2xy^2-\frac{4}{3}y^3)\vec{k})_{y=0}^{2-x}dx$$

$$= \int_0^2 (4(2-x)-2x(2-x)-(2-x)^2)\vec{j} - (4(2-x)^2-2x(2-x)^2-\frac{4}{3}(2-x)^3)\vec{k}dx$$

$$= \int_0^2 (8-4x-4x+2x^2-(2-x)^2)\vec{j} - (4(2-x)^2-2(4x+x^3-4x^2)-\frac{4}{3}(2-x)^3)\vec{k}dx$$

$$= \left[8(x)_0^2 - 8\left(\frac{x^2}{2}\right)_0^2 + 2\left(\frac{x^3}{3}\right)_0^2 - \left(\frac{(2-x)^3}{-3}\right)_0^2\right]\vec{j} - \vec{k}\left[4\left(\frac{(2-x)^3}{-3}\right)_0^2 - 8\left(\frac{x^2}{2}\right)_0^2\right.$$
$$\left. -2\left(\frac{x^4}{4}\right)_0^2 + 8\left(\frac{x^3}{3}\right)_0^2 - \frac{4}{3}\left(\frac{(2-x)^4}{-4}\right)_0^2\right]$$

$$= \left[8(2-0)-4(4-0)+\frac{2}{3}(8-0)+\frac{1}{3}(0-8)\right]\vec{j}$$
$$-\vec{k}\left[\frac{-4}{3}(0-8)-4(4-0)-\frac{1}{2}(16-0)+\frac{8}{3}(8-0)+\frac{1}{3}(0-16)\right]$$

$$= \left[16-16+\frac{16}{3}-\frac{8}{3}\right]\vec{j} - \vec{k}\left[\frac{32}{3}-16-8+\frac{64}{3}-\frac{16}{3}\right] = \frac{8}{3}\vec{j} - \frac{8}{3}\vec{k} = \frac{8}{3}\left[\vec{j}-\vec{k}\right].$$

2.7 Gauss Divergence theorem

[Dec 2012, Jun 2012, Jun 2010]

Gauss divergence theorem express a surface integral interms of a volume integral.

Statement. Let V be the volume bounded by a closed surface S. If a vector function $\vec{F}$ is continuous and has continuous partial derivatives inside and on S, then the surface integral of $\vec{F}$ over S is equal to the volume integral of the divergence of $\vec{F}$ taken throughout V.

i.e., $\iint_S \vec{F}\cdot\vec{n}ds = \iiint_V \nabla\cdot\vec{F}dv$.

—

Gauss Divergence theorem in cartesian coordinates

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$\vec{F} \cdot \vec{n} = F_1(\vec{i} \cdot \vec{n}) + F_2(\vec{j} \cdot \vec{n}) + F_3(\vec{k} \cdot \vec{n})$

$\vec{F} \cdot \vec{n}ds = F_1(\vec{i} \cdot \vec{n})ds + F_2(\vec{j} \cdot \vec{n})ds + F_3(\vec{k} \cdot \vec{n})ds = F_1dydz + F_2dxdz + F_3dxdy$

$\nabla \cdot \vec{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

$\therefore$ Gauss divergence theorem becomes

$$\iint_S (F_1dydz + F_2dzdx + F_3dxdy) = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dxdydz.$$

Worked Examples

Example 2.90. Evaluate $\iint_S (xdydz + ydzdx + zdxdy)$ where S is the surface of the sphere of radius a.

Solution. We know that by the cartesian form of Gauss divergence theorem $\iint\limits_S (F_1dydz + F_2dzdx + F_3dxdy) = \iiint\limits_V \left(\dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}\right)dV.$

Now $F_1 = x, F_2 = y, F_3 = z$

$\dfrac{\partial F_1}{\partial x} = 1, \dfrac{\partial F_2}{\partial y} = 1 \dfrac{\partial F_3}{\partial z} = 1$

$$\begin{aligned} \therefore \iint\limits_S (xdydz + ydzdx + zdxdy) &= \iiint\limits_V (1 + 1 + 1)dV = \iiint\limits_V 3dV = 3V \\ &= 3 \times \text{ Volume of the sphere.} \\ &= 3 \times \frac{4}{3}\pi a^3 = 4\pi a^3. \end{aligned}$$

Example 2.91. If S is any closed surface enclosing volume V and $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$, prove that $\iint_S \vec{F} \cdot \vec{n}ds = (a + b + c)V.$ [Jun 2004]

Solution. By Gauss Divergence theorem

$$\iint_S \vec{F} \cdot \vec{n}ds = \iiint_V \nabla \cdot \vec{F}dv$$

$$\begin{aligned} \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz) \\ &= a + b + c. \end{aligned}$$

—

$$\therefore \iint_S \vec{F}\cdot\vec{n}ds = \iiint_V (a+b+c)dv$$
$$= (a+b+c)V.$$

Example 2.92. Prove that $\iint\limits_S \vec{F}.\vec{n}ds = 0$ for any closed surface if $\vec{F} = curl\vec{F}$.

or

Show that $\iint\limits_S curl\vec{F}.\vec{n}ds = 0$ where S is any closed surface.

Solution. By Gauss divergence theorem we have

$$\iint\limits_S \vec{F}.\vec{n}ds = \iiint\limits_V div\vec{F}dV = \iiint\limits_V div(curl\vec{F})dV$$
$$= \iiint\limits_V 0dV[\because div(curl\vec{F}) = 0]$$
$$= 0.$$

Example 2.93. Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

[May 2015, Dec 2013, Dec 2012, May 2011, Dec 2010]

Solution. By Gauss divergence theorem, $\iint_S \vec{F}\cdot\vec{n}ds = \iiint_V \nabla\cdot\vec{F}dv$

From Example 2.87 we have $\iint_S \vec{F}\cdot\vec{n}ds = \frac{3}{2}$.

$$\text{Now, } \iiint_V \nabla\cdot\vec{F}dv = \int_0^1\int_0^1\int_0^1 \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dxdydz$$
$$= \int_0^1\int_0^1\int_0^1 (4z - 2y + y)dxdydz$$
$$= \int_0^1\int_0^1\int_0^1 (4z - y)dxdydz = \int_0^1\int_0^1 \left[4\frac{z^2}{2} - yz\right]_0^1 dxdy$$
$$= \int_0^1\int_0^1 (2-y)dxdy = \int_0^1 \left[2(y)_0^1 - \left(\frac{y^2}{2}\right)_0^1\right]dx$$
$$= \int_0^1 \left(2 - \frac{1}{2}\right)dx = \frac{3}{2}(x)_0^1 = \frac{3}{2}.$$

$\therefore$ LHS = RHS.

Hence, Gauss Divergence theorem is verified.

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Example 2.94. Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular paralleopiped bounded by $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.

[May 1994]

Solution. By Gauss divergence theorem, $\iint_S \vec{F} \cdot \vec{n} ds = \iiint_V \nabla \cdot \vec{F} dv$.

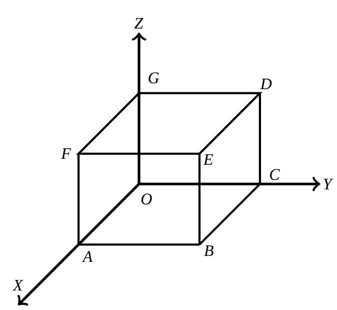
$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$
$$= 2x + 2y + 2z = 2(x + y + z).$$

$$\text{RHS} = \iiint_V \nabla \cdot \vec{F} dV = \int_0^a \int_0^b \int_0^c 2(x + y + z) dz dy dx$$
$$= 2 \int_0^a \int_0^b \left(xz + yz + \frac{z^2}{2}\right)_0^c dy dx$$
$$= 2 \int_0^a (cx + cy + \frac{c^2}{2}) dy dx$$
$$= 2 \int_0^a [cx(y)_0^b + c(\frac{y^2}{2})_0^b + \frac{c^2}{2}(y)_0^b] dx$$
$$= 2 \int_0^a (bcx + \frac{b^2 c}{2} + \frac{bc^2}{2}) dx$$
$$= 2[bc(\frac{x^2}{2})_0^a + \frac{b^2 c}{2}(x)_0^a + \frac{bc^2}{2}(x)_0^a]$$
$$= 2[\frac{a^2 bc}{2} + \frac{b^2 ac}{2} + \frac{c^2 ab}{2}] = abc(a + b + c). \quad (1)$$

Now we shall evaluate $\iint_S \vec{F} \cdot \vec{n} ds$.

The rectangular parallelopiped has six faces,

$S_1 \to ABEF$, $S_2 \to OCDG$, $S_3 \to BCDE$,
$S_4 \to OAFG$, $S_5 \to OABC$, $S_6 \to DEFG$.



$$\therefore \iint_S \vec{F} \cdot \vec{n} ds = \iint_{S_1} \vec{F} \cdot \vec{n} ds + \iint_{S_2} \vec{F} \cdot \vec{n} ds + \iint_{S_3} \vec{F} \cdot \vec{n} ds + \iint_{S_4} \vec{F} \cdot \vec{n} ds$$
$$+ \iint_{S_5} \vec{F} \cdot \vec{n} ds + \iint_{S_6} \vec{F} \cdot \vec{n} ds.$$

The following table gives the necessary quantities for the computation of the surface integral over each face.

Face	Equation	Outward normal	$\vec{F}\cdot\vec{n}$	ds
S_1	$x=a$	$\vec{i}$	x^2-yz $\boxed{a^2-yz}$	$dydz$
S_2	$x=0$	$-\vec{i}$	$-(x^2-yz)$ $\boxed{yz}$	$dydz$
S_3	$y=b$	$\vec{j}$	(y^2-zx) $\boxed{b^2-zx}$	$dxdz$
S_4	$y=0$	$-\vec{j}$	$-(y^2-zx)$ $\boxed{zx}$	$dxdz$
S_5	$z=0$	$-\vec{k}$	$-(z^2-xy)$ $\boxed{xy}$	$dxdy$
S_6	$z=c$	$\vec{k}$	(z^2-xy) $\boxed{c^2-xy}$	$dxdy$

$$\iint_{S_1}\vec{F}\cdot\vec{n}ds=\int_0^b\int_0^c\left(a^2-yz\right)dzdy=\int_0^b\left]a^2(z)_0^c-y\left(\frac{z^2}{2}\right)_0^c\right]dy$$

$$=\int_0^b\left(a^2c-\frac{yc^2}{2}\right)dy=a^2c(y)_0^b-\frac{c^2}{2}\left(\frac{y^2}{2}\right)_0^b=a^2bc-\frac{b^2c^2}{4}.$$

$$\iint_{S_2}\vec{F}\cdot\vec{n}ds=\int_0^c\int_0^b yzdydz=\int_0^c z\left(\frac{y^2}{2}\right)_0^b dz=\int_0^c z\frac{b^2}{2}dz=\frac{b^2}{2}\left(\frac{z^2}{2}\right)_0^c=\frac{b^2c^2}{4}.$$

$$\iint_{S_3}\vec{F}\cdot\vec{n}ds=\int_0^a\int_0^c\left(b^2-zx\right)dzdx=\int_0^a\left]b^2(z)_0^c-x\left(\frac{z^2}{2}\right)_0^c\right]dx$$

$$=\int_0^a\left\{\left(b^2c-\frac{c^2}{2}x\right)\right\}dx=b^2c(x)_0^a-\frac{c^2}{2}\left(\frac{x^2}{2}\right)_0^a=ab^2c-\frac{a^2c^2}{4}.$$

$$\iint_{S_4}\vec{F}\cdot\vec{n}ds=\int_0^a\int_0^c zxdzdx=\int_0^a x\left(\frac{z^2}{2}\right)_0^c dx=\frac{c^2}{2}\left(\frac{x^2}{2}\right)_0^a=\frac{a^2c^2}{4}.$$

$$\iint_{S_5}\vec{F}\cdot\vec{n}ds=\int_0^a\int_0^b xydydx=\int_0^a x\left(\frac{y^2}{2}\right)_0^b dx=\frac{a^2b^2}{4}.$$

—

$$\iint_{S_6} \vec{F}\cdot\vec{n}ds = \int_0^a\int_0^b \left(c^2 - xy\right)dydx = \int_0^a \left] c^2\,(y)_0^b - x\left(\frac{y^2}{2}\right)_0^b\right] dx$$

$$= \int_0^a \left(bc^2 - \frac{b^2x}{2}\right)dx = bc^2\,(x)_0^a - \frac{b^2}{2}\left(\frac{x^2}{2}\right)_0^a = abc^2 - \frac{a^2b^2}{4}.$$

$$\iint_S \{\vec{F}\cdot\vec{n}\}ds = a^2bc - \frac{b^2c^2}{4} + \frac{b^2c^2}{4} + ab^2c - \frac{a^2c^2}{4} + \frac{a^2c^2}{4} + \frac{a^2b^2}{4} + abc^2 - \frac{a^2b^2}{4}$$

$$= abc\,(a+b+c)\,. \qquad (2)$$

From (1) and (2) $\iint_S \{\vec{F}\cdot\vec{n}\}ds = \iiint\limits_V \nabla.\vec{F}dV.$

Hence, Gauss divergence theorem is verified.

Example 2.95. Verify divergence theorem for $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over the cube formed by the planes $x = \pm 1, y = \pm 1, z = \pm 1.$ [Jun 2013, May 2006]

Solution. By Gauss divergence theorem we have $\iint_S \vec{F}\cdot\vec{n}ds = \iiint_V div\vec{F}dv.$

$$\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$$

$$div\vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(yz) = 2x + y$$

$$\iiint\limits_V div\vec{F}dV = \int\limits_{-1}^{1}\int\limits_{-1}^{1}\int\limits_{-1}^{1}(2x+y)dzdydx$$

$$= \int\limits_{-1}^{1}\int\limits_{-1}^{1}(2x+y)(z)_{-1}^{1}dydx = 2\int\limits_{-1}^{1}\int\limits_{-1}^{1}(2x+y)dydx$$

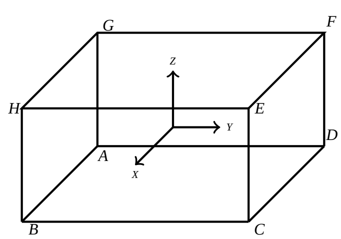
$$= 2\int\limits_{-1}^{1}\left(2x(y)_{-1}^{1} + \left(\frac{y^2}{2}\right)_{-1}^{1}\right)dx = 2\int\limits_{-1}^{1}4xdx = 8\left(\frac{x^2}{2}\right)_{-1}^{1} = 8\times 0 = 0. \qquad (1)$$

We shall evaluate $\iint_S \vec{F}\cdot\vec{n}ds.$

The given cube has 6 faces

$S_1 = BCEH,\ S_2 = ADFG,\ S_3 = CDEF,$

$S_4 = ABGH,\ S_5 = EFGH,\ S_6 = ABCD.$



Now $\iint_S \vec{F}\cdot\vec{n}ds = \sum_{i=1}^{6}\iint_{S_i}\vec{F}\cdot\vec{n}ds$

The following table gives the necessary quantities for the computation of the surface integral over each face.

Face	Equation	Outward normal	$\vec{F}\cdot\vec{n}$	ds
S_1	$x=1$	$\vec{i}$	$x^2=1$	$dydz$
S_2	$x=-1$	$-\vec{i}$	$-x^2=-1$	$dydz$
S_3	$y=1$	$\vec{j}$	z	$dxdz$
S_4	$y=-1$	$-\vec{j}$	$-z$	$dxdz$
S_5	$z=1$	$\vec{k}$	$yz=y$	$dxdy$
S_6	$z=-1$	$-\vec{k}$	$-yz=y$	$dxdy$

$$\iint_{S_1}\vec{F}\cdot\vec{n}ds = \int_{-1}^{1}\int_{-1}^{1}dydz = \int_{-1}^{1}(y)_{-1}^{1}dz = 2\int_{-1}^{1}dz = 2(z)_{-1}^{1} = 4.$$

$$\iint_{S_2}\vec{F}\cdot\vec{n}ds = -4$$

$$\iint_{S_3}\vec{F}\cdot\vec{n}ds = \int_{-1}^{1}\int_{-1}^{1}zdxdz = 0$$

$$\iint_{S_4}\vec{F}\cdot\vec{n}ds = \int_{-1}^{1}\int_{-1}^{1}(-z)dxdz = 0$$

$$\iint_{S_5}\vec{F}\cdot\vec{n}ds = \int_{-1}^{1}\int_{-1}^{1}ydxdy = 0$$

$$\iint_{S_6}\vec{F}\cdot\vec{n}ds = \int_{-1}^{1}\int_{-1}^{1}(-y)dxdy = 0.$$

$$\therefore \iint_{S}\vec{F}\cdot\vec{n}ds = 4-4+0+0=0. \qquad (2)$$

—

From (1) and (2) we get

$$\iint_S \vec{F}\cdot\vec{n}ds = \iiint_V div\vec{F}dV.$$

Hence, Gauss Divergence theorem is verified.

Example 2.96. Verify Gauss divergence theorem for the vector function $\vec{f} = (x^3 - yz)\vec{i} - 2x^2y\vec{j} + 2\vec{k}$ over the cube bounded by $x = 0, y = 0, z = 0$ and $x = a, y = a, z = a$.

[Dec 2011, Jun 2010]

Solution. By Gauss divergence theorem

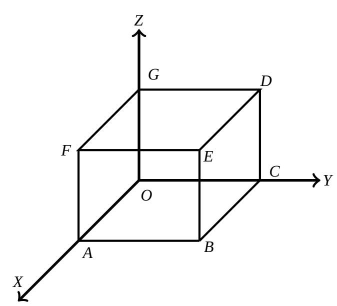
$$\iint_S \vec{f}\cdot\vec{n}ds = \iiint_V \nabla\cdot\vec{f}dv.$$

$$\nabla\cdot\vec{f} = \frac{\partial}{\partial x}(x^3 - yz) + \frac{\partial}{\partial y}(-2x^2y) + \frac{\partial}{\partial z}(2)$$
$$= 3x^2 - 2x^2 + 0 = x^2.$$

$$\text{Now, } \iiint_V \nabla\cdot\vec{f}dv = \int_{x=0}^{a}\int_{y=0}^{a}\int_{z=0}^{a} x^2dzdydx = \int_{x=0}^{a}\int_{y=0}^{a} x^2[z]_0^a dydx$$
$$= \int_{x=0}^{a}\int_{y=0}^{a} ax^2dydx = a\int_{x=0}^{a} x^2[y]_0^a dx$$
$$= a\int_0^a ax^2dx = a^2\left[\frac{x^3}{3}\right]_0^a = \frac{a^5}{3}. \qquad (1)$$

Now we shall evaluate $\iint_s \vec{f}\cdot\vec{n}ds.$

The cube has six faces namely S_1 : $ABEF$, S_2 : $OCDG$, S_3 : $BCDE$, S_4 : $OAFG$, S_5 : $OABC$, S_6 : $DEFG$.



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$$\therefore \iint_S \vec{f}\cdot\vec{n}ds = \iint_{S_1} \vec{f}\cdot\vec{n}ds + \iint_{S_2} \vec{f}\cdot\vec{n}ds + \iint_{S_3} \vec{f}\cdot\vec{n}ds + \iint_{S_4} \vec{f}\cdot\vec{n}ds + \iint_{S_5} \vec{f}\cdot\vec{n}ds + \iint_{S_6} \vec{f}\cdot\vec{n}ds.$$

The following table gives the necessary quantities for the computation of the surface integral over each face.

Face	Equation	Outward normal($\vec{n}$)	$\vec{F}\cdot\vec{n}$	ds
S_1	$x = a$	$\vec{i}$	$x^3 - yz = a^3 - yz$	$dydz$
S_2	$x = 0$	$-\vec{i}$	$-(x^3 - y^2) = yz$	$dydz$
S_3	$y = a$	$\vec{j}$	$-2x^2y = -2ax^2$	$dxdz$
S_4	$y = 0$	$-\vec{j}$	$2x^2y = 0$	$dxdz$
S_5	$z = 0$	$-\vec{k}$	-2	$dxdy$
S_6	$z = a$	$\vec{k}$	2	$dxdy$

$$\iint_{S_1} \vec{f}\cdot\vec{n}ds = \int_{y=0}^{a}\int_{z=0}^{a}(a^3 - yz)dzdy = \int_{y=0}^{a}\left[a^3 z - y\frac{z^2}{2}\right]_0^a dy$$

$$= \int_0^a \left(a^4 - \frac{a^2}{2}y\right)dy = \left[a^4 y - \frac{a^2}{2}\cdot\frac{y^2}{2}\right]_0^a = a^5 - \frac{a^4}{4}.$$

$$\iint_{S_2} \vec{f}\cdot\vec{n}ds = \int_{y=0}^{a}\int_{z=0}^{a} yzdzdy = \int_{y=0}^{a} y\left[\frac{z^2}{2}\right]_0^a dy$$

$$= \frac{1}{2}\int_0^a y\cdot a^2 dy = \frac{a^2}{2}\left[\frac{y^2}{2}\right]_0^a = \frac{a^2}{2}\cdot\frac{a^2}{2} = \frac{a^4}{4}.$$

$$\iint_{S_3} \vec{f}\cdot\vec{n}ds = \int_{x=0}^{a}\int_{z=0}^{a} -2ax^2dzdx = -2a\int_{x=0}^{a} x^2\cdot[z]_0^a dx$$

$$= -2a\int_0^a x^2\cdot adx = -2a^2\left[\frac{x^3}{3}\right]_0^a = -\frac{2a^5}{3}.$$

$$\iint_{S_4} \vec{f}\cdot\vec{n}ds = 0.$$

—

$$\iint_{S_5} \vec{f}\cdot\vec{n}ds = \int_{x=0}^{a}\int_{y=0}^{a} -2dxdy = -2\int_{x=0}^{a} dx \int_{y=0}^{a} dy = -2\cdot[x]_0^a[y]_0^a = -2\cdot a\cdot a = -2a^2.$$

$$\iint_{S_6} \vec{f}\cdot\vec{n}ds = \int_{x=0}^{a}\int_{y=0}^{a} 2dxdy = 2a^2.$$

$$\therefore \iint_{S} \vec{f}\cdot\vec{n}ds = a^5 - \frac{a^4}{4} + \frac{a^4}{4} - \frac{2a^5}{3} + 0 - 2a^2 + 2a^2 = a^5 - \frac{2a^5}{3} = \frac{a^5}{3}. \quad (2)$$

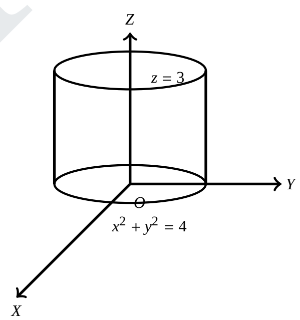
From (1) and (2) we have

$$\iint_{s} \vec{f}\cdot\vec{n}ds = \iiint_{V} \nabla\cdot\vec{f}dv$$

$\therefore$ Gauss divergence theorem is verified.

Example 2.97. Evaluate $\int_S \vec{F}\cdot d\vec{s}$, where $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ and S is the surface bounding the region $x^2+y^2=4$, $z=0$ and $z=3$.

Solution.



By Gauss divergence theorem,

$$\int_S \{\vec{F}\cdot d\vec{s}\} = \int_V \nabla\cdot\vec{F}dV$$

Now, $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$

$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z.$

$$\int_V \nabla\cdot\vec{F}\ dV = \iiint_V (4-4y+2z)\ dxdydz$$

$$= \iiint_V (4-4y+2z)\ dzdydx = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4-4y+2z)\ dzdydx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4z-4yz+z^2)_{z=0}^{3}\ dydx$$

—

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9)\ dydx = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y)\ dydx$$

$$= \int_{x=-2}^{2} \left(21y - 6y^2\right)_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\ dx$$

$$= \int_{x=-2}^{2} \left(21\left(\sqrt{4-x^2} + \sqrt{4-x^2}\right) - 6\left((4-x^2) - (4-x^2)\right)\right)\ dx$$

$$= \int_{x=-2}^{2} \left(42\sqrt{4-x^2}\right)\ dx = 42 \times 2 \int_{x=0}^{2} \left(\sqrt{4-x^2}\right)\ dx$$

$$= 84\left(\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right)_{x=0}^{2} = 84\left(2\sin^{-1}1 - 0\right) = 164 \times \frac{\pi}{2} = 84\pi.$$

Example 2.98. Using divergence theorem, evaluate $\int_S \left(yz\vec{i} + zx\vec{j} + xy\vec{k}\right)\ d\cdot\vec{S}$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant.

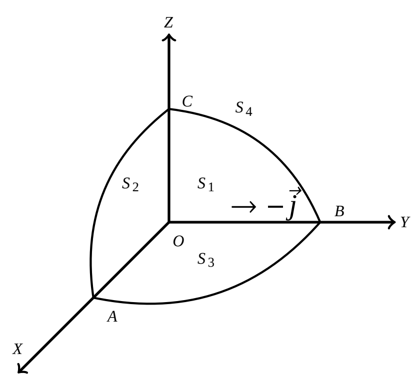
Solution. Let V be the volume bounded by the surface of the sphere. This region is bounded by 4 surfaces namely

S_1 : The circular quadrant OBC in the yz plane.

S_2 : The circular quadrant OAC in the xz plane.

S_3 : The circular quadrant OAB in the xy plane.

S : The surface ABC of the sphere in the first octant.



Here $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$.

—

$\therefore \quad \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0.$

By Gauss divergence theorem,

$\int_V \nabla \cdot \vec{F}\, dV = \int_S \vec{F} \cdot d\vec{S}$

$\int\limits_V \nabla \cdot \vec{F}\; dV = \int\limits_{S_1} \vec{F} \cdot d\vec{S} + \int\limits_{S_2} \vec{F} \cdot d\vec{S} + \int\limits_{S_3} \vec{F} \cdot d\vec{S} + \int\limits_{S} \vec{F} \cdot d\vec{S}$

$$0 = \int\limits_{S_1} \vec{F} \cdot d\vec{S} + \int\limits_{S_2} \vec{F} \cdot d\vec{S} + \int\limits_{S_3} \vec{F} \cdot d\vec{S} + \int\limits_{S} \vec{F} \cdot d\vec{S} \qquad (1)$$

On S_1, $x = 0$,

$$\therefore \int\limits_{S_1} \vec{F} \cdot d\vec{S} = \iint\limits_{S_1} (yz\vec{i})\ (dydz(-j))$$

$$= -\iint\limits_{S_1} yz\ dydz$$

$$= -\int\limits_{y=0}^{a} \int\limits_{z=0}^{\sqrt{a^2-y^2}} yz\ dzdy = -\int\limits_{y=0}^{a} \left(y\frac{z^2}{2}\right)_{z=0}^{\sqrt{a^2-y^2}} dy$$

$$= -\frac{1}{2}\int\limits_{y=0}^{a} \left(y(a^2-y^2)\right)\ dy = -\frac{1}{2}\int\limits_{y=0}^{a} \left(a^2y - y^3\right)\ dy$$

$$= -\frac{1}{2}\left(\frac{a^2y^2}{2} - \frac{y^4}{4}\right)_{y=0}^{a} = -\frac{1}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = -\frac{1}{2} \times \frac{a^4}{4} = -\frac{a^4}{8}.$$

In the same way we get

$\int\limits_{S_2} \vec{F} \cdot d\vec{S} = -\frac{a^4}{8}$ and $\int\limits_{S_3} \vec{F} \cdot d\vec{S} = -\frac{a^4}{8}.$

$$\therefore (1) \quad \Rightarrow 0 = -\frac{a^4}{8} - \frac{a^4}{8} - \frac{a^4}{8} + \int\limits_{S} \vec{F} \cdot d\vec{S}$$

$$0 = -\frac{3a^4}{8} + \int\limits_{S} \vec{F} \cdot d\vec{S}$$

$$\therefore \int\limits_{S} \vec{F} \cdot d\vec{S} = \frac{3a^4}{8}.$$

—

2.8 Stoke's theorem

Stoke's theorem is the relation between line and surface integrals.

[Dec 2014, Dec 2010]

Statement. If S is an open surface bounded by a simple closed curve C and if $\vec{F}$ is continuous having continuous partial derivatives in S and on C, then $\int\limits_C \vec{F}\cdot d\vec{r} = \iint\limits_S Curl\vec{F}\cdot\vec{n}ds$ where C is traversed in the positive direction.

Another Form. The line integral of the tangential component of a vector function $\vec{F}$ around a simple closed curve C is equal to the surface integral of the normal component of $curl\vec{F}$ over any surface S having C as its boundary.

Note. Green's theorem in the plane is a particular case of Stoke's theorem. If S is the region R in the xy plane bounded by the simple closed curve C then $\vec{n} = \vec{k}$ is the outward unit normal.

$\therefore$ Stoke's theorem in the plane is $\int_C \vec{F}\cdot d\vec{r} = \iint_R Curl\vec{F}\cdot\vec{k}dR$.

Cartesian form of Stoke's theorem

If $\vec{F} = F_1\vec{i}+F_2\vec{j}+F_3\vec{k}$, then $curl\vec{F} = \vec{i}\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big) - \vec{j}\Big(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\Big) + \vec{k}\Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)$.

$\vec{F}\cdot d\vec{r} = F_1dx + F_2dy + F_3dz.$

$\therefore$ The cartesian form of Stoke's theorem is

$$\int_C (F_1dx + F_2dy + F_3dz) = \iint_S \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)dydz + \iint_S \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)dzdx + \iint_S \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)dxdy.$$

Note: If $\vec{F} = P\vec{i}+Q\vec{j}, \vec{r} = x\vec{i}+y\vec{j}, d\vec{r} = dx\vec{i}+dy\vec{j}$.

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\vec{k} \quad \Rightarrow curl\vec{F}\cdot\vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

$\therefore$ Stoke's theorem in the plane is $\displaystyle\int_C (Pdx + Qdy) = \iint_R \Big\{\Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\Big\}dxdy$ which is the Green's theorem.

Worked Examples

Example 2.99. Evaluate $\int_C (e^x dx + 2ydy - dz)$ where C is the curve $x^2 + y^2 = 4, z = 2$ using Stokes theorem. [Dec 2005]

Solution. We have $\vec{F}.d\vec{r} = e^x dx + 2ydy - dz$.

$$\therefore \vec{F} = e^x\vec{i} + 2y\vec{j} - \vec{k}.$$

By Stoke's theorem, $\iint\limits_S curl\vec{F}.\vec{n}ds = \int\limits_C \vec{F}.d\vec{r}.$

$$\text{Now } curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$
$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$
$$= \vec{0}$$

$$\therefore \int\limits_C \vec{F}.d\vec{r} = 0.$$

Example 2.100. Prove that $\int_C \{\vec{r} \cdot d\vec{r}\} = 0$ where C is the simple closed curve.

Solution. Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Hence $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

Now $\vec{r} \cdot d\vec{r} = xdx + ydy + zdz$

$$curl\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{0}$$

By Stoke's theorem $\int_C \{\vec{r} \cdot d\vec{r}\} = \iint_R \{curl\vec{r} \cdot \vec{n}\}ds = 0.$

Example 2.101. Using Stoke's theorem prove that $curl(grad\phi) = \vec{0}$.

[Jun 2014, Dec 2008]

Solution. By Stoke's theorem $\int_C \{\vec{F} \cdot d\vec{r}\} = \iint_S \{curl\vec{F} \cdot \vec{n}\}ds$, where S is an open surface bounded by a simple closed curve C.

Let $\vec{F} = \nabla\phi$.

$\therefore$ By Stoke's theorem

$$\iint_S (\nabla \times \nabla\phi).\vec{n}ds = \int_C \nabla\phi.d\vec{r} = \int_C \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$
$$= \int_C d\phi = [\phi]_C$$
$$= 0[\because C \text{ is simple closed, both ends are same}]$$
$$\iint_S (\nabla \times \nabla\phi).\vec{n}ds = 0.$$

This is true for any surface S, we obtain $\nabla \times \nabla\phi = \vec{0}$.

$\implies curl(grad\phi) = \vec{0}$.

Example 2.102. Evaluate $\int_C \{(xydx + xy^2dy)\}$ by stoke's theorem where C is the square in the xy plane with vertices $(1, 0), (-1, 0), (0, 1), (0, -1)$. [Jun 2002]

Solution. By Stoke's theorem $\int_C \{\vec{F} \cdot d\vec{r}\} = \iint_S \{curl\vec{F} \cdot \vec{n}\}ds$.

Given, $\int_C \{(xydx + xy^2dy)\} = \int_C \{\vec{F} \cdot d\vec{r}\}$.

$\therefore \vec{F} = xy\vec{i} + xy^2\vec{j}$

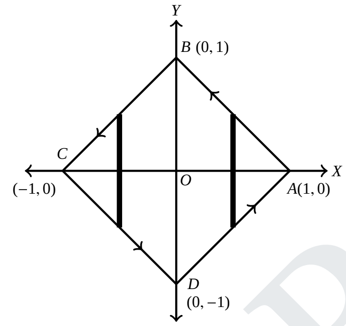
$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(y^2 - x)$$

$$curl\vec{F} = (y^2 - x)\vec{k}.$$

Since C is a square in the xy plane, we have $\vec{n} = \vec{k}$ and $ds = dxdy$

$curl\vec{F} \cdot \vec{n} = y^2 - x$.

$\iint_S \{curl\vec{F} \cdot \vec{n}\}ds = \iint_R \{(y^2 - x)\}dxdy$, where R is the region inside the square.



Equation to AB is $\frac{x}{1}+\frac{y}{1}=1 \Rightarrow x+y=1$.

Equation to BC is $\frac{x}{-1}+\frac{y}{1}=1$

$\Rightarrow -x+y=1 \Rightarrow x-y=-1$.

Equation to CD is $\frac{x}{-1}+\frac{y}{-1}=1$

$\Rightarrow -x-y=1 \Rightarrow x+y=-1$.

Equation to DA is $\frac{x}{1}+\frac{y}{-1}=1 \Rightarrow x-y=1$.

$$\iint_R curl\vec{F}\cdot\vec{n}ds = \iint_{ABD}+\iint_{BCD}$$

$$= \int_{x=0}^{1}\int_{y=x-1}^{y=1-x}(y^2-x)dydx + \int_{-1}^{0}\int_{y=-1-x}^{x+1}(y^2-x)dxdy$$

$$= \int_0^1\Big[\Big(\frac{y^3}{3}\Big)_{x-1}^{(1-x)} - x\big(y\big)_{x-1}^{1-x}\Big]dx + \int_{-1}^{0}\Big[\Big(\frac{y^3}{3}\Big)_{-1-x}^{x+1} - x\big(y\big)_{-1-x}^{x+1}\Big]dx$$

$$= \frac{1}{3}\int_0^1[(1-x)^3-(x-1)^3]dx - \int_0^1 x(1-x-x+1)dx$$

$$+\frac{1}{3}\int_{-1}^{0}[(x+1)^3+(1+x)^3]dx - \int_{-1}^{0}x(1+x+x+1)dx$$

$$= \frac{1}{3}\left[\frac{(1-x)^4}{-4}-\frac{(x-1)^4}{4}\right]_0^1 - \left[x^2-\frac{2x^3}{3}\right]_0^1 + \frac{2}{3}\left[\frac{(x+1)^4}{4}\right]_{-1}^{0} - \left[x^2+\frac{2x^3}{3}\right]_{-1}^{0}$$

$$= \frac{2}{12}-\frac{1}{3}+\frac{2}{12}+\frac{1}{3}=\frac{4}{12}=\frac{1}{3}.$$

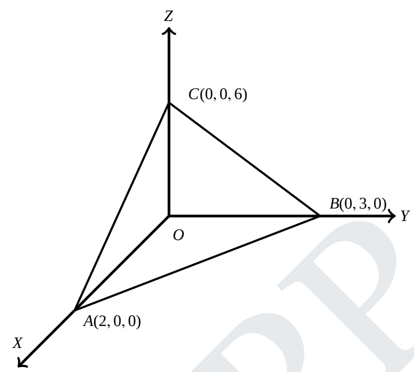
Example 2.103. Evaluate $\int_C (x+y)dx+(2x-z)dy+(y+z)dz$ where C is the boundary of the triangle with vertices $(2,0,0),(0,3,0),(0,0,6)$ using Stokes theorem.

Solution. By Stoke's theorem $\int_C\{\vec{F}\cdot d\vec{r}\}=\iint_S\{curl\vec{F}\cdot\vec{n}\}ds$.

Now, $\vec{F}=(x+y)\vec{i}+(2x-z)\vec{j}+(y+z)\vec{k}$

$$curl\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x+y & 2x-z & y+z\end{vmatrix}=\vec{i}(1+1)-\vec{j}(0-0)+\vec{k}(2-1)=2\vec{i}+\vec{k}.$$

—



Now, S is the surface of the triangle ABC bounded by the curve C.

Equation of the plane ABC is $\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1.$

$\therefore \phi=\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$

$$\nabla\phi=\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}=\frac{1}{2}\vec{i}+\frac{1}{3}\vec{j}+\frac{1}{6}\vec{k}$$

$$|\nabla\phi|=\sqrt{\frac{1}{4}+\frac{1}{9}+\frac{1}{36}}=\sqrt{\frac{9+4+1}{36}}=\frac{\sqrt{14}}{6}$$

$$\vec{n}=\frac{\nabla\phi}{|\nabla\phi|}=\frac{\frac{1}{2}\vec{i}+\frac{1}{3}\vec{j}+\frac{1}{6}\vec{k}}{\frac{\sqrt{14}}{6}}=\frac{1}{6}\frac{(3\vec{i}+2\vec{j}+\vec{k})}{\frac{\sqrt{14}}{6}}=\frac{3\vec{i}+2\vec{j}+\vec{k}}{\sqrt{14}}$$

$\vec{F}=(x+y)\vec{i}+(2x-z)\vec{j}+(y+z)\vec{k}.$

$$curl\vec{F}.\vec{n}=\frac{6}{\sqrt{14}}+\frac{1}{\sqrt{14}}=\frac{7}{\sqrt{14}}.$$

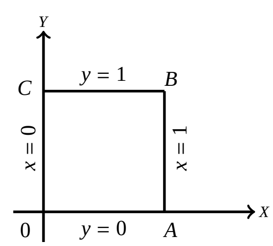
Now, $\iint_S\{curl\vec{F}\cdot\vec{n}\}ds=\iint_S\frac{7}{\sqrt{14}}ds=\frac{7}{\sqrt{14}}\iint_R\{\frac{dxdy}{|\vec{n}\cdot\vec{k}|}\}$

Where R is the orthogonal projection of S on the xy plane

$$\iint_S\{curl\vec{F}\cdot\vec{n}\}ds=\frac{7}{\sqrt{14}}\iint_R\frac{dxdy}{1/\sqrt{14}}=7\iint_R dxdy=7\times\text{Area of }\Delta\text{OAB}=7\times\frac{1}{2}\times2\times3=21.$$

Example 2.104. Evaluate $\iint_S\{curl\vec{F}\cdot\vec{n}ds\}$ where $\vec{F}=(y-z)\vec{i}+yz\vec{j}-xz\vec{k}$ and S is the open surface bounded by the planes $x=0,x=1,y=0,y=1,z=0,z=1$ above the xy plane.

Solution.



By Stoke's theorem $\iint_S\{curl\vec{F}\cdot\vec{n}\}ds=\int_C\{\vec{F}\cdot d\vec{r}\}$, where S is the open surface bounded by C and C is the square in the xy plane, $z=0$ bounded by $x=0,y=0,x=1,y=1$.

—

Now, $\iint_S curl\vec{F}\cdot\vec{n}ds = \int\limits_{OA}\vec{F}\cdot d\vec{r} + \int\limits_{AB}\vec{F}\cdot d\vec{r} + \int\limits_{BC}\vec{F}\cdot d\vec{r} + \int\limits_{CO}\vec{F}\cdot d\vec{r}.$

$\vec{F}\cdot d\vec{r} = (y-z)dx + yzdy - xzdz.$

Since $z = 0, dz = 0.$

$\therefore \vec{F}\cdot d\vec{r} = ydx.$

On OA: $y = 0, dy = 0, x$ varies from 0 to 1 and $\vec{F}\cdot d\vec{r} = 0.$

$\therefore \int_{OA}\vec{F}\cdot d\vec{r} = 0.$

On AB: $x = 1, dx = 0, y$ varies from 0 to 1 and $\vec{F}\cdot d\vec{r} = 0.$

$\therefore \int_{AB}\vec{F}\cdot d\vec{r} = 0.$

On BC: $y = 1, dy = 0, x$ varies from 1 to 0 and $\vec{F}\cdot d\vec{r} = dx.$

$\therefore \int_{BC}\vec{F}\cdot d\vec{r} = \int_1^0 dx = [x]_1^0 = 0 - 1 = -1.$

On CO: $x = 0, dx = 0, y$ varies from 1 to 0 and $\vec{F}\cdot d\vec{r} = 0.$

$\therefore \int_{CO}\vec{F}\cdot d\vec{r} = 0.$

$$\text{Now, } \int_c \vec{F}\cdot d\vec{r} = -1$$

$$\therefore \iint_s curl\vec{F}.\vec{n}ds = -1.$$

Example 2.105. Verify Stoke's theorem for $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}$ where S is the surface of the cube $x = 0, x = 2, y = 0, y = 2, z = 0, z = 2$ above the xy plane.

[Jun 2005]

Solution. By Stoke's theorem,

$$\int_C\{\vec{F}\cdot d\vec{r}\} = \iint_S\{Curl\vec{F}\cdot\vec{n}\}ds.$$

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z + 2 & yz + 4 & -xz \end{vmatrix}$$

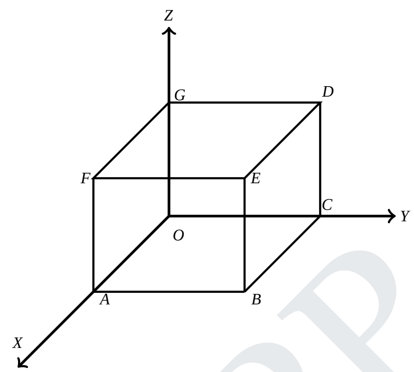
$$= \vec{i}(0 - y) - \vec{j}(-z + 1) + \vec{k}(0 - 1)$$

$$= -y\vec{i} + (z - 1)\vec{j} - \vec{k}.$$

—

We shall evaluate $\iint\limits_S curl\vec{F}.\vec{n}ds$ where S is the open surface consisting of 5 faces except the face $OABC$.

The following table gives the necessary quantities for the computation of the surface integral.



Face	Equation	Normal$\vec{n}$	curl$\vec{F}\cdot\vec{n}$	ds
$S_1 = ABEF$	$x = 2$	$\vec{i}$	$-y$	$dydz$
$S_2 = OCDG$	$x = 0$	$-\vec{i}$	y	$dydz$
$S_3 = BCDE$	$y = 2$	$\vec{j}$	$z-1$	$dxdz$
$S_4 = OAFG$	$y = 0$	$-\vec{j}$	$1-z$	$dxdz$
$S_5 = DEFG$	$z = 2$	$\vec{k}$	-1	$dxdy$

$$\iint_{S_1} curl\vec{F}\cdot\vec{n}ds = \int_0^2\int_0^2 -ydydz = -\int_0^2 \left(\frac{y^2}{2}\right)_0^2 dz = -2(z)_0^2 = -4.$$

$$\iint_{S_2} curl\vec{F}\cdot\vec{n}ds = \int_0^2\int_0^2 ydydz = \int_0^2 \left(\frac{y^2}{2}\right)_0^2 dz = 2(z)_0^2 = 4.$$

$$\iint_{S_3} curl\vec{F}\cdot\vec{n}ds = \int_0^2\int_0^2 (z-1)dzdx = \int_0^2 \left(\frac{z^2}{2}-z\right)_0^2 dx = \int_0^2 0dx = 0.$$

$$\iint_{S_4} curl\vec{F}\cdot\vec{n}ds = \int_0^2\int_0^2 (1-z)dxdz = 0.$$

$$\iint_{S_5} curl\vec{F}\cdot\vec{n}ds = \int_0^2\int_0^2 (-1)dydx = -\int_0^2 \left(y\right)_0^2 dx = -2(x)_0^2 = -4.$$

$$\iint_{S} curl\vec{F}\cdot\vec{n}ds = -4. \tag{1}$$

We shall now compute the line integral over the simple closed curve C where C is the boundary of the surface consisting of the edges OA,AB,BC & CO in the $z = 0$ plane.

—

$$\therefore \int_C \vec{F}\cdot\, d\vec{r} = \int_{OA} \vec{F}\cdot\, d\vec{r} + \int_{AB} \vec{F}\cdot\, d\vec{r} + \int_{BC} \vec{F}\cdot\, d\vec{r} + \int_{CO} \vec{F}\cdot\, d\vec{r}$$

$\vec{F}\cdot d\vec{r} = (y - z + 2)dx + (yz + 4)dy - xzdz = (y + 2)dx + 4dy \qquad [\because z = 0]$

On OA: $y = 0, dy = 0, x$ varies from 0 to 2

$\therefore \int_{OA} \{\vec{F}\cdot d\vec{r}\} = \int_0^2 2dx = 2(x)_0^2 = 4.$

On AB: $x = 2, dx = 0, y$ varies from 0 to 2.

$\therefore \int_{AB} \{\vec{F}\cdot d\vec{r}\} = \int_0^2 4dy = 4(y)_0^2 = 8.$

On BC: $y = 2, dy = 0, x$ varies from 2 to 0.

$\therefore \int_{BC} \{\vec{F}\cdot d\vec{r}\} = \int_2^0 4dx = -8.$

On CO: $x = 0, dx = 0, y$ varies from 2 to 0.

$\therefore \int_{CO} \{\vec{F}\cdot d\vec{r}\} = \int_2^0 4dy = 4(y)_2^0 = -8.$

Now, $\int_C \{\vec{F}\cdot d\vec{r}\} = 4 + 8 - 8 - 8 = -4.$ (2)

From (1) and (2) we obtain $\iint_S \{curl\vec{F}\cdot\vec{n}\}ds = \int_C \{\vec{F}\cdot d\vec{r}\}$

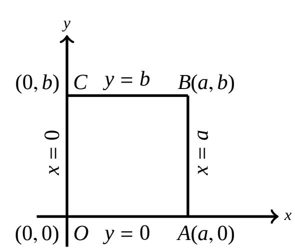
$\therefore$ Stoke's theorem is verified.

Example 2.106. Verify Stoke's theorem for $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region in the xy plane bounded by the lines $x = 0, x = a, y = 0, y = b.$ [Dec 2005]

Solution. We have $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$

By stoke's theorem, $\int_C \{\vec{F}\cdot d\vec{r}\} = \iint_S \{curl\vec{F}\cdot\vec{n}\}ds$

Now, $Curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(2y + 2y) = 4y\vec{k}.$



Since the surface is a rectangle in the xy plane, the normal is $\vec{n} = \vec{k}$.

$curl\vec{F}\cdot\vec{n} = 4y\vec{k}\cdot\vec{k} = 4y.$

$$\iint_S curl\vec{F}\cdot\vec{n}ds = \iint_R 4ydxdy$$
$$= \int_{x=0}^{a}\int_{y=0}^{b} 4ydydx = \int_0^a 4\Big(\frac{y^2}{2}\Big)_0^b dx = 2\int_0^a b^2dx = 2b^2\,[x]_0^a = 2ab^2. \quad (1)$$

We shall now evaluate $\int_C\{\vec{F}\cdot d\vec{r}\}$.

$$\int_C\{\vec{F}\cdot d\vec{r}\} = \int_{OA}\{\vec{F}\cdot d\vec{r}\} + \int_{AB}\{\vec{F}\cdot d\vec{r}\} + \int_{BC}\{\vec{F}\cdot d\vec{r}\} + \int_{CO}\{\vec{F}\cdot d\vec{r}\}$$

Now, $\vec{F}\cdot d\vec{r} = (x^2-y^2)dx + 2xydy$

Along OA: $y=0, dy=0, x$ varies from 0 to a.

$$\therefore \int_{OA}\{\vec{F}\cdot d\vec{r}\} = \int_0^a x^2dx = \Big(\frac{x^3}{3}\Big)_0^a = \frac{a^3}{3}.$$

Along AB: $x=a, dx=0, y$ varies from 0 to b.

$$\therefore \int_{AB}\{\vec{F}\cdot d\vec{r}\} = \int_0^b 2aydy = 2a\Big(\frac{y^2}{2}\Big)_0^b = ab^2.$$

Along BC: $y=b, dy=0, x$ varies from a to 0.

$$\therefore \int_{BC}\{\vec{F}\cdot d\vec{r}\} = \int_a^0 x^2-b^2dx = \Big(\frac{x^3}{3}\Big)_a^0 - b^2\Big(x\Big)_a^0 = -\frac{a^3}{3} + ab^2 = ab^2 - \frac{a^3}{3}.$$

Along CO: $x=0, dx=0, y$ varies from b to 0.

$$\int_{CO}\{\vec{F}\cdot d\vec{r}\} = \int_b^0 0dx = 0.$$

$$\therefore \int_C\{\vec{F}\cdot d\vec{r}\} = \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} + 0 = 2ab^2. \quad (2)$$

From (1) and (2) we obtain $\int_C\{\vec{F}\cdot d\vec{r}\} = \iint_S\{curl\vec{F}\cdot\vec{n}\}ds$.

Hence, Stoke's theorem is verified.

Example 2.107. Verify Stoke's theorem for $\vec{F} = y^2z\vec{i} + z^2x\vec{j} + x^2y\vec{k}$ where S is the open surface of the cube formed by the planes $x=-a, x=a, y=-a, y=a, z=-a, z=a$ in which $z=-a$ is cut open.

Solution. By Stoke's theorem,

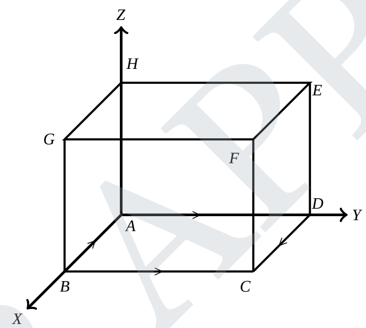
$\int_C\{\vec{F}\cdot d\vec{r}\} = \iint_S\{curl\vec{F}\cdot\vec{n}\}ds$

—

Now, $\vec{F} = y^2z\vec{i} + z^2x\vec{j} + x^2y\vec{k}$

$$Curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z & z^2x & x^2y \end{vmatrix} = \vec{i}(x^2 - 2zx) - \vec{j}(2xy - y^2) + \vec{k}(z^2 - 2yz) = \vec{F_1}$$

Now we shall complete $\iint_S \{curl\vec{F} \cdot \vec{n}\}ds$ where S is the open surface consisting of 5 faces of the cube except ABCD.



$$\iint_S \{curl\vec{F} \cdot \vec{n}\}ds = \iint_{s_1} \{\vec{F_1} \cdot \vec{n}\}ds + \iint_{s_2} \{\vec{F_1} \cdot \vec{n}\}ds + \iint_{s_3} \{\vec{F_1} \cdot \vec{n}\}ds + \iint_{s_4} \{\vec{F_1} \cdot \vec{n}\}ds + \iint_{s_5} \{\vec{F_1} \cdot \vec{n}\}ds$$

The following table gives the necessary quantities for the computation of the surface integral.

Face	Equation	normal $\vec{n}$	curl$\vec{F} \cdot \vec{n}$	ds
S_1:BCFG	$x = a$	$\vec{i}$	$a^2 - 2az$	$dydz$
S_2:ADEH	$x = -a$	$-\vec{i}$	$-(a^2 + 2az)$	$dydz$
S_3:EFGH	$z = a$	$\vec{k}$	$a^2 - 2ax$	$dxdy$
S_4:ABGH	$y = -a$	$-\vec{j}$	$-(a^2 + 2ax)$	$dzdx$
S_5:CDEF	$y = a$	$\vec{j}$	$a^2 - 2ay$	$dzdx$

$$\text{Now },\iint_{s_1} \{\vec{F_1} \cdot \vec{n}\}ds = \int_{-a}^{a}\int_{-a}^{a} \{a^2 - 2az\}dydz = \int_{-a}^{a} [a^2(y)_{-a}^{a} - 2az(y)_{-a}^{a}]\ dz$$

$$= \int_{-a}^{a} \{a^2 \cdot 2a - 2a \cdot 2az\}dz = \int_{-a}^{a} (2a^3 - 4a^2z)dz$$

$$= 2a^3(z)_{-a}^{a} - 4a^2 \cdot \left(\frac{z^2}{2}\right)_{-a}^{a} = 2a^3 \cdot 2a - 2a^2 \cdot 0 = 4a^4.$$

Similarly $\iint_{S_2} Curl\vec{F} \cdot \vec{n}ds = -4a^4.$

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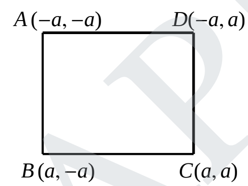
$$\iint_{S_3} Curl\vec{F}\cdot\vec{n}ds = 4a^4.$$
$$\iint_{S_4} Curl\vec{F}\cdot\vec{n}ds = -4a^4.$$
$$\iint_{S_5} Curl\vec{F}\cdot\vec{n}ds = 4a^4.$$
$$\iint_{S} Curl\vec{F}\cdot\vec{n}ds = 4a^4. \quad (1)$$

We shall now compute $\int_C \vec{F}\cdot d\vec{r}$ over the simple closed curve C, consisting of the edges AB, BC, CD, DA.



Here $z = -a, dz = 0$.

$\therefore \vec{F}\cdot d\vec{r} = -ay^2dx + a^2xdy$.

On AB: $y = -a, dy = 0, x$ varies from $-a$ to a.

$\therefore \int_{AB} \vec{F}\cdot d\vec{r} = \int_{-a}^{a} -a^3dx = -a^3(x)_{-a}^{a} = -2a^4$.

On BC: $x = a, dx = 0, y$ varies from $-a$ to a.

$\therefore \int_{BC} \vec{F}\cdot d\vec{r} = \int_{-a}^{a} a^3dy = 2a^4$.

On CD: $y = a, dy = 0, x$ varies from a to $-a$.

$\therefore \int_{CD} \vec{F}\cdot d\vec{r} = \int_{a}^{-a} -a^3dx = -a^3(x)_{a}^{-a} = 2a^4$.

On DA: $x = -a, dx = 0, y$ varies from a to $-a$.

$\therefore \int_{DA} \vec{F}\cdot d\vec{r} = \int_{a}^{-a} -a^3dy = 2a^4$.

$$\text{Hence, } \int_c \vec{F}\cdot d\vec{r} = 4a^4. \quad (2)$$

From (1) and (2) we obtain $\int_C \{\vec{F}\cdot d\vec{r}\} = \iint_S \{curl\vec{F}\cdot\vec{n}\}ds$

Hence, Stoke's theorem is verified.

Example 2.108. Verify Stoke's theorem for $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ taken around the rectangle bounded by the lines $x = \pm a, y = 0, y = b$. [Dec 2013, May 2001]

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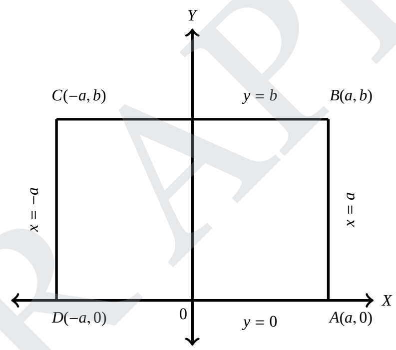
Solution. By Stoke's theorem we have $\int_C \{\vec{F}\cdot d\vec{r}\} = \iint_S \{curl\vec{F}\cdot\vec{n}\}ds.$

$$\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$$

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(-2y-2y) = -4y\vec{k}$$

The surface S is given as a rectangle in the xy plane.

Hence, $\vec{n} = \vec{k}$.



$$\iint_S \{curl\vec{F}\cdot\vec{n}\}ds = \iint_S -4y\vec{k}.\vec{k}dxdy$$

$$= -4\int_0^b\int_{-a}^a ydxdy = -4\left[\frac{y^2}{2}\right]_0^b [x]_{-a}^a$$

$$= -2[b^2][a+a] = -4ab^2. \quad (1)$$

Now, we shall evaluate the line integral $\int_C \{\vec{F}\cdot d\vec{r}\}$

$\vec{F}\cdot d\vec{r} = [(x^2+y^2)\vec{i} - 2xy\vec{j}].(dx\vec{i} + dy\vec{j}) = (x^2+y^2)dx - 2xydy.$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r} + \int_{DA} \vec{F}\cdot d\vec{r}.$$

<u>On AB</u>: $x = a, dx = 0, y$ varies from 0 to b and $\vec{F}\cdot d\vec{r} = -2aydy.$

$$\therefore \int_{AB} \vec{F}\cdot d\vec{r} = \int_0^b -2aydy = -2a\left(\frac{y^2}{2}\right)_0^b = -ab^2.$$

<u>On BC</u>: $y = b, dy = 0, x$ varies from a to $-a$ and $\vec{F}\cdot d\vec{r} = (x^2+b^2)dx.$

$$\therefore \int_{BC} \vec{F}\cdot d\vec{r} = \int_a^{-a} (x^2+b^2)dx = \left(\frac{x^3}{3}\right)_a^{-a} + b^2(x)_a^{-a} = \frac{1}{3}(-a^3-a^3) + b^2(-a-a) = -\frac{2a^3}{3} - 2ab^2.$$

<u>On CD</u>: $x = -a, dx = 0, y$ varies from b to 0 and $\vec{F}\cdot d\vec{r} = 2aydy.$

$$\therefore \int_{CD} \vec{F}\cdot d\vec{r} = \int_b^0 2aydy = 2a\left(\frac{y^2}{2}\right)_b^0 = a(0-b^2) = -ab^2.$$

<u>On DA</u>: $y = 0, dy = 0, x$ varies from $-a$ to a and $\vec{F}\cdot d\vec{r} = x^2dx.$

—

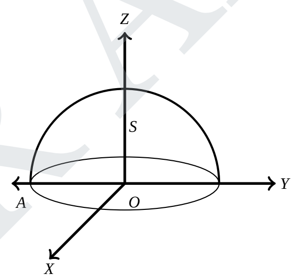
$$\therefore \int_{DA} \vec{F} \cdot d\vec{r} = \int_{-a}^{a} x^2 dx = \left(\frac{x^3}{3}\right)_{-a}^{a} = \frac{1}{3}(a^3 + a^3) = \frac{2a^3}{3}.$$

$$\text{Hence, } \int_c \vec{F} \cdot d\vec{r} = -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2. \quad (2)$$

From (1) and (2) we obtain $\int_C \{\vec{F} \cdot d\vec{r}\} = \iint_S \{curl\vec{F} \cdot \vec{n}\} ds$
Hence, Stoke's theorem is verified.

Example 2.109. Verify Stoke's theorem for the vector field $\vec{F} = (2x - y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ over the upper half surface of $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy plane.

Solution. The projection of the upper half of the given sphere on the xy plane is the circle $x^2 + y^2 = 1$ $[z = 0]$.



$$\int_C \vec{F} \cdot d\vec{r} = \int_C \left((2x - y)dx - yz^2 dy - y^2 z dz\right)$$
$$= \int_C (2x - y)dx.$$

Since C is the unit circle, change x and y interms of the parametric coordinates.
$x = \cos\theta$, $y = \sin\theta$, $dy = -\sin\theta d\theta$.
Since C is the full circle, θ varies from 0 to 2π.

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta \;\; d\theta)$$
$$= -\int_0^{2\pi} (2\sin\theta\cos\theta - \sin^2\theta) \;\; d\theta$$
$$= -\int_0^{2\pi} \left(\sin 2\theta - \frac{1 - \cos 2\theta}{2}\right) \;\; d\theta$$

$\therefore$ Cayley Hamilton theorem is verified.

Step 3. To find A^{-1}

By Cayley Hamilton theorem we have $A^3 - 6A^2 + 8A - 3I = 0$.

Multiply by A^{-1} we get

$$A^2 - 6A + 8I - 3A^{-1} = 0$$

$$3A^{-1} = A^2 - 6A + 8I$$

$$3A^{-1} = \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 8\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} + \begin{pmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$3A^{-1} = \begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}.$$

11. (b) Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2zx$ into canonical form.

Solution. Refer question no. 11. (b) (i) of Nov/Dec. 2014 [2.1.2]

2.2 Unit II Vector Calculus

2.2.1 May/June 2016 (R 2013)

Part A

1. Evaluate $\nabla^2 \log r$.

   Solution. We have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

   $$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}.$$

   $$\therefore r^2 = x^2 + y^2 + z^2.$$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}. \quad \text{Similarly, } \frac{\partial r}{\partial y} = \frac{y}{r}, \ \frac{\partial r}{\partial z} = \frac{z}{r}.$$

$$\nabla(\log r) = \vec{i}\frac{\partial}{\partial x}(\log r) + \vec{j}\frac{\partial}{\partial y}(\log r) + \vec{k}\frac{\partial}{\partial z}(\log r)$$

$$= \vec{i}\frac{1}{r}.\frac{\partial r}{\partial x} + \vec{j}\frac{1}{r}.\frac{\partial r}{\partial y} + \vec{k}\frac{1}{r}.\frac{\partial r}{\partial z} = \frac{1}{r}\left[\frac{x\vec{i}}{r} + \frac{y\vec{j}}{r} + \frac{z\vec{k}}{r}\right] = \frac{\vec{r}}{r^2}.$$

$$\nabla^2 \log r = \nabla \cdot \nabla \log r = \nabla \cdot \left(\frac{\vec{r}}{r^2}\right) = \nabla\left(\frac{1}{r^2}\right) \cdot \vec{r} + \frac{1}{r^2}(\nabla \cdot \vec{r})$$

$$= -2r^{-2-2}\vec{r} \cdot \vec{r} + \frac{1}{r^2} \cdot 3 = -2r^{-4}r^2 + \frac{3}{r^2} = \frac{-2}{r^2} + \frac{3}{r^2} = \frac{1}{r^2}.$$

2. State Stoke's theorem.

Solution. Refer page 98 of the main text book.

Part B

11. (a) (i) If $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ find $\phi(x, y, z)$ given that $\phi(1, -2, 2) = 4$.

Solution. Refer Example 1.23 on page 19 of the main text book.

11. (a) (ii) Using Green's theorem in a plane evaluate $\int_C x^2(1+y)dx + (x^3+y^3)dy$ where C is the square formed by $x = \pm 1$ and $y = \pm 1$.

Solution. Comparing the given integral with Green's theorem we have

$$P = x^2(1+y), \qquad Q = x^3 + y^3.$$

$$\frac{\partial P}{\partial y} = x^2, \qquad \frac{\partial Q}{\partial x} = 3x^2.$$

By Green's theorem,

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \iint_R (3x^2 - x^2)dxdy$$

$$= \iint_R 2x^2 dxdy = \int_{y=-1}^{1}\int_{x=-1}^{1} 2x^2 dxdy$$

$$= \int_{y=-1}^{1}\left[\frac{2x^3}{3}\right]_{x=-1}^{1} dy = \frac{2}{3}\int_{y=-1}^{1}[1+1]\,dy$$

$$= \frac{4}{3}[y]_{y=-1}^{1} = \frac{4}{3}[1+1] = \frac{8}{3}.$$

11. (b) (i) Find a and b so that the surfaces $ax^3 - by^2z = (a+3)x^2$ and $4x^2y - z^3 = 11$ cut orthogonally at $(2,-1,-3)$.

Solution. Refer Example 1.19 on page 16 of the main text book.

11. (b) (ii) Prove that $CurlCurl\vec{F} = graddiv\vec{F} - \nabla^2\vec{F}$.

Solution. Refer Identity VIII on page 15 of the main text book.

2.2.2 Dec.2015/Jan. 2016 (R 2013)

Part A

1. Prove that $Grad(1/r) = \frac{-\vec{r}}{r^3}$.

   Solution. Refer Example 1.26 subdivision (iii) on page 22 of the main text book.

2. Evaluate $\int_C (yz\vec{i} + xz\vec{j} + xy\vec{k})\cdot d\vec{r}$ where C is the boundary of a surface S.

   Solution. By Stoke's theorem we have
$$\int_c \vec{F}\cdot d\vec{r} = \int\int_S curl\vec{F}\cdot\vec{n}ds.$$
   Here, $\vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k}$.

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$
$$= \vec{i}(x-x) - \vec{j}(y-y) + \vec{k}(z-z) = \vec{0}.$$

$$\therefore \int_c \vec{F}\cdot d\vec{r} = \int\int_S 0\cdot\vec{n}ds = 0.$$

Part B

11. (a) Verify Green's theorem for $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$ taken round the rectangle bounded by the lines $x = \pm a$, $y = 0$ and $y = b$.

Solution. Refer Example 1.79 on page 76 of the main text book.

11. (b) (i) A fluid motion is given by $\vec{V} = (y + z)\vec{i} + (z + x)\vec{j} + (x + y)\vec{k}$. Is this motion irrotational and is this possible for an incompressible fluid?

Solution. Given, $\vec{V} = (y + z)\vec{i} + (z + x)\vec{j} + (x + y)\vec{k}$.

$$curl\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + z & z + x & x + y \end{vmatrix}$$

$$= \vec{i}(1 - 1) - \vec{j}(1 - 1) + \vec{k}(1 - 1) = \vec{0}.$$

$\therefore$ The motion is irrotational.

11. (b) (ii) Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$. And S is the surface of the rectangular parallelepiped bounded by $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$ and $z = c$.

Solution. Refer Example 1.91 on page 92 of the main text book.

2.2.3 April/May 2015 (R 2013)

Part A

1. In what direction form $(3, 1, -2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum? Find also the magnitude of this maximum.

Solution. Given $\phi = x^2y^2z^4$.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$
$$= (2xy^2z^4)\vec{i} + (2x^2yz^4)\vec{j} + (4x^2y^2z^3)\vec{k}.$$
$$(\nabla\phi)_{(3,1,-2)} = (2\times 3\times 1\times(-2)^4)\vec{i} + (2\times 9\times 1\times(-2)^4)\vec{j} + (4\times 9\times 1\times(-2)^3)\vec{k}$$
$$= 96\vec{i} + 288\vec{j} - 288\vec{k}.$$

Directional derivative is maximum in the direction of $\nabla\phi = 96\vec{i} + 288\vec{j} - 288\vec{k}$.

Maximum value of the directional derivative

$$|\nabla\phi| = \sqrt{96^2 + 288^2 + (-288)^2}$$
$$= \sqrt{96^2 + (96\times 3)^2 + (96\times 3)^2}$$
$$= \sqrt{(96)^2(1+9+9)} = 96\sqrt{19}.$$

2. Find α such that $\vec{F} = (3x - 2y + z)\vec{i} + (4x + \alpha y - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal.

Solution. Let $\vec{F} = (3x - 2y + z)\vec{i} + (4x + \alpha y - z)\vec{j} + (x - y + 2z)\vec{k}$.

Given that $\vec{F}$ is solenoidal.i.e.,$\nabla.\vec{F} = 0$.

$$\Rightarrow \frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + \alpha y - z) + \frac{\partial}{\partial z}(x - y + 2z) = 0.$$

i.e., $3 + \alpha + 2 = 0 \quad \Rightarrow \alpha + 5 = 0 \Rightarrow \alpha = -5.$

Part B

11. (a) (i) Show that $\nabla^2(r^n) = n(n+1)r^{n-2}$ where $r^2 = x^2+y^2+z^2$. Hence find the value of $\nabla^2\left(\frac{1}{r}\right)$.

Solution. Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$r^2 = x^2 + y^2 + z^2.$$
$$2r\frac{\partial r}{\partial x} = 2x \;\Rightarrow\; \frac{\partial r}{\partial x} = \frac{x}{r}. \quad \text{similarly}\frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}.$$

We have, $\nabla(r^n) = \vec{i}\frac{\partial}{\partial x}(r^n) + \vec{j}\frac{\partial}{\partial y}(r^n) + \vec{k}\frac{\partial}{\partial z}(r^n)$

$$= \vec{i}nr^{n-1}\frac{\partial r}{\partial x} + \vec{j}nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right)$$
$$= nr^{n-2}(x\vec{i} + y\vec{j} + z\vec{k}) = nr^{n-2}\vec{r}.$$

We have $\nabla^2 (r^n) = \nabla.\nabla (r^n) = \nabla.\left(nr^{n-2}\vec{r}\right)$

$$= n\left[\nabla\left(r^{n-2}\vec{r}\right) + r^{n-2}\nabla.\vec{r}\right]$$
$$= n\left[(n-2)r^{n-4}\vec{r}\cdot\vec{r} + r^{n-2}3\right]$$
$$= n\left[(n-2)r^{n-4}r^2 + 3r^{n-2}\right]$$
$$= n\left[(n-2)r^{n-2} + 3r^{n-2}\right]$$
$$= nr^{n-2}[n-2+3] = n(n+1)r^{n-2}.$$

Put $n = -1$ in the above result we get

$$\nabla^2\left(\frac{1}{r}\right) = (-1)(-1+1)r^{n-2} = 0.$$

11. (a) (ii) Using Green's theorem, evaluate $\int_C (y - \sin x)dx + \cos x\, dy$ where C is the triangle formed by $y = 0$, $x = \frac{\pi}{2}$, $y = \frac{2x}{\pi}$.

Solution. Comparing the given integral with Green's theorem we have

$$P = y - \sin x \qquad Q = \cos x.$$
$$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = -\sin x.$$

By Green's theorem we have

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy.$$

$$\text{i.e., } \int_C \{(y - \sin x)dx + \cos x dy\} = \iint_R (-\sin x - 1)dxdy$$

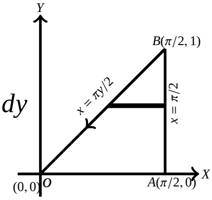
Where R is the triangle formed by $y = 0$, $x = \frac{\pi}{2}$, $y = \frac{2x}{\pi}$ (or) $x = \frac{\pi y}{2}$.

$$= -\int_0^1 \int_{x=\frac{\pi y}{2}}^{\frac{\pi}{2}} (\sin x + 1)dxdy$$

$$\therefore \oint_C \{(y - \sin x)dx + \cos x dy\}$$

$$= -\int_0^1 \left(-\cos x + x\right)_{\frac{\pi y}{2}}^{\frac{\pi}{2}} dy$$

$$= -\int_0^1 \left\{-\cos\frac{\pi}{2} + \frac{\pi}{2} - (-\cos\frac{\pi y}{2} + \frac{\pi y}{2})\right\} dy$$



$$= -\int_0^1 (\frac{\pi}{2} + \cos\frac{\pi y}{2} - \frac{\pi y}{2}) dy$$

$$= -\left[\frac{\pi}{2}(y)_0^1 + \left(\frac{\sin\frac{\pi y}{2}}{\frac{\pi}{2}}\right)_0^1 - \frac{\pi}{2}\left(\frac{y^2}{2}\right)_0^1\right]$$

$$= -\left[\frac{\pi}{2} + \frac{2}{\pi}(\sin\frac{\pi}{2} - \sin 0) - \frac{\pi}{4}(1-0)\right]$$

$$= -\left[\frac{\pi}{2} + \frac{2}{\pi}\cdot 1 - \frac{\pi}{4}\right] = -\left[\frac{\pi}{4} + \frac{2}{\pi}\right].$$

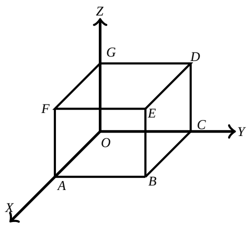
11. (b) Verify Gauss divergence theorem for $\vec{F} = (4xz)\vec{i} - (y^2)\vec{j} + (yz)\vec{k}$ taken over the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

Solution. By Gauss divergence theorem,

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

Given, $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$.

The given surface is the cube $OABCDEFG$. S is piecewise smooth which consists of six smooth surfaces namely $ABEF, OCDG, BCDE, OAFG, OABC$ and $DEFG$.



$$\therefore \iint_S \{\vec{F}\cdot\vec{n}\}ds = \iint_{ABEF}\{\vec{F}\cdot\vec{n}\}ds + \iint_{OCDG}\{\vec{F}\cdot\vec{n}\}ds + \iint_{OABC}\{\vec{F}\cdot\vec{n}\}ds$$

$$+ \iint_{DEFG}\{\vec{F}\cdot\vec{n}\}ds + \iint_{BCDE}\{\vec{F}\cdot\vec{n}\}ds + \iint_{OAFG}\{\vec{F}\cdot\vec{n}\}ds.$$

The following table gives the necessary quantities for the computation of the surface integral on each surface.

Face	Equation	Outward normal	$\vec{F}\cdot\vec{n}$	ds
ABEF	$x=1$	$\vec{i}$	$4z$	$dydz$
OCDG	$x=0$	$-\vec{i}$	$-4xz=0$	$dydz$
BCDE	$y=1$	$\vec{j}$	$-y^2=-1$	$dxdz$
OAFG	$y=0$	$-\vec{j}$	$-y^2=0$	$dxdz$
OABC	$z=0$	$-\vec{k}$	$-yz=0$	$dxdy$
DEFG	$z=1$	$\vec{k}$	$yz=y$	$dxdy$

$$\iint\limits_{ABEF} \vec{F}\cdot\vec{n}ds = \int\limits_0^1\int\limits_0^1 4zdydz = 4\int_0^1 z(y)_0^1 dz = 4\int\limits_0^1 zdz = 4\left(\frac{z^2}{2}\right)_0^1 = 2.$$

$$\iint\limits_{BCDE} \vec{F}\cdot\vec{n}ds = \int\limits_0^1\int\limits_0^1 (-1)dxdz = -\int\limits_0^1 (x)_0^1 dz = -\int\limits_0^1 dz = -(z)_0^1 = -1.$$

$$\iint\limits_{DEFG} \vec{F}\cdot\vec{n}ds = \int\limits_0^1\int\limits_0^1 ydxdy = \int\limits_{x=0}^1 \left(\frac{y^2}{2}\right)_0^1 dx = \frac{1}{2}(x)_0^1 = \frac{1}{2}.$$

All the other integrals will be zero.

$$\therefore \iint_S \vec{F}\cdot\vec{n}ds = 2 - 1 + \frac{1}{2} = \frac{3}{2}. \qquad (1)$$

Now, $\iiint_V \nabla\cdot\vec{F}dv$

$$= \int_0^1\int_0^1\int_0^1 \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dzdydx$$

$$= \int_0^1\int_0^1\int_0^1 (4z - 2y + y)dzdydx$$

$$= \int_0^1\int_0^1\int_0^1 (4z - y)dzdydx = \int_0^1\int_0^1 \left[4\frac{z^2}{2} - yz\right]_0^1 dydx$$

$$= \int_0^1\int_0^1 (2-y)dydx = \int_0^1 \left[2(y)_0^1 - \left(\frac{y^2}{2}\right)_0^1\right]dx$$

$$= \int_0^1 \left(2 - \frac{1}{2}\right)dx = \frac{3}{2}(x)_0^1 = \frac{3}{2}. \qquad (2)$$

From (1) and (2) we have $\iint_S \vec{F}\cdot\vec{n}ds = \iiint_V \nabla\cdot\vec{F}dv.$
Hence, Gauss Divergence theorem is verified.

2.2.4 November/December 2014 (R 2013)

Part A

1. Find $curl\vec{F}$ if $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$.

   Solution. Given, $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$.

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix}$$

$$= \vec{i}\Big(\frac{\partial}{\partial y}(zx) - \frac{\partial}{\partial z}(yz)\Big) - \vec{j}\Big(\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial z}(xy)\Big) + \vec{k}\Big(\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xy)\Big)$$

$$= \vec{i}(0-y) - \vec{j}(z-0) + \vec{k}(0-x) = -y\vec{i} - z\vec{j} - x\vec{k}.$$

2. State Stoke's theorem.

   Solution. If S is an open surface bounded by a simple closed curve C and if $\vec{F}$ is continuous having continuous partial derivatives in S and on C, then $\oint\limits_C \vec{F}\cdot d\vec{r} = \iint\limits_S Curl\vec{F}\cdot\vec{n}ds$ where C is traversed in the positive direction.

Part B

11. (a) (i) Find the angle between the normals to the surface $x^2 = yz$ at the points $(1,1,1)$ and $(2,4,1)$.

   Solution. Given $\phi = x^2 - yz$.

   Let $\vec{n}_1$ and $\vec{n}_2$ be the normals to the surface ϕ at (1,1,1) and (2,4,1) respectively.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = \vec{i}(2x) + \vec{j}(-z) + \vec{k}(-y).$$

$$\vec{n}_1 = (\nabla\phi)_{(1,1,1)} = 2\vec{i} - \vec{j} - \vec{k}.$$

$$\vec{n}_2 = (\nabla\phi)_{(2,4,1)} = 4\vec{i} - \vec{j} - 4\vec{k}.$$

Let θ be the angle between the two normals.

$$\therefore \cos\theta = \frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}.$$

$$= \frac{8+1+4}{\sqrt{4+1+1}\sqrt{16+1+16}}$$

$$= \frac{13}{\sqrt{6}\sqrt{33}} = \frac{13}{\sqrt{6\times 33}} = \frac{13}{\sqrt{198}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{13}{\sqrt{198}}\right).$$

11. (a) (ii) Using Green's theorem evaluate $\int_C [(3x^2-8y^2)dx + (4y-6xy)dy]$ where C is the boundary of the triangle formed by the lines $x = 0$, $y = 0$, $x + y = 1$ in the $xy-$plane.

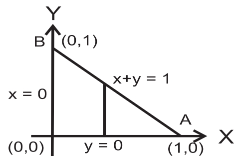
Solution. By Green's theorem,

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy. \qquad (1)$$

$$P = 3x^2 - 8y^2 \qquad Q = 4y - 6xy.$$

$$\frac{\partial P}{\partial y} = -16y \qquad \frac{\partial Q}{\partial x} = -6y.$$

Applying in (1) we get



$$\int_C \left(3x^2 - 8y^2\right) dx + (4y - 6xy)\ dy$$

$$= \iint_R (-6y + 16y)dxdy$$

$$= \iint_R 10ydxdy$$

$$= 10\iint_R ydydx = 10\int_{x=0}^{1}\int_{y=0}^{1-x} ydydx = 10\int_{x=0}^{1}\left(\frac{y^2}{2}\right)_0^{1-x} dx$$

$$= \frac{10}{2}\int_0^1 (1-x)^2 dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1 = -\frac{5}{3}[0-1] = \frac{5}{3}.$$

11. (b) (i) Verify divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ taken over the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$,

$z = 0, z = 1.$

Solution. Refer question 11. (b) of April/May 2015.

11. (b) (ii) Prove that $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ is irrational and hence find its scalar potential.

Solution. Given $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 + x & -2xy - y & 0 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(-2y + 2y) = \vec{0}.$$

$\therefore\ \vec{F}$ is irrotational.

Since $\nabla \times \vec{F} = \vec{0}, \vec{F} = \nabla\phi$, where ϕ is a scalar potential.

$$\therefore \vec{F} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}.$$

Comparing the components of $\vec{i}, \vec{j}$ & $\vec{k}$ we get

$$\frac{\partial\phi}{\partial x} = x^2 - y^2 + x \tag{1}$$

$$\frac{\partial\phi}{\partial y} = -2xy - y \tag{2}$$

$$\frac{\partial\phi}{\partial z} = 0. \tag{3}$$

Integrating (1), (2) and (3) partially w.r.t. x, y and z respectively we get

$$\phi = \int (x^2 - y^2 + x)dx + f_1(y,z)$$

$$\phi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} + f_1(y,z) \tag{4}$$

$$\phi = \int (-2xy - y)dy + f_2(x,z)$$

$$\phi = \frac{-2xy^2}{2} - \frac{y^2}{2} + f_2(x,z) = -xy^2 - \frac{y^2}{2} + f_2(x,z) \tag{5}$$

$$\phi = f_3(x, y) \tag{6}$$

From (4), (5) and (6) we get
$\phi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} - \frac{y^2}{2} + c$, where c is a complex constant.

2.2.5 May/June 2014 (R 2013)

Part A

1. Find the unit normal vector to the surface $x^2+y^2 = z$ at $(1, -2, 5)$.

Solution. $\phi = x^2 + y^2 - z$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = \vec{i}(2x) + \vec{j}(2y) + \vec{k}(-1).$$

$$(\nabla\phi)_{(1,-2,5)} = 2\vec{i} - 4\vec{j} - \vec{k}.$$

$$|\nabla\phi| = \sqrt{4+16+1} = \sqrt{21}.$$

$$\text{Unit normal } = \frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{\sqrt{21}}(2\vec{i} - 4\vec{j} - \vec{k}).$$

2. Prove that $curl(grad\ \phi) = \vec{0}$.

Solution. We have $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$.

$$\nabla\times\nabla\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}.$$

$$= \vec{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - \vec{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) + \vec{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)$$

$$= 0 \qquad \left[\text{assuming } \frac{\partial^2\phi}{\partial y\partial z} = \frac{\partial^2\phi}{\partial z\partial y} \text{ etc}\right]$$

$$\nabla\times\nabla\phi = 0 \quad \text{always.}$$

Part B

11. (a) Verify Gauss divergence theorem for $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ taken over the cube bounded by the planes $x = 0, y = 0, z = 0, x = 1,$

$y = 1$ and $z = 1$.

Solution. By Gauss divergence theorem

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_V \nabla \cdot \vec{F} dv.$$

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2x + 2y + 2z.$$

$$\text{Now, } \iiint_V \nabla \cdot \vec{F} dv = \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} (2x + 2y + 2z) dxdydz$$

$$= \int_0^1 \int_0^1 \left[2\left(\frac{x^2}{2}\right)_0^1 + 2(y)(x)_0^1 + 2z(x)_0^1 \right] dydz$$

$$= \int_0^1 \int_0^1 (1 + 2y + 2z) dydz$$

$$= \int_0^1 \left(y + 2 \cdot \frac{y^2}{2} + 2zy \right)_{y=0}^1 dz$$

$$= \int_0^1 (1 + 1 + 2z) dz = \int_0^1 (2 + 2z) dz$$

$$= 2 \int_0^1 (1 + z) dz = \left(\frac{2(1+z)^2}{2} \right)_0^1 = 4 - 1 = 3.$$

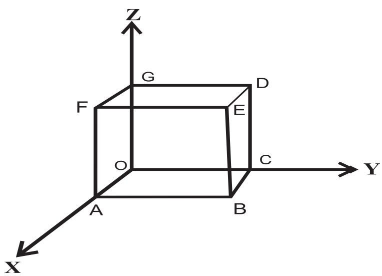
Now, we shall evaluate $\iint_S \vec{F} \cdot \vec{n} ds$.

The cube has six faces namely

$S_1 : ABEF$, $S_2 : OCDG$,

$S_3 : BCDE$, $S_4 : OAFG$,

$S_5 : OABC$, $S_6 : DEFG$.



$$\therefore \iint_S \vec{F}\cdot\vec{n}ds = \iint_{S_1} \vec{F}\cdot\vec{n}ds + \iint_{S_2} \vec{F}\cdot\vec{n}ds$$

$$+ \iint_{S_3} \vec{F}\cdot\vec{n}ds + \iint_{S_4} \vec{F}\cdot\vec{n}ds + \iint_{S_5} \vec{F}\cdot\vec{n}ds + \iint_{S_6} \vec{F}\cdot\vec{n}ds.$$

The following table gives the necessary quantities for the computation of the surface integral over each face.

Face	Equation	Outward normal $\vec{n}$	$\vec{F}\cdot\vec{n}$	ds
S_1	$x = 1$	$\vec{i}$	$x^2 = 1$	$dydz$
S_2	$x = 0$	$-\vec{i}$	$-x^2 = 0$	$dydz$
S_3	$y = 1$	$\vec{j}$	$y^2 = 1$	$dxdz$
S_4	$y = 0$	$-\vec{j}$	$-y^2 = 0$	$dxdz$
S_5	$z = 0$	$-\vec{k}$	$-z^2 = 0$	$dxdy$
S_6	$z = 1$	$\vec{k}$	$z^2 = 1$	$dxdy$

$$\iint_{S_1} \vec{F}\cdot\vec{n}ds = \int_{y=0}^{1}\int_{z=0}^{1} dydz = \int_{y=0}^{1} (y)_0^1 dz = \int_0^1 dz = (z)_0^1 = 1.$$

$$\iint_{S_2} \vec{F}\cdot\vec{n}ds = 0.$$

$$\iint_{S_3} \vec{F}\cdot\vec{n}ds = \int_{z=0}^{1}\int_{x=0}^{1} dxdz = \int_0^1 (x)_0^1 dz = \int_0^1 dz = (z)_0^1 = 1.$$

$$\iint_{S_4} \vec{F}\cdot\vec{n}ds = 0.$$

$$\iint_{S_5} \vec{F}\cdot\vec{n}ds = 0.$$

$$\iint_{S_6} \vec{F}\cdot\vec{n}ds = \int_{x=0}^{1}\int_{y=0}^{1} dxdy = \int_0^1 (y)_0^1 dx = \int_0^1 dx = (x)_0^1 = 1.$$

Now, $\iint_S \vec{F}\cdot\vec{n}ds = 1 + 0 + 1 + 0 + 0 + 1 = 3$

From (1) and (2) we have $\iint_S \vec{F}\cdot\vec{n}ds = \iiint_V \nabla\cdot\vec{F}dv$

$\therefore$ Gauss divergence theorem is verified.

11. (b) (i) Find the value of n such that the vector $r^n\vec{r}$ is both solenoidal and irrotational.

Solution. Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$r^2 = x^2 + y^2 + z^2. \quad \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

Let $\vec{F} = r^n\vec{r} = r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}$.

$$F_1 = r^n x, F_2 = r^n y, F_3 = r^n z$$

$$\begin{aligned}
\nabla\cdot\vec{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.\\
&= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z).\\
&= r^n + x\times n\times r^{(n-1)}\frac{\partial r}{\partial x} + r^n + y\times n\times r^{(n-1)}\frac{\partial r}{\partial y} + r^n + z\times n\times r^{(n-1)}\frac{\partial r}{\partial z}\\
&= 3r^n + nr^{(n-1)}\left(x\times\frac{x}{r} + y\times\frac{y}{r} + z\times\frac{z}{r}\right)\\
&= 3r^n + nr^{(n-2)}(x^2+y^2+z^2)\\
&= 3r^n + nr^{(n-2)}\times r^2 = 3r^n + nr^n = (n+3)r^n.
\end{aligned}$$

If $r^n\vec{r}$ is solenoidal then $\nabla\cdot\vec{F} = 0$.

$$\text{i.e., } (n+3)r^n = 0. \quad \Rightarrow n+3 = 0. \quad \Rightarrow n = -3.$$

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial(r^n z)}{\partial y} - \frac{\partial(r^n y)}{\partial z}\right) - \vec{j}\left(\frac{\partial(r^n z)}{\partial x} - \frac{\partial(r^n x)}{\partial z}\right) + \vec{k}\left(\frac{\partial(r^n y)}{\partial x} - \frac{\partial(r^n x)}{\partial y}\right).$$

$$= \vec{i}\left(z\times n\times r^{(n-1)}\times\frac{y}{r} - y\times n\times r^{n-1}\frac{z}{r}\right) - \vec{j}(0) + \vec{k}(0) = \vec{0}$$

$\therefore r^n\vec{r}$ is irrotational for all values of n.

11. (b) (ii) Verify Stokes theorem for $\vec{F} = (x^2 - y_2)\vec{i} + 2xy\vec{j}$ in the rectangular region of $z = 0$ plane bounded by the lines $x = 0$, $y = 0$, $x = a$ and $y = b$.

Solution. We have $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$

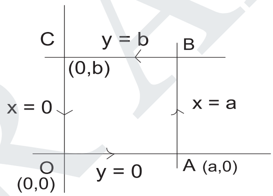
By stoke's theorem, $\oint_C \{\vec{F} \cdot d\vec{r}\} = \iint_S \{curl\vec{F} \cdot \vec{n}\}ds$

Now,

$$Curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(2y+2y) = 4y\vec{k}.$$

Since the surface is a rectangle in the xy plane, the normal is $\vec{n} = \vec{k}$.

$curl\vec{F} \cdot \vec{n} = 4y\vec{k} \cdot \vec{k} = 4y.$



$$\iint_S curl\vec{F} \cdot \vec{n}ds = \iint_R 4ydxdy$$

$$= \int_{x=0}^{a} \int_{y=0}^{b} 4ydydx$$

$$= \int_0^a 4\left(\frac{y^2}{2}\right)_0^b dx = 2\int_0^a b^2 dx = 2b^2 (x)_0^a = 2ab^2. \qquad (1)$$

We shall now evaluate $\int_C \{\vec{F} \cdot d\vec{r}\}$.

$$\int_C \{\vec{F} \cdot d\vec{r}\} = \int_{OA} \{\vec{F} \cdot d\vec{r}\} + \int_{AB} \{\vec{F} \cdot d\vec{r}\} + \int_{BC} \{\vec{F} \cdot d\vec{r}\} + \int_{CO} \{\vec{F} \cdot d\vec{r}\}$$

Now, $\vec{F} \cdot d\vec{r} = (x^2 - y^2)dx + 2xydy$

Along OA: $y = 0, dy = 0, x$ varies from 0 to a.

$$\therefore \int_{OA} \{\vec{F} \cdot d\vec{r}\} = \int_0^a x^2 dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}.$$

Along AB: $x = a, dx = 0, y$ varies from 0 to b.

$$\therefore \int_{AB} \{\vec{F} \cdot d\vec{r}\} = \int_0^b 2aydy = 2a\left(\frac{y^2}{2}\right)_0^b = ab^2.$$

Along BC: $y = b, dy = 0, x$ varies from a to 0.

$$\therefore \int_{BC} \{\vec{F}\cdot d\vec{r}\} = \int_a x^2 - b^2 dx = \left(\frac{x^3}{3}\right)_a^0 - b^2 (x)_a^0.$$

$$= -\frac{a^3}{3} + ab^2 = ab^2 - \frac{a^3}{3}.$$

Along CO: $x = 0, dx = 0, y$ varies from b to 0.

$$\int_{CO} \{\vec{F}\cdot d\vec{r}\} = \int_b^0 0dx = 0.$$

$$\therefore \int_C \{\vec{F}\cdot d\vec{r}\} = \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} + 0 = 2ab^2. \quad (2)$$

From (1) and (2) we obtain $\oint_C \{\vec{F}\cdot d\vec{r}\} = \iint_S \{curl\vec{F}\cdot\vec{n}\}ds.$

Hence, Stoke's theorem is verified.

2.2.6 November/December 2013(R 2008)

Part A

1. Define Solenoidal vector function. If $\vec{V} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+2\lambda z)\vec{k}$ is solenoidal, find the value of λ.

   Solution. Solenoidal vector. If $div\vec{F} = 0$ everywhere in a region R, then $\vec{F}$ is called a solenoidal vector in R.

   Given $\vec{V} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+2\lambda z)\vec{k}$.

   Given $\vec{V}$ is solenoidal.

$$\therefore \nabla\cdot\vec{V} = 0$$

$$i.e., \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+2\lambda z) = 0$$

$$1 + 1 + 2\lambda = 0$$

$$2\lambda = -2$$

$$\lambda = -1.$$

2. State Green's theorem.

   Statement: If $P(x,y)$ and $Q(x,y)$ are continuous functions with continuous partial derivatives in a region R in the xy plane and on its boundary C which is a simple closed curve, then $\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$ where C is described in the anticlockwise sense.

Part B

11. (a) (i) Show that the vector field $\vec{F} = (x^2 + xy^2)\vec{i} + (y^2 + x^2y)\vec{j}$ is irrotational. Find its scalar potential.

Solution. Given $\vec{F} = (x^2 + xy^2)\vec{i} + (y^2 + x^2y)\vec{j}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix}$$

$$= \vec{i}(0 - 0) - \vec{j}(0 - 0) + \vec{k}(2xy - 2xy)$$

$$= \vec{0}.$$

$\therefore$ $\vec{F}$ is irrotational.

Since $\nabla \times \vec{F} = \vec{0}, \vec{F} = \nabla\phi$, where ϕ is a scalar potential.

$$\therefore \vec{F} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}.$$

Comparing the components of $\vec{F}$, we get

$$\frac{\partial\phi}{\partial x} = x^2 + xy^2 \quad (1)$$

$$\frac{\partial\phi}{\partial y} = y^2 + x^2y \quad (2)$$

$$\frac{\partial\phi}{\partial z} = 0. \quad (3)$$

Integrating (1), (2) and (3) partially w.r.t. x, y and z respectively we get

$$\phi = \int (x^2 + xy^2)dx + f_1(y,z) \quad \Rightarrow \quad \phi = \frac{x^3}{3} + \frac{x^2y^2}{2} + f_1(y,z). \qquad (4)$$

$$\phi = \int (y^2 + x^2y)dy + f_2(x,z) \quad \Rightarrow \quad \phi = \frac{y^3}{3} + \frac{x^2y^2}{2} + f_2(x,z). \qquad (5)$$

$$\phi = f_3(x,y). \qquad (6)$$

From (4), (5) and (6) we get

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + c, \text{ where } c \text{ is a constant.}$$

11. (a) (ii) Verify Stoke's theorem for $\vec{F} = \left(x^2 + y^2\right)\vec{i} - 2xy\vec{j}$ taken around the rectangle formed by the lines $x = -a, x = a, y = 0, y = b$.

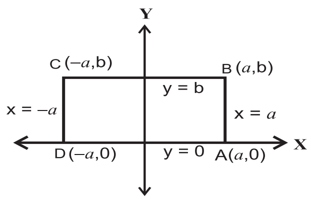
Solution. By Stoke's theorem we have $\oint_C \{\vec{F} \cdot d\vec{r}\} = \iint_S \{curl\vec{F} \cdot \vec{n}\}ds.$

Here, $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(-2y - 2y) = -4y\vec{k}$$

The surface S is given as a rectangle in the xy plane. Hence, $\vec{n} = \vec{k}$.



$$\iint_S \{curl\vec{F} \cdot \vec{n}\}ds = \iint_S -4y\vec{k}.\vec{k}dxdy = -4\int_0^b \int_{-a}^a ydxdy$$

$$= -4\left[\frac{y^2}{2}\right]_0^b [x]_{-a}^a = -2[b^2][a+a] = -4ab^2. \qquad (1)$$

Now, we shall evaluate the line integral $\oint_C \{\vec{F}\cdot d\vec{r}\}$

$\vec{F}\cdot d\vec{r} = [(x^2+y^2)\vec{i} - 2xy\vec{j}].(dx\vec{i} + dy\vec{j}) = (x^2+y^2)dx - 2xydy.$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r} + \int_{DA} \vec{F}\cdot d\vec{r}.$$

<u>On AB</u>: $x = a, dx = 0, y$ varies from 0 to b and $\vec{F}\cdot d\vec{r} = -2aydy$.

$$\therefore \int_{AB} \vec{F}\cdot d\vec{r} = \int_0^b -2aydy = -2a\left(\frac{y^2}{2}\right)_0^b = -ab^2.$$

<u>On BC</u>: $y = b, dy = 0, x$ varies from a to $-a$ and $\vec{F}\cdot d\vec{r} = (x^2+b^2)dx$.

$$\begin{aligned}\therefore \int_{BC} \vec{F}\cdot d\vec{r} = \int_a^{-a} (x^2+b^2)dx &= \left(\frac{x^3}{3}\right)_a^{-a} + b^2(x)_a^{-a}.\\ &= \frac{1}{3}(-a^3 - a^3) + b^2(-a-a)\\ &= -\frac{2a^3}{3} - 2ab^2.\end{aligned}$$

<u>On CD</u>: $x = -a, dx = 0, y$ varies from b to 0 and $\vec{F}\cdot d\vec{r} = 2aydy$.

$$\therefore \int_{CD} \vec{F}\cdot d\vec{r} = \int_b^0 2aydy = 2a\left(\frac{y^2}{2}\right)_b^0 = a(0-b^2) = -ab^2.$$

<u>On DA</u>: $y = 0, dy = 0, x$ varies from $-a$ to a and $\vec{F}\cdot d\vec{r} = x^2dx$.

$$\therefore \int_{DA} \vec{F}\cdot d\vec{r} = \int_{-a}^{a} x^2dx = \left(\frac{x^3}{3}\right)_{-a}^{a} = \frac{1}{3}(a^3+a^3) = \frac{2a^3}{3}.$$

$$\text{Hence, } \int_c \vec{F}\cdot d\vec{r} = -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2. \quad (2)$$

From (1) and (2) we obtain $\oint_C \{\vec{F}\cdot d\vec{r}\} = \iint_S \{curl\vec{F}\cdot\vec{n}\}ds$

Hence, Stoke's theorem is verified.

11. (b) (i) Find the values of a and b so that the surfaces $ax^3 - by^2z = (a+3)x^2$ and $4x^2y - z^3 = 11$ may cut orthogonally at (2,-1,-3).

Solution. Let ϕ_1 and ϕ_2 be the given surfaces.

$$\therefore \phi_1 = ax^3 - by^2z - (a+3)x^2, \text{ and } \phi_2 = 4x^2y - z^3 - 11.$$

$$\begin{aligned}\nabla\phi_1 &= \vec{i}\frac{\partial\phi_1}{\partial x} + \vec{j}\frac{\partial\phi_1}{\partial y} + \vec{k}\frac{\partial\phi_1}{\partial z}\\ &= [3ax^2 - 2(a+3)x]\vec{i} - 2byz\vec{j} - by^2\vec{k}\end{aligned}$$

$$(\nabla\phi_1)_{(2,-1,-3)} = (12a - 4(a+3))\vec{i} - 6b\vec{j} - b\vec{k}$$
$$= (8a-12)\vec{i} - 6b\vec{j} - b\vec{k}.$$
$$\nabla\phi_2 = \vec{i}\frac{\partial\phi_2}{\partial x} + \vec{j}\frac{\partial\phi_2}{\partial y} + \vec{k}\frac{\partial\phi_2}{\partial z}$$
$$\nabla\phi_2 = 8xy\vec{i} + 4x^2\vec{j} - 3z^2\vec{k}.$$
$$(\nabla\phi_2)_{(2,-1,-3)} = -16\vec{i} + 16\vec{j} - 27\vec{k}.$$

Given that the surfaces are orthogonal.

$$\therefore\ \nabla\phi_1.\nabla\phi_2 = 0$$
$$-16(8a-12) + 16(-6b) + 27b = 0$$
$$-128a + 192 - 96b + 27b = 0$$
$$-128a - 69b + 192 = 0$$
$$128a + 69b = 192. \qquad (1)$$
$$(2,-1,-3)\text{lies on } \phi_1.$$
$$\therefore\ 8a + 3b - 4(a+3) = 0$$
$$8a + 3b - 4a - 12 = 0$$
$$4a + 3b = 12 \qquad (2)$$
$$(2)\times 23 \Rightarrow 92a + 69b = 276. \qquad (3)$$
$$(1)-(3) \Rightarrow 36a = -84 \Rightarrow a = \frac{-84}{36} = \frac{-7}{3}.$$

Substituting in (2) we get

$$4\left(-\frac{7}{3}\right) + 3b = 12 \quad \Rightarrow 3b = 12 + \frac{28}{3} = \frac{36+28}{3} = \frac{64}{3} \quad \Rightarrow b = \frac{64}{9}.$$

11. (b) (ii) Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Solution. Refer question 11. (b) of April/May 2015.

2.2.7 May/June 2013 (R 2008)

1. Find the directional derivative of $\phi = xyz$ at $(1, 1, 1)$ in the direction of $\vec{i} + \vec{j} + \vec{k}$.

   Solution. $\phi = xyz$

   $$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = \vec{i}(yz) + \vec{j}(xz) + \vec{k}(xy)$$

   $$(\nabla\phi)_{(1,1,1)} = \vec{i} + \vec{j} + \vec{k}.$$

   Let $\vec{a} = \vec{i} + \vec{j} + \vec{k}$.

   Directional derivative in the direction of $\vec{a} = \nabla\phi.\frac{\vec{a}}{|\vec{a}|}$

   $$(\vec{i} + \vec{j} + \vec{k})\frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{1+1+1}} = \frac{1+1+1}{\sqrt{3}} = \sqrt{3}.$$

2. If $\vec{A}$ and $\vec{B}$ are irrotational, prove that $\vec{A} \times \vec{B}$ is solenoidal.

   Solution. Given that $\vec{A}$ and $\vec{B}$ are irrotational.

   Therefore, $\nabla \times \vec{A} = 0$ and $\nabla \times \vec{B} = 0$.

   Now we have,

   $$\nabla.(\vec{A} \times \vec{B}) = \vec{A}.(\nabla \times \vec{B}) - \vec{B}.(\nabla \times \vec{A}) = \vec{A}.\vec{0} - B.\vec{0} = 0.$$

   $\Longrightarrow \vec{A} \times \vec{B}$ is solenoidal.

Part B

11.(a) Using Stoke's theorem, evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$ where C is the boundary of the triangle with vertices $(0, 0, 0)$, $(1, 0, 0)$ & $(1, 1, 0)$.

Solution. By Stoke's theorem we have $\oint_C \vec{F} \cdot d\vec{r} = \iint_S curl\vec{F} \cdot \widehat{n}ds.$

Here, $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$

$$\text{crul } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(-1-0) + \vec{k}(2x-2y) = \vec{j} + 2(x-y)\vec{k}$$

Now S is the surface of the triangle OAB and since it lies in the xy plane, we have $\widehat{n} = \vec{k}$.

$$\text{Now, curl}\vec{F}\cdot\widehat{n} = \left[\vec{j} + 2(x-y)\vec{k}\right]\cdot\vec{k} = 2(x-y).$$

$$\therefore \iint_S \text{curl } \vec{F}\cdot\widehat{n}ds = 2\iint_S (x-y)dxdy$$

$$= 2\int_{y=0}^{1}\int_{x=y}^{1}(x-y)dxdy$$

$$= 2\int_0^1 \left(\frac{x^2}{2} - yx\right)_{x=y}^{1} dy$$

$$= 2\int_0^1 \left(\frac{1}{2} - y - \frac{y^2}{2} + y^2\right)dy$$

$$= 2\int_0^1 \left(\frac{1}{2} - y + \frac{y^2}{2}\right)dy$$

$$= 2\left[\frac{y}{2} - \frac{y^2}{2} + \frac{y^3}{6}\right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right] = \frac{2}{6} = \frac{1}{3}.$$

11. (b) Verify divergence theorem for $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over the cube formed by the planes $x = \pm1, y = \pm1, z = \pm1$.

Solution.

By Gauss divergence theorem we have $\iint_S \vec{F}\cdot\vec{n}ds = \iiint_V div\vec{F}dv$.

$$\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$$

$$div\vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(yz) = 2x + y$$

$$\iiint_V div\vec{F}dV = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(2x+y)dzdydx = \int_{-1}^{1}\int_{-1}^{1}(2x+y)(z)_{-1}^{1}dydx$$

$$= 2\int_{-1}^{1}\int_{-1}^{1}(2x+y)dydx = 2\int_{-1}^{1}\left(2x(y)_{-1}^{1} + \left(\frac{y^2}{2}\right)_{-1}^{1}\right)dx$$

$$= 2\int_{-1}^{1} 4xdx = 8\left(\frac{x^2}{2}\right)_{-1}^{1} = 8\times 0 = 0 \qquad (1)$$

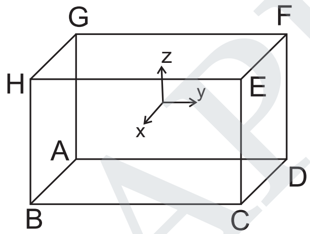
We shall evaluate $\iint_S \vec{F}\cdot\vec{n}ds$.

The given cube has 6 faces

$S_1 = BCFG, S_2 = ADEH,$

$S_3 = CDEF, S_4 = ABGH,$

$S_5 = EFGH, S_6 = ABCD.$



Now $\iint_S \vec{F}\cdot\vec{n}ds = \sum_{i=1}^{6}\iint_{S_i}\vec{F}\cdot\vec{n}ds$

The following table gives the necessary quantities for the computation of the surface integral over each face.

Face	Equation	Outward normal	$\vec{F}\cdot\vec{n}$	ds
S_1	$x=1$	$\vec{i}$	$x^2=1$	$dydz$
S_2	$x=-1$	$-\vec{i}$	$-x^2=-1$	$dydz$
S_3	$y=1$	$\vec{j}$	z	$dxdz$
S_4	$y=-1$	$-\vec{j}$	$-z$	$dxdz$
S_5	$z=1$	$\vec{k}$	$yz=y$	$dxdy$
S_6	$z=-1$	$-\vec{k}$	$-yz=-y$	$dxdy$

$$\iint_{S_1}\vec{F}\cdot\vec{n}ds = \int_{-1}^{1}\int_{-1}^{1}dydz = \int_{-1}^{1}(y)_{-1}^{1}dz$$

$$= 2\int_{-1}^{1}dz = 2(z)_{-1}^{1}$$

$$= 4.\ \iint_{S_2}\vec{F}\cdot\vec{n}ds = -4$$

$$\iint_{S_3} \vec{F}\cdot\vec{n}ds = \int_{-1}^{1}\int_{-1}^{1} zdxdz = 0$$

$$\iint_{S_4} \vec{F}\cdot\vec{n}ds = \int_{-1}^{1}\int_{-1}^{1} (-z)dxdz = 0$$

$$\iint_{S_5} \vec{F}\cdot\vec{n}ds = \int_{-1}^{1}\int_{-1}^{1} ydxdy = 0$$

$$\iint_{S_6} \vec{F}\cdot\vec{n}ds = \int_{-1}^{1}\int_{-1}^{1} (-y)dxdy = 0.$$

$$\therefore \iint_{S} \vec{F}\cdot\vec{n}ds = 4 - 4 + 0 + 0 = 0. \qquad (2)$$

From (1) and (2) we get

$$\iint_{S} \vec{F}\cdot\vec{n}ds = \iiint_{V} div\vec{F}dV.$$

Hence, Gauss Divergence theorem is verified.

2.2.8 November/December 2012 (R 2008)

Part A

1. Prove that $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is irrotational.

   Solution. $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$

   $= \vec{i}(x-x) - \vec{j}(y-y) + \vec{k}(z-z) = \vec{0}.$

   $\therefore \vec{F}$ is irrotational.

2. State Gauss Divergence theorem.

   Solution. Let V be the volume bounded by a closed surface S. If a vector function $\vec{F}$ is continuous and has continuous partial derivatives inside and on S, then the surface integral of $\vec{F}$ over S is equal to the volume integral of the divergence of $\vec{F}$ taken throughout V.

i.e., $\iint_S \vec{F}\cdot\vec{n}ds = \iiint_V \nabla\cdot\vec{F}dv.$

Part B

11. (a) (i) Show that $\vec{F} = (2xy-z^2)\vec{i}+(x^2+2yz)\vec{j}+(y^2-2zx)\vec{k}$ is irrotational and hence find its scalar potential.

Solution. Given $\vec{F} = (2xy-z^2)\vec{i}+(x^2+2yz)\vec{j}+(y^2-2zx)\vec{k}$.

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy-z^2 & x^2+2yz & y^2-2zx \end{vmatrix}$$

$$= \vec{i}(2y-2y) - \vec{j}(-2z+2z) + \vec{k}(2x-2x) = \vec{0}.$$

$\therefore$ $\vec{F}$ is irrotational.

Since $\nabla\times\vec{F} = \vec{0}, \vec{F} = \nabla\phi$, where ϕ is a scalar potential.

$$\therefore \vec{F} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}.$$

Comparing the components of $\vec{F}$ we get

$$\frac{\partial\phi}{\partial x} = 2xy - z^2. \qquad (1)$$

$$\frac{\partial\phi}{\partial y} = x^2 + 2yz. \qquad (2)$$

$$\frac{\partial\phi}{\partial z} = y^2 - 2zx. \qquad (3)$$

Integrating (1), (2) and (3) partially w.r.t. x, y and z respectively we get

$$\phi = \int (2xy - z^2)dx + f_1(y,z)$$
$$= 2y \cdot \frac{x^2}{2} - xz^2 + f_1(y,z)$$
$$= x^2y - xz^2 + f_1(y,z). \tag{4}$$
$$\phi = \int (x^2 + 2yz)dy + f_2(x,z)$$
$$= x^2y + 2z \cdot \frac{y^2}{2} + f_2(x,z) = x^2y + y^2z + f_2(x,z). \tag{5}$$
$$\phi = \int (y^2 - 2zx)dz + f_3(x,y)$$
$$= zy^2 - 2x \cdot \frac{z^2}{2} + f_3(x,y)$$
$$= zy^2 - xz^2 + f_3(x,y). \tag{6}$$

From (4), (5) and (6) we get

$\phi = x^2y - xz^2 + y^2z + c$, where c is a constant.

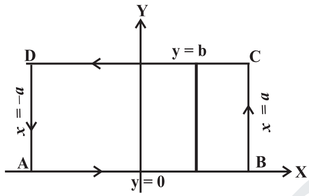
11. (a) (ii) Verify Green's theorem for $\vec{V} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$ taken around the rectangle bounded by the lines $x = \pm a, y = 0, y = b$.

Solution. Consider the line integral $\oint_C \vec{V} \cdot d\vec{r}$ where C is the given rectangle.

$$\therefore \oint_C \vec{V} \cdot d\vec{r} = \oint_C [(x^2 + y^2)dx - 2xydy]$$
$$= \oint_C (Pdx + Qdy) \quad \text{where } P = x^2 + y^2, Q = -2xy.$$

By Green's theorem $\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$

$$\frac{\partial P}{\partial y} = 2y, \ \frac{\partial Q}{\partial x} = -2y.$$



Now, $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

$$= \iint_R (-2y - 2y)dxdy$$

$$= -\iint_R 4ydxdy$$

$$= -\int_{x=-a}^{a}\int_{y=0}^{b} 4ydydx = -4\int_{x=-a}^{a}\left[\frac{y^2}{2}\right]_0^b dx$$

$$= -4\int_{-a}^{a}\frac{b^2}{2}dx = -2b^2[x]_{-a}^{a} = -2b^2[a+a] = -4ab^2. \qquad (1)$$

Now let us evaluate $\int_C (Pdx + Qdy)$.

$$\int_C (Pdx+Qdy) = \int_{AB}(Pdx+Qdy) + \int_{BC}(Pdx+Qdy)$$

$$+ \int_{CD}(Pdx + Qdy) + \int_{DA}(Pdx + Qdy).$$

Along $AB, y = 0, dy = 0$ and x varies from $-a$ to a.

$$\int_{AB}(Pdx + Qdy) = \int_{-a}^{a} x^2dx = \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{1}{3}(a^3 + a^3) = \frac{2a^3}{3}.$$

Along $BC, x = a, dx = 0$ and y varies from 0 to b.

$$\int_{BC}(Pdx + Qdy) = \int_0^b -2aydy = -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2.$$

Along $CD, y = b, dy = 0$ and x varies from a to $-a$.

$$\therefore \int_{CD} (Pdx + Qdy) = \int_{a}^{-a} (x^2 + b^2)dx = \left[\frac{x^3}{3} + b^2 x\right]_a^{-a}$$

$$= -\frac{a^3}{3} - ab^2 - \left(\frac{a^3}{3} + ab^2\right)$$

$$= -\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = -\frac{2a^3}{3} - 2ab^2.$$

Along $DA, x = -a, dx = 0$ and y varies from b to 0.

$$\therefore \int_{DA} (Pdx + Qdy) = \int_{b}^{0} 2aydy = 2a\left[\frac{y^2}{2}\right]_b^0 = a(0 - b^2) = -ab^2.$$

$$\therefore \int_C (Pdx + Qdy) = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 = -4ab^2. \quad (2)$$

From (1) and (2) we have

$$\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

$\therefore$ Green's theorem is verified.

11. (b) Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Solution. Refer question 11. (b) of April/May 2015.

2.2.9 May/June 2012 (R 2008)

Part A

1. Find λ such that $\vec{F} = (3x - 2y + z)\vec{i} + (4x + \lambda y - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal.

   Solution. Refer question 2. of April/May 2015.

2. State Gauss Divergence theorem.

   Statement. Refer question 2. of Nov./Dec. 2012.

Part B

11. (a) (i) Show that $\vec{F} = (y^2 + 2xz^2)\vec{i} + (2xy - z)\vec{j} + (2x^2z - y + 2z)\vec{k}$ is irrotational and hence find its scalar potential.

Solution.

Given that $\vec{F} = (y^2 + 2xz^2)\vec{i} + (2xy - z)\vec{j} + (2x^2z - y + 2z)\vec{k}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + 2xz^2 & 2xy - z & 2x^2z - y + 2z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(2x^2z - y + 2z) - \frac{\partial}{\partial z}(2xy - z)\right)$$
$$- \vec{j}\left(\frac{\partial}{\partial x}(2x^2z - y + 2z) - \frac{\partial}{\partial z}(y^2 + 2xz^2)\right)$$
$$+ \vec{k}\left(\frac{\partial}{\partial x}(2xy - z) - \frac{\partial}{\partial y}(y^2 + 2xz^2)\right)$$
$$= \vec{i}(-1 + 1) - \vec{j}(4xz - 4xz) + \vec{k}(2y - 2y)$$
$$= \vec{i}.0 - \vec{j}.0 + \vec{k}.0 = \vec{0}.$$

$\therefore \vec{F}$ is irrotational.

Since $\nabla \times \vec{F} = \vec{0}$, then $\vec{F} = \nabla\phi$ where ϕ is the scalar potential.

$$\therefore \vec{F} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}.$$

Comparing the components of $\vec{F}$ we obtain

$$\frac{\partial\phi}{\partial x} = y^2 + 2xz^2 \qquad (1)$$
$$\frac{\partial\phi}{\partial y} = 2xy - z \qquad (2)$$
$$\frac{\partial\phi}{\partial z} = 2x^2z - y + 2z. \qquad (3)$$

Integrating partially (1), (2), (3) w.r.t. x, y and z respectively we get

$$\phi = \int (y^2 + 2xz^2)dx + f_1(y,z)$$
$$= y^2x + 2z^2\frac{x^2}{2} + f_1(y,z) = xy^2 + x^2z^2 + f_1(y,z). \qquad (4)$$

From (2) we get

$$\phi = \int (2xy - z)dy + f_2(x,z)$$
$$= 2x\frac{y^2}{2} - zy + f_2(x,z)$$
$$= xy^2 - yz + f_2(x,z) \qquad (5)$$

From (3) we obtain

$$\phi = \int (2x^2z - y + 2z)dz + f_3(x,y)$$
$$= 2x^2\frac{z^2}{2} - yz + 2\frac{z^2}{2} + f_3(x,y)$$
$$= x^2z^2 - yz + z^2 + f_3(x,y). \qquad (6)$$

From (4), (5) and (6) we get

$\therefore \phi = xy^2 + x^2z^2 - yz + z^2 + c$, where c is a constant.

11. (a) (ii) Verify Green's theorem in the plane for $\int_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ where C is the boundary of the region bounded by $x = 0, y = 0, x + y = 1$.

Solution. By Green's theorem,

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$
$$P = 3x^2 - 8y^2 \qquad Q = 4y - 6xy.$$
$$\frac{\partial P}{\partial y} = -16y \qquad \frac{\partial Q}{\partial x} = -6y.$$

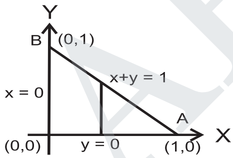
$$\text{Now, } \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint_R (-6y + 16y)dxdy = \iint_R 10ydxdy$$
$$= 10 \iint_R ydydx = 10 \int_{x=0}^{1} \int_{y=0}^{1-x} ydydx$$

$$= 10 \int_{x=0}^{1} \left(\frac{y^2}{2}\right)_0^{1-x} dx = \frac{10}{2} \int_0^1 (1-x)^2 dx$$

$$= 5\left[\frac{(1-x)^3}{-3}\right]_0^1 = -\frac{5}{3}[0-1] = \frac{5}{3}. \qquad (1)$$

Now, $\oint_C (Pdx + Qdy)$

$$= \int_{OA} (Pdx + Qdy) + \int_{AB} (Pdx + Qdy) + \int_{BO} (Pdx + Qdy).$$



Along OA, $y = 0 \Rightarrow dy = 0$ and x varies from 0 to 1.

$$\therefore \int_{OA} (Pdx + Qdy) = \int_0^1 3x^2 dx$$

$$= 3\left(\frac{x^3}{3}\right)_0^1 = 1.$$

Along AB, $y = 1 - x$, $dy = -dx$, x varies from 1 to 0

$$\therefore \int_{AB} (Pdx + Qdy)$$

$$= \int_1^0 (3x^2 - 8(1-x)^2)dx + (4(1-x) - 6x(1-x))(-dx)$$

$$= \int_1^0 \{3x^2 - 8 - 8x^2 + 16x - (4 - 4x - 6x + 6x^2)\}dx$$

$$= \int_1^0 (-5x^2 + 16x - 8 - 4 + 10x - 6x^2)dx$$

$$= \int_1^0 (-11x^2 + 26x - 12)dx = -11\left(\frac{x^3}{3}\right)_1^0 + 26\left(\frac{x^2}{2}\right)_1^0 - 12\,(x)_1^0$$

$$= -\frac{11}{3}(0-1) + 13(0-1) - 12(0-1) = \frac{11}{3} - 13 + 12 = \frac{11}{3} - 1 = \frac{8}{3}.$$

Along BO, $x = 0 \Rightarrow dx = 0$, y varies from 1 to 0.

$$\int_{BO} (Pdx + Qdy) = \int_1^0 4ydy = \left[4\frac{y^2}{2}\right]_1^0 = 2[0-1] = -2.$$

$$\therefore \oint_C (Pdx + Qdy) = 1 + \frac{8}{3} - 2 = \frac{8}{3} - 1 = \frac{5}{3}. \qquad (2)$$

From (1) and (2) The Green's theorem is verified.

11. (b) (i) Using Stoke's theorem, evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$ where C is the boundary of the triangle with vertices $(0, 0, 0)$, $(1, 0, 0)$ & $(1, 1, 0)$.

Solution. Refer problem 11. (a) of May/June 2013 (R2008).

11. (b) (ii) Find the work done in moving a particle in the vector field $\vec{A} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ along the straight line from $(0, 0, 0)$ to $(2, 1, 3)$.

Solution. Given $\vec{A} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$.

The equation of the straight line joining $(0, 0, 0)$ and $(2, 1, 3)$ is

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t\ (say)$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t.$$

$$\therefore\ x = 2t,\ y = t,\ z = 3t.$$

$$dx = 2dt,\ dy = dt,\ dz = 3dt.$$

$$\text{Now } \vec{A} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}.$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}.$$

$$\begin{aligned} \vec{A}.d\vec{r} &= 3x^2dx + (2xz - y)dy + zdz \\ &= 3.4t^2.2dt + (2.2t.3t - t)dt + 3t.3dt \\ &= 24t^2dt + (12t^2 - t)dt + 9tdt = (36t^2 + 8t)dt. \end{aligned}$$

To find the limits for t.

when $x = 0, t = 0$.

when $x = 2, t = 1$.

$$\begin{aligned} \therefore \text{ Now, work done } &= \int_c \vec{A}.d\vec{r} = \int_0^1 (36t^2 + 8t)dt \\ &= 36.\left(\frac{t^3}{3}\right)_0^1 + 8.\left(\frac{t^2}{2}\right)_0^1 \\ &= 36.\frac{1}{3} + 8.\frac{1}{2} = 12 + 4 = 16. \end{aligned}$$

UNIT 3

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3 Analytic Functions

3.1 Introduction

Any number of the form $x + iy$ where $i = \sqrt{-1}$ and $x, y \in \mathbb{R}$ is called complex number. x is called the real part and y is called the imaginary part of the complex number.

If $z = x + iy$ is any complex number, then $Re(z) = x, Im(z) = y$.

Results

1. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then $z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.
2. $z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$.
3. $z_1 - z_2 = x_1 - x_2 + i(y_1 - y_2)$.
4. $z_1z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$.
5. $\dfrac{z_1}{z_2} = \dfrac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\dfrac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$.
6. Comparison of complex numbers such as $z_1 < z_2$ or $z_1 > z_2$ is not possible.
7. Always $\mathbb{C}$ denotes the set of all complex numbers.

Conjugate of a Complex number. If $z = x + iy$ is any complex number then, $x - iy$ is defined to be the complex conjugate of z. It is denoted by $\bar{z}$. Hence, $\bar{z} = x - iy$.

Results

1. $z = \bar{z}$ if and only if z is real.

2. $\bar{\bar{z}} = z$.

3. $\dfrac{z + \bar{z}}{2} = Re(z)$.

4. $\dfrac{z - \bar{z}}{2i} = Im(z)$.

5. $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$.

6. $\overline{z_1 - z_2} = \bar{z_1} - \bar{z_2}$.

7. $\overline{z_1 z_2} = \bar{z_1}.\bar{z_2}$.

8. $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z_1}}{\bar{z_2}}$.

Geometrical representation of a complex number

If $z = x + iy$ is a complex number, then z can be represented as a point P with coordinates (x, y) in the Argand diagram. With this notation, we can represent addition, subtraction, multiplication, division and conjugate as follows. If $z = x + iy, z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

1. $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$.

2. $z_1 - z_2 = (x_1 - x_2, y_1 - y_2)$.

3. $z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$.

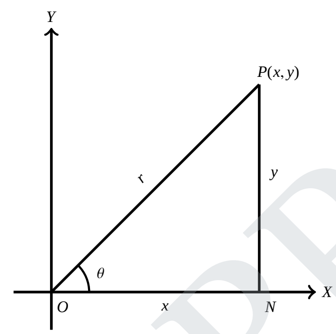
4. $\left(\dfrac{z_1}{z_2}\right) = \left(\dfrac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \dfrac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right)$.

5. $\bar{z} = (x, -y)$.

Modulus and amplitude form of a complex number

Let $z = x + iy$ be a complex number.

In the argand plane let $P(x, y)$ represents z. Join OP. Let $OP = r$. Let OP makes an angle θ with the positive direction of the real axis. Then, r is called the modulus and θ is called the amplitude (or argument) of z. The modulus of z is denoted by $|z|$ and amplitude of z is denoted by $arg(z)$.



i.e., $|z| = r, arg(z) = \theta$.

Let PN be drawn perpendicular to the real axis.

Then, $ON = x, NP = y$ and $OP = r$.

From the right angled triangle OPN,

$$\cos\theta = \frac{ON}{OP} = \frac{x}{r} \Longrightarrow x = r\cos\theta. \tag{1}$$

$$\sin\theta = \frac{NP}{OP} = \frac{y}{r} \Longrightarrow y = r\sin\theta. \tag{2}$$

Now, $z = x + iy = r\cos\theta + i(r\sin\theta)$.

$z = r(\cos\theta + i\sin\theta)$.

This expression represents the modulus amplitude form of the complex number z.

Squaring and adding (1) and (2) we obtain

$$x^2 + y^2 = r^2 \Longrightarrow r = \sqrt{x^2 + y^2}$$

$$\Longrightarrow |z| = \sqrt{x^2 + y^2}. \tag{3}$$

$$\frac{(2)}{(1)} \Longrightarrow \tan\theta = \frac{y}{x} \Longrightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right). \tag{4}$$

(3) and (4) gives a formula to find the modulus and argument of a given complex number z.

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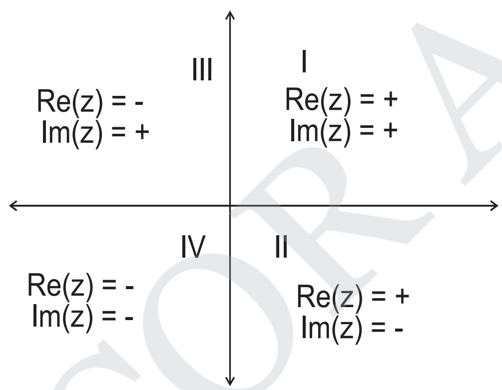
For finding the modulus of z, we can always use (3). But (4) is true only when x and y are positive. (i.e.,) if z lies in the first quadrant. If z lies in other quadrants we cannot use (4) to find $arg(z)$. One can adopt the following procedure to find the correct value of $arg(z)$.

Step (i). Without considering the sign of the real and imaginary parts, find $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

Step (ii). Depending on the quadrant in which z lies, evaluate $arg(z)$ in the following way.



Principal value. The value of the argument that lies between π and $-\pi$ is called the principal value of the argument.

The following results can be easily proved with respect to the modulus and argument of complex numbers.

Let z_1 and z_2 be two complex numbers. Then

1. $|z_1 z_2| = |z_1|.|z_2|$ and $arg(z_1 z_2) = arg(z_1) + arg(z_2)$.
2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ if $z_2 \neq 0$ and $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$.
3. $|\overline{z}| = |z|$.
4. $z\overline{z} = |z|^2$.

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5. $|z| \geq |Re(z)|$.

6. $|z| \geq |Im(z)|$.

Triangle inequality. The following two relations are true and called triangle inequalities

1. $|z_1 + z_2| \leq |z_1| + |z_2|$.

2. $|z_1 - z_2| \geq |z_1| - |z_2|$.

Result. Using the Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, the modulus - amplitude form of z can be expressed as $z = re^{i\theta}$.

Extended complex number system

Let $\mathbb{C}$ be the set of all complex numbers. The extended complex number system is $\mathbb{C}$ together with ∞ which satisfies the following properties.

(i) If $z \in \mathbb{C}$, then $z + \infty = \infty$, $z - \infty = -\infty$ and $\frac{z}{\infty} = 0$.

(ii) If z is a nonzero complex number, then $z.\infty = \infty$, $\frac{z}{0} = \infty$.

(iii) $\infty + \infty = \infty$, $\infty.\infty = \infty$.

(iv) $\frac{\infty}{z} = \infty$ if $z \in \mathbb{C}$.

The extended complex number system $\mathbb{C} \cup \{\infty\}$ when represented in the plane geometrically is called the extended complex plane and ∞ is known as the point at infinity.

Function of a complex variable. Let A be a set of complex numbers. Let f be a function from A to C be a rule that assigns to each $z \in A$, a unique complex number $w \in \mathbb{C}$. w is the value of f at z denoted by $w = f(z)$. Then, w is called a function of the complex variable z. A is called the domain of f and C is called the codomain of f. The collection of all values of f is called the range of f. Since $z = x + iy$ is a complex variable, the complex function $w = f(z)$ can be considered as having real and imaginary parts u and v respectively such that u and v are functions of x and y. Hence, we have $w = f(z) = u(x, y) + iv(x, y)$.

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Examples

1. $f(z) = z^2$ is a complex function. ie., $f(z) = (x+iy)^2 = x^2 - y^2 + 2ixy$.
   $\Longrightarrow u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$.

2. e^z is a complex function.
   $e^z = e^{x+iy} = e^x.e^{iy} = e^x(\cos y + i\sin y) \Longrightarrow u(x,y) = e^x\cos y, v(x,y) = e^x\sin y$.

3. $\sin z$ is a complex function.
   $\sin z = \sin(x+iy) = \sin x\cos(iy) + \cos x\sin(iy) = \sin x\cosh y + i\cos x\sinh y$.
   $u(x,y) = \sin x\cosh y, v = \cos x\sinh y$.

Neighbourhood of a point

Let z_0 be a point in the complex plane. The neighbourhood of z_0 is the set of all points in the complex plane inside the circle with centre z_0.
i.e. the interior of the circle $|z - z_0| = r$.
The interior is an open circular disc.
A δ neighbourhood of z_0 is the open circular disc $|z - z_0| < \delta$.
A set is open if it contains none of its boundary points. eg. $|z| < 1$.

Limit of a function. Let f be a function defined in some neighbourhood of z_0 except possibly at z_0. We say a complex number w_0 is the limit of $f(z)$ as z tends to z_0 if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ for $|z - z_0| < \delta$. Symbolically it can be written as $\lim\limits_{z\to z_0} f(z) = w_0 = f(z_0)$.

Result. State the basic difference between the limit of a real variable and that of a complex variable. [Jun 2012]
Solution. In the case of the real variable, x approaches x_0 only along the line, whereas in complex variable, z approaches z_0 in any direction.

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Note

1. When the limit exists it is unique in whatever direction z approaches z_0.

2. If $f(z) = u(x, y) + iv(x, y)$ is defined in a neighbourhoood of $z_0 = x_0 + iy_0$, then $\lim_{z \to z_0} f(z) = u_0 + iv_0$ if and only if $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$.

3. $\lim_{z\to\infty} f(z) = w$ if $\lim_{z\to 0} f(\frac{1}{z}) = w$.
   $\lim_{z\to z_0} f(z) = \infty$ if $\lim_{z\to z_0} \frac{1}{f(z)} = 0$.
   $\lim_{z\to\infty} f(z) = \infty$ if $\lim_{z\to 0} \frac{1}{f(\frac{1}{z})} = 0$.

Continuity of a function. Let f be a function defined in a neighbourhood of z_0 (including z_0). f is continuous at the point z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$.

A function f is continuous in a region R of the complex plane if f is continuous at each point of R.

Derivative of $f(z)$. Let f be a function defined in a neighbourhood of z_0. The derivative of f at z_0 is $f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ if the limit exists.

If $f'(z_0)$ exists, we say that the function is differentiable at z_0.

Note

(1) A function f is differentiable at z_0, then it is continuous at z_0. But the converse is not true.

i.e. A function f is continuous at z_0 then it need not be differentiable.

Example. Let $z = x + iy$.

Consider $f(z) = \bar{z} = x - iy, u = x, v = -y$.

Consider the origin $(0, 0)$.

$f(0) = 0$.

$\lim_{(x,y)\to(0,0)} u(x, y) = \lim_{(x,y)\to(0,0)} x = 0.$
$\lim_{(x,y)\to(0,0)} v(x, y) = \lim_{(x,y)\to(0,0)} (-y) = 0.$

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$\lim_{z\to 0} f(z) = \lim_{(x,y)\to 0} u(x,y) - i \lim_{(x,y)\to 0} v(x,y) = 0 - i0 = 0 = f(0).$

$\therefore$ $f(z)$ is continuous at the point $z = 0$.

Now $\lim_{z\to 0} \dfrac{f(z) - f(z_0)}{z - 0} = \lim_{z\to 0} \dfrac{\bar{z} - 0}{z - 0} = \lim_{z\to 0} \dfrac{\bar{z}}{z}$.

Let us choose the path $z \to 0$ along $y = mx$.

$$\begin{aligned}
\lim_{z\to 0} \frac{\bar{z}}{z} &= \lim_{x\to 0,\ y\to 0} \frac{x - im\,x}{x + im\,x} \\
&= \lim_{x\to 0\ y\to 0} \frac{1 - im}{1 + im} \\
&= \frac{1 - im}{1 + im},
\end{aligned}$$

which is depending on m.

$\therefore$ The limit is not unique.

$\therefore$ $f(z) = \bar{z}$ is not differentiable at $z = 0$.

$\therefore$ $f(z)$ continuous $\not\Rightarrow$ $f(z)$ is differentiable.

Differentiation Formulae. If $f(z)$ and $g(z)$ are differentiable at z and c is a constant then

1. $\dfrac{d}{dz}(c f(z)) = c f'(z).$

2. $\dfrac{d}{dz}(f(z) \pm g(z)) = f'(z) \pm g'(z).$

3. $\dfrac{d}{dz}(f(z).g(z)) = f(z)g'(z) + g(z)f'(z).$

4. $\dfrac{d}{dz}\left(\dfrac{f(z)}{g(z)}\right) = \dfrac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2}.$

5. $\dfrac{d}{dz}(f(g(z))) = f'(g(z))g'(z).$

6. $\dfrac{d}{dz}(z^n) = nz^{n-1}.$

3.2 Analytic Function

A complex function $f(z)$ is said to be analytic at a point z_0 if $f(z)$ is differentiable at z_0 and at every point of some neighbourhood of z_0.

A function is analytic in a domain D if it is analytic at each point of D.

Note

An analytic function is also known as a regular function or holomorphic function.

3.3 Necessary and sufficient conditions for $f(z)$ to be analytic

Theorem

The necessary and sufficient conditions for the function $f(z) = u + iv$ to be analytic in a domain D are

(i) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are continuous functions of x and y in the domain D.

(ii) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

i.e. $u_x = v_y$ and $u_y = -v_x$.

The conditions $u_x = v_y$ and $u_y = -v_x$ are called Cauchy - Riemann equations (or) C-R equations.

Proof. Necessary condition

Let $w = f(z) = u + iv$ be analytic. Then, by the definition of analytic function $\dfrac{dw}{dz} = f'(z)$ exists and it is unique at every point.

Let Δx and Δy be the small incremental changes in x and y respectively. Then the corresponding changes in u, v and z will be $\Delta u, \Delta v$ and Δz respectively.

$$\begin{aligned}
\therefore\ f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{u + \Delta u + i(v + \Delta v) - (u + iv)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{u + \Delta u + iv + i\Delta v - u - iv}{\Delta z}
\end{aligned}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta u + i\Delta v}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \left(\frac{\Delta u}{\Delta z} + i \frac{\Delta v}{\Delta z} \right).$$

Consider the path, a line parallel to the x-axis along which $\Delta z \to 0$. Then y = constant and hence $\Delta y = 0$.

$$\therefore \Delta z = \Delta x + i\Delta y = \Delta x.$$
$$\therefore f'(z) = \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x} + i \frac{\Delta v}{\Delta x} \right)$$
$$= \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + i \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$$
$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}. \tag{1}$$

Now, consider the path, a line parallel to the y- axis along which $\Delta z \to 0$. Then, x = constant and hence $\Delta x = 0$.

$$\therefore \Delta z = \Delta x + i\Delta y = i\Delta y.$$
$$\text{Now, } f'(z) = \lim_{\Delta y \to 0} \left(\frac{\Delta u}{i\Delta y} + i \frac{\Delta v}{i\Delta y} \right)$$
$$= \frac{1}{i} \lim_{\Delta y \to 0} \frac{\Delta u}{\Delta y} + \lim_{\Delta y \to 0} \frac{\Delta v}{\Delta y}$$
$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{2}$$

From (1) and (2) we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\text{i.e., } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Sufficient conditions

Let us assume that $f(z) = u + iv$ which satisfies

1. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous.

2. $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

We have to prove that $f(z)$ is analytic.

We have, $f(z + \Delta z) = u + \Delta u + i(v + \Delta v)$.

By Taylor's theorem we have

$$f(z+\Delta z) = u + iv + \left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right) + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right) + \cdots$$

Since Δx and Δy are very small, omitting higher powers of Δx and Δy, we obtain

$$\begin{aligned} f(z+\Delta z) &= u + iv + \left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right) + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right) \\ &= f(z) + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + i\frac{\partial v}{\partial x}\Delta x + i\frac{\partial v}{\partial y}\Delta y \\ &= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y \end{aligned}$$

$$f(z+\Delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y.$$

Using C-R equations, we replace $\dfrac{\partial u}{\partial y}$ by $-\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ by $\dfrac{\partial u}{\partial x}$ we obtain

$$\begin{aligned} f(z+\Delta z) - f(z) &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + i\left(\frac{\partial u}{\partial x} - \frac{1}{i}\frac{\partial v}{\partial x}\right)\Delta y \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta y \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta z \end{aligned}$$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Taking limit as $\Delta z \to 0$ we get

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

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$\therefore f'(x)$ exists and it is unique.

Hence, $f(z)$ is analytic.

C-R equations in polar form

We know that $x = r\cos\theta, y = r\sin\theta$

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$f(z) = u + iv = f(re^{i\theta}) \tag{1}$$

Differentiating (1) partially w.r.t. r we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})e^{i\theta}. \tag{2}$$

Differentiating (1) partially w.r.t. θ we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})rie^{i\theta}$$

$$\frac{1}{ir}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta} = f'(re^{i\theta})e^{i\theta}$$

$$\frac{-i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}.$$

Equating the real and imaginary parts we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \ \& \ \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \ \& \ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r},$$

which are the C-R equations in polar form.

Entire function

A complex function f is said to be an entire function if it is analytic in the entire complex plane.

Example

A polynomial function $f(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n, a_n \neq 0$ is an entire function, since it is differentiable everywhere in the plane.

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Worked Examples

Example 3.1. Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

[Dec 2012, Dec 2010, Jan 2001]

Solution. Let $f(z) = u + iv = \bar{z} = x - iy$.

$$\therefore u = x, \qquad v = -y$$

$$u_x = 1, \qquad v_x = 0$$

$$u_y = 0, \qquad v_y = -1.$$

Since $u_x \neq v_y$, C.R equations are not satisfied.

$\therefore$ $f(z)$ is not analytic anywhere.

$\Longrightarrow$ $f(z)$ is nowhere differentiable.

Example 3.2. Prove that $f(z) = \sin z$ is analytic. [Jun 2004]

Solution. $f(z) = \sin z$.

$$u + iv = \sin(x + iy)$$

$$= \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y$$

$$\therefore u = \sin x \cosh y, \qquad v = \cos x \sinh y$$

$$u_x = \cos x \cosh y, \qquad v_x = -\sin x \sinh y$$

$$u_y = \sin x \sinh y, \qquad v_y = \cos x \cosh y.$$

Since $u_x = v_y$ and $u_y = -v_x$, C-R equations are satisfied for all x and y.

$\therefore$ $f(z)$ is analytic.

Example 3.3. Test whether the function $f(z) = z^2$ is analytic or not. [May 2007]

Solution.

$$f(z) = u + iv = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

$$\therefore u = x^2 - y^2, \qquad v = 2xy.$$

$$u_x = 2x, \qquad v_x = 2y.$$

$$u_y = -2y, \qquad v_y = 2x.$$

The above equations imply $u_x = v_y$ and $u_y = -v_x$.

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$\Rightarrow$ C-R equations are satisfied for all x and y and the partial derivatives being polynomials in x and y are continuous everywhere in the complex plane.

$\therefore$ $f(z)$ is analytic in the entire plane. $f(z)$ is an entire function.

Example 3.4. Test whether the function $f(z) = 2xy + i(x^2 - y^2)$ is analytic or not.

[Apr 2002]

Solution. $f(z) = u + iv = 2xy + i(x^2 - y^2)$.

$u = 2xy \qquad v = x^2 - y^2.$

$u_x = 2y \qquad v_x = 2x.$

$u_y = 2x \qquad v_y = -2y.$

$u_x \neq v_y \qquad u_y \neq -v_x.$

$\therefore$ C-R equations are not satisfied.

Therefore, $f(z)$ is not analytic at any point.

Example 3.5. Test whether the function $f(z) = |z|^2$ is analytic or not.

[May 2015, Jan 2006]

Solution. $f(z) = u + iv = |z|^2 = x^2 + y^2$.

$\therefore u = x^2 + y^2 \qquad v = 0.$

$u_x = 2x \qquad v_x = 0.$

$u_y = 2y \qquad v_y = 0.$

At $(0,0)$ $u_x = v_y = 0$ and $u_y = -v_x = 0$.

$\Rightarrow$ $f(z)$ is differentiable only at the origin.

But for all $(x, y) \neq (0,0)$, C-R equations are not satisfied.

Therefore, $f(z)$ is not analytic.

Example 3.6. Check for analyticity of $\log z$.

Solution. Let $f(z) = \log z = \log re^{i\theta} = \log r + \log e^{i\theta} = \log r + i\theta$.

$$u = \log r, \qquad v = \theta.$$

$$u_r = \frac{1}{r}, \qquad v_r = 0.$$

$$u_\theta = 0, \qquad v_\theta = 1.$$
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

$\therefore$ C-R equations are satisfied and partial derivatives are continuous for $r \neq 0$.

$\therefore$ $f(z)$ is analytic except for $r = 0$ or $z = 0$.

$\therefore$ $\log z$ is analytic for all $z \neq 0$.

Example 3.7. Show that $f(z) = z^n$ is analytic.

Solution. Changing into polar coordinates we get,

$$f(z) = u + iv = \left(re^{i\theta}\right)^n = r^n e^{in\theta}$$
$$= r^n(\cos n\theta + i\sin n\theta)$$
$$\therefore u = r^n \cos n\theta \qquad v = r^n \sin n\theta.$$
$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta \qquad \frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta$$
$$\frac{\partial u}{\partial \theta} = -nr^n \sin n\theta \qquad \frac{\partial v}{\partial \theta} = nr^n \cos n\theta.$$

From the above equations we observe that,

(i) the first partial derivatives of u and v with respect to r and θ exist,

(ii) all the partial derivatives are continuous and

(iii) $r\dfrac{\partial u}{\partial r} = nr^n \cos n\theta = \dfrac{\partial v}{\partial \theta}$.

$\Rightarrow \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$.

$-r\dfrac{\partial v}{\partial r} = -nr^n \sin n\theta = \dfrac{\partial u}{\partial \theta}$.

Hence, C-R equations are satisfied.

$\therefore$ $f(z) = z^n$ is analytic.

Example 3.8. Prove that $f(z) = e^z$ is analytic and find its derivative. [Dec 2004]

Solution. $f(z) = u + iv = e^z = e^{x+iy}$

$$= e^x.e^{iy}$$
$$= e^x(\cos y + i\sin y)$$
$$= e^x \cos y + ie^x \sin y.$$

—

$$\therefore u = e^x \cos y, \quad v = e^x \sin y.$$
$$u_x = e^x \cos y, \quad v_x = e^x \sin y.$$
$$u_y = -e^x \sin y, \quad v_y = e^x \cos y.$$

Hence, $u_x = v_y$ and $u_y = -v_x$.

$\therefore$ C-R equations are satisfied.

$\therefore$ $f(z)$ is analytic.

$$\begin{aligned}
\text{Also } f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\
&= e^x \cos y + ie^x \sin y \\
&= e^x(\cos y + i \sin y) \\
&= e^x e^{iy} \\
&= e^z.
\end{aligned}$$

Example 3.9. Show that $f(z) = \dfrac{1}{z-1}$ is analytic at $z = 1 + i$.

Solution. We have to show that u_x, u_y, v_x, v_y are continuous in some neighbourhood of $z = (1, 1)$ and the C-R equations are satisfied in this neighbourhood.

$$f(z) = \frac{1}{z-1} = \frac{1}{x+iy-1} = \frac{1}{x-1+iy}$$
$$u + iv = \frac{x-1-iy}{(x-1)^2+y^2} = \frac{x-1}{(x-1)^2+y^2} - i\frac{y}{(x-1)^2+y^2}.$$
$$\therefore u = \frac{x-1}{(x-1)^2+y^2}, v = \frac{-y}{(x-1)^2+y^2}.$$

Since $u(x, y)$ and $v(x, y)$ are rational functions of the real variables x and y and are defined at $(1, 1)$ and in a neighbourhood of $(1, 1)$ u_x, u_y, v_x, v_y exist and are continuous.

$$u_x = \frac{(x-1)^2 + y^2 - (x-1)2(x-1)}{((x-1)^2+y^2)^2} = \frac{y^2-(x-1)^2}{((x-1)^2+y^2)^2}$$

$$u_y = \frac{\{(x-1)^2+y^2\}0 - (x-1)2y}{((x-1)^2+y^2)^2} = \frac{-2y(x-1)}{((x-1)^2+y^2)^2}$$

—

$$v_x = -y(-1)((x-1)^2 + y^2)^{-2} 2(x-1) = \frac{2y(x-1)}{((x-1)^2 + y^2)^2}$$

$$v_y = \frac{((x-1)^2 + y^2)(-1) + y2y}{((x-1)^2 + y^2)^2} = \frac{y^2 - (x-1)^2}{((x-1)^2 + y^2)^2}$$

$\therefore u_x = v_y$ and $u_y = -v_x$.

$\therefore$ C-R equations are satisfied.

$\therefore f(z)$ is analytic.

Example 3.10. Verify whether $w = x^2 - y^2 - 2xy + i(x^2 - y^2 + 2xy)$ is an analytic function of $z = x + iy$.

Solution. $w = u + iv = x^2 - y^2 - 2xy + i(x^2 - y^2 + 2xy)$.

$u = x^2 - y^2 - 2xy, \qquad v = x^2 - y^2 + 2xy.$

$u_x = 2x - 2y, \qquad v_x = 2y + 2x.$

$u_y = -2y - 2x, \qquad v_y = -2y + 2x.$

$u_x = v_y, \qquad u_y = -(2y + 2x) = -v_x.$

$\therefore$ C-R equations are satisfied.

$\therefore w$ is analytic.

Example 3.11. Find the analytic region of $f(z) = (x-y)^2 + 2i(x+y)$.

Solution. $f(z) = u + iv = (x-y)^2 + 2i(x+y)$

$u = (x-y)^2, \qquad v = 2(x+y).$

$u_x = 2(x-y), \qquad v_x = 2.$

$u_y = -2(x-y), \qquad v_y = 2.$

Since u and v are polynomials, their partial derivatives are continuous everywhere. For analyticity it should satisfy C-R equations.

i.e., $u_x = v_y$ and $u_y = -v_x$.

$\Longrightarrow 2(x-y) = 2$ and $-2(x-y) = -2$.

i.e., $x - y = 1$ and $x - y = 1$.

$\therefore$ For points on $x - y = 1$, C-R equations are satisfied.

$\therefore f(z)$ is analytic for points on $x - y = 1$.

—

Example 3.12. If $u + iv$ is analytic, show that $v - iu$ and $-v + iu$ are also analytic. [Apr 2005]

Solution. Given $f(z) = u + iv$ is analytic in a domain D.

$\therefore$ u and v have continuous partial derivatives and satisfy C-R equations.

$$\text{i.e., } u_x = v_y \tag{1}$$

$$\text{and } u_y = -v_x. \tag{2}$$

(i) Let $h + ig = v - iu$.

To prove $h_x = g_y, h_y = -g_x$.

Now, $h = v \qquad g = -u$

$\therefore h_x = v_x \qquad g_x = -u_x$

and $h_y = v_y \qquad g_y = -u_y$.

Now, $h_x = v_x = -u_y = g_y$

and $h_y = v_y = u_x = -g_x$.

$\therefore h + ig = v - iu$ satisfies C-R equations and first order partial derivatives are continuous.

$\therefore v - iu$ is analytic in D.

(ii) Let $p + iq = -v + iu$

$\implies p = -v, \qquad q = u$.

To prove $p_x = q_y, \qquad p_y = -q_x$.

Now, $p_x = -v_x, \qquad q_x = u_x$.

and $p_y = -v_y, \qquad q_y = u_y$.

Now, $p_x = -v_x = u_y = q_y$.

$p_y = -v_y = -u_x = -q_x$.

$\therefore p + iq = -v + iu$ satisfy C-R equations.

$\therefore -v + iu$ is analytic in D.

Example 3.13. If $f(z)$ and $\overline{f(z)}$ are analytic functions, prove that $f(z)$ is a constant.

Solution. Let $f(z) = u + iv$. Then, $\overline{f(z)} = u - iv$.

—

Given, $f(z)$ is analytic.

$\therefore$ It satisfies C-R equations.

$$\therefore u_x = v_y \text{ and } u_y = -v_x. \tag{1}$$

Given that $\overline{f(z)}$ is also analytic

$$\therefore u_x = -v_y \text{ and } u_y = v_x. \tag{2}$$

From (1) and (2) we obtain $2u_x = 0$ and $2u_y = 0$

$\implies u_x = 0$ and $u_y = 0$.

$\therefore u$ is a constant $\Rightarrow u = c_1$.

Also $v_x = 0$ and $v_y = 0$.

$\therefore v$ is a constant $\Rightarrow v = c_2$.

$\therefore f(z) = u + iv = c_1 + ic_2 = a$ constant.

Example 3.14. Prove that an analytic function with constant modulus is constant. [Dec 2007]

Solution. Let $f(z) = u + iv$ be the analytic function.

Given, $|f(z)| = \text{constant}$.

i.e., $|u + iv| = c$, c is a constant.

$$\Rightarrow u^2 + v^2 = c \tag{1}$$

Differentiating w.r. to x and y we get

$$2uu_x + 2vv_x = 0$$

$$2uu_y + 2vv_y = 0$$

$$\text{i.e., } uu_x + vv_x = 0 \tag{2}$$

$$\text{and } uu_y + vv_y = 0 \tag{3}$$

But $f(z)$ is analytic.

$\therefore$ It satisfies C-R equations.

—

i.e., $u_x = v_y$ and $u_y = -v_x$.

Applying C-R equations in (3) we obtain

$$-uv_x + vu_x = 0$$

$$\text{i.e., } vu_x - uv_x = 0. \tag{4}$$

We can solve for u_x and v_x using Cramer's rule for the equations (2) and (4).

$$\Delta = \begin{vmatrix} u & v \\ v & -u \end{vmatrix} = -u^2 - v^2 = -(u^2 + v^2) = -c \neq 0.$$

Since (2) and (4) are homogeneous equations and since $\Delta \neq 0$, we get only trivial solutions.

i.e., $u_x = 0$ and $v_x = 0$.

$\Longrightarrow u_y = 0$ and $v_y = 0$.

$u_x = 0, u_y = 0 \Rightarrow u = c_1$ a constant.

$v_x = 0, v_y = 0 \Rightarrow v = c_2$ a constant.

$\therefore\ f(z) = u + iv = c_1 + ic_2$ is a constant.

If $\Delta = 0$, then $u^2 + v^2 = 0 \Rightarrow u = 0$ and $v = 0$.

$\Rightarrow f(z) = 0$ a constant. Therefore $f(z)$ is always a constant.

Example 3.15. Show that an analytic function with constant imaginary part is constant. [Dec 2011]

Solution. Let $f(z) = u + iv$ be the given analytic function with $v = c$, a constant.

$\Rightarrow v_x = 0, v_y = 0.$

Since $f(z)$ is analytic, it satisfy C-R equations.

$\Rightarrow u_x = v_y$ and $u_y = -v_x$

$\Rightarrow u_x = 0$ and $u_y = 0$.

$\Rightarrow u = k$ a constant.

$\therefore\ f(z) = u + iv = k + ic$ a constant.

$\Rightarrow f(z) = $ constant.

—

Example 3.16. If $w = f(z)$ is analytic, prove that $\dfrac{\partial w}{\partial z} = \dfrac{\partial w}{\partial x} = -i\dfrac{\partial w}{\partial y}$, where $z = x + iy$ and prove that $\dfrac{\partial^2 w}{\partial z \partial \bar{z}} = 0$. [May 2011]

Solution. Given, $w = f(z)$ is analytic.

Let $w = u + iv$, then u, v satisfy C-R equations

$$\text{i.e., } u_x = v_y \text{ and } u_y = -v_x.$$

$$\text{Now } \frac{dw}{dz} = f'(z) = u_x + iv_x$$
$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(u + iv)$$
$$= \frac{\partial w}{\partial x}.$$

$$\text{Also } \frac{dw}{dz} = f'(z) = u_x + iv_x$$
$$= v_y - iu_y = -i(u_y + iv_y)$$
$$= -i\Big(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\Big)$$
$$= -i\frac{\partial}{\partial y}(u + iv) = -i\frac{\partial w}{\partial y}$$
$$\therefore \frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}.$$

Since $z = x + iy, \bar{z} = x - iy, z + \bar{z} = 2x, x = \dfrac{z + \bar{z}}{2}$.

and $z - \bar{z} = 2iy, y = \dfrac{z - \bar{z}}{2i}$.

Now, $\dfrac{\partial x}{\partial \bar{z}} = \dfrac{1}{2}, \dfrac{\partial y}{\partial \bar{z}} = \dfrac{-1}{2i}$.

$u(x, y)$ and $v(x, y)$ can be considered as functions of z and $\bar{z}$

$$\frac{\partial w}{\partial \bar{z}} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}} + i\Big(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\Big)$$
$$= \frac{\partial u}{\partial x}\frac{1}{2} + \frac{\partial u}{\partial y}\left(\frac{-1}{2i}\right) + i\Big(\frac{\partial v}{\partial x}\frac{1}{2} + \frac{\partial v}{\partial y}\left(\frac{(-1)}{2i}\right)\Big)$$
$$= \frac{1}{2}\Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big) + \frac{i}{2}\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) \qquad \left[\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right]$$

$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2}0 + \frac{i}{2}0 = 0. \implies \frac{\partial^2 w}{\partial z \partial \bar{z}} = 0.$$

—

Example 3.17. Show that the function defined by $f(z) = \sqrt{|xy|}$ is not analytic at the origin although C-R equations are satisfied. [Apr 2005]

Solution. Given $f(z) = \sqrt{|xy|}$

$$u + iv = \sqrt{|xy|}$$

$$u = \sqrt{|xy|} \qquad v = 0$$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{0-0}{x} = 0.$$

$$\frac{\partial u}{\partial y} = \lim_{x\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{x\to 0} \frac{0-0}{x} = 0.$$

Similarly, $\dfrac{\partial v}{\partial x} = 0, \dfrac{\partial v}{\partial y} = 0.$

$\Rightarrow$ C-R equations are satisfied at $(0,0)$.

Now

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{z\to 0} \frac{\sqrt{|xy|} - 0}{x + iy}.$$

If $z \to 0$ along the line $y = mx$ then $x \to 0, y \to 0$.

$$\text{Now, } f'(0) = \lim_{x\to 0} \frac{\sqrt{|mx^2|}}{x + imx} = \lim_{x\to 0} \frac{\sqrt{|m|}x}{(1+im)x} = \frac{\sqrt{|m|}}{(1+im)}.$$

Since the limit depends on m, for different paths we have different limits.

$\therefore$ The limit is not unique.

$\therefore$ $f'(0)$ does not exist.

$\therefore$ $f(z)$ is not analytic at $z = 0$ eventhough C-R equations are satisfied at the origin.

Example 3.18. Prove that every analytic function $w = u + iv$ can be expressed as a function of z alone, not as a function of $\bar{z}$. [Jun 2012, Jun 2010]

Solution. Let $z = x + iy$.

Then, $\bar{z} = x - iy$, which implies $x = \dfrac{z+\bar{z}}{2}$ and $y = \dfrac{z-\bar{z}}{2i}$.

Now, $w = u + iv$.

$$\begin{aligned}
\frac{\partial w}{\partial \bar{z}} &= \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}} \\
&= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}} + i\left[\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right] \\
&= \frac{\partial u}{\partial x}\frac{1}{2} + \frac{\partial u}{\partial y}\frac{-1}{2i} + i\left[\frac{\partial v}{\partial x}\frac{1}{2} + \frac{\partial v}{\partial y}\frac{-1}{2i}\right] \\
&= \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right] + \frac{i}{2}\left[\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right] \\
&= \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right] \\
&= \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \\
&= \frac{1}{2}[0 + i.0][\because u_x = v_y, u_y = -v_x] \\
\frac{\partial w}{\partial \bar{z}} &= 0.
\end{aligned}$$

$\Rightarrow w$ is independent of $\bar{z}$.

Hence, w can be written only as a function of z and not as a function of $\bar{z}$.

Example 3.19. If $u = x^2 - y^2, v = -\dfrac{y}{x^2+y^2}$ then prove that $u + iv$ is not an analytic function. [May 2005]

Solution. $u = x^2 - y^2, \qquad v = -\dfrac{y}{x^2+y^2}$.

$$u_x = 2x, \qquad v_x = -y(-1)(x^2+y^2)^{-2}.2x = \frac{2xy}{(x^2+y^2)^2}.$$

$$u_y = -2y, \qquad v_y = -\frac{x^2+y^2-y2y}{(x^2+y^2)^2} = -\frac{x^2-y^2}{(x^2+y^2)^2}.$$

We observe that $u_x \neq v_y$ and $u_y \neq -v_x$.

$\therefore$ C-R equations are not satisfied.

Hence, $u + iv$ is not analytic.

Example 3.20. Find the constants a, b if $f(z) = x + 2ay + i(3x + by)$ is analytic. [Dec 2013]

Solution. $f(z) = x + 2ay + i(3x + by)$.

$$u = x + 2ay, \qquad v = 3x + by.$$

$$u_x = 1 \qquad v_x = 3$$

$$u_y = 2a \qquad v_y = b.$$

Since $f(z)$ is analytic, C-R equations are satisfied.

$\therefore u_x = v_y$ and $u_y = -v_x$.

Taking $u_x = v_y$ we get $b = 1$.

Taking $u_y = -v_x$, we get $2a = -3 \Rightarrow a = -\frac{3}{2}$.

Example 3.21. Find the values of a and b such that the function $f(z) = x^2 + ay^2 - 2xy + i(bx^2 - y^2 + 2xy)$ is analytic. Also find $f'(z)$.

Solution. $f(z) = x^2 + ay^2 - 2xy + i(bx^2 - y^2 + 2xy)$.

$$\therefore u = x^2 + ay^2 - 2xy, \qquad v = bx^2 - y^2 + 2xy.$$

$$u_x = 2x - 2y, \qquad v_x = 2bx + 2y.$$

$$u_y = 2ay - 2x, \qquad v_y = -2y + 2x.$$

Since $f(z)$ is analytic, C-R equations are satisfied.

$\therefore u_x = v_y$ and $u_y = -v_x$.

Taking $u_x = v_y$ we get $2x - 2y = 2x - 2y$ which is true for all x and y.

Taking $u_y = -v_x$ we obtain, $2ay - 2x = -2bx - 2y$.

Comparing the coefficients we get $2a = -2$ and $2b = 2$.

$\Longrightarrow a = -1$ and $b = 1$.

$$\begin{aligned}
\text{Now, } f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\
&= 2x - 2y + i(2x + 2y) \\
&= 2x + 2ix + 2y(i - 1) \\
&= 2[x(1 + i) + iy(1 - \frac{1}{i})] \\
&= 2[x(1 + i) + iy(1 + i)] = 2(1 + i)(x + iy) = 2(1 + i)z.
\end{aligned}$$

—

Example 3.22. Find the constants a, b, c if $f(z) = x + ay + i(bx + cy)$ is analytic.

[Jun 2010]

Solution. $f(z) = u + iv = x + ay + i(bx + cy)$.

$\therefore u = x + ay \qquad v = bx + cy.$

$u_x = 1 \qquad v_x = b$

$u_y = a \qquad v_y = c.$

Since $f(z)$ is analytic, $u_x = v_y$ and $u_y = -v_x$.

$\Rightarrow 1 = c$ and $a = -b$. If $b = k$ then $a = -k$.

$\therefore$ For $a = -k$, $b = k$ and $c = 1$, the given function is analytic.

Exercise 3 A

1. Show that $f(z) = xy + iy$ is continuous every where but not differentiable any where.
2. Prove that $f(z) = \sinh z$ is analytic. [Dec 2003]
3. Prove that $f(z) = z^3$ is analytic. [May 2001]
4. Prove that $z^3 + z$ is analytic. [Jan 2006]
5. Test the analyticity of $f(z) = 2xy + i(x^2 + y^2)$. [Dec 2007]
6. Test the analyticity of the function $f(z) = \dfrac{1}{z}$. [Jan 2004]
7. Prove that $\cos z$ and $\cosh z$ are analytic functions.
8. Verify whether $\dfrac{x - iy}{x^2 + y^2}$ is an analytic function. [Jan 2005]
9. Show that the function $\dfrac{z}{z-1}$ is analytic except at $z = 1$.
10. Verify whether $w = (x^2 - y^2 - 2xy) + i(x^2 - y^2 + 2xy)$ is an analytic function of $z = x + iy$.

—

11. If $f(z) = \begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ prove that $f(z)$ is continuous and the C-R equations are satisfied at $z = 0$, yet $f'(0)$ does not exist.

12. Determine p such that the function $f(z) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{px}{y}$ is an analytic function.

13. Show that $f(z) = e^{-y}(\cos x + i\sin x)$ is differentiable everywhere in the finite plane and $f'(z) = f(z)$.

14. Show that $e^y(\cos x + i\sin x)$ is nowhere differentiable.

15. Find a such that the function $f(z) = r^2\cos 2\theta + ir^2\sin a\theta$ is analytic.

16. Find the constants a, b, c if $f(z) = x + ay + i(bx + cy)$ is analytic.

3.4 Harmonic functions and properties of analytic functions

Definition. A real function ϕ of two variables x and y is said to be harmonic in a domain D if it has continuous second order partial derivatives and satisfy the Laplace equation $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$.

Properties of Analytic functions

Property (1). If $f(z) = u + iv$ is analytic in a domain D, then u and v are harmonic in D.

Proof. Given $f(z)$ is analytic.

$\Rightarrow$ u and v have first partial derivatives and satisfy C-R equations in D.

$$u_x = v_y \text{ and } u_y = -v_x$$

$$\text{i.e., } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (1)$$

$$\text{and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (2)$$

Differentiating (1) w.r.t x we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{3}$$

Differentiating (2) w.r.t y we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \tag{4}$$

Since u_x, u_y, v_x, v_y are continuous, we have

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

$\therefore$ (3) + (4) $\Rightarrow \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$
$\therefore$ u is harmonic.

Differentiating (1) w.r.t y and (2) w.r.t x we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \text{ and } \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

$\Rightarrow$ v is harmonic.

Property (2). If $f(z) = u+iv$ is an analytic function, then the level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ form an orthogonal system of curves.

Proof. Given: $f(z) = u + iv$ is analytic.

$\therefore$ u and v has continuous partial derivatives and satisfy C-R equations.

$$\text{i.e., } u_x = v_y \text{ and } u_y = -v_x. \tag{1}$$

Let $u(x, y) = c_1'$ and $v(x, y) = c_2'$ be two members of the given families intersecting at $P(x_0, y_0)$.

Since $u(x, y) = c_1'$, we have $du = 0$.

$$\Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0.$$

$$\frac{\partial u}{\partial y}dy = -\frac{\partial u}{\partial x}dx$$

$$\implies \frac{dy}{dx} = -\frac{u_x}{u_y}.$$

At P, $\frac{dy}{dx} = -\frac{u_x}{u_y} = m_1$, the slope of the tangent at (x_0, y_0) to the curve $u(x,y) = c_1'$.
Similarly, the slope of the tangent at (x_0, y_0) to the curve $v(x,y) = c_2'$ is

$$\frac{dy}{dx} = -\frac{v_x}{v_y} = m_2.$$

$$\text{Now, } m_1 m_2 = -\frac{u_x}{u_y}\left(-\frac{v_x}{v_y}\right)$$

$$= \frac{u_x}{u_y}\left(\frac{v_x}{v_y}\right)$$

$$= \frac{u_x}{v_y}\left(\frac{v_x}{u_y}\right) = 1(-1) = -1$$

$\therefore$ The curves cut orthogonally.

$\Rightarrow$ The two system of curves are orthogonal.

Result

By Property (1), if $f(z) = u + iv$ is analytic, then u and v are harmonic. But for any two harmonic functions u and v, $u + iv$ need not be analytic.

For example, consider $u = x, v = -y$ then $u_x = 1, u_y = 0$, $v_x = 0, v_y = -1$, $u_{xx} = 0, u_{yy} = 0$, $v_{xx} = 0, v_{yy} = 0$

$u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$.

$\therefore$ u and v are harmonic functions.

But $u_x \neq v_y$.

$\Rightarrow$ C-R equations are not satisfied.

$\Rightarrow$ $u + iv$ is not analytic.

Definition. If two harmonic functions u and v satisfy the C-R equations in a domain D, then they are the real and imaginary parts of an analytic function f in D. Then v is said to be the conjugate harmonic function or harmonic conjugate function in D.

—

Polar form of the Laplace equation.

Proof. We know that if $f(z) = u + iv$ is analytic, then u and v satisfy C-R equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{1}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}. \tag{2}$$

Differentiating partially (1) w.r.t. r we get

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\cdot\frac{\partial^2 v}{\partial r \partial \theta} \tag{3}$$

(2) can also be written as $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$.
Differentiating this partially w.r.t. θ we get

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial \theta \partial r} = -r\frac{\partial^2 v}{\partial r \partial \theta} \tag{4}$$

$$\text{Now, } \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} + \frac{1}{r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right) + \frac{1}{r^2}\left(-r\frac{\partial^2 v}{\partial r \partial \theta}\right)$$
$$= -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial v}{\partial \theta} - \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} = 0.$$

This is the Laplace equation in Polar Coordinates. □

Result. We know that if $f(z) = u + iv$ is analytic then u and v satisfy Laplace equation.

i.e., $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$.

In polar form, the harmonic function $u(r, \theta)$ satisfy the Laplace equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

In the same way, the harmonic function v satisfy the Laplace equation

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0.$$

Worked Examples

Example 3.23. Show that $u = 2x - x^3 + 3xy^2$ is harmonic. [May 2011, Jun 2000]

Solution. $u = 2x - x^3 + 3xy^2$.

$u_x = 2 - 3x^2 + 3y^2$.

$u_{xx} = -6x$.

$u_y = 6xy$.

$u_{yy} = 6x$.

$u_{xx} + u_{yy} = -6x + 6x = 0$.

$\therefore u$ is harmonic.

Example 3.24. Show that $x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic.

[Jun 2010, Jun 1996]

Solution. Let $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

$u_x = 3x^2 - 3y^2 + 6x$.

$u_{xx} = 6x + 6$.

$u_y = -6xy - 6y$.

$u_{yy} = -6x - 6$.

$u_{xx} + u_{yy} = 6x + 6 - 6x - 6 = 0$.

$\therefore u$ is harmonic.

Example 3.25. Prove that $v = x^2 - y^2 + 2xy - 3x - 2y$ is harmonic. [Jun 2004]

Solution. Let $v = x^2 - y^2 + 2xy - 3x - 2y$.

$v_x = 2x + 2y - 3$.

$v_{xx} = 2$.

$v_y = -2y + 2x - 2$.

$v_{yy} = -2$.

$v_{xx} + v_{yy} = 2 - 2 = 0$.

$\therefore v$ is harmonic.

Example 3.26. Prove that $3x^2y - y^3$ is harmonic. [Jun 2007]

Solution. Let $u = 3x^2y - y^3$.

$u_x = 6xy.$

$u_{xx} = 6y.$

$u_y = 3x^2 - 3y^2.$

$u_{yy} = -6y.$

$u_{xx} + u_{yy} = 6y - 6y = 0.$

$\therefore u$ is harmonic.

Example 3.27. Show that $u = 2x(1-y)$ is harmonic. [Jun 2008]

Solution. Let $u = 2x(1-y)$.

$u_x = 2(1-y).$

$u_{xx} = 0.$

$u_y = -2x.$

$u_{yy} = 0.$

$u_{xx} + u_{yy} = 0.$

$\therefore u$ is harmonic.

Example 3.28. Show that $u = e^x(\cos y - \sin y)$ is harmonic. [May 2003]

Solution. Let $u = e^x(\cos y - \sin y)$.

$u_x = e^x(\cos y - \sin y).$

$u_{xx} = e^x(\cos y - \sin y).$

$u_y = e^x(-\sin y - \cos y).$

$u_{yy} = e^x(-\cos y + \sin y) = -e^x(\cos y - \sin y).$

$u_{xx} + u_{yy} = 0.$

$\therefore u$ is harmonic.

Example 3.29. If $u(x,y)$ and $v(x,y)$ are harmonic functions in a region R, prove that the function $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is an analytic function of $z = x + iy$.

[Dec 2011]

Solution. Given, u and v are harmonic.

$$\therefore u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0.$$

$$\begin{aligned}
\text{Let } h + ig &= \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\
&= u_y - v_x + i(u_x + v_y). \\
\therefore h &= u_y - v_x \text{ and } g = u_x + v_y. \\
\text{Now, } h_x &= u_{xy} - v_{xx}. \\
h_y &= u_{yy} - v_{yx}. \\
g_x &= u_{xx} + v_{xy}. \\
g_y &= u_{yx} + v_{yy}. \\
h_x - g_y &= u_{xy} - v_{xx} - u_{yx} - v_{yy} \\
&= -v_{xx} - v_{yy} \quad [\because u_{xy} = u_{yx}] \\
&= -(v_{xx} + v_{yy}) = 0. \\
\Rightarrow h_x &= g_y. \\
\text{Also, } h_y + g_x &= u_{yy} - v_{yx} + u_{xx} + v_{xy} \\
&= u_{xx} + u_{yy} \quad [\because v_{xy} = v_{yx}] \\
&= 0. \\
\Rightarrow h_y &= -g_x.
\end{aligned}$$

$\therefore h$ and g satisfy C-R equations.

$\Rightarrow h + ig$ is analytic.

$\Rightarrow \dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x} + i\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right)$ is analytic.

Exercise 3 B

1. Prove that $u = \dfrac{1}{2}\log(x^2 + y^2)$ is harmonic. [Dec 2006]

—

2. Prove that $u = e^x \cos y$ is harmonic.

3. Prove that $v(x,y) = e^{-x}(2xy\cos y + (y^2 - x^2)\sin y)$ is a harmonic function.

4. Show that the function $y + e^x \cos y$ is harmonic. [May 2001]

5. Prove that for all real values of the function $u = \dfrac{ay}{x^2+y^2}$ is harmonic.

3.5 Construction of an analytic function

Milne-Thomson method. This method is useful to find the analytic function when the real or imaginary part alone is given.

$$\text{Let } f(z) = u(x,y) + iv(x,y) \tag{1}$$

We have $z = x + iy$. Then, $\bar{z} = x - iy$.

$$\text{Now, } x = \frac{z+\bar{z}}{2} \text{ and } y = \frac{z-\bar{z}}{2i}.$$

$$\text{Now we have, } f(z) = u\Big(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\Big) + iv\Big(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\Big).$$

Putting $z = \bar{z}$ we get

$$f(z) = u(z,0) + iv(z,0). \tag{2}$$

(2) is same as (1) if we replace x by z and y by 0. This is valid for any function of the form $f(x+iy)$.

(i) Suppose, the real part $u(x,y)$ is given.

Let $u(x,y)$ be the given real part of an analytic function $f(z)$.

We have to find $f(z)$ and $v(x,y)$.

From $u(x,y)$, find u_x and u_y.

Since $f(z)$ is analytic, we have

$$f'(z) = u_x + iv_x = u_x - iu_y$$

$$= u_x(x,y) - iu_y(x,y).$$

—

By Milne-Thomson method

$$f'(z) = u_x(z,0) - iu_y(z,0).$$

$$\text{Now, } f(z) = \int (u_x(z,0) - iu_y(z,0))dz + c,$$

where c is an arbitrary complex constant.

Then, separating the real and imaginary parts, we can find $v(x,y)$.

(ii) Suppose, the imaginary part $v(x,y)$ is given. Find v_x and v_y.

$$\text{We have, } f'(z) = u_x + iv_x = v_y + iv_x$$

$$\text{i.e, } f'(z) = v_y(x,y) + iv_x(x,y).$$

By Milne-Thomson method

$$f'(z) = v_y(z,0) + iv_x(z,0).$$

$$\text{Now, } f(z) = \int (v_y(z,0) + iv_x(z,0))dz + c.$$

By equating the real parts we can find $u(x,y)$.

Complex form of the Laplace equation [May 2015]

Let $u(x,y)$ be a harmonic function.

Since $z = x + iy, \bar{z} = x - iy$, we have, $x = \dfrac{z+\bar{z}}{2}, y = \dfrac{z-\bar{z}}{2i}$.

$\Rightarrow u$ is a function of z and $\bar{z}$.

$$\begin{aligned}
\frac{\partial u}{\partial z} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial z} \\
&= \frac{\partial u}{\partial x}\frac{1}{2} + \frac{\partial u}{\partial y}\frac{1}{2i} \\
&= \frac{1}{2}\frac{\partial u}{\partial x} - \frac{i}{2}\frac{\partial u}{\partial y} \\
\frac{\partial u}{\partial z} &= \frac{1}{2}\Big(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\Big). \\
\frac{\partial^2 u}{\partial \bar{z}\partial z} &= \frac{\partial}{\partial \bar{z}}\Big(\frac{1}{2}(u_x - iu_y)\Big)
\end{aligned}$$

$$= \frac{1}{2}\Big(\frac{\partial}{\partial x}(u_x - iu_y)\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}(u_x - iu_y)\frac{\partial y}{\partial \bar{z}}\Big)$$
$$= \frac{1}{2}\Big((u_{xx} - iu_{xy})\frac{1}{2} + (u_{yx} - iu_{yy})\Big(\frac{-1}{2i}\Big)\Big)$$
$$= \frac{1}{4}(u_{xx} - iu_{xy} + iu_{yx} + u_{yy}) = \frac{1}{4}(u_{xx} + u_{yy})$$

$$u_{xx} + u_{yy} = 4\frac{\partial^2 u}{\partial \bar{z}\partial z}. \tag{1}$$

$$\text{i.e., } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\frac{\partial^2 u}{\partial \bar{z}\partial z}.$$

Since u is harmonic, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

$$\therefore \frac{\partial^2 u}{\partial \bar{z}\partial z} = 0.$$

This is the complex form of the Laplace equation.

From (1), we get the Laplacian operator

$$\boxed{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial \bar{z}\partial z}.}$$

Worked Examples

Example 3.30. Find a function w such that $w = u + iv$ is analytic if $u = e^x \sin y$.

[Jun 2000]

Solution. We have $u = e^x \sin y$.

$u_x = e^x \sin y$.

$u_y = e^x \cos y$.

By Milne Thomson Method

$$f'(z) = u_x(z, 0) - iu_y(z, 0)$$
$$= e^z \sin 0 - ie^z \cos 0$$
$$\text{i.e., } f'(z) = -ie^z.$$

Integrating both sides w.r.t. z we get, $f(z) = -ie^z + C$.

Example 3.31. Construct an analytic function $f(z)$ for which the real part is $e^x \cos y$. [Dec 2001]

Solution. Let $f(z) = u + iv$ be the required function.

$\therefore u = e^x \cos y$

$u_x = e^x \cos y$

$u_y = -e^x \sin y.$

By Milne Thomson method

$f'(z) = u_x(z,0) - iu_y(z,0) = e^z \cos 0 - ie^z \sin 0$

$f'(z) = e^z.$

Integrating both sides w.r.t. z we get $f(z) = e^z + c.$

Example 3.32. If $\phi = 3x^2y - y^3$, find ψ such that $w = \phi + i\psi$ is an analytic function. [May 1996]

Solution. $\phi = 3x^2y - y^3.$

$\phi_x = 6xy.$

$\phi_y = 3x^2 - 3y^2.$

By Milne's Thomson method

$$\begin{aligned} w'(z) &= \phi_x(z,0) - i\phi_y(z,0) \\ &= 0 - i3z^2 \\ &= -i3z^2. \end{aligned}$$

Integrating both sides w. r. t. z we get

$$\int w'(z)dz = -3i\int z^2 dz + c, \text{ where } c \text{ is a complex constant.}$$

$$w(z) = -3i\frac{z^3}{3} + c$$

$$\begin{aligned} \phi + i\psi &= -iz^3 + c \\ &= -i(x+iy)^3 + c_1 + ic_2, \text{ where } c_1 \text{ and } c_2 \text{ are real} \\ &= -i(x^3 + 3ix^2y - 3xy^2 - iy^3) + c_1 + ic_2. \end{aligned}$$

Equating the imaginary parts we get, $\psi = -x^3 + 3xy^2 + c_1.$

—

Example 3.33. Find the analytic function whose real part is $e^{2x}\sin 2y$. [Dec 1996]

Solution. Let $f(z) = u + iv$ be the required analytic function.

Given that $u = e^{2x}\sin 2y$.

$u_x = 2e^{2x}\sin 2y, u_y = 2e^{2x}\cos 2y$.

By Milne's Thomson method

$$f'(z) = u_x(z,0) - iu_y(z,0) = 2e^{2z}\sin 0 - i2e^{2z}\cos 0 = -2ie^{2z}.$$

Integrating w.r.t. z we get

$$f(z) = -2i\int e^{2z}dz + c, \text{ where } c \text{ is complex.}$$
$$= -2i\frac{e^{2z}}{2} + c = -ie^{2z} + c.$$

Example 3.34. Construct an analytic function $f(z) = u+iv$, given that $v = y+e^x\cos y$. [May 1999]

Solution. $v = y + e^x\cos y$.

$v_x = e^x\cos y, v_y = 1 - e^x\sin y$.

By Milne's Thomson method,

$$f'(z) = v_y(z,0) + iv_x(z,0) = 1 - e^z\sin 0 + ie^z\cos 0 = 1 + ie^z.$$

Integrating w.r.t. z, we obtain,

$f(z) = \int(1 + ie^z)dz + c$ where c is complex.

$= z + ie^z + c.$

Example 3.35. Find the analytic function $u + iv$ whose real part is
$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1.$ [Jun 2006]

Solution. Given: $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1.$

$$u_x = 3x^2 - 3y^2 + 6x, \qquad u_y = -6xy - 6y.$$

$$u_x(z,0) = 3z^2 + 6z, \qquad u_y(z,0) = 0.$$

—

By Milne Thomson method,

$$f'(z) = u_x(z,0) - iu_y(z,0)$$
$$f'(z) = 3z^2 + 6z.$$

Integrating w.r.t. z, we get

$$\int f'(z)dz = \int (3z^2 + 6z)dz$$
$$= 3\frac{z^3}{3} + 6\frac{z^2}{2} + c.$$
$$= z^3 + 3z^2 + c.$$

Example 3.36. If $u = \dfrac{\sin 2x}{\cos h2y + \cos 2x}$ find the corresponding analytic function $f(z) = u + iv$. [Dec 2006]

Solution. Given, $u = \dfrac{\sin 2x}{\cos h2y + \cos 2x}$.

$$u_x = \frac{(\cos h2y + \cos 2x)2\cos 2x - \sin 2x(-\sin 2x)\times 2}{(\cos h2y + \cos 2x)^2}.$$
$$u_x(z,0) = \frac{(1+\cos 2z)2\cos 2z + 2\sin^2 2z}{(1+\cos 2z)^2}$$
$$= \frac{2\cos 2z + 2\cos^2 2z + 2\sin^2 2z}{(1+\cos 2z)^2}$$
$$= \frac{2\cos 2z + 2}{(1+\cos 2z)^2}$$
$$= \frac{2(1+\cos 2z)}{(1+\cos 2z)^2} = \frac{2}{2\cos^2 z} = \sec^2 z.$$
$$u_y = \sin 2x(-1)(\cos h2y + \cos 2x)^{-2} 2\sin h2y.$$
$$u_y(z,0) = 0.$$

By Milne Thomson method,

$f'(z) = u_x(z,0) - iu_y(z,0) = \sec^2 z.$

Integrating w.r.t. z, we get

$\int f'(z)dz = \int \sec^2 z dz + c$

$f(z) = \tan z + c.$

—

Example 3.37. Find the analytic function $w = u + iv$ where
$v = e^{-2y}(y\cos 2x + x\sin 2x)$ and find u. [Dec 2011]

Solution. $v = e^{-2y}(y\cos 2x + x\sin 2x)$.

$$\frac{\partial v}{\partial x} = e^{-2y}[-2y\sin 2x + 2x\cos 2x + \sin 2x].$$

$$v_x(z,0) = 2z\cos 2z + \sin 2z.$$

$$v_y = e^{-2y}[\cos 2x] + (y\cos 2x + x\sin 2x)(-2e^{-2y}).$$

$$v_y(z,0) = \cos 2z + z\sin 2z(-2) = \cos 2z - 2z\sin 2z.$$

By Milne Thomson method

$$\begin{aligned} f'(z) &= v_y(z,0) + iv_x(z,0) \\ &= \cos 2z - 2z\sin 2z + i(2z\cos 2z + \sin 2z) \\ &= \cos 2z + i\sin 2z + 2iz(\cos 2z + i\sin 2z) = e^{i2z} + 2ize^{i2z} = e^{i2z}(1 + 2iz). \end{aligned}$$

Integrating w.r.t.z we get

$$\begin{aligned} f(z) &= \int e^{i2z}(1 + 2iz)dz + c \\ &= \int e^{i2z}dz + 2i\int ze^{i2z}dz + c \\ &= \frac{e^{i2z}}{2i} + 2i\int zd\left(\frac{e^{i2z}}{2i}\right) + c \\ &= \frac{e^{i2z}}{2i} + \left[ze^{i2z} - \int e^{i2z}dz\right] + c = \frac{e^{i2z}}{2i} + ze^{i2z} - \frac{e^{i2z}}{2i} + c = ze^{i2z} + c. \end{aligned}$$

$$\begin{aligned} u + iv &= (x+iy)e^{2i(x+iy)} + c_1 + ic_2 \\ &= (x+iy)e^{2ix-2y} + c_1 + ic_2 \\ &= (x+iy)e^{2ix}\cdot e^{-2y} + c_1 + ic_2 \\ &= (x+iy)e^{-2y}(\cos 2x + i\sin 2x) + c_1 + ic_2. \end{aligned}$$

Equating the real parts we get,
$u = e^{-2y}(x\cos 2x - y\sin 2x) + c_1$.

—

Example 3.38. Show that the function $u(x, y) = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic. Find the conjugate harmonic function v and express $u + iv$ as an analytic function of z.

[Dec 2014, May 2004]

Solution. To prove u to be harmonic, we have to prove that $u_{xx} + u_{yy} = 0$.

Given, $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$$u_x = 6xy + 4x$$
$$u_y = 3x^2 - 3y^2 - 4y$$
$$u_x(z, 0) = 4z$$
$$u_y(z, 0) = 3z^2.$$

By Milne Thompson method,

$$f'(z) = u_x(z, 0) - iu_y(z, 0)$$
$$= 4z - i3z^2.$$

Integrating w.r.t. z we get

$$f(z) = \int (4z - i3z^2)dz + c$$
$$= 4\frac{z^2}{2} - 3i\frac{z^3}{3} + c$$
$$= 2z^2 - iz^3 + c, \text{ c is a complex constant}$$

i.e., $f(x + iy) = 2(x + iy)^2 - i(x + iy)^3 + c$

$$u + iv = 2(x^2 - y^2 + 2ixy) - i(x^3 + 3x^2iy - 3xy^2 - iy^3) + c$$
$$= 2(x^2 - y^2) + 4ixy - ix^3 + 3x^2y + i3xy^2 - y^3 + c$$
$$= 2(x^2 - y^2) + 3x^2y - y^3 + i(3xy^2 + 4xy - x^3) + c_1 + ic_2$$

where c_1, c_2 are real.

Equating the imaginary parts we get,

$v = 4xy - x^3 + 3xy^2 + c_2.$

—

Example 3.39. Prove that the function $u = e^x(x\cos y - y\sin y)$ satisfies Laplace equation and find the corresponding analytic function $f(z) = u + iv$.

[Jun 2013, Dec 2010, May 2005]

Solution. Let $f(z) = u + iv$ be the analytic function.

Given $u = e^x(x\cos y - y\sin y)$ is the real part of $f(z)$.

$$\begin{aligned}
u_x &= e^x \cos y + e^x(x\cos y - y\sin y) \\
&= e^x(\cos y + x\cos y - y\sin y). \\
u_y &= e^x(-x\sin y - y\cos y - \sin y). \\
u_{xx} &= e^x(\cos y + x\cos y - y\sin y) + e^x \cos y \\
&= e^x(2\cos y + x\cos y - y\sin y). \\
u_{yy} &= e^x(-x\cos y + y\sin y - \cos y - \cos y) \\
&= e^x(-x\cos y + y\sin y - 2\cos y).
\end{aligned}$$

$$u_{xx} + u_{yy} = e^x(2\cos y + x\cos y - y\sin y) + e^x(-x\cos y + y\sin y - 2\cos y) = 0.$$

$\Rightarrow u$ satisfies Laplace equation.

Now, $u_x(z,0) = e^z(1+z)$

and $u_y(z,0) = 0$.

By Milne-Thomson method, $f'(z) = u_x(z,0) - iu_y(z,0) = e^z(1+z)$.

Integrating w.r.t z we get

$$\begin{aligned}
f(z) &= \int e^z(1+z)dz + c \\
&= \int (1+z)d(e^z) + c \\
&= (1+z)e^z - \int e^z dz + c \\
&= (1+z)e^z - e^z + c = e^z + ze^z - e^z + c
\end{aligned}$$

$$\therefore f(z) = ze^z + c.$$

Example 3.40. Prove that the function $v = e^{-x}(x\cos y + y\sin y)$ is harmonic and determine the corresponding analytic function $f(z) = u + iv$. [May 2004, May 2014]

Solution. We first prove that $v_{xx} + v_{yy} = 0$.

$$\begin{aligned}
v_x &= e^{-x}(\cos y) + (x\cos y + y\sin y)(-e^{-x})\\
&= e^{-x}(\cos y - x\cos y - y\sin y)\\
v_{xx} &= e^{-x}(-\cos y) - e^{-x}(\cos y - x\cos y - y\sin y)\\
&= e^{-x}(-\cos y - \cos y + x\cos y + y\sin y)\\
&= e^{-x}(-2\cos y + x\cos y + y\sin y)\\
v_y &= e^{-x}(-x\sin y + y\cos y + \sin y)\\
v_{yy} &= e^{-x}(-x\cos y - y\sin y + \cos y + \cos y)\\
&= e^{-x}(2\cos y - x\cos y - y\sin y).
\end{aligned}$$

$v_{xx} + v_{yy} = 0. \qquad \therefore v$ is harmonic

Now, $v_x(z,0) = e^{-z}(1-z)$

and $v_y(z,0) = e^{-z}0 = 0$.

By Milne Thomson method,

$f'(z) = v_y(z,0) + iv_x(z,0) = ie^{-z}(1-z)$.

Integrating we get,

$$\begin{aligned}
f(z) &= i\int e^{-z}(1-z)dz + c\\
&= i\int (1-z)d(-e)^{-z}dz + c\\
&= i\int (z-1)d(e)^{-z} + c\\
&= i\Big[(z-1)e^{-z} - \int e^{-z}dz\Big] + c\\
&= i\Big((z-1)e^{-z} + e^{-z}\Big) + c\\
&= ie^{-z}(z-1+1) + c\\
f(z) &= ize^{-z} + c.
\end{aligned}$$

Example 3.41. Prove that $u = e^{-2xy}\sin(x^2-y^2)$ is harmonic. Find the corresponding analytic function and the imaginary part. [Dec 2013, Jan 2016]

—

Solution. $u = e^{-2xy} \sin(x^2 - y^2)$

$$
\begin{aligned}
u_x &= e^{-2xy} \cos(x^2 - y^2) \cdot 2x + \sin(x^2 - y^2) \cdot e^{-2xy}(-2y) \\
&= 2e^{-2xy} \left[x \cos(x^2 - y^2) - y \sin(x^2 - y^2) \right]. \\
u_{xx} &= 2e^{-2xy} \left[-x \sin(x^2 - y^2) \cdot 2x + \cos(x^2 - y^2) - y \cos(x^2 - y^2) \cdot 2x \right] + \\
&\qquad \left[x \cos(x^2 - y^2) - y \sin(x^2 - y^2) \right] 2e^{-2xy}(-2y) \\
u_{xx} &= 2e^{-2xy} \left[-2x^2 \sin(x^2 - y^2) + \cos(x^2 - y^2) - 2xy \cos(x^2 - y^2) - 2xy \cos(x^2 - y^2) \right. \\
&\qquad \left. +2y^2 \sin(x^2 - y^2) \right]. \\
u_y &= e^{-2xy} \cos(x^2 - y^2)(-2y) + \sin(x^2 - y^2) e^{-2xy}(-2x) \\
&= -2e^{-2xy} \left[y \cos(x^2 - y^2) + x \sin(x^2 - y^2) \right]. \\
u_{yy} &= -2e^{-2xy} \left[-y \sin(x^2 - y^2)(-2y) + \cos(x^2 - y^2) + x \sin(x^2 - y^2)(-2y) \right] + \\
&\qquad \left[y \cos(x^2 - y^2) + x \sin(x^2 - y^2) \right] (-2e^{-2xy})(-2x) \\
&= -2e^{-2xy} \left[2y^2 \sin(x^2 - y^2) + \cos(x^2 - y^2) - 2xy \sin(x^2 - y^2) - 2xy \cos(x^2 - y^2) \right. \\
&\qquad \left. -2x^2 \sin(x^2 - y^2) \right] \\
&= 2e^{-2xy} \left[-2y^2 \sin(x^2 - y^2) - \cos(x^2 - y^2) + 2xy \sin(x^2 - y^2) + 2xy \cos(x^2 - y^2) \right. \\
&\qquad \left. +2x^2 \sin(x^2 - y^2) \right].
\end{aligned}
$$

Now, $u_{xx} + u_{yy} = 0$.

$\therefore$ u is harmonic.

Now $u_x(z, 0) = 2z \cos z^2$.

$u_y(z, 0) = -2z \sin z^2$.

By Milne Thomson method

$$
\begin{aligned}
f'(z) &= u_x(z, 0) + i u_y(z, 0) \\
&= 2z \cos z^2 + 2iz \sin z^2 \\
&= 2z(\cos z^2 + i \sin z^2) \\
&= 2ze^{iz^2}
\end{aligned}
$$

—

Integrating w.r.t.z we get

$$
\begin{aligned}
f(z) &= \int 2ze^{iz^2}dz + c \qquad t = z^2\\
&= \int e^{it}dt + c \qquad dt = 2zdz\\
&= \frac{e^{it}}{i} + c = \frac{1}{i}e^{iz^2} + c\\
f(z) &= -ie^{iz^2} + c\\
u + iv &= -ie^{i(x+iy)^2} + c_1 + ic_2\\
&= -ie^{i(x^2-y^2+2ixy)} + c_1 + ic_2\\
&= -ie^{i(x^2-y^2)-2xy} + c_1 + ic_2\\
&= -ie^{i(x^2-y^2)} \cdot e^{-2xy} + c_1 + ic_2\\
&= -ie^{-2xy}[\cos(x^2-y^2) + i\sin(x^2-y^2)] + c_1 + ic_2\\
&= -ie^{-2xy}\cos(x^2-y^2) + e^{-2xy}\sin(x^2-y^2)
\end{aligned}
$$

Equating the imaginary parts we get
$v = -e^{-2xy}\cos(x^2-y^2)$.

Example 3.42. Show that $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic. Determine its analytic function. Find also its Conjugate. [May 2011]

Solution. $u = \frac{1}{2}\log(x^2+y^2)$.

$$
\begin{aligned}
u_x &= \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2}.\\
u_{xx} &= \frac{x^2+y^2-x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.\\
u_y &= \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2y = \frac{y}{x^2+y^2}.\\
u_{yy} &= \frac{x^2+y^2-y\cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.
\end{aligned}
$$

Now, $u_{xx} + u_{yy} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0.$

$\therefore u$ is harmonic.

—

$$u_x(z,0) = \frac{z}{z^2} = \frac{1}{z}.$$

$$u_y(z,0) = 0.$$

By Milne-Thomson method

$f'(z) = u_x(z,0) - iu_y(z,0) = \frac{1}{z}.$

Integrating w.r.t.z we get

$$f(z) = \int \frac{1}{z}dz + c = \log z + c.$$

$$u + iv = \log(x+iy) + c = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right) + c_1 + ic_2.$$

Equating the imaginary parts we get

$$v = \tan^{-1}\left(\frac{y}{x}\right) + c_2.$$

Example 3.43. Prove that $u = x^2 - y^2$ and $v = -\frac{y}{x^2+y^2}$ are harmonic, but $u + iv$ is not regular. [Dec 2010, Dec 2014, Jan 2016]

Solution. $u = x^2 - y^2$.

$$u_x = 2x.$$

$$u_{xx} = 2.$$

$$u_y = -2y.$$

$$u_{yy} = -2.$$

$$u_{xx} + u_{yy} = 2 - 2 = 0.$$

$\therefore u$ is harmonic.

$$v = -\frac{y}{x^2+y^2} = -y(x^2+y^2)^{-1}.$$

$$v_x = -y(-1)(x^2+y^2)^{-2}\cdot 2x = \frac{2xy}{(x^2+y^2)^2}.$$

$$v_{xx} = 2y\left[\frac{(x^2+y^2)^2 - x2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}\right]$$

$$= \frac{2y(x^2+y^2)[x^2+y^2-4x^2]}{(x^2+y^2)^4}$$

$$v_{xx} = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}.$$

$$v_y = -\frac{x^2+y^2-y\cdot 2y}{(x^2+y^2)^2}$$

$$= -\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

$$v_{yy} = \frac{(x^2+y^2)^2\cdot 2y-(y^2-x^2)\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}.$$

$$= \frac{2y(x^2+y^2)[x^2+y^2-2(y^2-x^2)]}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2-2y^2+2x^2)}{(x^2+y^2)^3}$$

$$v_{yy} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}.$$

$$v_{xx}+v_{yy} = \frac{2y}{(x^2+y^2)^3}[y^2-3x^2+3x^2-y^2] = 0.$$

$\therefore$ v is harmonic.

We have to prove that, if $u+iv$ is regular then $u+iv$ must satisfy C-R equations.

But here, $u_x = 2x \neq v_y$ and $u_y = -2y \neq -v_x$.

$\therefore$ C-R equations are not satisfied.

$\therefore$ $u+iv$ is not regular.

Example 3.44. Find the Conjugate harmonic of $v(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2$. Show that v is harmonic.

Solution. $v = r^2\cos 2\theta - r\cos\theta + 2.$

$$\frac{\partial v}{\partial r} = 2r\cos 2\theta - \cos\theta.$$

$$\frac{\partial^2 v}{\partial r^2} = 2\cos 2\theta.$$

$$\frac{\partial v}{\partial \theta} = -2r^2\sin 2\theta + r\sin\theta.$$

$$\frac{\partial^2 v}{\partial \theta^2} = -4r^2\cos 2\theta + r\cos\theta$$

$$\text{Now, } \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 2\cos 2\theta + \frac{1}{r}(2r\cos 2\theta - \cos\theta) + \frac{1}{r^2}(-4r^2\cos 2\theta + r\cos\theta)$$

$$= 2\cos 2\theta + 2\cos 2\theta - \frac{1}{r}\cos\theta - 4\cos 2\theta + \frac{1}{r}\cos\theta$$

$$= 0.$$

$\therefore$ v is harmonic.

Using C-R equations we have

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = -2r^2\sin 2\theta + r\sin\theta.$$

$$\frac{\partial u}{\partial r} = -2r\sin 2\theta + \sin\theta.$$

Integrating w.r.t.r we get

$$u = -2\sin 2\theta\int r dr + \sin\theta\int dr + f(\theta)$$

$$= -2\sin 2\theta\left(\frac{r^2}{2}\right) + r\sin\theta + f(\theta)$$

$$u = -r^2\sin 2\theta + r\sin\theta + f(\theta). \tag{1}$$

Also by C-R equations we have

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = 2r\cos 2\theta - \cos\theta.$$

$$\frac{\partial u}{\partial \theta} = -2r^2\cos 2\theta + r\cos\theta.$$

Integrating w.r.t θ we get

$$u = -2r^2\int \cos 2\theta d\theta + r\int \cos\theta d\theta + g(r)$$

$$= -2r^2\cdot\frac{\sin 2\theta}{2} + r\sin\theta + g(r)$$

$$u = -r^2\sin 2\theta + r\sin\theta + g(r). \tag{2}$$

From (1) and (2) we get

$u = -r^2\sin 2\theta + r\sin\theta + c$ where c is a constant, which is the conjugate harmonic of v.

—

Example 3.45. Find the analytic function $f(z) = u + iv$ where $v = 3r^2 \sin 2\theta - 2r \sin\theta$. Verify that u is a harmonic function. [Dec 2013]

Solution. $v = 3r^2 \sin 2\theta - 2r \sin\theta$.

$$\frac{\partial v}{\partial r} = 6r \sin 2\theta - 2\sin\theta.$$
$$\frac{\partial^2 v}{\partial r^2} = 6\sin 2\theta.$$
$$\frac{\partial v}{\partial \theta} = 6r^2 \cos 2\theta - 2r\cos\theta.$$
$$\frac{\partial^2 v}{\partial \theta^2} = -12r^2 \sin 2\theta + 2r\sin\theta.$$

$$\text{Now, } \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 6\sin 2\theta + \frac{1}{r}[6r\sin 2\theta - 2\sin\theta] + \frac{1}{r^2}[-12r^2 \sin 2\theta + 2r\sin\theta]$$
$$= 6\sin 2\theta + 6\sin 2\theta - \frac{2}{r}\sin\theta - 12\sin 2\theta + \frac{2}{r}\sin\theta$$
$$= 0.$$

$\therefore$ v is harmonic.

Using C-R equations we have

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = 6r^2\cos 2\theta - 2r\cos\theta.$$
$$\frac{\partial u}{\partial r} = 6r\cos 2\theta - 2\cos\theta.$$

Integrating w.r.t. r we get

$$u = 6\cos 2\theta \int r\,dr - 2\cos\theta \int dr + f(\theta)$$
$$= 6\cos 2\theta \cdot \frac{r^2}{2} - 2r\cos\theta + f(\theta)$$
$$u = 3r^2 \cos 2\theta - 2r\cos\theta + f(\theta). \qquad (1)$$

—

Again by C-R equations we have

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = 6r\sin 2\theta - 2\sin\theta.$$
$$\frac{\partial u}{\partial \theta} = -6r^2\sin 2\theta + 2r\sin\theta.$$

Integrating w.r.t.θ we get

$$\begin{aligned} u &= -6r^2\int \sin 2\theta d\theta + 2r\int \sin\theta d\theta + g(r) \\ &= -6r^2\left(\frac{-\cos 2\theta}{2}\right) + 2r(-\cos\theta) + g(r) \\ u &= 3r^2\cos 2\theta - 2r\cos\theta + g(r). \end{aligned} \tag{2}$$

From (1) and (2) we get

$u = 3r^2\cos 2\theta - 2r\cos\theta + c$, where c is a constant,

which is the harmonic conjugate of v.

To verify whether u is harmonic

We have $\frac{\partial u}{\partial r} = 6r\cos 2\theta - 2\cos\theta.$

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} &= 6\cos 2\theta. \\ \frac{\partial u}{\partial \theta} &= -6r^2\sin 2\theta + 2r\sin\theta. \\ \frac{\partial^2 u}{\partial \theta^2} &= -6r^2\cdot(\cos 2\theta)\times 2 + 2r\cos\theta. \\ &= -12r^2\cos 2\theta + 2r\cos\theta. \end{aligned}$$

$$\begin{aligned} \text{Now, } \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= 6\cos 2\theta + \frac{1}{r}(6r\cos 2\theta - 2\cos\theta) \\ &\quad + \frac{1}{r^2}(-12r^2\cos 2\theta + 2r\cos\theta) \\ &= 6\cos 2\theta + 6\cos 2\theta - \frac{2}{r}\cos\theta - 12\cos 2\theta + \frac{2}{r}\cos\theta = 0. \end{aligned}$$

$\therefore u$ is harmonic.

—

Example 3.46. Determine the analytic function $f(z) = u + iv$ such that $u - v = e^x(\cos y - \sin y)$. [April 1997]

Solution. Given $f(z) = u + iv$.

$i f(z) = iu - v$

Adding we get, $(1 + i) f(z) = u - v + i(u + v)$.

Put $U = u - v, V = u + v$.

$F(z) = (1 + i) f(z) = U + iV$.

Since $f(z)$ is analytic, $F(z)$ is also analytic.

$U = u - v = e^x(\cos y - \sin y)$

$U_x = e^x(\cos y - \sin y)$

$U_y = e^x(-\sin y - \cos y)$

$U_x(z, 0) = e^z, U_y(z, 0) = -e^z$.

By Milne-Thomson method,

$$F'(z) = U_x(z, 0) - iU_y(z, 0) = e^z + ie^z = (1 + i)e^z$$

Integrating we get

$$F(z) = \int (1 + i)e^z dz + c$$

$$= (1 + i)e^z + c$$

$$(1 + i) f(z) = (1 + i)e^z + c$$

$$f(z) = e^z + c' \text{ where } c' = \frac{c}{1 + i}.$$

Example 3.47. If $u + v = \dfrac{2 \sin 2x}{e^{2y} + e^{-2y} - 2 \cos 2x}$ and $f(z) = u + iv$ is an analytic function of z find $f(z)$ in terms of z. [Dec 2012]

Solution. $f(z) = u + iv$

$i f(z) = iu - v$

Adding we get

$(1 + i) f(z) = u - v + i(u + v)$

Let $U = u - v, V = u + v, (1 + i) f(z) = F(z)$.

—

$\therefore F(z) = U + iV, U = u - v, V = u + v.$

Since $f(z)$ is analytic, $F(z)$ is also analytic.

$$\text{Now, } u + v = V = \frac{2\sin 2x}{2\cosh 2y - 2\cos 2x} = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$V_x = \frac{(\cosh 2y - \cos 2x)2\cos 2x - \sin 2x.2\sin 2x}{(\cosh 2y - \cos 2x)^2}$$

$$V_x(z,0) = \frac{(1-\cos 2z)2\cos 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1-\cos 2z)^2} = \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2} = \frac{-2}{2\sin^2 z} = -cosec^2 z$$

$$V_y = \sin 2x(-1)(\cosh 2y - \cos 2x)^{-2} 2\sinh 2y$$

$$V_y(z,0) = 0.$$

By Milne Thomson method,

$$F'(z) = V_y(z,0) + iV_x(z,0) = 0 + i(-cosec^2 z) = -i\ cosec^2 z.$$

Integrating we get,

$$F(z) = -i\int cosec^2 z dz + c. = -i(-\cot z) + c.$$

$$F(z) = i\cot z + c.$$

$$(1+i)f(z) = i\cot z + c.$$

$$f(z) = \frac{i}{1+i}\cot z + \frac{c}{1+i}$$

$$= \frac{i(1-i)}{2}\cot z + c_1 \quad \text{where } c_1 = \frac{c}{1+i}$$

$$f(z) = \frac{1+i}{2}\cot z + c_1.$$

Example 3.48. Find the analytic function $f(z) = u + iv$ given that $2u + 3v = e^x(\cos y - \sin y)$. [Dec 2005]

Solution. $2u + 3v = e^x(\cos y - \sin y)$

$$\text{Given } f(z) = u + iv$$

$$3f(z) = 3u + i3v$$

$$i2f(z) = 2iu - 2v$$

Adding we get

$$(3+2i)f(z) = 3u - 2v + i(2u+3v).$$

Let $U = 3u - 2v, \qquad V = 2u + 3v.$

and $F(z) = (3+2i)f(z) = U + iV.$

Since $f(z)$ is analytic, $F(z)$ is also analytic.

$$\text{Now, } V = e^x(\cos y - \sin y)$$

$$V_x = e^x(\cos y - \sin y) \qquad V_x(z,0) = e^z$$

$$V_y = e^x(-\sin y - \cos y) \qquad V_y(z,0) = -e^z$$

By Milne Thomson method

$$F'(z) = V_y(z,0) + iV_x(z,0) = -e^z + ie^z$$

$$F'(z) = (-1+i)e^z$$

Integrating we get, $F(z) = (-1+i)\int e^z + c$

$(3+2i)f(z) = (-1+i)e^z + c$

$$\begin{aligned} f(z) &= \frac{-1+i}{3+2i}e^z + \frac{c}{3+2i} \\ &= \frac{(-1+i)(3-2i)}{13}e^z + c_1 \text{ Where } c_1 = \frac{c}{3+2i} \\ &= \frac{-3+2i+3i+2}{13}e^z + c_1 \\ f(z) &= \frac{-1+5i}{13}e^z + c. \end{aligned}$$

Example 3.49. Determine the analytic function $f(z) = u + iv$ given that $3u + 2v = y^2 - x^2 + 16xy$. [May 2007]

Solution. Given $f(z) = u + iv$

$$2f(z) = 2u + i2v$$

—

$3i f(z) = 3iu - 3v$

$(2+3i) f(z) = 2u - 3v + i(3u + 2v)$

Let $U = 2u - 3v, V = 3u + 2v$ and $F(z) = U + iV$.

where $F(z) = (2+3i) f(z)$.

Since $f(z)$ is analytic, $F(z)$ is also analytic.

Now, $V = y^2 - x^2 + 16xy$

$V_x = -2x + 16y$

$V_y = 2y + 16x$

By Milne Thomson method,

$$F'(z) = V_y(z,0) + iV_x(z,0) = 16z - i2z = (16-2i)z$$

Integrating we get,

$$\begin{aligned}
F(z) &= (16-2i)\frac{z^2}{2} + c \\
(2+3i) f(z) &= (8-i)z^2 + c \\
f(z) &= \frac{8-i}{2+3i}z^2 + \frac{c}{2+3i} \\
&= \frac{(8-i)(2-3i)}{13}z^2 + c_1, \quad c_1 = \frac{c}{2+3i} \\
&= (1-2i)z^2 + c_1.
\end{aligned}$$

Example 3.50. Find the analytic function $f(z) = u + iv$ given that $5u - 3v = e^x(5\cos y - 3\sin y)$.

Solution. $f(z) = u + iv$.

$5 f(z) = 5u + 5iv$.

$3i f(z) = 3iu - 3v$.

Adding $5 f(z) + 3i f(z) = 5u - 3v + i(5v + 3u)$

$(5+3i) f(z) = 5u - 3v + i(5v + 3u)$

—

$F(z) = U + iV$, where

$F(z) = (5 + 3i) f(z)$.

$U = 5u - 3v$ and

$V = 5v + 3u$.

Since $f(z)$ is analytic, $F(z)$ is also analytic.

Given $U = 5u - 3v = e^x[5\cos y - 3\sin y]$.

$U_x = e^x[5\cos y - 3\sin y]$.

$U_x(z, 0) = 5e^z$.

$U_y = e^x[-5\sin y - 3\cos y]$.

$U_y(z, 0) = -3e^z$.

By Milne-Thomson method,

$F'(z) = U_x(z, 0) - iU_y(z, 0)$.

$= 5e^z + 3ie^z = (5 + 3i)e^z$.

Integrating w.r.t. z we get

$F(z) = \int (5 + 3i)e^z + c$

$= (5 + 3i)e^z + c$

$(5 + 3i) f(z) = (5 + 3i)e^z + c$

$f(z) = e^z + \frac{c}{5 + 3i}$

$f(z) = e^z + c'$ where $c' = \frac{c}{5 + 3i}$.

Example 3.51. Find the analytic function $f(z) = u + iv$ given that $4u - 3v = e^x[(4x - 3y)\cos y - (3x + 4y)\sin y]$.

Solution. $f(z) = u + iv$.

$$4 f(z) = 4u + i4v.$$

$$i3 f(z) = 3iu - 3v.$$

—

$$4f(z) + 3if(z) = 4u - 3v + i(4v + 3u)$$

$$(4 + 3i)f(z) = 4u - 3v + i(4v + 3u).$$

$$F(z) = U + iV \quad \text{where}$$

$$F(z) = (4 + 3i)f(z).$$

$$U = 4u - 3v \quad \text{and} \quad V = 3u + 4v.$$

Since $f(z)$ is analytic, $F(z)$ is also analytic.

Now, $U = 4u - 3v = e^x[(4x - 3y)\cos y - (3x + 4y)\sin y]$.

$$U_x = e^x[4\cos y - 3\sin y] + [(4x - 3y)\cos y - (3x + 4y)\sin y]e^x.$$

$$U_x(z, 0) = 4e^z + e^z(4z) = 4e^z(1 + z).$$

$$U_y = e^x[-4x\sin y - 3(-y\sin y + \cos y) - 3x\cos y + 4(y\cos y + \sin y)].$$

$$U_y(z, 0) = e^z[-3 - 3z] = -3e^z(1 + z).$$

By Milne-Thomson method,

$$F'(z) = U_x(z, 0) - iU_y(z, 0)$$

$$= 4e^z(1 + z) + i3e^z(1 + z) = e^z(1 + z)(4 + 3i).$$

Integrating w.r.t.z, we get

$$F(z) = \int e^z(1 + z)(4 + 3i)dz + c$$

$$= (4 + 3i)\int (1 + z)d(e^z) + c$$

$$= (4 + 3i)\left[e^z(1 + z) - \int e^z dz\right] + c$$

$$= (4 + 3i)\left[e^z(1 + z) - e^z\right] + c = (4 + 3i)e^z(1 + z - 1) + c$$

$$(4 + 3i)f(z) = (4 + 3i)ze^z + c$$

$$f(z) = ze^z + \frac{c}{4 + 3i}$$

$$f(z) = ze^z + c' \quad \text{where} \quad c' = \frac{c}{4 + 3i}.$$

—

Example 3.52. If $f(z)$ is an analytic function of z, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$. [Dec 2014, Jun 2013, May 2011, May 2009]

Solution. Given $f(z)$ is analytic.

Let $f(z) = u + iv$.

$\Rightarrow$ u and v have continuous partial derivatives and they satisfy C-R equations.

$\therefore u_x = v_y$ and $u_y = -v_x$ and $f'(z) = u_x + iv_x$, $|f'(z)|^2 = u_x^2 + v_x^2$.

Since u and v are harmonic functions, they satisfy Laplace equation.

The complex form of the Laplace operator is

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial \bar{z}\partial z}.$$

$$\text{Now, } \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\frac{\partial^2}{\partial \bar{z}\partial z}|f(z)|^2 = 4\frac{\partial^2}{\partial \bar{z}\partial z}(f(z)\overline{f(z)}).$$

Since $f(z)$ is an analytic function, it is independent of $\bar{z}$.

i.e., $f(z)$ is a function of z only.

Similarly its conjugate $\overline{f(z)}$ is an analytic function of $\bar{z}$ only.

$\therefore$ We can denote $\overline{f(z)}$ by $\bar{f}(\bar{z})$.

i.e., $\overline{f(x+iy)} = \bar{f}(x-iy)$.

$$\begin{aligned}\text{Hence, } \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 &= 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}(f(z)\bar{f}(\bar{z}))\\ &= 4\frac{\partial}{\partial \bar{z}}(\bar{f}(\bar{z}))\frac{\partial}{\partial z}(f(z))\\ &= 4\overline{f'(z)}f'(z) = 4|f'(z)|^2.\end{aligned}$$

Example 3.53. If $w = f(z)$ is a regular function of z, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$. [May 2015, Dec 2001]

Solution. Since $f(z)$ is analytic, if $f(z) = u + iv$, then u and v are harmonic functions. Hence, they satisfy Laplace equation. The complex form of the Laplace operator is

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial \bar{z}\partial z}.$$

$$\begin{aligned}\text{Now, }\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| &= 4\frac{\partial^2}{\partial \bar{z}\partial z}\left(\log|f'(z)|\right)\\ &= 2\frac{\partial^2}{\partial \bar{z}\partial z}\log|f'(z)|^2\\ &= 2\frac{\partial^2}{\partial \bar{z}\partial z}\left(\log(f'(z)\overline{f'(z)})\right)\\ &= 2\frac{\partial^2}{\partial \bar{z}\partial z}\left(\log f'(z)+\log\overline{f'(z)}\right)\\ &= 2\frac{\partial^2}{\partial \bar{z}\partial z}\left(\log f'(z)+\log \bar{f}'(\bar{z})\right)\\ &= 2\frac{\partial}{\partial \bar{z}}\left(\frac{1}{f'(z)}f''(z)\right)=2.0=0.\end{aligned}$$

Example 3.54. If $f(z)$ is analytic, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2|f(z)|^{p-2}|f'(z)|^2$. [May 2001]

Solution. Let $f(z)=u+iv$.

Since $f(z)$ is analytic, u and v are harmonic functions.

Hence, they satisfy Laplace equation.

The complex form of the Laplace operator is

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial \bar{z}\partial z}$$

$$\begin{aligned}\text{Now, }\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p &= 4\frac{\partial^2}{\partial \bar{z}\partial z}(|f(z)|^p)\\ &= 4\frac{\partial^2}{\partial \bar{z}\partial z}(|f(z)|^2)^{\frac{p}{2}}\\ &= 4\frac{\partial^2}{\partial \bar{z}\partial z}\left(f(z)\overline{f(z)}\right)^{\frac{p}{2}}\\ &= 4\frac{\partial^2}{\partial \bar{z}\partial z}\left((f(z))^{\frac{p}{2}}(\bar{f}(\bar{z}))^{\frac{p}{2}}\right)\\ &= 4\frac{\partial}{\partial \bar{z}}(\bar{f}(\bar{z}))^{\frac{p}{2}}\frac{\partial}{\partial z}(f(z))^{\frac{p}{2}}\\ &= 4\frac{p}{2}(\bar{f}(\bar{z}))^{\frac{p}{2}-1}\bar{f}'(\bar{z})\frac{p}{2}(f(z))^{\frac{p}{2}-1}f'(z)\end{aligned}$$

$$= p^2(f(z)\overline{f}(\overline{z}))^{\frac{p}{2}-1}\overline{f}'(\overline{z})f'(z)$$
$$= p^2\left(|f(z)|^2\right)^{\frac{p}{2}-1}\overline{f}'(\overline{z}).f(z)$$
$$= p^2|f(z)^2|^{\frac{p-2}{2}}|f'(z)|^2$$
$$= p^2|f'(z)|^2|f(z)|^{p-2}.$$

Example 3.55. If $f(z)$ is an analytic function of z, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Re } f(z)|^2 = 2|f'(z)|^2$. [Jun 2003]

Solution. Let $f(z) = u + iv$.

Given $f(z)$ is analytic.

Since $Re\ f(z) = u$, we have to prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2) = 2|f'(z)|^2$.

We have $f'(z) = u_x + iv_x$.

$|f'(z)|^2 = u_x^2 + v_x^2$

$$\text{Now, }\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^2 = \frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2} = \frac{\partial}{\partial x}(2uu_x)+\frac{\partial}{\partial y}(2uu_y)$$
$$= 2(uu_{xx}+u_x^2)+2(uu_{yy}+u_y^2)$$
$$= 2u(u_{xx}+u_{yy})+2(u_x^2+u_y^2).$$

Since $f(z)$ is analytic, u is harmonic.

$\therefore u_{xx}+u_{yy} = 0.$

Hence, $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^2 = 2(u_x^2+u_y^2) = 2(u_x^2+v_x^2) = 2|f'(z)|^2$.

Example 3.56. If $f(z)$ is an analytic function, prove that $\left(\frac{\partial}{\partial x}|f(z)|\right)^2+\left(\frac{\partial}{\partial y}|f(z)|\right)^2 = |f'(z)|^2$.

Solution. Given $f(z) = u + iv$.

Since $f(z)$ is analytic, u and v are harmonic.

Also, $f(z)$ satisfies C-R equations.

—

i.e., $u_x = v_y, u_y = -v_x \Rightarrow f'(z) = u_x + iv_x, |f'(z)|^2 = u_x^2 + v_x^2$.

$$\begin{aligned}
|f(z)| &= \sqrt{u^2+v^2} \\
\frac{\partial}{\partial x}|f(z)| &= \frac{1}{2\sqrt{u^2+v^2}}(2uu_x + 2vv_x) \\
&= \frac{uu_x + vv_x}{\sqrt{u^2+v^2}} \\
\left(\frac{\partial}{\partial x}|f(z)|\right)^2 &= \frac{(uu_x+vv_x)^2}{u^2+v^2}
\end{aligned}$$

$$\begin{aligned}
\text{Similarly, } \left(\frac{\partial}{\partial y}|f(z)|\right)^2 &= \frac{(uu_y+vv_y)^2}{u^2+v^2} \\
\text{Adding, } \left(\frac{\partial}{\partial x}|f(z)|\right)^2 + \left(\frac{\partial}{\partial y}|f(z)|\right)^2 &= \frac{u^2u_x^2 + v^2v_x^2 + 2uvu_xv_x + u^2u_y^2 + v^2v_y^2 + 2uvu_yv_y}{u^2+v^2} \\
&= \frac{u^2(u_x^2+u_y^2) + v^2(v_x^2+v_y^2) + 2uv(u_xv_x + u_yv_y)}{u^2+v^2} \\
&= \frac{u^2(u_x^2+v_x^2) + v^2(u_y^2+u_x^2) + 2uv(-u_xu_y + u_xu_y)}{u^2+v^2} \\
&= \frac{u^2|f'(z)|^2 + v^2|f'(z)|^2}{u^2+v^2} \\
&= \frac{|f'(z)|^2(u^2+v^2)}{u^2+v^2} = |f'(z)|^2.
\end{aligned}$$

Example 3.57. If $f(z)$ is analytic, prove that $\nabla^2 \log|f(z)| = 0$. [Jun 2012]

Solution. $\nabla^2 \log|f(z)| = \frac{1}{2}\nabla^2 \log|f(z)|^2$

$$\begin{aligned}
&= \frac{1}{2}\nabla^2 \log(f(z)\overline{f(z)}) \\
&= \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\log f(z) + \log \bar{f}(\bar{z})\right) \\
&= \frac{1}{2}4\frac{\partial^2}{\partial \bar{z}\partial z}\left(\log f(z) + \log \bar{f}(\bar{z})\right) \\
&= 2\frac{\partial}{\partial \bar{z}}\left(\frac{1}{f(z)}f'(z)\right) = 0.
\end{aligned}$$

Exercise 3 C

1. Construct the analytic function whose imaginary part is $v = x^2 - y^2 + 2xy - 3x - 2y$. [Dec 2001]

2. If $u = x^3 - 3xy^2 + y + 1$, find $f = u + iv$ is analytic. [Dec 2004]

3. Construct the analytic function whose real part is $u = \dfrac{y}{x^2 + y^2}$. [April 2004]

4. Find the analytic function whose imaginary part is given by $v = e^x(x \sin y + y \cos y)$. [Jan 2008]

5. Find the analytic function whose real part is given by $u = e^{-x}(x \sin y - y \cos y)$. [May 2005]

6. If $u = \log(x^2 + y^2)$, find v and $f(z)$ such that $f(z) = u + iv$ is analytic. [May 2005]

7. Find the analytic function $w = u + iv$ given $v = e^{-2xy} \sin(x^2 - y^2)$. [May 1997]

8. If $w = u + iv$ is an analytic function and $v = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$ find u. [Dec 1997]

9. Find the analytic function $f(z) = u + iv$ if $u - v = (x - y)(x^2 + 4xy + y^2)$.

10. If $f(z) = u + iv$ and $u + v = (x - y)(x^2 + 4xy + y^2)$ find $f(z)$.

11. Find the analytic function $f(z)$ given that $u + v = e^x(\cos y - \sin y)$.

12. If $f(z) = u + iv$ is an analytic function of z and $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cos hy)}$, find $f(z)$ given that $f(\frac{\pi}{2}) = 0$.

13. Show that the following functions are harmonic

    i. $u = 2x - x^3 + 3xy^2$. [May 2000]

    ii. $v = \dfrac{-y}{x^2 + y^2}$. [Dec 2001]

    iii. $u = 3x^2y - y^3$. [Jun 2001]

—

14. Show that the function $u = y^3 - 3x^2y$ is a harmonic function. Find its harmonic conjugate and the corresponding analytic function.

15. Show that $u = y + e^x \cos y$ is harmonic. Find its harmonic conjugate and also find the function with u as its real part.

16. If $u - v = x^3 - y^3 + 3xy(x - y)$, find $f(z) = u + iv$ interms of z.

17. Find the analytic function $f(z) = u + iv$, given that $u - 2v = e^x(\cos y - \sin y)$.

18. If $2u + v = e^x(\cos y - \sin y)$ find $f(z) = u + iv$.

19. Find the analytic function $f(z) = u + iv$ given that $2u + v = e^{2x}((2x + y)\cos 2y + (x - 2y)\sin 2y)$.

20. If $f(z) = u + iv$ is an analytic function of z, find $f(z)$ if $2u + v = e^x(\cos y - \sin y)$.

21. Find the analytic function $f(z) = u + iv$ given that $u + v = \frac{1}{2}\log(x^2 + y^2) + \tan^{-1}\left(\frac{y}{x}\right)$.

22. Find the analytic function $f(z) = u + iv$ given that $u + v = \dfrac{\sin 2x}{\cos h2y + \cos 2x}$.

23. Find the analytic function $f(z) = u + iv$, given that $u + v = \dfrac{\sin 2x}{\cos h2y - \cos 2x}$.

24. Find the analytic function $f(z) = u + iv$ given that $3u + 7v = e^x(3\cos y + 7\sin y)$.

25. Find the analytic function $f(z) = u + iv$, given that $3u + 5v = e^{x^2 - y^2}(3\cos 2xy + 5\sin 2xy)$.

26. Find the analytic function $f(z) = u + iv$ given (i) $u = a(1 + \cos\theta)$ (ii) $v = \left(r - \frac{1}{r}\right)\sin\theta,\ r \neq 0$.

—

3.6 Bilinear Transformation (Mobius Transformation)

Definition. The Transformation

$$w = \frac{az+b}{cz+d} \tag{1}$$

where a, b, c, d are complex constants such that $ad - bc \neq 0$ is called a **bilinear transformation or Mobius transformation**.

Results

1. If $ad - bc = 0$ then $ad = bc \Rightarrow \frac{d}{c} = \frac{b}{a}$.
   Now $w = \frac{a(z+\frac{b}{a})}{c(z+\frac{d}{c})} = \frac{a}{c}$ which is a constant.
   $\therefore ad - bc \neq 0$ is the necessary condition for (1) to be bilinear.

2. When $z = \frac{-d}{c}$, then $w = \infty$.
   Therefore, there is a one to one correspondence between the extended $z-$plane and the extended $w-$plane. When $w = \frac{a}{c}, z = \infty$.

3. Special cases.
   (i) If $c = 0, a = d \neq 0$ then $w = \frac{a}{d}z + \frac{b}{d} = z + \frac{b}{a}$, which is a translation.
   (ii) If $c = 0, b = 0, d \neq 0$ then $w = \frac{a}{d}z$ which is rotation and magnification.
   (iii) If $c = 0, d \neq 0$ then $w = \frac{a}{d}z + \frac{b}{d}$ which is rotation, magnification and translation.

3.6.1 Fixed points or Invariant points

Result. Prove that a bilinear transformation has atmost two fixed points.

[Jun 2012]

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The fixed points of the bilinear transformation are given by $w = z$

$$\text{i.e., } z = \frac{az+b}{cz+d}$$

$$\implies cz^2 + dz - az - b = 0$$

$$\text{i.e., } cz^2 + (d-a)z - b = 0$$

If $c \neq 0$, it is a quadratic in z.

$\therefore$ It has two roots which are the fixed points of the transformation.

If $c = 0, d \neq a$ then there is one fixed point.

In this case, it is a linear transformation.

$\therefore$ A bilinear transformation has atmost two fixed points in the extended plane.

Note

1. If a bilinear transformation has three or more fixed points, then it must be the identity mapping.

2. If a bilinear mapping has exactly two fixed points z_1 and z_2, then for some nonzero k, they satisfy the equation.

$$\frac{w-z_1}{w-z_2} = k\frac{z-z_1}{z-z_2}$$

3. If a bilinear transformation has only one fixed point z_1, then it can be written as $\dfrac{1}{w-z_1} = k + \dfrac{1}{z-z_1}$ for $k \neq 0$.
   These two forms are called normal forms or canonical forms of a bilinear transformation.

Cross ratio

The cross ratio of four points z_1, z_2, z_3, z_4 is defined as $\dfrac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$ and it is denoted by (z_1, z_2, z_3, z_4).

Note. The four letters can be arranged in $4! = 24$ ways and hence we can write 24 cross ratios. But, only six of them are different. In each of them z_1 is fixed and the remaining 3 are arranged in $3! = 6$ ways.

—

If λ is any cross ratio of 4 points then the 6 different cross ratios are

$$\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, 1-\frac{1}{\lambda}, \frac{1}{1-\frac{1}{\lambda}} = \frac{\lambda}{\lambda-1}$$

Theorem. The bilinear transformation that maps the points z_1, z_2, z_3 of the $z-$plane on to the points w_1, w_2, w_3 of the w plane is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

This equation defines a bilinear transformation that maps the distinct points z_1, z_2, z_3 onto the distinct points w_1, w_2, w_3 of the finite $w-$plane.

Note. If one of the given points is ∞, say $z_1 = \infty$. Then $\frac{1}{z_1} = 0$.

Now

$$\begin{aligned} \text{RHS} &= \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \\ &= \frac{z_1\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)z_1\left(\frac{z_2}{z_1}-1\right)} = \frac{-1(z_2-z_3)}{(z-z_3)(-1)} = \frac{z_2-z_3}{z-z_3} \end{aligned}$$

$\therefore$ The modified form of the bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{z_2-z_3}{z-z_3}$$

i.e., The equation is obtained formally by omitting the factors involving z_1.

In the same way, we can rewrite the equation of the bilinear transformation if any other prescribed point is ∞ in the $z-$plane or $w-$plane.

Properties of Bilinear transformation

1. A bilinear transformation always transforms circles into circles and lines into lines.

2. Bilinear transformation preserves cross ratio of four points.
   i.e., cross ratio of four points is invariant under bilinear transformation.

—

Worked Examples

Example 3.58. Find the bilinear map which maps the points $\infty, i, 0$ of the z plane onto $0, i, \infty$ of the w–plane. [May 2005]

Solution. Let $z_1 = \infty, z_2 = i, z_3 = 0$ and $w_1 = 0, w_2 = i, w_3 = \infty$.

The bilinear transformation is given by

$$\frac{w - w_1}{w_2 - w_1} = \frac{z_2 - z_3}{z - z_3}$$

$$\text{i.e., } \frac{w - 0}{i - 0} = \frac{i - 0}{z - 0}$$

$$\frac{w}{i} = \frac{i}{z}$$

$$w = \frac{-1}{z}.$$

Example 3.59. Find the invariant points of the transformation $w = \dfrac{2z + 6}{z + 7}$. [May 2009]

Solution. The fixed points are given by $w = z$

$$\text{i.e., } \frac{2z + 6}{z + 7} = z$$

$$z(z + 7) = 2z + 6$$

$$z^2 + 7z - 2z - 6 = 0$$

$$z^2 + 5z - 6 = 0$$

$$(z + 6)(z - 1) = 0$$

$$z = 1, -6.$$

Example 3.60. Find the fixed points of the transformation $w = \dfrac{3 - z}{1 + z}$.

Solution. The fixed points are given by $w = z$

$$\text{i.e., } \frac{3 - z}{1 + z} = z$$

$$z(1 + z) = 3 - z$$

$$z + z^2 - 3 + z = 0$$
$$z^2 + 2z - 3 = 0$$
$$(z+3)(z-1) = 0 \quad \Rightarrow z = 1, -3.$$

Example 3.61. Find the invariant points of the transformation $w = \dfrac{1+z}{1-z}$.
[Dec 2011, Jun 2007]

Solution. The fixed points are given by $w = z$

$$\text{i.e., } \frac{1+z}{1-z} = z$$
$$z(1-z) = 1+z$$
$$z - z^2 - 1 - z = 0$$
$$z^2 + 1 = 0$$
$$z^2 = -1 \quad \Rightarrow z = \pm i.$$

Example 3.62. Find the bilinear transformation which maps the points $-2, 0, 2$ into the points $w = 0, i, -i$. [May 2002]

Solution. Let $z_1 = -2, z_2 = 0, z_3 = 2$ and $w_1 = 0, w_2 = i, w_3 = -i$.

The bilinear transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$
$$\frac{(w-0)(i+i)}{(w+i)(i-0)} = \frac{(z+2)(0-2)}{(z-2)(0+2)}$$
$$\frac{w2i}{(w+i)i} = -\frac{2(z+2)}{2(z-2)}.$$
$$2w(z-2) = -(z+2)(w+i)$$
$$2w(z-2) + w(z+2) = -i(z+2)$$
$$w(2z - 4 + z + 2) = -i(z+2)$$
$$w(3z-2) = -i(z+2)$$
$$w = -\frac{i(z+2)}{3z-2}.$$

—

Example 3.63. Find the Mobius transformation that maps $z = 0, 1, \infty$ into points $w = -5, -1, 3$ respectively. What are the invariant points of the transformation. [May 2004]

Solution. Let $z_1 = 0, z_2 = 1, z_3 = \infty$, and $w_1 = -5, w_2 = -1, w_3 = 3$.

The Mobius transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)}{(z_2-z_1)}$$

$$\text{i.e., } \frac{(w+5)(-4)}{(w-3)4} = \frac{z-0}{1}$$

$$\Rightarrow -(w+5) = z(w-3)$$

$$\Rightarrow -w-5 = zw-3z$$

$$\Rightarrow -w-zw = 5-3z$$

$$\Rightarrow -w(1+z) = -(3z-5)$$

$$w = \frac{3z-5}{1+z}.$$

The invariant points are given by $w = z$.

$$z = \frac{3z-5}{1+z}$$

$$z + z^2 - 3z + 5 = 0$$

$$z^2 - 2z + 5 = 0$$

$$(z-1)^2 + 5 - 1 = 0$$

$$(z-1)^2 + 4 = 0$$

$$(z-1)^2 = -4$$

$$z - 1 = \pm 2i$$

$$z = 1 \pm 2i.$$

Example 3.64. Find the bilinear transformation that maps the points $z = \infty, i, 0$ onto $w = 0, i, \infty$ respectively. [Dec 2012]

Solution. Given, $z_1 = \infty, z_2 = i, z_3 = 0$.

$$w_1 = 0, w_2 = i, w_3 = \infty.$$

The bilinear transformation is

$$\frac{(w - w_1)}{w_2 - w_1} = \frac{z_2 - z_3}{z - z_3}$$
$$\frac{w - 0}{i - 0} = \frac{i - 0}{z - 0}$$
$$\frac{w}{i} = \frac{i}{z}$$
$$w = -\frac{1}{z}.$$

Example 3.65. Find the bilinear transformation mapping the points $z = 1, i, -1$ into the points $w = 2, i, -2$. [Dec 2005]

Solution. Let $z_1 = 1, z_2 = i, z_3 = -1$ and $w_1 = 2, w_2 = i, w_3 = -2$.

The bilinear transformation is given by

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$\text{i.e., } \frac{(w - 2)(i + 2)}{(w + 2)(i - 2)} = \frac{(z - 1)(i + 1)}{(z + 1)(i - 1)}$$

$$\frac{w - 2}{w + 2} = \frac{z - 1}{z + 1}\frac{(i + 1)(i - 2)}{(i - 1)(i + 2)}$$

$$= \left(\frac{z - 1}{z + 1}\right)\frac{(-1 - 2i + i - 2)}{(-1 + 2i - i - 2)}$$

$$= \left(\frac{z - 1}{z + 1}\right)\frac{(-3 - i)}{(-3 + i)}$$

$$\frac{w - 2}{w + 2} = \left(\frac{z - 1}{z + 1}\right)\frac{(3 + i)}{(3 - i)}$$

$$\frac{w - 2 + w + 2}{w - 2 - w - 2} = \frac{(z - 1)(3 + i) + (z + 1)(3 - i)}{(z - 1)(3 + i) - (z + 1)(3 - i)}$$

$$\frac{2w}{-4} = \frac{3z + iz - 3 - i + 3z - iz + 3 - i}{3z + iz - 3 - i - 3z + iz - 3 + i}$$

$$= \frac{6z - 2i}{2iz - 6}$$

$$= \frac{2(3z - i)}{2(iz - 3)}$$

$$w = -\frac{(6z - 2i)}{iz - 3}.$$

Example 3.66. Find the bilinear transformation which maps the points $i, -1, 1$ of the z plane into the points $0, 1, \infty$ of the $w-$plane respectively. [Jun 2008]

Solution. Let $z_1 = i, z_2 = -1, z_3 = 1$ and $w_1 = 0, w_2 = 1, w_3 = \infty$.

The required bilinear transformation is

$$\frac{(w-w_1)}{(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\text{i.e., } w = \frac{(z-i)(-2)}{(z-1)(-1-i)} = \frac{-2(z-i)}{-(z-1)(1+i)} = \frac{2(z-i)}{(1+i)(z-1)}.$$

Example 3.67. Find the bilinear transformation which maps $z_1 = 0, z_2 = 1, z_3 = \infty$ into the points $w_1 = i, w_2 = 1, w_3 = -i$. [Jun 2013, Jun 2012, Jun 2010, Jun 2008]

Solution. Let $z_1 = 0, z_2 = 1, z_3 = \infty$, and $w_1 = i, w_2 = 1, w_3 = -i$.

The mobius transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)}{(z_2-z_1)}$$

$$\text{i.e., } \frac{(w-i)(1+i)}{(w+i)(1-i)} = z$$

$$\frac{w-i}{w+i} = \frac{z(1-i)}{1+i} = \frac{z(1-i)^2}{2}$$

$$= \frac{z(1-1-2i)}{2} = \frac{z(-2i)}{2} = -iz$$

$$\frac{w-i+w+i}{w-i-w-i} = \frac{-iz+1}{-iz-1}$$

$$\frac{2w}{-2i} = \frac{1-iz}{-(1+iz)}$$

$$w = \frac{i(1-iz)}{1+iz} = \frac{z+i}{1+iz}.$$

Example 3.68. Find the bilinear map if 1 and i are fixed points and the origin goes into -1.

Solution. Let $z_1 = 1, z_2 = i, z_3 = 0$ and $w_1 = 1, w_2 = i, w_3 = -1$.

The bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-1)(i+1)}{(w+1)(i-1)} = \frac{(z-1)(i-0)}{(z-0)(i-1)}$$

$$\frac{(w-1)(i+1)}{w+1} = \frac{(z-1)i}{z}$$
$$\frac{w-1}{w+1} = \frac{(z-1)i}{z(1+i)}$$
$$\frac{w-1+w+1}{w-1-(w+1)} = \frac{(z-1)i+z(1+i)}{(z-1)i-z(1+i)}$$
$$\frac{2w}{-2} = \frac{z(1+2i)-i}{-(z+i)}$$
$$w = \frac{z(1+2i)-i}{z+i}.$$

Example 3.69. Show that $w = \frac{i-z}{i+z}$ maps the real axis of the $z-$plane into the circle $|w| = 1$ and the half plane $y > 0$ into the interior of the unit circle $|w| = 1$ in the w plane.

Solution. Given $w = \frac{i-z}{i+z}$.

$|w| = 1$ becomes

$$\left|\frac{i-z}{i+z}\right| = 1$$
$$\Rightarrow \frac{|i-z|}{|i+z|} = 1$$
$$\Rightarrow |i-z| = |i+z|$$

i.e., $|i-x-iy| = |i+x+iy|$

$$|-x+i(1-y)| = |x+i(1+y)|$$
$$|-x+i(1-y)|^2 = |x+i(1+y)|^2$$
$$\cancel{x^2}+(1-y)^2 = \cancel{x^2}+(1+y)^2$$
$$(1-y)^2 = (1+y)^2$$
$$1-y = 1+y$$

i.e., $2y = 0$

$\Rightarrow y = 0,$ which is the real axis.

$\therefore$ The real axis of the $z-$plane is mapped onto the circle $|w| = 1$.

—

Now, for the interior of the circle $|w| = 1$ we have

$$|w| < 1$$
$$\Rightarrow \frac{|i-z|}{|i+z|} < 1$$
$$|i-z| < |i+z|$$
$$(1-y)^2 < (1+y)^2$$
$$\Rightarrow 1-y < 1+y$$
$$\Rightarrow -2y < 0$$
$$2y > 0$$
$$\text{i.e., } y > 0.$$

Hence, the half plane $y > 0$ is mapped onto the interior of the circle $|w| < 1$.

Example 3.70. Find the bilinear map which maps the points $z = 0, -1, i$ onto the points $w = i, 0, \infty$. Also find the image of the unit circle of the $z-$plane.

[May 2015, Dec 2013]

Solution. Given, $z_1 = 0, z_2 = -1, z_3 = i.$

$w_1 = i, w_2 = 0, w_3 = \infty.$

The bilinear transformation is

$$\frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$
$$\frac{w-i}{0-i} = \frac{(z-0)(-i-1)}{(z-i)(-1-0)}$$
$$\frac{w-i}{-i} = -\frac{z(1+i)}{(z-i)(-1)}$$
$$\frac{w-i}{-i} = \frac{z(1+i)}{z-i}$$
$$w-i = \frac{-iz(1+i)}{z-i} = \frac{-iz+z}{z-i}$$
$$w = i + \frac{-iz+z}{z-i} = \frac{iz+1-iz+z}{z-i}$$
$$w = \frac{1+z}{z-i}.$$

$$w(z-i) = 1+z$$
$$wz - wi = 1+z$$
$$wz - z = 1+wi$$
$$z(w-1) = 1+wi$$
$$z = \frac{1+wi}{w-1}.$$

Given $|z| = 1$

$$\left|\frac{1+wi}{w-1}\right| = 1$$
$$\frac{|1+wi|}{|w-1|} = 1$$
$$|1+(u+iv)i| = |u+iv-1|$$
$$|1+iu-v|^2 = |(u-1)+iv|^2$$
$$(1-v)^2+u^2 = (u-1)^2+v^2$$
$$1+v^2-2v+u^2 = u^2+1-2u+v^2$$
$$-2v = -2v$$
$$u = v,$$

which is a straight line in the w–plane.

Example 3.71. Prove that $w = \dfrac{z}{1-z}$ maps the upper half of the z–plane to the upper half of the w–plane and also find the image of the unit circle of the z–plane.

[Dec 2013, Dec 2012, Jun 2010]

Solution. Given $w = \dfrac{z}{1-z}$.

$$u+iv = \frac{x+iy}{1-x-iy}$$
$$= \frac{x+iy}{(1-x)-iy} \times \frac{(1-x)+iy}{(1-x)+iy}$$
$$= \frac{x(1-x)-y^2+i[y(1-x)+xy]}{(1-x)^2+y^2}$$

—

$$= \frac{x(1-x)-y^2}{(1-x)^2+y^2} + i\frac{y(1-x)+xy}{(1-x)^2+y^2}.$$

$$\therefore u = \frac{x(1-x)-y^2}{(1-x)^2+y^2} \text{ and } v = \frac{y(1-x)+xy}{(1-x)^2+y^2}.$$

Let $v > 0$.

$$\Rightarrow \frac{y(1-x)+xy}{(1-x)^2+y^2} > 0$$

$$y - xy + xy > 0$$

$$y > 0.$$

This shows that the upper half of the $z-$plane is mapped onto the upper half of the $w-$plane.

<u>To find the image of the unit circle.</u>

$$\text{we have } w = \frac{z}{1-z}$$

$$w(1-z) = z$$

$$w - wz = z$$

$$w = wz + z$$

$$z(1+w) = w$$

$$z = \frac{w}{1+w}.$$

Given $|z| = 1$.

$$\Rightarrow \left|\frac{w}{1+w}\right| = 1$$

$$|w| = |1+w|$$

$$|w|^2 = |1+w|^2$$

$$|u+iv|^2 = |1+u+iv|^2$$

$$u^2 + v^2 = (1+u)^2 + v^2 = 1 + u^2 + 2u + v^2$$

$$0 = 2u + 1$$

$$u = -\frac{1}{2}.$$

The image of the unit circle $|z| = 1$ is the straight line $u = -\frac{1}{2}$.

Example 3.72. Find the bilinear transformation that transforms $1, i, -1$ of the $z-$plane onto $0, 1, \infty$ of the $w-$plane. Also show that the transformation maps interior of the unit circle of the $z-$plane onto upper half of the $w-$plane.[Apr 2014]

Solution. Let $z_1 = 1, z_2 = i, z_3 = -1$.

$w_1 = 0, w_2 = 1, w_3 = \infty$.

The bilinear transformation is

$$\frac{w - w_1}{w_2 - w_1} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$\frac{w - 0}{1 - 0} = \frac{(z - 1)(i + 1)}{(z + 1)(i - 1)}$$

$$w = -\frac{z - 1}{z + 1}\frac{1 + i}{1 - i} \times \frac{1 + i}{1 + i}$$

$$= -\frac{z - 1}{z + 1}\frac{(1 + i)^2}{1 + 1}$$

$$= -\frac{z - 1}{z + 1}\left(\frac{1 - 1 + 2i}{2}\right) = -i\frac{z - 1}{z + 1}$$

$$w(z + 1) = -iz + i$$

$$wz + w = -iz + i$$

$$wz + iz = i - w$$

$$z(w + i) = i - w$$

$$z = \frac{i - w}{w + i}.$$

Given $|z| < 1$

$$\Rightarrow \left|\frac{i - w}{w + i}\right| < 1$$

$$|i - w| < |w + i|$$

$$|i - u - iv| < |u + iv + i|$$

$$| - u + i(1 - v)| < |u + i(1 + v)|$$

$$
\begin{aligned}
|-u+i(1-v)|^2 &< |u+i(1+v)|^2 \\
u^2+(1-v)^2 &< u^2+(1+v)^2 \\
(1-v)^2 &< (1+v)^2 \\
1-v < 1+v &\Rightarrow -2v < 0 \\
v &> 0,
\end{aligned}
$$

which is the upper half of the w−plane.

Hence, the interior of the unit circle of the z−plane is mapped onto the upper half of the w−plane.

Exercise 3 D

1. Find the bilinear transformation that maps $z_1 = -1, z_2 = 0, z_3 = 1$ onto $w_1 = -1, w_2 = -i, w_3 = 1$. [May 2006]

2. Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$. [Jun 2005]

3. Find the bilinear transformation that maps $z_1 = 0, z_2 = 1, z_3 = \infty$ onto $w_1 = -1, w_2 = -i, w_3 = 1$. [Apr 2006]

4. Find the bilinear transformation that maps the points $-1, 0, 1$ onto $1, -1, \infty$. [Jan 2007]

5. Find the bilinear transformation which maps the points $1, i, -1$ of the z−plane onto $0, 1, \infty$ of the w−plane. [May 2007]

6. Find the bilinear transformation which maps $z = -1, i, 1$ onto the points $w = 1, i, -1$ respectively. [Dec 2004]

7. Find the fixed points of the bilinear map $w = \dfrac{2i-6z}{iz-3}$.

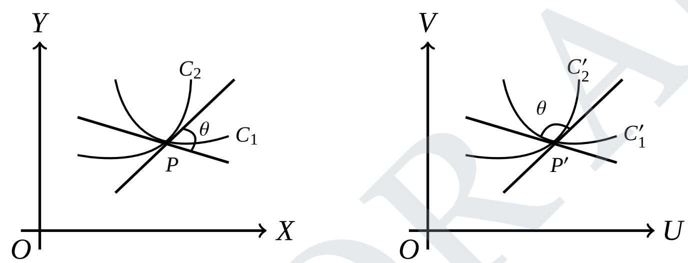
—

8. Find the bilinear transformation that maps the points $-1, 0, 1$ in the $z-$plane onto the points $0, i, 3i$ in the $w-$plane. [Apr 1999]

9. Obtain the bilinear transformation which maps $z = 0$ onto $w = -i$ and has $-1, 1$ as fixed points. [Apr 2000]

10. Find the bilinear transformation which maps $z = -i, 0, i$ onto the points $w = \infty, -1, 0$.

3.7 Conformal mapping



Let C_1 and C_2 be two curves in the $z-$plane intersecting at P. Let C_1' and C_2' be the corresponding curves in the w plane intersecting at P' under the transformation $w = f(z)$. If the angle of intersection of the curves at P is the same as the angle of intersection of the curves at P', both in magnitude and direction, then the transformation is said to be conformal at P.

Result A mapping $w = f(z)$ is said to be conformal in a domain D if it is conformal at each point of the domain.

Theorem

A mapping $w = f(z)$ is conformal at a point z_0 if it is analytic at z_0 and $f'(z_0) \neq 0$.

Critical Points

Let $f(z)$ be a non constant analytic function in a domain D. If $f'(z_0) = 0$ for some z_0 in D, then z_0 is called a critical point of the transformation $w = f(z)$.

—

Fixed point or invariant point

A fixed point or invariant point of a transformation $w = f(z)$ is a point z_0 such that $f(z_0) = z_0$.

Scale factor

Scalar factor of a conformal mapping $w = f(z)$ at z_0 is $|f'(z_0)|$.

If $|f'(z_0)| > 1$ then $|f'(z_0)|$ represents an expansion.

If $|f'(z_0)| < 1$ then $|f'(z_0)|$ represents an contraction.

Mapping of Elementary functions

3.7.1 The transformation $w = z + k$

The mapping $w = z + k$ where k is a complex constant represents a translation.

Let $z = x + iy, w = u + iv, k = k_1 + ik_2$

$u + iv = x + iy + k_1 + ik_2 = (x + k_1) + i(y + k_2)$.

The image of any point (x, y) in the $z-$plane is the point $(x + k_1, y + k_2)$ in the $w-$plane.

Under translation any figure is shifted through a distance given by the vector k. But the size and shape remain the same.

$\therefore$ Circles are transformed into circles, squares are transformed into squares etc.

In general, under the transformation, the image region is geometrically congruent to the original region.

If $k = 0$, then the mapping becomes $w = z$ which is the identity mapping.

Worked Examples

Example 3.73. Find the image of the circle $|z| = 2$ under the transformation $w = z + 3 + 2i$. [May 2011, May 2008]

Solution. $w = z + 3 + 2i$

$u + iv = x + iy + 3 + 2i = x + 3 + i(y + 2)$.

Equating the real and imaginary parts, we get $u = x + 3, v = y + 2$.

The equation of the circle is $|z| = 2$.

i.e., $\sqrt{x^2 + y^2} = 2$

—

$\implies x^2 + y^2 = 4$

i.e., $(u-3)^2 + (v-2)^2 = 4$

which represents a circle with centre $(3, 2)$ and radius 2 in the $w-$plane.

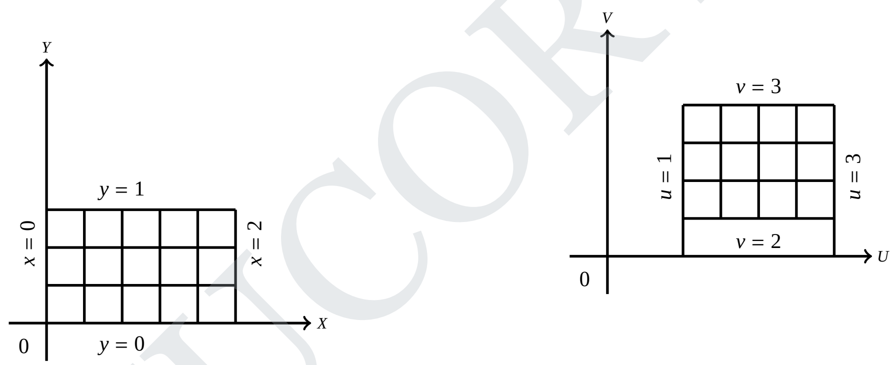
Example 3.74. What is the region of the $w-$plane of the rectangular region in the $z-$plane bounded by the lines $x = 0, x = 2, y = 0$ and $y = 1$ in the region mapped under the transformation $w = z + (1 + 2i)$. [Dec 2006]

Solution. $w = z + 1 + 2i$

$$u + iv = x + iy + 1 + 2i = x + 1 + i(y + 2).$$

Equating the real and imaginary parts, we get $u = x + 1, v = y + 2$.

The lines $x = 0, x = 2, y = 0$ and $y = 1$ have images $u = 1, u = 3, v = 2, v = 3$ respectively in the w plane.



Example 3.75. Find the map of the circle $|z| = 3$ by the transformation $w = z + 1 + i$. [May 2007]

Solution. $w = z + 1 + i$

$u + iv = x + iy + 1 + i = x + 1 + i(y + 1)$.

Equating the real and imaginary parts, we get $u = x + 1, v = y + 1$.

The circle in the $z-$plane is $|z| = 3$.

$$\text{i.e., } \sqrt{x^2 + y^2} = 3$$

$$\Longrightarrow x^2 + y^2 = 9$$

$$\text{i.e., } (u-1)^2 + (v-1)^2 = 9$$

which represents a circle with centre $(1, 1)$ and radius 3 in the w–plane.

Example 3.76. Find the image of the circle $|z| = 1$ under the map $w = z + (2 + 2i)$.

Solution. $w = z + 2 + 2i$

i.e., $u + iv = x + iy + 2 + 2i = x + 2 + i(y + 2)$

Equating the real and imaginary parts we obtain

$u = x + 2 \qquad v = y + 2$

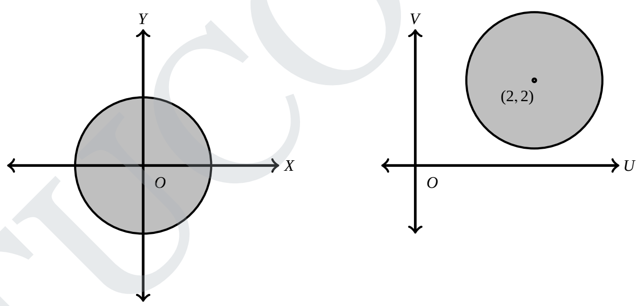
$x = u - 2 \qquad y = v - 2.$

The circle in the z–plane is

$|z| = 1 \Rightarrow x^2 + y^2 = 1$

$\Rightarrow (u-2)^2 + (v-2)^2 = 1.$

i.e., The unit circle $|z| = 1$ is mapped onto an equal circle with centre $(2, 2)$ and radius 1.



3.7.2 The transformation $w = az$

Here a is a nonzero complex constant and $z \neq 0$.

Let $z = re^{i\theta}, w = Re^{i\phi}, a = a_1 e^{i\alpha}$ where $|a| = a_1$.

Now, the given transformation is $w = az$

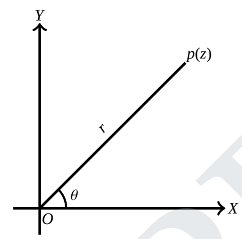
$$Re^{i\phi} = a_1 e^{i\alpha} re^{i\theta} = a_1 re^{i(\alpha+\theta)}$$

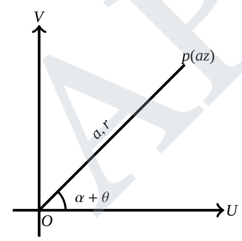
$\therefore$ The transformation equations are $R_1 = a_1 r, \phi = \alpha + \theta$.

$\therefore$ The point (r, θ) in the $z-$plane is mapped onto $(a_1 r, \alpha + \theta)$ in the w plane.

i.e., The magnitude of the vector representing z in the plane is magnified or contracted by $|a| = a_1$ in the $w-$plane and rotated through an angle $\alpha = \arg a$.

$\therefore$ This transformation consists of magnification and rotation. Hence, the image of the given region is geometrically similar to that of the original.





Note

1. If a is real then $\alpha = 0$. $\Rightarrow w = az$ represents magnification or contraction by $|a|$.

2. The general transformation $w = az + b$ where $a \neq 0, a$ and b are complex numbers is a composition of the transformation $w = az$ and $w = z + b$.
   $\therefore$ The general linear transformation represents expansion or contraction by $|a|$ and rotation followed by a transaction by vector b.

Worked Examples

Example 3.77. Find the image of the circle $|z| = 2$, under the transformation $w = 3z$. [Jun 2008]

Solution. Given $w = 3z$

Taking modulus both sides we get

$|w| = |3z| = 3|z| = 3 \times 2 = 6.$

—

$$\implies \sqrt{u^2+v^2} = 6$$

or $u^2+v^2 = 36.$

$\therefore$ The image of $|z| = 2$ under the transformation $w = 3z$ is a circle with centre $(0,0)$ and radius 6.

Example 3.78. Find the map of the circle $|z| = 3$ under the transformation $w = 2z$. [Dec 2012, May 2011]

Solution. Given $w = 2z$

$$|w| = |2z| = 2|z| = 2\times 3 = 6.$$

It is a circle in the $w-$plane with center at the origin and radius 6 units.

Example 3.79. Find the image of the circle $|z| = \lambda$ under the transformation $w = 5z$. [May 2001]

Solution. Given $w = 5z$

Taking modulus both sides we get

$$|w| = |5z| = 5|z| = 5\lambda.$$

$$\sqrt{u^2+v^2} = 5\lambda$$

$$u^2+v^2 = 25\lambda^2.$$

This is a circle in the w plane with centre $(0,0)$ and radius 5λ.

Example 3.80. Find the image of the rectangular region bounded by $x = 0$, $y = 0, x = 1, y = 2$ under the map $w = (1+i)z+2$.

Solution. Given $w = (1+i)z+2$

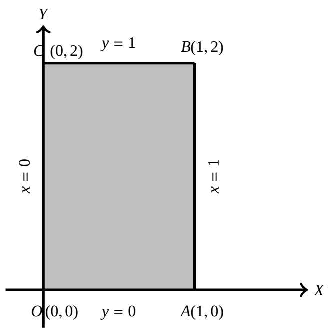
$$\text{ie., } u+iv = (1+i)(x+iy)+2 = x+iy+ix-y+2$$

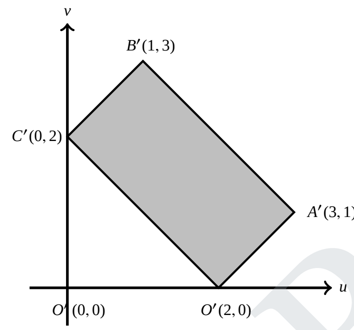
$$u+iv = x-y+2+i(x+y)$$

Equating the real and imaginary parts we get

$$u = x-y+2 \qquad v = x+y$$

—





Let us find the image of the points O, A, B, C.

$$x = 0, y = 0 \Rightarrow u = 2, v = 0 \Rightarrow O(0,0) \text{ has image } O'(2,0)$$

$$x = 1, y = 0 \Rightarrow u = 3, v = 1 \Rightarrow A(1,0) \text{ is mapped onto } A'(3,1)$$

$$x = 1, y = 2 \Rightarrow u = 1, v = 3 \Rightarrow B(1,2) \text{ has the image } B'(1,3)$$

$$x = 0, y = 2 \Rightarrow u = 0, v = 2 \Rightarrow C(0,2) \text{ is mapped onto } C'(0,2).$$

The image of the given rectangular region and its image are as presented above.

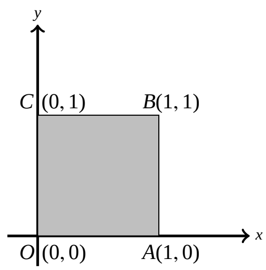
Example 3.81. Draw the image of the square whose vertices are at $(0,0), (1,0), (1,1), (0,1)$ in the $z-$ plane under the transformation $w = (1+i)z$. What has this transformation done to the original square. [Dec 2004]

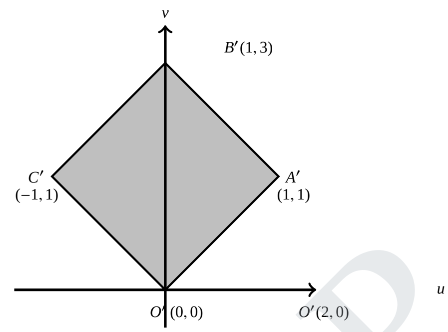
Solution. Given $w = (1+i)z$

$$\text{i.e., } u + iv = (1+i)(x+iy) = x + iy + ix - y = x - y + i(x+y).$$

Equating the real and imaginary parts we get $u = x - y, v = x + y$.

We shall find the images of the corners $(x, y) \to (u, v)$.





Let the given points be $O(0,0), A(1,0), B(1,1)$ and $C(0,1)$.

Let O', A', B', C' be the images of O, A, B, C respectively.

Now O' is $(0,0)$, A' is $(1,1)$, B' is $(0,2)$ and C' is $(-1,1)$.

In the original square, the radius vector of each corner is extended $\sqrt{2}$ times and rotated through an angle of $\frac{\pi}{4}$ about the origin in the anticlockwise sense.

$$OA' = \sqrt{2} = \sqrt{2}OA$$

$$OB' = 2 = \sqrt{2}OB$$

$$OC' = \sqrt{2} = \sqrt{2}OC.$$

Example 3.82. Find the image of the region $y > 1$ under the transformation $w = (1-i)z$. [May 1999]

Solution. Given $w = (1-i)z$

i.e., $u + iv = (1-i)(x+iy) = x + iy - ix + y = x + y + i(y-x)$

Equating the real and imaginary parts we get

$$u = x + y \tag{1}$$

$$v = y - x \tag{2}$$

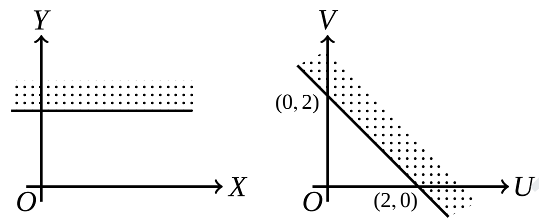
$(1) + (2) \Longrightarrow u + v = 2y \Rightarrow y = \frac{1}{2}(u+v)$

—

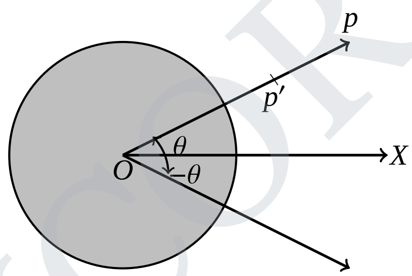
$$(1) - (2) \Longrightarrow u - v = 2x \Rightarrow x = \frac{1}{2}(u - v)$$

$$y > 1 \Rightarrow \frac{1}{2}(u + v) > 1 \Rightarrow u + v > 2.$$

The regions corresponding to the original and image are presented below.



Definition. The inverse of a point P with respect to a circle O and radius r is the point P' on OP such that $OP.OP' = r^2$.



3.7.3 The transformation $w = \frac{1}{z}$

Given $w = \frac{1}{z}$

$\Rightarrow |w| = \left|\frac{1}{z}\right|$

$|w||z| = 1.$

It represents inversion w.r.t the unit circle $|z| = 1$ followed by reflection on the real axis.

Let $z = re^{i\theta}, w = Re^{i\phi}$

Now $w = \frac{1}{z}$

$\Longrightarrow Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$.

Equating the modulii and amplitudes we get

$$R = \frac{1}{r}, \phi = -\theta, \text{i.e., } \arg w = -\arg z$$

$$|w| = \frac{1}{|z|}$$

The image of $P(r,\theta)$ is $Q(\frac{1}{r}, -\theta)$ under the transformation $w = \frac{1}{z}$.

If P' is the inverse of P w.r.t $|z| = 1$ then P' is $(\frac{1}{r}, \theta)$. The reflection of P' in the real axis is $(\frac{1}{r}, -\theta)$.

$\therefore w = \frac{1}{z}$ consists of inversion w.r.t $|z| = 1$ followed by reflection in the real axis.

If $|z| < 1$, then $|w| > 1$.

$\therefore$ Interior of the circle $|z| = 1$ is mapped onto the exterior of the circle $|w| = 1$ and conversely.

The circle $|z| = 1$ is mapped onto $|w| = 1$.

Now $\lim_{z\to 0} \frac{1}{z} = \infty$ and $\lim_{z\to\infty} \frac{1}{z} = 0$, the centre of the circle is mapped onto ∞.

Hence, $w = \frac{1}{z}$ defines a one to one correspondence between the extended $z-$ plane and the extended w-plane.

Result. When $w = u + iv$ is the image of a nonzero point $z = x + iy$, under the transformation $w = \frac{1}{z}$ then

$$w = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}$$

$$u = \frac{x}{x^2 + y^2}, v = \frac{-y}{x^2 + y^2}.$$

Also, $z = \frac{1}{w}$

$$\Rightarrow z = \frac{\bar{w}}{w\bar{w}} = \frac{\bar{w}}{|w|^2} = \frac{u - iv}{u^2 + v^2}$$

$$x = \frac{u}{u^2 + v^2}, y = \frac{-v}{u^2 + v^2}.$$

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Worked Examples

Example 3.83. Find the image of the line $x = k$, under the transformation $w = \frac{1}{z}$. [Jun 2013]

Solution. Given $w = \frac{1}{z}$.

$$z = \frac{1}{w} \times \frac{\overline{w}}{\overline{w}} = \frac{\overline{w}}{w\overline{w}} = \frac{\overline{w}}{|w|^2}$$

$$x + iy = \frac{u - iv}{u^2 + v^2} = \frac{u}{u^2 + v^2} - i\frac{v}{u^2 + v^2}$$

$$\therefore x = \frac{u}{u^2 + v^2}, y = -\frac{v}{u^2 + v^2}.$$

Given $x = k$.

$$i.e., k = \frac{u}{u^2 + v^2}$$

$$k(u^2 + v^2) - u = 0.$$

It represents a circle with centre $\left(\frac{1}{2k}, 0\right)$ and radius $\frac{1}{2k}$.

Example 3.84. Find the image of $x = 2$ under the transformation $w = \frac{1}{z}$. [Dec 1998]

Solution. We have $w = \frac{1}{z}$

$$\Longrightarrow z = \frac{1}{w} = \frac{\overline{w}}{w\overline{w}}$$

$$x + iy = \frac{u - iv}{(u + iv)(u - iv)} = \frac{u - iv}{u^2 + v^2}.$$

$$\therefore x = \frac{u}{u^2 + v^2}, y = -\frac{v}{u^2 + v^2}.$$

When $x = 2$, we have $\frac{u}{u^2 + v^2} = 2$.

$$\Rightarrow u^2 + v^2 - \frac{u}{2} = 0,$$

which is a circle in the w−plane.

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Example 3.85. Find the image of $|z-2i| = 2$ under the transformation $w = \frac{1}{z}$.

[Jun 2013, May 2005]

Solution. $w = \frac{1}{z}$

$$\Rightarrow x = \frac{u}{u^2+v^2}, y = \frac{-v}{u^2+v^2}.$$

The given circle is $|z-2i| = 2$.

$$\Rightarrow |z-2i|^2 = 4$$

$$|x+iy-2i|^2 = 4$$

$$x^2+(y-2)^2 = 4$$

$$x^2+(y-2)^2 = 2^2.$$

It represents circle center at (0, 2) radius 2.

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} - 4\frac{-v}{u^2+v^2} = 0$$

$$\frac{1}{u^2+v^2} + \frac{4v}{u^2+v^2} = 0$$

$$4v+1 = 0 \quad \Rightarrow v = -\frac{1}{4},$$

which is a straight line in the w plane.

$\therefore$ The image of the circle $|z-2i| = 2$ in the $z-$plane is a straight line in the $w-$plane.

Example 3.86. Find the image of the strip $1 < x < 2$ under the map $w = \frac{1}{z}$.

[May 2001]

Solution. Given $w = \frac{1}{z}$

$$\Rightarrow x = \frac{u}{u^2+v^2}, y = \frac{-v}{u^2+v^2}$$

$$1 < x < 2 \Rightarrow 1 < \frac{u}{u^2+v^2} < 2$$

$$\Rightarrow u^2+v^2 < u < 2(u^2+v^2).$$

Taking $u^2+v^2 < u \Rightarrow u^2+v^2-u < 0$, which is the interior of the circle

$$u^2+v^2-u = 0. \tag{1}$$

The centre is $(\frac{-1}{2}, 0)$ and radius $= \frac{1}{2}$.

Taking $u < 2(u^2 + v^2)$ we get

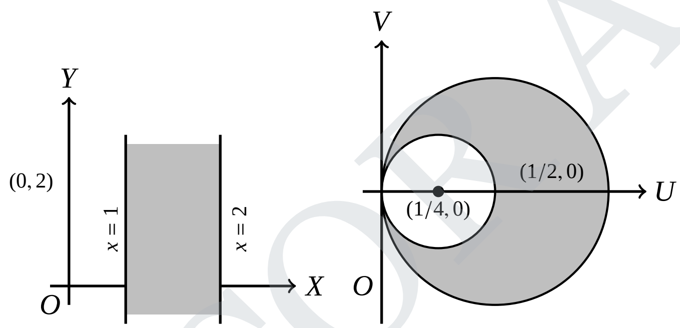
$$2u^2 + 2v^2 - u > 0$$

$$u^2 + v^2 - \frac{1}{2}u > 0. \qquad (2)$$

which is the exterior of the circle $u^2 + v^2 - \frac{1}{2}u = 0$.

The centre is $(\frac{1}{4}, 0)$ and radius $= \frac{1}{4} = r$.

$\therefore$ The strip $1 < x < 2$ is mapped onto the region between the circles (1) and (2).



Example 3.87. Find the image of the infinite strip $0 < y < \frac{1}{2}$ under $w = \frac{1}{z}$.

[May 2005]

Solution. Given $w = \frac{1}{z}$

$$\Rightarrow x = \frac{u}{u^2 + v^2}, y = \frac{-v}{u^2 + v^2}$$

$$0 < y < \frac{1}{2} \Rightarrow 0 < \frac{-v}{u^2 + v^2} < \frac{1}{2}$$

$$\Rightarrow 0 < -v < \frac{1}{2}(u^2 + v^2)$$

Taking $0 < -v \Rightarrow -v > 0 \Rightarrow v < 0 \qquad (1)$

Taking $-v < \frac{1}{2}(u^2 + v^2)$

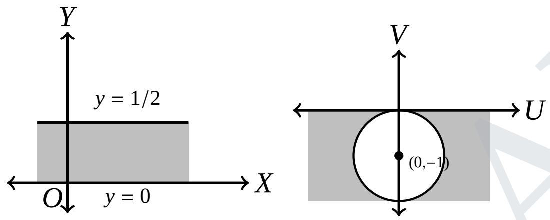
$$\Rightarrow -2v < u^2 + v^2$$

$$u^2 + v^2 > -2v$$

$$u^2 + v^2 + 2v > 0 \tag{2}$$

It is a circle with centre $(0, -1), r = 1$.

The shaded region below represents the original strip and its image in the $w-$plane.



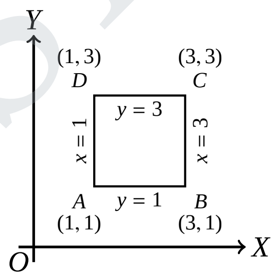
Example 3.88. Find the image of the square whose vertices are $z = 1+i, 3+i, 1+3i$ and $3+3i$ under the transformation $w = \frac{1}{z}$. [Dec 2001]

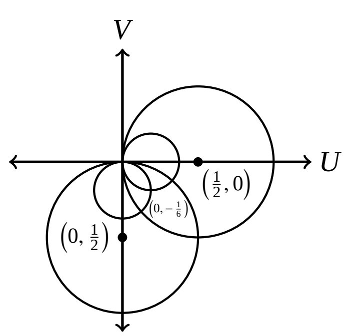
Solution. Let A, B, C, D represent the complex numbers. $1+i, 3+i, 1+3i$ and $3+3i$ respectively in the $z-$plane.

$$w = \frac{1}{z}$$

$$\Rightarrow x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

$$u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}$$





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Equation of AB is $y = 1$.

Equation of CD is $y = 3$.

Equation of AC is $x = 1$.

Equation of AD is $x = 3$.

Image of $y = 1$ is $1 = \frac{-v}{u^2 + v^2} \Rightarrow u^2 + v^2 + v = 0$.

It is a circle with centre $(0, \frac{-1}{2}), r = \frac{1}{2}$.

Image of $y = 3$ is $u^2 + v^2 + \frac{1}{3}v = 0$. It is a circle with centre $(0, \frac{-1}{6}), r = \frac{1}{6}$.

Image of $x = 1$ is $u^2 + v^2 - u = 0$. It is a circle with centre $(\frac{1}{2}, 0), r = \frac{1}{2}$.

Image of $x = 3$ is $u^2 + v^2 - \frac{1}{3}u = 0$. It is a circle with centre $(\frac{1}{6}, 0), r = \frac{1}{6}$.

The point $A(1, 1)$ is mapped onto $A'(\frac{1}{2}, \frac{-1}{2})$.

The point $B(3, 1)$ is mapped onto $B'(\frac{3}{10}, \frac{-1}{10})$.

$C(3, 3)$ is mapped onto $C'(\frac{1}{6}, \frac{-1}{6})$.

$D(1, 3)$ is mapped onto $D'(\frac{1}{10}, \frac{-3}{10})$.

$\therefore$ The image of the interior of the square $ABCD$ is the region $A'B'C'D'$ bounded by the 4 circles which are the images of the sides of the square.

Example 3.89. Show that the transformation $w = \frac{1}{z}$ maps a circle in the z–plane into a circle in the w–plane or to a straight line. [Dec 2011, Jun 2009]

Solution. Given $w = \frac{1}{z}$

Let $w = u + iv, z = x + iy$

$$z = \frac{1}{w} = \frac{\overline{w}}{w\overline{w}} = \frac{u - iv}{u^2 + v^2}$$

$$x = \frac{u}{u^2 + v^2}, y = \frac{-v}{u^2 + v^2}.$$

The general equation of the circle in the z–plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (1)$$

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + 2g\frac{u}{u^2 + v^2} + 2f\frac{-v}{u^2 + v^2} + c = 0$$

$$\frac{u^2+v^2}{(u^2+v^2)^2}+\frac{2gu}{u^2+v^2}-\frac{2fv}{u^2+v^2}+c=0.$$

$$\frac{1}{u^2+v^2}+\frac{2gu}{u^2+v^2}-\frac{2fv}{u^2+v^2}+c=0$$

$$1+2gu-2fv+c(u^2+v^2)=0$$

$$c(u^2+v^2)+2gu-2fv+1=0. \qquad (2)$$

If (1) does not pass through the origin then $c \neq 0$.

(2) is a circle in the w plane.

$\therefore w = \frac{1}{z}$ transforms a circle in the z–plane not passing through the origin into a circle not passing through the origin in the w plane.

If (1) passes through the origin then $c = 0$.

(2)$\Rightarrow$ $2gu - 2fv + 1 = 0$ which is a straight line not passing through the origin.

$\therefore$ A circle not passing through the origin in the z-plane is mapped onto a circle in the w-plane and a circle through the origin in the z-plane is mapped onto a straight line in the w-plane.

Example 3.90. Find the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = \frac{1}{z}$ and prove that it is the lemniscate of Bernouillie $r^2 = \cos 2\theta$.

[Dec 2012, Jun 2012, Jun 2010]

Solution. Let $z = x + iy$ and $w = re^{i\theta}$.

Given $w = \frac{1}{z}$ or $z = \frac{1}{w} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$.

$$i.e., x + iy = \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$\therefore x = \frac{1}{r}\cos\theta, y = -\frac{1}{r}\sin\theta.$$

The given curve is $x^2 - y^2 = 1$

$$\frac{1}{r^2}\cos^2\theta - \frac{1}{r^2}\sin^2\theta = 1$$

$$\frac{1}{r^2}(\cos^2\theta - \sin^2\theta) = 1$$

$$\frac{1}{r^2}\cos 2\theta = 1$$
$$\Rightarrow r^2 = \cos 2\theta.$$

Example 3.91. Under the transformation $w = \frac{1}{z}$, find the image of the region (i) $x > c$ where $c > 0$ (ii) $y > c$ where $c < 0$. Find also the fixed points of w. [Dec 2010]

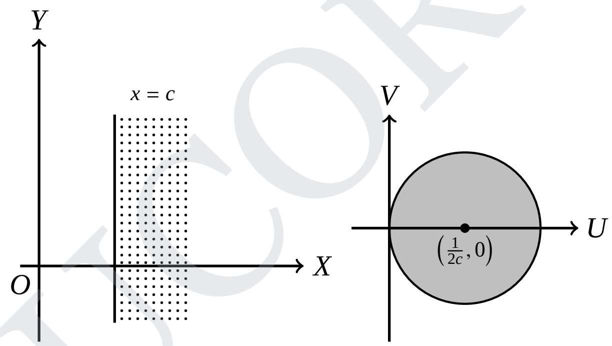
Solution. $w = \frac{1}{z}$

$$\Rightarrow x = \frac{u}{u^2+v^2}, y = \frac{-v}{u^2+v^2}$$

(i) $x > c, c > 0 \Rightarrow \frac{u}{u^2+v^2} > c \Rightarrow \frac{u}{c} > u^2 + v^2$

$$u^2 + v^2 - \frac{u}{c} < 0.$$

It is a circle with centre $(\frac{1}{2c}, 0)$ and radius $\frac{1}{2c}$.
The image is the interior of the circle.



(ii) $y > c, c < 0$

$$\frac{-v}{u^2+v^2} > c$$
$$\frac{-v}{c} < u^2 + v^2$$
$$u^2 + v^2 + \frac{v}{c} > 0.$$

The image is the exterior of the circle $u^2 + v^2 + \frac{v}{c} = 0$ in the $w-$plane with centre $(0, \frac{-1}{2c})$ and radius $\frac{1}{2|c|}$.

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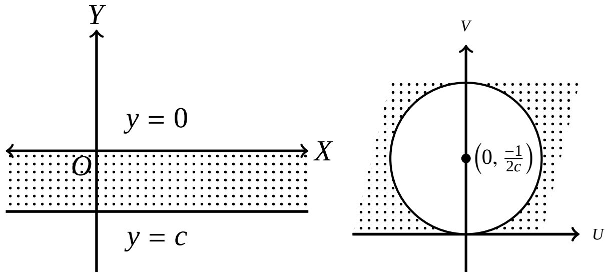


The fixed points are given by $w = z$.

i.e., $z = \dfrac{1}{z} \Rightarrow z^2 = 1 \Rightarrow z = \pm 1$.

3.7.4 The transformation $w = z^2$

Let $z = x + iy$ and $w = u + iv$.

Given $w = z^2$

$u + iv = (x + iy)^2 = x^2 - y^2 + i2xy$

$$\therefore u = x^2 - y^2 \tag{1}$$

$$\text{and} \quad v = 2xy \tag{2}$$

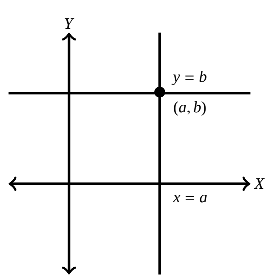
Case(i). Image of lines parallel to the axes.

Consider the line $y = b$, parallel to the x-axis.

Now $u = x^2 - b^2$, $v = 2bx$ $\Rightarrow x = \dfrac{v}{2b}$.

$\therefore u = \dfrac{v^2}{4b^2} - b^2$

$$\frac{v^2}{4b^2} = u + b^2 \quad \Rightarrow \quad v^2 = 4b^2(u + b^2) \tag{3}$$



This is a parabola with vertex $(-b^2, 0)$, axis u-axis.

Hence, the image of a line parallel to the x-axis in the z-plane is a parabola in the w-plane.

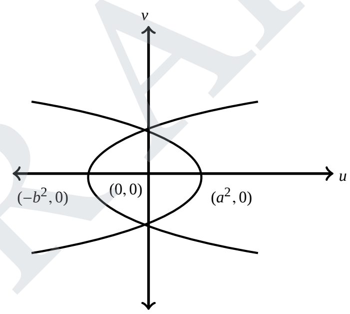
Consider the line $x = a$, parallel to the y-axis.

Then, (1) and (2) are reduced to $u = a^2 - y^2, \quad v = 2ay \quad \Rightarrow ie., y = \dfrac{v}{2a}$.

$$\therefore u = a^2 - \frac{v^2}{4a^2}$$

$$\frac{v^2}{4a^2} = -(u - a^2) \qquad v^2 = -4a^2(u - a^2) \tag{4}$$

which represents a parabola with vertex $(a^2, 0)$, axis as u-axis and focus $(0, 0)$. We shall prove that these two parabolas are orthogonal. Let us find the slopes of the tangents to the parabolas (3) and (4) corresponding to the point $x = a$ and $y = b$.



ie., $u = a^2 - b^2$ and $v = 2ab$

Differentiating (3) w.r.to u, we get

$$2v\frac{dv}{du} = 4b^2$$

$$\frac{dv}{du} = \frac{2b^2}{v}$$

$$\left(\frac{dv}{du}\right)_{(a^2-b^2, 2ab)} = \frac{2b^2}{2ab} = \frac{b}{a}$$

i.e., Slope of the tangent at $(a^2 - b^2, 2ab)$ to the parabola (3) is $m_1 = \dfrac{b}{a}$.

Differentiating (4) w.r.to u, we get

$$2v\frac{dv}{du} = -4a^2$$

—

$$\frac{dv}{du} = \frac{-2a^2}{v}$$

$$\left(\frac{dv}{du}\right)_{(a^2-b^2,2ab)} = \frac{-2a^2}{2ab} = \frac{-a}{b}$$

i.e., Slope of the tangent at $(a^2 - b^2, 2ab)$ to the parabola (4) is $m_2 = \frac{-a}{b}$.

Now, $m_1 m_2 = \frac{b}{a} \times \frac{-a}{b} = -1$

Hence, the tangents are perpendicular and therefore the parabolas are orthogonal.

Hence, lines parallel to the axes in the z-plane are mapped on to orthogonal parabolas in the w-plane.

Case(ii). Image of circles.

We shall now study how circles in the z-plane are transformed by the mapping $w = z^2$.

Changing into polar coordinates, let $z = re^{i\theta}$ and $w = Re^{i\phi}$.

We have $w = z^2 \quad \Rightarrow \quad Re^{i\phi} = r^2 e^{i2\theta}$.

Equating the modulii and arguments on both sides, we get

$$R = r^2 \qquad , \phi = 2\theta.$$

This shows that the modulus is squared and the amplitude is doubled.

Along the real axis in the z-plane θ takes values 0 and π.

When $\theta = 0 \quad \Rightarrow \quad \phi = 0$

When $\theta = \pi \quad \Rightarrow \quad \phi = 2\pi$

Hence, ϕ assumes values 0 and 2π, which is the positive side of the u-axis.

Hence, the real axis in the z-plane is mapped onto the positive real axis in the w-plane. Along the imaginary axis, θ takes values $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ while ϕ assumes values π and 3π, which is only the negative side of the imaginary axis.

i.e., The imaginary axis of the z-plane is mapped on to the negative real axis in the w-plane.In the first quadrant of the z-plane θ varies from 0 to $\frac{\pi}{2}$ while ϕ varies

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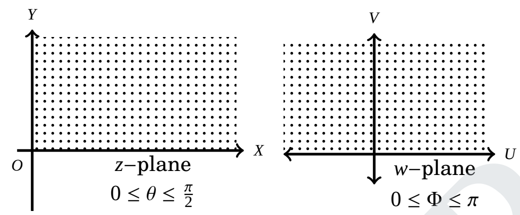
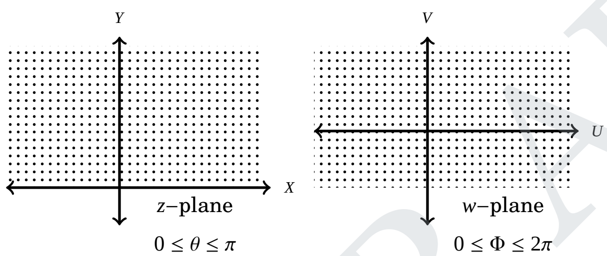
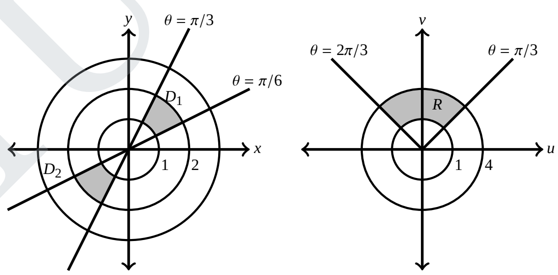
from 0 to π.

i.e., the first quadrant of the z-plane is mapped on to the entire upper half of the w-plane. Similarly, the upper half of the z-plane is mapped on to the whole of the w-plane.





In the same way the lower half of the z-plane is mapped onto the entire w-plane. Hence, two distinct points z_0 and $-z_0 = (z_0 e^{i\pi})$ in the z-plane correspond to a single point $w = z_0^2$ in the w-plane.
In the same way two domains D_1 and D_2 in z-plane will be mapped onto the single domain $\mathbb{R}$ in the w-plane.



Case (iii). The images of $u(x, y) = c$ and $v(x, y) = d$ where c and d are constants.
We shall study the images if the lines $u(x, y) = c$ and $v(x, y) = d$ under the mapping $w = z^2$.

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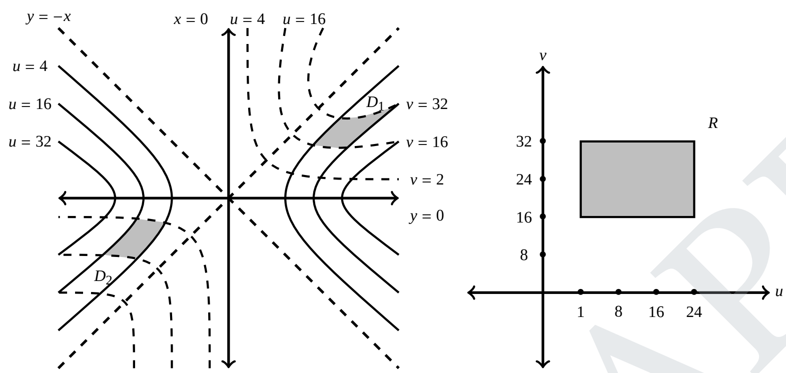
$$u = c \quad \Rightarrow \quad x^2 - y^2 = c \tag{5}$$

$$v = d \quad \Rightarrow \quad 2xy = d \tag{6}$$

(5) and (6) represent a family of rectangular hyperbolas.



For the family of R.H represented by (5), the asymptotes are the lines $y = x$ and $y = -x$. [Taking $u = 0$]

For the family of R.H represented by (6), the asymptotes are the axes given by $y = 0$ and $x = 0$. Infact, the families of rectangular hyperbolas (5) and (6) are orthogonal.

For, differentiating (5) w.r.to x, we get

$$2x - 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x}{y}$$

Differentiating (6) w.r.to x, we get

$$2\left[x\frac{dy}{dx} + y\right] = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-y}{x}$$

Hence, at any point (x, y), the slope of the tangent to (5) is $m_1 = \dfrac{x}{y}$ and the slope of the tangent to (6) is $m_2 = \dfrac{-y}{x}$.

$$\therefore m_1 \times m_2 = -1.$$

$\therefore$ (5) and (6) are orthogonal.

Also we observe that, (x, y) and $(-x, -y)$ satisfy both (5) and (6). Thus, the regions

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D_1 and D_2 in the z-plane correspond to only one region R in the w-plane. This shows that the transformation $w = z^2$ is two-to-one.

Application. If we consider the axes to represent two walls, a single quadrant could be used to represent the flow of a fluid at a corner wall. This transformation can also represent the electrostatic field in the vicinity of a corner conductor.

Worked Examples

Example 3.92. Find the image of the circle $|z| = a$ by the mapping $w = z^2$.

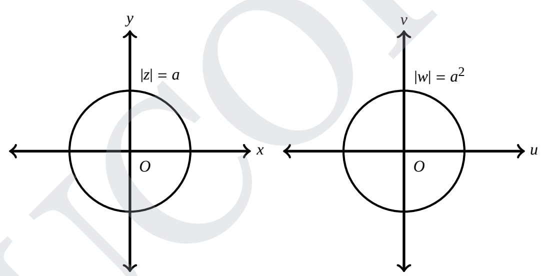
Solution. Let $z = re^{i\theta}$ and $w = Re^{i\phi}$.

$$w = z^2 \Rightarrow \quad Re^{i\phi} = r^2 e^{i2\theta}.$$

$$\therefore \quad R = r^2 \text{ and } \phi = 2\theta.$$

Now, $|z| = a$ is a circle with centre $(0, 0)$ and radius $r = a$. Along this circle, θ varies form 0 to 2π. In the w-plane, we have $R = a^2$ and ϕ varies from 0 to 4π.

$\therefore$ $|z| = a$ in the z-plane is mapped onto the circle $|w| = a^2$, whose centre is the origin and radius $= a^2$ in the w-plane.



Note: As θ varies from 0 to π, ϕ varies from 0 to 2π. Hence, the upper semicircle $|z| = a$ is also mapped onto the full circle $|w| = a^2$.

Example 3.93. Find the image of the semi circular area $r \leq \theta$, $0 \leq \theta \leq \pi$ by the map $w = z^2$.

Solution. By the transformation $w = z^2$, if $z = re^{i\theta}$ and $w = Re^{i\phi}$, we get, $R = r^2$ and $\phi = 2\theta$. $r \leq \theta$, $0 \leq \theta \leq \pi$ is the interior of the upper semi circle $|z| = a$. When $r \leq a$, we have, $R \leq a^2$ and as $0 \leq \theta \leq \pi$,we obtain $0 \leq \phi \leq 2\pi$. This is the interior of the whole circular area, $|w| = a^2$ in the w-plane.

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Example 3.94. Find the image of the hyperbola $x^2 - y^2 = 10$ under the transformation $w = z^2$ if $w = u + iv$. [May 1997].

Solution. For $w = z^2$ we have

$$u = x^2 - y^2 \tag{1}$$

and $v = 2xy$ (2)

Given $x^2 - y^2 = 10$

$\Rightarrow u = 10$

i.e., the image of the hyperbola $x^2 - y^2 = 10$ in the z-plane is mapped onto the line $u = 10$ which is parallel to the v-axis which is at a distance 10 from it.

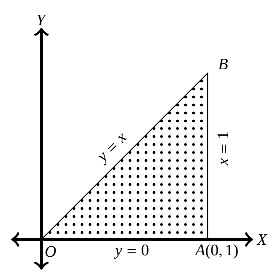
Example 3.95. Find the image of the region of the z-plane bounded by the line $y = 0$, $x = 1$ and $y = x$ under the mapping $w = z^2$.

Solution. Under the mapping $w = z^2$, we have

$$u = x^2 - y^2. \tag{1}$$

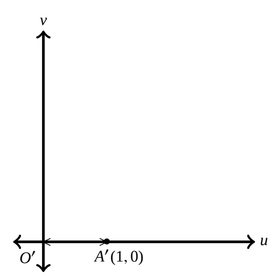
$$v = 2xy. \tag{2}$$

The region bounded by $y = 0$, $x = 1$ and $y = x$ is the right angled triangle OAB where A is $(0, 1)$, B is $(1, 1)$ and O is $(0, 0)$.



(i) Consider $y = 0$.

The transformations (1) and (2) become $u = x^2$ and $v = 0$. $v = 0$ is the u-axis and since $0 \le x \le 1$, we have $0 \le u \le 1$. Therefore $y = 0$ is mapped on the line segment $O'A'$ on the u-axis.

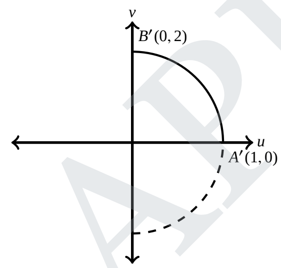


(ii) Consider $x = 1$.

(1) and (2) become $u = 1 - y^2$ and $v = 2y$ i.e., $y = \frac{v}{2}$.

$\therefore u = 1 - \frac{v^2}{4} \quad \Rightarrow \quad \frac{v^2}{4} = 1 - u$

$v^2 = 4(1 - u) = -4(u - 1)$.



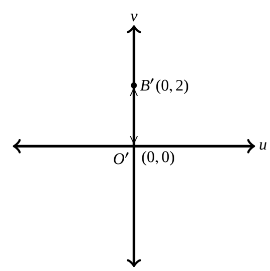
This is a parabola with vertex $(1, 0)$ and open to the left.

On $x = 1$, y varies form 0 to 1. $\therefore u$ varies from 1 to 0 and v varies from 0 to 2.

$\therefore x = 1$ is mapped onto the portion $A'B'$ of the parabola $v^2 = -4(u - 1)$.

(iii) Consider $y = x$ where $0 \le x \le 1$, $0 \le y \le 1$.

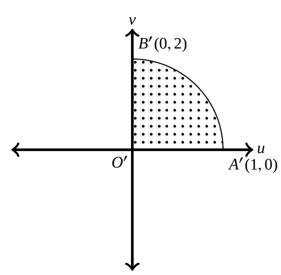
(1) and (2) take the form $u = 0$ and $v = 2x^2$.



As $0 \le x \le 1$, $0 \le v \le 2$.

$u = 0$ is the v-axis and along this Varies from 0 to 2.

$\therefore$ The image of the line $y = x$ is the portion of the v-axis form $O'(0, 0)$ to $B'(0, 2)$.



$\therefore$ The triangular region OAB in the z-plane is mapped onto the region, $O'A'B'$ enclosed by $u = 0$, $v = 0$ and $v^2 = -4(u-1)$, in the w-plane and this region is lying in the first quadrant of the w-plane.

—

Example 3.96. Plot the image under the mapping $w = z^2$ of the triangular region bounded by $y = 1$, $x = 1$ and $x + y = 1$.

Solution. The region bounded by the line $x = 1$, $y = 1$ and $x + y = 1$ is the triangle ABC where $A(1,0)$, $B(0,1)$ and $C(1,1)$.

From the transformation $w = z^2$, we have

$$u = x^2 - y^2 \tag{1}$$

$$v = 2xy. \tag{2}$$

To find the image of $A(1,0)$.

When $x = 1, y = 0, u = 1, v = 0$.

The image of A is $A'(1,0)$.

To find the image of $B(0,1)$.

When $x = 0, y = 1, u = -1, v = 0$.

The image of B is $B'(-1,0)$.

To find the image of $C(1,1)$

When $x = 1, y = 1, u = 0, v = 2$.

$\therefore$ The image of C is $C'(0,2)$.

(i) Consider $y = 1$.

(1) and (2) take the form

$u = x^2 - 1$ and $v = 2x \Rightarrow x = \dfrac{v}{2}$

$$\therefore u = \frac{v^2}{4} - 1$$

$$\frac{v^2}{4} = u + 1$$

$$\Rightarrow \quad v^2 = 4(u+1)$$

which is a parabola with vertex $B'(-1,0)$ and passing through $C'(0,2)$.

(ii) Consider $x = 1$

(1) and (2) take the form

$u = 1 - y^2$ and $v = 2y \Rightarrow y = \dfrac{v}{2}$

$$\therefore u = 1 - \frac{v^2}{4} \quad \Rightarrow \quad \frac{v^2}{4} = 1 - u$$

$\therefore v^2 = 4(1-u) = -4(u-1)$.

It is a parabola whose vertex is $A'(1,0)$ and open to the left. It passes through $C'(0,2)$ and $C_1(0,-2)$.

(iii) Consider $x + y = 1$.

(1) and (2) take the form

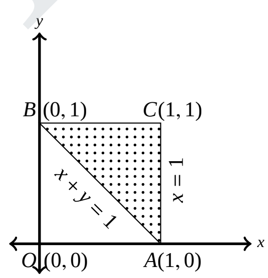
$u = x^2 - (1-x)^2 = x^2 - 1 - x^2 + 2x = 2x - 1$

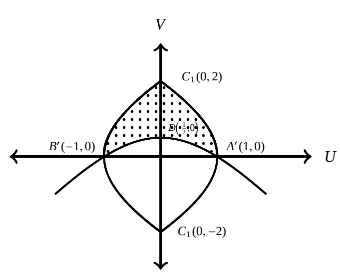
$\Rightarrow \quad 2x = u + 1$ or $x = \dfrac{u+1}{2}$

$$\begin{aligned} v &= 2x(1-x) \\ &= (u+1)\left(1 - \frac{u+1}{2}\right) \\ &= (u+1)\left(\frac{2-u-1}{2}\right) \\ &= \frac{(u+1)(1-u)}{2} \\ &= \frac{1-u^2}{2} \end{aligned}$$

$$\begin{aligned} 1 - u^2 &= 2v \\ u^2 &= 1 - 2v \\ &= -2\left(v - \frac{1}{2}\right). \end{aligned}$$

It is a parabola with vertex $D(\frac{1}{2}, 0)$ and passing through the points $A'(1,0)$ and $B'(-1,0)$. The required region is the shaded portion.





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Example 3.97. By the map $w = z^2$, show that the straight lines through the origin of the z-plane are mapped onto the straight lines through the origin of the w-plane.

Solution. The transformation $w = z^2$ is equivalent to

$$u = x^2 - y^2 \quad (1)$$

$$v = 2xy \quad (2)$$

Let $y = mx$ be te straight line passing through the origin.

(1) and (2) become

$$u = x^2 - m^2x^2 \quad \text{and} \quad v = 2mx^2$$

$$u = x^2(1 - m^2) \quad \text{and} \quad x^2 = \frac{v}{2m}$$

$$u = \frac{v}{2m}(1 - m^2) \text{ (or) } v = \frac{2m}{1 - m^2}u.$$

This represents a straight line in the w-plane with slope $\frac{2m}{1 - m^2}$ and passing through the origin if $m \neq \pm 1$.

If $m = 1$, $u = 0$ and $v = 2x^2 > 0$.

If $m = -1$, $u = 0$ and $v = -2x^2 < 0$

and if $m = -1$, the image is the negative V-axis.

$\therefore$ For all m, the image of a line through the origin in the z-plane is a straight line through the origin in the w-plane.

3.7.5 The transformation $w = e^z$

Let $z = x + iy$ and $w = re^{i\theta}$.

$\therefore$ $w = e^z$ becomes

$$re^{i\theta} = e^{x+iy} = e^x e^{iy}.$$

Equating the modulii and amplitudes, we obtain

$$r = e^x \quad (1)$$

$$\theta = y \quad (2)$$

Discussion

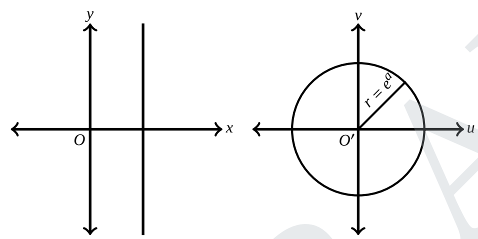
Case(i). Image of lines parallel to the axes

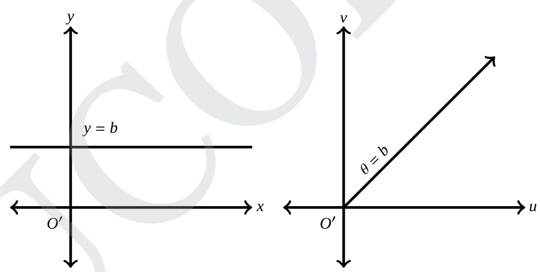
Consider the line $x = a$, which is parallel to the y-axis.

$\therefore$ (1) $\Rightarrow$ $r = e^a$, which is a circle with center as origin and radius e^a.

$\therefore$ Any line parallel to the y-axis in the z-plane is mapped onto a circle in the w-plane.

Similarly, the line $y = b$, parallel to the x-axis has the image $\theta = b$, which is a radius vector in the w-plane.





Case(ii). Image of a rectangular region

Consider the rectangular region bounded by the line $a_1 \leq x \leq a_2$ and $b_1 \leq y \leq b_2$. These are mapped onto circles $e^{a_1} \leq r \leq e^{a_2}$ and radius vectors $b_1 \leq \theta \leq b_2$. If D is the rectangular region bounded by $a_1 \leq x \leq a_2$ and $b_1 \leq y \leq b_2$ is mapped onto the region R in the w-plane defined by $e^{a_1} \leq r \leq e^{a_2}$, $b_1 \leq \theta \leq b_2$.

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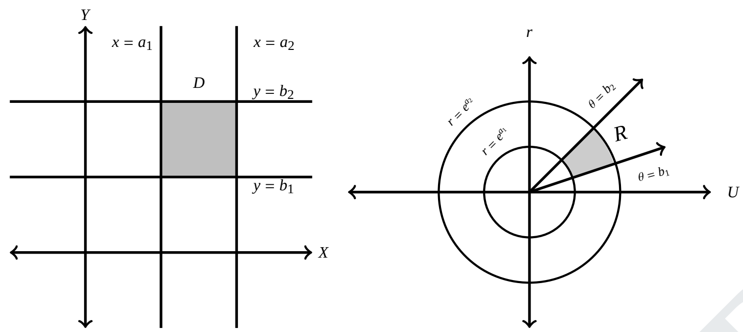
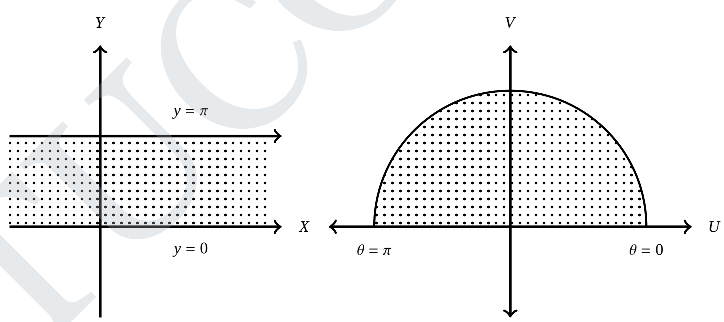


Case(iii). Consider the image of the x-axis.

i.e., when $y = 0$, we have $\theta = 0$; when $y = \pi$, $\theta = \pi$.

i.e., The line $y = 0$ and $y = \pi$ are mapped into the positive U-axis and the negative U-axis respectively. The region bounded by $y = 0$ and $y = \pi$ are mapped into the upper half of the W-plane.



In the same way, the region bounded by the lines $y = 0$ and $y = -\pi$ is mapped into the entire lower half of the W-plane. Hence, the fundamental strip $-\pi < y \leq \pi$ is mapped into the entire W-plane.

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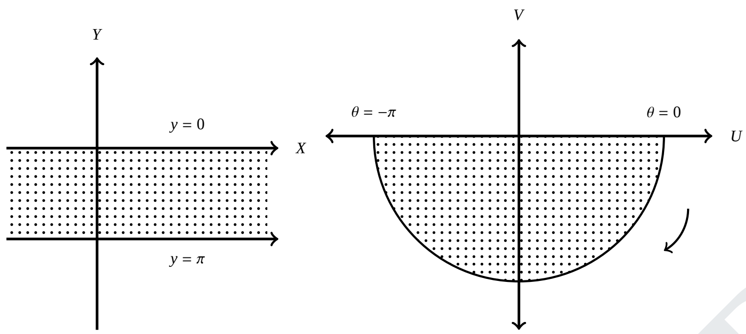
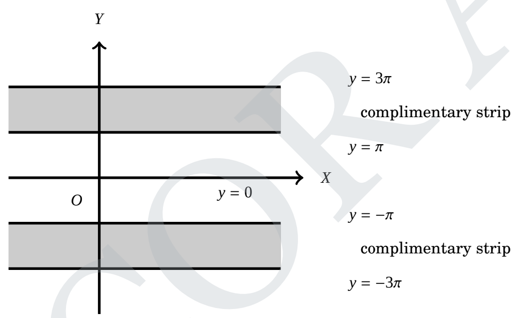


There are an infinite number of strips of width 2π namely $(2n-1)\pi \le y \le (2n+1)\pi$ are mapped into the whole w-plane. These strips of width 2π are called complementary strips.



This shows the many to one correspondence and the periodic nature of the the exponential function. The fundamental period is $2i\pi$. More generally, each of the fundamental strips bounded by $\alpha \le y \le \alpha + 2\pi$ where α is any constant is mapped into the entire w-plane.
Hence,

$$e^z = e^{z+i2n\pi}, \qquad n \quad \text{is an integer.}$$

From the relations $r = e^x$; $\theta = y$, we notice that the imaginary axis $x = 0$ is mapped into the unit circle, the right half plane $x = Re(z) > 0$ is mapped into the outside of the unit circle and $x = Re(Z) < 0$ is mapped inside the unit circle.

—

Application

This transformation can be used to obtain the circulation of a liquid around a cylindrical obstacle, the electrostatic field due to a changed circular cylinder etc.

Worked Examples

Example 3.98. Find the critical points of the map $w = 1 + \frac{2}{z}$. [Dec 2013]

Solution. Critical points are given by $\frac{dw}{dz} = 0$.

$$w = 1 + \frac{2}{z} = 1 + 2 \cdot z^{-1}$$

$$\frac{dw}{dz} = 0 \text{ gives}$$

$$2(-1)z^{-2} = 0$$

$$-\frac{2}{z^2} = 0$$

$$-2 = 0, \text{ which is not possible.}$$

$\therefore$ w has no critical points.

Example 3.99. Find the critical points of the map $w = \frac{2i - 6z}{iz - 3}$.

Solution. Critical points are given by $\frac{dw}{dz} = 0$.

$$\text{i.e., } \frac{(iz-3)(-6) - (2i - 6z)i}{(iz-3)^2} = 0$$

$$\Longrightarrow -6iz + 18 + 2 + 6iz = 0.$$

$\Longrightarrow$ $20 = 0$ which is not possible.

Hence, there is no critical point for this transformation.

Example 3.100. For the conformal mapping $w = z^2$, find the scale factor at $z = i$.

Solution. $f(z) = w = z^2$.

$f'(z) = 2z$ scale factor at $(z = i) = |f'(i)| = |2i| = 2$.

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Exercise 3 E

1. Find the image of the region bounded by $x = 0, y = 0, x = 1$ and $y = 2$ under the map $w = z + 2 - i$.

2. Find the region in the $w-$plane into which the rectangular region in the $z-$plane bounded by the lines $x = 0, y = 0, x = 3, x = 1$ is mapped by the map $w = z + (3 + 2i)$.

3. Find the image of the circle $|z| = 1$ under the map $w = z + (2 + 2i)$.

4. Find the image of $2x + y - 3 = 0$ under the transformation $w = z + 2i$. [May 2005]

5. Determine the region of the $w-$plane into which the region in the $z-$plane bounded by the lines $x = 0, y = 0, y = 1, x = 1$ is mapped under the transformation $w = (1 + i)z$. [Dec 2004]

6. Find the image of the circle $|z| = 9$ under the transformation $w = 5z$.

7. Find the image of $|z + 1| = 1$ under the mapping $w = \frac{1}{z}$. [Apr 2006]

8. Under the transformation $w = \frac{1}{z}$, determine the region in $w-$plane of the infinite strip R bounded by $\frac{1}{4} \le y \le \frac{1}{2}$. [Jan 2005]

9. Find the line $y - x + 1 = 0$ under the mapping $w = \frac{1}{z}$.

10. Find the mapping of $|z - 3| = 5$ under the transformation $w = \frac{1}{z}$. [Apr 2006]

11. Find and draw the image of the infinite horizontal strip $2 < y < 4$ under $w = \frac{1}{z}$. [Jan 2004]

12. Find the critical point of $w = e^z$.

13. Find the critical point of $w = z^2 + az + b$.

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2.3 Unit III

Analytic Function

2.3.1 May/June 2016 (R 2013)

Part A

7. Give an example of a function where u and v are harmonic but $u + iv$ is not analytic.

Solution. Let $u = x$, $v = -y$.

$u_x = 1$, $u_y = 0$, $v_x = 0$, $v_y = -1$.

$u_{xx} = 0$, $u_{yy} = 0$, $v_{xx} = 0$, $v_{yy} = 0$.

$u_{xx} + u_{yy} = 0$, $v_{xx} + v_{yy} = 0$.

$\Rightarrow$ u and v are harmonic. But $u_x \neq v_y$.

$\therefore$ C. R. equations are not satisfied.

$\Rightarrow$ $f(z) = u + iv$ is not analytic.

8. Find the critical points of the map $w^2 = (z - \alpha)(z - \beta)$.

Solution. Given $w^2 = (z - \alpha)(z - \beta)$

Differentiating w.r.to z we get

$$2w\frac{dw}{dz} = (z - \alpha) \times 1 + (z - \beta) \times 1 = z - \alpha + z - \beta = 2z - (\alpha + \beta)$$

$$\frac{dw}{dz} = \frac{2z - (\alpha + \beta)}{2w}.$$

Critical points are given by $\frac{dw}{dz} = 0$.

$$\frac{2z - (\alpha + \beta)}{2w} = 0$$

$$2z - (\alpha + \beta) = 0$$

$$2z = \alpha + \beta \quad \Rightarrow z = \frac{\alpha + \beta}{2}.$$

Part B

14. (a) (i) If $u + iv$ is an analytic function in $z = x + iy$ then prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|u|^2 = 2|f'(z)|^2$

Solution. Refer Example 4.55 on page 368 of the main text book.

14. (a) (ii) Prove that $w = \dfrac{z}{z+a}$ where $a \neq 0$ is analytic whereas $w = \dfrac{\bar{z}}{\bar{z}+a}$ is not analytic.

Solution. $w = \dfrac{z}{z+a}$.

$$u + iv = \frac{x+iy}{x+iy+a} = \frac{x+iy}{(x+a)+iy} = \frac{(x+iy)(x+a-iy)}{(x+a)^2+y^2}.$$

$$= \frac{x(x+a)+y^2+i(-xy+y(x+a))}{(x+a)^2+y^2} = \frac{x^2+ax+y^2+i(ay)}{(x+a)^2+y^2}.$$

$$= \frac{x^2+ax+y^2}{(x+a)^2+y^2} + i\frac{ay}{(x+a)^2+y^2}.$$

$$u = \frac{x^2+ax+y^2}{(x+a)^2+y^2}, \qquad v = \frac{ay}{(x+a)^2+y^2}.$$

$$u_x = \frac{(x^2+2ax+a^2+y^2)(2x+a)-(x^2+ax+y^2)(2x+2a)}{(x^2+2ax+a^2+y^2)^2}$$

$$= \frac{2x(x^2+2ax+a^2+y^2-x^2-ax-y^2)+a(x^2+2ax+a^2+y^2-2x^2-2ax-2y^2)}{(x^2+2ax+a^2+y^2)^2}$$

$$= \frac{2ax^2+2a^2x-ax^2-ay^2+a^3}{(x^2+2ax+a^2+y^2)^2}$$

$$= \frac{ax^2+2a^2x-ay^2+a^3}{(x^2+2ax+a^2+y^2)^2}$$

$$= \frac{a(x^2+2ax-y^2+a^2)}{(x^2+2ax+a^2+y^2)^2}.$$

$$u_y = \frac{(x^2+2ax+a^2+y^2)(2y)-(x^2+ax+y^2)(2y)}{(x^2+2ax+a^2+y^2)^2}$$

$$= \frac{2y(x^2+2ax+a^2+y^2-x^2-ax-y^2)}{(x^2+2ax+a^2+y^2)^2}$$

$$= \frac{2y(ax+a^2)}{(x^2+2ax+a^2+y^2)^2}$$

$$= \frac{2ay(x+a)}{(x^2+2ax+a^2+y^2)^2}. \qquad (2)$$

$$v_x = \frac{-ay(2x+2a)}{(x^2+2ax+a^2+y^2)^2}$$
$$= \frac{-2ay(x+a)}{(x^2+2ax+a^2+y^2)^2}. \qquad (3)$$

$$v_y = \frac{(x^2+2ax+a^2+y^2)a - ay(2y)}{(x^2+2ax+a^2+y^2)^2}$$
$$= \frac{a(x^2+2ax+a^2+y^2-2y^2)}{(x^2+2ax+a^2+y^2)^2}$$
$$= \frac{a(x^2+2ax+a^2-y^2)}{(x^2+2ax+a^2+y^2)^2}. \qquad (4)$$

From (1) and (4) we find $u_x = v_y$

From (2) and (3) we find $u_y = -v_x$

$\therefore$ C.R. equations are satisfied.

$\therefore f(z) = \dfrac{z}{z+a}$ is analytic.

Now $w = \dfrac{\bar{z}}{\bar{z}+a}$

$$u + iv == \frac{x-iy}{x-iy+a} = \frac{x-iy}{(x+a)-iy} = \frac{(x-iy)(x+a+iy)}{(x+a)^2+y^2}.$$
$$= \frac{x(x+a)+y^2+i(xy-y(x+a))}{(x+a)^2+y^2} = \frac{x^2+ax+y^2-i(ay)}{(x+a)^2+y^2}.$$
$$= \frac{x^2+ax+y^2}{(x+a)^2+y^2} + i\frac{-ay}{(x+a)^2+y^2}.$$

$$u = \frac{x^2+ax+y^2}{(x+a)^2+y^2}, \qquad v = \frac{-ay}{(x+a)^2+y^2}.$$

As before we get

$$u_x = \frac{a(x^2+2ax-y^2+a^2)}{(x^2+2ax+a^2+y^2)^2}. \qquad (5)$$
$$u_y = \frac{2ay(x+a)}{(x^2+2ax+a^2+y^2)^2}. \qquad (6)$$
$$v_x = \frac{2ay(x+a)}{(x^2+2ax+a^2+y^2)^2}. \qquad (7)$$
$$v_y = \frac{-a(x^2+2ax+a^2-y^2)}{(x^2+2ax+a^2+y^2)^2}. \qquad (8)$$

From (5), (6), (7) and (8) we note that C.R. equation are not satisfied

$\therefore w = \dfrac{\bar{z}}{\bar{z} + a}$ is not analytic.

14. (b) (i) Can $v = \tan^{-1}\left(\dfrac{y}{x}\right)$ be the imaginary part of an analytic function? If so construct an analytic function $f(z) = u + iv$, taking v as the imaginary part and hence find u.

Solution. Given $v = \tan^{-1}\left(\dfrac{y}{x}\right)$.

$$v_x = \frac{1}{1 + \frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2 + y^2}.$$

$$v_{xx} = -y(-1)(x^2 + y^2)^{-2}.2x = \frac{2xy}{(x^2 + y^2)^2}.$$

$$v_y = \frac{1}{1 + \frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}.$$

$$v_{yy} = x(-1)(x^2 + y^2)^{-2}.2y = \frac{-2xy}{(x^2 + y^2)^2}.$$

$$v_{xx} + v_{yy} = 0.$$

$\therefore v$ is harmonic.

Since f is analytic, v can be the imaginary part of f.

By Milne Thomson method.

$$f'(z) = v_y(z, 0) + iv_x(z, 0) = \tfrac{z}{z^2} + i(0) = \tfrac{1}{z}.$$

$$\therefore f(z) = \int \tfrac{1}{z}dz + c = \log z + c = \log(x + iy) + c$$

$$u + iv = \tfrac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\left(\tfrac{y}{x}\right) + c.$$

$\therefore u = \frac{1}{2}\log(x^2 + y^2)$, which is the real part of $f(z)$.

14. (b) (ii) Find the bilinear transformation that transforms the points $z = 1, i, -1$of the $z-$plane into the points $w = 2, i, -2$ of the $w-$plane.

Solution. Refer Example 4.65 on page 378 of the main text book.

2.3.2 Dec.2015/Jan. 2016 (R 2013)

Part A

7. Prove that the family of curves $u = c$, $v = k$ cuts orthogonally for an analytic function $f(z) = u + iv$.

   Solution. Refer property 2 on page 337 of the main text book.

8. Find the invariant points of the function $f(z) = \dfrac{z^3 + 7z}{7 - 6zi}$.

   Solution. The invariant points are givey by $f(z) = z$.

$$\text{i.e., } \frac{z^3 + 7z}{7 - 6zi} = z$$
$$z^3 + 7z = 7z - 6z^2 i$$
$$z^3 + 6z^2 i = 0$$
$$z^2(z + 6i) = 0 \quad \Rightarrow z = 0, z = -6i.$$

Part B

14. (a) (i) If $u = x^2 - y^2$, $v = \dfrac{y}{x^2 + y^2}$, prove that u and v are harmonic functions but $f(z) = u + iv$ is not an analytic function.

   Solution. Refer Example 4.43 on page 355 of the main text book.

14. (a) (ii) Show that the function $u = e^{-2xy} \sin\left(x^2 - y^2\right)$ is a real part of an analytic function. Also find its conjugate harmonic function v and express $f(t) = u + iv$ as function of z.

   Solution. $u = e^{-2xy} \sin\left(x^2 - y^2\right)$.

$$u_x = e^{-2xy} \cos\left(x^2 - y^2\right)(2x) + \sin\left(x^2 - y^2\right) e^{-2xy}(-2y)$$
$$= 2xe^{-2xy} \cos\left(x^2 - y^2\right) - 2ye^{-2xy} \sin\left(x^2 - y^2\right).$$
$$u_{xx} = 2xe^{-2xy}(-\sin\left(x^2 - y^2\right))(2x) + 2xe^{-2xy}(-2y)\cos\left(x^2 - y^2\right)$$
$$+ 2e^{-2xy} \cos\left(x^2 - y^2\right) - 2ye^{-2xy}(\cos\left(x^2 - y^2\right)(2x)$$
$$- 2ye^{-2xy}(-2y) \sin\left(x^2 - y^2\right)$$
$$= -4x^2e^{-2xy}(\sin\left(x^2 - y^2\right) - 4xye^{-2xy} \cos\left(x^2 - y^2\right)$$

$$+ 2e^{-2xy}\cos\left(x^2-y^2\right) - 4xye^{-2xy}(\cos\left(x^2-y^2\right)$$

$$+ 4y^2e^{-2xy}\sin\left(x^2-y^2\right) \qquad (1)$$

$$u_y = e^{-2xy}\cos\left(x^2-y^2\right)(-2y) + \sin\left(x^2-y^2\right)e^{-2xy}(-2x)$$

$$= -2ye^{-2xy}\cos\left(x^2-y^2\right) - 2xe^{-2xy}\sin\left(x^2-y^2\right).$$

$$u_{yy} = -2ye^{-2xy}(-\sin\left(x^2-y^2\right))(-2y) - 2ye^{-2xy}(-2x)\cos\left(x^2-y^2\right)$$

$$- 2e^{-2xy}\cos\left(x^2-y^2\right) - 2xe^{-2xy}(\cos\left(x^2-y^2\right)(-2y)$$

$$- 2xe^{-2xy}(-2x)\sin\left(x^2-y^2\right)$$

$$= -4y^2e^{-2xy}\sin\left(x^2-y^2\right) + 4xye^{-2xy}\cos\left(x^2-y^2\right)$$

$$- 2e^{-2xy}\cos\left(x^2-y^2\right) + 4xye^{-2xy}(\cos\left(x^2-y^2\right)$$

$$+ 4x^2e^{-2xy}\sin\left(x^2-y^2\right) \qquad (2)$$

$$(1) + (2) \Rightarrow u_{xx} + u_{yy} = 0.$$

$\Rightarrow u$ is harmonic.

Since f is analytic, u can be the real part of f.

By Milne Thomson method.

$$f'(z) = u_x(z,0) - iu_y(z,0)$$

$$= 2z\cos z^2 - i(-2z\sin z^2)$$

$$= 2z\cos z^2 + i2z\sin z^2$$

$$= 2z(\cos z^2 + i\sin z^2) = 2ze^{iz^2}$$

$$f(z) = \int e^{iz^2} 2zdz + c. \quad \text{Let } t = z^2 \quad dt = 2zdz.$$

$$= \int e^{it}dt + c = \frac{e^{it}}{i} = -ie^{iz^2} = -ie^{i(x+iy)^2} = -ie^{i(x^2-y^2+2ixy)}$$

$$= -ie^{i(x^2-y^2)-2xy} = -ie^{-2xy}\left[\cos\left(x^2-y^2\right) + i\sin\left(x^2-y^2\right)\right]$$

$$u + iv = e^{-2xy}\sin\left(x^2-y^2\right) - ie^{-2xy}\cos\left(x^2-y^2\right).$$

$$\therefore v = -e^{-2xy}\cos\left(x^2-y^2\right).$$

Also $f(z) = -ie^{iz^2}$.

14. (b) (i) Is $f(z) = z^n$ analytic function everywhere?

Solution. Refer Example 4.7 on page 325 of the main text book.

14. (b) (ii) Find the image of the line $u = a$ and $v = b$ in $w-$plane into $z-$plane under the transformation $z = \sqrt{w}$.

Solution. The given transformation is $z = \sqrt{w}$.

$$\text{i.e., } w = z^2$$
$$u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy.$$
$$\therefore u = x^2 - y^2, \qquad v = 2xy.$$

$$u = a \quad \Rightarrow x^2 - y^2 = a, \qquad v = b \qquad 2xy = b \Rightarrow xy = \tfrac{b}{2}.$$

Both are rectangular parabolas in the $z-$ plane.

Hence, the lines $u = a$ and $v = b$ in the $w-$plane are mapped onto rectangular hyperbolas in the $z-$plane.

14. (b) (iii) Find the bilinear transformation which maps $i, -i, 1$ in $z-$plane into $0, 1, \infty$ of the $w-$plane respectively.

Solution. Given, $z_1 = i, z_2 = -i, z_3 = 1.$

$$w_1 = 0, w_2 = 1, w_3 = \infty.$$

The bilinear transformation is

$$\frac{w - w_1}{w_2 - w_1} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$
$$\frac{w - 0}{1 - 0} = \frac{(z - i)(-i - 1)}{(z - 1)(-i - i)}$$
$$w = \frac{(z - i)(-i - 1)}{(z - 1)(-2i)}$$
$$w = \frac{(z - i)(-i - 1)i}{2(z - 1)}$$
$$w = \frac{(z - i)(1 - i)}{2(z - 1)}.$$

2.3.3 April/May 2015 (R 2013)

Part A

7. Show that $|z|^2$ is not analytic at any point.

Solution. $f(z) = u + iv = |z|^2 = x^2 + y^2$.

$$\therefore u = x^2 + y^2 \qquad v = 0.$$

$$u_x = 2x \qquad v_x = 0.$$

$$u_y = 2y \qquad v_y = 0.$$

At $(0,0)$ $u_x = v_y = 0$ and $u_y = -v_x = 0$.

$\Rightarrow f(z)$ is differentiable only at the origin.

But for all $(x,y) \neq (0,0)$, C-R equations are not satisfied.

Therefore, $f(z)$ is not analytic.

8. Find the invariant points of the transformation $w = \dfrac{z-1}{z+1}$.

Solution. The invariant points are given by $w = z$

$$\text{i.e., } \frac{z-1}{z+1} = z$$

$$z(z+1) = z-1$$

$$z^2 + z - z + 1 = 0$$

$$z^2 + 1 = 0$$

$$z^2 = -1 \quad \Rightarrow z = \pm i.$$

Part B

14. (a) (i) Determine the analytic function $w = u + iv$ if $u = e^{2x}(x\cos 2y - y\sin 2y)$.

Solution. Let $w = f(z) = u + iv$ be the analytic function.

Given $u = e^{2x}(x\cos 2y - y\sin 2y)$ is the real part of $f(z)$.

$$u_x = e^{2x}\cos 2y + 2e^{2x}(x\cos 2y - y\sin 2y)$$

$$= e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin 2y).$$

$$u_x(z,0) = e^{2z}(1 + 2z).$$

$$u_y = e^{2x}\,[x(-2\sin 2y) - [y(\cos 2y)\cdot 2 + \sin 2y]]$$

$$u_y = e^{2x}(-2x\sin 2y - 2y\cos 2y - \sin 2y)$$

$$u_y(z,0) = e^{2z}[0] = 0.$$

By Miline Thomson method

$$f'(z) = u_x(z,0) - iu_y(z,0) = e^{2z}(1+z).$$

Integrating w.r.t Z we get

$$\begin{aligned}
f(z) &= \int e^{2z}(1+z)dz + c \\
&= \int (1+z)d\left(\frac{e^z}{2}\right) + c \\
&= (1+z)\frac{e^{2z}}{2} - \int \frac{e^{2z}}{2}dz + c \\
&= (1+z)\frac{e^{2z}}{2} - \frac{e^{2z}}{4}dz + c \\
&= \frac{e^{2z}}{2}\left[(1+z) - \frac{1}{2}\right] + c \\
&= \frac{e^{2z}}{2}\left[\frac{1}{2} + z\right] + c.
\end{aligned}$$

14. (a) (ii) Show that a harmonic function $'u'$ satisfies the formal differential equation $\dfrac{\partial^2 u}{\partial z \partial \overline{z}} = 0$ and hence prove that $\log|f'(z)|$ is harmonic, where $f(z)$ is a regular function.

Solution. Given u is harmonic.

$\therefore$ u satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

we have $z = x + iy$, $\overline{z} = x - iy$.

Hence, $x = \frac{z+\overline{z}}{2}$, $y = \frac{z-\overline{z}}{2i}$.

$\Rightarrow$ u is a function of z and $\overline{z}$.

$$\begin{aligned}
\text{Now } \frac{\partial u}{\partial \overline{z}} &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \overline{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \overline{z}} \\
&= \frac{\partial u}{\partial x}\cdot\left(\frac{1}{2}\right) + \frac{\partial u}{\partial y}\cdot\left(\frac{-1}{2i}\right) \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{1}{i}\frac{\partial u}{\partial y}\right)
\end{aligned}$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) \qquad \left[\text{since, } \frac{1}{i} = -i\right]$$

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z}\left(\frac{1}{2}\left(u_x + iu_y\right)\right)$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial x}\left(u_x + iu_y\right) \cdot \frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\left(u_x + iu_y\right) \cdot \frac{\partial y}{\partial z}\right]$$

$$= \frac{1}{2}\left[\left(u_{xx} + iu_{xy}\right)\frac{1}{2} + \left(u_{yx} + iu_{yy}\right)\frac{1}{2i}\right]$$

$$= \frac{1}{4}\left[\left(u_{xx} + iu_{xy}\right) + \frac{1}{i}\left(u_{yx} + iu_{yy}\right)\right]$$

$$= \frac{1}{4}\left[u_{xx} + u_{yy} + iu_{xy} - iu_{xy}\right] \qquad \left(\text{since, } \frac{1}{i} = -i,\ u_{yx} = u_{xy}\right)$$

$$4\frac{\partial^2 u}{\partial z \partial \bar{z}} = u_{xx} + u_{yy}. (2)$$

Since, u is harmonic, $u_{xx} + u_{yy} = 0$, which implies

$$\Rightarrow \quad \frac{\partial^2 u}{\partial z \partial \bar{z}} = 0.$$

14. (b) (i) Find the image in the w−plane of the infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$ under the transformation $w = \frac{1}{z}$.

Solution. Given , $w = \frac{1}{z}$

$$\Rightarrow x = \frac{u}{u^2 + v^2}, y = \frac{-v}{u^2 + v^2}$$

$$\text{Consider} \frac{1}{4} \le y$$

$$\Rightarrow \frac{1}{4} \le \frac{-v}{u^2 + v^2}$$

$$i.e., u^2 + v^2 \le -4v$$

$$i.e., u^2 + v^2 + 4v \le 0.$$

This is a circle with centre $(0, -2)$ and radius 2.

$$\text{Consider } \ y \le \frac{1}{2}$$

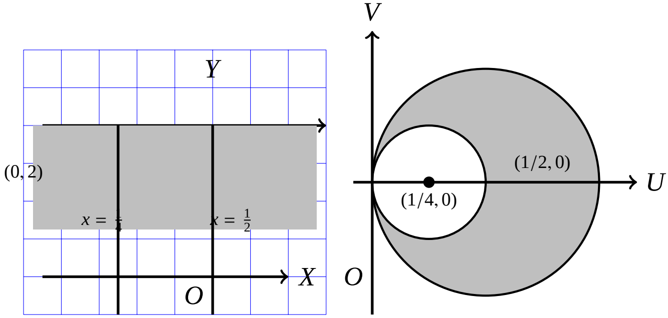
$$\Rightarrow \frac{-v}{u^2 + v^2} \le \frac{1}{2}$$

$$i.e., -2v \leq u^2 + v^2$$

$$i.e., u^2 + v^2 + 2v \geq 0.$$

This is a circle with centre $(0, -1)$ and radius 1.

The original and the image of the given transformation are given below.



14. (b) (ii) Find the bilinear transformation that maps the points $z = 0, -1, i$ into the points $w = i, 0, \infty$ respectively.

Solution. Given, $z_1 = 0, z_2 = -1, z_3 = i.$

$$w_1 = i, w_2 = 0, w_3 = \infty.$$

The bilinear transformation is

$$\frac{w - w_1}{w_2 - w_1} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$\frac{w - i}{0 - i} = \frac{(z - 0)(-i - 1)}{(z - i)(-1 - 0)}$$

$$\frac{w - i}{-i} = -\frac{z(1 + i)}{(z - i)(-1)}$$

$$\frac{w - i}{-i} = \frac{z(1 + i)}{z - i}$$

$$w - i = \frac{-iz(1 + i)}{z - i} = \frac{-iz + z}{z - i}$$

$$w = i + \frac{-iz + z}{z - i} = \frac{iz + 1 - iz + z}{z - i}$$

$$w = \frac{1 + z}{z - i}.$$

which is a straight line in the $w-$plane.

2.3.4 November/December 2014 (R 2013)

Part A

7. Verify $f(z) = z^3$ is analytic or not.

Solution. $f(z) = u+iv = z^3 = (x+iy)^3 = x^3+3(x^2)(iy)+3(x)(iy)^2+(iy)^3$.

$u + iv = x^3 + 3ix^2y - 3xy^2 - iy^3$.

$\therefore u = x^3 - 3xy^2, \qquad v = 3x^2y - y^3$.

$u_x = 3x^2 - 3y^2, \qquad v_x = 6xy$.

$u_y = -6xy, \qquad v_y = 3x^2 - 3y^2$.

we observe that $u_x = v_y$ and $u_y = -v_x$.

C-R equations are satisfied.

$\therefore$ $f(z)$ is an entire function.

8. Find the critical points of the transformation $w^2 = (z-\alpha)(z-\beta)$.

Solution. $w^2 = (z-\alpha)(z-\beta)$

Differentiating w.r.to z.

$$2w\frac{dw}{dz} = (z-\alpha)\times 1 + (z-\beta)\times 1$$
$$= z-\alpha+z-\beta$$
$$= 2z-(\alpha+\beta)$$
$$\frac{dw}{dz} = \frac{2z-(\alpha+\beta)}{2w}.$$

Critical points are given by $\frac{dw}{dz} = 0$.

$$\frac{2z-(\alpha+\beta)}{2w} = 0$$
$$2z-(\alpha+\beta) = 0$$

$$2z = \alpha + \beta$$
$$z = \frac{\alpha + \beta}{2}.$$

Part B

14. (a) (i) If $f(z)$ is an analytic function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

Solution. Given $f(z)$ is analytic.

Let $f(z) = u + iv$.

$\Rightarrow$ u and v have continuous partial derivatives and they satisfy C-R equations.

$\therefore$ $u_x = v_y$ and $u_y = -v_x$ and $f'(z) = u_x + iv_x$, $|f'(z)|^2 = u_x^2 + v_x^2$.

Since u and v are harmonic functions, they satisfy Laplace equation.

The complex form of the Laplace operator is

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial \bar{z}\partial z}.$$

$$\text{Now, } \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\frac{\partial^2}{\partial \bar{z}\partial z}|f(z)|^2 = 4\frac{\partial^2}{\partial \bar{z}\partial z}(f(z)\overline{f(z)}).$$

Since $f(z)$ is an analytic function, it is independent of $\bar{z}$.

i.e., $f(z)$ is a function of z only.

Similarly its conjugate $\overline{f(z)}$ is an analytic function of $\bar{z}$ only.

$\therefore$ We can denote $\overline{f(z)}$ by $\bar{f}(\bar{z})$.

i.e., $\overline{f(x+iy)} = \bar{f}(x - iy)$.

$$\begin{aligned}\text{Hence, } \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 &= 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}(f(z)\bar{f}(\bar{z})) \\ &= 4\frac{\partial}{\partial \bar{z}}(\bar{f}(\bar{z}))\frac{\partial}{\partial z}(f(z)) \\ &= 4\overline{f'(z)}f'(z) = 4|f'(z)|^2.\end{aligned}$$

14. (a) (ii) Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$.

Solution. Given, $z_1 = 0, z_2 = i, z_3 = -1$.

$$w_1 = i, w_2 = 0, w_3 = -i.$$

The bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w-i)(i)}{(w+i)(-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w-i)}{(w+i)} = \frac{(1-z)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w-i)+(w+i)}{(w-i)-(w+i)} = \frac{(1-z)(i+1)+(z+1)(i-1)}{(1-z)(i+1)-(z+1)(i-1)}$$

$$\frac{w-i+w+i}{w-i-w-i} = \frac{i-zi+1-z+zi+i-z-1}{i-zi+1-z-zi-i+z+1}$$

$$\frac{2w}{-2i} = \frac{2i-2z}{-2zi+2}$$

$$\frac{w}{-i} = \frac{i-z}{1-zi}$$

$$w = \frac{zi+1}{1-zi}.$$

14. (b) (i) Prove that $u = x^2 - y^2$ and $v = -\frac{y}{x^2+y^2}$ are harmonic functions but not harmonic conjugates.

Solution. $u = x^2 - y^2$.

$u_x = 2x.$

$u_{xx} = 2.$

$u_y = -2y.$

$u_{yy} = -2.$

$u_{xx} + u_{yy} = 2 - 2 = 0.$

$\therefore$ u is harmonic.

$$v = -\frac{y}{x^2+y^2} = -y(x^2+y^2)^{-1}.$$

$$v_x = -y(-1)(x^2+y^2)^{-2} \cdot 2x = \frac{2xy}{(x^2+y^2)^2}.$$

$$v_{xx} = 2y\left[\frac{(x^2+y^2)^2 - x2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}\right]$$

$$= \frac{2y(x^2+y^2)[x^2+y^2-4x^2]}{(x^2+y^2)^4}$$

$$v_{xx} = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}.$$

$$v_y = -\frac{x^2+y^2-y\cdot 2y}{(x^2+y^2)^2}$$

$$= -\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

$$v_{yy} = \frac{(x^2+y^2)^2\cdot 2y - (y^2-x^2)\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}.$$

$$= \frac{2y(x^2+y^2)[x^2+y^2-2(y^2-x^2)]}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2-2y^2+2x^2)}{(x^2+y^2)^3}$$

$$v_{yy} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}.$$

$$v_{xx}+v_{yy} = \frac{2y}{(x^2+y^2)^3}[y^2-3x^2+3x^2-y^2] = 0.$$

$\therefore$ v is harmonic.

We have to prove that, if $u+iv$ is regular then $u+iv$ must satisfy C-R equations.

But here, $u_x = 2x \neq v_y$ and $u_y = -2y \neq -v_x$.

$\therefore$ C-R equations are not satisfied.

$\therefore$ u and v are not harmonic conjugates.

14. (b) (ii) Given that $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$, find the analytic function $f(z) = u + iv$. (8)

2.3.5 May/June 2014 (R 2013)

Part A

7. Is the function $f(z) = \bar{z}$ analytic?

Solution. Let $f(z) = u + iv = \bar{z} = x - iy$.

$$\therefore u = x, \qquad v = -y$$

$$u_x = 1, \qquad v_x = 0$$

$$u_y = 0, \qquad v_y = -1.$$

Since $u_x \neq v_y$, C.R equations are not satisfied.

$\therefore$ $f(z)$ is not analytic anywhere.

$\Longrightarrow$ $f(z)$ is nowhere differentiable.

8. Find the invariant points of $f(x) = z^2$.

Solution. Invariant points are given by $f(z) = z$

i.e., $z^2 = z \Rightarrow z^2 - z = 0 \Rightarrow z(z-1) = 0 \Rightarrow z = 0, z = 1$.

Part B

14. (a) (i) Prove that the real and imaginary parts of an analytic function are harmonic functions.

Solution. Given $f(z)$ is analytic.

$\Rightarrow$ u and v have first partial derivatives and satisfy C-R equations in D.

$$u_x = v_y \text{ and } u_y = -v_x$$

$$\text{i.e., } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$\text{and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2}$$

Differentiating (1) w.r.t x we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{3}$$

Differentiating (2) w.r.t y we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \tag{4}$$

Since u_x, u_y, v_x, v_y are continuous, we have

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

$\therefore$ (3) + (4) $\Rightarrow \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

$\therefore$ u is harmonic.

Differentiating (1) w.r.t y and (2) w.r.t x we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \text{ and } \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

$$\therefore \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2} \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

$\Rightarrow$ v is harmonic.

14. (a) (ii) Find the bilinear transformation that maps $1, i$ and -1 of the $z-$plane onto $0, 1$ and ∞ of the $w-$plane.

Solution. Let $z_1 = 1, z_2 = i, z_3 = -1.$

$w_1 = 0, w_2 = 1, w_3 = \infty.$

The bilinear transformation is

$$\frac{w - w_1}{w_2 - w_1} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$\frac{w - 0}{1 - 0} = \frac{(z - 1)(i + 1)}{(z + 1)(i - 1)}$$

$$w = -\frac{z - 1}{z + 1}\frac{1 + i}{1 - i} \times \frac{1 + i}{1 + i}$$

$$= -\frac{z - 1}{z + 1}\frac{(1 + i)^2}{1 + 1}$$

$$= -\frac{z - 1}{z + 1}\left(\frac{1 - 1 + 2i}{2}\right)$$

$$= -i\frac{z - 1}{z + 1}$$

$$w(z+1) = -iz + i$$

$$wz + w = -iz + i$$

$$wz + iz = i - w$$

$$z(w+i) = i - w \quad \Rightarrow z = \frac{i-w}{w+i}.$$

14. (b) (i) Show that $v = e^{-x}(x\cos y + y\sin y)$ is a harmonic function. Hence find the analytic function $f(z) = u + iv$.

Solution. We first prove that $v_{xx} + v_{yy} = 0$.

$$\begin{aligned}
v_x &= e^{-x}(\cos y) + (x\cos y + y\sin y)(-e^{-x}) \\
&= e^{-x}(\cos y - x\cos y - y\sin y) \\
v_{xx} &= e^{-x}(-\cos y) - e^{-x}(\cos y - x\cos y - y\sin y) \\
&= e^{-x}(-\cos y - \cos y + x\cos y + y\sin y) \\
&= e^{-x}(-2\cos y + x\cos y + y\sin y) \\
v_y &= e^{-x}(-x\sin y + y\cos y + \sin y) \\
v_{yy} &= e^{-x}(-x\cos y - y\sin y + \cos y + \cos y) \\
&= e^{-x}(2\cos y - x\cos y - y\sin y).
\end{aligned}$$

$$v_{xx} + v_{yy} = 0. \qquad \therefore v \text{ is harmonic}$$

Now, $v_x(z,0) = e^{-z}(1-z)$

and $v_y(z,0) = e^{-z}0 = 0$.

By Milne Thomson method,

$f'(z) = v_y(z,0) + iv_x(z,0) = ie^{-z}(1-z)$.

Integrating we get,

$$\begin{aligned}
f(z) &= i\int e^{-z}(1-z)dz + c = i\int (1-z)d(-e)^{-z}dz + c \\
&= i\int (z-1)d(e)^{-z} + c = i\left[(z-1)e^{-z} - \int e^{-z}dz\right] + c \\
&= i\left((z-1)e^{-z} + e^{-z}\right) + c = ie^{-z}(z-1+1) + c
\end{aligned}$$

$$f(z) = ize^{-z} + c.$$

14. (b) (ii) Find the image of $|z+1| = 1$ under the map $w = \frac{1}{z}$.

Solution. $w = \frac{1}{z}$.

$$\Rightarrow x = \frac{u}{u^2+v^2}, y = -\frac{v}{u^2+v^2}.$$

The given figure is

$$|z+1| = 1$$

$$i.e., |x+iy+1| = 1$$

$$i.e., (x+1)^2 + y^2 = 1$$

$$x^2 + y^2 + 2x + 1 = 1$$

$$x^2 + y^2 + 2x = 0$$

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + 2\frac{u}{u^2+v^2} = 0$$

$$\frac{u^2+v^2}{(u^2+v^2)^2} + \frac{2u}{u^2+v^2} = 0$$

$$\frac{1+2u}{u^2+v^2} = 0 \quad \Rightarrow 2u+1 = 0$$

which is a straight line in the w–plane.

2.3.6 November/December 2013(R 2008)

Part A

7. Find the constants a, b if $f(z) = x + 2ay + i(3x + by)$ is analytic.

Solution. $f(z) = x + 2ay + i(3x + by)$.

$$u = x + 2ay, \qquad v = 3x + by.$$

$$u_x = 1 \qquad v_x = 3$$

$$u_y = 2a \qquad v_y = b.$$

Since $f(z)$ is analytic, C-R equations are satisfied.

$\therefore u_x = v_y$ and $u_y = -v_x$.

Taking $u_x = v_y$ we get $b = 1$.

Taking $u_y = -v_x$, we get $2a = -3 \Rightarrow a = -\frac{3}{2}$.

8. Find the critical points of the map $w = 1 + \frac{2}{z}$.

Solution. Critical points are given by $\frac{dw}{dz} = 0$.

$$w = 1 + \frac{2}{z} = 1 + 2 \cdot z^{-1}$$

$$\frac{dw}{dz} = 0 \text{ gives}$$

$$2(-1)z^{-2} = 0$$

$$\Rightarrow -\frac{2}{z^2} = 0$$

$\Rightarrow -2 = 0$ which is not possible.

$\therefore$ w has no critical points.

Part B

14. (a) (i) Prove that $u = e^{-2xy}\sin(x^2 - y^2)$ is harmonic. Find the corresponding analytic function and the imaginary part.

Solution. $u = e^{-2xy}\sin(x^2 - y^2)$

$$u_x = e^{-2xy}\cos(x^2 - y^2) \cdot 2x + \sin(x^2 - y^2) \cdot e^{-2xy}(-2y)$$

$$= 2e^{-2xy}\left[x\cos(x^2 - y^2) - y\sin(x^2 - y^2)\right].$$

$$u_{xx} = 2e^{-2xy}\left[-x\sin(x^2 - y^2) \cdot 2x + \cos(x^2 - y^2) - y\cos(x^2 - y^2) \cdot 2x\right]$$
$$+ \left[x\cos(x^2 - y^2) - y\sin(x^2 - y^2)\right]2e^{-2xy}(-2y)$$

$$u_{xx} = 2e^{-2xy}\left[-2x^2\sin(x^2 - y^2) + \cos(x^2 - y^2) - 2xy\cos(x^2 - y^2)\right.$$
$$\left.-2xy\cos(x^2 - y^2) + 2y^2\sin(x^2 - y^2)\right].$$

$$u_y = e^{-2xy}\cos(x^2 - y^2)(-2y) + \sin(x^2 - y^2)e^{-2xy}(-2x)$$

$$= -2e^{-2xy}\left[y\cos(x^2 - y^2) + x\sin(x^2 - y^2)\right].$$

$$u_{yy} = -2e^{-2xy}\left[-y\sin(x^2 - y^2)(-2y) + \cos(x^2 - y^2) + x\sin(x^2 - y^2)(-2y)\right]$$
$$+ \left[y\cos(x^2 - y^2) + x\sin(x^2 - y^2)\right](-2e^{-2xy})(-2x)$$

$$= -2e^{-2xy}\left[2y^2\sin(x^2 - y^2) + \cos(x^2 - y^2) - 2xy\sin(x^2 - y^2)\right.$$

$$-2xy\cos(x^2-y^2)-2x^2\sin(x^2-y^2)\Big]$$
$$= 2e^{-2xy}\Big[-2y^2\sin(x^2-y^2)-\cos(x^2-y^2)+2xy\sin(x^2-y^2)$$
$$+2xy\cos(x^2-y^2)+2x^2\sin(x^2-y^2)\Big].$$

Now, $u_{xx}+u_{yy}=0$.

$\therefore u$ is harmonic.

Now $u_x(z,0)=2z\cos z^2$ and $u_y(z,0)=-2z\sin z^2$.

By Milne Thomson method

$$\begin{aligned} f'(z) &= u_x(z,0)+iu_y(z,0)\\ &= 2z\cos z^2+2iz\sin z^2\\ &= 2z(\cos z^2+i\sin z^2)=2ze^{iz^2}\end{aligned}$$

Integrating w.r.t. z we get

$$\begin{aligned} f(z) &= \int 2ze^{iz^2}dz+c \qquad\qquad t=z^2\\ &= \int e^{it}dt+c \qquad\qquad dt=2zdz\\ &= \frac{e^{it}}{i}+c\\ &= \frac{1}{i}e^{iz^2}+c\\ f(z) &= -ie^{iz^2}+c\\ u+iv &= -ie^{i(x+iy)^2}+c_1+ic_2\\ &= -ie^{i(x^2-y^2+2ixy)}+c_1+ic_2\\ &= -ie^{i(x^2-y^2)-2xy}+c_1+ic_2\\ &= -ie^{i(x^2-y^2)}\cdot e^{-2xy}+c_1+ic_2\\ &= -ie^{-2xy}[\cos(x^2-y^2)+i\sin(x^2-y^2)]+c_1+ic_2\\ &= -ie^{-2xy}\cos(x^2-y^2)+e^{-2xy}\sin(x^2-y^2)\end{aligned}$$

Equating the imaginary parts we get $v=-e^{-2xy}\cos(x^2-y^2)$.

14. (a) (ii) Find the bilinear map which maps the points $z = 0, -1, i$ onto the points $w = i, 0, \infty$. Also find the image of the unit circle of the $z-$plane.

Solution. Given, $z_1 = 0, z_2 = -1, z_3 = i$.

$$w_1 = i, w_2 = 0, w_3 = \infty.$$

The bilinear transformation is

$$\frac{w - w_1}{w_2 - w_1} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$\frac{w - i}{0 - i} = \frac{(z - 0)(-i - 1)}{(z - i)(-1 - 0)}$$

$$\frac{w - i}{-i} = -\frac{z(1 + i)}{(z - i)(-1)}$$

$$\frac{w - i}{-i} = \frac{z(1 + i)}{z - i}$$

$$w - i = \frac{-iz(1 + i)}{z - i} = \frac{-iz + z}{z - i}$$

$$w = i + \frac{-iz + z}{z - i} = \frac{iz + 1 - iz + z}{z - i}$$

$$w = \frac{1 + z}{z - i}.$$

$$w(z - i) = 1 + z$$

$$wz - wi = 1 + z$$

$$wz - z = 1 + wi$$

$$z(w - 1) = 1 + wi$$

$$z = \frac{1 + wi}{w - 1}.$$

Given $|z| = 1$

$$\left|\frac{1 + wi}{w - 1}\right| = 1$$

$$\frac{|1 + wi|}{|w - 1|} = 1$$

$$|1 + (u + iv)i| = |u + iv - 1|$$

$$|1 + iu - v|^2 = |(u - 1) + iv|^2$$

$$(1-v)^2 + u^2 = (u-1)^2 + v^2$$

$$1 + v^2 - 2v + u^2 = u^2 + 1 - 2u + v^2$$

$$-2v = -2u$$

$$u = v,$$

which is a straight line in the $w-$plane.

14. (b) (i) Prove that $w = \dfrac{z}{1-z}$ maps the upper half of the $z-$plane to the upper half of the $w-$plane and also find the image of the unit circle of the $z-$plane.

Solution.

$$\text{Given } w = \frac{z}{1-z}$$

$$u + iv = \frac{x+iy}{1-x-iy}$$

$$= \frac{x+iy}{(1-x)-iy} \times \frac{(1-x)+iy}{(1-x)+iy}$$

$$= \frac{x(1-x) - y^2 + i[y(1-x)+xy]}{(1-x)^2 + y^2}$$

$$= \frac{x(1-x)-y^2}{(1-x)^2+y^2} + i\frac{y(1-x)+xy}{(1-x)^2+y^2}.$$

$$\therefore u = \frac{x(1-x)-y^2}{(1-x)^2+y^2} \text{ and}$$

$$v = \frac{y(1-x)+xy}{(1-x)^2+y^2}.$$

Let $v > 0$.

$$\Rightarrow \frac{y(1-x)+xy}{(1-x)^2+y^2} > 0$$

$$y - xy + xy > 0$$

$$y > 0.$$

This shows that the upper half of the $z-$plane is mapped onto the upper half of the $w-$plane.

To find the image of the unit circle.

$$\text{we have } w = \frac{z}{1-z}$$

$$w(1-z) = z$$

$$w - wz = z$$

$$w = wz + z$$

$$z(1+w) = w$$

$$z = \frac{w}{1+w}.$$

Given $|z| = 1$.

$$\Rightarrow \left|\frac{w}{1+w}\right| = 1$$

$$|w| = |1+w|$$

$$|w|^2 = |1+w|^2$$

$$|u+iv|^2 = |1+u+iv|^2$$

$$u^2 + v^2 = (1+u)^2 + v^2$$

$$= 1 + u^2 + 2u + v^2$$

$$0 = 2u + 1 \qquad \Rightarrow u = -\frac{1}{2}.$$

The image of the unit circle $|z| = 1$ is the straight line $u = -\dfrac{1}{2}$.

14. (b) (ii) Find the analytic function $f(z) = u+iv$ where $v = 3r^2 \sin 2\theta - 2r \sin\theta$. Verify that u is a harmonic function.

Solution. $v = 3r^2 \sin 2\theta - 2r \sin\theta$.

$$\frac{\partial v}{\partial r} = 6r \sin 2\theta - 2\sin\theta.$$

$$\frac{\partial^2 v}{\partial r^2} = 6\sin 2\theta.$$

$$\frac{\partial v}{\partial \theta} = 6r^2 \cos 2\theta - 2r\cos\theta.$$

$$\frac{\partial^2 v}{\partial \theta^2} = -12r^2 \sin 2\theta + 2r\sin\theta.$$

$$\text{Now,} \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}$$
$$= 6\sin 2\theta + \frac{1}{r}[6r\sin 2\theta - 2\sin\theta] + \frac{1}{r^2}[-12r^2\sin 2\theta + 2r\sin\theta]$$
$$= 6\sin 2\theta + 6\sin 2\theta - \frac{2}{r}\sin\theta - 12\sin 2\theta + \frac{2}{r}\sin\theta = 0.$$

$\therefore v$ is harmonic.

Using C-R equations we have

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = 6r^2\cos 2\theta - 2r\cos\theta.$$
$$\frac{\partial u}{\partial r} = 6r\cos 2\theta - 2\cos\theta.$$

Integrating w.r.t.r we get

$$u = 6\cos 2\theta \int r dr - 2\cos\theta \int dr + f(\theta)$$
$$= 6\cos 2\theta \cdot \frac{r^2}{2} - 2r\cos\theta + f(\theta)$$
$$u = 3r^2\cos 2\theta - 2r\cos\theta + f(\theta). \tag{1}$$

Again by C-R equations we have

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = 6r\sin 2\theta - 2\sin\theta.$$
$$\frac{\partial u}{\partial \theta} = -6r^2\sin 2\theta + 2r\sin\theta.$$

Integrating w.r.t.θ we get

$$u = -6r^2\int \sin 2\theta d\theta + 2r\int \sin\theta d\theta + g(r)$$
$$= -6r^2\left(\frac{-\cos 2\theta}{2}\right) + 2r(-\cos\theta) + g(r)$$
$$u = 3r^2\cos 2\theta - 2r\cos\theta + g(r). \tag{2}$$

From (1) and (2) we get, $u = 3r^2\cos 2\theta - 2r\cos\theta + c$, where c is a constant, which is the harmonic conjugate of v.

To verify whether u is harmonic

$$\text{We have } \frac{\partial u}{\partial r} = 6r\cos 2\theta - 2\cos\theta.$$

$$\frac{\partial^2 u}{\partial r^2} = 6\cos 2\theta.$$

$$\frac{\partial u}{\partial \theta} = -6r^2 \sin 2\theta + 2r\sin\theta.$$

$$\frac{\partial^2 u}{\partial \theta^2} = -6r^2 \cdot (\cos 2\theta) \times 2 + 2r\cos\theta.$$

$$= -12r^2 \cos 2\theta + 2r\cos\theta.$$

$$\text{Now,} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= 6\cos 2\theta + \frac{1}{r}(6r\cos 2\theta - 2\cos\theta) + \frac{1}{r^2}(-12r^2\cos 2\theta + 2r\cos\theta)$$

$$= 6\cos 2\theta + 6\cos 2\theta - \frac{2}{r}\cos\theta - 12\cos 2\theta + \frac{2}{r}\cos\theta = 0.$$

$\therefore u$ is harmonic.

2.3.7 May/June 2013 (R 2008)

Part A

7. Find the image of $x = k$ under the transformation $w = \frac{1}{z}$.

Solution. We have $w = \dfrac{1}{z}$

$$x = \frac{u}{u^2+v^2} \text{ and } y = \frac{-v}{u^2+v^2}$$

When $x = k$, $k = \dfrac{u}{u^2+v^2}$ or $u^2 + v^2 = \dfrac{u}{k}$

$$u^2 + v^2 = \frac{u}{k}.$$

It is a circle with centre $\left(\dfrac{1}{2k}, 0\right)$ and radius $\dfrac{1}{2k}$.

8. Find the fixed points of the mapping $w = \dfrac{6z-9}{z}$.

Solution. The fixed points are given by $w = z$.

$$\therefore z = \frac{6z-9}{z}$$

$$\Rightarrow z^2 = 6z - 9$$

$$\Rightarrow z^2 - 6z + 9 = 0$$

$$\Rightarrow (z-3)^2 = 0 \Rightarrow z = 3, 3.$$

Part B

14. (a) (i) Prove that the function $u = e^x(x\cos y - y\sin y)$ satisfies Laplace equation and find the corresponding analytic function $f(z) = u + iv$.

Solution. Let $f(z) = u + iv$ be the analytic function.

Given $u = e^x(x\cos y - y\sin y)$ is the real part of $f(z)$.

$$u_x = e^x\cos y + e^x(x\cos y - y\sin y) = e^x(\cos y + x\cos y - y\sin y).$$

$$u_y = e^x(-x\sin y - y\cos y - \sin y).$$

$$u_{xx} = e^x(\cos y + x\cos y - y\sin y) + e^x\cos y = e^x(2\cos y + x\cos y - y\sin y).$$

$$u_{yy} = e^x(-x\cos y + y\sin y - \cos y - \cos y) = e^x(-x\cos y + y\sin y - 2\cos y).$$

$$u_{xx} + u_{yy} = e^x(2\cos y + x\cos y - y\sin y) + e^x(-x\cos y + y\sin y - 2\cos y)$$

$$= 0.$$

$\Rightarrow u$ satisfies Laplace equation.

Now, $u_x(z,0) = e^z(1+z)$ and $u_y(z,0) = 0$.

By Milne-Thomson method, $f'(z) = u_x(z,0) - iu_y(z,0) = e^z(1+z)$.

Integrating w.r.t z we get

$$f(z) = \int e^z(1+z)dz + c$$

$$= \int (1+z)d(e^z) + c$$

$$= (1+z)e^z - \int e^z dz + c$$

$$= (1+z)e^z - e^z + c = e^z + ze^z - e^z + c$$

$$\therefore f(z) = ze^z + c.$$

14. (a) (ii) Find the bilinear transformation which maps $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$ into the points $w_1 = i, w_2 = 1, w_3 = -i$.

Solution. Let $z_1 = 0, z_2 = 1, z_3 = \infty$, and $w_1 = i, w_2 = 1, w_3 = -i$.

The mobius transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)}{(z_2-z_1)}$$

i.e., $\dfrac{(w-i)(1+i)}{(w+i)(1-i)} = z$

$$\frac{w-i}{w+i} = \frac{z(1-i)}{1+i} = \frac{z(1-i)^2}{2}$$

$$= \frac{z(1-1-2i)}{2} = \frac{z(-2i)}{2} = -iz$$

$$\frac{w-i+w+i}{w-i-w-i} = \frac{-iz+1}{-iz-1}$$

$$\frac{2w}{-2i} = \frac{1-iz}{-(1+iz)}$$

$$w = \frac{i(1-iz)}{1+iz} = \frac{z+i}{1+iz}.$$

14. (b) (i) Find the image of $|z-2i| = 2$ under the transformation $w = \dfrac{1}{z}$.

Solution. $w = \dfrac{1}{z}$

$$\Rightarrow x = \frac{u}{u^2+v^2}, y = \frac{-v}{u^2+v^2}.$$

The given circle is $|z-2i| = 2$.

$$\Rightarrow |z-2i|^2 = 4$$

$$|x+iy-2i|^2 = 4$$

$$x^2 + (y-2)^2 = 4$$

$$x^2 + y^2 - 4y = 0.$$

It is a circle passing through the origin.

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} - 4\frac{-v}{u^2+v^2} = 0$$
$$\frac{1}{u^2+v^2} + \frac{4v}{u^2+v^2} = 0$$
$$4v + 1 = 0,$$

which is a straight line in the w plane.

$\therefore$ The image of the circle $|z - 2i| = 2$ in the $z-$plane is a straight line in the $w-$plane.

14. (b) (ii) If $f(z)$ is an analytic function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

Solution. Given $f(z)$ is analytic.

Let $f(z) = u + iv$.

$\Rightarrow$ u and v have continuous partial derivatives and they satisfy C-R equations.

$\therefore u_x = v_y$ and $u_y = -v_x$ and $f'(z) = u_x + iv_x$, $|f'(z)|^2 = u_x^2 + v_x^2$.

Since u and v are harmonic functions, they satisfy Laplace equation.

The complex form of the Laplace operator is

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial \bar{z}\partial z}.$$

Now, $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\frac{\partial^2}{\partial \bar{z}\partial z}|f(z)|^2 = 4\frac{\partial^2}{\partial \bar{z}\partial z}(f(z)\overline{f(z)}).$$

Since $f(z)$ is an analytic function, it is independent of $\bar{z}$.

i.e., $f(z)$ is a function of z only.

Similarly its conjugate $\overline{f(z)}$ is an analytic function of $\bar{z}$ only.

$\therefore$ We can denote $\overline{f(z)}$ by $\bar{f}(\bar{z})$.

i.e., $\overline{f(x+iy)} = \bar{f}(x-iy)$.

Hence, $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\dfrac{\partial}{\partial \bar{z}}\dfrac{\partial}{\partial z}(f(z)\bar{f}(\bar{z}))$

$$= 4\frac{\partial}{\partial \bar{z}}(\bar{f}(\bar{z}))\frac{\partial}{\partial z}(f(z))$$

$$= 4\overline{f'(z)}f'(z) = 4|f'(z)|^2.$$

2.3.8 November/December 2012 (R 2008)

Part A

7. Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

Solution. Let $f(z) = u + iv = \bar{z} = x - iy$.

$$\therefore u = x, \qquad v = -y$$

$$u_x = 1, \qquad v_x = 0$$

$$u_y = 0, \qquad v_y = -1.$$

Since $u_x \neq v_y$, C.R equations are not satisfied.

$\therefore$ $f(z)$ is not analytic anywhere.

$\Longrightarrow$ $f(z)$ is nowhere differentiable.

8. Find the map of the circle $|z| = 3$ under the transformation $w = 2z$.

Solution. Let $z = x + iy$ and $w = u + iv$.

$$w = 2z$$

$$u + iv = 2(x+iy) = 2x + i2y$$

$$u = 2x, v = 2y$$

$$\text{Now } |z| = 3$$

$$\Rightarrow x^2 + y^2 = 9$$

$$\left(\frac{u}{2}\right)^2 + \left(\frac{v}{2}\right)^2 = 9$$

$$\frac{u^2}{4} + \frac{v^2}{4} = 9$$

$$u^2 + v^2 = 36.$$

It is a circle with centre $(0, 0)$ and radius 6.

Part B

14. (a) (i) Find the bilinear map which maps the points $\infty, i, 0$ of the z plane onto $0, i, \infty$ of the w-plane.

Solution. Let $z_1 = \infty, z_2 = i, z_3 = 0$ and $w_1 = 0, w_2 = i, w_3 = \infty$.

The bilinear transformation is given by

$$\frac{w - w_1}{w_2 - w_1} = \frac{z_2 - z_3}{z - z_3}$$

$$\text{i.e., } \frac{w - 0}{i - 0} = \frac{i - 0}{z - 0}$$

$$\frac{w}{i} = \frac{i}{z}$$

$$w = \frac{-1}{z}.$$

14. (a) (ii) Determine the analytic function whose real part is $u = \dfrac{\sin 2x}{\cos h2y - \cos 2x}$.

Solution. Given, $u = \dfrac{\sin 2x}{\cos h2y - \cos 2x}$.

$$u_x = \frac{(\cos h2y - \cos 2x)2\cos 2x - \sin 2x(2\sin 2x)}{(\cos h2y - \cos 2x)^2}.$$

$$u_x(z, 0) = \frac{(1 - \cos 2z)2\cos 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1 - \cos 2z)^2}$$

$$= \frac{-2(1 - \cos 2z)}{(1 - \cos 2z)^2} = \frac{-2}{2\cos^2 z} = -\sec^2 z.$$

$$u_y = \sin 2x(-1)(\cos h2y - \cos 2x)^{-2} 2\sin h2y.$$

$u_y(z, 0) = 0.$

By Milne Thomson method,

$f'(z) = u_x(z, 0) - iu_y(z, 0) = -\sec^2 z.$

Integrating w.r.t. z, we get

$\int f'(z)dz = \int -\sec^2 z dz + c$

$f(z) = -\tan z + c.$

14. (b) (i) Find the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = \frac{1}{z}$ and prove that it is the lemniscate of Bernouillie $r^2 = \cos 2\theta$.

Solution. Let $z = x + iy$ and $w = re^{i\theta}$.

$$\text{Given } w = \frac{1}{z} \text{ or } z = \frac{1}{w} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}.$$

$$i.e., x + iy = \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$\therefore x = \frac{1}{r}\cos\theta, y = -\frac{1}{r}\sin\theta.$$

The given curve is $x^2 - y^2 = 1$

$$\frac{1}{r^2}\cos^2\theta - \frac{1}{r^2}\sin^2\theta = 1$$

$$\frac{1}{r^2}(\cos^2\theta - \sin^2\theta) = 1$$

$$\frac{1}{r^2}\cos 2\theta = 1$$

$$\Rightarrow r^2 = \cos 2\theta.$$

14. (b) (ii) Prove that $w = \frac{z}{1-z}$ maps the upper half of the z−plane to the upper half of the w−plane and also find the image of the unit circle of the z−plane.

Solution. Given $w = \frac{z}{1-z}$.

$$u + iv = \frac{x + iy}{1 - x - iy}$$

$$= \frac{x+iy}{(1-x)-iy} \times \frac{(1-x)+iy}{(1-x)+iy}$$

$$= \frac{x(1-x)-y^2+i[y(1-x)+xy]}{(1-x)^2+y^2}$$

$$= \frac{x(1-x)-y^2}{(1-x)^2+y^2} + i\frac{y(1-x)+xy}{(1-x)^2+y^2}.$$

$$\therefore u = \frac{x(1-x)-y^2}{(1-x)^2+y^2} \text{ and}$$

$$v = \frac{y(1-x)+xy}{(1-x)^2+y^2}.$$

Let $v > 0$.

$$\Rightarrow \frac{y(1-x)+xy}{(1-x)^2+y^2} > 0$$

$$y - xy + xy > 0$$

$$y > 0.$$

This shows that the upper half of the z−plane is mapped onto the upper half of the w−plane.

To find the image of the unit circle.

we have $w = \dfrac{z}{1-z}$

$$w(1-z) = z$$

$$w - wz = z$$

$$w = wz + z$$

$$z(1+w) = w$$

$$z = \frac{w}{1+w}.$$

Given $|z| = 1$.

$$\Rightarrow \left|\frac{w}{1+w}\right| = 1$$

$$|w| = |1+w|$$

$$|w|^2 = |1+w|^2$$

$$|u+iv|^2 = |1+u+iv|^2$$
$$u^2+v^2 = (1+u)^2+v^2$$
$$= 1+u^2+2u+v^2$$
$$0 = 2u+1$$
$$u = -\frac{1}{2}.$$

The image of the unit circle $|z| = 1$ is the straight line $u = -\frac{1}{2}$.

2.3.9 May/June 2012 (R 2008)

Part A

7. State the basic difference between the limit of a function of a real variable and that of the complex variable.

   Solution. Let f be a function defined in some neighbourhood of z_0 except possibly at z_0. We say a complex number w_0 is the limit of $f(z)$ as z tends to z_0 if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ for $|z - z_0| < \delta$. Symbolically it can be written as $\lim_{z\to z_0} f(z) = w_0 = f(z_0)$.

8. Prove that a bilinear transformation has at most two fixed points.

   Solution. The bilinear transformation is $w = \dfrac{az+b}{cz+d}$.

   The fixed points are given by $w = z$.

   i.e., $z = \dfrac{az+b}{cz+d}$

   $$z(cz+d) = az+b$$
   $$cz^2+dz-az-b = 0$$
   $$cz^2+(d-a)z-b = 0.$$

   This is a quadratic in z. It has two roots which are the fixed points of the given bilinear transformation.

   Hence, a bilinear transformation has at most two fixed points.

Part B

14. (a) (i) Prove that every analytic function $w = u + iv$ can be expressed as a function of z alone, not as a function of $\overline{z}$.

Solution. Let $z = x + iy$.

Then, $\overline{z} = x - iy$, which implies $x = \dfrac{z+\overline{z}}{2}$ and $y = \dfrac{z-\overline{z}}{2i}$.

Now, $w = u + iv$.

$$\begin{aligned}
\frac{\partial w}{\partial \overline{z}} &= \frac{\partial u}{\partial \overline{z}} + i\frac{\partial v}{\partial \overline{z}} \\
&= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \overline{z}} + i\left[\frac{\partial v}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \overline{z}}\right] \\
&= \frac{\partial u}{\partial x}\frac{1}{2} + \frac{\partial u}{\partial y}\frac{-1}{2i} + i\left[\frac{\partial v}{\partial x}\frac{1}{2} + \frac{\partial v}{\partial y}\frac{-1}{2i}\right] \\
&= \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right] + \frac{i}{2}\left[\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right] \\
&= \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right] \\
&= \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \\
&= \frac{1}{2}[0 + i.0][\because u_x = v_y, u_y = -v_x] \\
\frac{\partial w}{\partial \overline{z}} &= 0.
\end{aligned}$$

$\Rightarrow w$ is independent of $\overline{z}$.

Hence, w can be written only as a function of z and not as a function of $\overline{z}$

14. (a) (ii) Find the bilinear transformation which maps $z_1 = 0, z_2 = 1, z_3 = \infty$ into the points $w_1 = i, w_2 = 1, w_3 = -i$.

Solution. Let $z_1 = 0, z_2 = 1, z_3 = \infty$, and $w_1 = i, w_2 = 1, w_3 = -i$.

The mobius transformation is

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)}{(z_2 - z_1)}$$

i.e., $\dfrac{(w - i)(1 + i)}{(w + i)(1 - i)} = z$

$$\frac{w-i}{w+i} = \frac{z(1-i)}{1+i} = \frac{z(1-i)^2}{2}$$
$$= \frac{z(1-1-2i)}{2} = \frac{z(-2i)}{2} = -iz$$
$$\frac{w-i+w+i}{w-i-w-i} = \frac{-iz+1}{-iz-1}$$
$$\frac{2w}{-2i} = \frac{1-iz}{-(1+iz)}$$
$$w = \frac{i(1-iz)}{1+iz} = \frac{z+i}{1+iz}.$$

14. (b) (i) If $f(z)$ is analytic, prove that $\nabla^2 \log|f(z)| = 0$.

Solution.
$$\nabla^2 \log|f(z)| = \frac{1}{2}\nabla^2 \log|f(z)|^2$$
$$= \frac{1}{2}\nabla^2 \log(f(z)\overline{f(z)})$$
$$= \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\log f(z) + \log \bar{f}(\bar{z})\right)$$
$$= \frac{1}{2}4\frac{\partial^2}{\partial \bar{z}\partial z}\left(\log f(z) + \log \bar{f}(\bar{z})\right)$$
$$= 2\frac{\partial}{\partial \bar{z}}\left(\frac{1}{f(z)} f'(z)\right) = 0.$$

14. (b) (ii) Find the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = \frac{1}{z}$ and prove that it is the lemniscate of Bernouillie $r^2 = \cos 2\theta$.

Solution. Let $z = x + iy$ and $w = re^{i\theta}$.

$$\text{Given } w = \frac{1}{z} \text{ or } z = \frac{1}{w} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}.$$

$$i.e., x + iy = \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$\therefore x = \frac{1}{r}\cos\theta, y = -\frac{1}{r}\sin\theta.$$

The given curve is $x^2 - y^2 = 1$

$$\frac{1}{r^2}\cos^2\theta - \frac{1}{r^2}\sin^2\theta = 1$$
$$\frac{1}{r^2}(\cos^2\theta - \sin^2\theta) = 1$$

UNIT 4

By EasyEngineering.net

4 Complex Integration

4.1 Introduction

Contour

A curve that is constructed by joining finitely many smooth curves end to end is called a contour. A curve is given by the equation $z(t) = x(t) + iy(t), a \le t \le b$ where $x(t)$ and $y(t)$ are continuous functions of t.

Contour Integral. If $f(z)$ is a function of the complex variable z which is defined on a given curve C in the complex plane, then the complex line integral is written as $\int_C f(z)dz$.

If the equation of C is $z(t) = x(t) + iy(t), a \le t \le b$ and C is the contour from $z_1 = z(a)$ to $z_2 = z(b)$ then we write $\int_C f(z)dz$ as $\int_{z_1}^{z_2} f(z)dz$.

If $f(z) = u(x, y) + iv(x, y)$ then

$$\int_C f(z)dz = \int_C (u + iv)(dx + idy) = \int_C (udx - vdy) + i\int_C (vdx + udy).$$

Properties of Contour Integrals

1. $\int_C z_0 f(z)dz = z_0 \int_C f(z)dz$, where z_0 is a constant.
2. $\int_C (f(z) \pm g(z))dz = \int_C f(z)dz \pm \int_C g(z)dz$.

3. $\int\limits_{-C} f(z)dz = -\int\limits_{C} f(z)dz.$
4. If C is broken up into two parts C_1 and C_2, then
$\int\limits_{C} f(z)dz = \int\limits_{C_1} f(z)dz + \int\limits_{C_2} f(z)dz.$

Note. The value of the integral is independent of the path of integration, when the integrand is analytic.

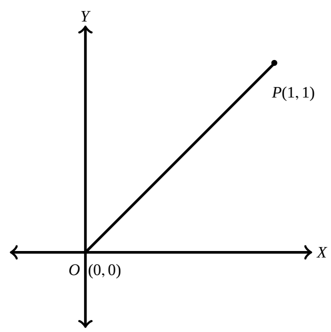
Worked Examples

Example 4.1. Evaluate $\int\limits_{C}(x - y + ix^2)dz$ where C is the line joining the points from $z = 0$ to $z = 1 + i$. [Apr 2009]

Solution. Let $z = x + iy$.

$$dz = dx + idy.$$

The points $z = 0$ and $z = 1 + i$ represent O and P respectively in the complex plane. Hence O is $(0, 0)$ and P is $(1, 1)$. Equation to OP is $y = x$. Along $OP, y = x, dy = dx$ and x varies from 0 to 1.



$$\therefore \int\limits_{C}(x - y + ix^2)dz = \int\limits_{0}^{1}(x - x + ix^2)(dx + idy)$$

$$= \int\limits_{0}^{1} ix^2(dx + idx) = i\int\limits_{0}^{1} x^2(1 + i)dx$$

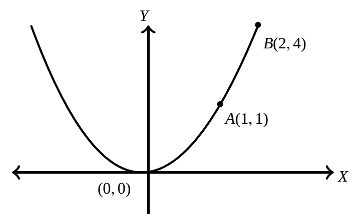
$$= i(1 + i)\left(\frac{x^3}{3}\right)_0^1 = \frac{i(1 + i)}{3} = \frac{-1 + i}{3} = -\frac{1}{3} + \frac{1}{3}i.$$

—

Example 4.2. Evaluate $\int_C z^2 dz$ where C is the arc from $A(1,1)$ to $B(2,4)$ along (i) $y = x^2$ (ii) $y = 3x - 2$. [Jun 2010]

Solution. Let $z = x + iy$.

$dz = dx + idy.$



(i) Let us evaluate $\int_C z^2 dz$ where C is $y = x^2$.

Along $y = x^2, dy = 2xdx, x$ varies from 1 to 2.

Now, $dz = dx+idy = dx+i2xdx = (1+2ix)dx$.

$$\therefore \int_C z^2 dz = \int_C (x+iy)^2(dx+idy) = \int_C (x^2 - y^2 + 2ixy)(dx+idy)$$

$$= \int_1^2 (x^2 - x^4 + 2ix^3)(1 + i2x)dx$$

$$= \int_1^2 \Big((x^2 - x^4 - 4x^4) + i(2x^3 - 2x^5 + 2x^3)\Big)dx$$

$$= \int_1^2 \Big((x^2 - 5x^4) + i(4x^3 - 2x^5)\Big)dx$$

$$= \left(\frac{x^3}{3}\right)_1^2 - 5\left(\frac{x^5}{5}\right)_1^2 + i\left(4\left(\frac{x^4}{4}\right)_1^2 - 2\left(\frac{x^6}{6}\right)_1^2\right) = \frac{1}{3}(8-1) - (32-1) + i(16 - 1 - \frac{1}{3}(64-1))$$

$$= \frac{7}{3} - 31 + i(15 - 21) = \frac{7-93}{3} - 6i = -\left[\frac{86}{3} + 6i\right].$$

(ii) Let us evaluate $\int_C z^2 dz$ along $y = 3x - 2$.

Along $y = 3x - 2, dy = 3dx, dz = dx + idy = dx + i3dx = (1+3i)dx$

$$\text{Now} \int_C z^2 dz = \int_1^2 (x^2 - (3x-2)^2 + 2ix(3x-2))(1+3i)dx$$

$$= (1+3i)\int_1^2 x^2 - 9x^2 - 4 + 12x + i(6x^2 - 4x)dx$$

$$= (1+3i)\int_1^2 \Big[-8x^2 - 4 + 12x + i(6x^2 - 4x)\Big]dx$$

$$= (1+3i)\Big[-8\Big(\frac{x^3}{3}\Big)_1^2 + 12\Big(\frac{x^2}{2}\Big)_1^2 - 4(x)_1^2 + i\Big(6\Big(\frac{x^3}{3}\Big)_1^2 - 4\Big(\frac{x^2}{2}\Big)_1^2\Big)\Big]$$

$$= (1+3i)[\frac{-8}{3}7 + 6(3) - 4 + i(2(7) - 2(3))]$$

$$= (1+3i)[\frac{-56}{3} + 14 + 8i] = \frac{(1+3i)}{3}[-56 + 42 + 24i]$$

$$= \frac{(1+3i)}{3}[-14 + 24i] = \frac{1}{3}[-14 + 24i - 42i - 72]$$

$$= \frac{1}{3}[-86 - 18i] = -\Big(\frac{86}{3} + 6i\Big).$$

Example 4.3. Evaluate $\int_0^{2+i} (\overline{z})^2 dz$ along the line $y = \frac{x}{2}$. [Dec 2010]

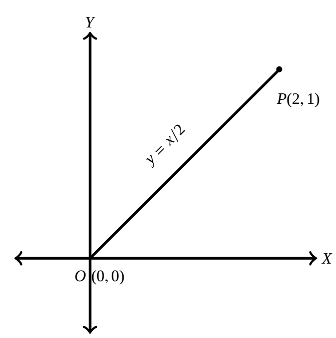
Solution. Let $z = x + iy$.

$dz = dx + idy.$

$\overline{z} = x - iy.$

$(\overline{z})^2 = (x-iy)^2 = x^2 - y^2 - 2ixy.$

We have to evaluate the integral along $y = \frac{x}{2}$ from $(0,0)$ to $(2,1)$. Along $y = \frac{x}{2}, dy = \frac{1}{2}dx$, x varies from 0 to 2.



$$(\overline{z})^2 dz = (x^2 - y^2 - 2ixy)(dx + idy)$$

$$= \Big(x^2 - \frac{x^2}{4} - 2ix\frac{x}{2}\Big)\Big(dx + i\frac{dx}{2}\Big)$$

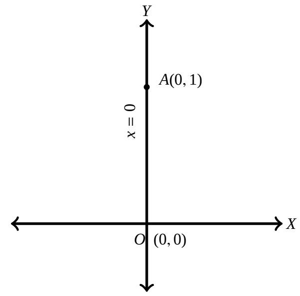
$$= \Big(\frac{3x^2}{4} - ix^2\Big)\Big(1 + \frac{i}{2}\Big)dx$$

$$\therefore \int_0^{2+i} (\overline{z})^2 dz = \int_0^2 \Big(\frac{3x^2}{4} - ix^2\Big)\Big(1 + \frac{i}{2}\Big)dx = \Big(1 + \frac{i}{2}\Big)\Big[\frac{3}{4}\Big(\frac{x^3}{3}\Big)_0^2 - i\Big(\frac{x^3}{3}\Big)_0^2\Big]$$

$$= \left(1+\frac{i}{2}\right)\left[2 - i\frac{8}{3}\right] = 2 - \frac{8}{3}i + i + \frac{4}{3} = \frac{10}{3} - \frac{5}{3}i.$$

Example 4.4. Evaluate $\int_C \sin z dz$ along the line $z = 0$ to $z = i$. [Apr 2010]

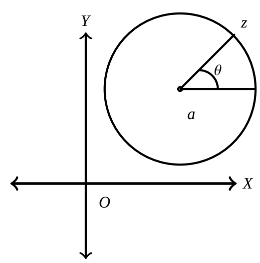
Solution.



$$\int_C \sin z \, dz = \int_0^i \sin z \, dz$$
$$= [-\cos z]_0^i$$
$$= -[\cos i - \cos 0]$$
$$= 1 - \cos i.$$

Example 4.5. Prove that (i) $\int_C \frac{dz}{z-a} = 2\pi i$ (ii) $\int_C (z-a)^n dz = 0$, n any integer $\neq -1$, where C is the circle $|z-a| = r$.

Solution. The equation of C is $z - a = re^{i\theta}$, where θ varies from 0 to 2π as z describes once in the positive sense.



(i) $z - a = re^{i\theta}$.

$$dz = re^{i\theta} \cdot i d\theta.$$

$$\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{1}{re^{i\theta}} \cdot re^{i\theta} \cdot i d\theta$$
$$= \int_0^{2\pi} i d\theta$$
$$= i[\theta]_0^{2\pi}$$
$$= i(2\pi - 0) = 2\pi i.$$

(ii) $$\int_C (z-a)^n dz = \int_0^{2\pi} \left(re^{i\theta}\right)^n \cdot re^{i\theta} i d\theta = \int_0^{2\pi} r^n e^{in\theta} \cdot re^{i\theta} i d\theta$$

—

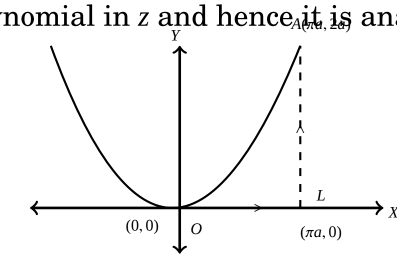
$$= \int_0^{2\pi} ir^{n+1}e^{i(n+1)\theta}d\theta = ir^{n+1}\left(\frac{e^{i(n+1)\theta}}{i(n+1)}\right)_0^{2\pi}$$

$$= \frac{r^{n+1}}{n+1}\left[e^{i2(n+1)\pi} - e^0\right] = \frac{r^{n+1}}{n+1}[1-1] = 0.$$

Example 4.6. Evaluate $\int_C (z^2 + 3z + 2)dz$ where C is the arc of the cycloid $x = a(\theta + \sin\theta)$, $\quad y = a(1 - \cos\theta)$ between the points $(0, 0)$ and $(\pi a, 2a)$.

Solution. Let $f(z) = z^2 + 3z + 2$ which is a polynomial in z and hence it is analytic.

$\therefore \int_C (z^2 + 3z + 2)dz$ is independent of the path of integration.



Let O be $(0, 0)$ and A be $(\pi a, 2a)$ on the cycloid. Draw AL perpendicular to the x-axis.

Now, $\int_C f(z)dz = \int_{OL} f(z)dz + \int_{LA} f(z)dz.$

Along OL, $y = 0 \Rightarrow dy = 0$, x varies from 0 to πa.

$$\therefore \int_{OL} f(z)dz = \int_0^{\pi a} (x^2 + 3x + 2)dx$$

$$= \left[\frac{x^3}{3} + 3 \cdot \frac{x^2}{2} + 2 \cdot x\right]_0^{\pi a}$$

$$= \frac{\pi^3 a^3}{3} + \frac{3}{2}\pi^2 a^2 + 2\pi a.$$

$\because z = x + iy$

$dz = dx + iy$

$y = 0.$

Along LA, $x = \pi a$, $dx = 0$, y varies from 0 to $2a$.

$$z = x + iy,\ dz = dx + idy = idy$$

$$f(z) = z^2 + 3z + 2$$

$$= (x + iy)^2 + 3(x + iy) + 2$$

$$= (\pi a + iy)^2 + 3(\pi a + iy) + 2.$$

—

$$\int_{LA} f(z)dz = \int_0^{2a} \left[(\pi a + iy)^2 + 3(\pi a + iy) + 2\right] idy = i\left[\frac{(\pi a + iy)^3}{3i} + \frac{3(\pi a + iy)^2}{2i} + 2y\right]_0^{2a}$$

$$= \frac{(\pi a + 2ai)^3}{3} + \frac{3}{2}(\pi a + 2ai)^2 + 4ai - \frac{\pi^3 a^3}{3} - \frac{3}{2}\pi^2 a^2$$

$$= \frac{a^3}{3}(\pi + 2i)^3 + \frac{3a^2}{2}(\pi + 2i)^2 + 4ai - \frac{\pi^3 a^3}{3} - \frac{3\pi^2 a^2}{2}.$$

$$\therefore \int_C f(z)dz = \cancel{\frac{\pi^3 a^3}{3}} + \cancel{\frac{3\pi^2 a^2}{2}} + 2\pi a + \frac{a^3}{3}(\pi + 2i)^3 + \frac{3a^2}{2}(\pi + 2i)^2 + 4ai - \cancel{\frac{\pi^3 a^3}{3}} - \cancel{\frac{3\pi^2 a^2}{2}}$$

$$= \frac{a^3}{3}(\pi + 2i)^3 + \frac{3a^2}{2}(\pi + 2i)^2 + 2a(\pi + 2i).$$

Exercise 4 A

1. Evaluate $\int_C (z+1)dz$ where C is the boundary of the square whose vertices are at the points $z = 0, z = 1, z = 1 + i$ and $z = i$. [Mar 2010]

2. Evaluate $\int_0^{1+i} (x^2 - iy)dz$ along the paths (i) $y = x$ (ii) $y = x^2$.

3. Evaluate $\int_C f(z)dz$ where $f(z) = y - x - i3x^2$ from $z = 0$ to $z = 1 + i$ along the path (i) from $(0, 0)$ to $A(1, 0)$ and to $B(1, 1)$. (ii) $y = x$.

4. Evaluate $\int_C e^z dz$ where C is the circle $|z| = 1$.

5. Evaluate $\int_C \log z dz$ where C is the circle $|z| = 1$.

6. Prove that $\int_C \frac{1}{z-a} dz = 2\pi i$ where C is the circle $|z - a| = r$.

7. Prove that $\int_C (z - a)^n dz = 0 [n \neq -1]$ where C is the circle $|z - a| = r$.

—

8. Evaluate $\int_{1-i}^{2+3i} (z^2 + z)dz$ along the line joining the points $(1, -1)$ and $(2, 3)$.

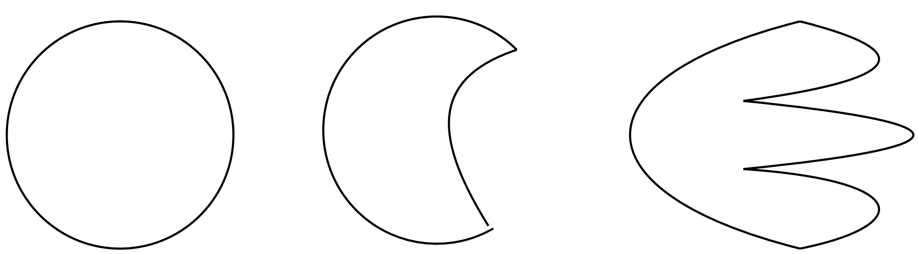
9. Evaluate the integral $\int_{0}^{2+i} (x^2 - iy)dz$ (i) along the straight line $y = x$. (ii) along the curve $y = x^2$.

10. Evaluate $\int_{-2}^{-2+i} (2 + z)^2 dz$.

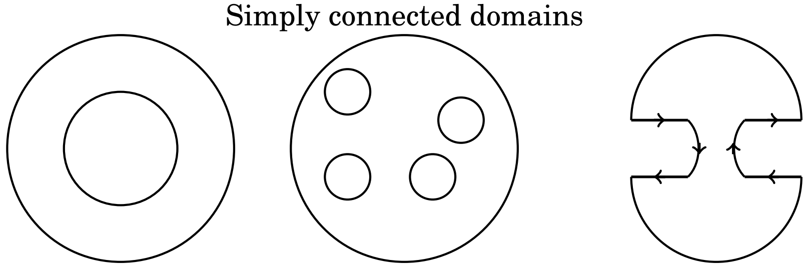
Simply Connected and multiply connected domains

A domain D in the complex plane is called simply connected if every closed curve in D encloses points of D only.

Example. Interior of a circle, square and ellipse are simply connected domains. A domain D which is not simply connected is called multiply connected.



Simply connected domains



Multiply connected domains.

4.2 Cauchy's Integral theorem or Cauchy's fundamental theorem

Statement. If $f(z)$ is analytic and $f'(z)$ is continuous at each point within and on a simple closed curve C, then $\oint_C f(z)dz = 0$. [May 2015]

Proof. Let $f(z) = u(x, y) + iv(x, y)$.

$$z = x + iy.$$

$$dz = dx + idy.$$

Now $\int_C f(z)dz = \int_C (u + iv)(dx + idy)$

$$= \int_C (udx - vdy) + i \int_C (udy + vdx). \tag{1}$$

Given, $f'(z)$ is continuous.

$\therefore \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in the domain D enclosed by C.

$\therefore$ Applying Green's theorem for the integrals on the RHS of (1) we get

$$\int_C f(z)dz = -\iint_D \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dxdy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy. \tag{2}$$

Since $f(z)$ is analytic, u and v satisfy C-R equations.

ie., $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$\Longrightarrow \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$.

$$\therefore \int_C f(z)dz = 0.$$

Extension of Cauchy's theorem. If $f(z)$ is analytic in the region D between two simple closed curves C and C_1, then $\oint_C f(z)dz = \oint_{C_1} f(z)dz$.

Proof. Let AB be the cross cut between C and C_1. Now, by the path indicated by arrows, $\oint_C f(z)dz = 0$.

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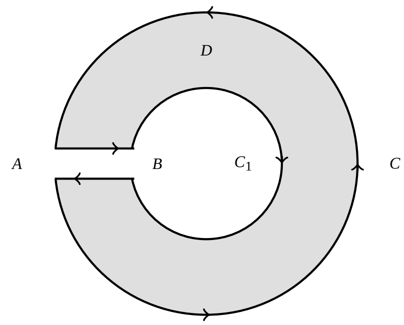
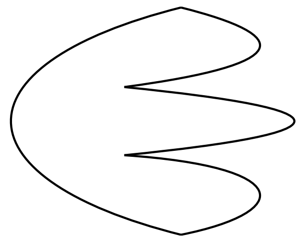




The path described is as follows. Along $AB \rightarrow$ along C_1 in the clockwise sense $\rightarrow$ along $BA \rightarrow$ along C in the anticlockwise sense.

$$\text{i.e.,} \int_{AB} f(z)dz + \int_{C_1} f(z)dz + \int_{BA} f(z)dz + \int_{C} f(z)dz = 0.$$

The integrals along AB and BA are equal and opposite, $\therefore \int_{AB} f(z)dz + \int_{BA} f(z)dz = 0.$

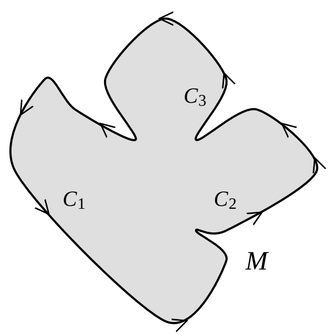
Hence we have $\int_{C_1} f(z)dz + \int_{C} f(z)dz = 0.$

Reversing the direction of the integral along C_1 and transposing, we obtain $\int_{C} f(z)dz = \int_{C_1} f(z)dz$, each integration is being taken in the anticlockwise sense.

Hence the theorem.

Corollary. If $C_1, C_2, C_3, \ldots$ are any number of closed curves within C, then

$$\oint_{C} f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz + \cdots$$



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4.3 Singular points

If a function $'f'$, is not analytic at a point z_0, but is analytic at some point in every neighbourhood of z_0, then z_0 is a singular point of f.

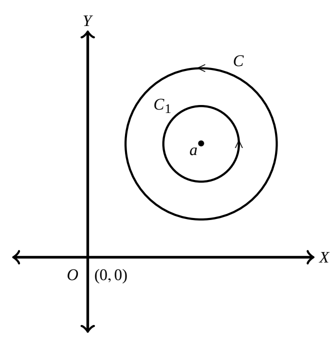
Example. If $f(z) = \dfrac{1}{z}, z = 0$ is a singular point.

4.4 Cauchy's Integral Formula

If $f(z)$ is an analytic function within and on a simple closed contour C taken in the positive sense and if a is any interior point of C, then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz.$$

[Dec 2014]



Proof. Consider the function $\dfrac{f(z)}{z-a}$. This is analytic at all points within C except at $z = a$. Draw a small circle C_1 with a as centre and radius r lying entirely within C.

Now, $\dfrac{f(z)}{z-a}$ is analytic in the region enclosed by C and C_1.

$\therefore$ By Cauchy's integral theorem,

$$\oint_C \frac{f(z)}{z-a} dz = \oint_{C_1} \frac{f(z)}{z-a} dz \quad [z - a = re^{i\theta},\ dz = rie^{i\theta} d\theta]$$

$$= \oint_{C_1} \frac{f(a + re^{i\theta})}{re^{i\theta}} rie^{i\theta} d\theta = i \oint_{C_1} f(a + re^{i\theta}) d\theta.$$

In the limiting form, as the circle C_1 tends to the point a, $r \to 0$. Hence, the integral is reduced to

$$\oint_C \frac{f(z)}{z-a}dz = i\int_0^{2\pi} f(a)d\theta = if(a)(\theta)_0^{2\pi} = 2\pi i f(a)$$

$$\therefore\ f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz,$$

which is the required Cauchy's integral formula.

Corollary. Cauchy's Integral formula for derivatives

Statement. $\displaystyle\int_C \frac{f(z)}{(z-a)^{n+1}}dz = 2\pi i\frac{f^n(a)}{n!}.$

Proof. By Cauchy's integral formula we have

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz.$$

Differentiating w.r.t. a we get

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial a}\left[\frac{f(z)}{z-a}\right]dz = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}dz.$$

Similarly, $\displaystyle f''(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}dz.$

In general, $\displaystyle f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz.$

$$\therefore \oint_C \frac{f(z)}{(z-a)^{n+1}}dz = 2\pi i\frac{f^{(n)}(a)}{n!}.$$

Worked Examples

Example 4.7. Evaluate $\displaystyle\int_C e^z dz$ where C is $|z| = 1$. [Jun 2003]

Solution. Let $f(z) = e^z$.

e^z is analytic everywhere in the complex plane.

—

Hence it is analytic in $|z| = 1$.

$\therefore$ By Cauchy's Integral theorem $\int_C f(z)dz = 0$.

i.e., $\int_C e^z dz = 0$.

Example 4.8. Evaluate $\int_C \frac{z^2 - z + 1}{z - 1} dz$ where C is the circle (i) $|z| = 1$ (ii) $|z| = \frac{1}{2}$.

Solution. (i) Consider the circle $|z| = 1$.

The singular point is $z = 1$ which lies on C.

$\therefore$ By Cauchy's integral formula

$$\int_C \frac{f(z)}{z-1} dz = 2\pi i f(1) \quad \text{where } f(z) = z^2 - z + 1$$

$$= 2\pi i \times 1 = 2\pi i.$$

(ii) C is $|z| = \frac{1}{2}$.

In this case, $z = 1$ lies outside C.

$\therefore \frac{f(z)}{z-1}$ is analytic every where within C.

$$\therefore \int_C \frac{f(z)}{z-1} dz = 0$$

$$\int_C \frac{z^2 - z + 2}{z - 1} dz = 0.$$

Example 4.9. Evaluate $\int_C \frac{z+4}{z^2 + 2z} dz$, where C is $|z - \frac{1}{2}| = \frac{1}{3}$. [Dec 2013]

Solution. C is the circle $|z - \frac{1}{2}| = \frac{1}{3}$.

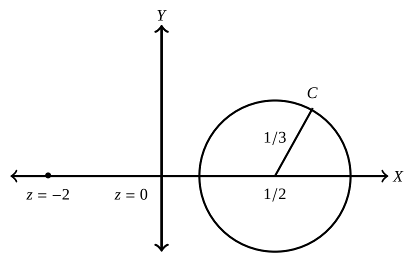
$$\int_C \frac{z+4}{z^2 + 2z} dz = \int_C \frac{z+4}{z(z+2)} dz.$$

—

The singular points are $z = 0$ and $z = -2$, which lie completely outside C.

$\therefore \dfrac{z+4}{z^2+2z}$ is analytic everywhere inside C.

$\therefore \displaystyle\int_C \frac{z+4}{z^2+2z}dz = 0.$



Example 4.10. Evaluate $\displaystyle\int_C \frac{3z^2+7z+1}{z+1}dz$ where C is $|z| = \frac{1}{2}$. [Jun 2013, Jun 2010]

Solution. C is $|z| = \frac{1}{2}$.

For the integrand $\dfrac{3z^2+7z+1}{z+1}$, $z = -1$ is a singular point which lies completely outside $|z| = \frac{1}{2}$.

$\therefore \dfrac{3z^2+7z+1}{z+1}$ is analytic everywhere inside C.

$\therefore$ By Cauchy's integral theorem $\displaystyle\int_C \frac{3z^2+7z+1}{z+1}dz = 0.$

Example 4.11. Evaluate $\displaystyle\int_C \frac{zdz}{(z-1)(z-2)}$ where C is the circle $|z| = \frac{1}{2}$.

[Dec 2012, Dec 2011]

Solution. C is $|z| = \frac{1}{2}$.

The function $\dfrac{z}{(z-1)(z-2)}$ has singular points $z = 1$ and $z = 2$ which lie completely outside $|z| = \frac{1}{2}$.

$\therefore \dfrac{z}{(z-1)(z-2)}$ is analytic everywhere inside C.

$\therefore \displaystyle\int_C \frac{zdz}{(z-1)(z-2)} = 0$, by Cauchy's integral theorem.

Example 4.12. Evaluate $\displaystyle\int_C \frac{e^z}{z-1}dz$ if C is $|z| = 2$. [Dec 2010]

Solution. C is the circle $|z| = 2$.

It is a circle with centre $(0,0)$ and radius $= 2$ units.

The singular point is at $z = 1$ which is an interior point of C.

By Cauchy's Integral formula,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz.$$

Here $f(z) = e^z$ and $a = 1$.

$$\therefore\ f(1) = \frac{1}{2\pi i}\int_C \frac{e^z}{z-1}dz$$

$$e = \frac{1}{2\pi i}\int_C \frac{e^z}{z-1}dz$$

$$\int_C \frac{e^z}{z-1}dz = 2\pi i e.$$

Example 4.13. Using Cauchy's integral formula, evaluate $\int_C \frac{z}{z-2}dz$ where C is the circle $|z-2| = \frac{3}{2}$. [May 2001]

Solution. The singular point is $z = 2$.

Let $f(z) = z$.

C is $|z-2| = \frac{3}{2}$.

When $z = 2, |z-2| = |2-2| = 0 < \frac{3}{2}$.

$\therefore z = 2$ lies inside C.

By Cauchy's integral formula

$$f(2) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-2}dz$$

$$2 = \frac{1}{2\pi i}\int_C \frac{z}{z-2}dz$$

$$\int_C \frac{z}{z-2}dz = 4\pi i.$$

—

Example 4.14. Evaluate $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$ where C is $|z| = \frac{3}{2}$. [Jun 2001]

Solution. By Cauchy's Integral formula $\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$.

The circle is $|z| = \frac{3}{2}$.

$z = 1$ and $z = 2$ are singular points.

When $z = 1$, $|z| = |1| < \frac{3}{2}$.

$\therefore\ z = 1$ lies inside the circle.

When $z = 2$, $|z| = |2| > \frac{3}{2}$.

$\therefore\ z = 2$ lies outside the circle.

Now, $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = \int_C \frac{\frac{\cos \pi z^2}{(z-2)}}{(z-1)} dz = \int_C \frac{f(z)}{(z-1)} dz$

where $f(z) = \frac{\cos \pi z^2}{z-2}$

By Cauchy's Integral formula $f(1) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-1} dz$

$$\frac{\cos \pi}{1} = \frac{1}{2\pi i} \int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$$

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i.$$

Example 4.15. Evaluate $\oint_C \frac{z^2+1}{z^2-1} dz$ where C is the circle $|z-1| = 1$. [Dec 2001]

Solution. The singular points are given by $z^2 - 1 = 0$.

$(z-1)(z+1) = 0.$

$z = 1, -1.$

The contour is $|z-1| = 1$.

When $z = 1, |z-1| = |1-1| = 0 < 1$.

$\Longrightarrow z = 1$ lies inside C.

When $z = -1, |z-1| = |-1-1| = |-2| > 1$.

$\implies z = -1$ lies outside C.

$$\text{Now, } \int_C \frac{z^2+1}{z^2-1}dz = \int_C \frac{\frac{z^2+1}{z+1}}{z-1}dz = \int_C \frac{f(z)}{z-1}dz \text{ where } f(z) = \frac{z^2+1}{z+1}$$

By Cauchy's integral formula, $f(1) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z-1}dz$

$$\frac{1+1}{1+1} = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-1}dz$$

$$2\pi i = \int_C \frac{z^2+1}{z^2-1}dz.$$

Example 4.16. Evaluate $\displaystyle\oint_C \frac{z+2}{z}dz$ where C is the circle $|z| = 2$ in the z plane.

[Jun 2007]

Solution. The singular point is $z = 0$.

C is the circle $|z| = 2$.

When $z = 0, |z| = |0| = 0 < 2$.

$\implies z = 0$ lies inside C.

Now, $\displaystyle\oint_C \frac{z+2}{z}dz = \oint_C \frac{f(z)}{z}dz$ where $f(z) = z+2$.

By Cauchy's integral formula, $f(0) = \dfrac{1}{2\pi i}\displaystyle\oint_C \frac{f(z)}{z}dz$

$$2 = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z}dz$$

$$4\pi i = \oint_C \frac{z+2}{z}dz.$$

Example 4.17. Evaluate $\displaystyle\int_C \frac{z+4}{z^2+2z+5}dz$ where C is the circle (i) $|z+1+i| = 2$ (ii) $|z+1-i| = 2$ (iii) $|z+1| = 1$. [Dec 2012, Dec 2011, Dec 2010, Jun 2008]

Solution. Let $I = \displaystyle\int_C \frac{z+4}{z^2+2z+5}dz.$

The singular points are given by $z^2+2z+5 = 0$.

$(z+1)^2+5-1=0$

$(z+1)^2+4=0$

$(z+1)^2=-4$

$z+1=\pm 2i$

$z=-1\pm 2i.$

(i) The circle is $|z+1+i|=2$.

When $z=-1-2i, |z+1+i|=|-1-2i+1+i|=|-i|=1<2.$

$\therefore\ -1-2i$ is a point inside the circle.

When $z=-1+2i, |z+1+i|=|-1+2i+1+i|=|3i|=3>2.$

$\therefore\ z=-1+2i$ lies outside the circle.

Now,

$$\int_C \frac{z+4}{z^2+2z+5}dz=\int_C \frac{z+4}{(z+1-2i)(z+1+2i)}dz=\int_C \frac{\frac{z+4}{z+1-2i}}{z+1+2i}dz=\int_C \frac{f(z)}{z+1+2i}dz.$$

Where $f(z)=\dfrac{z+4}{z+1-2i}$.

By Cauchy's integral formula

$$f(-1-2i)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z+1+2i}dz$$

$$\frac{-1-2i+4}{-1-2i+1-2i}=\frac{1}{2\pi i}\int_C \frac{f(z)}{z+1-2i}dz$$

$$\frac{3-2i}{-4i}=\frac{1}{2\pi i}\int_C \frac{z+4}{z^2+2z+5}dz$$

$$-\frac{\pi}{2}(3-2i)=\int_C \frac{z+4}{z^2+2z+5}dz$$

$$\int_C \frac{z+4}{z^2+2z+5}dz=\frac{\pi}{2}(2i-3).$$

(ii) The circle is $|z+1-i|=2$.

When $z=-1-2i$, $|z+1-i|=|-1-2i+1-i|=|-3i|=3>2.$

$\therefore\ z=-1-2i$ lies outside the circle.

When $z=-1+2i$, $|z+1-i|=|-1+2i+1-i|=|i|=1<2.$

$\therefore\ z = -1+2i$ lies inside the circle.

Now $\displaystyle\int_C \frac{z+4}{z^2+2z+5}dz = \int_C \frac{\frac{z+4}{z+1+2i}}{z+1-2i}dz = \int_C \frac{f(z)}{z+1-2i}dz,$

$$\text{where, } f(z) = \frac{z+4}{z+1+2i}, a = -1+2i.$$

$$f(a) = \frac{-1+2i+4}{-1+2i+1+2i} = \frac{3+2i}{4i}.$$

By Cauchy's integral formula

$$\int_C \frac{f(z)}{z-a}dz = 2\pi i f(a) = 2\pi i \frac{3+2i}{4i} = \frac{3+2i}{2}\pi.$$

(iii) C is $|z+1| = 1$.

When $z = -1-2i, |z+1| = |-1-2i+1| = |-2i| = 2 > 1$.

$\therefore -1-2i$ lies outside C.

When $z = -1+2i, |z+1| = |-1+2i+1| = |2i| = 2 > 1$.

$\therefore -1+2i$ lies outside C.

$\therefore\ f(z) = \dfrac{z+4}{z^2+2z+5}$ is analytic inside C.

$\therefore$ By Cauchy's theorem $\displaystyle\int_C f(z)dz = 0$.

Example 4.18. Evaluate $\displaystyle\int_C \frac{z}{(z-1)^3}dz$ where C is $|z| = 2$ using Cauchy's integral formula. [May 1998]

Solution. $z = 1$ is a singular point.

C is $|z| = 2$.

When $z = 1, |z| = |1| < 2$.

$\therefore\ z = 1$ is an interior point.

Now $\displaystyle\int_C \frac{z}{(z-1)^3}dz = \int_C \frac{f(z)}{(z-1)^3}dz$

where $f(z) = z, n+1 = 3 \Rightarrow n = 2$.

$f'(z) = 1, f''(z) = 0, f''(1) = 0.$

—

By Cauchy's integral formula, $\int\limits_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i f^{(n)}(a)}{n!}$.

i.e., $\int\limits_C \frac{z}{(z-1)^3}dz = \frac{2\pi i f''(1)}{2} = \pi i.0 = 0.$

Example 4.19. If $f(a) = \int\limits_C \frac{3z^2+7z+1}{z-a}dz$ where C is the circle $|z| = 2$, find $f(3), f(1), f'(1-i), f''(1-i)$. [Dec 2014, Dec 2011]

Solution. Given

$$f(a) = \int\limits_C \frac{3z^2+7z+1}{z-a}dz = \frac{1}{2\pi i}\int\limits_C \frac{(2\pi i)(3z^2+7z+1)}{z-a}dz.$$

By Cauchy's integral formula $f(a) = \frac{1}{2\pi i}\int\limits_C \frac{f(z)}{z-a}dz.$

$$\therefore\ f(z) = 2\pi i(3z^2+7z+1).$$

C is the circle $|z| = 2$.

(i) When $z = 3, |z| = 2 \Rightarrow |3| > 2$.

$\therefore\ z = 3$ lies outside the circle.

$\therefore\ \frac{f(z)}{z-3}$ is analytic inside and on C.

$\therefore\ \int\limits_C \frac{f(z)}{z-3}dz = 0.$

By Cauchy's integral theorem,

$f(3) = \int\limits_C \frac{f(2)}{z-3}dz = 0 \qquad \Rightarrow\ f(3) = 0.$

(ii) When $z = 1, |z| = |1| < 2$.

$\therefore\ z = 1$ is an interior point of C.

$\therefore\ f(1) = 2\pi i(3+7+1) = 22\pi i.$

(iii) $f(z) = 2\pi i(3z^2+7z+1).$

$f'(z) = 2\pi i(6z+7).$

When $z = 1 - i, |z| = |1 - i| = \sqrt{2} < 2.$

$\therefore$ $1 - i$ is an interior point.

$$f'(1-i) = 2\pi i(6 - 6i + 7) = 2\pi i(13 - 6i) = 2\pi(6 - 13i)$$

$$f''(z) = 2\pi i.6 = 12\pi i$$

$$f''(1-i) = 12\pi i.$$

Example 4.20. Evaluate $\displaystyle\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is $|z| = 3$.

[Jun 2013, Dec 2011]

Solution. $z = 1$ and $z = 2$ are singular points.

The circle is $|z| = 3$.

When $z = 1, |z| = |1| < 3$.

$\therefore z = 1$ lies inside the circle.

When $z = 2, |z| = |2| < 3$.

$\therefore z = 2$ also lies inside the circle.

Let $\displaystyle\frac{1}{(z-1)(z-2)} = \frac{(z-1)-(z-2)}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$

$$\text{Now } I = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz$$

$$= \int_C \frac{f(z)}{z-2} dz - \int_C \frac{f(z)}{z-1} dz$$

$$I = 2\pi i f(2) - 2\pi i f(1) \tag{1}$$

where $f(z) = \sin \pi z^2 + \cos \pi z^2$.

When $a = 2$. $f(2) = \sin 4\pi + \cos 4\pi = 1$.

When $a = 1$, $f(1) = \sin \pi + \cos \pi = -1$.

$(1) \implies I = 2\pi i f(2) - 2\pi i f(1) = 2\pi i.1 - 2\pi i(-1) = 4\pi i.$

Example 4.21. Evaluate $\displaystyle\int_C \frac{dz}{(z+1)^2(z-2)}$ where C is the circle $|z| = \frac{3}{2}$.

Solution. $z = -1$ and $z = 2$ are singular points.

C is $|z| = \frac{3}{2}$.

—

When $z = -1, |z| = |-1| = 1 < \frac{3}{2}$.

$\therefore\ z = -1$ is an interior point of C.

When $z = 2, |z| = 2 > \frac{3}{2}$.

$\therefore\ z = 2$ lies outside C.

$$\int_C \frac{dz}{(z+1)^2(z-2)} = \int_C \frac{f(z)}{(z+1)^2}dz \text{ where}$$

$$f(z) = \frac{1}{z-2},\ a = -1,\ n+1 = 2.$$

$$f'(z) = -\frac{1}{(z-2)^2},\quad n = 1.$$

$$f'(-1) = -\frac{1}{9}.$$

By Cauchy's integral formula

$$\int_C \frac{f(z)}{(z+1)^2}dz = 2\pi i f'(-1) = 2\pi i\left(\frac{-1}{9}\right) = -\frac{2\pi i}{9}.$$

Example 4.22. Evaluate $\int_C \frac{z+1}{z^2(z-2)}dz$ where C is the circle $|z-2-i| = 2$.

Solution. Let $I = \int_C \frac{z+1}{z^2(z-2)}dz$.

$z = 0$ and $z = 2$ are singular points.

C is the circle $|z-2-i| = 2$.

When $z = 0, |z-2-i| = |-2-i| = \sqrt{4+1} = \sqrt{5} > 2$.

$\Rightarrow\ z = 0$ lies outside the circle.

When $z = 2, |z-2-i| = |2-2-i| = |-i| = 1 < 2$.

$\Rightarrow\ z = 2$ lies inside the circle.

Now, $\int_C \frac{z+1}{z^2(z-2)}dz = \int_C \frac{\frac{z+1}{z^2}}{z-2}dz = \int_C \frac{f(z)}{z-2}dz$

where $f(z) = \frac{z+1}{z^2}$ and $a = 2$.

$f(a) = f(2) = \frac{3}{4}$.

By Cauchy's Integral formula $\int\limits_C \frac{f(z)}{z-2}dz = 2\pi i f(2) = 2\pi i \frac{3}{4} = \frac{3\pi i}{2}$.

Example 4.23. By Cauchy's integral formula, evaluate $\int\limits_C \frac{z+1}{z^4-4z^3+4z^2}dz$ where C is the circle $|z-2-i| = 2$.

Solution. $z^4 - 4z^3 + 4z^2 = z^2(z^2-4z+4) = z^2(z-2)^2$.

$z = 0$ and $z = 2$ are singular points.

C is the circle $|z-2-i| = 2$.

When $z = 0, |z-2-i| = |-2-i| = \sqrt{4+1} = \sqrt{5} > 2$.

$\therefore\ z = 0$ lies outside the circle.

When $z = 2, |z-2-i| = |2-2-i| = |-i| = 1 < 2$.

$\therefore\ z = 2$ is an interior point.

$$\text{Now, } \int\limits_C \frac{z+1}{z^4-4z^3+4z^2}dz = \int\limits_C \frac{z+1}{z^2(z-2)^2}dz = \int\limits_C \frac{\frac{z+1}{z^2}}{(z-2)^2}dz = \int\limits_C \frac{f(z)}{(z-2)^2}dz,$$

where, $f(z) = \frac{z+1}{z^2}$.

By Cauchy's integral formula $\int\limits_C \frac{f(z)}{(z-2)^2}dz = 2\pi i f'(2)$.

We have $f(z) = \frac{z+1}{z^2} = \frac{1}{z} + \frac{1}{z^2}$.

$$f'(z) = -\frac{1}{z^2} - \frac{2}{z^3}.$$

$$f'(2) = -\frac{1}{4} - \frac{2}{8} = \frac{-2-2}{8} = \frac{-1}{2}.$$

$$\therefore \int\limits_C \frac{f(z)}{(z-2)^2}dz = 2\pi i f'(2) = 2\pi i\left(\frac{-1}{2}\right) = -\pi i.$$

Example 4.24. Evaluate $\int\limits_C \frac{z+3}{2z+5}dz$ where C is $|z| = 3$. [May 2005]

Solution. Let $I = \int\limits_C \frac{z+3}{2z+5}dz = \frac{1}{2}\int\limits_C \frac{z+3}{z+\frac{5}{2}}dz$.

—

$z = \frac{-5}{2}$ is a singular point.

C is $|z| = 3$. When $z = -\frac{5}{2}$, $|z| = \left|\frac{-5}{2}\right| = \frac{5}{2} < 3$.

$z = \frac{-5}{2}$ is an interior point.

$I = \frac{1}{2}\int_C \frac{z+3}{z+\frac{5}{2}}dz = \frac{1}{2}\int_C \frac{f(z)}{z+\frac{5}{2}}dz$ where $f(z) = z+3$.

By Cauchy's integral formula

$I = \frac{1}{2}2\pi i f\left(\frac{-5}{2}\right) = i\pi\left(-\frac{5}{2}+3\right)$

$= \pi i \frac{1}{2} = \frac{\pi i}{2}$.

Example 4.25. Evaluate $\int_C \frac{dz}{z-2}$ where C is the circle whose centre is $(2, 0)$ and radius 4.

Solution. Let $I = \int_C \frac{dz}{z-2}$

$z = 2$ is a singular point.

The circle is $|z-2| = 4$.

If $z = 2$, then $|z-2| = |2-2| = 0 < 4$.

$\therefore z = 2$ lies inside the circle C.

Now $f(z) = 1, a = 2$.

$f(a) = f(2) = 1$.

By Cauchy's Integral formula

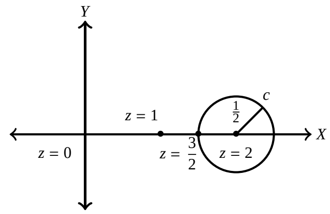
$$\int_C f(z)dz = 2\pi i f(a) = 2\pi i.$$

Example 4.26. Evaluate $\int_C \frac{z}{(z-1)(z-2)^2}dz$, where C is $|z-2| = \frac{1}{2}$ using Cauchy's integral formula [Jun 2012]

Solution. C is $|z-2| = \frac{1}{2}$.

—

C is a circle with $(2,0)$ as centre and radius $\frac{1}{2}$ units.
The singular points are $z = 1$ and $z = 2$.
$z = 1$ lies outside C and $z = 2$ lie within C.



$$\therefore \int_C \frac{z}{(z-1)(z-2)^2}dz = \int_C \frac{\frac{z}{z-1}}{(z-2)^2}dz$$

$$= \int_C \frac{f(z)}{(z-2)^2}dz \qquad \text{where } f(z) = \frac{z}{z-1}$$

$$f'(z) = \frac{z-1-z}{(z-1)^2} = -\frac{1}{(z-1)^2}$$

$$= 2\pi i \times f'(2) \qquad f'(2) = \frac{-1}{1} = -1.$$

$$= 2\pi i \times (-1) = -2\pi i.$$

Example 4.27. Evaluate $\int_C \frac{z+1}{(z^2+2z+4)^2}dz$ where C is $|z+1+i| = 2$ using Cauchy's integral formula. [Dec 2013, May 2011]

Solution. $z^2 + 2z + 4 = (z+1)^2 + 4 - 1$

$$= (z+1)^2 + 3$$

$$= (z + 1 + \sqrt{3}i)(z + 1 - \sqrt{3}i).$$

The singular points are $-1 - \sqrt{3}i$ and $-1 + \sqrt{3}i$.

When $z = -1 - \sqrt{3}i$, $|z + 1 + i| = |-1 - \sqrt{3}i + 1 + i|$

$$= |i(1 - \sqrt{3})|$$

$$= |\sqrt{3} - 1| < 2.$$

$\therefore z = -1 - \sqrt{3}i$ lies inside C.

—

$$\text{When } z = -1 + \sqrt{3}i,\ |z+1+i| = |-1+\sqrt{3}i+1+i|$$
$$= |i(\sqrt{3}+1)| = \sqrt{3}+1$$
$$> 2.$$
$$\therefore z = -1+\sqrt{3}i \text{ lies out side C.}$$

$$\therefore \frac{z+1}{(z^2+2z+4)^2} = \frac{z+1}{[(z+1\sqrt{3}i)(z+1)-\sqrt{3}i]^2}$$
$$= \frac{\frac{z+1}{(z+1-\sqrt{3}i)^2}}{(z+1+\sqrt{3}i)^2}$$
$$= \frac{f(z)}{(z+1+\sqrt{3}i)^2} \text{ where } f(z) = \frac{z+1}{(z+1-\sqrt{3}i)^2}.$$
$$\therefore \int \frac{z+1}{(z^2+2z+4)^2}dz = \int \frac{f(z)}{(z+1+\sqrt{3}i)^2}dz$$
$$= 2\pi i \times f'(-1-\sqrt{3}i) \text{ [By Cauchy's integral formula]}$$
$$\text{Now, } f(z) = \frac{z+1}{(z+1-\sqrt{3}i)^2}$$
$$f'(z) = \frac{(z+1-\sqrt{3}i)^2 - (z+1)\cdot 2(z+1-\sqrt{3}i)}{(z+1-\sqrt{3}i)^4}$$
$$f'(-1-\sqrt{3}i) = \frac{(-1-\sqrt{3}i-\sqrt{3}i)^2 - (-1-\sqrt{3}i+1)\cdot 2(-1-\sqrt{3}i+1-\sqrt{3}i)}{(-1-\sqrt{3}i+1-\sqrt{3}i)^4}$$
$$= \frac{(-2\sqrt{3}i)^2 - 2(-\sqrt{3}i)(-2\sqrt{3}i)}{(-2\sqrt{3}i)^4}$$
$$= \frac{4\times 3\times(-1) - 4\times 3(-1)}{16\times 9}$$
$$= \frac{-12+12}{144} = 0.$$
$$\therefore \int_C \frac{z+1}{(z^2+2z+4)^2}dz = 2\pi i \times 0 = 0.$$

Example 4.28. Using Cauchy's integral formula evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)}dz$, where C is the circle $|z| = \frac{3}{2}$. [Jun 2010]

—

Solution. The singular points are $z = 0, 1, 2$.

C is $|z| = \dfrac{3}{2}$.

$z = 0$ and $z = 1$ lie inside C and $z = 2$ lies outside C.

$$\therefore \frac{4-3z}{z(z-1)(z-2)} = \frac{\frac{4-3z}{z-2}}{z(z-1)} = \frac{f(z)}{z(z-1)} \quad \text{where} \quad f(z) = \frac{4-3z}{z-2}.$$

$$= f(z)\left[\frac{1}{z-1} - \frac{1}{z}\right] = \frac{f(z)}{z-1} - \frac{f(z)}{z}.$$

$$\therefore \int_C \frac{4-3z}{z(z-1)(z-2)}dz = \int_C \frac{f(z)}{z-1}dz - \int_C \frac{f(z)}{z}dz$$

$$= 2\pi i \times f(1) - 2\pi i f(0) \quad \text{[By Cauchy's integral formula]}.$$

Now, $f(z) = \dfrac{4-3z}{z-2}$.

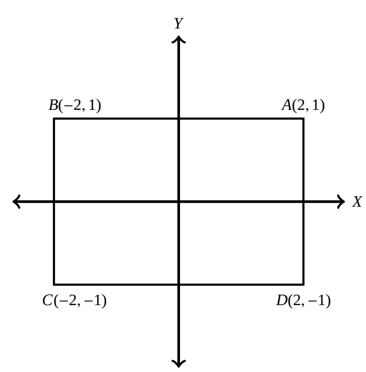
$$f(0) = \frac{4}{-2} = -2.$$

$$f(1) = \frac{1}{-1} = -1.$$

$$\therefore \int_C \frac{4-3z}{z(z-1)(z-2)}dz = 2\pi i(-1) - 2\pi i(-2) = -2\pi i + 4\pi i = 2\pi i.$$

Example 4.29. Evaluate using Cauchy's integral formula $\oint_C \dfrac{\cos \pi z}{z^2-1}dz$ around the rectangle with vertices $2 \pm i, -2 \pm i$.

Solution. $z^2 - 1 = (z+1)(z-1)$.



C is the rectangle with vertices $(2, 1), (2, -1), (-2, 1)$ and (-2,-1).

The singular points are $z = 1 \& -1$, both lie inside C.

$$\text{Now,} \quad \frac{\cos \pi z}{z^2-1} = \frac{\cos \pi z}{(z-1)(z+1)}$$

$$= \frac{1}{2}\cos \pi z\left[\frac{1}{z-1} - \frac{1}{z+1}\right]$$

$$= \frac{1}{2}\left[\frac{\cos \pi z}{z-1} - \frac{\cos \pi z}{z+1}\right].$$

$$\therefore \int_C \frac{\cos \pi z}{z^2-1} dz = \frac{1}{2}\left[\int_C \frac{\cos \pi z}{z-1} dz - \int_C \frac{\cos \pi z}{z+1} dz\right]$$

$$= \frac{1}{2}\left[\int_C \frac{f(z)}{z-1} dz - \int_C \frac{f(z)}{z+1} dz\right] \quad \text{where } f(z) = \cos \pi z.$$

$$= \frac{1}{2}[2\pi i \times f(1) - 2\pi i f(-1)] \quad \text{[by Cauchy's integral formula]}$$

$$= \frac{1}{2} 2\pi i [f(1) - f(-1)]$$

$$= \pi i [\cos \pi - \cos(-\pi)]$$

$$= \pi i [-1 + 1] = 0.$$

Example 4.30. Evaluate $\int_C \frac{\sin^2 z}{\left(z - \frac{\pi}{6}\right)^3} dz$ where C is the cicle $|z| = 1$.

Solution. The singular point is $z = \frac{\pi}{6}$, which lies inside C.

$$\therefore \int_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3} dz = \int_C \frac{f(z)}{\left(z-\frac{\pi}{6}\right)^3} dz \quad \text{where } f(z) = \sin^2 z.$$

$$= 2\pi i \times \frac{f''\left(\frac{\pi}{6}\right)}{2!} \quad \text{[by Cauchy's integral formula]}$$

$$= \pi i \times f''\left(\frac{\pi}{6}\right).$$

Now $f(z) = \sin^2 z.$

$$f'(z) = 2 \sin z \cos z = \sin 2z.$$

$$f''(z) = 2 \cos 2z.$$

$$f''\left(\frac{\pi}{6}\right) = 2 \cos\left(\frac{2\pi}{6}\right) = 2 \cos\left(\frac{\pi}{3}\right) = 2 \times \frac{1}{2} = 1.$$

$$\therefore \int_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3} dz = \pi i \times 1 = \pi i.$$

Example 4.31. Evaluate $\int_C \frac{e^z}{(z^2+\pi^2)^2} dz$ where C is the circle $|z| = 4$.

—

Solution. The singular points are given by

$$z^2 + \pi^2 = 0$$

$$z^2 = -\pi^2$$

$z = \pm \pi i$, both lie within the circle $|z| = 4$.

Now, $\dfrac{1}{(z^2+\pi^2)^2} = \dfrac{1}{[(z+\pi i)(z-\pi i)]^2} = \dfrac{1}{(z+\pi i)^2(z-\pi i)^2}$.

Let $\dfrac{1}{(z+\pi i)^2(z-\pi i)^2} = \dfrac{A}{z+\pi i} + \dfrac{B}{(z+\pi i)^2} + \dfrac{C}{z-\pi i} + \dfrac{D}{(z-\pi i)^2}$

$$= \frac{A(z+\pi i)(z-\pi i)^2 + B(z-\pi i)^2 + C(z-\pi i)(z+\pi i)^2 + D(z+\pi i)^2}{(z+\pi i)^2(z-\pi i)^2}$$

$$\therefore\ 1 = A(z+\pi i)(z-\pi i)^2 + B(z-\pi i)^2 + C(z-\pi i)(z+\pi i)^2 + D(z+\pi i)^2.$$

When $z = \pi i, D(2\pi i)^2 = 1$

$$D(-4\pi^2) = 1$$

$$D = -\frac{1}{4\pi^2}.$$

When $z = -\pi i, B(-2\pi i)^2 = 1$

$$4\pi^2(-1)B = 1$$

$$B = -\frac{1}{4\pi^2}.$$

Equating the coefficients of z^3, we get

$$A + C = 0. \tag{1}$$

When $z = 0$,

$$\pi i(-\pi i)^2 A + (-\pi i)^2 B + (-\pi i)(\pi i)^2 C + (\pi i)^2 D = 1$$

$$-\pi^3 iA - \pi^2 B + \pi^3 iC - \pi^2 D = 1$$

$$-\pi^3 iA - \pi^2\left(-\frac{1}{4\pi^2}\right) + \pi^3 iC - \pi^2\left(-\frac{1}{4\pi^2}\right) = 1$$

$$-\pi^3 iA + \frac{1}{4} + \pi^3 iC + \frac{1}{4} = 1$$

$$-\pi^3 iA + \pi^3 iC = \frac{1}{2} \tag{2}$$

From (1) $C = -A$.

$$\therefore -\pi^3 iA + \pi^3 i(-A) = \frac{1}{2}$$

$$-2\pi^3 iA = \frac{1}{2}$$

$$A = -\frac{1}{4\pi^3 i}.$$

$$C = \frac{1}{4\pi^3 i}.$$

$$\frac{1}{(z+\pi i)^2(z-\pi i)^2} = -\frac{1}{4\pi^3 i}\cdot\frac{1}{z+\pi i} - \frac{1}{4\pi^2}\frac{1}{(z+\pi i)^2} + \frac{1}{4\pi^3 i}\cdot\frac{1}{z-\pi i} - \frac{1}{4\pi^2}\frac{1}{(z-\pi i)^2}.$$

$$\therefore \int\limits_C \frac{e^z}{(z^2+\pi^2)}dz = -\frac{1}{4\pi^3 i}\int\limits_C \frac{e^z}{z+\pi i}dz - \frac{1}{4\pi^2}\int\limits_C \frac{e^z}{(z+\pi i)^2}dz + \frac{1}{4\pi^3 i}\int\limits_C \frac{e^z}{z-\pi i}dz - \frac{1}{4\pi^2}\int\limits_C \frac{e^z}{(z-\pi i)^2}dz.$$

$$= -\frac{1}{4\pi^3 i}\times 2\pi i f(-\pi i) - \frac{1}{4\pi^2}\times 2\pi i f'(-\pi i) + \frac{1}{4\pi^3 i}\times 2\pi i f(\pi i) - \frac{1}{4\pi^2}\times 2\pi i \times f'(\pi i), \text{ where } f(z) = e^z. \quad (1)$$

Now, $f(z) = e^z$.

$$f'(z) = e^z.$$

$$f(-\pi i) = e^{-\pi i}.$$

$$f(\pi i) = e^{\pi i}.$$

$$f'(\pi i) = e^{\pi i}.$$

$$f'(-\pi i) = e^{-\pi i}.$$

Now from (1) we have

$$\int\limits_C \frac{e^z}{(z^2+\pi^2)^2}dz = -\frac{1}{4\pi^3 i}\times 2\pi i e^{-\pi i} - \frac{1}{4\pi^2}\times 2\pi i e^{-\pi i} + \frac{1}{4\pi^3 i}\times 2\pi i e^{\pi i} - \frac{1}{4\pi^2}\times 2\pi i e^{\pi i}$$

$$= -\frac{1}{2\pi^2}e^{-\pi i} - \frac{i}{2\pi}e^{-\pi i} + \frac{1}{2\pi^2}e^{\pi i} - \frac{i}{2\pi}e^{\pi i}$$

$$= \frac{1}{2\pi^2}\left(e^{\pi i} - e^{-\pi i}\right) - \frac{i}{2\pi}\left(e^{\pi i} + e^{-\pi i}\right)$$

—

$$= \frac{1}{2\pi^2} 2i \sin \pi - \frac{i}{2\pi} \cdot 2 \cos \pi$$
$$= -\frac{i}{\pi}(-1)$$
$$= \frac{i}{\pi}.$$

Example 4.32. Evaluate $\oint_C \frac{e^{tz}}{z^2+1}$ where C is the circle $|z| = 3$.

Solution. The singular points are given by

$$z^2 + 1 = 0$$
$$z^2 = -1$$

$z = \pm i$, both lie inside $|z| = 3$.

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$
$$\frac{e^{tz}}{z^2+1} = \frac{1}{2i}\left[\frac{e^{tz}}{z-i} - \frac{e^{tz}}{z+i}\right]$$
$$\therefore \int_C \frac{e^{tz}}{z^2+1} dz = \frac{1}{2i}\left[\int_C \frac{e^{tz}}{z-i} dz - \int_C \frac{e^{tz}}{z+i} dz\right]$$
$$= \frac{1}{2i}[2\pi i \times f(i) - 2\pi i \times f(-i)] \quad \text{where } f(z) = e^{tz}.$$
$$= \frac{1}{2i} \times 2\pi i[f(i) - f(-i)]$$
$$= \pi[f(i) - f(-i)]. \tag{1}$$

Now $f(z) = e^{tz}$.

$$f(i) = e^{it}$$
$$f(-i) = e^{-it}.$$
$$f(i) - f(-i) = e^{it} - e^{-it}$$
$$= 2i \sin t.$$

Now, (1) becomes

$$\int_C \frac{e^{tz}}{z^2+1} dz = \pi[2i \sin t] = 2\pi i \sin t.$$

—

Example 4.33. Evaluate $\oint_C \frac{e^{2z}}{(z+1)^4}dz$, where C is the circle $|z| = 3$. [Jan 2016]

Solution. The singular point is $z = -1$, which lies completely inside the circle C, $|z| = 3$.

By Cauchy's integral formula

$$\int_C \frac{e^{2z}}{(z+1)^4}dz = \int_C \frac{f(z)}{(z+1)^4}dz \quad \text{where } f(z) = e^{2z}.$$

$$= 2\pi i \times \frac{f'''(-1)}{3!} \tag{1}$$

Now, $f(z) = e^{2z}$.

$$f'(z) = 2e^{2z}.$$

$$f''(z) = 4e^{2z}.$$

$$f'''(z) = 8e^{2z}.$$

$$f'''(-1) = 8e^{-2}$$

Now, (1) becomes

$$\int_C \frac{e^{2z}}{(z+1)^4}dz = 2\pi i \times \frac{8e^{-2}}{6} = \frac{8}{3}\pi i e^{-2}.$$

Exercise 4 B

1. Evaluate $\int_C \frac{dz}{z+4}$ where C is the circle $|z| = 2$. [Dec 2004]

2. Evaluate $\int_C \frac{\cos \pi z}{z-1}dz$ where C is $|z| = 2$. [May 2003]

3. Evaluate $\int_C \frac{3z^2+7z+1}{z+1}dz$ where C is $|z| = \frac{1}{2}$. [Dec 2003]

4. Use Cauchy's integral formula to evaluate $\int_C \frac{1}{z^2-1}dz$ where C is the circle with centre 1 and radius = 1. [May 2000]

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5. Evaluate $\int_C \frac{z}{(z-1)^3}dz$ where C is the circle $|z| = 2$. [May 1999]

6. Evaluate $\int_C \frac{e^{-z}}{z+1}dz$ where C is the circle $|z| = 2$.

7. Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-1)}dz$ where C is $|z| = 4$. [May 2000]

8. Use Cauchy's integral formula, evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)}dz$ where C is $|z| = \frac{\pi}{2}$. [Dec 2011]

9. Evaluate $\int_C \frac{1}{(z^2+4)^2}dz$ where C is $|z - i| = 4$. [May 2003]

10. Evaluate $\int_C \frac{z}{(z-1)(z-2)}dz$ where C is $|z-2| = \frac{1}{2}$. [May 2004]

11. Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}dz$ where C is $|z| = 3$. [Dec 2003]

12. Evaluate $\int_C \frac{e^z}{(z+2)(z+1)^2}dz$ where C is $|z| = 3$. [May 2007]

13. Evaluate $\int_C \frac{z}{(z-1)(z-2)^2}dz$ where C is $|z-2| = \frac{1}{2}$. [Dec 2009, Jun 2012]

14. Evaluate $\int_C \frac{1}{2z-3}dz$ where C is the circle $|z| = 1$. [May 2009]

15. Evaluate $\int_C \frac{1}{ze^z}dz$ where C is the circle $|z| = 1$. [Dec 2009]

16. Evaluate using Cauchy's integral formula $\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}dz$ on the circle (i) $|z| = 4$ and (ii) $|z| = 1$. [May 2010]

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17. Evaluate using Cauchy's integral formula $\int_C \frac{\sin 3z}{z+\frac{\pi}{2}}dz$ if C is the circle $|z| = 5$. [Apr 2007]

18. Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-2)}dz$ where C is the circle $|z| = 3$. [Nov 2011]

19. Evaluate $\int_C \frac{3z^2+z}{z^2-1}dz$ where C is the circle $|z-1| = 1$. [Apr 2008]

20. Evaluate $\int_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3}dz$ if C is the circle $|z| = 1$. [Nov 2010]

21. Evaluate $\int_C \frac{z+1}{z^3-2z^2}dz$ where C is the unit circle $|z| = 1$. [May 2010]

22. Evaluate $\int_C \frac{\tan\frac{z}{2}}{(z-a)^2}dz, -2 < a < 2$ where C is the boundary of the square whose sides are $x = \pm 2, y = \pm 2$. [Apr 2010]

23. Evaluate using Cauchy's integral formula $\int_C \frac{z^2+7}{z-2}dz$ on the circle (i) $|z| = 3$ and (ii) $|z| = 1$. [May 2003]

24. Evaluate $\int_C \frac{1}{z^2+4z+1}dz$ where C is $|z| = 1$. [Apr 2008]

4.5 Taylor's and Laurent's series

4.5.1 Power series

A series of the form

$$\sum_{n=0}^{\infty} a_n(z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$

where z is a complex variable, $a_0, a_1, a_2, \ldots$ and a are complex constants is called a power series in powers of $z-a$ or a power series about the point a.

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$a_0, a_1, a_2, \ldots$ are called the coefficients of the series and a is called the centre of the series.

Result 1. The power series converges at all points inside a circle $|z-a| = R$ for some positive number R and diverges outside the circle. This circle is called the circle of convergence and its radius R is called the radius of convergence.

Result 2. Let $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \ell$. Then, the radius of convergence of the series $\sum_{n=0}^{\infty} a_n(z-a)^n$ is $R = \frac{1}{\ell}$.

Result 3. If $\ell = 0$, then $R = \infty$ and so, the power series converges for all z in the finite plane.

Result 4. If $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$, then $R = 0$ and hence the series converges only at the centre $z = a$.

Result 5. If $a = 0$, then we get a particular power series in powers of z or power series about the origin.

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

Taylor's Series

If $f(z)$ is analytic inside a circle C with centre at a then at each point z inside the circle, $f(z)$ has the power series representation

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^n(a)}{n!}(z-a)^n + \ldots\infty.$$

Note. If $a = 0$, we get the Taylor's series of $f(z)$ about the origin which is called the Maclaurin's series.

$$\text{i.e., } f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \cdots + \frac{f^{(n)}}{n!}z^n + \cdots.$$

Maclaurin's series for some elementary functions

1. $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$ if $|z| < \infty$.

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2. $(1-z)^{-1} = 1 + z + z^2 + z^3 + + \cdots$ if $|z| < 1$.

3. $(1+z)^{-1} = 1 - z + z^2 - z^3 + + \cdots$ if $|z| < 1$.

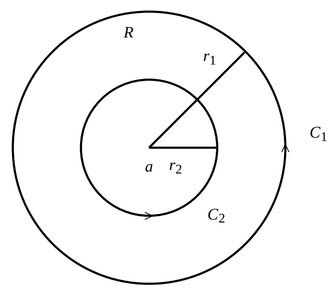
4. $(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + + \cdots$ if $|z| < 1$.

5. $\log(1+z) = z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \dfrac{z^4}{4} + + \cdots$ if $|z| < 1$.

6. $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \cdots$ if $|z| < \infty$.

7. $\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \cdots$ if $|z| < \infty$.

Laurent's series. If $f(z)$ is analytic in the ring shaped region R bounded by two concentric circles C_1 and C_2 of radii r_1 and r_2 $(r_1 > r_2)$ and with centre at a, then for all z in R, $f(z)$ has the power series representation.



$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}, r_2 < |z-a| < r_1 \qquad (1)$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\oint_{C_1} \dfrac{f(w)}{(w-a)^{n+1}} dw, n = 1, 2, 3, \ldots.$

$b_n = \dfrac{1}{2\pi i} \displaystyle\oint_{C_2} \dfrac{f(w)}{(w-a)^{1-n}} dw, n = 1, 2, 3, \ldots.$

The series (1) is called Laurent's series about $z = a$. Laurent's series is a series with positive and negative integral powers of $(z-a)$. The part of the series with positive powers of $z-a$ is called the analytic part or regular part and the part with negative powers of $z-a$ is called the principal part of the Laurent's series.

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Worked Examples

Example 4.34. Expand $f(z) = \sin z$ in a Taylor's series about the origin.

Solution.

$$\begin{aligned} f(z) &= \sin z. & f(0) &= \sin 0 = 0. \\ f'(z) &= \cos z. & f'(0) &= 1. \\ f''(z) &= -\sin z. & f''(0) &= 0. \\ f'''(z) &= -\cos z. & f'''(0) &= -1. \\ f^{iv}(z) &= \sin z. & f^{iv}(0) &= 0. \\ \ldots & & \ldots \end{aligned}$$

By Taylor's series we have

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \cdots$$

$$\sin z = 0 + \frac{1}{1!}z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

Example 4.35. Find the Taylor's series for $f(z) = \sin z$ about $z = \frac{\pi}{4}$. [Jun 2009]

Solution.

$$\begin{aligned} f(z) &= \sin z, & f\left(\frac{\pi}{4}\right) &= \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \\ f'(z) &= \cos z, & f'\left(\frac{\pi}{4}\right) &= \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \\ f''(z) &= -\sin z, & f''\left(\frac{\pi}{4}\right) &= -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}} \\ f'''(z) &= -\cos z, & f'''\left(\frac{\pi}{4}\right) &= -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}} \\ f^{iv}(z) &= -\sin z, & f^{iv}\left(\frac{\pi}{4}\right) &= \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}. \\ \ldots & & \ldots \end{aligned}$$

By Taylor's series we have

$$f(z) = f\left(\frac{\pi}{4}\right) + \frac{f'\left(\frac{\pi}{4}\right)}{1!}\left(z - \frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(z - \frac{\pi}{4}\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(z - \frac{\pi}{4}\right)^3 + \cdots$$

$$\sin z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(z - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(z - \frac{\pi}{4}\right)^2 - \frac{1}{6\sqrt{2}}\left(z - \frac{\pi}{4}\right)^3 + \cdots$$

—

Example 4.36. Obtain the expansion of $\log(1+z)$ when $|z| < 1$. [May 2006]

Solution. Since $|z| < 1, z = 0$ is a regular point of $f(z)$.

$$f(z) = \log(1+z), \qquad f(0) = \log 1 = 0$$
$$f'(z) = \frac{1}{1+z}, \qquad f'(0) = 1$$
$$f''(z) = -\frac{1}{(1+z)^2}, \qquad f''(0) = -1$$
$$f'''(z) = \frac{2}{(1+z)^3}, \qquad f'''(0) = 2$$
$$f^{iv}(z) = -\frac{6}{(1+z)^4}, \qquad f^{iv}(0) = -6.$$
$$\cdots \qquad \cdots$$

The Taylor's series expansion about $z = 0$ is

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \cdots$$

$$\log(1+z) = 0 + \frac{1}{1}z - \frac{1}{2!}z^2 + \frac{2}{3!}z^3 - \frac{6}{4!}z^4 + \cdots$$

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots.$$

Example 4.37. Expand $\frac{1}{z-2}$ at $z = 1$ in Taylor's series.

Solution. Let $f(z) = \frac{1}{z-2}$.

$z = 1$ is a regular point.

$\therefore$ We can find Taylor's series about $z = 1$.

$\therefore$ The series is

$$f(z) = f(1) + \frac{f'(1)}{1!}(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \frac{f'''(1)}{3!}(z-1)^3 + \cdots$$

$$\text{Now, } f(z) = \frac{1}{z-2} = (z-2)^{-1} \qquad f(1) = -1$$
$$f'(z) = -1(z-2)^{-2} \qquad f'(1) = -1$$
$$f''(z) = 2(z-2)^{-3} \qquad f''(1) = -2$$
$$f'''(z) = -6(z-2)^{-4} \qquad f'''(1) = -6$$

$$f(z) = -1 - \frac{1}{1!}(z-1) - \frac{2}{2!}(z-1)^2 - \frac{6}{3!}(z-1)^3 - \dots$$
$$\text{t}= -[1 + z - 1 + (z-1)^2 + (z-1)^3 + \dots].$$

Example 4.38. Obtain the Taylor's series to represent the function $\frac{1}{(z+2)(z+3)}$ in the region $|z| < 2$. [Apr 2006]

Solution. Let $f(z) = \frac{1}{(z+2)(z+3)}$

$$= \frac{z+3-(z+2)}{(z+2)(z+3)}$$
$$= \frac{z+3}{(z+2)(z+3)} - \frac{z+2}{(z+2)(z+3)}$$
$$f(z) = \frac{1}{z+2} - \frac{1}{z+3}.$$

Since $|z| < 2$ we have $\left|\frac{z}{2}\right| < 1$ and $\left|\frac{z}{3}\right| < \frac{2}{3} < 1$.

Hence $\frac{1}{z+2}$ and $\frac{1}{z+3}$ can be expanded as powers of $\frac{z}{2}$ and $\frac{z}{3}$.

$$\therefore f(z) = \frac{1}{2\left(1+\frac{z}{2}\right)} - \frac{1}{3\left(1+\frac{z}{3}\right)}$$
$$= \frac{1}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}$$
$$= \frac{1}{2}\left[1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right] - \frac{1}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right]$$
$$= \frac{1}{2} - \frac{z}{4} + \frac{z^2}{8} - \frac{z^3}{16} + \cdots - \frac{1}{3} + \frac{z}{9} - \frac{z^2}{27} + \frac{z^3}{81} + \cdots$$
$$f(z) = \frac{1}{6} - \frac{5}{36}z + \frac{9}{216}z^2 - \frac{65}{1296}z^3 + \cdots.$$

Since the expansion contains only positive powers of z, this is the required Taylor's series of $f(z)$.

Example 4.39. Find the Taylor's series to represent the function $\frac{z^2-1}{(z+2)(z+3)}$ in $|z| < 2$. [Jun 2010, Dec 2015]

Solution. Let $f(z) = \frac{z^2-1}{(z+2)(z+3)}$.

The singular points are $z = -2$ and $z = -3$.

Both the singular points lie outside $|z| < 2$.

$\therefore$ $f(z)$ is analytic in the open disc $|z| < 2$.

Hence $f(z)$ can be expanded as a power series about $z = 0$.

Splitting $f(z)$ into partial fractions we get

$$\frac{z^2 - 1}{(z+2)(z+3)} = A + \frac{B}{z+2} + \frac{C}{z+3}$$

$$z^2 - 1 = A(z+2)(z+3) + B(z+3) + C(z+2).$$

Put $z = -2$, we obtain $B = 3$.

Put $z = -3$, we obtain $C = -8$

Equating the coeff. of z^2 we get $A = 1$.

$$\begin{aligned}
\therefore \frac{z^2 - 1}{(z+2)(z+3)} &= 1 + \frac{3}{z+2} - \frac{8}{z+3} \\
f(z) &= 1 + \frac{3}{2\left(1 + \frac{z}{2}\right)} - 8\frac{1}{3\left(1 + \frac{z}{3}\right)} \\
&= 1 + \frac{3}{2}\left(1 + \frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1} \\
&= 1 + \frac{3}{2}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} \cdots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} \cdots\right) \\
&= -\frac{1}{6} + \frac{5}{36}z + \frac{17}{216}z^2 - \frac{115}{1296}z^3 + \cdots .
\end{aligned}$$

Example 4.40. Expand $f(z) = \dfrac{z^2 - 1}{(z+2)(z+3)}$ as a Laurent's series if (i) $2 < |z| < 3$ (ii) $|z| > 3$. [May 2015, Jun 2013, Dec 2011, May 2011, May 2009]

Solution. By the previous problem we have $\dfrac{z^2 - 1}{(z+2)(z+3)} = 1 + \dfrac{3}{z+2} - \dfrac{8}{z+3}$.

Given $2 < |z| < 3$. This region is annular about $z = 0$ and $f(z)$ is analytic in this region.

$\therefore$ $f(z)$ has a Laurent's series about $z = 0$.

$|z| < 3 \Rightarrow \left|\dfrac{z}{3}\right| < 1.$

$$|z| > 2 \Rightarrow \left|\frac{z}{2}\right| > 1 \Rightarrow \left|\frac{2}{z}\right| < 1$$

$$\text{Now, } f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \cdots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^3}{9} - \frac{z^3}{27} + \cdots\right)$$

$$= 1 + 3(z^{-1} - 2z^{-2} + 4z^{-3} - 8z^{-4} + \cdots) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right).$$

(ii) $|z| > 3$. This region is annular about $z = 0$ where $f(z)$ is analytic.

$$|z| > 3 \Rightarrow \left|\frac{z}{3}\right| > 1 \Rightarrow \left|\frac{3}{z}\right| < 1 \Rightarrow \left|\frac{1}{z}\right| < \frac{1}{3}$$

$$\frac{2}{|z|} < \frac{2}{3} < 1$$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \cdots\right) - \frac{8}{2}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right)$$

$$= 1 + \frac{3}{2} - \frac{6}{z^2} + \frac{12}{z^3} - \frac{24}{z^4} + \cdots - \frac{8}{z} + \frac{24}{z^2} - \frac{72}{z^3} + \cdots$$

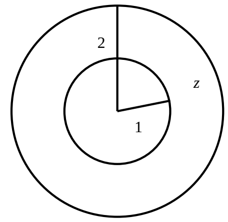
$$= 1 - 5z^{-1} + 16z^{-2} - 60z^{-3} \cdots.$$

Example 4.41. Expand $\dfrac{1}{z^2 - 3z + 2}$ in the region (i) $1 < |z| < 2$ (ii) $0 < |z - 1| < 2$.

[May 2002]

Solution. $\dfrac{1}{z^2 - 3z + 2} = \dfrac{1}{(z-1)(z-2)} = \dfrac{z - 1 - (z-2)}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}.$

Since $1 < |z| < 2$, then z lies in the annular region about $z = 0$ where $f(z)$ is analytic.



Hence, we can expand $f(z)$ as a Laurent's series about $z = 0$.

Since $|z| < 2$, $\left|\frac{z}{2}\right| < 1$ and $1 < |z| \Rightarrow |z| > 1 \Rightarrow \frac{1}{|z|} < 1$.

Hence, $f(z)$ can be written as,

$$\begin{aligned}
f(z) &= \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} \\
&= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} \\
&= \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) \\
&= -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right) - \left(z^{-1}+z^{-2}+z^{-3}+z^{-4}+\cdots\right).
\end{aligned}$$

(ii) We have $f(z) = \frac{1}{z-2} - \frac{1}{z-1}$.

Given $0 < |z-1| < 2$.

This region is annular about $z = 1$.

$\therefore$ The expansion of $f(z)$ in this region is a Laurent's series about $z = 1$.

Put $t = z - 1 \Rightarrow z = t + 1$.

$\therefore$ The region is $o < |t| < 2$.

$$\begin{aligned}
f(z) &= \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{t+1-2} - \frac{1}{t+1-1} \\
&= \frac{1}{t-1} - \frac{1}{t} = -t^{-1} - \frac{1}{1-t} \\
&= -t^{-1} - (1-t)^{-1} \qquad [\text{ Since } |t| < 2 \Rightarrow |t| < 1] \\
&= -(z-1)^{-1} - [1+t+t^2+t^3+\cdots] \\
&= -\left((z-1)^{-1} - (1+z-1+(z-1)^2+(z-1)^3)+\cdots\right).
\end{aligned}$$

—

Example 4.42. Expand $\dfrac{1}{z^2-3z+2}$ in the region $|z|>2$.

Solution. By the previous problem we have $f(z)=\dfrac{1}{z-2}-\dfrac{1}{z-1}$ Given $|z|>2$.

This region is annular about $z=0$ where $f(z)$ is analytic.

$\therefore$ The expansion is a Laurent's series about $z=0$.

Since $|z|>2, \left|\dfrac{z}{2}\right|>1 \Rightarrow \left|\dfrac{2}{z}\right|<1$.

Also $\dfrac{1}{|z|}<\dfrac{1}{2}<1$.

$$\begin{aligned}
\text{Hence, } f(z) &= -\frac{1}{z-1}+\frac{1}{z-2}\\
&= -\frac{1}{z\left(1-\frac{1}{z}\right)}+\frac{1}{z\left(1-\frac{2}{z}\right)}\\
&= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}+\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}\\
&= -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)+\frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right)\\
&= -\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}-\cdots+\frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\frac{8}{z^4}\cdots\\
&= z^{-2}+3z^{-3}+7z^{-4}+\ldots.
\end{aligned}$$

Example 4.43. Expand $f(z)=\dfrac{1}{(z+1)(z+3)}$ in Laurent's series valid for the regions $1<|z|<3$ and $|z|>3$. [Jun 2012]

Solution. Let $f(z)=\dfrac{1}{(z+1)(z+3)}=\dfrac{A}{z+1}+\dfrac{B}{z+3}$.

$1=A(z+3)+B(z+1)$.

When $z=-1, 2A=1, A=\dfrac{1}{2}$.

When $z=-3, -2B=1, B=-\dfrac{1}{2}$.

$\therefore \dfrac{1}{(z+1)(z+3)}=\dfrac{1}{2}\cdot\dfrac{1}{z+1}-\dfrac{1}{2}\cdot\dfrac{1}{z+3}$.

(i) Consider $1<|z|<3$.

This is annular about $z=0$ and $f(z)$ is analytic in the region.

—

$\therefore$ $f(z)$ has a Laurent's series expansion about $z = 0$.

$$|z| < 3 \Rightarrow |\frac{z}{3}| < 1.$$

$$|z| > 1 \Rightarrow \frac{1}{|z|} < 1.$$

$$\therefore f(z) = \frac{1}{2} \cdot \frac{1}{z\left(1 + \frac{1}{z}\right)} - \frac{1}{2} \cdot \frac{1}{3\left(1 + \frac{z}{3}\right)}$$

$$= \frac{1}{2z}\left(1 + \frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1 + \frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} \cdots\right] - \frac{1}{6}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \frac{z^4}{81} \cdots\right]$$

(ii) Consider the region $|z| > 3$.

This region is annular about $z = 0$, where $f(z)$ is analytic.

$|z| > 3 \Rightarrow \frac{1}{|z|} < \frac{1}{3} \Rightarrow \frac{3}{|z|} < 1$.

Since $\frac{1}{|z|} < \frac{1}{3} < 1$, we have

$$f(z) = \frac{1}{2} \cdot \frac{1}{z\left(1 + \frac{1}{z}\right)} - \frac{1}{2} \cdot \frac{1}{z\left(1 + \frac{3}{z}\right)}$$

$$= \frac{1}{2z}\left(1 + \frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1 + \frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} \cdots\right] - \frac{1}{2z}\left[1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \frac{81}{z^4} \cdots\right].$$

Example 4.44. Obtain the series for $\frac{1}{z-3}$ valid in (i) $|z| < 3$ (ii) $|z| > 3$.

Solution. Let $f(z) = \frac{1}{z-3}$.

$z = 3$ is a singular point.

(i) Given $|z| < 3$

$f(z)$ is analytic in the region $|z| < 3$.

$\therefore$ $f(z)$ has a Taylor's series about $z = 0$.

Since $|z| < 3$, we have $\left|\frac{z}{3}\right| < 1$.

—

Hence, the Taylor's series about $z = 0$ is

$$f(z) = \frac{1}{z-3} = \frac{1}{-3\left(1-\frac{z}{3}\right)}$$
$$= \frac{-1}{3}\left(1-\frac{z}{3}\right)^{-1}$$
$$= -\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}\cdots\right).$$

(ii) Given $|z| > 3$.

$f(z)$ is analytic in the annular region $3 < |z| < \infty$

$\therefore$ $f(z)$ has a Laurent's series expansion.

As $|z| > 3$, $\frac{1}{|z|} < \frac{1}{3} < 1 \Longrightarrow \frac{3}{|z|} < 1$.

$$f(z) = \frac{1}{z-3} = \frac{1}{z\left(1-\frac{3}{z}\right)}$$
$$= \frac{1}{z}\left(1-\frac{3}{z}\right)^{-1} = \frac{1}{z}\left(1+\frac{3}{z}+\frac{9}{z^2}+\frac{27}{z^3}+\cdots\right)$$
$$= z^{-1} + 3z^{-2} + 9z^{-3} + 27z^{-4} + \cdots .$$

Example 4.45. Find the Laurent's series expansion of $f(z) = \frac{7z-2}{z(z-2)(z+1)}$ in $1 < |z+1| < 3$. [Jun 2010, May 2005]

Solution. The singular points are $z = 0, z = -1, z = 2$, which lies outside the annular region $1 < |z+1| < 3$.

$\therefore$ We can expand $f(z)$ as a Laurent's series about $z = -1$ or interms of $z+1$.

Let $f(z) = \frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$.

$$7z - 2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

When $z = 0$ we obtain $-2 = -2A \Rightarrow A = 1$.

When $z = -1$ we have $-9 = 3C \Rightarrow C = -3$.

When $z = 2$ we obtain $6B = 12 \Rightarrow B = 2$.

Let $t = z + 1$.

—

$\therefore\ 1 < |z+1| < 3 \quad \Rightarrow \quad \therefore\ 1 < |t| < 3,$

$|t| > 1 \Rightarrow \left|\frac{1}{t}\right| < 1.$

$|t| < 3 \Rightarrow \left|\frac{t}{3}\right| < 1.$

$$
\begin{aligned}
f(z) &= \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1} = \frac{1}{t-1} + \frac{2}{t-3} - \frac{3}{t} \\
&= \frac{1}{t\left(1-\frac{1}{t}\right)} + \frac{2}{-3\left(1-\frac{t}{3}\right)} - \frac{3}{t} = \frac{1}{t}\left(1-\frac{1}{t}\right)^{-1} - \frac{2}{3}\left(1-\frac{t}{3}\right)^{-1} - \frac{3}{t} \\
&= \frac{1}{t}\left(1 + \frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \cdots\right) - \frac{2}{3}\left(1 + \frac{t}{3} + \frac{t^2}{9} + \frac{t^3}{27} + \cdots\right) - \frac{3}{t} \\
&= -\frac{2}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \frac{1}{t^4} + \cdots - \frac{2}{3}\left(1 + \frac{t}{3} + \frac{t^2}{9} + \frac{t^3}{27} + \cdots\right) \\
&= -2(z+1)^{-1} + (z+1)^{-2} + (z+1)^{-3} + \cdots \\
&\qquad - \frac{2}{3}\left(1 + \frac{z+1}{3} + \frac{(z+1)^2}{9} + \frac{(z+1)^3}{27} + \cdots\right).
\end{aligned}
$$

Example 4.46. Expand $f(z) = \dfrac{z}{(z-1)(z-3)}$ as Laurent's series valid in the regions (i) $1 < |z| < 3$ (ii) $0 < |z-1| < 2$ (iii) $|z| > 3$. [Dec 2001, Apr 1998]

Solution. The singular points are $z = 1$ and $z = 3$.

Let $\dfrac{z}{(z-1)(z-3)} = \dfrac{A}{z-1} + \dfrac{B}{z-3}$

$z = A(z-3) + B(z-1).$

Put $z = 1 \Longrightarrow 1 = -2A \Rightarrow A = \dfrac{-1}{2}.$

Put $z = 3 \Longrightarrow 3 = 2B \Rightarrow B = \dfrac{3}{2}.$

$$\therefore\ f(z) = \frac{-\frac{1}{2}}{z-1} + \frac{\frac{3}{2}}{z-3}.$$

(i) Consider $1 < |z| < 3$

This region is annular about $z = 0$ and $f(z)$ is analytic in $1 < |z| < 3$

$|z| > 1 \Longrightarrow \left|\frac{1}{z}\right| < 1.$ Also $|z| < 3 \Longrightarrow \left|\frac{z}{3}\right| < 1$

Now, $f(z) = -\dfrac{1}{2}\dfrac{1}{z\left(1-\frac{1}{z}\right)} + \dfrac{3}{2(-3)}\dfrac{1}{\left(1-\frac{z}{3}\right)}$

$$= -\frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1-\frac{z}{3}\right)^{-1}$$
$$= -\frac{1}{2z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \frac{1}{2}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\cdots\right)$$
$$= -\frac{1}{2}\left(z^{-1}+z^{-2}+z^{-3}+z^{-4}+\cdots\right) - \frac{1}{2}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\cdots\right).$$

(ii) Consider $0 < |z-1| < 2$.

This region is annular about $z = 1$ which does not contain $z = 1$ and $z = 3$.

$\therefore\ f(z)$ is analytic in $0 < |z-1| < 2$.

Let $t = z - 1$. Now, $0 < |z-1| < 2 \Longrightarrow 0 < |t| < 2 \Rightarrow \left|\frac{t}{2}\right| < 1$

$$\therefore\ f(z) = \frac{-1}{2}\frac{1}{t} + \frac{3}{2}\frac{1}{t+1-3}$$
$$= \frac{-1}{2}\frac{1}{t} + \frac{3}{2}\frac{1}{t-2}$$
$$= -\frac{1}{2}\frac{1}{t} - \frac{3}{4}\frac{1}{\left(1-\frac{t}{2}\right)}$$
$$= -\frac{1}{2t} - \frac{3}{4}\left(1-\frac{t}{2}\right)^{-1}$$
$$= -\frac{t^{-1}}{2} - \frac{3}{4}\left(1+\frac{t}{2}+\frac{t^2}{4}+\frac{t^3}{8}+\cdots\right)$$
$$= -\frac{(z-1)^{-1}}{2} - \frac{3}{4}\left(1+\frac{z-1}{2}+\frac{(z-1)^2}{4}+\frac{(z-1)^3}{8}+\cdots\right).$$

(iii) Consider $|z| > 3 \Rightarrow \left|\frac{z}{3}\right| > 1$

$\left|\frac{1}{z}\right| < \frac{1}{3} < 1 \Rightarrow \left|\frac{3}{z}\right| < 1.$

This region is annular about $z = 0$.

$f(z)$ is analytic in $|z| > 3$.

$$f(z) = -\frac{1}{2}\frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{3}{2}\frac{1}{z\left(1-\frac{3}{z}\right)}$$
$$= -\frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1} + \frac{3}{2z}\left(1-\frac{3}{z}\right)^{-1}$$
$$= -\frac{1}{2z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) + \frac{3}{2z}\left(1+\frac{3}{z}+\frac{9}{z^2}+\frac{27}{z^3}+\cdots\right)$$

—

$$= -\frac{1}{2z} - \frac{1}{2z^2} - \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots + \frac{3}{2z} + \frac{9}{2z^2} + \frac{27}{2z^3} + \frac{81}{2z^4} + \cdots$$
$$= \frac{1}{z} + \frac{4}{z^2} + \frac{13}{z^3} + \frac{40}{z^4} + \cdots.$$

Example 4.47. Find the Laurent's series of $f(z) = \dfrac{1}{z(1-z)}$ valid in the region (i) $|z+1| < 1$ (ii) $|z+1| > 2$. [Dec 2011, Jun 2008]

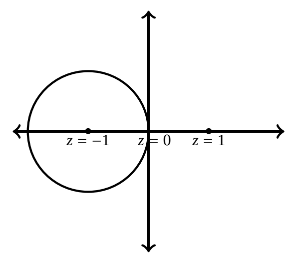
Solution. $f(z) = \dfrac{1}{z(1-z)} = \dfrac{1-z+z}{z(1-z)} = \dfrac{1-z}{z(1-z)} + \dfrac{z}{z(1-z)} = \dfrac{1}{z} + \dfrac{1}{1-z}$.

The singular points are $z = 0, z = 1$.

(i) Consider $|z+1| < 1$

The singular points lie outside the open disc with centre $z = -1$.

$\therefore$ $f(z)$ is analytic in $|z+1| < 1$.



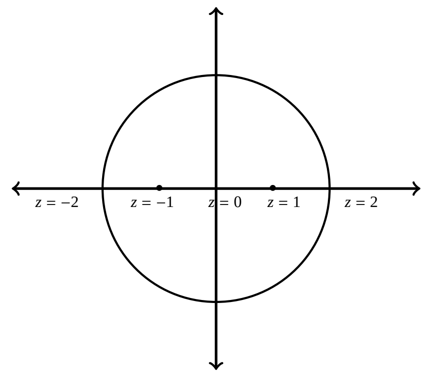
$\therefore$ We can expand $f(z)$ as a Laurent's series about $z = -1$.

Let $t = z + 1$.

$\therefore$ The region becomes $|t| < 1 \implies \left|\frac{t}{2}\right| < \frac{1}{2} < 1$.

$$f(z) = \frac{1}{t-1} + \frac{1}{2-t}$$
$$= -\frac{1}{1-t} + \frac{1}{2\left(1-\frac{t}{2}\right)} = -(1-t)^{-1} + \frac{1}{2}\left(1-\frac{t}{2}\right)^{-1}$$
$$= -(1 + t + t^2 + t^3 + \cdots) + \frac{1}{2}\left(1 + \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{8} + \cdots\right)$$
$$= -1 - (z+1) - (z+1)^2 - (z+1)^3 + \frac{1}{2} + \frac{z+1}{4} + \frac{(z+1)^2}{8} + \frac{(z+1)^3}{16} + \ldots$$
$$= -\frac{1}{2} - \frac{3}{4}(z+1) - \frac{7}{8}(z+1)^2 - \frac{15}{16}(z+1)^3 + \cdots$$
$$= -\frac{1}{2}\left(1 + \frac{3}{2}(z+1) + \frac{7}{4}(z+1)^2 + \frac{15}{8}(z+1)^3 \cdots\right).$$

—



(ii) Consider the region $|z+1| > 2$.

If $|z+1| > 2$, $f(z)$ is analytic in the annular region about $z=-1$.

$\therefore$ $f(z)$ can be expanded as a Laurent's series about $z=-1$.

Let $t = z+1$.

$\therefore$ The region is $|t| > 2$.

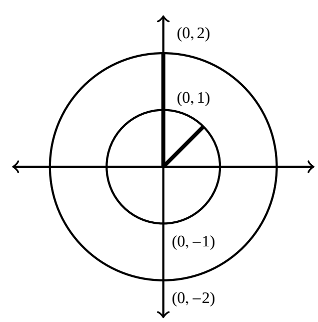
$\Rightarrow \left|\frac{t}{2}\right| > 1.$

$\Rightarrow \left|\frac{2}{t}\right| < 1, \left|\frac{1}{t}\right| < \frac{1}{2} < 1.$

$$\begin{aligned}
f(z) &= \frac{1}{t-1} + \frac{1}{2-t}\\
&= \frac{1}{t\left(1-\frac{1}{t}\right)} + \frac{1}{t\left(\frac{2}{t}-1\right)} = \frac{1}{t}\left(1-\frac{1}{t}\right)^{-1} - \frac{1}{t}\left(1-\frac{2}{t}\right)^{-1}\\
&= \frac{1}{t}\left(1+\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\cdots\right) - \frac{1}{t}\left(1+\frac{2}{t}+\frac{4}{t^2}+\frac{8}{t^3}+\cdots\right)\\
&= \frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\frac{1}{t^4}+\cdots - \frac{1}{t} - \frac{2}{t^2} - \frac{4}{t^3} - \frac{8}{t^4}\cdots\\
&= -\frac{1}{t^2} - \frac{3}{t^3} - \frac{7}{t^4} - \cdots = -\left((z+1)^{-2} + 3(z+1)^{-3} + 7(z+1)^{-4} + \cdots\right).
\end{aligned}$$

Example 4.48. Find the Laurent series expansion of $f(z) = \frac{z}{(z^2+1)(z^2+4)}$ in $1 < |z| < 2$. [Dec 2005]

Solution.



The singular points are $z = \pm i, z = \pm 2i$ which lies outside $1 < |z| < 2$.

$\therefore$ $f(z)$ is analytic in this annular region $z = 0$.

Let $\frac{z}{(z^2+1)(z^2+4)} = \frac{Az+B}{z^2+1} + \frac{Cz+D}{z^2+4}$.

$z = (Az+B)(z^2+4)+(Cz+D)(z^2+1)$.

—

Equating the coeff. of $z^3 \implies A + C = 0.$ (1)

Equating the coeff. of $z^2 \implies B + D = 0.$ (2)

Equating the coeff. of $z \implies 4A + C = 1.$ (3)

Equating the constants $\implies 4B + D = 0.$ (4)

From (2) and (4) we obtain $B = D = 0$

(1) $\Rightarrow C = -A$

(3) $\Rightarrow 4A - A = 1 \Rightarrow 3A = 1 \Rightarrow A = \frac{1}{3}, C = \frac{-1}{3}$

$\therefore \quad \dfrac{z}{(z^2+1)(z^2+4)} = \dfrac{1}{3}\dfrac{z}{z^2+1} - \dfrac{1}{3}\dfrac{z}{z^2+4}$

Consider $|z| > 1 \Rightarrow \dfrac{1}{|z|} < 1 \implies \dfrac{1}{|z^2|} < 1$

$|z| < 2 \Rightarrow \left|\dfrac{z}{2}\right| < 1 \Rightarrow \left|\dfrac{z^2}{4}\right| < 1$

$$= \frac{1}{3}\frac{z}{z^2\left(1+\frac{1}{z^2}\right)} - \frac{1}{3(4)}\left(\frac{z}{1+\frac{z^2}{4}}\right)$$

$$= \frac{1}{3z}\left(1+\frac{1}{z^2}\right)^{-1} - \frac{z}{12}\left(1+\frac{z^2}{4}\right)^{-1}$$

$$= \frac{1}{3z}\left(1-\frac{1}{z^2}+\frac{1}{z^4}\cdots\right) - \frac{z}{12}\left(1-\frac{z^2}{4}+\frac{z^4}{16}-\cdots\right).$$

Example 4.49. Find the Laurent's series of (i) $f(z) = z^2 e^{\frac{1}{z}}$ about $z = 0$.
(ii) $f(z) = \dfrac{e^{2z}}{(z-1)^3}$ about $z = 1$.

Solution. $f(z) = z^2 e^{\frac{1}{z}}$

$z = 0$ is a singular point.

$f(z)$ is analytic in $|z| > 0$.

$$f(z) = z^2\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots\right)$$

(iii) $f(z) = \dfrac{e^{2z}}{(z-1)^3}$.

$z = 1$ is a singular point.

$f(z)$ is analytic in $|z - 1| > 0$.

$\therefore \quad f(z)$ has a Laurent's series about $z = 1$.

—

Put $t = z - 1$.

$$\begin{aligned} f(z) &= \frac{e^{2(t+1)}}{t^3} = \frac{e^{2t}e^2}{t^3} = \frac{e^2}{t^3}\Big(1 + 2t + \frac{4t^2}{2} + \frac{8t^3}{6} + \cdots\Big) \\ &= \frac{e^2}{t^3}\Big(1 + 2t + 2t^2 + \frac{4t^3}{3} + \cdots\Big) = e^2\Big(\frac{1}{t^3} + \frac{2}{t^2} + \frac{2}{t} + \frac{4}{3} + \frac{2}{3}t + \cdots\Big) \\ &= e^2\Big(\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)} + \frac{4}{3} + \frac{2}{3}(z-1) + \cdots\Big). \end{aligned}$$

Example 4.50. Find the Taylor's series expansion of $f(z) = \dfrac{1}{(z+1)^2}$ about $z = -i$.

Solution. To expand $f(z)$ about $z = -i$, we have to express $f(z)$ in powers of $z + i$.

Let $z + i = t$.

$$\begin{aligned} \text{Now, } f(z) &= \frac{1}{(t-i+1)^2} = \frac{1}{(1-i)^2\left(1+\frac{t}{1-i}\right)^2} \\ &= \frac{1}{1+i^2-2i}\left(1+\frac{t}{1-i}\right)^{-2} \\ &= \frac{i}{2}\left[1 - \frac{2t}{1-i} + \frac{3t^2}{(1-i)^2} - \frac{4t^3}{(1-i)^3}\cdots\right] \\ &= \frac{i}{2}\left[1 - \frac{2}{1-i}(z+i) + \frac{3}{(1-i)^2}(z+i)^2 - \frac{4}{(1-i)3}(z+i)^3\cdots\right] \end{aligned}$$

Example 4.51. Find the Taylor's series expansion of $f(z) = \dfrac{2z^3+1}{z^2+z}$ about the point $z = i$.

Solution. By actual division we get

$$\begin{aligned} \frac{2z^3+1}{z^2+z} &= 2z - 2 + \frac{2z+1}{z(z+1)} \\ &= 2z - 2 + (2z+1)\frac{1}{z(z+1)} \\ &= (2z-2) + (2z+1)\left[\frac{1}{z} - \frac{1}{z+1}\right]. \end{aligned} \tag{1}$$

Let us expand $\dfrac{1}{z}$ and $\dfrac{1}{z+1}$ about $z = i$.

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i.e., We have to expand $\frac{1}{z}$ and $\frac{1}{z+1}$ in powers of $z-i$.

Let $t = z - i$

i.e., $z = t + i$

$$\text{Now } \frac{1}{z} = \frac{1}{t+i} = \frac{1}{i\left(1+\frac{t}{i}\right)} = -i\left(1+\frac{t}{i}\right)^{-1}$$

$$= -i\left[1 - \frac{t}{i} + \frac{t^2}{i^2} - \frac{t^3}{i^3} + \frac{t^4}{i^4} \cdots\right]$$

$$= -i\left[1 - \frac{z-i}{i} + \frac{(z-i)^2}{i^2} - \frac{(z-i)^3}{i^3} + \frac{(z-i)^4}{i^4} \cdots\right]$$

$$= -i + (z-i) - \frac{(z-i)^2}{i} + \frac{(z-i)^3}{i^2} - \frac{(z-i)^4}{i^3} \cdots$$

$$\text{Now, } \frac{1}{z+1} = \frac{1}{t+i+1} = \frac{1}{(1+i)\left(1+\frac{t}{1+i}\right)}$$

$$= \frac{1}{1+i}\left(1+\frac{t}{1+i}\right)^{-1}$$

$$= \frac{1}{1+i}\left[1 - \frac{t}{1+i} + \frac{t^2}{(1+i)^2} - \frac{t^3}{(1+i)^3} \cdots\right]$$

$$= \frac{1}{1+i}\left[1 - \frac{z-i}{1+i} + \frac{(z-i)^2}{(1+i)^2} - \frac{(z-i)^3}{(1+i)^3} \cdots\right]$$

Substituting the expansion of $\frac{1}{z}$ and $\frac{1}{z+1}$ in (1) we get

$$\frac{2z^3+1}{z^2+z} = (2z-2) + (2z+1)\left[-i + (z-i) - \frac{(z-i)^2}{i} + \frac{(z-i)^3}{i^2} \cdots + \frac{1}{1+i}\{1 - \frac{z-i}{1+i} + \frac{(z-i)^2}{(1+i)^2} - \frac{(z-i)^3}{(1+i)^3} \cdots\}\right].$$

Exercise 4 C

1. Obtain the Laurent's series for $f(z) = \frac{1}{(z+1)(z+3)}$ for (i) $1 < |z| < 3$ (ii) $|z| < 1$. [May 2005]

2. Expand $f(z) = \frac{z^2}{(z+2)(z-3)}$ in a Laurent's series expansion if (i) $|z| < 2$ (ii) $2 < |z| < 3$. [Nov 2005]

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3. Expand $f(z) = \dfrac{z-1}{(z+2)(z+3)}$ as a Laurent's series valid in $2 < |z| < 3$. [May 2006]

4. Expand $f(z) = \dfrac{z}{(z-1)(z-3)}$ in a series of positive and negative powers of $z-1$ if $0 < |z-1| < 2$. [Nov 2010]

5. Obtain a Laurent's series expansion for $f(z) = \dfrac{(z-2)(z+2)}{(z+1)(z+4)}$ valid in (i) $1 < |z| < 4$ (ii) $|z| > 4$.

6. Expand $f(z) = \dfrac{1}{z(z-1)}$ for $0 < |z| < 1$ and $0 < |z-1| < 1$. [May 1999]

7. Expand $f(z) = \dfrac{z-1}{z^2}$ valid in $|z-1| > 1$. [Dec 1997]

8. Expand $f(z) = \dfrac{z+3}{z(z^2-z-2)}$ if $1 < |z| < 2$. [May 2003]

9. Expand $f(z) = \dfrac{1}{(z-1)(z-2)}$ in the region (i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$ (iv) $0 < |z-1| < 1$.

10. Expand $f(z) = \dfrac{12}{z(2-z)(z+1)}$ if (i) $0 < |z| < 1$ (ii) $1 < |z| < 2$.

4.5.2 Singularities

Zero of a function. Let $f(z)$ be analytic at $z = z_0$. If $f(z_0) = 0$, then z_0 is called a zero of $f(z)$. If there is a positive integer m such that $f^{(m)}(z_0) \neq 0$ and $f'(z_0) = 0 = f''(z_0) = f'''(z_0) = \cdots = f^{(m-1)}(z_0)$, then $f(z)$ is said to have a zero for order m at z_0.

$$\therefore\ f(z) = (z-z_0)^m g(z), g(z_0) \neq 0.$$

Note. $f(z) = \dfrac{p(z)}{q(z)}$ where $p(z)$ and $q(z)$ are analytic at $z = a$ and $p(a) \neq 0$ and if a is a zero of order m for $q(z)$ then a is a pole of order m for $f(z)$.

Regular point. A point at which a complex function $f(z)$ is analytic is called a

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regular point or ordinary point of $f(z)$.

Singular point. A point $z = a$ is a singular point of $f(z)$ if $f(z)$ is not analytic (or not even defined) but is analytic at some point in every deleted neighbourhood of a. [Jun 2012, May 2011]

Example.

1. $f(z) = \frac{1}{z-2}$ has $z = 2$ as a singular point.
2. $f(z) = \frac{1}{z^2(z-1)}$ has $z = 0$ & $z = 1$ as singular points.

Isolated Singularity. A singular point $z = a$ is called an isolated singularity of $f(z)$ if there exists a neighbourhood of a in which there is no other singularity.

A singular point which is not isolated is called a nonisolated singularity.

Example.

$f(z) = \frac{1}{z(z+2)}$. $z = 0$ and $z = -2$ are isolated singular points.

Pole. An isolated singular point a of $f(z)$ is said to be a pole of order m if the principal part of the Laurent's Series of $f(z)$ about a contains m terms where m is finite.

i.e., there exists a positive integer m such that $b_m \neq 0$ and $b_{m+1} = 0 = b_{m+2} = \ldots$

$\therefore$ The Laurents's series becomes

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}, \quad r_2 < |z-a| < r_1 \text{ where } b_m \neq 0.$$

If $m = 1$, then a is called a simple pole.

Example. Consider $f(z) = \frac{1}{z^2(z-1)}$. The poles of $f(z)$ are $z = 0$ and $z = 1$. $z = 0$ is a pole of order 2 and $z = 1$ is a simple pole.

Essential Singularity. An isolated singular point a is said to be an essential singularity of $f(z)$ if the principal part of the Laurent's series of $f(z)$ about a contains infinitely many terms.

Example. Consider $f(z) = e^{\frac{1}{z}}$

$$= 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots \text{ if } 0 < |z| < \infty$$

$$= 1 + z^{-1} + \frac{1}{2!}z^{-2} + \frac{1}{3!}z^{-3} + \ldots$$

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Since $f(z)$ has infinitely many terms in the principal part, $z = 0$ is an essential Singularity.

Removable Singularity. An isolated singular point $z = a$ of $f(z)$ is called a removable singularity of $f(z)$ if in some neighbourhood of a, the Laurent's series expansion of $f(z)$ has no principal part.

Example. Consider $f(z) = \frac{\sin z}{z}, \ z \neq 0$

$$= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots \quad 0 < |z| < \infty.$$

It has no principal part.

$\therefore \quad z = 0$ is a removable singularity if $\frac{\sin z}{z}$ is not defined at $z = 0$.

Suppose $f(z) = \begin{cases} \frac{\sin z}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0. \end{cases}$ Then the series converges to 1 at $z = 0$ and hence $f(z)$ is analytic at $z = 0$.

Results

1. A singularity $z = a$ is an essential singularity if there is no integer n such that $\lim_{z\to a}(z-a)^n f(z) = A \neq 0$.
2. A singularity $z = a$ is a pole of order m if $\lim_{z\to a}(z-a)^m f(z) = A \neq 0$.
3. A singularity $z = a$ is a removable singularity of $f(z)$ if $\lim_{z\to a} f(z) = A, \ A$ finite.

Working rule for detecting the type of singularity

1. If $\lim_{z\to a} f(z)$ exists and finite then $z = a$ is a removable singularity.
2. If $\lim_{z\to a} f(z) = \infty$ then $z = a$ is a pole of $f(z)$.
3. If $\lim_{z\to a} f(z)$ does not exist, then $z = a$ is an essential singularity.

Worked Examples

Example 4.52. Classify the singularity of $e^{e^{\frac{1}{z^2}}}$. [Dec 2011]

Solution. Let $f(z) = e^{e^{\frac{1}{z^2}}}$.

$z = 0$ is a singular point.

$\lim_{z\to 0} f(z) = \lim_{z\to 0} e^{e^{\frac{1}{z^2}}}$ which does not exist.

$\therefore z = 0$ is an essential singularity.

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Example 4.53. Discuss the nature of the singularity of $\dfrac{e^z}{z^2+4}$. [May 2001]

Solution. Let $f(z) = \dfrac{e^z}{z^2+4}$.

The singularities are given by $z^2 + 4 = 0$.

i.e., $z^2 = -4$

$z = \pm 2i$.

$\therefore\ z - 2i$ and $z = -2i$ are the singular points.

$$\lim_{z\to 2i} f(z) = \lim_{z\to 2i} \frac{e^z}{(z+2i)(z-2i)} = \infty.$$

Also $\displaystyle\lim_{z\to -2i} f(z) = \lim_{z\to -2i} \frac{e^z}{(z+2i)(z-2i)} = \infty.$

Hence, $z = 2i$ and $z = -2i$ are simple poles.

Example 4.54. Classify the singularity of $\dfrac{\sin z - z}{z^3}$. [May 2000]

Solution. Let $f(z) = \dfrac{\sin z - z}{z^3}$.

The singularities are given by $z^3 = 0 \Longrightarrow z = 0$.

$$\text{Now, } f(z) = \frac{1}{z^3}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots - z\right)$$

$$= \frac{1}{z^3}\left(-\frac{z^3}{6} + \frac{z^5}{120} - \cdots\right)$$

$$= -\frac{1}{6} + \frac{z^2}{120} - \cdots$$

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}\left(-\frac{1}{6} + \frac{z^2}{120} - \cdots\right)$$

$$= \frac{-1}{6} = \text{finite}.$$

Hence, $z = 0$ is a removable singularity.

Example 4.55. Find the nature and location of the singularities of the function $f(z) = \dfrac{z - \sin z}{z^2}$.

Solution. Let $f(z) = \dfrac{z - \sin z}{z^2}$.

$z = 0$ is a singularity.

$$f(z) = \frac{1}{z^2}\left[z - \left\{z - \frac{z^3}{3!} + \frac{z^5}{5!}\cdots\right\}\right]$$

$$= \frac{1}{z^2}\left[z - z + \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!}\cdots\right] = \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!}\cdots$$

Since there is no negative powers of z in the expansion of $f(z), z = 0$ is a removable singularity.

Example 4.56. Find the nature and location of the singularity of the function $f(z) = (z+1)\sin\left(\dfrac{1}{z-2}\right)$.

Solution. $f(z) = (z+1)\sin\left(\dfrac{1}{z-2}\right)$.

Let $z - 2 = t$.

$$\begin{aligned}
\therefore\ f(z) &= (t+2+1)\sin\left(\frac{1}{t}\right)\\
&= (t+3)\left[\frac{1}{t} - \frac{1}{3!}\frac{1}{t^3} + \frac{1}{5!}\frac{1}{t^5}\cdots\right]\\
&= 1 - \frac{1}{3!t^2} + \frac{1}{5!t^4}\cdots + \frac{3}{t} - \frac{1}{2t^3} + \frac{3}{5!t^5}\cdots\\
&= 1 + \frac{3}{t} - \frac{1}{3!t^2} - \frac{1}{2t^3} + \frac{1}{5!}t^4 + \frac{3}{5!}\frac{1}{t^5}\cdots\\
&= 1 + \frac{3}{z-2} - \frac{1}{3!}\frac{1}{(z-2)^2} - \frac{1}{2(z-2)^3} + \frac{1}{5!}\frac{1}{(z-2)^4}\cdots
\end{aligned}$$

Since there are infinite number of terms in the negative powers of $z-2$, $z = 2$ is an essential singularity.

Example 4.57. Find the nature and location of the singularity of $\dfrac{1}{\cos z - \sin z}$.

Solution. Let $f(z) = \dfrac{1}{\cos z - \sin z}$.

The singularities are given by

$$\begin{aligned}
\sin z - \cos z &= 0\\
\sin z = \cos z &\Rightarrow \frac{\sin z}{\cos z} = 1\\
\text{i.e., } \tan z = 1 &\quad \Rightarrow z = \frac{\pi}{4}.\\
\therefore z &= \frac{\pi}{4} \text{ is a singularity.}
\end{aligned}$$

$$\text{Now } \lim_{z\to\frac{\pi}{4}} f(z) = \lim_{z\to\frac{\pi}{4}} \frac{1}{(\cos z - \sin z)} = \frac{1}{\cos\frac{\pi}{4} - \sin\frac{\pi}{4}} = \infty.$$

$\therefore z = \dfrac{\pi}{4}$ is a simple pole.

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Example 4.58. Discuss the singularities of $\dfrac{e^{\frac{1}{z}}}{(z-a)^2}$. [May 2001]

Solution. Let $f(z) = \dfrac{e^{\frac{1}{z}}}{(z-a)^2}$.

The singular points are given by $z = 0$ and $(z-a)^2 = 0$.

i.e., $z = a, a$.

$\lim\limits_{z\to 0} f(z) = \lim\limits_{z\to 0} \dfrac{e^{\frac{1}{z}}}{(z-a)^2}$ = does not exist.

$\therefore$ $z = 0$ is an essential singularity.

Now $\lim\limits_{z\to a} f(z) = \lim\limits_{z\to a} \dfrac{e^{\frac{1}{z}}}{(z-a)^2} = \infty$.

$\therefore$ $z = a$ is a pole of order 2.

Example 4.59. Find the singularities of the function $\dfrac{1-e^{2z}}{z^4}$. [May 1999]

Solution. Let $f(z) = \dfrac{1-e^{2z}}{z^4}$.

$$= \frac{1}{z^4}\left[1 - \left(1 + 2z + \frac{4z^2}{2!} + \frac{8z^3}{3!} + \cdots\right)\right]$$

$$= -\frac{1}{z^4}\left[2z + 2z^2 + \frac{4z^3}{3} + \cdots\right]$$

$$= -\frac{2 + 2z + \frac{4z^2}{3} + \cdots}{z^3}$$

The singular points are given by $z^3 = 0 \Longrightarrow z = 0, 0, 0$.

$\lim\limits_{z\to 0} f(z) = \infty$.

$\therefore$ $z = 0$ is a pole of order 3.

Example 4.60. Find the nature of the singularities of the function $\dfrac{\cot \pi z}{(z-a)^3}$. [May 2005]

Solution. Let $f(z) = \dfrac{\cot \pi z}{(z-a)^3}$

$$= \frac{\cos \pi z}{\sin \pi z (z-a)^3}.$$

The singularities of $f(z)$ are given by $\sin \pi z = 0$ and $(z-a)^3 = 0$.

$\Longrightarrow \pi z = n\pi, n \in \mathbb{Z}$ and $z = a, a, a$.

$\Longrightarrow z = n, n \in \mathbb{Z}$ and $z = a$.

Now, $\lim\limits_{z\to n} f(z) = \infty$

$\therefore z = 0, \pm 1, \pm 2, \ldots$ are simple poles.

Also $\lim\limits_{z \to a} f(z) = \infty$.

$z = a$ is a pole of order 3.

Example 4.61. What type of singularity has the function $f(z) = \dfrac{1}{1 - e^z}$.[May 2008]

Solution. The poles are given by

$$1 - e^z = 0$$
$$e^z = 1 = e^{2n\pi i}, \ n \in z.$$
$$\therefore z = 2n\pi i, n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

When $n = 1, z = 2\pi i$ is a simple pole.

Similarly, we can prove that all the poles are simple.

Example 4.62. Discuss the nature of the singularity of the function $\dfrac{e^{\frac{1}{z}}}{z^2}$.

Solution. Let $f(z) = \dfrac{e^{\frac{1}{z}}}{z^2}$

$$= \frac{1}{z^2}\left[1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots\right]$$
$$= z^{-2} + z^{-3} + \frac{1}{2}z^{-4} + \frac{1}{6}z^{-5} \cdots$$

Since there are infinite number of terms in the negative powers of $z, z = 0$ is an essential singularity, of $f(z)$.

Example 4.63. Find the nature of the singularity of the function $\dfrac{e^{2z}}{(z-1)^4}$.

Solution. The singularity is at $z = 1$.

Let $z - 1 = t$

$\therefore z = 1 + t$.

$$\frac{e^{2z}}{(z-1)^4} = \frac{e^{2(1+t)}}{t^4} = \frac{e^{2+2t}}{t^4} = \frac{e^2}{t^4} \cdot e^{2t}$$
$$= \frac{e^2}{t^4}\left[1 + 2t + \frac{4t^2}{2!} + \frac{8t^3}{3!} + \frac{16t^4}{4!} + \frac{32t^5}{5!} \cdots\right]$$
$$= e^2\left[t^{-4} + 2t^{-3} + 2t^{-2} + \frac{4}{3}t^{-1} + \frac{2}{3} + \frac{4}{15}t + \cdots\right]$$

$$= e^2\left[(z-1)^{-4} + 2(z-1)^{-3} + 2(z-1)^{-2} + \frac{4}{3}(z-1)^{-1} + \frac{2}{3} + \frac{4}{5}(z-1)\cdots\right]$$

Since the expansion contains only 4 terms with negative powers of $z-1$, $z = 1$ is a pole of order 4.

Exercise 4 D

1. Discuss the nature of singularities of the following functions.

(i). $\frac{z-2}{z^2}\sin\frac{1}{z-1}$.

(ii). $\sin\frac{1}{z-1}$.

(iii). $\frac{1-e^z}{1+e^z}$.

(iv). $\frac{1}{z(e^z-1)}$.

(v). $\frac{z}{e^z-1}$.

(vi). $\frac{\sin z}{z^5}$.

(vii). $\frac{\tan z}{z}$.

(viii). $\sin\frac{1}{z-2}$.

(ix). $e^{\frac{1}{z}}$.

(x). $\frac{1}{(z-5)^3(z-4)^2}$.

(xi). $\frac{e^z}{z^2+\pi^2}$.

(xii). $\frac{z^2}{(1+z^2)^3}$.

(xiii). $\frac{\cos 2z}{(z^2+1)^2(z^2+16)}$.

(xiv). $\frac{\sin z}{z\cos z}$.

(xv). $\frac{1}{(z^2+a^2)^2}$.

4.6 Residue

Let $z = a$ be an isolated singular point of $f(z)$. The coefficient b_1 of $(z-a)^{-1}$ in the Laurent's series expansion of $f(z)$ about a is called the residue of $f(z)$ at $z = a$. The residue of $f(z)$ at $z = a$ is denoted by $b_1 = [\text{Res } f(z)]_{z=a}$ or $R(a)$.

The residue at $z = a$ is given by $b_1 = R(a) = \frac{1}{2\pi i}\int_C f(z)dz$.

—

Methods of finding Residue

1. If $z = a$ is a simple pole, then $R(a) = \lim_{z\to a}(z-a)f(z)$.

2. If $z = a$ is a pole of order m then $R(a) = \dfrac{1}{(m-1)!}\lim_{z\to a}\left(\dfrac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)\right)$.

3. Let $f(z) = \dfrac{g(z)}{h(z)}$, where $g(z)$ and $h(z)$ are analytic functions at $z = a$.

If $h(a) = 0, h'(a) \neq 0$ and $g(a) \neq 0$ and finite, then $z = a$ is a simple pole of $f(z)$ and $R(a) = \lim_{z\to a}\dfrac{g(z)}{h'(z)}$.

Cauchy's residue theorem

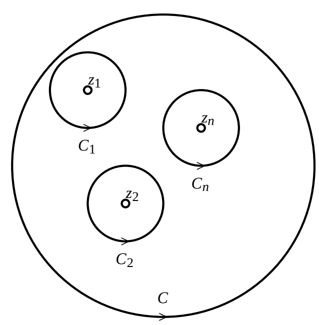
If $f(z)$ is analytic inside and on a simple closed curve C, except at a finite number of singular points $z_1, z_2, \ldots, z_n$ lying inside C, then

$$\int_C f(z)dz = 2\pi i(R(z_1) + R(z_2) + \cdots + R(z_n))$$

where the integral over C is taken in the anticlockwise sense.

Proof. Let C be the closed curve and $z_1, z_2, \ldots, z_n$ be the isolated singular points of $f(z)$ which lie inside C.

Let $C_1, C_2, \ldots, C_n$ be non overlapping neighbourhoods of $z_1, z_2, \ldots, z_n$ respectively. Hence, $f(z)$ is analytic in the multiply connected region bounded by $C_1, C_2, \ldots, C_n$ and C.



By Cauchy's theorem we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \cdots + \int_{C_n} f(z)dz$$

$$\frac{1}{2\pi i}\int_C f(z)dz = \frac{1}{2\pi i}\int_{C_1} f(z)dz + \frac{1}{2\pi i}\int_{C_2} f(z)dz + \frac{1}{2\pi i}\int_{C_3} f(z)dz + \cdots + \frac{1}{2\pi i}\int_{C_n} f(z)dz$$

—

$$= R[z_1] + R[z_2] + \cdots + R[z_n]$$

$$\therefore \int_C f(z)dz = 2\pi i\,(R[z_1] + R[z_2] + \cdots + R[z_n])\,.$$

Result. According to Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i$ [Sum of the residues at the singular points].

Worked Examples

Example 4.64. Find the residue of $f(z) = \dfrac{4}{z^3(z-2)}$ at a simple pole.

[Dec 2012, Jun 2012]

Solution. $f(z) = \dfrac{4}{z^3(z-2)}$.

The poles are given by $z = 0, z = 2$.

$z = 2$ is a simple pole.

$$\text{Now } R(2) = \lim_{z\to 2}(z-2)f(z)$$

$$= \lim_{z\to 2}(z-2)\frac{4}{z^3(z-2)} = \lim_{z\to 2}\frac{4}{z^3} = \frac{4}{z^3} = \frac{4}{8} = \frac{1}{2}.$$

Example 4.65. Find the singular point of $\dfrac{z+2}{(z+1)^2}$ and hence find its residue.

[May 2001]

Solution. Let $f(z) = \dfrac{z+2}{(z+1)^2}$

$z = -1$ is a pole of order 2.

$$\therefore R(-1) = \frac{1}{(2-1)!}\lim_{z\to -1}\frac{d}{dz}[(z+1)^2 f(z)]$$

$$= \lim_{z\to -1}\frac{d}{dz}(z+1)^2\frac{z+2}{(z+1)^2}$$

$$= \lim_{z\to -1}\frac{d}{dz}(z+2) = \lim_{z\to -1} 1 = 1.$$

Example 4.66. Calculate the residue of $f(z) = \dfrac{e^{2z}}{(z+1)^2}$ at its pole.

[Apr 2004, Dec 2011]

—

Solution. $f(z) = \dfrac{e^{2z}}{(z+1)^2}$.

$z = -1$ is a pole of order 2.

$$\begin{aligned}
\therefore R(-1) &= \frac{1}{(2-1)!} \lim_{z\to -1} \frac{d}{dz}[(z+1)^2 f(z)] \\
&= \lim_{z\to -1} \frac{d}{dz}\left[(z+1)^2 \frac{e^{2z}}{(z+1)^2}\right] = \lim_{z\to -1} \frac{d}{dz} e^{2z} \\
&= \lim_{z\to -1} 2e^{2z} = 2e^{-2} = \frac{2}{e^2}.
\end{aligned}$$

Example 4.67. Test the singularity of $\dfrac{1}{z^2+1}$ and find the corresponding residues. [Apr 1997]

Solution. Let $f(z) = \dfrac{1}{z^2+1}$.

The poles are given by $z^2 + 1 = 0$.

i.e., $z^2 = -1$

$z = \pm i$.

$z = i$ and $-i$ are simple poles.

$$\begin{aligned}
R(i) &= \lim_{z\to i} (z-i) f(z) \\
&= \lim_{z\to i} (z-i) \frac{1}{(z+i)(z-i)} \\
&= \lim_{z\to i} \frac{1}{z+i} = \frac{1}{2i} = \frac{-i}{2}.
\end{aligned}$$

$$\begin{aligned}
R(-i) &= \lim_{z\to -i} (z+i) f(z) \\
&= \lim_{z\to -i} (z+i) \frac{1}{(z+i)(z-i)} \\
&= \lim_{z\to i} \frac{1}{z-i} = \frac{-1}{2i} = \frac{i}{2}.
\end{aligned}$$

Example 4.68. Determine the residue at the simple pole of $\dfrac{z+2}{(z+1)^2(z-2)}$. [May 2000]

Solution. Let $f(z) = \dfrac{z+2}{(z+1)^2(z-2)}$.

—

$z = 2$ is a simple pole of $f(z)$.

$$\begin{aligned}
R(2) &= \lim_{z\to 2}(z-2)f(z)\\
&= \lim_{z\to 2}(z-2)\frac{z+2}{(z+1)^2(z-2)}\\
&= \lim_{z\to 2}\frac{(z+2)}{(z+1)^2} = \frac{4}{9}.
\end{aligned}$$

Example 4.69. Find the residue of $f(z) = \dfrac{z}{(z-1)^2}$ at its pole.

Solution. $f(z) = \dfrac{z}{(z-1)^2}$.

$z = 1$ is a pole of order 2.

$$\begin{aligned}
\therefore R(1) &= \frac{1}{(2-1)!}\lim_{z\to 1}\frac{d}{dz}(z-1)^2 f(z)\\
&= \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{z}{(z-1)^2}\right]\\
&= \lim_{z\to 1}\frac{d}{dz}z = 1.
\end{aligned}$$

Example 4.70. Find the residue of $\dfrac{1-e^{2z}}{z^4}$ at $z = 0$. [Jun 2013]

Solution. Let $f(z) = \dfrac{1-e^{2z}}{z^4}$.

$z = 0$ is a pole of order 3. [Expand e^{2z} and simplify]

$$\begin{aligned}
\therefore R(0) &= \frac{1}{(3-1)!}\lim_{z\to 0}\frac{d^2}{dz^2}(z-0)^3 f(z)\\
&= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left[z^3\frac{1-e^{2z}}{z^4}\right]\\
&= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left[\frac{1-e^{2z}}{z}\right]\\
&= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left[\frac{1-1-2z-\frac{4z^2}{2!}-\frac{8z^3}{3!}-\cdots}{z}\right]\\
&= \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left[2-\frac{8z}{3}-\cdots\right]\\
&= \frac{1}{2}\lim_{z\to 0}\left[\frac{-8}{3}+\text{powers of } z\right]
\end{aligned}$$

$$= \frac{1}{2}\left(\frac{-8}{3}\right) = -\frac{4}{3}.$$

Example 4.71. Find the residue of $f(z) = \dfrac{1-e^{-z}}{z^3}$ at $z = 0$. [Dec 2013]

Solution. $f(z) = \dfrac{1-e^{-z}}{z^3}$

$$= \frac{1-\left(1-z+\frac{z^2}{2!}-\frac{z^3}{3!}+\frac{z^4}{4!}-\frac{z^5}{5!}\cdots\right)}{z^3}$$

$$= \frac{1}{z^3}\left[1-1+z-\frac{z^2}{2!}+\frac{z^3}{3!}-\frac{z^4}{4!}+\frac{z^5}{5!}\cdots\right]$$

$$= \frac{1}{z^3}\left[z-\frac{z^2}{2!}+\frac{z^3}{3!}-\frac{z^4}{4!}+\frac{z^5}{5!}\cdots\right]$$

$$= \frac{1}{z^2}-\frac{1}{2!}\frac{1}{z}+\frac{1}{3!}-\frac{z}{4!}+\frac{z^2}{5!}\cdots$$

z is the pole of order 2.

$\therefore$ The residues of $f(z)$ at $z = 0$ is the coefficient of $\dfrac{1}{z} = -\dfrac{1}{2}$.

Example 4.72. Find the residues of $f(z) = \dfrac{z^2}{(z+2)(z-1)^2}$ at its isolated singularities using Laurent's series expansion. [Dec 2013]

Solution. $\dfrac{z^2}{(z+2)(z-1)^2} = \dfrac{A}{z+2} + \dfrac{B}{z-1} + \dfrac{C}{(z-1)^2}$

$$z^2 = A(z-1)^2 + B(z-1)(z+2) + c(z+2).$$

When $z = 1, 3C = 1 \Rightarrow C = \dfrac{1}{3}$.

When $z = -2, 9A = 4 \Rightarrow A = \dfrac{4}{9}$.

Equating the coefficients of z^2 we get

$$A + B = 1$$

$$B = 1 - A = 1 - \frac{4}{9} = \frac{5}{9}.$$

$$\therefore \frac{z^2}{(z+2)(z-1)^2} = \frac{4}{9}\frac{1}{z+2} + \frac{5}{9}\frac{1}{z-1} + \frac{1}{3}\frac{1}{(z-1)^2}.$$

—

The residues at the singularity $z=-2$ is the coefficient of $\frac{1}{z}$ in the expansion of $\frac{4}{9}\frac{1}{z+2}$.

$$\text{Now, } \frac{4}{9}\cdot\frac{1}{z+2}=\frac{4}{9}\cdot\frac{1}{z\left(1+\frac{2}{z}\right)}=\frac{4}{9}\frac{1}{z}\left(1+\frac{2}{z}\right)^{-1}$$

$$=\frac{4}{9}\frac{1}{z}\left[1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}\cdots\right]=\frac{4}{9}\left[\frac{1}{z}-\frac{2}{z^2}+\frac{4}{z^3}\cdots\right]$$

$\therefore$ Residue at $z=-2$ = Coefficient of $\frac{1}{z}=\frac{4}{9}$.

The residue at the singularity $z=1$ is the coefficient of $\frac{1}{z}$ in the expansion of $\frac{5}{9}\frac{1}{z-1}+\frac{1}{3}\frac{1}{(z-1)^2}$.

$$\text{Now, } \frac{5}{9}\frac{1}{z-1}+\frac{1}{3}\frac{1}{(z-1)^2}=\frac{5}{9z\left(1-\frac{1}{z}\right)}+\frac{1}{3z^2\left(1-\frac{1}{z}\right)^2}$$

$$=\frac{5}{9z}\left(1-\frac{1}{z}\right)^{-1}+\frac{1}{3z^2}\left(1-\frac{1}{z}\right)^{-2}$$

$$=\frac{5}{9z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]+\frac{1}{3z^2}\left[1+\frac{2}{z}+\frac{3}{z^2}+\cdots\right].$$

Residues at $z=1$ = Coefficient of $\frac{1}{z}=\frac{5}{9}$.

Example 4.73. Find the residues of $f(z)=\frac{z^2}{(z-1)^2(z+2)^2}$ at its isolated singularities using Laurent's series expansions. Also state the valid region.

[Dec 2012, Dec 2010]

Solution. Let $\frac{z^2}{(z-1)^2(z+2)^2}=\frac{A}{z-1}+\frac{B}{(z-1)^2}+\frac{C}{z+2}+\frac{D}{(z+2)^2}$.

$$\therefore z^2=A(z-1)(z+2)^2+B(z+2)^2+C(z+2)(z-1)^2+D(z-1)^2.$$

When $z=1, 9B=1$

$\Rightarrow B=\frac{1}{9}$.

When $z=-2, 9D=4 \quad \Rightarrow D=\frac{4}{9}$.

Equating the coefficients of z^3 we get

$A + C = 0 \Rightarrow C = -A.$

When $z = 0, -4A + 4B + 2C + D = 0$

$$-4A + \frac{4}{9} - 2A + \frac{4}{9} = 0$$

$$6A = \frac{8}{9}$$

$$A = \frac{8}{6 \times 9} = \frac{4}{27}.$$

$$\therefore C = -\frac{4}{27}.$$

$$\begin{aligned}
f(z) &= \frac{4}{27} \cdot \frac{1}{z-1} + \frac{1}{9} \cdot \frac{1}{(z-1)^2} - \frac{4}{27} \cdot \frac{1}{z+2} + \frac{4}{9} \cdot \frac{1}{(z+2)^2}. \\
&= \frac{4}{27} \cdot \frac{1}{z-1} + \frac{1}{9} \cdot \frac{1}{(z-1)^2} - \frac{4}{27} \cdot \frac{1}{(z-1)+3} + \frac{4}{9} \cdot \frac{1}{((z-1)+3)^2}. \\
&= \frac{4}{27} \cdot \frac{1}{z-1} + \frac{1}{9} \cdot \frac{1}{(z-1)^2} - \frac{4}{81}\left(1 + \frac{z-1}{3}\right)^{-1} + \frac{4}{81}\left(1 + \frac{z-1}{3}\right)^{-2}. \\
&= \frac{4}{27} \cdot \frac{1}{z-1} + \frac{1}{9} \cdot \frac{1}{(z-1)^2} - \frac{4}{81}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{3}\right)^n + \frac{4}{81}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{3}\right)^n (n+1). \qquad (1)
\end{aligned}$$

Expansion is valid if $|\frac{z-1}{3}| < 1$. i.e., $|z-1| < 3$.

Residue of $f(z)$ at $z = 1$ = co-efficient of $\frac{1}{z-1}$ in (1) = $\frac{4}{27}$.

The validity of the region is $|\frac{z-1}{3}| < 1 \Rightarrow 0 < |z-1| < 3$.

The find the residue of $f(z)$ at $z = -2$, we have to expand $f(z)$ in series of powers of $z + 2$ valid $0 < |z+2| < r$.

$$\begin{aligned}
f(z) &= \frac{4}{27} \cdot \frac{1}{(z+2)-3} + \frac{1}{9} \cdot \frac{1}{((z+2)-3)^2} - \frac{4}{27} \cdot \frac{1}{z+2} + \frac{4}{9} \cdot \frac{1}{(z+2)^2}. \\
&= -\frac{4}{81}\left(1 - \frac{z+2}{3}\right)^{-1} + \frac{1}{81}\left(1 - \frac{z+2}{3}\right)^{-2} - \frac{4}{27} \cdot \frac{1}{z+2} + \frac{4}{9} \cdot \frac{1}{(z+2)^2}. \qquad (2)
\end{aligned}$$

Expansion is valid if $|\frac{z+2}{3}| < 1$. i.e., $|z+2| < 3$.

Residue of $f(z)$ at $z = 2$ = co-efficient of $\frac{1}{z+2}$ in (2) = $-\frac{4}{27}$.

The validity of the region $0 < |z+2| < 3$.

—

Example 4.74. Find the sum of the residues of $f(z) = \dfrac{\sin z}{z \cos z}$ at its poles inside the circle $|z| = 2$.

Solution. The poles are given by $z \cos z = 0$

i.e., $z = 0, \cos z = 0$

i.e., $z = 0, z = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \cdots$

Given c is $|z| = 2$.

When $z = 0$, $|z| = |0| = 0 < 2$.

$\therefore z = 0$ lies inside C,

When $z = \pm\dfrac{\pi}{2}$, $|z| = |\dfrac{\pi}{2}| < 2$.

$\therefore z = \dfrac{\pi}{2}$ and $-\dfrac{\pi}{2}$ lie inside C.

When $z = \pm\dfrac{3\pi}{2}$, $|z| = |\dfrac{3\pi}{2}| > 2$.

$\therefore$ All other singular points lie out side C.

$z = 0$ is a simple pole.

$\therefore R(0) = \lim\limits_{z\to 0} z f(z) = \lim\limits_{z\to 0} z \cdot \dfrac{\sin z}{z \cos z} = \lim\limits_{z\to 0} \dfrac{\sin z}{\cos z} = 0.$

$z = \frac{\pi}{2}$ is a simple pole.

$$\begin{aligned}
\therefore R\left(\frac{\pi}{2}\right) &= \lim_{z\to\frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) f(z) \\
&= \lim_{z\to\frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) \cdot \frac{\sin z}{z \cos z} \quad [\frac{0}{0} \text{ form}] \\
&= \lim_{z\to\frac{\pi}{2}} \frac{\left(z - \frac{\pi}{2}\right) \cdot \cos z + \sin z}{-z \sin z + \cos z} \quad [\text{by L'hospital rule}] \\
&= \frac{\left(\frac{\pi}{2} - \frac{\pi}{2} \cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right)}{-\frac{\pi}{2} \cdot \sin\frac{\pi}{2} + \cos\frac{\pi}{2}} = \frac{1}{-\frac{\pi}{2}} = -\frac{2}{\pi}.
\end{aligned}$$

$z = -\dfrac{\pi}{2}$ is also a simple pole.

$$\begin{aligned}
R\left(-\frac{\pi}{2}\right) &= \lim_{z\to(-\frac{\pi}{2})} \left(z + \frac{\pi}{2}\right) f(z) \\
&= \lim_{z\to(-\frac{\pi}{2})} \frac{\left(z + \frac{\pi}{2}\right) \sin z}{z \cos z} \quad [\frac{0}{0} \text{ form}]
\end{aligned}$$

—

$$= \lim_{z\to(-\frac{\pi}{2})} \frac{\left(z+\frac{\pi}{2}\right)\cos z + \sin z}{-z\sin z + \cos z}$$

$$= \frac{\left(-\frac{\pi}{2}+\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{2}\right) + \sin\left(-\frac{\pi}{2}\right)}{-\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) + \cos\left(-\frac{\pi}{2}\right)}$$

$$= \frac{0-\sin\frac{\pi}{2}}{\frac{\pi}{2}\left(-\sin\frac{\pi}{2}\right)} = -\frac{\sin\left(\frac{\pi}{2}\right)}{-\frac{\pi}{2}} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}.$$

$$\therefore \text{ Sum of the residues} = R(0) + R\left(\frac{\pi}{2}\right) + R\left(-\frac{\pi}{2}\right)$$

$$= 0 + \frac{2}{\pi} - \frac{2}{\pi} = 0.$$

Example 4.75. If $f(z) = -\dfrac{1}{z-1} - 2\left[1 + (z-1) + (z-1)^2 + \cdots\right]$ find the residue of $f(z)$ at $z = 1$. [Dec 2012, Dec 2010, Jun 2010]

Solution. In the Laurent's series expansion, residue at $1 =$ Coefficient of $\dfrac{1}{z-1}\left((z-1)^{-1}\right) = -1.$

Example 4.76. If C is the circle $|z| = 2$, evaluate $\displaystyle\int_C \tan z dz.$ [May 2009]

Solution. Let $f(z) = \tan z = \dfrac{\sin z}{\cos z}.$

The poles are given by $\cos z = 0$

$\implies z = (2n+1)\frac{\pi}{2}, n \in Z.$

Since C is $|z| = 2, z = \dfrac{\pi}{2}$ and $\dfrac{-\pi}{2}$ lie inside $|z| = 2$ which are simple poles.

$$\text{Now } R\left(\frac{\pi}{2}\right) = \lim_{z\to\frac{\pi}{2}} \frac{g(z)}{h'(z)}$$

$$= \lim_{z\to\frac{\pi}{2}} \frac{\sin z}{-\sin z} = -1$$

$$R\left(\frac{-\pi}{2}\right) = \lim_{z\to\frac{\pi}{2}} \frac{\sin z}{-\sin z} = -1.$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i[\text{ sum of the residues }] = 2\pi i[-1-1] = -4\pi i.$$

—

Example 4.77. Evaluate $\int_C \frac{1}{(z^2+4)^2} dz$ where C is the circle $|z-i| = 2$. [May 2008]

Solution. Let $f(z) = \frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$.

$z = 2i$ and $z = -2i$ are poles of order 2.

C is the circle $|z-2i| = 2$.

When $z = 2i, |z-i| = |2i-i| = |i| = 1 < 2$.

$\therefore z = 2i$ lies inside C.

When $z = -2i, |z-i| = |-2i-i| = |-3i| = 3 > 2$.

$\therefore z = -2i$ lies outside C.

$\therefore$ By Cauchy's residue theorem $\int_C f(z)dz = 2\pi i R(2i)$

$$R(2i) = \frac{1}{1!}\lim_{z\to 2i}\frac{d}{dz}\left\{(z-2i)^2\frac{1}{(z+2i)^2(z-2i)^2}\right\}$$

$$= \lim_{z\to 2i}\frac{d}{dz}(z+2i)^{-2}$$

$$= \lim_{z\to 2i}(-2)(z+2i)^{-3}$$

$$= (-2)\frac{1}{(4i)^3} = \frac{-2}{64(-i)} = \frac{1}{32i}$$

$$\therefore \int_C f(z)dz = 2\pi i\frac{1}{32i} = \frac{\pi}{16}.$$

Example 4.78. Evaluate using Cauchy's residue theorem $\int_C \frac{2z+3}{z(z-1)(z-2)} dz$ where C is $|z| = 3$. [May 2004]

Solution. Let $f(z) = \frac{2z+3}{z(z-1)(z-2)}$.

The poles are given by $z = 0, z = 1$ and $z = 2$.

C is $|z| = 3$.

When $z = 0, |z| = |0| = 0 < 3$.

$\therefore z = 0$ lies inside C.

When $z = 1, |z| = 1 < 3$.

$\therefore z = 1$ lies inside C.

When $z = 2, |z| = |2| = 2 < 3$.

$\therefore z = 2$ lies inside C.

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i[R(0) + R(1) + R(2)]$

$z = 0$ is a simple pole.

$$\therefore R(0) = \lim_{z\to 0}(z-0)f(z)$$
$$= \lim_{z\to 0} z\frac{2z+3}{z(z-1)(z-2)}$$
$$= \lim_{z\to 0} z\frac{2z+3}{(z-1)(z-2)}$$
$$= \frac{3}{(-1)(-2)}$$
$$R(0) = \frac{3}{2}.$$

$z = 1$ is a simple pole.

$$\therefore R(1) = \lim_{z\to 1}(z-1)f(z)$$
$$= \lim_{z\to 1}(z-1)\frac{2z+3}{z(z-1)(z-2)} = \lim_{z\to 1}\frac{2z+3}{z(z-2)}$$
$$= \frac{5}{1(-1)} = -5.$$

$z = 2$ is a simple pole.

$$\therefore R(2) = \lim_{z\to 2}(z-2)f(z)$$
$$= \lim_{z\to 2}(z-2)\frac{2z+3}{z(z-1)(z-2)}$$
$$= \lim_{z\to 2}\frac{2z+3}{z(z-1)}$$
$$= \frac{7}{2.1} = \frac{7}{2}.$$
$$\int_C f(z)dz = 2\pi i\left(\frac{3}{2} - 5 + \frac{7}{2}\right)$$
$$= 2\pi i\left(\frac{3-10+7}{2}\right) = 0.$$

Example 4.79. Determine the poles of $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ and the residue at each pole. Hence evaluate $\displaystyle\int_C \frac{z^2}{(z-1)^2(z+2)}dz$ where C is $|z| = 3$. [May 2000]

Solution. Let $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$.

The poles are given by $z = 1, z = -2$.

$z = -2$ is a simple pole, and $z = 1$ is a pole of order 2.

C is $|z| = 3$.

When $z = -2, |z| = |-2| = 2 < 3$.

$\therefore\ z = -2$ lies inside C. When $z = 1, |z| = |1| = 1 < 3$.

$\therefore\ z = 1$ also lies inside C.

$\therefore$ By Cauchy's residue theorem $\displaystyle\int_C f(z)dz = 2\pi i(R(1) + R(-2))$.

Since $z = 1$ is a pole of order 2,

$$\begin{aligned} R(1) &= \frac{1}{1!}\lim_{z\to1}\frac{d}{dz}(z-1)^2 f(z) = \lim_{z\to1}\frac{d}{dz}\frac{(z-1)^2z^2}{(z-1)^2(z+2)} \\ &= \lim_{z\to1}\frac{d}{dz}\left(\frac{z^2}{z+2}\right) = \lim_{z\to1}\frac{(z+2)2z - z^2}{(z+2)^2} = \frac{6-1}{9} = \frac{5}{9}. \end{aligned}$$

$z = -2$ is a simple pole.

$$R(-2) = \lim_{z\to-2}(z+2)f(z) = \lim_{z\to-2}(z+2)\frac{z^2}{(z-1)^2(z+2)} = \frac{4}{9}.$$

Now, $\displaystyle\int_C f(z)dz = 2\pi i\left(\frac{5}{9} + \frac{4}{9}\right) = 2\pi i.$

Example 4.80. Find the residue of $f(z) = \dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ at its poles and hence evaluate $\oint_C f(z)dz$ where C is the circle $|z| = 2\cdot 5$.

Solution. The poles are given by $(z-1)^4(z-2)(z-3) = 0$

i.e., $z = 1, z = 2, z = 3$.

$z = 1$ and $z = 2$ lie inside C whereas, $z = 3$ lies outside C.

—

$\therefore z = 1$ is a pole of order 4.

$$R(1) = \frac{1}{3!}\lim_{z\to 1}\frac{d^3}{dz^3}(z-1)^4 \cdot f(z)$$

$$= \frac{1}{6}\lim_{z\to 1}\frac{d^3}{dz^3}\left\{(z-1)^4\frac{z^3}{(z-1)^4(z-2)(z-3)}\right\}$$

$$= \frac{1}{6}\lim_{z\to 1}\frac{d^3}{dz^3}\left\{\frac{z^3}{(z-2)(z-3)}\right\}$$

Let $\dfrac{z^3}{(z-2)(z-3)} = Az + B + \dfrac{C}{z-2} + \dfrac{D}{z-3}$

$$z^3 = (Az + B)(z-2)(z-3) + C(z-3) + D(z-2).$$

When $z = 2$, $-C = 8 \Rightarrow C = -8$.

When $z = 3$, $D = 27$.

Equating the coefficients of z^3 we get $A = 1$.

When $z = 0, 6B - 3C - 2D = 0$

$6B = 3C + 2D = -24 + 54$

$6B = 30 \Rightarrow B = 5.$

$$\therefore \frac{z^3}{(z-2)(z-3)} = z + 5 - \frac{8}{z-2} + \frac{27}{z-3}$$

$$\therefore R(1) = \frac{1}{6}\lim_{z\to 1}\frac{d^3}{dz^3}\left\{z + 5 - \frac{8}{z-2} + \frac{27}{z-3}\right\}$$

$$= \frac{1}{6}\lim_{z\to 1}\frac{d^3}{dz^3}\left\{z + 5 - 8(z-2)^{-1} + 27(z-3)^{-1}\right\}$$

$$= \frac{1}{6}\lim_{z\to 1}\frac{d^2}{dz^2}\left\{1 - 8(-1)(z-2)^{-2} + 27(-1)(z-3)^{-2}\right\}$$

$$= \frac{1}{6}\lim_{z\to 1}\frac{d}{dz}\left\{8(-2)(z-2)^{-3} - 27(-2)(z-3)^{-3}\right\}$$

$$= \frac{1}{6}\lim_{z\to 1}\left\{-16(-3)(z-2)^{-4} + 54(-3)(z-3)^{-4}\right\}$$

$$= \frac{1}{6}\left[\frac{48}{(1-2)^4} - \frac{162}{(1-3)^4}\right] = \frac{1}{6}\left[48 - \frac{162}{16}\right]$$

$$= 8 - \frac{27}{16} = \frac{128-27}{16} = \frac{101}{16}.$$

—

$z = 2$ is a simple pole.

$$\therefore R(2) = \lim_{z\to 2}(z-2)f(z)$$
$$= \lim_{z\to 2}(z-2)\frac{z^3}{(z-1)^4(z-2)(z-3)}$$
$$= \lim_{z\to 2}\frac{z^3}{(z-1)^4(z-3)} = \frac{8}{1\times(-1)} = -8.$$
$$\therefore \oint_C f(z)dz = 2\pi i \times [\text{sum of the residues}]$$
$$= 2\pi i\left[\frac{101}{16} - 8\right] = 2\pi i\left[\frac{101-128}{16}\right] = 2\pi i \times \left[-\frac{27}{16}\right] = -\frac{27\pi i}{8}.$$

Example 4.81. Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$ where C is $|z-i| = 2$.

[June 2014, June 2012]

Solution. Let $f(z) = \frac{z-1}{(z+1)^2(z-2)}$.

The poles are given by $z = -1$ and $z = 2$.

$z = -1$ is a pole of order 2 and $z = 2$ is a simple pole.

C is $|z-i| = 2$.

When $z = -1, |z-i| = |-1-i| = \sqrt{1+1} = \sqrt{2} < 2$.

$\Rightarrow z = -1$ lies inside C.

When $z = 2, |z-i| = |2-i| = \sqrt{4+1} = \sqrt{5} > 2$.

$\therefore z = 2$ lies outside C.

$\therefore$ By Cauchy's residue theorem $\int_C f(z)dz = 2\pi i R(-1)$.

Since $z = -1$ is a pole of order 2,

$$R(-1) = \frac{1}{(2-1)!}\lim_{z\to -1}\frac{d}{dz}((z+1)^2 f(z)) = \lim_{z\to -1}\frac{d}{dz}\Big((z+1)^2\frac{z-1}{(z+1)^2(z-2)}\Big)$$
$$= \lim_{z\to -1}\frac{d}{dz}\Big(\frac{z-1}{z-2}\Big) = \lim_{z\to -1}\frac{z-2-(z-1)}{(z-2)^2} = \frac{-3+2}{9} = \frac{-1}{9}.$$

$$\text{Hence, } \int_C \frac{z-1}{(z+1)^2(z-2)}dz = 2\pi i R(-1) = 2\pi i\Big(\frac{-1}{9}\Big) = -\frac{2\pi i}{9}.$$

Example 4.82. Evaluate $\oint_C \frac{z-3}{z^2+2z+5}dz$ where C is the circle (i) $|z|=1$, (ii) $|z+1-i|=2$ (iii) $|z+1+i|=2$.

Solution. Let $f(z)=\frac{z-3}{z^2+2z+5}$.

The singularities are given by

$$z^2+2z+5=0$$

$$(z+1)^2+5-1=0$$

$$(z+1)^2+4=0$$

$$(z+1)^2=-4$$

$$z+1=\pm 2i$$

$$z=-1\pm 2i.$$

The singular points are $z=-1+2i$ and $z=-1-2i$.

(i) Consider C as $|z|=1$.

When $z=-1+2i$, $|z|=|-1+2i|=\sqrt{1+4}=\sqrt{5}>1$.

$\therefore -1+2i$ lie outside C.

When $z=-1-2i$, $|z|=|-1-2i|=\sqrt{1+4}=\sqrt{5}>1$.

$\therefore -1-2i$ lies outside C.

$\therefore f(z)$ is analytic inside C.

$\therefore \int_C f(z)dz=0$.

(ii) C is $|z+1-i|=2$.

When $z=-1+2i$, $|z|=|-1+2i+1-i|=|i|=1<2$.

$\therefore -1+2i$ lies inside C.

When $z=-1-2i$, $|z|=|-1-2i+1-i|=|-3i|=3>2$.

$\therefore -1-2i$ lie outside C.

$$\begin{aligned} R(-1+2i) &= \lim_{z\to(-1+2i)}(z+1-2i)f(z) \\ &= \lim_{z\to(-1+2i)}(z+1-2i)\frac{z-3}{(z+1+2i)(z+1-2i)} \end{aligned}$$

$$= \lim_{z \to (-1+2i)} \frac{z-3}{(z+1+2i)}$$
$$= \frac{-1+2i-3}{-1+2i+1+2i} = \frac{-4+2i}{4i}$$
$$= \frac{1}{4}(-4+2i)(-i)$$
$$= \frac{1}{4}(4i+2) = \frac{1+2i}{2} = \frac{1}{2} + i.$$
$$\therefore \int_C f(z)dz = 2\pi i R(-1+2i) = 2\pi i \left(\frac{1}{2} + i\right) = \pi(i-2).$$

(iii) C is $|z+1+i| = 2$

When $z = -1+2i$, $|z+1+i| = |-1+2i+1+i| = |3i| = 3 > 2$.

$\therefore z = -1+2i$ lies outside C.

When $z = -1-2i$, $|z+1+i| = |-1-2i+1+i| = |-i| = 1 < 2$.

$\therefore z = (-1-2i)$ lies inside C.

$$R(-1-2i) = \lim_{z \to -1-2i} (z+1+2i) f(z)$$
$$= \lim_{z \to -1-2i} (z+1+2i) \frac{z-3}{(z+1+2i)(z+1-2i)}$$
$$= \lim_{z \to -1-2i} \frac{z-3}{z+1-2i} = \frac{-1-2i-3}{-1-2i+1-2i}$$
$$= \frac{-4-2i}{-4i} = \frac{4+2i}{4i}.$$
$$\therefore \oint_C f(z)dz = 2\pi i R(-1-2i)$$
$$= 2\pi i \left(\frac{4+2i}{4i}\right) = \pi(2+i).$$

Example 4.83. Evaluate $\int_C \frac{e^z dz}{(z^2+\pi^2)^2}$ where C is the circle $|z| = 4$ using Cauchy's residue theorem. [May 2006]

Solution. Let $f(z) = \frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z+\pi i)^2(z-\pi i)^2}$.

The poles are $z = \pi i$ and $z = -\pi i$ both are of order 2.

C is $|z| = 4$.

—

When $z = \pi i, |z| = |\pi i| = \pi < 4$.

When $z = -\pi i, |z| = |-\pi i| = \pi < 4$.

$\therefore$ Both the poles lie inside C.

$\therefore$ By Cauchy's residue theorem $\int_C f(z)dz = 2\pi i[R(\pi i) + R(-\pi i)]$

$z = \pi i$ is a pole of order 2.

$$R(\pi i) = \frac{1}{1!}\lim_{z\to\pi i}\frac{d}{dz}(z-\pi i)^2\frac{e^z}{(z+\pi i)^2(z-\pi i)^2} = \lim_{z\to\pi i}\frac{d}{dz}\left(\frac{e^z}{(z+\pi i)^2}\right)$$
$$= \lim_{z\to\pi i}\frac{(z+\pi i)^2 e^z - e^z 2(z+\pi i)}{(z+\pi i)^4}$$
$$= \frac{(2\pi i)^2 e^{\pi i} - e^{\pi i}4\pi i}{(2\pi i)^4} = \frac{e^{\pi i}(-4\pi^2 - 4\pi i)}{16\pi^4}$$
$$= \frac{-4\pi e^{\pi i}(\pi + i)}{16\pi^4} = -\frac{(\cos\pi + i\sin\pi)(\pi + i)}{4\pi^3}$$
$$R(\pi i) = \frac{\pi + i}{4\pi^3}.$$

Changing i to $-i$

$$R(-\pi i) = \frac{\pi - i}{4\pi^3}$$
$$\therefore \int_C f(z)dz = 2\pi i\left(\frac{\pi + i}{4\pi^3} + \frac{\pi - i}{4\pi^3}\right)$$
$$= \frac{2\pi i}{4\pi^3}2\pi = \frac{i}{\pi}.$$

Example 4.84. Evaluate $\oint_C \frac{e^z}{\cos\pi z}dz$ where C is the unit circle $|z| = 1$.

Solution. Let $f(z) = \frac{e^z}{\cos\pi z}$.

The singular points are given by

$\cos\pi z = 0$

i.e., $\pi z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \cdots$

i.e., $z = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots$

Among all the singular points, $z = \frac{1}{2}$ and $z = -\frac{1}{2}$ lie inside $|z| = 1$.

$z = \frac{1}{2}$ is a simple pole.

$$R\left(\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}} \left(z - \frac{1}{2}\right) f(z)$$

$$= \lim_{z\to\frac{1}{2}} \left(z - \frac{1}{2}\right) \frac{e^z}{\cos \pi z} \quad [\tfrac{0}{0} \text{ form}]$$

$$= \lim_{z\to\frac{1}{2}} \frac{\left(z - \frac{1}{2}\right) e^z + e^z}{-\pi \sin \pi z} \quad \text{[By L'Hospital rule]}$$

$$= \frac{e^{\frac{1}{2}}}{-\pi \sin \frac{\pi}{2}} = -\frac{\sqrt{e}}{\pi}.$$

$$R\left(-\frac{1}{2}\right) = \lim_{z\to-\frac{1}{2}} \left(z + \frac{1}{2}\right) f(z)$$

$$= \lim_{z\to-\frac{1}{2}} \left(z + \frac{1}{2}\right) \frac{e^z}{\cos \pi z} \quad [\tfrac{0}{0} \text{ form}]$$

$$= \lim_{z\to-\frac{1}{2}} \frac{\left(z + \frac{1}{2}\right) e^z + e^z}{-\pi \sin \pi z}$$

$$= \frac{e^{-\frac{1}{2}}}{-\pi \sin\left(-\frac{\pi}{2}\right)} = \frac{1}{\pi \sqrt{e}}.$$

$$\therefore \oint_C f(z)dz = 2\pi i \times \text{Sum of the residues.}$$

$$= 2\pi i \left[\frac{-\sqrt{e}}{\pi} + \frac{1}{\pi \sqrt{e}}\right]$$

$$= 2i \left[\frac{1}{\sqrt{e}} - \sqrt{e}\right] = 2i \left[e^{-\frac{1}{2}} - e^{\frac{1}{2}}\right]$$

$$= -4i \left[\frac{e^{\frac{1}{2}} - e^{-\frac{1}{2}}}{2}\right] = -4i \sin h \left(\frac{1}{2}\right).$$

Example 4.85. Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}$ where C is $|z| = 4$.

[Dec 2011, May 2009]

Solution. $z = 1$ is a pole of order 2 and $z = 2$ is a simple pole.

Let $f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}$

C is $|z| = 4$.

When $z = 1, |z| = |1| = 1 < 4$.

$\Rightarrow\ z = 1$ lies inside C.

When $z = 2, |z| = |2| = 2 < 4$.

$\Rightarrow\ z = 2$ also lies inside C.

$\therefore$ By Cauchy's residue theorem, $\displaystyle\int_C f(z)dz = 2\pi i[R(1) + R(2)]$.

$z = 1$ is a pole of order 2.

$$\therefore R(1) = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\{(z-1)^2 f(z)\}$$
$$= \lim_{z\to 1}\frac{d}{dz}\left\{\frac{\sin\pi z^2 + \cos\pi z^2}{z-2}\right\}$$
$$= \lim_{z\to 1}\frac{(z-2)\{\cos\pi z^2(\pi 2z) - \sin\pi z^2(\pi 2z)\} - (\sin\pi z^2 + \cos\pi z^2)}{(z-2)^2}$$
$$= \frac{-1(-2)\pi + 1}{1} = 2\pi + 1.$$

$z = 2$ is a simple pole.

$$\therefore R(2) = \lim_{z\to 2}(z-2)f(z)$$
$$= \lim_{z\to 2}\frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2} = 1.$$

$$\therefore \int_C f(z)dz = 2\pi i\big[R(1) + R(2)\big] = 2\pi i[2\pi + 1 + 1] = 2\pi i[2\pi + 2] = 4\pi i(\pi + 1).$$

Example 4.86. Evaluate $\displaystyle\int_C z^2 e^{\frac{1}{z}}dz$ where C is the unit circle. [May 2001]

Solution. $f(z) = z^2 e^{\frac{1}{z}} = z^2\left(1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots\right)$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6}z^{-1} + \cdots.$$

Since the principal part contains infinite number of terms, $z = 0$ is an essential singularity.

$\therefore\ R(0) =$ Coeff. of $z^{-1} = \frac{1}{6}$.

By Cauchy's residue theorem $\displaystyle\int_C f(z)dz = 2\pi i R(0) = 2\pi i\frac{1}{6} = \frac{\pi i}{3}$.

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Example 4.87. Use Cauchy's residue theorem to evaluate $\int_C \frac{3z^2+z-1}{(z^2-1)(z-3)}dz$ where C is the circle $|z| = 2$. [May 2007]

Solution. Let $f(z) = \frac{3z^2+z-1}{(z^2-1)(z-3)}$.

The poles are given by $(z^2-1)(z-3) = 0$

$(z-1)(z+1)(z-3) = 0$

i.e., $z = 1, -1, 3$.

All the poles are simple poles.

C is $|z| = 2$.

When $z = -1, |z| = |-1| = 1 < 2$.

$\therefore z = -1$ lies inside C.

When $z = 1, |z| = |1| = 1 < 2$.

$\therefore z = 1$ lies inside C.

When $z = 3, |z| = |3| = 3 > 2$.

$\therefore z = 3$ lies outside C.

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i[R(-1) + R(1)]$

$z = -1$ is a simple pole.

$$\begin{aligned}
\therefore R(-1) &= \lim_{z\to-1}(z+1)f(z) \\
&= \lim_{z\to-1}(z+1)\frac{3z^2+z-1}{(z+1)(z-1)(z-3)} \\
&= \lim_{z\to-1}\frac{3z^2+z-1}{(z-1)(z-3)} = \frac{1}{8}.
\end{aligned}$$

$z = 1$ is a simple pole.

$$\begin{aligned}
\therefore R(1) &= \lim_{z\to1}(z-1)f(z) \\
&= \lim_{z\to1}(z-1)\frac{3z^2+z-1}{(z+1)(z-1)(z-3)} = \lim_{z\to1}\frac{3z^2+z-1}{(z+1)(z-3)} = -\frac{3}{4}.
\end{aligned}$$

$$\therefore \int_C f(z)dz = 2\pi i\left[\frac{1}{8} - \frac{3}{4}\right] = 2\pi i\left[\frac{-5}{8}\right] = -\frac{5\pi i}{4}.$$

—

Example 4.88. Evaluate $\oint\limits_C \tan z dz$ where C is the circle $|z| = 2$.

Solution. Let $f(z) = \tan z = \dfrac{\sin z}{\cos z}$.

The poles are given by $\cos z = 0$.

$\therefore z = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \cdots$

Among the poles $z = \dfrac{\pi}{2}$ and $z = -\dfrac{\pi}{2}$ lie inside C and all others lie outside C.

$$\begin{aligned}
\text{Now, } R\left(\frac{\pi}{2}\right) &= \lim_{z\to\frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) f(z) \\
&= \lim_{z\to\frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) \frac{\sin z}{\cos z} \quad \left(\frac{0}{0} \text{ form}\right) \\
&= \lim_{z\to\frac{\pi}{2}} \frac{\left(z - \frac{\pi}{2}\right)\cos z + \sin z}{-\sin z} \quad \text{[By L'Hospital rule]} \\
&= \frac{\sin\frac{\pi}{2}}{-\sin\frac{\pi}{2}} = -1.
\end{aligned}$$

$$\begin{aligned}
R\left(-\frac{\pi}{2}\right) &= \lim_{z\to-\frac{\pi}{2}} \left(z + \frac{\pi}{2}\right) \frac{\sin z}{\cos z} \quad \left(\frac{0}{0} \text{ form}\right) \\
&= \lim_{z\to-\frac{\pi}{2}} \frac{\left(z + \frac{\pi}{2}\right)\cos z + \sin z}{-\cos z} = \frac{\sin\left(-\frac{\pi}{2}\right)}{-\cos\left(-\frac{\pi}{2}\right)} = -1.
\end{aligned}$$

$$\therefore \oint\limits_C f(z)dz = 2\pi i \times \text{Sum of the residues} = 2\pi i[-1 - 1] = -4\pi i.$$

Exercise 4 E

1. Find the residue of $f(z) = \dfrac{1}{z^2 e^z}$ at is poles. [Nov 2010]

2. Find the residue of $f(z) = \dfrac{z^2}{(z-1)^2(z-2)}$ at each of the poles. [May 2010]

3. Find the residue of $f(z) = \dfrac{1}{(z^2+1)^2}$ about each singularity. [Apr 2009]

4. Find the residue of $f(z) = \dfrac{z+1}{z^2(z-2)}$ at each of its poles. [May 2010]

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5. Find the residue of $f(z) = \frac{\tan z}{z}$ at each of its poles inside $|z| = 2$. [Dec 2010]

6. Evaluate $\int_C \frac{2z-1}{z(z+1)(z-3)}dz$ where C is the circle $|z| = 2$, using Cauchy's residue theorem. [May 2011]

7. Evaluate $\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)}dz$ where C is the circle $|z| = 3$. [Dec 2011]

8. Evaluate $\int_C \frac{2\sec z}{(1-z^2)}dz$ where C is the ellipse $4x^2 + 9y^2 = 9$. [Dec 2010]

9. Evaluate $\int_C \frac{z^2+4}{(z-1)(z^2-1)}dz$ where C is $|z-1| = 3$. [Apr 2005]

10. Evaluate $\int_C \frac{4z^2-4z+1}{(z-2)(z^2+4)}dz$ where C is the circle $|z| = 1$.

11. Evaluate $\int_C \frac{12z-7}{(z^2-1)(2z+3)}dz$ where C is $|z-i| = \sqrt{3}$.

12. Evaluate $\int_C \frac{\tan \frac{z}{2}}{(z-1-i)^2}dz$ where C is the boundary of the square whose sides are the lines $x = \pm 2$ and $y = \pm 2$.

13. Evaluate $\int_C \frac{z-3}{z^2+2z+5}dz$ where C is $|z+1-i| = 2$.

14. Evaluate $\int_C \frac{1}{(z^2+1)(z^2-4)}dz$ where C is $|z| = \frac{3}{2}$.

15. Evaluate $\int_C \frac{e^{2z}}{(z+1)^3}dz$ where C is $|z| = 2$.

16. Evaluate $\int_C \frac{e^{-z}}{(z-1)^4}dz$ where C is $|z| = 2$.

—

17. Evaluate $\int\limits_C \frac{1}{z^3(z+4)}dz$ where C is $|z| = 2$.

18. Evaluate $\int\limits_C \frac{z+1}{(z-1)^2(z+2)}dz$ where C is $|z| = \frac{3}{2}$.

19. Evaluate $\int\limits_C \frac{3z^2+5z-1}{(z-1)^2}dz$ where C is $|z| = 2$.

20. Evaluate $\int\limits_C \frac{z-1}{(z+1)^2(z-2)}dz$ where C is the circle $|z-i| = 2$.

21. Evaluate $\int\limits_C \frac{1}{(z^2+4)^2}dz$ where C is the circle $|z-i| = 2$ using residue theorem.

22. Evaluate $\int\limits_C \frac{z-2}{z^2-z}dz$ where C is the circle $x^2+y^2 = 4$ using residue theorem.

23. Evaluate $\int\limits_C \frac{4-3z}{z(z-1)(z-2)}dz$ where C is the circle $|z| = \frac{3}{2}$. [Jun 2010]

24. Evaluate $\int\limits_C \frac{z-2}{z(z-1)}dz$ where C is $|z| = 2$.

25. Evaluate $\int\limits_C \frac{e^z}{(z+1)^2}dz$ around the circle $C : |z-1| = 3$.

4.7 Evaluation of real integrals using residue theorem

Type 1. Real definite integrals of the form $\int_0^{2\pi} F(\sin\theta, \cos\theta)d\theta$ where $F(\sin\theta, \cos\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$ and is finite on the interval of integration.

Working rule.

$$\text{Put } z = e^{i\theta} \Rightarrow \frac{1}{z} = e^{-i\theta}$$

$$z = \cos\theta + i\sin\theta, \frac{1}{z} = \cos\theta - i\sin\theta$$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

$\frac{dz}{d\theta} = ie^{i\theta} = iz \Rightarrow d\theta = \frac{dz}{iz}$.

As θ varies from 0 to 2π, z moves on the unit circle $|z| = 1$.

$\therefore$ The contour C is the unit circle $|z| = 1$.

$$\int_0^{2\pi} F(\sin\theta, \cos\theta) d\theta = \int_C f(z)\frac{dz}{iz} \text{ where } C \text{ is } |z| = 1.$$

The integral can be evaluated using Cauchy's residue theorem.

Worked Examples

Example 4.89. Evaluate $\int_0^{2\pi} \frac{1}{2+\cos\theta} d\theta$ by contour integration.

[May 2011, Jun 2010, Dec 2010, May 2001]

Solution. Let $z = e^{i\theta}$. Then $d\theta = \frac{dz}{iz}$.

Also, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$.

As θ varies from 0 to $2\pi, z$ moves along the unit circle $|z| = 1$.

$\therefore C$ is $|z| = 1$.

$$\begin{aligned}
\text{Now, } \int_0^{2\pi} \frac{1}{2+\cos\theta} d\theta &= \int_C \frac{1}{2 + \frac{1}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz} = \int_C \frac{2}{4 + z + \frac{1}{z}} \frac{dz}{iz} \\
&= \frac{2}{i}\int_C \frac{z}{4z + z^2 + 1}\frac{dz}{z} = \frac{2}{i}\int_C \frac{1}{(z+2)^2 + 1 - 4} dz \\
&= \frac{2}{i}\int_C \frac{1}{(z+2)^2 - (\sqrt{3})^2} dz \\
&= \frac{2}{i}\int_C \frac{1}{(z+2+\sqrt{3})(z+2-\sqrt{3})} dz = \frac{2}{i}\int_C f(z)dz
\end{aligned}$$

where $f(z) = \frac{1}{(z+2+\sqrt{3})(z+2-\sqrt{3})}$.

The simple poles are $-(2+\sqrt{3})$ and $-(2-\sqrt{3})$. C is $|z| = 1$.

When $z = -(2+\sqrt{3}), |z| = |-(2+\sqrt{3})| = 2+\sqrt{3} > 1$.

$\therefore z = -(2+\sqrt{3})$ lies outside C.

—

When $z = -(2 - \sqrt{3})$, $|z| = |\sqrt{3} - 2| < 1$.

$\therefore z = -(2 - \sqrt{3})$ lies inside C.

By Cauchy's residue theorem, $\displaystyle\int_C f(z)dz = 2\pi i R(-(2 - \sqrt{3}))$.

$$\text{Now, } R(-(2-\sqrt{3})) = \lim_{z\to-(2-\sqrt{3})} (z+2-\sqrt{3})\frac{1}{(z+2+\sqrt{3})(z+2-\sqrt{3})}$$

$$= \frac{1}{(-2+\sqrt{3}+2+\sqrt{3})} = \frac{1}{2\sqrt{3}}$$

$$\therefore \int_0^{2\pi} \frac{1}{2+\cos\theta}d\theta = \frac{2}{i}2\pi \times R(-(2-\sqrt{3})) = 4\pi\frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

Example 4.90. Evaluate $\displaystyle\int_0^{2\pi} \frac{1}{13+5\sin\theta}d\theta$ by using contour integration.

[Apr 2010]

Solution. Let $z = e^{i\theta}$, then $d\theta = \dfrac{dz}{iz}$.

Also, $\sin\theta = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right)$.

As θ varies from 0 to 2π, z moves, on the unit circle $|z| = 1$.

$$\therefore \int_0^{2\pi} \frac{1}{13+5\sin\theta}d\theta = \int_C \frac{1}{13+\frac{5}{2i}\left(z-\frac{1}{z}\right)}\frac{dz}{iz}$$

$$= \int_C \frac{2i}{26i+5\left(z-\frac{1}{z}\right)}\frac{dz}{iz}$$

$$= 2\int_C \frac{1}{26iz+5z^2-5}dz$$

$$= 2\int_C \frac{1}{5\left(z^2+\frac{26i}{5}z-1\right)}dz$$

$$= \frac{2}{5}\int_C \frac{1}{\left(z+\frac{13}{5}i\right)^2-1+\frac{169}{25}}dz$$

$$= \frac{2}{5}\int_C \frac{1}{\left(z+\frac{13}{5}i\right)^2+\frac{144}{25}}dz$$

$$= \frac{2}{5}\int_C \frac{1}{\left(z+\frac{13}{5}i\right)^2+\left(\frac{12}{5}\right)^2}dz$$

$$= \frac{2}{5}\int_C \frac{1}{\left(z+\frac{13}{5}i\right)^2-\left(\frac{i12}{5}\right)^2}dz$$
$$= \frac{2}{5}\int_C \frac{1}{\left(z+\frac{13i}{5}+\frac{12i}{5}\right)\left(z+\frac{13i}{5}-\frac{12i}{5}\right)}dz$$
$$= \frac{2}{5}\int_C \frac{1}{(z+5i)\left(z+\frac{i}{5}\right)}dz$$

Here $f(z) = \dfrac{1}{(z+5i)\left(z+\frac{i}{5}\right)}$.

The poles are $z = -5i$ and $z = \dfrac{-i}{5}$, which are simple poles.

Now, C is $|z| = 1$.

When $z = -5i, |z| = |-5i| = 5 > 1$.

$\therefore\ z = -5i$ lies outside C.

When $z = \dfrac{-i}{5}, |z| = \left|\dfrac{-i}{5}\right| = \dfrac{1}{5} < 1$.

$\therefore\ z = \dfrac{i}{5}$ lies inside C.

By Cauchy's residue theorem, $\displaystyle\int_C f(z)dz = 2\pi i R\left(\frac{-i}{5}\right)$.

Now, $R\left(\dfrac{-i}{5}\right) = \displaystyle\lim_{z\to\frac{-i}{5}}\left(z+\frac{i}{5}\right)\frac{1}{(z+5i)\left(z+\frac{i}{5}\right)} = \frac{1}{-i+5i} = \frac{5}{24i}$.

$$\therefore \int_0^{2\pi}\frac{1}{13+5\sin\theta}d\theta = \frac{2}{5}2\pi i R\left(\frac{-i}{5}\right) = \frac{4\pi i}{5}\frac{5}{24i} = \frac{\pi}{6}.$$

Example 4.91. Evaluate $\displaystyle\int_0^{2\pi}\frac{\cos 2\theta}{5+4\cos\theta}d\theta$ using contour integration. [Dec 2013]

Solution. Let $z = e^{i\theta}$, then $d\theta = \dfrac{dz}{iz}$ and $\cos\theta = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$.

As θ varies from 0 to 2π, z moves along $|z| = 1$.

$\therefore C$ is $|z| = 1$.

Now $z^2 = e^{i2\theta} = \cos 2\theta + i\sin 2\theta$.

$\therefore \cos 2\theta = \text{R.P.}e^{i2\theta}$.

$$\therefore \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos 2\theta}d\theta = \text{R.P}\int_C \frac{z^2}{5+4\times\frac{1}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{iz}$$

$$= \text{R.P}\left(\frac{1}{i}\right)\int_C \frac{z}{5+2\left(\frac{z^2+1}{z}\right)}dz$$

$$= \text{R.P}(-i)\int_C \frac{z^2}{2z^2+5z+2}dz$$

$$= \text{R.P}\left(\frac{-i}{2}\right)\int_C \frac{z^2}{z^2+\frac{5}{2}z+1}dz$$

$$= \text{R.P}\left(\frac{-i}{2}\right)\int_C \frac{z^2}{\left(z+\frac{5}{4}\right)^2+1-\frac{25}{16}}dz$$

$$= \text{R.P}\left(\frac{-i}{2}\right)\int_C \frac{z^2}{\left(z+\frac{5}{4}\right)^2-\frac{9}{16}}dz$$

$$= \text{R.P}\left(\frac{-i}{2}\right)\int_C \frac{z^2}{\left(z+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2}dz$$

$$= \text{R.P}\left(\frac{-i}{2}\right)\int_C \frac{z^2}{\left(z+\frac{5}{4}-\frac{3}{4}\right)\left(z+\frac{5}{4}+\frac{3}{4}\right)}dz$$

$$= \text{R.P}\left(\frac{-i}{2}\right)\int_C \frac{z^2}{\left(z+\frac{1}{2}\right)(z+2)}dz$$

$$= \text{R.P}\left(\frac{-i}{2}\right)\int_C f(z)dz \quad \text{where } f(z) = \frac{z^2}{\left(z+\frac{1}{2}\right)(z+2)}$$

The singular points are $z = -\dfrac{1}{2}$ and $z = -2$.

Now C is $|z| = 1$.

When $z = -\dfrac{1}{2}$, $|z| = |-\dfrac{1}{2}| = \dfrac{1}{2} < 1$.

$\therefore z = -\frac{1}{2}$ lies inside C.

When $z = -2$, $|z| = |-2| = 2 > 1$.

$\therefore z = -2$ lies outside C.

$z = -\frac{1}{2}$ is a simple pole.

$$R\left(-\frac{1}{2}\right) = \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) f(z) = \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) \frac{z^2}{\left(z + \frac{1}{2}\right)(z+2)}$$

$$= \lim_{z \to -\frac{1}{2}} \frac{z^2}{z+2} = \frac{\frac{1}{4}}{-\frac{1}{2}+2} = \frac{\frac{1}{4}}{\frac{3}{2}} = \frac{1}{4} \times \frac{2}{3} = \frac{1}{6}.$$

By Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i \times R\left(-\frac{1}{2}\right) = 2\pi i \times \frac{1}{6} = \frac{\pi i}{3}.$$

$$\therefore \int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta = \text{R.P}\left(-\frac{i}{2}\right) \times \left(\frac{\pi i}{3}\right) = \text{R.P}\left(\frac{\pi}{6}\right) = \frac{\pi}{6}.$$

Example 4.92. Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta$ using contour integration.

[Jun 2013, May 2005]

Solution. Let $z = e^{i\theta}$, then $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$.

As θ varies from 0 to 2π, z moves along $|z| = 1$.

$\therefore C$ is $|z| = 1$.

Now $z^3 = e^{i3\theta} = \cos 3\theta + i \sin 3\theta$.

$\therefore \cos 3\theta =$ R.P of z^3.

$$\therefore \int_C \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta = R.P \int_C \frac{z^3}{5 - 4\frac{1}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz} = R.P \int_C \frac{z^2}{5 - 2z - \frac{2}{z}} \frac{dz}{i}$$

$$= R.P \frac{1}{i} \int_C \frac{z^3}{5z - 2z^2 - 2} dz = R.P\left(\frac{-1}{i}\right) \int_C \frac{z^3}{2z^2 - 5z + 2} dz$$

$$= R.P\left(\frac{-1}{i}\right) \int_C \frac{z^3}{2\left(z^2 - \frac{5}{2}z + 1\right)} dz$$

$$= R.P\left(\frac{i}{2}\right) \int_C \frac{z^3}{\left(z - \frac{5}{4}\right)^2 + 1 - \frac{25}{16}} dz$$

$$= R.P\left(\frac{i}{2}\right) \int_C \frac{z^3}{\left(z - \frac{5}{4}\right)^2 - \left(\frac{3}{4}\right)^2} dz$$

$$= R.P\left(\frac{i}{2}\right)\int_C \frac{z^3}{\left(z-\frac{5}{4}+\frac{3}{4}\right)\left(z-\frac{5}{4}-\frac{3}{4}\right)}dz$$

$$= R.P\left(\frac{i}{2}\right)\int_C \frac{z^3}{\left(z-\frac{1}{2}\right)(z-2)}dz$$

$$= R.P\left(\frac{i}{2}\right)\int_C f(z)dz \text{ where } f(z) = \frac{z^3}{\left(z-\frac{1}{2}\right)(z-2)}.$$

Singular points are $\frac{1}{2}$ and 2, which are simple poles.
Now, C is $|z| = 1$.
When $z = \frac{1}{2}$, $|z| = \frac{1}{2} < 1 \implies z = \frac{1}{2}$ lies inside C.
When $z = 2$, $|z| = 2 > 1 \implies z = 2$ lies outside C.
By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i R\left(\frac{1}{2}\right)$.

$$\text{Now } R\left(\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)f(z)$$

$$R\left(\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z^3}{\left(z-\frac{1}{2}\right)(z-2)} = \frac{\frac{1}{8}}{\frac{-3}{2}} = -\frac{1}{8}\frac{2}{3} = -\frac{1}{12}.$$

$$\therefore \int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}d\theta = R.P\left(\frac{i}{2}\right)2\pi i R\left(\frac{1}{2}\right) = -\pi\left(\frac{-1}{12}\right) = \frac{\pi}{12}.$$

Example 4.93. Show that $\int_0^{2\pi}\frac{\cos 2\theta}{1-2a\cos\theta+a^2}d\theta = \frac{2\pi a^2}{1-a^2} \quad (a^2 < 1)$.

Solution. Let $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$.

$2\cos\theta = z + \frac{1}{z}$.

$\cos 2\theta = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)$.

As θ varies from 0 to 2π, z varies along the unit circle C, $|z| = 1$.

$$\therefore \int_0^{2\pi}\frac{\cos 2\theta}{1-2a\cos\theta+a^2}d\theta = \int_C \frac{\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)}{1-a\left(z+\frac{1}{z}\right)+a^2}\frac{dz}{iz}$$

$$= \frac{1}{2i}\int_C \frac{z^4+1}{z^2\left(1-\frac{a(z^2+1)}{z}+a^2\right)}\frac{dz}{z}$$

$$= \frac{1}{2i}\int_C \frac{z(z^4+1)}{z^2(z-az^2-a+a^2z)}\frac{dz}{z}$$

$$= \frac{1}{2i}\int_C \frac{z^4+1}{z^2(\{(z-a)-az(z-a)\})}dz$$

$$= \frac{1}{2i}\int_C \frac{z^4+1}{z^2(z-a)(1-az)}dz$$

$$= \frac{1}{2i}\int_C f(z)dz \quad \text{where } f(z) = \frac{z^4+1}{z^2(z-a)(1-az)}.$$

The singular points are $z = 0, z = a, z = \frac{1}{a}$.

C is $|z| = 1$.

When $z = 0, |z| = |0| = 0 < 1$.

$\therefore z = 0$ lie inside C.

When $z = a, |z| = |a| = a < 1 \quad [\because a^2 < 1 \Rightarrow a < 1]$.

$\therefore z = a$ lies inside C

When $z = \frac{1}{a}, |z| = |\frac{1}{a}| = \frac{1}{a} > 1 \quad [\because a < 1 \Rightarrow \frac{1}{a} > 1]$.

$\therefore z = \frac{1}{a}$ lies outside C.

By Cauchy's residue theorem,

$\int_C f(z)dz = 2\pi i \times$ Sum of the residues.

Now, $z = 0$ is a pole of order 2.

$$\therefore R(0) = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left((z-0)^2 f(z)\right)$$

$$= \lim_{z\to 0}\frac{d}{dz}\left(z^2\frac{z^4+1}{z^2(z-a)(1-az)}\right)$$

$$= \lim_{z\to 0}\frac{d}{dz}\left[\frac{z^4+1}{(z-a)(1-az)}\right]$$

$$= \lim_{z\to 0}\frac{(z-a)(1-az)4z^3-(z^4+1)\{(z-a)(-a)+(1-az)\cdot 1\}}{[(z-a)(1-az)]^2}$$

$$= \frac{0 - 1\{a^2 + 1\}}{a^2} = -\frac{a^2 + 1}{a^2}$$

$z = a$ is a simple pole.

$$\begin{aligned}
R(a) &= \lim_{z \to a} (z - a) f(z) \\
&= \lim_{z \to a} (z - a) \frac{z^4 + 1}{z^2(z - a)(1 - az)} \\
&= \lim_{z \to a} \frac{z^4 + 1}{z^2(1 - az)} \\
&= \frac{a^4 + 1}{a^2(1 - a^2)}
\end{aligned}$$

$$\begin{aligned}
\therefore \int_C f(z) dz &= 2\pi i[R(0) + R(a)] \\
&= 2\pi i \left[\frac{a^4 + 1}{a^2(1 - a^2)} - \frac{a^2 + 1}{a^2} \right] \\
&= 2\pi i \left[\frac{a^4 + 1 - (1 - a^2)(1 + a^2)}{a^2(1 - a^2)} \right] \\
&= 2\pi i \left[\frac{a^4 + 1 - (1 - a^4)}{a^2(1 - a^2)} \right] \\
&= 2\pi i \left[\frac{2a^4}{a^2(1 - a^2)} \right] \\
&= 2\pi i \frac{2a^2}{1 - a^2} = \frac{4\pi i a^2}{1 - a^2}
\end{aligned}$$

$$\therefore \int_0^{2\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} d\theta = \frac{1}{2i} \cdot \frac{4\pi i a^2}{1 - a^2} = \frac{2\pi a^2}{1 - a^2}.$$

Example 4.94. Evaluate $\int_0^{2\pi} \frac{1}{1 - 2p\sin\theta + p^2} d\theta$, $|p| < 1$. [May 2009]

Solution. Let $z = e^{i\theta}$, then $d\theta = \frac{dz}{iz}$.

Also, $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$.

As θ varies from 0 to 2π, z moves along the unit circle $|z| = 1$.

$$\text{Now, } \int_0^{2\pi} \frac{1}{1 - 2p\sin\theta + p^2} d\theta = \int_C \frac{1}{1 - 2p\frac{1}{2i}\left(z - \frac{1}{z}\right) + p^2} \frac{dz}{iz}$$

$$= -i\int_C \frac{1}{1-\frac{p}{i}\left(\frac{z^2-1}{z}\right)+p^2}\frac{dz}{z}$$

$$= -i\int_C \frac{iz}{iz-pz^2+p+p^2iz}\frac{dz}{z}$$

$$= -\int_C \frac{1}{pz^2-iz-p^2iz-p}dz$$

$$= -\int_C \frac{1}{pz^2-i(1+p^2)z-p}dz$$

$$= -\frac{1}{p}\int_C \frac{1}{z^2-i\left(\frac{1+p^2}{p}\right)z-1}dz$$

$$= -\frac{1}{p}\int_C \frac{1}{\left(z-i\left(\frac{p^2+1}{2p}\right)\right)^2-1+\left(\frac{p^2+1}{2p}\right)^2}dz$$

$$= -\frac{1}{p}\int_C \frac{1}{\left(z-i\left(\frac{p^2+1}{2p}\right)\right)^2+\frac{(1+p^2)^2-4p^2}{4p^2}}dz$$

$$= -\frac{1}{p}\int_C \frac{1}{\left(z-i\left(\frac{p^2+1}{2p}\right)\right)^2+\left(\frac{1-p^2}{2p}\right)^2}dz$$

$$= -\frac{1}{p}\int_C \frac{1}{\left(z-i\left(\frac{p^2+1}{2p}\right)\right)^2-\left(\frac{i(1-p^2)}{2p}\right)^2}dz$$

$$= -\frac{1}{p}\int_C \frac{1}{\left(z-i\left(\frac{p^2+1}{2p}\right)+i\left(\frac{1-p^2}{2p}\right)\right)\left(z-i\left(\frac{p^2+1}{2p}\right)-i\left(\frac{1-p^2}{2p}\right)\right)}dz$$

$$= -\frac{1}{p}\int_C \frac{1}{\left(z+\frac{i}{2p}\left(1-p^2-1-p^2\right)\right)\left(z-\frac{i}{2p}\left(1+p^2+1-p^2\right)\right)}dz$$

$$= -\frac{1}{p}\int_C \frac{1}{\left(z+\frac{i}{2p}(-2p^2)\right)\left(z-\frac{i}{2p}2\right)}dz$$

$$= -\frac{1}{p}\int_C \frac{1}{(z-ip)\left(z-\frac{i}{p}\right)}dz = -\frac{1}{p}\int_C f(z)dz$$

where $f(z) = \dfrac{1}{(z-ip)\left(z-\frac{i}{p}\right)}$

The simple poles are ip and $\dfrac{i}{p}$.

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C is $|z| = 1$.

When $z = ip, |z| = |ip| = |p| < 1$.

$\therefore\ z = ip$ lies inside C.

When $z = \dfrac{i}{p}, |z| = \left|\dfrac{i}{p}\right| = \dfrac{1}{|p|} > 1$.

$\therefore\ z = \frac{i}{p}$ lies outside C.

By Cauchy's residue theorem, $\displaystyle\int_C f(z)dz = 2\pi i R\big(ip\big)$.

$$\text{Now, } R(ip) = \lim_{z\to ip}(z-ip)\frac{1}{(z-ip)\big(z-\frac{i}{p}\big)}$$

$$= \frac{1}{ip-\frac{i}{p}} = \frac{1}{i\big(\frac{p^2-1}{p}\big)} = \frac{p}{i(p^2-1)}$$

$$\therefore \int_0^{2\pi}\frac{1}{1-2p\sin\theta+p^2}d\theta = -\frac{1}{p}2\pi i\times R(ip) = -\frac{1}{p}2\pi i\frac{p}{i(p^2-1)} = \frac{2\pi}{1-p^2}.$$

Example 4.95. Evaluate $\displaystyle\int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}d\theta,\ a > b > 0.$ [Dec 2012, May 1998]

Solution. Let, $\displaystyle I = \int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}d\theta = \int_0^{2\pi}\frac{1-\cos 2\theta}{2(a+b\cos\theta)}d\theta.$

Put $z = e^{i\theta}$ then $d\theta = \dfrac{dz}{iz}$.

Now, $z^2 = e^{i2\theta} = \cos 2\theta + i\sin 2\theta$.

$1 - z^2 = 1 - \cos 2\theta - i\sin 2\theta$.

$\therefore 1 - \cos 2\theta =$ R.P $(1 - z^2)$.

As θ varies from 0 to 2π, C varies along the unit circle $|z| = 1$.

$$\text{Hence, } I = R.P\int_C\frac{1-z^2}{2\Big(a+\frac{b}{2}\big(z+\frac{1}{z}\big)\Big)}\frac{dz}{iz} = R.P\frac{1}{2}\int_C\frac{1-z^2}{2a+b(z+\frac{1}{z})}\frac{dz}{iz}$$

$$= R.P\frac{1}{2i}\int_C\frac{z(1-z^2)}{(2az+bz^2+b)}\frac{dz}{z} = R.P\frac{1}{2i}\int_C\frac{1-z^2}{2az+bz^2+b}dz$$

$$= R.P\frac{1}{2i}\int_C\frac{1-z^2}{b(z^2+\frac{2a}{b}z+1)}dz = R.P\frac{1}{2bi}\int_C\frac{1-z^2}{(z+\frac{a}{b})^2+1-\frac{a^2}{b^2}}dz$$

$$= R.P\frac{1}{2bi}\int_C\frac{1-z^2}{(z+\frac{a}{b})^2-\frac{a^2-b^2}{b^2}}dz$$

—

$$= R.P \frac{1}{2bi} \int_C \frac{1 - z^2}{(z + \frac{a}{b})^2 - (\frac{\sqrt{a^2-b^2}}{b})^2} dz$$

$$= R.P \frac{1}{2bi} \int_C \frac{1 - z^2}{(z + \frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b})(z + \frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b})} dz$$

$$= R.P \frac{1}{2bi} \int_C \frac{1 - z^2}{(z - z_1)(z - z_2)} dz$$

$$= R.P \frac{1}{2bi} \int_C f(z) dz \text{ where } f(z) = \frac{1 - z^2}{(z - z_1)(z - z_2)} \text{ where}$$

$z_1 = -\dfrac{a}{b} - \dfrac{\sqrt{a^2 - b^2}}{b}$ and $z_2 = -\dfrac{a}{b} + \dfrac{\sqrt{a^2 - b^2}}{b}$.

z_1 and z_2 are simple poles.

When $z = z_1$, $|z| = |z_1| = \left| -\dfrac{a}{b} - \dfrac{\sqrt{a^2 - b^2}}{b} \right| = \dfrac{a}{b} + \dfrac{\sqrt{a^2 - b^2}}{b} > \dfrac{a}{b} > 1.$

As $a > b \Longrightarrow \dfrac{a}{b} > 1 \Longrightarrow \dfrac{a^2}{b^2} > 1, \; a^2 > b^2, \; a^2 - b^2 > 0.$

$\therefore$ z_1 lies outside C.

Also, $z_1 z_2 = 1$ (products of the roots)

Now, $z_2 = \dfrac{1}{z_1} \Longrightarrow |z_2| = \dfrac{1}{|z_1|} < 1.$

$\therefore$ z_2 lies inside C.

By Cauchy's residue theorem, $\displaystyle\int_C f(z) dz = 2\pi i R(z_2).$

Now, $R(z_2) = \lim\limits_{z \to z_2} (z - z_2) f(z)$

$$R\Big(-\frac{a}{b} + \frac{\sqrt{a^2 - b^2}}{b} \Big) = \lim_{z \to z_2} (z - z_2) \frac{1 - z^2}{(z - z_1)(z - z_2)} = \frac{1 - z_2^2}{z_2 - z_1}.$$

Now, $1 - z_2^2 = 1 - \Big(\dfrac{-a + \sqrt{a^2 - b^2}}{b} \Big)^2$

$$= \frac{b^2 - (a^2 + a^2 - b^2 - 2a\sqrt{a^2 - b^2})}{b^2}$$

$$= \frac{b^2 - a^2 - a^2 + b^2 + 2a\sqrt{a^2 - b^2}}{b^2}$$

$$= \frac{2b^2 - 2a^2 + 2a\sqrt{a^2 - b^2}}{b^2}$$

$$= \frac{2a\sqrt{a^2-b^2} - 2(a^2-b^2)}{b^2} = \frac{2\sqrt{a^2-b^2}(a-\sqrt{a^2-b^2})}{b^2}$$

$$z_2 - z_1 = \frac{2\sqrt{a^2-b^2}}{b}$$

$$\therefore R(z_2) = \frac{2\sqrt{a^2-b^2}(a-\sqrt{a^2-b^2})}{b^2 2\sqrt{a^2-b^2}} \times b = \frac{a-\sqrt{a^2-b^2}}{b}$$

$$\therefore I = \text{R.P}\ \frac{1}{2bi}\cdot 2\pi i\left(\frac{a-\sqrt{a^2-b^2}}{b}\right) = \frac{\pi}{b^2}(a-\sqrt{a^2-b^2})$$

Example 4.96. Evaluate $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}d\theta.$ [Dec 2011]

Solution. Let $Z = e^{i\theta}$. Then $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$.

As θ varies from 0 to 2π, z moves along $|z| = 1$.

By the properties of definite integrals,

$$\therefore \int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}d\theta$$

$$= \frac{1}{2}\int_C \frac{1+2\frac{1}{2}(z+\frac{1}{z})}{5+4\frac{1}{2}(z+\frac{1}{z})}\frac{dz}{iz}$$

$$= \frac{1}{2i}\int_C \frac{1+z+\frac{1}{z}}{5+2(z+\frac{1}{z})}\frac{dz}{z}$$

$$= \frac{1}{2i}\int_C \frac{(z^2+z+1)z}{z(2z^2+5z+2)}\cdot\frac{dz}{z}$$

$$= \frac{1}{2i}\int_C \frac{z^2+z+1}{z(2z^2+5z+2)}dz$$

$$= \frac{1}{2i}\int_C \frac{z^2+z+1}{2z(z^2+\frac{5}{2}z+1)}dz$$

$$= \frac{1}{4i}\int_C \frac{z^2+z+1}{z\left((z+\frac{5}{4})^2+1-\frac{25}{16}\right)}dz$$

$$= \frac{1}{4i}\int_C \frac{z^2+z+1}{z\left((z+\frac{5}{4})^2-(\frac{3}{4})^2\right)}dz$$

—

$$= \frac{1}{4i}\int_C \frac{z^2+z+1}{z\left(z+\frac{5}{4}+\frac{3}{4}\right)\left(z+\frac{5}{4}-\frac{3}{4}\right)}dz$$

$$= \frac{1}{4i}\int_C \frac{z^2+z+1}{z(z+2)\left(z+\frac{1}{2}\right)}dz$$

$$= \frac{1}{4i}\int_C f(z)dz, \text{ where } f(z) = \frac{z^2+z+1}{z(z+2)\left(z+\frac{1}{2}\right)}.$$

The simple poles are $z = 0, z = -2$ and $z = -\frac{1}{2}$.

C is $|z| = 1$.

When $z = 0, |z| = 0 < 1$.

$\therefore z = 0$ lies inside C.

When $z = -2, |z| = |-2| = 2 > 1$.

$\therefore z = -2$ lies outside C.

When $z = \frac{-1}{2}, |z| = \left|\frac{-1}{2}\right| = \frac{1}{2} < 1$.

$\therefore z = -\frac{1}{2}$ lies inside C.

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i[R(0) + R(-\frac{1}{2})]$.

$$\text{Now, } R(0) = \lim_{z\to 0}(z-0)f(z) = \lim_{z\to 0} z\frac{z^2+z+1}{z(z+2)\left(z+\frac{1}{2}\right)} = \frac{1}{1} = 1.$$

$$R(\frac{-1}{2}) = \lim_{z\to -\frac{1}{2}}(z+\frac{1}{2})f(z)$$

$$= \lim_{z\to -\frac{1}{2}}(z+\frac{1}{2})\frac{z^2+z+1}{z(z+2)\left(z+\frac{1}{2}\right)}$$

$$= \lim_{z\to -\frac{1}{2}}\frac{z^2+z+1}{z(z+2)} = \frac{\frac{1}{4}-\frac{1}{2}+1}{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}+2\right)} = \frac{3}{4}\left(\frac{-4}{3}\right) = -1.$$

$$\therefore \int_C f(z)dz = 2\pi i[-1+1] = 0.$$

$$\int_0^{\pi}\frac{1+2\cos\theta}{5+4\cos\theta}d\theta = \frac{1}{4i}0 = 0.$$

—

Evaluation of improper integrals of rational functions

We consider integrals of the type $\int_{-\infty}^{\infty} f(x)dx$. We define

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R_1\to\infty}\int_{-R_1}^{0} f(x)dx + \lim_{R_2\to\infty}\int_{0}^{R_2} f(x)dx \tag{1}$$

when both limits exists on the RHS.

The Cauchy principal value is defined by

$$PV\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx \tag{2}$$

if the limit exists.

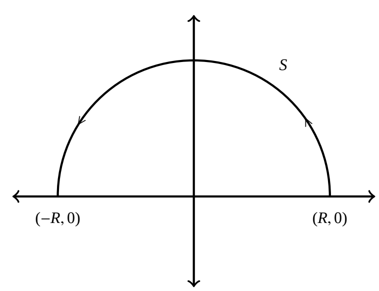
When $f(x)$ is even $\int_{-\infty}^{\infty} f(x)dx = 2\int_{0}^{\infty} f(x)dx$ or $\int_{0}^{\infty} f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx$.

Cauchy's lemma

If $f(z)$ is a continuous function such that $|zf(z)| \to 0$ as $|z| \to \infty$ on the upper semicircle $S : |z| = R$, then $\int_S f(z)dz \to 0$ as $R \to \infty$.

Type II. We consider integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx$ where $P(x)$ and $Q(x)$ are polynomials in x such that the degree of $Q(x)$ is atleast 2 more than the degree of $P(x)$ and $Q(x)$ does not vanish for any real x.

To evaluate this, consider $\int_C \frac{P(z)}{Q(z)}dz$ where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semicircle, $S : |z| = R$ for large R taken in the anticlockwise sense.



Now $f(z) = \frac{P(z)}{Q(z)}$. Then $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_S f(z)dz$ (1)

By Cauchy's residue theorem $\int_C f(z)dz = 2\pi i[$ sum of the residues $]$.

By Cauchy's lemma $\int_S f(z)dz \to 0$ as $R \to \infty$

—

$\therefore$ From (1) $\lim\limits_{R\to\infty} \int_{-R}^{R} f(x)dx = 2\pi i[$ sum of the residues].
i.e., $\int_{-\infty}^{\infty} f(x)dx = 2\pi i[$ sum of the residues].
$\Longrightarrow \displaystyle\int_{-\infty}^{\infty} \frac{p(x)}{Q(x)}dx = 2\pi i[$ sum of the residues].

Worked Examples

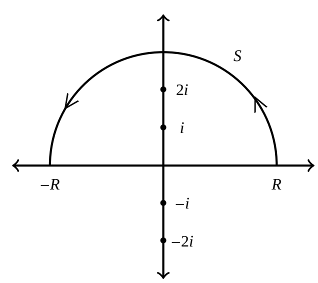
Example 4.97. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2dx}{(x^2+1)(x^2+4)}$ using contour integration.

[Dec 2010, May 2007]

Solution. Let $I = \displaystyle\int_{-\infty}^{\infty} \frac{x^2dx}{(x^2+1)(x^2+4)}$.
The integrand is of the form $\dfrac{P(x)}{Q(x)}$ where degree of $Q(x)$ is 2 more than that of $P(x)$ and $Q(x)$ does not vanish for any real x. Consider the integral $\displaystyle\int_C \frac{z^2dz}{(z^2+1)(z^2+4)}$ where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi circle $S : |z| = R$ taken in the anticlockwise sense.
Let $f(z) = \dfrac{z^2dz}{(z^2+1)(z^2+4)}$.
Now

$$\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_S f(z)dz. \qquad (1)$$

Evaluation of $\int_C f(z)dz$.
Poles are given by $(z^2+1)(z^2+4) = 0$.
i.e., $(z+i)(z-i)(z+2i)(z-2i) = 0$
i.e., $z = i, -i, 2i$ and $-2i$.
$z = i$ and $z = 2i$ lie inside C.



By Cauchy's residue theorem, $\displaystyle\int_C f(z)dz = 2\pi i[R(i) + R(2i)]$.

Now, $R(i) = \lim\limits_{z\to i}(z-i)f(z) = \lim\limits_{z\to i}(z-i)\dfrac{z^2}{(z+i)(z-i)(z^2+4)} = \dfrac{-1}{2i(3)} = -\dfrac{1}{6i}$

$$R(2i) = \lim_{z \to 2i} (z - 2i) \frac{z^2}{(z+2i)(z-2i)(z^2+1)} = \frac{4}{3(4i)} = \frac{1}{3i}.$$

$$\therefore \int_C f(z)dz = 2\pi i \Big(\frac{-1}{6i} + \frac{1}{3i} \Big) = \frac{2\pi i}{i} \Big(\frac{1}{3} + \frac{1}{6} \Big) = 2\pi \Big(\frac{2-1}{6} \Big) = 2\pi \frac{1}{6} = \frac{\pi}{3}.$$

$$(1) \Rightarrow \int_{-R}^{R} f(x)dx + \int_S f(z)dz = \frac{\pi}{3}.$$

Now, $\lim_{z\to\infty} z f(z) = \lim_{z\to\infty} \dfrac{z^3}{(z^2+1)(z^2+4)} = \lim_{z\to\infty} \dfrac{z^3}{z^4(1+\frac{4}{z^2})(1+\frac{1}{z^2})} = \dfrac{1}{\infty} = 0.$

$\therefore$ By Cauchy's lemma

$$\int_S f(z)dz = 0 \text{ as } R \to \infty$$

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \frac{\pi}{3} \Rightarrow \int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{3}.$$

Example 4.98. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3} dx.$ [Dec 2009]

Solution. Let $I = \displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3} dx.$

The integrand is of the form $\dfrac{P(x)}{Q(x)}$ where the degree of $Q(x)$ is at least 2 more than that of $P(x)$ and $Q(x)$ does not vanish for any real x.

Consider the integral $\displaystyle\int_C \frac{1}{(z^2+1)^3} dz$ where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi circle $S : |Z| = R$ taken in the anti clockwise sense.

Let $f(z) = \dfrac{1}{(z^2+1)^3}$.

$$\text{Now, } \int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_S f(z)dz \qquad (1)$$

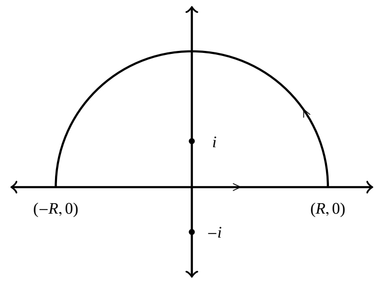
Let us evaluate $\int_C f(z)dz$.

The poles are given by $(z^2+1)^3 = 0$.

$(z+i)^3(z-i)^3 = 0$

i and $-i$ are poles of order 3. i will lie inside C and $-i$ lie outside C.

By Cauchy's residue theorem $\int\limits_C f(z)dz = 2\pi i R(i)$.



Now, $R(i) = \dfrac{1}{2!}\lim\limits_{z\to i}\dfrac{d^2}{dz^2}(z-i)^3 f(z)$

$$= \frac{1}{2!}\lim_{z\to i}\frac{d^2}{dz^2}\left[(z-i)^3\frac{1}{(z+i)^3(z-i)^3}\right]$$

$$= \frac{1}{2!}\lim_{z\to i}\frac{d^2}{dz^2}\left[(z+i)^{-3}\right]$$

$$= \frac{1}{2!}\lim_{z\to i}\frac{d}{dz}\left[(-3)(z+i)^{-4}\right]$$

$$= \frac{1}{2!}\lim_{z\to i}(-3)(-4)(z+i)^{-5} = \frac{1}{2}12(2i)^{-5} = \frac{6}{2^5 i^5} = \frac{6}{32i} = \frac{3}{16i}$$

$$\therefore \int\limits_C f(z)dz = 2\pi i \times \frac{3}{16i} = \frac{3\pi}{8}.$$

$$\text{Now (1)} \Longrightarrow \int\limits_{-R}^{R} f(x)dx + \int\limits_S f(z)dz = \frac{3\pi}{8}. \qquad (2)$$

Also $\lim\limits_{z\to\infty} zf(z) = \lim\limits_{z\to\infty}\dfrac{z}{(z^2+1)^3} = \lim\limits_{z\to\infty}\dfrac{z}{z^6\left(1+\frac{1}{z^2}\right)^3} = 0.$

$\therefore$ By Cauchy's lemma $\int_S f(z)dz = 0$ as $R\to\infty$.

$$\therefore (2) \Longrightarrow \lim_{R\to\infty}\int\limits_{-R}^{R} f(x)dx + \lim_{R\to\infty}\int\limits_S f(z)dz = \frac{3\pi}{8}$$

$$\int\limits_{-\infty}^{\infty} f(x)dx = \frac{3\pi}{8}.$$

Example 4.99. Evaluate the integral $\displaystyle\int_0^\infty \frac{x^2dx}{x^4+1}$ using contour integration.

[Jun 2013, May 2009]

Solution. Let $f(x) = \dfrac{x^2dx}{x^4+1}$.

—

Since $f(x)$ is even $\int_0^\infty \frac{x^2dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2dx}{x^4+1}$.

We shall evaluate $f(z) = \int_{-\infty}^\infty \frac{x^2dx}{x^4+1}$.

Let $I = \int_{-\infty}^\infty \frac{x^2dx}{x^4+1}$.

The integrand is of the form $\frac{P(x)}{Q(x)}$ where degree of $Q(x)$ is 2 more than that of $P(x)$ and $Q(x)$ does not vanish for any real x.

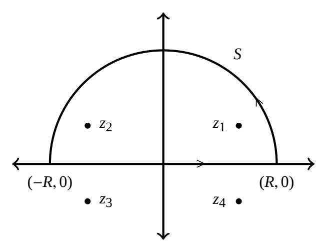
Consider the integral $\int_{-\infty}^\infty \frac{x^2dx}{x^4+1}$ where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi circle $S : |z| = R$ taken in the anticlockwise sense.

Let $f(z) = \int_{-\infty}^\infty \frac{z^2dz}{z^4+1}$.

The poles are given by $z^4 + 1 = 0$.

i.e., $z^4 = -1$

$z^4 = (-1)^{\frac{1}{4}} = e^{i(2n+1)\frac{\pi}{4}}, n = 0, 1, 2, 3.$



$\therefore z = e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}$.

Let the poles be z_1, z_2, z_3 and z_4.

$z_1 = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$.

$z_2 = e^{i\frac{3\pi}{4}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = \frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$.

$z_3 = e^{i\frac{5\pi}{4}} = \frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$.

$z_4 = e^{i\frac{7\pi}{4}} = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$.

Only z_1 and z_2 lie inside C.

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i\left[R(e^{i\frac{\pi}{4}}) + R(e^{i\frac{3\pi}{4}})\right]$.

Now, $R(e^{i}\frac{\pi}{4}) = R(z_1) = \lim_{z\to z_1}(z - z_1)f(z)$

—

$$= \lim_{z\to z_1}(z-z_1)\frac{z^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$$

$$= \frac{z_1^2}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}$$

$$= \frac{(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}})^2}{\sqrt{2}(\sqrt{2}+i\sqrt{2})(\sqrt{2}i)}$$

$$= \frac{(1+i)^2}{2\sqrt{2}\sqrt{2}(1+i)\sqrt{2}i} = \frac{1+i}{i4\sqrt{2}}$$

$$R(e^{i}\frac{3\pi}{4}) = R(z_2) = \lim_{z\to z_2}(z-z_2)f(z)$$

$$= \lim_{z\to z_2}(z-z_2)\frac{z^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$$

$$= \frac{z_2^2}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}$$

$$= \frac{(\frac{-1}{\sqrt{2}}+i\frac{1}{\sqrt{2}})^2}{-\sqrt{2}(-\sqrt{2}+i\sqrt{2})(\sqrt{2}i)}$$

$$= \frac{(-1+i)^2}{2(-2)i\sqrt{2}(-1+i)} = -\frac{1}{4\sqrt{2}}\frac{(-1+i)}{i} = \frac{1}{4\sqrt{2}}(\frac{1-i}{i})$$

$$\therefore \int_C f(z)dz = 2\pi i\left[\frac{1}{4\sqrt{2}}\frac{1+i}{i} + \frac{1}{4\sqrt{2}}(\frac{1-i}{i})\right]$$

$$= \frac{2\pi i}{4\sqrt{2}i}[1+i+1-i]$$

$$= \frac{\pi}{2\sqrt{2}}2 = \frac{\pi}{\sqrt{2}}.$$

$$\therefore\Rightarrow (1)\int_{-R}^{R} f(x)dx + \int_S f(z)dz = \frac{\pi}{\sqrt{2}} \qquad (2)$$

Now, $\lim_{z\to\infty} zf(z) = \lim_{z\to\infty}\frac{z^3}{z^4+1} \to 0.$

$\therefore$ By Cauchy's lemma, $\int_S f(z)dz \to \infty$ as $R\to\infty$.

$$\therefore (2) \Longrightarrow \lim_{R\to\infty}\int_{-R}^{R} f(x)dx + \lim_{R\to\infty}\int_S f(z)dz = \frac{\pi}{\sqrt{2}}$$

—

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{\sqrt{2}}$$

$$\therefore \int_{0}^{\infty} f(x)dx = \frac{\pi}{2\sqrt{2}}$$

$$\text{i.e.,} \int_{0}^{\infty} \frac{x^2}{x^4+1}dx = \frac{\pi}{2\sqrt{2}}.$$

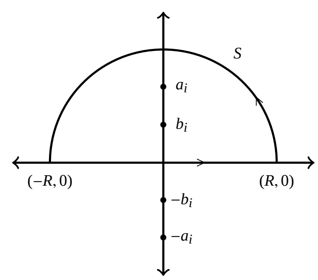
Example 4.100. Evaluate $\int_0^\infty \frac{x^2dx}{(x^2+a^2)(x^2+b^2)}$ $a > 0, b > 0$.[June 2013, May 2009]

Solution. Let $f(x) = \frac{x^2}{(x^2+a^2)(x^2+b^2)}$.

Since $f(x)$ is even, $\int_0^\infty f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx$.

Let $I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}$.

The integrand is of the form $\frac{P(x)}{Q(x)}$ where degree of $Q(x)$ is 2 more than that of $P(x)$ and $Q(x)$ does not vanish for any real x.



Consider the integral $\int_C \frac{z^2}{(z^2+a^2)(z^2+b^2)}dz$ where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi circle $S : |z| = R$ taken in the anticlockwise sense.

$$\text{Now, } \int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_S f(z)dz \qquad (1)$$

We shall evaluate $\int_C f(z)dz$.

We have, $f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)} = \frac{z^2}{(z+ai)(z-ai)(z+bi)(z-bi)}$

—

$\pm ai$ and $\pm bi$ are the simple poles.

But a_i and b_i only lie inside C.

$\therefore$ By Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i[R(ai) + R(bi)].$$

$$R(ai) = \lim_{z\to ai}(z-ai)\frac{z^2}{(z+ai)(z-ai)(z^2+b^2)}$$

$$= \frac{-a^2}{2ai(b^2-a^2)} = \frac{a}{2i(a^2-b^2)}.$$

$$R(bi) = \frac{b}{2i(b^2-a^2)}$$

Now, $\displaystyle\int_C f(z)dz = 2\pi i\Big(\frac{a}{2i(a^2-b^2)} + \frac{b}{2i(b^2-a^2)}\Big)$

$$= \frac{2\pi i}{2i}\Big(\frac{a-b}{(a-b)(a+b)}\Big) = \frac{\pi}{a+b}.$$

$$\therefore (1) \implies \int_{-R}^{R} f(x)dx + \int_S f(z)dz = \frac{\pi}{a+b}. \tag{2}$$

Also, $\displaystyle\lim_{z\to\infty} zf(z) = \lim_{z\to\infty}\frac{z^3}{(z^2+a^2)(z^2+b^2)} \to 0$ as $z\to\infty$

$\therefore$ By Cauchy's lemma, $\displaystyle\int_S f(z)dz \to 0$ as $R\to\infty$.

Now (2) $\implies \displaystyle\lim_{R\to\infty}\int_{-R}^{R} f(x)dx + \lim_{R\to\infty}\int_S f(z)dz = \frac{\pi}{a+b}.$

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx = \frac{\pi}{a+b}$$

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{a+b}$$

$$\int_0^{\infty} f(x)dx = \frac{\pi}{2(a+b)}.$$

Example 4.101. Show that $\displaystyle\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx = \frac{5\pi}{12}.$

[Dec 2013, May 2011, Jun 2010]

—

Solution. Let $I = \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$.

The integrand is of the form $\frac{P(x)}{Q(x)}$ where, the degree of $Q(x)$ is atleast 2 more than that of $P(x)$ and $Q(x)$ does not vanish for any real x.

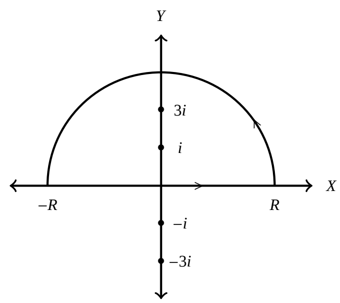
Consider the integral $\int_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz$, where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi circle $S : |z| = R$, taken in the anticlockwise sense.

Let $f(z) = \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}$.

Now $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_S f(z)dz.$ (1)

Let us evaluate $\int_C f(z)dz$.

The poles are given by



$$z^4 + 10z^2 + 9 = 0$$
$$(z^2 + 1)(z^2 + 9) = 0$$
$$z^2 + 1 = 0, \quad z^2 + 9 = 0$$
$$z^2 = -1, \quad z^2 = -9$$
$$z = \pm i, \quad z = \pm 3i.$$

$z = i$ and $3i$ lies inside C.

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i[R(i) + R(3i)]$$

$$\begin{aligned} R(i) &= \lim_{z \to i}(z - i) f(z) \\ &= \lim_{z \to i}(z - i)\frac{z^2 - z + 2}{(z + i)(z - i)(z^2 + 9)} \\ &= \frac{-1 - i + 2}{2i(-1 + 9)} = \frac{1 - i}{16i}. \end{aligned}$$

$$R(3i) = \lim_{z\to 3i}(z-3i)f(z)$$

$$= \lim_{z\to 3i}\cancel{(z-3i)}\frac{z^2-z+2}{(z^2+1)\cancel{(z-3i)}(z+3i)}$$

$$= \frac{-9-3i+2}{(-9+1)6i} = \frac{-7-3i}{-48i} = -\frac{7+3i}{-48i} = \frac{7+3i}{48i}.$$

$$\therefore \int\limits_C f(z)dz = 2\pi i\left[\frac{1-i}{16i}+\frac{7+3i}{48i}\right] = 2\pi i\left[\frac{3-3i+7+3i}{48i}\right] = 2\pi\times\frac{10}{48} = \frac{5\pi}{12}.$$

By Cauchy's lemma $\int_S f(z)dz = 0$ as $R\to\infty$.

$$\therefore (1) \Rightarrow \frac{5\pi}{12} = \int\limits_{-R}^{R} f(x)dx + \int\limits_{s} f(z)dz$$

Taking lim as $R\to\infty$ we get

$$\frac{5\pi}{12} = \lim_{R\to\infty}\int\limits_{-R}^{R} f(x)dx + \lim_{R\to\infty}\int\limits_{s} f(z)dz$$

$$\frac{5\pi}{12} = \int\limits_{-\infty}^{\infty} f(x)dx + 0$$

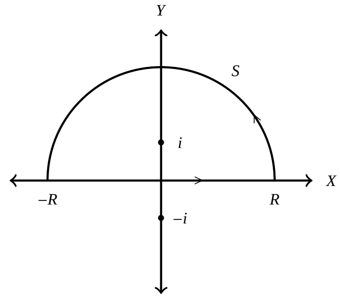
$$\therefore \int\limits_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx = \frac{5\pi}{12}.$$

Example 4.102. Evaluate using contour integration $\int\limits_{-\infty}^{\infty}\frac{x^2}{(x^2+1)^2}dx.$ [Dec 2011]

Solution. Let $I = \int\limits_{-\infty}^{\infty}\frac{x^2}{(x^2+1)^2}dx.$

The integrand is of the form $\frac{P(x)}{Q(x)}$ where the degree of $Q(x)$ is atleast 2 more than that of $P(x)$ and $Q(x)$ does not vanish for any real x.

Consider the integral $\int\limits_C \frac{z^2}{(z^2+1)^2}dz$ where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi circle $S : |z| = R$ taken in the anticlockwise direction.



Let $f(z) = \dfrac{z^2}{(z^2+1)^2}$

$$= \frac{z^2}{[(z+i)(z-i)]^2}$$

$$= \frac{z^2}{(z+i)^2(z-i)^2}.$$

Now $\displaystyle\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_S f(z)dz \qquad (1)$

Let us evaluate $\displaystyle\int_C f(z)dz.$

The poles are given by $(z+i)^2(z-i)^2 = 0$

i.e., $z = i$ and $-i$.

i and $-i$ are poles of order 2. Only i lies inside C.

$\therefore$ By Cauchy's residue theorem $\displaystyle\int_C f(z)dz = 2\pi i R(i).$

$$\text{Now, } R(i) = \frac{1}{1!}\lim_{z\to i}\frac{d}{dz}\{(z-i)^2 f(z)\}$$

$$= \lim_{z\to i}\frac{d}{dz}\left\{(z-i)^2\frac{z^2}{(z+i)^2(z-i)^2}\right\} = \lim_{z\to i}\frac{d}{dz}\left\{\frac{z^2}{(z+i)^2}\right\}$$

$$= \lim_{z\to i}\frac{(z+i)^2\cdot 2z - z^2 2(z+i)}{(z+i)^4} = \frac{(-4)2i-(-4)2(2i)}{16} = \frac{-8i+16i}{16} = \frac{8i}{16} = \frac{i}{2}.$$

$$\therefore \int_C f(z)dz = 2\pi i \times R(i) = 2\pi i\left(\frac{i}{2}\right) = -\pi.$$

By Cauchy's lemma, $\displaystyle\int_S f(z)dz = 0$ as $R\to\infty$.

$$\therefore (1) \Rightarrow -\pi = \int_{-R}^{R} f(x)dx + \int_S f(z)dz$$

$$-\pi = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx + \lim_{R\to\infty}\int_S f(z)dz$$

—

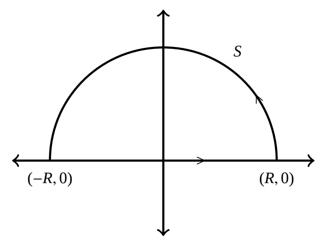
$$-\pi = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx + 0$$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx = -\pi.$$

Jordan's lemma. If $f(z)$ is a continuous function such that $|f(z)| \to 0$ uniformly as $|z| \to \infty$, then $\int_S f(z)e^{imz}dz \to 0$ as $R \to \infty$, where S is the upper semi circle $|z| = R$.

Type III

Evaluation of integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin mx dx$ and $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos mx dx$, $m > 0$, where $P(x)$ and $Q(x)$ are polynomials in x such that the degree of $Q(x)$ is greater than the degree of $P(x)$ and $Q(x)$ does not vanish for any real x.



To evaluate this integral we consider $\int_C f(z)e^{imz}dz$ where $f(z) = \frac{P(z)}{Q(z)}$ and C is a simple closed curve consisting of the real axis from $-R \to R$ and the upper semi circle $S : |z| = R$ taken in the anticlockwise sense. Now

$$\int_C f(z)e^{imz}dz = \int_{-R}^{R} f(x)e^{imx}dx + \int_S f(z)e^{imz}dz.$$

We evaluate $\int_C f(z)e^{imz}dz$ using Cauchy's residue theorem.

By Jordan's lemma $\int_S f(z)e^{imz}dz \to 0$ as $R \to \infty$.

$$\text{Hence, (1)} \Longrightarrow \lim_{R\to\infty} \int_{-R}^{R} f(x)e^{imx}dx = \int_C f(z)e^{imz}dz.$$

$$\int_{-\infty}^{\infty} f(x)e^{imx}dx = \int_C f(z)e^{imz}dz.$$

—

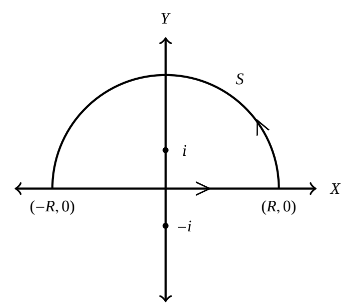
Equating the real and imaginary parts we get the values of
$\int_{-\infty}^{\infty} f(x)\cos mx dx$ and $\int_{-\infty}^{\infty} f(x)\sin mx dx$ respectively.

Worked Examples

Example 4.103. Evaluate $\int_0^{\infty} \frac{\cos ax}{x^2+1}dx, a > 0.$ [Nov 2010]

Solution.

Let us consider the integral $\int_C \frac{e^{iaz}}{z^2+1}dz$ where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semicircle $S : |z| = R$ taken in the anticlockwise sense.



Let $f(z) = \frac{1}{z^2+1}$.

$$\text{Now, } \int_C e^{iaz} f(z)dz = \int_{-R}^{R} e^{iax} f(x)dx + \int_S e^{iaz} f(z)dz \tag{1}$$

Let us evaluate $\int_C e^{iaz} f(z)dz$.

The poles are given by $z^2 + 1 = 0$.

i.e., $z^2 = -1$

$\Longrightarrow z = \pm i$ are the simple poles.

Only $z = i$ lies inside C.

$\therefore$ By Cauchy's residue theorem, $\int_C e^{iaz} f(z)dz = 2\pi i R(i)$

$$\text{Now } R(i) = \lim_{z\to i}(z-i)e^{iaz} f(z) = \lim_{z\to i}(z-i)\frac{e^{iaz}}{(z+i)(z-i)} = \frac{e^{-a}}{2i}$$

$$\therefore \int_C e^{iaz} f(z)dz = 2\pi i \frac{e^{-a}}{2i} = \pi e^{-a}.$$

—

Also $|f(z)| = \left|\dfrac{1}{z^2+1}\right| \to 0$ as $|z| \to \infty$.

$\therefore$ By Jordan's Lemma, $\displaystyle\int_S e^{iz} f(z)dz \to 0$ as $R \to \infty$.

Now, (1) $\Longrightarrow \pi e^{-a} = \displaystyle\int_{-R}^{R} e^{iax} f(x)dx + \int_S e^{iaz} f(z)dz.$

Taking limit as $R \to \infty$ on both sides we get

$$\pi e^{-a} = \lim_{R\to\infty} \int_{-R}^{R} e^{iax} f(x)dx + \lim_{R\to\infty} \int_S e^{iaz} f(z)dz.$$

i.e., $\pi e^{-a} = \displaystyle\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}dx$

$$\int_{-\infty}^{\infty} \frac{\cos ax + i\sin ax}{x^2+1}dx = \pi e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}dx + i\int_{-\infty}^{\infty} \frac{\sin ax}{x^2+1}dx = \pi e^{-a}$$

Equating the real parts on both sides, we get

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}dx = \pi e^{-a}$$

$$2\int_{0}^{\infty} \frac{\cos ax}{x^2+1}dx = \pi e^{-a} \left[\because \left(\frac{\cos ax}{x^2+1}\right) \text{ is even}\right]$$

$$\therefore \int_{0}^{\infty} \frac{\cos ax}{x^2+1}dx = \frac{\pi e^{-a}}{2}.$$

Example 4.104. Evaluate $\displaystyle\int_0^\infty \frac{x\sin mx}{x^2+a^2}dx,\ a > 0,\ m > 0.$ [May 2005]

Solution. Consider the integral $\displaystyle\int_C \frac{z\sin mz}{z^2+a^2}dz$, where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi circle $S : |z| = R$, taken in the anticlockwise.

Let $f(z) = \dfrac{z}{z^2+a^2}dz.$

—

$$\text{Now, } \int_C \frac{ze^{imz}}{z^2+a^2}dz = \int_{-R}^{R} \frac{xe^{imx}}{x^2+a^2}dx + \int_S \frac{ze^{imz}}{z^2+a^2}dz.$$

$$\text{i.e., } \int_C e^{imz} f(z)dz = \int_{-R}^{R} e^{imx} f(x)dx + \int_S e^{imz} f(z)dz. \quad (1)$$

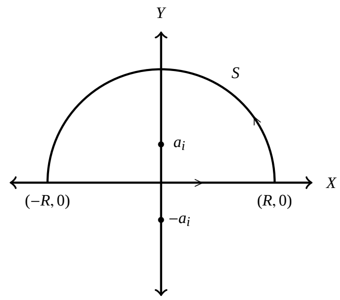
Let us evaluate $\int_C e^{imz} f(z)dz$.

The poles are given by $z^2 + a^2 = 0$.

i.e., $z^2 = -a^2$

$z = \pm ai$ which are simple.

But $z = ai$ is the only pole which lies inside C.



$$\therefore \text{ By Cauchy's residue theorem, } \int_C e^{imz} f(z)dz = 2\pi i \times R(ai)$$

$$\begin{aligned}
\text{Now } R(ai) &= \lim_{z\to ai}(z-ai)e^{imz} f(z) \\
&= \lim_{z\to ai}(z-ai)e^{imz}\frac{z}{(z+ai)(z-ai)} \\
&= e^{imai}\frac{ai}{2ai} = \frac{e^{-ma}}{2}
\end{aligned}$$

$$\therefore \int_C e^{imz} f(z)dz = 2\pi i\frac{e^{-ma}}{2} = \pi i e^{-ma}.$$

Also $|f(z)| = \left|\frac{1}{z^2+a^2}\right| \to 0$ as $|z| \to \infty$.

$\therefore$ By Jordan's Lemma, $\int_S e^{imz} f(z)dz \to 0$ as $R \to \infty$.

$$\text{Now, (1)} \Longrightarrow \pi i e^{-ma} = \int_{-R}^{R} e^{imx} f(x)dx + \int_S e^{imz} f(z)dz.$$

Taking limit as $R \to \infty$ on both sides we get

—

$$\pi i e^{-ma} = \lim_{R\to\infty} \int_{-R}^{R} e^{imx} f(x)dx = \lim_{R\to\infty} \int_{S} e^{imz} f(z)dz.$$

$$= \int_{-\infty}^{\infty} \frac{x(\cos mx + i\sin mx)}{x^2+a^2}dx = \int_{-\infty}^{\infty} \frac{x\cos mx}{x^2+a^2}dx + i\int_{-\infty}^{\infty} \frac{x\sin mx}{x^2+a^2}dx$$

Equating the imaginary parts on both sides, we get

$$\int_{-\infty}^{\infty} \frac{x\sin mx}{x^2+a^2}dx = \pi e^{-ma}$$

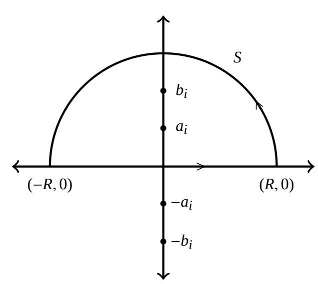
$$2\int_{0}^{\infty} \frac{x\sin mx}{x^2+a^2}dx = \pi e^{-ma} \ [\because \left(\frac{x\sin mx}{x^2+a^2}\right) \text{ is even}]$$

$$\therefore \int_{0}^{\infty} \frac{x\sin mx}{x^2+a^2}dx = \frac{\pi e^{-ma}}{2}.$$

Example 4.105. Evaluate $\int_{0}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx$ using contour integration.

[Dec 2011]

Solution. Let us consider the integral $\int_{C} \frac{e^{iz}}{(x^2+a^2)(x^2+b^2)}dz$ where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semicircle $S : |z| = R$ taken in the anticlockwise sense.



Let $f(z) = \frac{1}{(x^2+a^2)(x^2+b^2)}$.

Now, $$\int_{C} e^{iz} f(z)dz = \int_{-R}^{R} e^{ix} f(x)dx + \int_{S} e^{iz} f(z)dz \quad (1)$$

Let us evaluate $\int_{C} e^{iz} f(z)dz$.

The poles are given by $(x^2+a^2)(x^2+b^2) = 0$.

i.e., $(z+ai)(z-ai)(z+bi)(z-bi) = 0$

$\Rightarrow z = ai, -ai, bi, -bi$ are the simple poles.

But only $z = ai$ and $z = bi$ lie inside C.

—

$\therefore$ By Cauchy's residue theorem, $\displaystyle\int_C e^{iaz} f(z)dz = 2\pi i[R(ai) + R(bi)]$

$$\text{Now } R(ai) = \lim_{z\to ai}(z-ai)e^{iz} f(z)$$
$$= \lim_{z\to ai}(z-ai)\frac{e^{iz}}{(z+ai)(z-ai)(z+bi)(z-bi)}$$
$$= \frac{e^{i\times ai}}{2ai(ai-bi)(ai+bi)} = \frac{e^{-a}}{-2ai(a^2-b^2)} = -\frac{e^{-a}}{2ai(a^2-b^2)}$$
$$R(bi) = \lim_{z\to bi}(z-bi)\frac{e^{iz}}{(z+ai)(z-ai)(z+bi)(z-bi)}$$
$$= \frac{e^{i\times bi}}{2bi(bi-ai)(bi+ai)}$$
$$= \frac{e^{-b}}{2bii(b-a)i(b+a)} = -\frac{e^{-b}}{2bi(b^2-a^2)} = \frac{e^{-b}}{2bi(a^2-b^2)}$$
$$\therefore \int_C e^{iz} f(z)dz = 2\pi i\left[-\frac{e^{-a}}{2ai(a^2-b^2)} + \frac{e^{-b}}{2bi(a^2-b^2)}\right]$$
$$= \frac{2\pi i}{2i(a^2-b^2)}\left[-\frac{e^{-a}}{a} + \frac{e^{-b}}{b}\right]$$
$$= \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

$$\text{Now, (1)} \Longrightarrow \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right] = \int_{-R}^{R} e^{ix} f(x)dx + \int_S e^{iz} f(z)dz \tag{2}$$

Also $|f(z)| = \left|\dfrac{1}{(x^2+a^2)(x^2+b^2)}\right| \to 0$ as $|z| \to \infty$.

$\therefore$ By Jordan's Lemma, $\displaystyle\int_S e^{iz} f(z)dz \to 0$ as $R \to \infty$.

$$\text{Now, (2)} \Longrightarrow \int_{-R}^{R} e^{ix} f(x)dx + \int_S e^{iz} f(z)dz = \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]. \tag{3}$$

Taking limit as $R \to \infty$ on both sides we get

$$\int_{-\infty}^{\infty} e^{ix} f(x)dx = \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

—

$$\int_{-\infty}^{\infty} \frac{\cos bx + i\sin bx}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

$$\int_{-\infty}^{\infty} \frac{\cos bx}{(x^2+a^2)(x^2+b^2)}dx + i\int_{-\infty}^{\infty} \frac{\sin bx}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

Equating the real parts on both sides, we get

$$\int_{-\infty}^{\infty} \frac{\cos bx}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

$$2\int_{0}^{\infty} \frac{\cos bx}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right] \left[\because \left(\frac{\cos bx}{(x^2+a^2)(x^2+b^2)}\right) \text{ is even}\right]$$

$$\therefore \int_{0}^{\infty} \frac{\cos bx}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{2(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right].$$

Example 4.106. Evaluate $\int_{0}^{\infty} \frac{\cos mx}{x^2+a^2}dx$ using contour integration. [Jun 2012]

Solution. Let us consider the integral $\int_C \frac{e^{imz}}{z^2+a^2}dz$, where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi circle $S : |z| = R$ taken in the anticlockwise sense.

Let $f(z) = \frac{1}{z^2+a^2}$

$$\text{Now } \int_C e^{imz} f(z)dz = \int_{-R}^{R} e^{imx} f(x)dx + \int_S e^{imz} f(z)dz \tag{1}$$

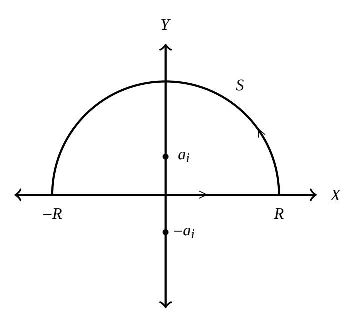
Let us evaluate $\int_C e^{imz} f(z)dz$.

The poles are given by $z^2 + a^2 = 0$

$(z + ai)(z - ai) = 0$

i.e., $z = ai, -ai$

$z = ai$ lies inside C.



By Cauchy's residue theorem, $\int\limits_C e^{imz} f(z)dz = 2\pi i R(ai)$.

$z = ai$ is a simple pole.

$$\therefore R(ai) = \lim_{z\to ai}(z-ai)e^{imz} f(z)$$

$$= \lim_{z\to ai} \cancel{(z-ai)}\frac{e^{imz}}{(z+ai)\cancel{(z-ai)}} = \frac{e^{-am}}{2ai}.$$

$$\therefore \int\limits_C e^{imz} f(z)dz = \frac{2\pi i e^{-am}}{2ai} = \frac{\pi}{a}e^{-am}.$$

Also $|f(z)| = |\frac{1}{z^2+a^2}| \to 0$ as $|z| \to \infty$.

$\therefore$ By Jordon's lemma, $\int_S e^{imz} f(z)dz \to 0$ as $R \to \infty$.

Now (1)$\Rightarrow$ $\frac{\pi}{a}e^{-am} = \int\limits_{-R}^{R} e^{imx} f(x)dx + \int_S e^{imz} f(z)dz.$

Taking limit as $R \to \infty$ we get

$$\frac{\pi}{a}e^{-am} = \lim_{R\to\infty} \int\limits_{-R}^{R} e^{imx} f(x)dx + \lim_{R\to\infty} \int_S e^{imz} f(z)dz$$

$$\frac{\pi}{a}e^{-am} = \int\limits_{-\infty}^{\infty} \frac{e^{imx}}{x^2+a^2}dx + 0$$

$$= \int\limits_{-\infty}^{\infty} \frac{\cos mx + i\sin mx}{x^2+a^2}dx.$$

Equating the real parts we get

$$\int\limits_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2}dx = \frac{\pi e^{-am}}{a}$$

$$2\int\limits_{0}^{\infty} \frac{\cos mx}{x^2+a^2}dx = \frac{\pi e^{-am}}{a}$$

$$\int\limits_{0}^{\infty} \frac{e^{mx}}{x^2+a^2}dx = \frac{\pi e^{-am}}{2a}.$$

—

Exercise 4 F

Evaluate the following integrals using contour integration

1. $\displaystyle\int_0^{2\pi} \frac{1}{5+4\cos\theta} d\theta.$ [Jun 2008]

2. $\displaystyle\int_0^{2\pi} \frac{1}{a+b\cos\theta} d\theta.$ [Apr 2011]

3. $\displaystyle\int_0^{2\pi} \frac{1}{5-4\cos\theta} d\theta.$ [Jun 2008]

4. $\displaystyle\int_0^{\pi} \frac{1}{a+b\cos\theta} d\theta.$ [Jun 2010]

5. $\displaystyle\int_0^{2\pi} \frac{1}{(a+b\cos\theta)^2} d\theta.$

6. $\displaystyle\int_0^{\pi} \frac{a}{a^2+\sin^2\theta} d\theta, a > 0.$

7. $\displaystyle\int_0^{2\pi} \frac{1+\cos 2\theta}{5+4\cos\theta} d\theta.$ [May 2005]

8. $\displaystyle\int_0^{\pi} \frac{1+\cos 2\theta}{5+4\cos\theta} d\theta.$ [May 2003]

9. $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2} d\theta, |a| < 1.$

10. $\displaystyle\int_0^{\pi} \frac{1}{a^2+\sin^2\theta} d\theta, a > 0.$

11. $\displaystyle\int_0^{\infty} \frac{1}{(x^2+1)^2} dx.$ [Jun 2011]

12. $\displaystyle\int_0^{\infty} \frac{1}{(x^4+1)} dx.$

13. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9} dx.$

14. $\displaystyle\int_0^{\infty} \frac{1}{(1+x^2)^2} dx.$

15. $\displaystyle\int_0^{\infty} \frac{x^2}{(x^2+1)^2} dx.$

16. $\displaystyle\int_0^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2} dx.$ [Dec 2009]

17. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9} dx.$ [Dec 2010]

18. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)} dx.$

19. $\displaystyle\int_0^{\infty} \frac{x\sin x}{x^2+a^2} dx.$

20. $\displaystyle\int_0^{\infty} \frac{\cos ax}{x^2+1} dx.a > 0.$

—

$$\frac{1}{r^2}\cos 2\theta = 1 \Rightarrow r^2 = \cos 2\theta.$$

2.4 Unit IV Complex Integration

2.4.1 May/June 2016 (R 2013)

Part A

9. Expand $f(z) = \dfrac{1}{z^2}$ as a Taylor series about the point $z = 2$.

Solution. Let $f(z) = \dfrac{1}{z^2}$.

$z = 2$ is a regular point.

$\therefore$ We can find Taylor's series about $z = 2$.

$\therefore$ The series is

$$f(z) = f(2) + \frac{f'(2)}{1!}(z-2) + \frac{f''(2)}{2!}(z-2)^2 + \frac{f'''(2)}{3!}(z-2)^3 + \cdots$$

Now, $f(z) = \dfrac{1}{z^2}$ $\qquad f(2) = \dfrac{1}{4}$

$f'(z) = \dfrac{-2}{z^3}$ $\qquad f'(2) = \dfrac{-1}{4}$

$f''(z) = \dfrac{6}{z^4}$ $\qquad f''(2) = \dfrac{3}{8}$

$f'''(z) = \dfrac{-24}{z^5}$ $\qquad f'''(2) = \dfrac{-3}{4}$

$$f(z) = -\frac{1}{4} - \frac{1}{4\times 1!}(z-2) + \frac{3}{8\times 2!}(z-2)^2 - \frac{3}{4\times 3!}(z-2)^3 - \dots$$

$$f(z) = -\frac{1}{4} - \frac{1}{4}(z-2) + \frac{3}{16}(z-2)^2 - \frac{1}{8}(z-2)^3 - \dots.$$

10. Express the residue of $f(z) = \tan z$ at its singularities.

Solution. $f(z) = \tan z = \dfrac{\sin z}{\cos z}$.

The singularities are given by $\cos z = 0$.

$$\Rightarrow z = (2n+1)\tfrac{\pi}{2},\ n \in \mathbb{Z}.$$

All the poles are simple poles.

We know that if $f(z) = \dfrac{g(z)}{h(z)}$ where $g(z)$ and $h(z)$ are analytic at $z = a$ and if $h(a) = 0$, $h'(a) \neq 0$ and $g(a) \neq 0$ and finite then $z = a$ is a simple pole of $f(z)$ and $R(a) = \lim\limits_{z \to a} \dfrac{g(z)}{h'(z)}$.

Using the above result, we have

$$R\left((2n+1)\tfrac{\pi}{2}\right) = \lim_{z \to (2n+1)\frac{\pi}{2}} \left(\frac{\sin z}{\cos z}\right) = \lim_{z \to (2n+1)\frac{\pi}{2}} \left(\frac{\sin z}{-\sin z}\right) = -1.$$

Part B

15. (a) (i) Evaluate using Cauchy's integral formula $\displaystyle\int_C \frac{(z+1)}{(z-3)(z-1)} dz$ where C is the circle $|z| = 2$.

Solution. The singular points are $z = 3, 1$.

Given C is $|z| = 2$.

$z = 1$ lie inside C and $z = 3$ lies outside C.

$$\therefore \frac{z+1}{(z-3)(z-1)} = \frac{\frac{z+1}{z-3}}{(z-1)} = \frac{f(z)}{(z-1)} \quad \text{where} \quad f(z) = \frac{z+1}{z-3}.$$

$$\therefore \int_C \frac{z+1}{(z-3)(z-1)} dz = \int_C \frac{f(z)}{z-1} dz$$

$$= 2\pi i \times f(1)$$

[By Cauchy's integral formula].

$$\text{Now,} \quad f(z) = \frac{z+1}{z-3}.$$

$$f(1) = \frac{2}{-2} = -1.$$

$$\therefore \int_C \frac{z+1}{(z-3)(z-1)} dz = 2\pi i(-1) = -2\pi i.$$

15. (a) (ii) Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{13 + 12\sin\theta}$ using contour integration.

Solution. Let $z = e^{i\theta}$, then $d\theta = \dfrac{dz}{iz}$.

$$\text{Also,} \quad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

As θ varies from 0 to $2\pi, z$ moves, on the unit circle $|z| = 1$.

$$\therefore \int_0^{2\pi} \frac{1}{13 + 12\sin\theta} d\theta = \int_C \frac{1}{13 + \frac{12}{2i}\left(z - \frac{1}{z}\right)} \frac{dz}{iz}$$

$$= \int_C \frac{i}{13i + 6\left(z - \frac{1}{z}\right)} \frac{dz}{iz}$$

$$= \int_C \frac{1}{13iz + 6z^2 - 6} dz$$

$$= \int_C \frac{1}{6\left(z^2 + \frac{13i}{6}z - 1\right)} dz$$

$$= \frac{1}{6} \int_C \frac{1}{\left(z + \frac{13}{12}i\right)^2 - 1 + \frac{169}{144}} dz$$

$$= \frac{1}{6} \int_C \frac{1}{\left(z + \frac{13}{12}i\right)^2 + \frac{25}{144}} dz$$

$$= \frac{1}{6} \int_C \frac{1}{\left(z + \frac{13}{12}i\right)^2 + \left(\frac{5}{12}\right)^2} dz$$

$$= \frac{1}{6} \int_C \frac{1}{\left(z + \frac{13}{12}i\right)^2 - \left(\frac{5i}{12}\right)^2} dz$$

$$= \frac{1}{6} \int_C \frac{1}{\left(z + \frac{13i}{12} + \frac{5i}{12}\right)\left(z + \frac{13i}{12} - \frac{5i}{12}\right)} dz$$

$$= \frac{1}{6} \int_C \frac{1}{\left(z + \frac{3i}{2}\right)\left(z + \frac{2i}{3}\right)} dz$$

Here $f(z) = \dfrac{1}{\left(z + \frac{3i}{2}\right)\left(z + \frac{2i}{3}\right)}$.

The poles are $z = \dfrac{-3i}{2}$ and $z = \dfrac{-2i}{3}$, which are simple poles.

Now, C is $|z| = 1$.

When $z = \dfrac{-3i}{2}, |z| = |\dfrac{-3i}{2}| = \dfrac{3i}{2} > 1$.

$\therefore \; z = \dfrac{-3i}{2}i$ lies outside C.

When $z = \dfrac{-2i}{3}, |z| = \left|\dfrac{-2i}{3}\right| = \dfrac{2}{3} < 1$.

$$\therefore\ z = \frac{-2i}{3} \text{ lies inside } C.$$

By Cauchy's residue theorem, $\displaystyle\int_C f(z)dz = 2\pi i R\left(\frac{-2i}{3}\right)$.

$$\text{Now, } R\left(\frac{-2i}{3}\right) = \lim_{z\to\frac{-2i}{3}} \left(z + \frac{2i}{3}\right) \frac{1}{\left(z+\frac{3i}{2}\right)\left(z+\frac{2i}{3}\right)}$$

$$= \frac{1}{\frac{2i}{3}+\frac{3i}{2}} = \frac{6}{4i+9i} = \frac{6}{13i}.$$

$$\therefore \int_0^{2\pi} \frac{1}{13+12\sin\theta}d\theta = \frac{1}{6}2\pi i R\left(\frac{-2i}{3}\right) = \frac{\pi i}{3} \times \frac{6}{13i} = \frac{2\pi}{13}.$$

15. (b) (i) Expand as a Laurent's series the function $f(z) = \dfrac{z}{(z^2-3z+2)}$ in the regions

(1) $|z| < 1$

(2) $1 < |z| < 2$

(3) $|z| > 2$.

Solution. $\dfrac{z}{z^2-3z+2} = \dfrac{z}{(z-1)(z-2)}$.

Let $\dfrac{z}{(z-1)(z-2)} = \dfrac{A}{(z-1)} + \dfrac{B}{(z-2)}$

$\therefore z = A(z-2) + B(z-1)$. When $z = 1$ $\quad A = -1$

When $z = 2$ $\quad A = 2$

$\therefore \frac{z}{(z-1)(z-2)} = \dfrac{2}{z-2} - \frac{1}{z-1}$. (1) When $|z| < 1, \left|\dfrac{z}{2}\right| < \dfrac{1}{2} < 1$.

$$\text{Hence, } f(z) = -\frac{1}{z-1} + \frac{2}{z-2}$$

$$= -\frac{1}{-(1-z)} + \frac{2}{-2\left(1-\frac{z}{2}\right)}$$

$$= (1-z)^{-1} + \frac{2}{-2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \left(1+z+z^2+z^3+\cdots\right) - \left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)$$

(2) Consider $1 < |z| < 2$.

When $1 < |z|$ we have $|z| > 1 \Rightarrow \dfrac{1}{|z|} < 1$.

When $|z| < 2, \left|\dfrac{z}{2}\right| < 1$.

Hence, $f(z)$ can be written as,

$$f(z) = \frac{2}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$
$$= -\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$
$$= -\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$
$$= -\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right) - \left(z^{-1}+z^{-2}+z^{-3}+z^{-4}+\cdots\right).$$

(3) Consider $|z| > 2$, then $\left|\frac{z}{2}\right| > 1 \Rightarrow \left|\frac{2}{z}\right| < 1.$

Also $\frac{1}{|z|} < \frac{1}{2} < 1.$

$$f(z) = -\frac{1}{z-1} + \frac{2}{z-2}$$
$$= -\frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{2}{z\left(1-\frac{2}{z}\right)}$$
$$= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1}$$
$$= -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) + \frac{2}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right)$$
$$= -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \cdots + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \frac{16}{z^4}\cdots$$
$$= z^{-1} + 3z^{-2} + 7z^{-3} + 15z^{-4} + \ldots.$$

15. (b) (ii) Evaluate $\int_0^{\infty} \frac{x \sin mx}{x^2+a^2}\,dx$ where $a > 0$, $m > 0$.

Solution. Refer Example 5.104 on page 530 of the main text book.

2.4.2 Dec.2015/Jan. 2016 (R 2013)

Part A

9. Define and give an example of essential singular points.

Solution. Refer page 474 of the main text book.

10. Express $\int_0^{2\pi} \frac{d\theta}{2\cos\theta + \sin\theta}$ as complex integration.

Solution. Let $z = e^{i\theta}$. Then $d\theta = \frac{dz}{iz}$.

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

As θ varies from 0 to 2π, z moves along the unit circle $|z| = 1$.

$$\therefore \int_0^{2\pi} \frac{1}{2\cos\theta + \sin\theta} d\theta = \int_c \frac{1}{z + \frac{1}{z} + \frac{1}{2i}\left(z - \frac{1}{z}\right)} \times \frac{dz}{iz} \text{ where } c \text{ is } |z| = 1.$$

Part B

15. (a) (i) Using Cauchy's integral formula evaluate $\oint_C \frac{e^{2z}}{(z+1)^4} dz$

where C is $|z| = 2$.

Solution. Given, C is $|z| = 2$.

The singular points are given by $(z+1)^4 = 0 \Rightarrow$ i.e., $z = -1$.

$z = -1$ is a singular point which lies inside C.

By Cauchy's integral formula.

$$\int_c \frac{f(z)}{(z+1)^4} dz = 2\pi i \frac{f'''(-1)}{3!} = \frac{2\pi i}{6} \times \frac{8}{e^2} = \frac{8\pi i}{3e^2}.$$

where $f(z) = e^{2z}$

$$f'(z) = 2e^{2z}$$

$$f''(z) = 4e^{2z}$$

$$f'''(z) = 8e^{2z}$$

$$\Rightarrow f'''(-1) = 8e^{-2} = \frac{8}{e^2}.$$

15. (a) (ii) Evaluate $\int_0^{\infty} \frac{dx}{x^4 + a^4}$ using contour integration.

Solution. Let $f(x) = \frac{1}{x^4 + a^4}$.

Since $f(x)$ is even $\int_0^{\infty} \frac{1dx}{x^4 + a^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4}$.

We shall evaluate $f(z) = \int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4}$.

Let $I = \int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4}$.

The integrand is of the form $\frac{P(x)}{Q(x)}$ where degree of $Q(x)$ is 2 more than that of $P(x)$ and $Q(x)$ does not vanish for any real x.

Consider the integral $\int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4}$ where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi circle $S : |z| = R$ taken in the anticlockwise sense.

Let $f(z) = \int_{-\infty}^{\infty} \frac{dz}{z^4 + a^4}$.

The poles are given by $z^4 + a^4 = 0$.

$$\text{i.e., } z^4 = -a^4$$

$$z = (-a^4)^{\frac{1}{4}} = a(-1)^{\frac{1}{4}} = ae^{i(2n+1)\frac{\pi}{4}}, n = 0, 1, 2, 3.$$

$\therefore z = ae^{i\frac{\pi}{4}}, ae^{i\frac{3\pi}{4}}, ae^{i\frac{5\pi}{4}}, ae^{i\frac{7\pi}{4}}$.

Let the poles be z_1, z_2, z_3 and z_4.

$$z_1 = ae^{i\frac{\pi}{4}} = a\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{a}{\sqrt{2}} + i\frac{a}{\sqrt{2}}.$$

$$z_2 = ae^{i\frac{3\pi}{4}} = a\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \frac{-a}{\sqrt{2}} + i\frac{a}{\sqrt{2}}.$$

$$z_3 = ae^{i\frac{5\pi}{4}} = \frac{-a}{\sqrt{2}} - i\frac{a}{\sqrt{2}}.$$

$$z_4 = ae^{i\frac{7\pi}{4}} = a\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \frac{a}{\sqrt{2}} - i\frac{a}{\sqrt{2}}.$$

Only z_1 and z_2 lie inside C.

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\left[R(e^{i\frac{\pi}{4}}) + R(e^{i\frac{3\pi}{4}})\right].$$

$$\text{Now, } R(e^{i\frac{\pi}{4}}) = R(z_1) = \lim_{z\to z_1}(z - z_1)f(z)$$

$$= \lim_{z\to z_1}(z - z_1)\frac{1}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)}$$

$$= \frac{1}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)}$$

$$= \frac{1}{a^3\sqrt{2}(\sqrt{2}+i\sqrt{2})(\sqrt{2}i)}$$

$$= \frac{1}{a^3\sqrt{2}\sqrt{2}(1+i)\sqrt{2}i} = \frac{1}{2(i-1)a^3\sqrt{2}}$$

$$R(e^{i\frac{3\pi}{4}}) = R(z_2) = \lim_{z\to z_2}(z-z_2)f(z)$$

$$= \lim_{z\to z_2}(z-z_2)\frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$$

$$= \frac{1}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}$$

$$= \frac{1}{-a^3\sqrt{2}(-\sqrt{2}+i\sqrt{2})(\sqrt{2}i)}$$

$$= \frac{1}{a^3(-2)i\sqrt{2}(-1+i)} = \frac{1}{2a^3\sqrt{2}(1+i)}$$

$$\therefore \int_C f(z)dz = 2\pi i\left[\frac{1}{2a^3\sqrt{2}(i-1)} + \frac{1}{2a^3\sqrt{2}(i+1)}\right]$$

$$= \frac{2\pi i}{2a^3\sqrt{2}}\left[\frac{1+i+1-i}{-2}\right]$$

$$= \frac{-\pi}{a^3\sqrt{2}}.$$

$$\therefore\Rightarrow (1)\int_{-R}^{R} f(x)dx + \int_S f(z)dz = \frac{-\pi}{a^3\sqrt{2}} \qquad (2)$$

Now, $\lim_{z\to\infty} zf(z) = \lim_{z\to\infty}\frac{z}{z^4+1} \to 0.$

$\therefore$ By Cauchy's lemma, $\int_S f(z)dz \to \infty$ as $R\to\infty$.

$$\therefore (2) \Longrightarrow \lim_{R\to\infty}\int_{-R}^{R} f(x)dx + \lim_{R\to\infty}\int_S f(z)dz = \frac{\pi}{a^3\sqrt{2}}$$

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{a^3\sqrt{2}}$$

$$\therefore \int_0^{\infty} f(x)dx = \frac{\pi}{2a^3\sqrt{2}}$$

$$\text{i.e.,} \int_0^\infty \frac{1}{x^4+1}\,dx = \frac{\pi}{2a^3\sqrt{2}}.$$

15. (b) (i) Obtain the Laurent's expansion of $f(z) = \dfrac{z^2-4z+2}{z^3-2z^2-5z+6}$ in $3 < |z+2| < 5$.

Solution. Given, $f(z) = \dfrac{z^2-4z+2}{z^3-2z^2-5z+6}$

$$= \frac{z^2-4z+2}{(z-1)(z-3)(z+2)}.$$

Let $\dfrac{z^2-4z+2}{(z-1)(z-3)(z+2)} = \dfrac{A}{z-1} + \dfrac{B}{z-3} + \dfrac{C}{z+2}.$

$z^2-4z+2 = A(z-3)(z+2) + B(z-1)(z+2) + C(z-1)(z-3).$

When $z = 1$	When $z = 3$	When $z = -2$
$1-4+2 = A(-2)(3)$	$9-12+2 = B(2)(5)$	$4+8+2 = C(-3)(-5)$
$A = \dfrac{1}{6}.$	$B = \dfrac{-1}{10}.$	$C = \dfrac{14}{15}.$

$$f(z) = \frac{z^2-4z+2}{(z-1)(z-3)(z+2)} = \frac{1}{6}\frac{1}{z-1} - \frac{1}{10}\frac{1}{z-3} + \frac{14}{15}\frac{1}{z+2}.$$

C is $3 < |z+2| < 5$.

Let $z+2 = t$.

C is $3 < |t| < 5$.

$$\therefore f(z) = \frac{1}{6}\frac{1}{t-3} - \frac{1}{10}\frac{1}{t-5} + \frac{14}{15}\frac{1}{t}.$$

When $3 < |t| \quad \Rightarrow \dfrac{3}{|t|} < 1.$

When $|t| < 5 \quad \Rightarrow \dfrac{|t|}{5} < 1.$

$$\therefore f(z) = \frac{1}{6}\frac{1}{t\left(1-\frac{3}{t}\right)} + \frac{1}{50}\frac{1}{\left(1-\frac{t}{5}\right)} + \frac{14}{15}\frac{1}{t}$$

$$= \frac{1}{6t}\left(1-\frac{3}{t}\right)^{-1} + \frac{1}{50}\left(1-\frac{t}{5}\right)^{-1} + \frac{14}{15t}$$

$$= \frac{1}{6t}\left(1+\frac{3}{t}+\frac{9}{t^2}+\cdots\right) + \frac{1}{50}\left(1+\frac{t}{5}+\frac{t^2}{25}+\cdots\right) + \frac{14}{15t}$$

$$= \frac{1}{6t} + \frac{1}{2t^2} + \frac{3}{2t^3} + \cdots + \frac{1}{50}\left(1 + \frac{t}{5} + \frac{t^2}{25} + \cdots\right) + \frac{14}{15t}$$

$$= \frac{7}{30t} + \frac{1}{2t^2} + \frac{3}{2t^3} + \cdots + \frac{1}{50}\left(1 + \frac{t}{5} + \frac{t^2}{25} + \cdots\right)$$

$$\therefore\ f(z) = \frac{7}{30(z+2)} + \frac{1}{2(z+2)^2} + \frac{3}{2(z+2)^3} + \cdots$$
$$+ \frac{1}{50}\left(1 + \frac{(z+2)}{5} + \frac{(z+2)^2}{25} + \cdots\right).$$

15. (b) (ii) Evaluate $\int\limits_C \frac{z^3\, dz}{(z-1)^4(z-2)(z-3)}$ where C is $|z| = 2.5$; using residue theorem.

Solution. Refer Example 5.80 on page 492 of the main text book.

2.4.3 April/May 2015 (R 2013)

Part A

9. State Cauchy's integral theorem.

Solution. If $f(z)$ is an analytic function within and on a simple closed contour C taken in the positive sense and if a is any interior point of C, then

$$f(a) = \frac{1}{2\pi i}\oint\limits_C \frac{f(z)}{z-a}dz.$$

10. Identify the type of singularity of function $\sin\left(\frac{1}{1-z}\right)$.

Solution. we know that $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$

$$\sin\left(\frac{1}{1-z}\right) = \frac{1}{1-z} - \frac{1}{6}\frac{1}{(1-z)^3} + \frac{1}{120}\frac{1}{(1-z)^5} - \cdots$$

$$\sin\left(\frac{1}{1-z}\right) = (1-z)^{-1} - \frac{(1-z)^{-3}}{6} + \frac{(1-z)^{-5}}{120} - \cdots$$

Since the Laurent's series expansion of $\sin\left(\frac{1}{1-z}\right)$ contains infinite number of negative powers of $1-z$, $z = 1$ is an essential singularity.

Part B

15. (a) (i) Evaluate $\int_C \frac{z^2}{(z-1)^2(z+2)}dz$ where C is $|z| = 3$.

Solution. The singular points are given by $(z-1)^2(z+2) = 0$.

$\Rightarrow z = 1$ and $z = -2$.

C is the circle $|z| = 3$

The center is $(0, 0)$ and radius 3.

$z = 1$ and $z = -2$ both lie inside the circle $|z| = 3$.

Consider $z = 1$

$$\int_C \frac{z^2}{(z-1)^2(z+2)}dz = \int_C \frac{\frac{z^2}{(z+2)}}{(z-1)^2}dz$$

$$= \int_C \frac{f(z)}{(z-1)^2}dz \quad \text{Where } f(z) = \frac{z^2}{(z+2)}$$

$$= 2\pi i \frac{f'(1)}{1!} \quad \text{by Cauchy's integral formula.}$$

$$= 2\pi i \frac{5}{9} \qquad f'(z) = \frac{(z+2)\cdot 2z - z^2}{(z+2)^2}$$

$$= \frac{10\pi i}{9} \qquad f'(1) = \frac{(3)(2)-1}{3^2} = \frac{5}{9}.$$

Consider $z = -2$

$$\int_C \frac{z^2}{(z-1)^2(z+2)}dz = \int_C \frac{\frac{z^2}{(z-1)^2}}{(z-2)}dz$$

$$= \int_C \frac{f(z)}{(z-2)}dz \quad \text{Where } f(z) = \frac{z^2}{(z-1)^2}$$

$$= 2\pi i f(2) \quad \text{by Cauchy's integral formula.}$$

$$= 2\pi i \frac{4}{9} \qquad f(-2) = \frac{4}{9}$$

$$= \frac{8\pi i}{9}.$$

$$\therefore \int_C \frac{z^2}{(z-1)^2(z+2)} dz = \frac{10\pi i}{9} + \frac{8\pi i}{9}$$

$$= \frac{18\pi i}{9} = 2\pi i.$$

15. (a) (ii) Find the Laurent's series expansion of $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ valid in the regions $|z| < 2$ and $2 < |z| < 3$.

Solution. Let $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$.

The singular points are $z = -2$ and $z = -3$.

Both the singular points lie outside $|z| < 2$.

$\therefore$ $f(z)$ is analytic in the open disc $|z| < 2$.

Hence $f(z)$ can be expanded as a power series about $z = 0$.

Splitting $f(z)$ into partial fractions we get

$$\frac{z^2-1}{(z+2)(z+3)} = A + \frac{B}{z+2} + \frac{C}{z+3}$$

$$z^2 - 1 = A(z+2)(z+3) + B(z+3) + C(z+2).$$

Put $z = -2$, we obtain $B = 3$.

Put $z = -3$, we obtain $C = -8$

Equating the coeff. of z^2 we get $A = 1$.

$$\therefore \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

Given $2 < |z| < 3$. This region is annular about $z = 0$ and $f(z)$ is analytic in this region.

$\therefore$ $f(z)$ has a Laurent's series about $z = 0$.

$$|z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1.$$

$$|z| > 2 \Rightarrow \left|\frac{z}{2}\right| > 1 \Rightarrow \left|\frac{2}{z}\right| < 1$$

Now, $f(z) = 1 + \dfrac{3}{z+2} - \dfrac{8}{z+3}$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \cdots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^3}{9} - \frac{z^3}{27} + \cdots\right)$$

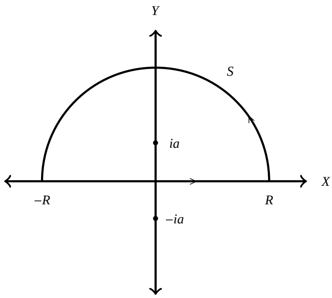
$$= 1 + 3(z^{-1} - 2z^{-2} + 4z^{-3} - 8z^{-4} + \cdots) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right).$$

15. (b) Evaluate $\displaystyle\int_0^\infty \frac{dx}{(x^2 + a^2)^2}$, $(a > 0)$ using contour integration.

Solution. Let $I = \displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)^2} dx.$

The integrand is of the form $\dfrac{P(x)}{Q(x)}$ where the degree of $Q(x)$ is atleast 2 more than that of $P(x)$ and $Q(x)$, does not vanish for any real x.

Consider the integral $\displaystyle\int_C \frac{z^2}{(z^2 + a^2)^2} dz$ where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi circle $S : |z| = R$ taken in the anticlockwise direction.



$$\text{Let } f(z) = \frac{z^2}{(z^2 + a^2)^2}$$

$$= \frac{z^2}{[(z + ia)(z - ia)]^2}$$

$$= \frac{z^2}{(z + ia)^2(z - ia)^2}.$$

Now $\displaystyle\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_S f(z)dz \qquad (1)$

Let us evaluate $\displaystyle\int_C f(z)dz.$

The poles are given by $(z + ia)^2(z - ia)^2 = 0$

i.e., $z = ai$ and $-ai$.

ai and $-ai$ are poles of order 2. Only ai lies inside C.

$\therefore$ By Cauchy's residue theorem $\int_C f(z)dz = 2\pi i R(ai)$.

$$\text{Now, } R(ai) = \frac{1}{1!} \lim_{z \to ai} \frac{d}{dz} \left\{ (z - ai)^2 f(z) \right\}$$

$$= \lim_{z \to ai} \frac{d}{dz} \left\{ (z - ai)^2 \frac{z^2}{(z + ai)^2 (z - ai)^2} \right\} = \lim_{z \to ai} \frac{d}{dz} \left\{ \frac{z^2}{(z + ai)^2} \right\}$$

$$= \lim_{z \to ai} \frac{(z + ai)^2 \cdot 2z - z^2 2(z + ai)}{(z + ai)^4}$$

$$= \frac{(ai + ai)^2 \cdot 2ai - (ai)^2 2(ai + ai)}{(ai + ai)^4}$$

$$= \frac{(2ai)^2 \cdot 2ai + a^2 \times 2(2ai)}{(2ai)^4}$$

$$= \frac{(-4a^2 \times 2ai) + 4ia^3}{16a^4}$$

$$= \frac{(-8a^3 i) + 4a^3 i}{16a^4}$$

$$= \frac{-4a^3 i}{16a^4} = \frac{-i}{4a}$$

$$\therefore \int_C f(z)dz = 2\pi i \times R(ai) = 2\pi i \left(\frac{-i}{4a} \right) = \frac{\pi}{2a}.$$

By Cauchy's lemma, $\int_S f(z)dz = 0$ as $R \to \infty$.

$$\therefore (1) \Rightarrow \frac{\pi}{2a} = \int_{-R}^{R} f(x)dx + \int_s f(z)dz$$

$$\frac{\pi}{2a} = \lim_{R \to \infty} \int_{-R}^{R} f(x)dx + \lim_{R \to \infty} \int_s f(z)dz$$

$$\frac{\pi}{2a} = \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)^2} dx + 0$$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)^2} dx = \frac{\pi}{2a}.$$

2.4.4 November/December 2014 (R 2013)

Part A

9. Evaluate $\int_C \frac{zdz}{z-2}$ where C is the circle $|z| = 1$.

Solution. $\int_C \frac{z}{z-2}dz = \int_C \frac{f(z)}{z-2}dz$ where $f(z) = z$.
C is $|z| = 1$.

The singular point is $z = 2$, which lies outside C.

Hence, $\frac{z}{z-2}$ is analytic inside C.

$\therefore$ By Cauchy's integral theorem $\int_C \frac{z}{z-2}dz = 0$.

10. Find the residue of $f(z) = \frac{z^2}{(z-2)(z+1)^2}$ at $z = 2$.

Solution. $z = 2$ is a simple pole.

$$\begin{aligned} R(2) &= \lim_{z\to 2}(z-2)f(z) \\ &= \lim_{z\to 2}\cancel{(z-2)}\frac{z^2}{\cancel{(z-2)}(z+1)^2} \\ &= \lim_{z\to 2}\frac{z^2}{(z+1)^2} = \frac{4}{9}. \end{aligned}$$

Part B

15. (a) (i) Evaluate $\int_C \frac{z+1}{z^2+2z+4}dz$ where C is the circle $|z+1+i| = 2$ using Cauchy's integral formula.

Solution. The singular points are given by

$$\begin{aligned} z^2 + 2z + 4 &= 0 \\ z^2 + 2z + 1 + 3 &= 0 \\ (z+1)^2 + 3 &= 0 \\ (z+1)^2 &= -3 \\ z + 1 &= \pm\sqrt{3}i \\ z &= -1 \pm \sqrt{3}i \end{aligned}$$

C is the circle $|z+1+i| = 2$

$z = -1 + i\sqrt{3}$ lies inside C where as $z = -1 - i\sqrt{3}$ lies out side C.

$$\int_C \frac{z+1}{z^2+2z+4}dz = \int_C \frac{z+1}{(z+1+i\sqrt{3})(z+1-i\sqrt{3})}dz$$

$$= \int_C \frac{\frac{z+1}{z+1-i\sqrt{3}}}{(z+1+i\sqrt{3})}dz$$

$$= \int_C \frac{f(z)}{(z+1+i\sqrt{3})}dz \quad \text{Where} \frac{z+1}{z+1-i\sqrt{3}}$$

$$= 2\pi i f(-1-i\sqrt{3}) \quad \text{By Cauchy's integral formula.}$$

$$= 2\pi i \times \frac{-1-i\sqrt{3}+1}{-1-i\sqrt{3}+1-i\sqrt{3}}$$

$$= 2\pi i \times \frac{-i\sqrt{3}}{-2i\sqrt{3}} = \pi i.$$

15. (a) (ii) Find the Laurent's series expansion of $f(z) = \dfrac{1}{(z-1)(z-2)}$ valid in the regions $|z| > 2$ and $0 < |z-1| < 1$.

Solution. $f(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{(z-1)-(z-2)}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$.

Since $|z| > 2$, $\left|\dfrac{1}{z}\right| < \dfrac{1}{2} < 1$ Also $1 > \dfrac{2}{|z|} \Rightarrow \dfrac{2}{|z|} < 1$.

Hence, $f(z)$ can be written as,

$$\therefore f(z) = \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

$$= \frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

$$= \frac{1}{z}\left(\frac{1}{z}+\frac{3}{z^2}+\frac{7}{z^3}+\cdots\right).$$

(ii) We have $f(z) = \dfrac{1}{z-2} - \dfrac{1}{z-1}$.

Given $0 < |z-1| < 1$.

Let $t = z - 1 \Rightarrow z = t + 1$.

$\therefore$ The region is $0 < |t| < 1$.

$$\begin{aligned}
f(z) &= \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{t+1-2} - \frac{1}{t+1-1} \\
&= \frac{1}{t-1} - \frac{1}{t} \\
&= -t^{-1} - \frac{1}{1-t} \\
&= -t^{-1} - (1-t)^{-1} \\
&= -(t)^{-1} - [1 + t + t^2 + t^3 + \cdots] \\
&= -\left[\frac{1}{t} + 1 + t + t^2 + t^3 + \cdots\right] \\
&= -\left[\frac{1}{(z-1)} + 1 + (z-1) + (z-1)^2 + (z-1)^3) + \cdots\right].
\end{aligned}$$

15. (b) (i) Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{13 + 5\cos\theta}$ using contour integration.

Solution. Let $z = e^{i\theta}$, then $d\theta = \dfrac{dz}{iz}$.

Also, $\cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$.

As θ varies from 0 to 2π, z moves along the unit circle $|z| = 1$.

$$\begin{aligned}
\therefore \int_0^{2\pi} \frac{1}{13 + 5\cos\theta} d\theta &= \int_C \frac{1}{13 + \frac{5}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz} \\
&= \frac{1}{i} \int_C \frac{2}{26 + 5\left(\frac{z^2+1}{z}\right)} \frac{dz}{z} \\
&= \frac{2}{i} \int_C \frac{\cancel{z}}{26z + 5z^2 + 5} \frac{dz}{\cancel{z}} \\
&= \frac{2}{i} \int_C \frac{1}{5(z^2 + \frac{26}{5}z + 1)} dz
\end{aligned}$$

$$= \frac{2}{5i}\int_C \frac{1}{\left(z+\frac{13}{5}\right)^2 + 1 - \frac{169}{25}}dz$$

$$= \frac{2}{5i}\int_C \frac{1}{\left(z+\frac{13}{5}\right)^2 + \frac{25-169}{25}}dz$$

$$= \frac{2}{5i}\int_C \frac{1}{\left(z+\frac{13}{5}\right)^2 - \frac{144}{25}}dz$$

$$= \frac{2}{5i}\int_C \frac{1}{\left(z+\frac{13}{5}\right)^2 - \left(\frac{12}{5}\right)^2}dz$$

$$= \frac{2}{5i}\int_C \frac{1}{\left(z+\frac{13}{5}+\frac{12}{5}\right)\left(z+\frac{13}{5}-\frac{12}{5}\right)}dz$$

$$= \frac{2}{5i}\int_C \frac{1}{(z+5)\left(z+\frac{1}{5}\right)}dz$$

$$= \frac{2}{5i}\int_C f(z)dz \quad \text{where} \quad f(z) = \frac{1}{(z+5)\left(z+\frac{1}{5}\right)}$$

The singular points are given by

$$(z+5)\left(z+\frac{1}{5}\right) = 0$$

$$z = -5,\ z = \frac{-1}{5}$$

Since C is $|z| = 1$, $z = -5$ lies outside C and $z = \frac{-1}{5}$, lies inside C.

$z = \frac{-1}{5}$ is a simple pole.

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i R\left(\frac{-1}{5}\right)$.

$$R\left(\frac{-1}{5}\right) = \lim_{z\to\frac{-1}{5}}\left(z+\frac{1}{5}\right)\frac{1}{(z+5)\left(z+\frac{1}{5}\right)} = \lim_{z\to\frac{-1}{5}}\frac{1}{(z+5)} = \frac{1}{\frac{-1}{5}+5} = \frac{5}{24}.$$

$$\therefore \int_0^{2\pi}\frac{1}{13+5\cos\theta}d\theta = \frac{2}{5i}2\pi i R\left(\frac{-1}{5}\right) = \frac{4\pi i}{5i}\frac{5}{24} = \frac{\pi}{6}.$$

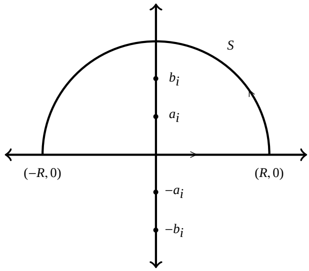
15. (b) (ii) Evaluate $\int_0^\infty \frac{x^2dx}{(x^2+9)(x^2+4)}$ using contour integration.

Solution. Let $f(x) = \frac{x^2}{(x^2+9)(x^2+4)}$.

Since $f(x)$ is even, $\int\limits_{0}^{\infty} f(x)dx = \frac{1}{2}\int\limits_{-\infty}^{\infty} f(x)dx$.

Let $I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)}$.

The integrand is of the form $\frac{P(x)}{Q(x)}$ where degree of $Q(x)$ is 2 more than that of $P(x)$ and $Q(x)$ does not vanish for any real x.



Consider the integral $\int\limits_C \frac{z^2}{(z^2+9)(z^2+4)}dz$ where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi circle $S : |z| = R$ taken in the anticlockwise sense.

$$\text{Now, } \int\limits_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_S f(z)dz \tag{1}$$

We shall evaluate $\int\limits_C f(z)dz$.

We have, $f(z) = \frac{z^2}{(z^2+9)(z^2+4)} = \frac{z^2}{(z+3i)(z-3i)(z+2i)(z-2i)}$

$\pm 3i$ and $\pm 2i$ are the simple poles.

But $3i$ and $2i$ only lie inside C.

$\therefore$ By Cauchy's residue theorem

$$\int\limits_C f(z)dz = 2\pi i[R(3i) + R(2i)].$$

$$R(3i) = \lim_{z\to 3i}(z-3i)\frac{z^2}{(z+3i)(z-3i)(z^2+4)}$$

$$= \frac{-9}{2\times 3i(4-9)} = \frac{3}{2i(9-4)} = \frac{3}{10i}.$$

$$R(bi) = \frac{2}{2i(4-9)} = \frac{-2}{10i}$$

Now, $\displaystyle\int_C f(z)dz = 2\pi i\left(\frac{3}{10i} - \frac{2}{10i}\right)$

$$= 2\pi i \frac{1}{10i} = \frac{\pi}{5}.$$

$$\therefore (1) \Longrightarrow \int_{-R}^{R} f(x)dx + \int_S f(z)dz = \frac{\pi}{5}. \tag{2}$$

$$\text{Also, } \lim_{z\to\infty} z f(z) = \lim_{z\to\infty} \frac{z^3}{(z^2+9)(z^2+4)} \to 0 \text{ as } z \to \infty$$

$$\therefore \text{ By Cauchy's lemma, } \int_S f(z)dz \to 0 \text{ as } R \to \infty.$$

$$\text{Now (2)} \Longrightarrow \lim_{R\to\infty} \int_{-R}^{R} f(x)dx + \lim_{R\to\infty} \int_S f(z)dz = \frac{\pi}{5}.$$

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \frac{\pi}{5}$$

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{5}$$

$$\int_{0}^{\infty} f(x)dx = \frac{\pi}{10}.$$

2.4.5 May/June 2014 (R 2013)

Part A

9. Evaluate $\displaystyle\int_C \frac{z}{z-2}dz$ where C is (a) $|z| = 1$ (b) $|z| = 3$.

Solution. $\displaystyle\int_C \frac{z}{z-2}dz = \int_C \frac{f(z)}{z-2}dz$ where $f(z) = z$.

(a) C is $|z| = 1$.

The singular point is $z = 2$, which lies outside C.

Hence, $\dfrac{z}{z-2}$ is analytic inside C.

$\therefore$ By Cauchy's integral theorem $\displaystyle\int_C \frac{z}{z-2}dz = 0$.

(b) C is $|z| = 3$.

The singular point $z = 2$ lies inside C.

By Cauchy's integral formula

$$\int_C \frac{z}{z-2}dz = \int_C \frac{f(z)}{z-2}dz \qquad \text{where } f(z) = z \Rightarrow f(2) = 2.$$

$$= 2\pi i \times f(2) = 2\pi i \times 2 = 4\pi i.$$

10. State Cauchy's residue theorem.

Solution. If $f(z)$ is analytic inside and on a simple closed curve C, except at a finite number of singular points $z_1, z_2, \ldots, z_n$ lying inside C, then

$$\int_C f(z)dz = 2\pi i(R(z_1) + R(z_2) + \cdots + R(z_n))$$

where the integral over C is taken in the anticlockwise sense.

Part B

15. (a) (i) Obtain the Laurent's series expansion of $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ in $2 < |z| < 3$.

Solution. Let $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$.

The singular points are $z = -2$ and $z = -3$.

Both the singular points lie outside $|z| < 2$.

$\therefore$ $f(z)$ is analytic in the open disc $|z| < 2$.

Hence $f(z)$ can be expanded as a power series about $z = 0$.

Splitting $f(z)$ into partial fractions we get

$$\frac{z^2-1}{(z+2)(z+3)} = A + \frac{B}{z+2} + \frac{C}{z+3}$$

$$z^2 - 1 = A(z+2)(z+3) + B(z+3) + C(z+2).$$

Put $z = -2$, we obtain $B = 3$.

Put $z = -3$, we obtain $C = -8$

Equating the coeff. of z^2 we get $A = 1$.

$$\therefore \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

Given $2 < |z| < 3$. This region is annular about $z = 0$ and $f(z)$ is analytic in this region.

$\therefore$ $f(z)$ has a Laurent's series about $z = 0$.

$|z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1.$

$|z| > 2 \Rightarrow \left|\frac{z}{2}\right| > 1 \Rightarrow \left|\frac{2}{z}\right| < 1$

$$\text{Now, } f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \cdots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^3}{9} - \frac{z^3}{27} + \cdots\right)$$

$$= 1 + 3(z^{-1} - 2z^{-2} + 4z^{-3} - 8z^{-4} + \cdots) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right).$$

15. (a) (ii) Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta}$.

Solution. Let $z = e^{i\theta}$, then $d\theta = \dfrac{dz}{iz}$.

Also, $\sin\theta = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right)$.

As θ varies from 0 to $2\pi, z$ moves, on the unit circle $|z| = 1$.

$$\therefore \int_0^{2\pi} \frac{1}{13 + 5\sin\theta} d\theta = \int_C \frac{1}{13 + \frac{5}{2i}\left(z - \frac{1}{z}\right)} \frac{dz}{iz}$$

$$= \int_C \frac{2i}{26i + 5\left(z - \frac{1}{z}\right)} \frac{dz}{iz}$$

$$= 2\int_C \frac{1}{26iz + 5z^2 - 5} dz$$

$$= 2\int_C \frac{1}{5\left(z^2 + \frac{26i}{5}z - 1\right)}dz$$

$$= \frac{2}{5}\int_C \frac{1}{\left(z + \frac{13}{5}i\right)^2 - 1 + \frac{169}{25}}dz$$

$$= \frac{2}{5}\int_C \frac{1}{\left(z + \frac{13}{5}i\right)^2 + \frac{144}{25}}dz$$

$$= \frac{2}{5}\int_C \frac{1}{\left(z + \frac{13}{5}i\right)^2 + \left(\frac{12}{5}\right)^2}dz$$

$$= \frac{2}{5}\int_C \frac{1}{\left(z + \frac{13}{5}i\right)^2 - \left(\frac{i12}{5}\right)^2}dz$$

$$= \frac{2}{5}\int_C \frac{1}{\left(z + \frac{13i}{5} + \frac{12i}{5}\right)\left(z + \frac{13i}{5} - \frac{12i}{5}\right)}dz$$

$$= \frac{2}{5}\int_C \frac{1}{(z + 5i)\left(z + \frac{i}{5}\right)}dz$$

Here $f(z) = \dfrac{1}{(z + 5i)\left(z + \frac{i}{5}\right)}$.

The poles are $z = -5i$ and $z = \dfrac{-i}{5}$, which are simple poles.

Now, C is $|z| = 1$.

When $z = -5i, |z| = |-5i| = 5 > 1$.

$\therefore\ z = -5i$ lies outside C.

When $z = \dfrac{-i}{5}, |z| = \left|\dfrac{-i}{5}\right| = \dfrac{1}{5} < 1$.

$\therefore\ z = \dfrac{i}{5}$ lies inside C.

By Cauchy's residue theorem, $\displaystyle\int_C f(z)dz = 2\pi i R\left(\frac{-i}{5}\right)$.

Now, $R\left(\dfrac{-i}{5}\right) = \displaystyle\lim_{z\to\frac{-i}{5}} \left(z + \frac{i}{5}\right)\frac{1}{(z + 5i)\left(z + \frac{i}{5}\right)} = \frac{1}{-i + 5i} = \frac{5}{24i}$.

$$\therefore \int_0^{2\pi} \frac{1}{13 + 5\sin\theta}d\theta = \frac{2}{5}2\pi i R\left(\frac{-i}{5}\right) = \frac{4\pi i}{5}\frac{5}{24i} = \frac{\pi}{6}.$$

15. (b) (i) Using Cauchy's residue theorem evaluate

$$\int_C \frac{(z-1)}{(z+1)^2(z-2)}dz, \text{ where } C \text{ is } |z-i| = 2.$$

Solution. Let $f(z) = \dfrac{z-1}{(z+1)^2(z-2)}$.

The poles are given by $z = -1$ and $z = 2$.

$z = -1$ is a pole of order 2 and $z = 2$ is a simple pole.

C is $|z-i| = 2$.

When $z = -1, |z-i| = |-1-i| = \sqrt{1+1} = \sqrt{2} < 2$.

$\Rightarrow$ $z = -1$ lies inside C.

When $z = 2, |z-i| = |2-i| = \sqrt{4+1} = \sqrt{5} > 2$.

$\therefore$ $z = 2$ lies outside C.

$\therefore$ By Cauchy's residue theorem $\int\limits_C f(z)dz = 2\pi i R(-1)$.

Since $z = -1$ is a pole of order 2,

$$R(-1) = \frac{1}{(2-1)!}\lim_{z\to-1}\frac{d}{dz}\left((z+1)^2 f(z)\right) = \lim_{z\to-1}\frac{d}{dz}\left((z+1)^2\frac{z-1}{(z+1)^2(z-2)}\right)$$

$$= \lim_{z\to-1}\frac{d}{dz}\left(\frac{z-1}{z-2}\right) = \lim_{z\to-1}\frac{z-2-(z-1)}{(z-2)^2} = \frac{-3+2}{9} = \frac{-1}{9}.$$

$$\text{Hence, } \int\limits_C \frac{z-1}{(z+1)^2(z-2)}dz = 2\pi i R(-1) = 2\pi i\left(\frac{-1}{9}\right) = -\frac{2\pi i}{9}.$$

15. (b) (ii) Evaluate by using contour integration $\displaystyle\int_0^\infty \frac{dx}{(1+x^2)^2}$.

Solution. Let $f(x) = \dfrac{1}{(1+x^2)^2}$

Since $f(x)$ is even, $\displaystyle\int_0^\infty f(x)dx = \frac{1}{2}\int_{-\infty}^\infty f(x)dx.$

Let $I = \displaystyle\int_{-\infty}^\infty \frac{1}{(1+x^2)^2}dx.$

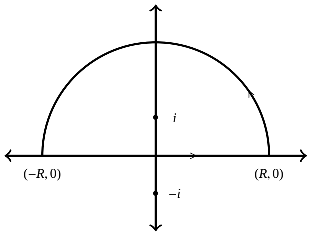
The integrand is of the form $\dfrac{P(x)}{Q(x)}$, where degree of $Q(x)$ is atleast 2 more than that of $P(x)$ and $Q(x)$ does not vanish for any real x. Consider the integral $\displaystyle\int_C \frac{1}{(1+z^2)^2}dz$ where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi circle $S : |z| = R$ taken in the anticlockwise sense.

$$\text{Now, } \int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_S f(z)dz. \tag{1}$$

We shall evaluate $\int_C f(z)dz$.

We have $f(z) = \dfrac{1}{(z^2+1)^2} = \dfrac{1}{[(z+i)(z-i)]^2} = \dfrac{1}{(z+i)^2(z-i)^2}$.

The poles are at i and $-i$ which are of order 2. Only i will lie inside C.



By Cauchy's residue theorem $\int_C f(z)dz = 2\pi i R(i)$.

$$R(i) = \frac{1}{1!}\lim_{z\to i}\frac{d}{dz}(z-i)^2 f(z)$$

$$= \lim_{z\to i}\frac{d}{dz}(z-i)^2\frac{1}{(z+i)^2(z-i)^2}$$

$$= \lim_{z\to i}\frac{d}{dz}(z+i)^{-2} = \lim_{z\to i} -2(z+i)^{-3}$$

$$= -\frac{2}{(2i)^3} = -\frac{2}{-8i} = \frac{1}{4i}.$$

$$\therefore \int_C f(z)dz = 2\pi i \times \frac{1}{4i} = \frac{\pi}{2}.$$

Also $\lim\limits_{z\to\infty} z f(z) = \lim\limits_{z\to\infty}\dfrac{z}{(z^2+1)^2} = \lim\limits_{z\to\infty}\dfrac{z}{z^4\left(1+\frac{1}{z^2}\right)^2} = \lim\limits_{z\to\infty}\dfrac{1}{z^3\left(1+\frac{1}{z^2}\right)^2} = 0.$

$\therefore$ By Cauchy's lemma, $\int_S f(z)dz = 0$.

Now (1)$\Rightarrow \int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_S f(z)dz.$

Taking limits as $R \to \infty$ we get

$$\int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx + \lim_{z\to\infty} \int_S f(z)dz$$

$$\frac{\pi}{2} = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}dx + 0$$

$$\frac{\pi}{2} = 2\int_{0}^{\infty} \frac{1}{(1+x^2)^2}dx \Rightarrow \int_{0}^{\infty} \frac{1}{(1+x^2)^2}dx = \frac{\pi}{4}.$$

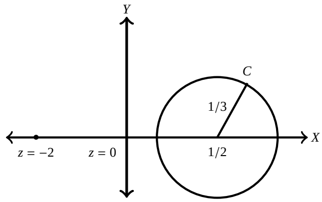
2.4.6 November/December 2013(R 2008)

Part A

9. Evaluate $\int_C \frac{z+4}{z^2+2z}dz$, where C is $|z - \frac{1}{2}| = \frac{1}{3}$.

Solution.

C is the circle $|z - \frac{1}{2}| = \frac{1}{3}$.
The singular points are $z = 0$ and $z = -2$, which lie completely outside C.
$\therefore \frac{z+4}{z^2+2z}$ is analytic everywhere inside C.



$$\int_C \frac{z+4}{z^2+2z}dz = \int_C \frac{z+4}{z(z+2)}dz = 0.$$

10. Find the residue of $\frac{1-e^{2z}}{z^4}$ at $z = 0$.

Solution. Let $f(z) = \frac{1-e^{2z}}{z^4}$.

$z = 0$ is a pole of order 3. [Expand e^{2z} and simplify]

$$\begin{aligned}
\therefore R(0) &= \frac{1}{(3-1)!} \lim_{z\to 0} \frac{d^2}{dz^2}(z-0)^3 f(z) \\
&= \frac{1}{2} \lim_{z\to 0} \frac{d^2}{dz^2}\left[z^3 \frac{1-e^{2z}}{z^4}\right] \\
&= \frac{1}{2} \lim_{z\to 0} \frac{d^2}{dz^2}\left[\frac{1-e^{2z}}{z}\right] \\
&= \frac{1}{2} \lim_{z\to 0} \frac{d^2}{dz^2}\left[\frac{1-1-2z-\frac{4z^2}{2!}-\frac{8z^3}{3!}-\cdots}{z}\right] \\
&= \frac{1}{2} \lim_{z\to 0} \frac{d}{dz}\left[2-\frac{8z}{3}-\cdots\right] \\
&= \frac{1}{2} \lim_{z\to 0}\left[\frac{-8}{3} + \text{powers of } z\right] = \frac{1}{2}\left(\frac{-8}{3}\right) = -\frac{4}{3}.
\end{aligned}$$

Part B

15. (a) (i) Find the residues of $f(z) = \dfrac{z^2}{(z+2)(z-1)^2}$ at its isolated singularities using Laurent's series expansion.

Solution. $$\frac{z^2}{(z+2)(z-1)^2} = \frac{A}{z+2} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$z^2 = A(z-1)^2 + B(z-1)(z+2) + c(z+2).$$

When $z = 1, 3C = 1 \Rightarrow C = \dfrac{1}{3}$.

When $z = -2, 9A = 4 \Rightarrow A = \dfrac{4}{9}$.

Equating the coefficients of z^2 we get

$$A + B = 1$$

$$B = 1 - A = 1 - \frac{4}{9} = \frac{5}{9}.$$

$$\therefore \frac{z^2}{(z+2)(z-1)^2} = \frac{4}{9}\frac{1}{z+2} + \frac{5}{9}\frac{1}{z-1} + \frac{1}{3}\frac{1}{(z-1)^2}.$$

The residues at the singularity $z = -2$ is the coefficient of $\dfrac{1}{z}$ in the expansion of $\frac{4}{9}\frac{1}{z+2}$.

$$\text{Now, } \frac{4}{9}\cdot\frac{1}{z+2} = \frac{4}{9}\cdot\frac{1}{z\left(1+\frac{2}{z}\right)} = \frac{4}{9}\frac{1}{z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= \frac{4}{9}\frac{1}{z}\left[1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}\cdots\right] = \frac{4}{9}\left[\frac{1}{z}-\frac{2}{z^2}+\frac{4}{z^3}\cdots\right]$$

$\therefore$ Residue at $z=-2$ = Coefficient of $\frac{1}{z} = \frac{4}{9}$.

The residue at the singularity $z=1$ is the coefficient of $\frac{1}{z}$ in the expansion of $\frac{5}{9}\frac{1}{z-1}+\frac{1}{3}\frac{1}{(z-1)^2}$.

$$\text{Now, } \frac{5}{9}\frac{1}{z-1}+\frac{1}{3}\frac{1}{(z-1)^2} = \frac{5}{9z\left(1-\frac{1}{z}\right)}+\frac{1}{3z^2\left(1-\frac{1}{z}\right)^2}$$

$$= \frac{5}{9z}\left(1-\frac{1}{z}\right)^{-1}+\frac{1}{3z^2}\left(1-\frac{1}{z}\right)^{-2}$$

$$= \frac{5}{9z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]+\frac{1}{3z^2}\left[1+\frac{2}{z}+\frac{3}{z^2}+\cdots\right].$$

Residues at $z=1$ = Coefficient of $\frac{1}{z} = \frac{5}{9}$.

15. (a) (ii) Evaluate $\displaystyle\int_0^{2\pi}\frac{\cos 2\theta}{5+4\cos\theta}d\theta$ using contour integration.

Solution. Let $z=e^{i\theta}$, then $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$.

As θ varies from 0 to 2π, z moves along $|z|=1$.

$\therefore C$ is $|z|=1$.

Now $z^2 = e^{i2\theta} = \cos 2\theta + i\sin 2\theta$.

$\therefore \cos 2\theta = \text{R.P.}e^{i2\theta}$.

$$\therefore \int_0^{2\pi}\frac{\cos 2\theta}{5+4\cos 2\theta}d\theta = \text{R.P}\int_C \frac{z^2}{5+4\times\frac{1}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{iz}$$

$$= \text{R.P}\left(\frac{1}{i}\right)\int_C \frac{z}{5+2\left(\frac{z^2+1}{z}\right)}dz$$

$$= \text{R.P}(-i)\int_C \frac{z^2}{2z^2+5z+2}dz$$

$$= \text{R.P}\left(\frac{-i}{2}\right)\int_C \frac{z^2}{z^2+\frac{5}{2}z+1}dz$$

$$= \text{R.P}\left(\frac{-i}{2}\right)\int_C \frac{z^2}{\left(z+\frac{5}{4}\right)^2+1-\frac{25}{16}}dz$$

$$= \text{R.P}\left(\frac{-i}{2}\right)\int_C \frac{z^2}{\left(z+\frac{5}{4}\right)^2-\frac{9}{16}}dz$$

$$= \text{R.P}\left(\frac{-i}{2}\right)\int_C \frac{z^2}{\left(z+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2}dz$$

$$= \text{R.P}\left(\frac{-i}{2}\right)\int_C \frac{z^2}{\left(z+\frac{5}{4}-\frac{3}{4}\right)\left(z+\frac{5}{4}+\frac{3}{4}\right)}dz$$

$$= \text{R.P}\left(\frac{-i}{2}\right)\int_C \frac{z^2}{\left(z+\frac{1}{2}\right)(z+2)}dz$$

$$= \text{R.P}\left(\frac{-i}{2}\right)\int_C f(z)dz \quad \text{where } f(z) = \frac{z^2}{\left(z+\frac{1}{2}\right)(z+2)}$$

The singular points are $z = -\dfrac{1}{2}$ and $z = -2$.

Now C is $|z| = 1$.

When $z = -\dfrac{1}{2}$, $|z| = |-\dfrac{1}{2}| = \dfrac{1}{2} < 1$.

$\therefore z = -\frac{1}{2}$ lies inside C.

When $z = -2$, $|z| = |-2| = 2 > 1$.

$\therefore z = -2$ lies outside C.

$z = -\dfrac{1}{2}$ is a simple pole.

$$R\left(-\frac{1}{2}\right) = \lim_{z\to-\frac{1}{2}}\left(z+\frac{1}{2}\right)f(z) = \lim_{z\to-\frac{1}{2}}\left(z+\frac{1}{2}\right)\frac{z^2}{\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \lim_{z\to-\frac{1}{2}}\frac{z^2}{z+2} = \frac{\frac{1}{4}}{-\frac{1}{2}+2} = \frac{\frac{1}{4}}{\frac{3}{2}} = \frac{1}{4}\times\frac{2}{3} = \frac{1}{6}.$$

By Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i \times R\left(-\frac{1}{2}\right) = 2\pi i \times \frac{1}{6} = \frac{\pi i}{3}.$$

$$\therefore \int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta = \text{R.P}\left(-\frac{i}{2}\right) \times \left(\frac{\pi i}{3}\right) = \text{R.P}\left(\frac{\pi}{6}\right) = \frac{\pi}{6}.$$

15. (b) (i) Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$.

Solution. Let $I = \displaystyle\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$.

The integrand is of the form $\dfrac{P(x)}{Q(x)}$ where, the degree of $Q(x)$ is atleast 2 more than that of $P(x)$ and $Q(x)$ does not vanish for any real x.

Consider the integral $\displaystyle\int_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz$, where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi circle

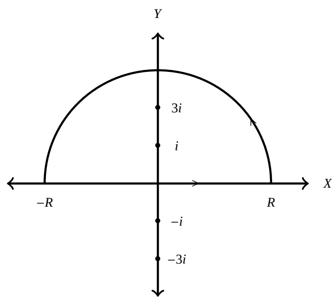
$S : |z| = R$, taken in the anticlockwise sense.

Let $f(z) = \dfrac{z^2 - z + 2}{z^4 + 10z^2 + 9}$.

Now $\displaystyle\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_S f(z)dz.$ (1)

Let us evaluate $\displaystyle\int_C f(z)dz$.

The poles are given by



$$z^4 + 10z^2 + 9 = 0$$

$$(z^2 + 1)(z^2 + 9) = 0$$

$$z^2 + 1 = 0, \quad z^2 + 9 = 0$$

$$z^2 = -1, \quad z^2 = -9$$

$$z = \pm i, \quad z = \pm 3i.$$

$z = i$ and $3i$ lies inside C.

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i[R(i) + R(3i)]$$

$$\begin{aligned}
R(i) &= \lim_{z\to i}(z-i)f(z) \\
&= \lim_{z\to i}(z-i)\frac{z^2-z+2}{(z+i)(z-i)(z^2+9)} \\
&= \frac{-1-i+2}{2i(-1+9)} = \frac{1-i}{16i}.
\end{aligned}$$

$$\begin{aligned}
R(3i) &= \lim_{z\to 3i}(z-3i)f(z) \\
&= \lim_{z\to 3i}\cancel{(z-3i)}\frac{z^2-z+2}{(z^2+1)\cancel{(z-3i)}(z+3i)} \\
&= \frac{-9-3i+2}{(-9+1)6i} = \frac{-7-3i}{-48i} = -\frac{7+3i}{-48i} = \frac{7+3i}{48i}.
\end{aligned}$$

$$\therefore \int_C f(z)dz = 2\pi i\left[\frac{1-i}{16i} + \frac{7+3i}{48i}\right]$$

$$= 2\pi i\left[\frac{3-3i+7+3i}{48i}\right] = 2\pi \times \frac{10}{48} = \frac{5\pi}{12}.$$

By Cauchy's lemma $\int_S f(z)dz = 0$ as $R \to \infty$.

$$\therefore (1) \Rightarrow \frac{5\pi}{12} = \int_{-R}^{R} f(x)dx + \int_S f(z)dz$$

Taking lim as $R \to \infty$ we get

$$\frac{5\pi}{12} = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx + \lim_{R\to\infty}\int_S f(z)dz$$

$$\frac{5\pi}{12} = \int_{-\infty}^{\infty} f(x)dx + 0$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}dx = \frac{5\pi}{12}.$$

15. (b) (ii) Evaluate $\int_C \frac{z+1}{(z^2+2z+4)^2} dz$ where C is $|z+1+i| = 2$ using Cauchy's integral formula.

Solution. $z^2 + 2z + 4 = (z+1)^2 + 4 - 1$

$$= (z+1)^2 + 3 = (z+1+\sqrt{3}i)(z+1-\sqrt{3}i).$$

The singular points are $-1-\sqrt{3}i$ and $-1+\sqrt{3}i$.

When $z = -1-\sqrt{3}i$, $|z+1+i| = |-1-\sqrt{3}i+1+i|$

$$= |i(1-\sqrt{3})| = |\sqrt{3}-1| < 2.$$

$\therefore z = -1-\sqrt{3}i$ lies inside C.

When $z = -1+\sqrt{3}i$, $|z+1+i| = |-1+\sqrt{3}i+1+i|$

$$= |i(\sqrt{3}+1)| = \sqrt{3}+1 > 2.$$

$\therefore z = -1+\sqrt{3}i$ lies out side C.

$$\therefore \frac{z+1}{(z^2+2z+4)^2} = \frac{z+1}{[(z+1\sqrt{3}i)(z+1)-\sqrt{3}i]^2}$$

$$= \frac{\frac{z+1}{(z+1-\sqrt{3}i)^2}}{(z+1+\sqrt{3}i)^2}$$

$$= \frac{f(z)}{(z+1+\sqrt{3}i)^2} \text{ where } f(z) = \frac{z+1}{(z+1-\sqrt{3}i)^2}.$$

$$\therefore \int \frac{z+1}{(z^2+2z+4)^2} dz = \int \frac{f(z)}{(z+1+\sqrt{3}i)^2} dz = 2\pi i \times f'(-1-\sqrt{3}i)$$

[By Cauchy's integral formula]

Now, $f(z) = \frac{z+1}{(z+1-\sqrt{3}i)^2}$

$$f'(z) = \frac{(z+1-\sqrt{3}i)^2 - (z+1)\cdot 2(z+1-\sqrt{3}i)}{(z+1-\sqrt{3}i)^4}$$

$$f'(-1-\sqrt{3}i) = \frac{(-1-\sqrt{3}i-\sqrt{3}i)^2 - (-1-\sqrt{3}i+1)\cdot 2(-1-\sqrt{3}i+1-\sqrt{3}i)}{(-1-\sqrt{3}i+1-\sqrt{3}i)^4}$$

$$= \frac{(-2\sqrt{3}i)^2 - 2(-\sqrt{3}i)(-2\sqrt{3}i)}{(-2\sqrt{3}i)^4}$$

$$= \frac{4 \times 3 \times (-1) - 4 \times 3(-1)}{16 \times 9} = \frac{-12 + 12}{144} = 0.$$

$$\therefore \int_C \frac{z+1}{(z^2+2z+4)^2} dz = 2\pi i \times 0 = 0.$$

2.4.7 May/June 2013 (R 2008)

Part A

9. Evaluate $\int_C \frac{3z^2+7z+1}{z+1} dz$ where C is $|z| = \frac{1}{2}$.

Solution. The singular point is $z = -1$.

Since C is $|z| = \frac{1}{2}$, $z = -1$ lies outside C.

$\therefore \frac{3z^2+7z+1}{z+1}$ is analytic inside C.

$\therefore$ By Cauchy's theorem, $\int_C \frac{3z^2+7z+1}{z+1} dz = 0$.

10. Find the residue of $\frac{1-e^{2z}}{z^4}$ at $z = 0$.

Solution. Let $f(z) = \frac{1-e^{2z}}{z^4}$.

$z = 0$ is a pole of order 3. [Expand e^{2z} and simplify]

$$\therefore R(0) = \frac{1}{(3-1)!} \lim_{z \to 0} \frac{d^2}{dz^2} (z-0)^3 f(z)$$

$$= \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \left[z^3 \frac{1-e^{2z}}{z^4} \right]$$

$$= \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \left[\frac{1-e^{2z}}{z} \right]$$

$$= \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \left[\frac{1 - 1 - 2z - \frac{4z^2}{2!} - \frac{8z^3}{3!} - \cdots}{z} \right]$$

$$= \frac{1}{2} \lim_{z \to 0} \frac{d}{dz} \left[2 - \frac{8z}{3} - \cdots \right]$$

$$= \frac{1}{2} \lim_{z \to 0} \left[\frac{-8}{3} + \text{powers of } z \right] = \frac{1}{2}\left(\frac{-8}{3} \right) = -\frac{4}{3}.$$

Part B

15. (a) (i) Expand the function $f(z) = \dfrac{z^2 - 1}{z^2 + 5z + 6}$ in Laurent's series for $|z| > 3$.

Solution. Let $f(z) = \dfrac{z^2 - 1}{z^2 + 5z + 6} = \dfrac{z^2 - 1}{(z + 2)(z + 3)}$.

$$|z| > 3 \Rightarrow \left|\frac{z}{3}\right| > 1 \Rightarrow \left|\frac{3}{z}\right| < 1 \Rightarrow \left|\frac{1}{z}\right| < \frac{1}{3}$$

$$\frac{2}{|z|} < \frac{2}{3} < 1$$

$$\begin{aligned}
f(z) &= 1 + \frac{3}{z+2} - \frac{8}{z+3} \\
&= 1 + \frac{3}{z\left(1 + \frac{2}{z}\right)} - \frac{8}{z\left(1 + \frac{3}{z}\right)} \\
&= 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1 + \frac{3}{z}\right)^{-1} \\
&= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \cdots\right) - \frac{8}{2}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right) \\
&= 1 + \frac{3}{2} - \frac{6}{z^2} + \frac{12}{z^3} - \frac{24}{z^4} + \cdots - \frac{8}{z} + \frac{24}{z^2} - \frac{72}{z^3} + \cdots \\
&= 1 - 5z^{-1} + 16z^{-2} - 60z^{-3} \cdots .
\end{aligned}$$

15. (a) (ii) Evaluate $\displaystyle\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is $|z| = 3$.

Solution. $z = 1$ and $z = 2$ are singular points.

The circle is $|z| = 3$.

When $z = 1, |z| = |1| < 3$.

$\therefore z = 1$ lies inside the circle.

When $z = 2, |z| = |2| < 3$.

$\therefore z = 2$ also lies inside the circle.

Let $\dfrac{1}{(z-1)(z-2)} = \dfrac{(z-1)-(z-2)}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$.

$$\text{Now } I = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$$

$$= \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz$$

$$= \int_C \frac{f(z)}{z-2} dz - \int_C \frac{f(z)}{z-1} dz$$

$$I = 2\pi i f(2) - 2\pi i f(1) \tag{1}$$

where $f(z) = \sin \pi z^2 + \cos \pi z^2$.

When $a = 2$. $f(2) = \sin 4\pi + \cos 4\pi = 1$.

When $a = 1$, $f(1) = \sin \pi + \cos \pi = -1$.

$(1) \implies I = 2\pi i f(2) - 2\pi i f(1) = 2\pi i.1 - 2\pi i(-1) = 4\pi i$.

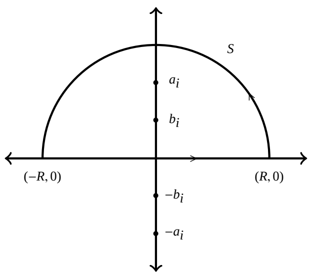
15. (b) (i) Evaluate $\displaystyle\int_0^\infty \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$ $a > 0, b > 0$.

Solution. Let $f(x) = \dfrac{x^2}{(x^2+a^2)(x^2+b^2)}$.

Since $f(x)$ is even, $\displaystyle\int_0^\infty f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx$.

Let $I = \displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}$.

The integrand is of the form $\dfrac{P(x)}{Q(x)}$ where degree of $Q(x)$ is 2 more than that of $P(x)$ and $Q(x)$ does not vanish for any real x.



Consider the integral $\displaystyle\int_C \frac{z^2}{(z^2+a^2)(z^2+b^2)} dz$ where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi circle $S : |z| = R$ taken in the anticlockwise sense.

Now, $\int\limits_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_S f(z)dz$ (1)

We shall evaluate $\int\limits_C f(z)dz$.

We have, $f(z) = \dfrac{z^2}{(z^2+a^2)(z^2+b^2)} = \dfrac{z^2}{(z+ai)(z-ai)(z+bi)(z-bi)}$

$\pm ai$ and $\pm bi$ are the simple poles.

But a_i and b_i only lie inside C.

$\therefore$ By Cauchy's residue theorem

$$\int\limits_C f(z)dz = 2\pi i[R(ai) + R(bi)].$$

$$R(ai) = \lim_{z\to ai}(z-ai)\frac{z^2}{(z+ai)(z-ai)(z^2+b^2)}$$

$$= \frac{-a^2}{2ai(b^2-a^2)} = \frac{a}{2i(a^2-b^2)}.$$

$$R(bi) = \frac{b}{2i(b^2-a^2)}$$

Now, $\int\limits_C f(z)dz = 2\pi i\left(\dfrac{a}{2i(a^2-b^2)} + \dfrac{b}{2i(b^2-a^2)}\right)$

$$= \frac{2\pi i}{2i}\left(\frac{a-b}{(a-b)(a+b)}\right) = \frac{\pi}{a+b}.$$

$$\therefore (1) \Longrightarrow \int_{-R}^{R} f(x)dx + \int_S f(z)dz = \frac{\pi}{a+b}. \quad (2)$$

Also, $\lim\limits_{z\to\infty} zf(z) = \lim\limits_{z\to\infty} \dfrac{z^3}{(z^2+a^2)(z^2+b^2)} \to 0$ as $z \to \infty$

$\therefore$ By Cauchy's lemma, $\int_S f(z)dz \to 0$ as $R \to \infty$.

$$\text{Now (2)} \Longrightarrow \lim_{R\to\infty} \int_{-R}^{R} f(x)dx + \lim_{R\to\infty} \int_{S} f(z)dz = \frac{\pi}{a+b}.$$

$$\begin{aligned}
\lim_{R\to\infty} \int_{-R}^{R} f(x)dx &= \frac{\pi}{a+b} \\
\int_{-\infty}^{\infty} f(x)dx &= \frac{\pi}{a+b} \\
\int_{0}^{\infty} f(x)dx &= \frac{\pi}{2(a+b)}.
\end{aligned}$$

15. (b) (ii) Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta$ using contour integration.

Solution. Let $z = e^{i\theta}$, then $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$.

As θ varies from 0 to $2\pi, z$ moves along $|z| = 1$.

$\therefore C$ is $|z| = 1$.

Now $z^3 = e^{i3\theta} = \cos 3\theta + i\sin 3\theta$.

$\therefore \cos 3\theta =$ R.P of z^3.

$$\begin{aligned}
\therefore \int_C \frac{\cos 3\theta}{5-4\cos\theta} d\theta &= R.P \int_C \frac{z^3}{5-4\frac{1}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{iz} = R.P \int_C \frac{z^2}{5-2z-\frac{2}{z}} \frac{dz}{i} \\
&= R.P \frac{1}{i} \int_C \frac{z^3}{5z-2z^2-2} dz = R.P\left(\frac{-1}{i}\right) \int_C \frac{z^3}{2z^2-5z+2} dz \\
&= R.P\left(\frac{-1}{i}\right) \int_C \frac{z^3}{2\left(z^2-\frac{5}{2}z+1\right)} dz \\
&= R.P\left(\frac{i}{2}\right) \int_C \frac{z^3}{\left(z-\frac{5}{4}\right)^2+1-\frac{25}{16}} dz \\
&= R.P\left(\frac{i}{2}\right) \int_C \frac{z^3}{\left(z-\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2} dz \\
&= R.P\left(\frac{i}{2}\right) \int_C \frac{z^3}{\left(z-\frac{5}{4}+\frac{3}{4}\right)\left(z-\frac{5}{4}-\frac{3}{4}\right)} dz \\
&= R.P\left(\frac{i}{2}\right) \int_C \frac{z^3}{\left(z-\frac{1}{2}\right)(z-2)} dz
\end{aligned}$$

$$= R.P\left(\frac{i}{2}\right)\int_C f(z)dz \text{ where } f(z) = \frac{z^3}{\left(z-\frac{1}{2}\right)(z-2)}.$$

Singular points are $\frac{1}{2}$ and 2, which are simple poles.
Now, C is $|z| = 1$.
When $z = \frac{1}{2}$, $|z| = \frac{1}{2} < 1 \Longrightarrow z = \frac{1}{2}$ lies inside C.
When $z = 2$, $|z| = 2 > 1 \Longrightarrow z = 2$ lies outside C.
By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i R\left(\frac{1}{2}\right)$.

$$\text{Now } R\left(\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)f(z)$$

$$R\left(\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z^3}{\left(z-\frac{1}{2}\right)(z-2)} = \frac{\frac{1}{8}}{\frac{-3}{2}} = -\frac{1}{8}\frac{2}{3} = -\frac{1}{12}.$$

$$\therefore \int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}d\theta = R.P\left(\frac{i}{2}\right)2\pi i R\left(\frac{1}{2}\right) = -\pi\left(\frac{-1}{12}\right) = \frac{\pi}{12}.$$

2.4.8 November/December 2012 (R 2008)

Part A

9. Evaluate $\int_C \frac{z}{(z-1)(z-2)}dz$ where C is the circle $|z| = \frac{1}{2}$.
Solution. The singular points are $z = 1$ and $z = 2$.

   Since C is $|z| = \frac{1}{2}, z = 1$ and $z = 2$ lie outside C.

   $\therefore \frac{z}{(z-1)(z-2)}$ is analytic inside C.

   $\therefore$ By Cauchy's theorem, $\int_C \frac{z}{(z-1)(z-2)}dz = 0$.

10. If $f(z) = -\frac{1}{z-1} - 2\left[1 + (z-1) + (z-1)^2 + \cdots\right]$ find the residue of $f(z)$ at $z = 1$.

    Solution. In the Laurent's series expansion,

    residue at 1 = Coefficient of $\frac{1}{z-1} = -1$.

Part B

15. (a) (i) Evaluate $\int_C \frac{z+4}{z^2+2z+5}dz$ where C is the circle (i) $|z+1+i| = 2$ (ii) $|z+1-i| = 2$ (iii) $|z+1| = 1$.

Solution. Let $I = \int_C \frac{z+4}{z^2+2z+5}dz.$

The singular points are given by $z^2 + 2z + 5 = 0$.

$(z+1)^2 + 5 - 1 = 0$

$(z+1)^2 + 4 = 0$

$(z+1)^2 = -4$

$z + 1 = \pm 2i$

$z = -1 \pm 2i.$

(i) The circle is $|z+1+i| = 2$.

When $z = -1-2i$, $|z+1+i| = |-1-2i+1+i| = |-i| = 1 < 2$.

$\therefore$ $-1-2i$ is a point inside the circle.

When $z = -1+2i$, $|z+1+i| = |-1+2i+1+i| = |3i| = 3 > 2$.

$\therefore$ $z = -1+2i$ lies outside the circle.

Now, $\int_C \frac{z+4}{z^2+2z+5}dz = \int_C \frac{z+4}{(z+1-2i)(z+1+2i)}dz$

$$= \int_C \frac{\frac{z+4}{z+1-2i}}{z+1+2i}dz = \int_C \frac{f(z)}{z+1+2i}dz.$$

Where $f(z) = \frac{z+4}{z+1-2i}$.

By Cauchy's integral formula

$$f(-1-2i) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z+1+2i}dz$$

$$\frac{-1-2i+4}{-1-2i+1-2i} = \frac{1}{2\pi i}\int_C \frac{f(z)}{z+1-2i}dz$$

$$\frac{3-2i}{-4i} = \frac{1}{2\pi i}\int_C \frac{z+4}{z^2+2z+5}dz$$

$$-\frac{\pi}{2}(3-2i) = \int_C \frac{z+4}{z^2+2z+5}dz$$

$$\int_C \frac{z+4}{z^2+2z+5}dz = \frac{\pi}{2}(2i-3).$$

(ii) The circle is $|z+1-i| = 2$.

When $z = -1-2i$, $|z+1-i| = |-1-2i+1-i| = |-3i| = 3 > 2$.

$\therefore$ $z = -1-2i$ lies outside the circle.

When $z = -1+2i$, $|z+1-i| = |-1+2i+1-i| = |i| = 1 < 2$.

$\therefore$ $z = -1+2i$ lies inside the circle.

Now $\displaystyle\int_C \frac{z+4}{z^2+2z+5}dz = \int_C \frac{\frac{z+4}{z+1+2i}}{z+1-2i}dz = \int_C \frac{f(z)}{z+1-2i}dz,$

where, $f(z) = \dfrac{z+4}{z+1+2i}$, $a = -1+2i$.

$$f(a) = \frac{-1+2i+4}{-1+2i+1+2i} = \frac{3+2i}{4i}.$$

By Cauchy's integral formula

$$\int_C \frac{f(z)}{z-a}dz = 2\pi i f(a) = 2\pi i\frac{3+2i}{4i} = \frac{3+2i}{2}\pi.$$

(iii) C is $|z+1| = 1$.

When $z = -1-2i$, $|z+1| = |-1-2i+1| = |-2i| = 2 > 1$.

$\therefore -1-2i$ lies outside C.

When $z = -1+2i$, $|z+1| = |-1+2i+1| = |2i| = 2 > 1$.

$\therefore -1+2i$ lies outside C.

$\therefore$ $f(z) = \dfrac{z+4}{z^2+2z+5}$ is analytic inside C.

$\therefore$ By Cauchy's theorem $\displaystyle\int_C f(z)dz = 0$.

15. (a) (ii) Find the residues of $f(z) = \dfrac{z^2}{(z-1)^2(z+2)^2}$ at its isolated singularities using Laurent's series expansions. Also state the valid region.

Solution. Let $\dfrac{z^2}{(z-1)^2(z+2)^2} = \dfrac{A}{z-1} + \dfrac{B}{(z-1)^2} + \dfrac{C}{z+2} + \dfrac{D}{(z+2)^2}$.

$$\therefore z^2 = A(z-1)(z+2)^2 + B(z+2)^2 + C(z+2)(z-1)^2 + D(z-1)^2.$$

When $z = 1, 9B = 1 \quad \Rightarrow B = \dfrac{1}{9}$.

When $z = -2, 9D = 4 \quad \Rightarrow D = \dfrac{4}{9}$.

Equating the coefficients of z^3 we get

$A + C = 0 \Rightarrow C = -A.$

When $z = 0, -4A + 4B + 2C + D = 0$

$$-4A + \frac{4}{9} - 2A + \frac{4}{9} = 0$$

$$6A = \frac{8}{9}$$

$$A = \frac{8}{6\times 9} = \frac{4}{27}.$$

$$\therefore C = -\frac{4}{27}.$$

$$\therefore \frac{z^2}{(z-1)^2(z+2)^2} = \frac{4}{27}\cdot\frac{1}{z-1} + \frac{1}{9}\cdot\frac{1}{(z-1)^2} - \frac{4}{27}\cdot\frac{1}{z+2} + \frac{4}{9}\cdot\frac{1}{(z+2)^2}.$$

Consider $\dfrac{4}{27}\cdot\dfrac{1}{z-1} + \dfrac{1}{9}\cdot\dfrac{1}{(z-1)^2} = \dfrac{4}{27z\left(1-\frac{1}{z}\right)} + \dfrac{1}{9z^2\left(1-\frac{1}{z}\right)^2}$

$$= \frac{4}{27z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{9z^2}\left(1-\frac{1}{z}\right)^{-2}$$

$$= \frac{4}{27z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right] + \frac{1}{9z^2}\left[1+\frac{2}{z}+\frac{3}{z^2}+\cdots\right]$$

Residue at $z = 1$ = Coefficient of $\dfrac{1}{z} = \dfrac{4}{27}$.

Also, $\dfrac{4}{9}\dfrac{1}{(z+2)^2} - \dfrac{4}{27}\dfrac{1}{z+2} = \dfrac{4}{9z^2\left(1+\frac{2}{z}\right)^2} - \dfrac{4}{27z\left(1+\frac{2}{z}\right)}$

$$= \frac{4}{9z^2}\left(1+\frac{2}{z}\right)^{-2} - \frac{4}{27z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= \frac{4}{9z^2}\left[1-2\cdot\frac{2}{z}+3\left(\frac{2}{z}\right)^2\cdots\right] - \frac{4}{27z}\left[1-\frac{2}{z}+\frac{4}{z^2}\right].$$

Residue at $z = -2$ = Coefficient of $\frac{1}{z} = -\frac{4}{27}$

The valid region is $|z| < 2 \cdot 5$.

15. (b) Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta,\ a > b > 0.$

Solution. Let, $I = \displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta} d\theta = \int_0^{2\pi} \frac{1-\cos 2\theta}{2(a+b\cos\theta)} d\theta.$

Put $z = e^{i\theta}$ then $d\theta = \frac{dz}{iz}$.

Now, $z^2 = e^{i2\theta} = \cos 2\theta + i\sin 2\theta.$

$1 - z^2 = 1 - \cos 2\theta - i\sin 2\theta.$

$\therefore 1 - \cos 2\theta =$ R.P $(1 - z^2)$.

As θ varies from 0 to 2π, C varies along the unit circle $|z| = 1$.

$$\text{Hence, } I = R.P \int_C \frac{1-z^2}{2\left(a + \frac{b}{2}\left(z + \frac{1}{z}\right)\right)} \frac{dz}{iz} = R.P \frac{1}{2} \int_C \frac{1-z^2}{2a + b\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$

$$= R.P \frac{1}{2i} \int_C \frac{z(1-z^2)}{(2az + bz^2 + b)} \frac{dz}{z} = R.P \frac{1}{2i} \int_C \frac{1-z^2}{2az + bz^2 + b} dz$$

$$= R.P \frac{1}{2i} \int_C \frac{1-z^2}{b\left(z^2 + \frac{2a}{b}z + 1\right)} dz = R.P \frac{1}{2bi} \int_C \frac{1-z^2}{\left(z + \frac{a}{b}\right)^2 + 1 - \frac{a^2}{b^2}} dz$$

$$= R.P \frac{1}{2bi} \int_C \frac{1-z^2}{\left(z + \frac{a}{b}\right)^2 - \frac{a^2-b^2}{b^2}} dz$$

$$= R.P \frac{1}{2bi} \int_C \frac{1-z^2}{\left(z + \frac{a}{b}\right)^2 - \left(\frac{\sqrt{a^2-b^2}}{b}\right)^2} dz$$

$$= R.P \frac{1}{2bi} \int_C \frac{1-z^2}{\left(z + \frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b}\right)\left(z + \frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b}\right)} dz$$

$$= R.P \frac{1}{2bi} \int_C \frac{1-z^2}{(z-z_1)(z-z_2)} dz$$

$$= R.P \frac{1}{2bi} \int_C f(z) dz \text{ where } f(z) = \frac{1-z^2}{(z-z_1)(z-z_2)} \text{ where}$$

$$z_1 = -\frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b} \text{ and } z_2 = -\frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b}.$$

z_1 and z_2 are simple poles.

When $z = z_1$, $|z| = |z_1| = \left|-\frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b}\right| = \frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b} > \frac{a}{b} > 1.$

As $a > b \implies \frac{a}{b} > 1 \implies \frac{a^2}{b^2} > 1,\ a^2 > b^2,\ a^2 - b^2 > 0.$

$\therefore$ z_1 lies outside C.

Also, $z_1 z_2 = 1$ (products of the roots)

Now, $z_2 = \frac{1}{z_1} \implies |z_2| = \frac{1}{|z_1|} < 1.$

$\therefore$ z_2 lies inside C.

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i R(z_2)$.

Now, $R(z_2) = \lim_{z\to z_2}(z-z_2)f(z)$

$$R\left(-\frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b}\right) = \lim_{z\to z_2}(z-z_2)\frac{1-z^2}{(z-z_1)(z-z_2)} = \frac{1-z_2^2}{z_2-z_1}.$$

$$\text{Now, } 1 - z_2^2 = 1 - \left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)^2$$

$$= \frac{b^2 - (a^2 + a^2 - b^2 - 2a\sqrt{a^2-b^2})}{b^2}$$

$$= \frac{b^2 - a^2 - a^2 + b^2 + 2a\sqrt{a^2-b^2}}{b^2}$$

$$= \frac{2b^2 - 2a^2 + 2a\sqrt{a^2-b^2}}{b^2}$$

$$= \frac{2a\sqrt{a^2-b^2} - 2(a^2-b^2)}{b^2}$$

$$= \frac{2\sqrt{a^2-b^2}(a - \sqrt{a^2-b^2})}{b^2}$$

$$z_2 - z_1 = \frac{2\sqrt{a^2-b^2}}{b}$$

$$\therefore R(z_2) = \frac{2\sqrt{a^2-b^2}(a-\sqrt{a^2-b^2})}{b^2 2\sqrt{a^2-b^2}} \times b$$

$$= \frac{a - \sqrt{a^2-b^2}}{b}$$

$$\therefore I = \text{R.P}\ \frac{1}{2bi} \cdot 2\pi i \left(\frac{a - \sqrt{a^2 - b^2}}{b} \right)$$

$$= \frac{\pi}{b^2}(a - \sqrt{a^2 - b^2})$$

2.4.9 May/June 2012 (R 2008)

Part A

9. Define singular point.

   If a function fails to be analytic at a point z_0, but is analytic at some point in every neighborhood of z_0, then z_0 is a singular point of f.

10. Find the residue of $f(z) = \dfrac{4}{z^3(z-2)}$ at a simple pole.

    Solution. $f(z) = \dfrac{4}{z^3(z-2)}$.

    The poles are given by $z = 0, z = 2$.

    $z = 2$ is a simple pole.

    $$\text{Now } R(2) = \lim_{z\to 2}(z-2)f(z)$$

    $$= \lim_{z\to 2}(z-2)\frac{4}{z^3(z-2)} = \lim_{z\to 2}\frac{4}{z^3} = \frac{4}{z^3} = \frac{4}{8} = \frac{1}{2}.$$

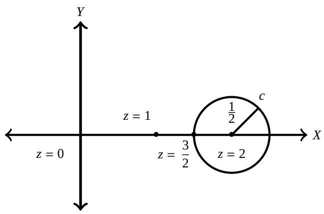
Part B

15. (a) (i) Evaluate $\displaystyle\int_C \frac{z}{(z-1)(z-2)^2}dz$, where C is $|z-2| = \dfrac{1}{2}$ using Cauchy's integral formula.

    Solution. C is $|z-2| = \dfrac{1}{2}$.

    C is a circle with $(2, 0)$ as centre and radius $\dfrac{1}{2}$ units.

    The singular points are $z = 1$ and $z = 2$. $z = 1$ lies outside C and $z = 2$ lie within C.



$$\therefore \int_C \frac{z}{(z-1)(z-2)^2}dz = \int_C \frac{\frac{z}{z-1}}{(z-2)^2}dz$$

$$= \int_C \frac{f(z)}{(z-2)^2}dz \qquad \text{where } f(z) = \frac{z}{z-1}$$

$$= 2\pi i \times f'(2) \qquad f'(z) = \frac{z-1-z}{(z-1)^2}$$

$$= 2\pi i \times (-1) \qquad = -\frac{1}{(z-1)^2}$$

$$= -2\pi i \qquad f'(2) = \frac{-1}{1} = -1.$$

15. (a) (ii) Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in Laurent's series valid for the regions $1 < |z| < 3$ and $|z| > 3$.

Solution. Let $f(z) = \dfrac{1}{(z+1)(z+3)} = \dfrac{A}{z+1} + \dfrac{B}{z+3}$.

$$1 = A(z+3) + B(z+1).$$

When $z = -1, 2A = 1, A = \dfrac{1}{2}$.

When $z = -3, -2B = 1, B = -\dfrac{1}{2}$.

$$\therefore \frac{1}{(z+1)(z+3)} = \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2}\cdot\frac{1}{z+3}.$$

(i) Consider $1 < |z| < 3$.

This is annular about $z = 0$ and $f(z)$ is analytic in the region.

$\therefore$ $f(z)$ has a Laurent's series expansion about $z = 0$.

$$|z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1.$$

$$|z| > 1 \Rightarrow \frac{1}{|z|} < 1.$$

$$\therefore f(z) = \frac{1}{2}\cdot\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{2}\cdot\frac{1}{3\left(1+\frac{z}{3}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$
$$= \frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\frac{1}{z^4}\cdots\right] - \frac{1}{6}\left[1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\frac{z^4}{81}\cdots\right]$$

(ii) Consider the region $|z| > 3$.

This region is annular about $z = 0$, where $f(z)$ is analytic.

$$|z| > 3 \Rightarrow \frac{1}{|z|} < \frac{1}{3} \Rightarrow \frac{3}{|z|} < 1.$$

Since $\frac{1}{|z|} < \frac{1}{3} < 1$, we have

$$f(z) = \frac{1}{2}\cdot\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{2}\cdot\frac{1}{z\left(1+\frac{3}{z}\right)}$$
$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$
$$= \frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\frac{1}{z^4}\cdots\right] - \frac{1}{2z}\left[1-\frac{3}{z}+\frac{9}{z^2}-\frac{27}{z^3}+\frac{81}{z^4}\cdots\right].$$

15. (b) (i) Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$ where C is $|z-i| = 2$.

Solution. Let $f(z) = \frac{z-1}{(z+1)^2(z-2)}$.

The poles are given by $z = -1$ and $z = 2$.

$z = -1$ is a pole of order 2 and $z = 2$ is a simple pole.

C is $|z-i| = 2$.

When $z = -1, |z-i| = |-1-i| = \sqrt{1+1} = \sqrt{2} < 2$.

$\Rightarrow$ $z = -1$ lies inside C.

When $z = 2, |z-i| = |2-i| = \sqrt{4+1} = \sqrt{5} > 2$.

$\therefore$ $z = 2$ lies outside C.

$\therefore$ By Cauchy's residue theorem $\int_C f(z)dz = 2\pi i R(-1)$.

Since $z = -1$ is a pole of order 2,

$$R(-1) = \frac{1}{(2-1)!}\lim_{z\to-1}\frac{d}{dz}\left((z+1)^2 f(z)\right) = \lim_{z\to-1}\frac{d}{dz}\left((z+1)^2\frac{z-1}{(z+1)^2(z-2)}\right)$$

$$= \lim_{z\to -1} \frac{d}{dz}\left(\frac{z-1}{z-2}\right) = \lim_{z\to -1} \frac{z-2-(z-1)}{(z-2)^2} = \frac{-3+2}{9} = \frac{-1}{9}.$$

$$\text{Hence, } \int_C \frac{z-1}{(z+1)^2(z-2)}dz = 2\pi i R(-1) = 2\pi i\left(\frac{-1}{9}\right) = -\frac{2\pi i}{9}.$$

15. (b) (ii) Evaluate $\int_0^\infty \frac{\cos mx}{x^2+a^2}dx$ using contour integration.

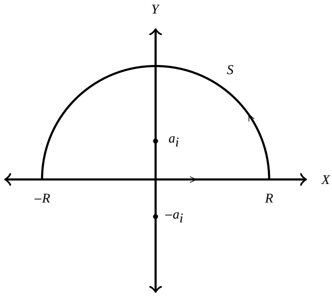
Solution. Let us consider the integral $\int_C \frac{e^{imz}}{z^2+a^2}dz$, where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi circle

$S : |z| = R$ taken in the anticlockwise sense.

Let $f(z) = \dfrac{1}{z^2+a^2}$

$$\text{Now } \int_C e^{imz} f(z)dz = \int_{-R}^{R} e^{imx} f(x)dx + \int_S e^{imz} f(z)dz \qquad (1)$$

Let us evaluate $\int_C e^{imz} f(z)dz.$



The poles are given by

$z^2 + a^2 = 0$

$(z+ai)(z-ai) = 0$

i.e., $z = ai, -ai$

$z = ai$ lies inside C.

By Cauchy's residue theorem, $\int_C e^{imz} f(z)dz = 2\pi i R(ai).$

$z = ai$ is a simple pole.

$$\therefore R(ai) = \lim_{z\to ai}(z-ai)e^{imz} f(z)$$

$$= \lim_{z\to ai} \cancel{(z-ai)} \frac{e^{imz}}{(z+ai)\cancel{(z-ai)}} = \frac{e^{-am}}{2ai}.$$

$$\therefore \int_C e^{imz} f(z)dz = \frac{2\pi i e^{-am}}{2ai} = \frac{\pi}{a}e^{-am}.$$

Also $|f(z)| = |\frac{1}{z^2+a^2}| \to 0$ as $|z| \to \infty$.

$\therefore$ By Jordon's lemma, $\int_s e^{imz} f(z)dz \to 0$ as $R \to \infty$.

Now (1)$\Rightarrow \frac{\pi}{a}e^{-am} = \int_{-R}^{R} e^{imx} f(x)dx + \int_s e^{imz} f(z)dz.$

Taking limit as $R \to \infty$ we get

$$\frac{\pi}{a}e^{-am} = \lim_{R\to\infty} \int_{-R}^{R} e^{imx} f(x)dx + \lim_{R\to\infty} \int_s e^{imz} f(z)dz$$

$$\frac{\pi}{a}e^{-am} = \int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+a^2}dx + 0$$

$$= \int_{-\infty}^{\infty} \frac{\cos mx + i\sin mx}{x^2+a^2}dx.$$

Equating the real parts we get

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2}dx = \frac{\pi e^{-am}}{a}$$

$$2\int_{0}^{\infty} \frac{\cos mx}{x^2+a^2}dx = \frac{\pi e^{-am}}{a}$$

$$\int_{0}^{\infty} \frac{e^{mx}}{x^2+a^2}dx = \frac{\pi e^{-am}}{2a}.$$

2.5 Unit V Laplace Transforms.

2.5.1 May/June 2016 (R 2013)

Part A

5. State convolution theorem on laplace transforms.

   Solution. Refer page 289 of the main text book.

UNIT 5

By EasyEngineering.net

5 Laplace Transform

5.1 Introduction

The method of solving a differential equation involves many steps, namely 1) Computation of complementary function. 2) Computation of particular integral. 3) Evaluation of the arbitrary constants using the initial conditions. But Laplace transform is a powerful tool for solving differential equations without involving all the above steps. By the application of Laplace transforms, the given differential equation can be converted into an algebraic equation and the process of finding the solution becomes easy.

Laplace transforms was first used by the French Mathematician Pierre Simon De Laplace in the 18^{th} century. Laplace transform has wide applications in the Engineering fields like network analysis, control systems, circuit theory and many other areas.

Definition. Let $f(t)$ be defined for all $t \geq 0$, then $\int_0^\infty e^{-st} f(t)dt$ is defined as the Laplace transform of $f(t)$, if the integral exists.

It is denoted by $L[f(t)]$. It is a function of s.

$$\text{Hence, } L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s).$$

Result. The parameter s may be real or complex.

5.2 Sufficient condition for existence of Laplace Transforms

[May2015, Dec 2014, May 2011]

Let $f(t)$ be defined for all $t \geq 0$ such that (i) $f(t)$ is piecewise continuous in the interval $[0, \infty)$ and (ii) $f(t)$ is of exponential order $\alpha > 0$, then the Laplace transform of $f(t)$ exists for $s > \alpha$.

Note

1. Piecewise continuous on $[0, \infty)$ means that the function is continuous on every finite subinterval $0 \leq t \leq \alpha$ except possibly at a finite number of points where they have jumps i.e., $f(x+)$ and $f(x-)$ exist but not equal.

2. $f(t)$ is of exponential order $\alpha > 0$ if $|f(t)| \leq Me^{\alpha t}$ for all $t \geq 0$ and M is a constant. In other words, $\lim\limits_{t\to\infty}(e^{-\alpha t} f(t))$ is finite.

Example. t^n is of exponential order as $t \to \infty$.

$$\lim_{t\to\infty}(e^{-\alpha t}t^n) = \lim_{t\to\infty}\frac{t^n}{e^{\alpha t}} = \lim_{t\to\infty}\frac{n!}{\alpha^n e^{\alpha t}} = 0.$$

Result. The above conditions are sufficient but not necessary.

Example. $L\left[\frac{1}{\sqrt{t}}\right]$ exists but it is not continuous at $t = 0$.

Linearity Property

If $f(t)$ and $g(t)$ are two continuous functions of t and k is a constant, then

(i) $L[f(t) \pm g(t)] = L[f(t)] \pm L[g(t)]$.

(ii) $L[kf(t)] = kL[f(t)]$ where $k \neq 0$.

Laplace Transform of Standard functions

1. $L[1] = \displaystyle\int_0^\infty e^{-st}1dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}[0-1] = \frac{1}{s}.$

2. $L[e^{at}] = \displaystyle\int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{-(s-a)t}dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = -\frac{1}{s-a}[0-1] = \frac{1}{s-a}.$

3. $L[e^{-at}] = \dfrac{1}{s+a}.$

—

4. $L[t] = \int_0^\infty e^{-st} t\,dt = \int_0^\infty t\,d\Big(\frac{e^{-st}}{-s}\Big) = \Big[t\frac{e^{-st}}{-s}\Big]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}dt = \frac{1}{s}\Big[\frac{e^{-st}}{-s}\Big]_0^\infty = \frac{1}{s^2}.$

5. $$L[t^n] = \int_0^\infty e^{-st} t^n dt = \int_0^\infty t^n d\Big(\frac{e^{-st}}{-s}\Big) = \Big[t^n\frac{e^{-st}}{-s}\Big]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} nt^{n-1}dt$$
$$= \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}dt = \frac{n}{s}L[t^{n-1}] = \frac{n}{s}\frac{n-1}{s}L[t^{n-2}]$$
$$= \frac{n}{s}\frac{n-1}{s}\frac{n-2}{s}\cdots L[t]$$
$$= \frac{n}{s}\frac{n-1}{s}\frac{n-2}{s}\cdots\frac{1}{s^2} = \frac{n!}{s^{n+1}}.$$

6. $$L[\cosh at] = L\Big[\frac{e^{at}+e^{-at}}{2}\Big] = \frac{1}{2}\Big[L[e^{at}] + L[e^{-at}]\Big] = \frac{1}{2}\Big[\frac{1}{s-a} + \frac{1}{s+a}\Big]$$
$$= \frac{1}{2}\Big[\frac{s+a+s-a}{s^2-a^2}\Big] = \frac{1}{2}\Big[\frac{2s}{s^2-a^2}\Big] = \frac{s}{s^2-a^2}.$$

7. $$L[\sinh at] = L\Big[\frac{e^{at}-e^{-at}}{2}\Big] = \frac{1}{2}\Big[L[e^{at}] - L[e^{-at}]\Big]$$
$$= \frac{1}{2}\Big[\frac{1}{s-a} - \frac{1}{s+a}\Big] = \frac{a}{s^2-a^2}.$$

8. $$L[\cos at] = L\Big[\frac{e^{iat}+e^{-iat}}{2}\Big] = \frac{1}{2}\Big[\frac{1}{s-ai} + \frac{1}{s+ai}\Big]$$
$$= \frac{1}{2}\Big[\frac{s+ai+s-ai}{s^2+a^2}\Big] = \frac{s}{s^2+a^2}.$$

9. $$L[\sin at] = L\Big[\frac{e^{iat}-e^{-iat}}{2i}\Big] = \frac{1}{2i}\Big[\frac{1}{s-ai} - \frac{1}{s+ai}\Big]$$
$$= \frac{1}{2i}\Big[\frac{s+ai-s+ai}{s^2+a^2}\Big] = \frac{1}{2i}\Big[\frac{2ai}{s^2+a^2}\Big] = \frac{a}{s^2+a^2}.$$

Worked Examples

Example 5.1. Write a function for which Laplace transform does not exist. Explain why Laplace transform does not exist? [May 2007]

Solution. Let $f(t) = e^{t^2}$.

$$\lim_{t\to\infty} e^{-st} f(t) = \lim_{t\to\infty} e^{-st}e^{t^2}$$
$$= \lim_{t\to\infty} e^{-st+t^2}$$
$$= \lim_{t\to\infty} e^{t(-s+t)} = e^{\infty} = \infty.$$

$\therefore \lim_{t\to\infty} e^{-st}e^{t^2}$ is not finite.

i.e., e^{t^2} is not of exponential order.

$\therefore$ Laplace transform of $f(t)$ does not exist.

Example 5.2. Is the linearity property applicable to $L\left[\dfrac{1-\cos t}{t}\right]$? Reason out. [Dec 2012]

Solution. Linearity property can be applied when $\dfrac{1}{t}$ and $\dfrac{\cos t}{t}$ are continuous. Here, $\dfrac{1}{t}$ and $\dfrac{\cos t}{t}$ are not even defined at $t = 0$. Hence, they are not continuous. Therefore, linearity property is not applicable to $\dfrac{1-\cos t}{t}$.

Example 5.3. Find $L[3e^{5t} + 5\cos t]$. [Jan 2009]

Solution. $L[3e^{5t} + 5\cos t] = 3L[e^{5t}] + 5L[cost]$

$$= 3\frac{1}{s-5} + 5\frac{s}{s^2+1} = \frac{3}{s-5} + \frac{5s}{s^2+1}.$$

Example 5.4. Find $L[\cos \pi t + e^{-\frac{2}{3}t} + \sin 8t]$. [Jun 2009]

Solution. $L[\cos \pi t + e^{-\frac{2}{3}t} + \sin 8t] = L[\cos \pi t] + L[e^{-\frac{2}{3}t}] + L[\sin 8t]$

$$= \frac{s}{s^2+\pi^2} + \frac{1}{s+\frac{2}{3}} + \frac{8}{s^2+64}.$$

Example 5.5. Find $L[t^2 + e^{-5t} + 8 + \sinh 5t]$. [Jun 2007]

Solution. $L[t^2 + e^{-5t} + 8 + \sinh 5t] = L[t^2] + L[e^{-5t}] + 8L[1] + L[\sinh 5t]$

$$= \frac{2!}{s^3} + \frac{1}{s+5} + 8\frac{1}{s} + \frac{5}{s^2-25} = \frac{2}{s^3} + \frac{1}{s+5} + \frac{8}{s} + \frac{5}{s^2-25}.$$

Example 5.6. Find $L[(t+1)^2]$. [Jun 2002]

Solution. $L[(t+1)^2] = L[t^2 + 2t + 1]$

$$= L[t^2] + 2L[t] + L[1] = \frac{2!}{s^3} + 2\frac{1}{s^2} + \frac{1}{s}$$

$$= \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} = \frac{2+2s+s^2}{s^3}.$$

Example 5.7. Find $L[\cos^2 2t]$. [Dec 2005]

Solution. $L[\cos^2 2t] = L\left[\dfrac{1+\cos 4t}{2}\right] = \dfrac{1}{2}\left[L[1] + L[\cos 4t]\right] = \dfrac{1}{2}\left[\dfrac{1}{s} + \dfrac{s}{s^2+16}\right]$

$$= \frac{1}{2}\left[\frac{s^2+16+s^2}{s(s^2+16)}\right] = \frac{1}{2}\left[\frac{2s^2+16}{s(s^2+16)}\right] = \frac{s^2+8}{s(s^2+16)}.$$

Example 5.8. Find $L[\sin 2t \sin 3t]$. [Dec 2004]

Solution.
$$L[\sin 2t \sin 3t] = \frac{1}{2}L[\cos t - \cos 5t] = \frac{1}{2}\Big[\frac{s}{s^2+1} - \frac{s}{s^2+25}\Big]$$
$$= \frac{s}{2}\Big[\frac{s^2+25-s^2-1}{(s^2+1)(s^2+25)}\Big] = \frac{s}{2}\Big[\frac{24}{(s^2+1)(s^2+25)}\Big]$$
$$= \frac{12s}{(s^2+1)(s^2+25)}.$$

Example 5.9. Find $L[\sin^3 2t]$. [May 2004]

Solution.
$$L[\sin^3 2t] = L\Big[\frac{1}{4}(3\sin 2t - \sin 6t)\Big] = \frac{1}{4}\Big[3\frac{2}{s^2+4} - \frac{6}{s^2+36}\Big]$$
$$= \frac{6}{4}\Big[\frac{s^2+36-s^2-4}{(s^2+4)(s^2+36)}\Big] = \frac{3}{2}\Big[\frac{32}{(s^2+4)(s^2+36)}\Big] = \frac{48}{(s^2+4)(s^2+36)}.$$

Example 5.10. Find $L[t^{\frac{-1}{2}}]$. [May 2005]

Solution. $L[t^{-\frac{1}{2}}] = \int_0^\infty e^{-st}t^{-\frac{1}{2}}dt$

Let $st = x \Rightarrow t = \frac{x}{s} \Rightarrow dt = \frac{1}{s}dx.$

When $t = 0, x = 0,$ when $t = \infty, x = \infty$

$$\therefore L[t^{\frac{-1}{2}}] = \int_0^\infty e^{-x}\Big(\frac{x}{s}\Big)^{-\frac{1}{2}}\frac{dt}{s} = \frac{1}{\sqrt{s}}\int_0^\infty e^{-x}x^{-\frac{1}{2}}dx$$
$$= \frac{1}{\sqrt{s}}\int_0^\infty e^{-x}x^{\frac{1}{2}-1}dx$$
$$= \frac{1}{\sqrt{s}}\Gamma\left(\frac{1}{2}\right) \qquad \Big[\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx\Big]$$
$$= \frac{1}{\sqrt{s}}\sqrt{\pi} = \sqrt{\frac{\pi}{s}}.$$

Example 5.11. Find $L\left[t^{\frac{1}{2}}\right]$. [Dec 1996]

Solution. $L[t^{\frac{1}{2}}] = \int_0^\infty e^{-st}t^{\frac{1}{2}}dt$

Put $st = x \Rightarrow t = \frac{x}{s} \Rightarrow dt = \frac{1}{s}dx.$

When $t=0, x=0$, when $t=\infty, x=\infty$

$$\begin{aligned}
L\left[t^{\frac{1}{2}}\right] &= \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^{\frac{1}{2}}\frac{dx}{s} = \frac{1}{s^{\frac{3}{2}}}\int_0^\infty e^{-x}x^{\frac{1}{2}}dx \\
&= \frac{1}{s^{\frac{3}{2}}}\int_0^\infty e^{-x}x^{\frac{3}{2}-1}dx = \frac{1}{s^{\frac{3}{2}}}\Gamma\frac{3}{2} \\
&= \frac{1}{s^{\frac{3}{2}}}\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right) = \frac{1}{s^{\frac{3}{2}}}\frac{1}{2}\Gamma\frac{1}{2} = \frac{1}{2s^{\frac{3}{2}}}\sqrt{\pi} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}.
\end{aligned}$$

Example 5.12. Find $L[\cos 4t.\sin 2t]$. [Dec 2009]

Solution. $L[\cos 4t.\sin 2t] = L\left[\frac{\sin 6t - \sin 2t}{2}\right] = \frac{1}{2}\left[L[\sin 6t] - L[\sin 2t]\right]$

$$= \frac{1}{2}\left[\frac{6}{s^2+36} - \frac{2}{s^2+4}\right] = \frac{3}{s^2+36} - \frac{1}{s^2+4}.$$

Example 5.13. Find $L[\sin^2 t\cos^3 t]$. [May 2008]

Solution. We have $\sin^2 t\cos^3 t = \left(\frac{1-\cos 2t}{2}\right)\left(\frac{\cos 3t + 3\cos t}{4}\right)$

$$\begin{aligned}
&= \frac{1}{8}[\cos 3t + 3\cos t - \cos 2t\cos 3t - 3\cos 2t\cos t] \\
&= \frac{1}{8}\left[\cos 3t + 3\cos t - \frac{1}{2}(\cos 5t + \cos t) - \frac{3}{2}(\cos 3t + \cos t)\right] \\
&= \frac{1}{8}\left[\cos 3t + 3\cos t - \frac{1}{2}\cos 5t - \frac{1}{2}\cos t - \frac{3}{2}\cos 3t - \frac{3}{2}\cos t\right] \\
&= \frac{1}{8}\left[-\frac{1}{2}\cos 3t + \cos t - \frac{1}{2}\cos 5t\right] \\
L[\sin^2 t\cos^3 t] &= L\left[\frac{1}{8}\left[-\frac{1}{2}\cos 3t + \cos t - \frac{1}{2}\cos 5t\right]\right] \\
&= \frac{1}{16}L[2\cos t - \cos 3t - \cos 5t] \\
&= \frac{1}{16}\left[\frac{2s}{s^2+1} - \frac{s}{s^2+9} - \frac{s}{s^2+25}\right].
\end{aligned}$$

Example 5.14. Find the Laplace transform of the function defined by $f(t) = \begin{cases} t & 0<t<1 \\ 0 & t>1 \end{cases}$. [Dec 2007]

Solution. $L[f(t)] = \int\limits_0^\infty e^{-st} f(t)dt$

$$= \int\limits_0^1 e^{-st} f(t)dt + \int\limits_1^\infty e^{-st} f(t)dt$$

$$= \int\limits_0^1 e^{-st} \cdot tdt + \int\limits_1^\infty e^{-st} \times 0dt$$

$$= \int\limits_0^1 e^{-st}tdt = \int\limits_0^1 td\left(\frac{e^{-st}}{-s}\right)$$

$$= \left[t\frac{e^{-st}}{-s}\right]_0^1 - \int\limits_0^1 \frac{e^{-st}}{-s}dt.$$

$$= -\frac{1}{s}[e^{-s} - 0] + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^1$$

$$= -\frac{e^{-s}}{s} - \frac{1}{s^2}(e^{-s} - 1)$$

$$= -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}.$$

Example 5.15. Find the Laplace transform of $f(t)$ defined by $f(t) = \begin{cases} e^t & 0 < t < 1 \\ 0 & t > 1. \end{cases}$

Solution. $L[f(t)] = \int\limits_0^\infty e^{-st} f(t)dt$

$$= \int\limits_0^1 e^{-st} f(t)dt + \int\limits_1^\infty e^{-st} f(t)dt$$

$$= \int\limits_0^1 e^{-st}e^t dt = \int\limits_0^1 e^{-(s-1)t}dt$$

$$= \left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^1 = -\frac{1}{s-1}\left[e^{-(s-1)} - 1\right] = \frac{1-e^{-(s-1)}}{s-1}.$$

Example 5.16. Find the Laplace transform of $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ 0 & t > \pi. \end{cases}$

Solution. $L[f(t)] = \int\limits_0^\infty e^{-st} f(t)dt$

$$= \int\limits_0^\pi e^{-st} f(t)dt + \int\limits_\pi^\infty e^{-st} f(t)dt$$

$$= \int\limits_0^\pi e^{-st} \cos t dt$$

$$= \left[\frac{e^{-st}}{s^2+1}(-s\cos t + \sin t)\right]_0^\pi$$

$$= \frac{e^{-\pi s}}{s^2+1}(-s\cos\pi + \sin\pi) - \frac{1}{s^2+1}(-s\cos 0 + \sin 0)$$

$$= \frac{e^{-\pi s}}{s^2+1}(s) + \frac{s}{s^2+1}$$

$$= \frac{s}{s^2+1}(1 + e^{-\pi s}).$$

Example 5.17. Find the Laplace transform of the function defined by $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & t > \pi \end{cases}$. [May 2006]

Solution. $L[f(t)] = \int\limits_0^\infty e^{-st} f(t)dt$

$$= \int\limits_0^\pi e^{-st} f(t)dt + \int\limits_\pi^\infty e^{-st} f(t)dt$$

$$= \int\limits_0^\pi e^{-st} \sin t dt$$

$$= \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_0^\pi$$

$$= \frac{e^{-\pi s}}{s^2+1}(-s\sin\pi - \cos\pi) - \frac{1}{s^2+1}(-s\sin 0 - \cos 0)$$

$$= \frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}.$$

Exercise 5 A

Find the Laplace transform of the following functions.

1. $\sin^2 2t$.
2. $\cos^2 3t$.
3. $\sin^3 2t$.
4. $\cos^3 3t$.
5. $\sin 3t \sin 2t$.
6. $\sin 3t \cos t$.
7. $(\sin t - \cos t)^2$.
8. $\cos^4 t$.
9. $\sin(\omega t + \alpha)$.
10. $(\alpha + \beta t)^2$.
11. $a + bt + \frac{c}{\sqrt{t}}$.
12. $\sinh^2 2t$.
13. $\cosh(5t+2)$.
14. $\cosh 2t - \cosh 3t$.
15. $\sinh \omega t - \sin \omega t$.
16. e^{3t+5}.
17. $t^2 + \cos 2t + \sin^2 2t$.
18. $e^{-bt} - e^{-at}$.
19. $(t^2+1)^2$.
20. $\cosh at + \sin 2t + t^3$.
21. $f(t) = \begin{cases} t & 0 < t < 2 \\ 0 & t > 2. \end{cases}$ [Jun 2012]
22. $f(t) = \begin{cases} e^{-t} & 0 < t < 4 \\ 0 & t \geq 4. \end{cases}$
23. $f(t) = \begin{cases} 0 & 0 < t < 2. \\ 3 & t \geq 2. \end{cases}$

Shifting Property - First Shifting theorem

1. **Statement.** If $L[f(t)] = F(s)$ then $L[e^{-at} f(t)] = F(s+a)$.

Proof. $L[f(t)] = \int_0^\infty e^{-st} f(t) dt = F(s)$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-st} e^{-at} f(t) dt = \int_0^\infty e^{-(st+at)} f(t) dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t) dt = F(s+a) = [L[f(t)]]_{s \to s+a}$$

2. $L[e^{at} f(t)] = F(s-a) = L[f(t)]_{s \to s-a}$.

—

Change of scale property

Statement. If $L[f(t)] = F(s)$ then $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right), a > 0.$

Proof. $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s)$

$$L[f(at)] = \int_0^\infty e^{-st} f(at)dt$$

Let $at = x \quad \Rightarrow t = \frac{x}{a} \Rightarrow dt = \frac{1}{a}dx$

When $t = 0, x = 0,$ when $t = \infty, x = \infty$

$$\therefore L[f(at)] = \int_0^\infty e^{-s\frac{x}{a}} f(x)\frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x)dx = \frac{1}{a}F\left(\frac{s}{a}\right) = \frac{1}{a}[F(s)]_{s\to\frac{s}{a}} = \frac{1}{a}[L[f(t)]]_{s\to\frac{s}{a}}.$$

Example 5.18. Find $L[e^t t^{-\frac{1}{2}}]$. [May 1996]

Solution. $L[e^t t^{-\frac{1}{2}}] = L\left[t^{-\frac{1}{2}}\right]_{s\to s-1} = \left[\sqrt{\frac{\pi}{s}}\right]_{s\to s-1} = \sqrt{\frac{\pi}{s-1}}.$

Example 5.19. Find the Laplace transform of $\frac{t}{e^t}$. [Jun 2013]

Solution. $L\left[\frac{t}{e^t}\right] = L[e^{-t}t] = L[t]_{s\to s+1} = \left[\frac{1}{s^2}\right]_{s\to s+1} = \frac{1}{(s+1)^2}.$

Example 5.20. Find $L\left[e^{-3t}\sin t\cdot\cos t\right]$ [Dec 2011]

Solution. $L\left[e^{-3t}\sin t\cos t\right] = L\left[e^{-3t}\frac{\sin 2t}{2}\right] = \frac{1}{2}L\left[e^{-3t}\sin 2t\right]$

$$= \frac{1}{2}L[\sin 2t]_{s\to s+3} = \frac{1}{2}\left(\frac{2}{s^2+4}\right)_{s\to s+3}$$

$$= \frac{1}{(s+3)^2+4} = \frac{1}{s^2+6s+13}.$$

Example 5.21. Find $L[e^{-3t}\sin^2 t]$. [Jun 2008]

Solution. $L[e^{-3t}\sin^2 t] = L[\sin^2 t]_{s\to s+3} = L\left[\frac{1-\cos 2t}{2}\right]_{s\to s+3}$

$$= \frac{1}{2}[L[1] - L[\cos 2t]]_{s\to s+3} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]_{s\to s+3}$$

$$= \frac{1}{2}\left[\frac{1}{s+3} - \frac{s+3}{(s+3)^2+4}\right].$$

Example 5.22. Find $L[\cosh t \sin 2t]$. [Dec 2009]

Solution. $L[\cosh t \sin 2t] = L\left[\frac{e^t + e^{-t}}{2}\sin 2t\right] = \frac{1}{2}\left[L(e^t \sin 2t + e^{-t}\sin 2t)\right]$

$$\begin{aligned}
&= \frac{1}{2}\left[L[\sin 2t]_{s\to s-1} + L[\sin 2t]_{s\to s+1}\right] \\
&= \frac{1}{2}\left[\left[\frac{2}{s^2+4}\right]_{s\to s-1} + \left[\frac{2}{s^2+4}\right]_{s\to s+1}\right] \\
&= \frac{1}{2}\left[\frac{2}{(s-1)^2+4} + \frac{2}{(s+1)^2+4}\right] \\
&= \frac{1}{(s-1)^2+4} + \frac{1}{(s+1)^2+4}.
\end{aligned}$$

Example 5.23. Find $L[e^{2t}\cos 5t]$. [May 2008]

Solution. $L[e^{2t}\cos 5t] = L[\cos 5t]_{s\to s-2} = \left[\frac{s}{s^2+25}\right]_{s\to s-2} = \frac{s-2}{(s-2)^2+25}.$

Example 5.24. Find the Laplace transforms of (i) $\cosh at \sinh bt$ (ii) $(1+te^{-t})^3$.

Solution.

(*i*) $L[\cosh at \sinh bt] = L\left[\frac{e^{at}+e^{-at}}{2}\sinh bt\right] = \frac{1}{2}L[e^{at}\sinh bt + e^{-at}\sinh bt]$

$$\begin{aligned}
&= \frac{1}{2}\left[L[e^{at}\sinh bt] + L[e^{-at}\sinh bt]\right] \\
&= \frac{1}{2}\left[L[\sinh bt]_{s\to s-a} + L[\sinh bt]_{s\to s+a}\right] \\
&= \frac{1}{2}\left[\left[\frac{b}{s^2+b^2}\right]_{s\to s-a} + \left[\frac{b}{s^2+b^2}\right]_{s\to s+a}\right] \\
&= \frac{1}{2}\left[\frac{b}{(s-a)^2+b^2} + \frac{b}{(s+a)^2+b^2}\right].
\end{aligned}$$

(*ii*) $L[(1+te^{-t})^3] = L[1 + 3te^{-t} + 3t^2e^{-2t} + t^3e^{-3t}]$

$$\begin{aligned}
&= L[1] + 3L[te^{-t}] + 3L[t^2e^{-2t}] + L[t^3e^{-3t}] \\
&= \frac{1}{s} + 3L[t]_{s\to s+1} + 3L[t^2]_{s\to s+2} + L[t^3]_{s\to s+3} \\
&= \frac{1}{s} + 3\left[\frac{1}{s^2}\right]_{s\to s+1} + 3\left[\frac{2}{s^3}\right]_{s\to s+2} + \left[\frac{6}{s^4}\right]_{s\to s+3} \\
&= \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}.
\end{aligned}$$

Example 5.25. Find $L[\cosh at \cos at]$. [Dec 2008]

Solution. $L[\cosh at \cos at] = \left[\dfrac{e^{at}+e^{-at}}{2}\cos at\right] = \dfrac{1}{2}\left[L[e^{at}\cos at] + L[e^{-at}\cos at]\right]$

$$= \frac{1}{2}\left[L[\cos at]_{s\to s-a} + L[\cos at]_{s\to s+a}\right]$$
$$= \frac{1}{2}\left[\left(\frac{s}{s^2+a^2}\right)_{s\to s-a} + \left(\frac{s}{s^2+a^2}\right)_{s\to s+a}\right]$$
$$= \frac{1}{2}\left[\left(\frac{s-a}{(s-a)^2+a^2}\right) + \left(\frac{s+a}{(s+a)^2+a^2}\right)\right].$$

Example 5.26. Find $L[e^{-3t}(2\cos 5t - 3\sin 5t)]$. [May 1999]

Solution. $L[e^{-3t}(2\cos 5t - 3\sin 5t)] = L[2e^{-3t}\cos 5t - 3e^{-3t}\sin 5t]$

$$= 2L[e^{-3t}\cos 5t] - 3L[e^{-3t}\sin 5t]$$
$$= 2L[\cos 5t]_{s\to s+3} - 3L[\sin 5t]_{s\to s+3}$$
$$= 2\left[\frac{s}{s^2+25}\right]_{s\to s+3} - 3\left[\frac{5}{s^2+25}\right]_{s\to s+3}$$
$$= \frac{2(s+3)}{(s+3)^2+25} - \frac{15}{(s+3)^2+25}.$$

Exercise 5 B

Find the Laplace Transform of the following functions.

1. $\sum_{n=0}^{N} a_n e^{-bt}\cos nt$. [Jan 2006]
2. $t^2 e^{-2t}$.
3. $e^t(\cosh 2t + \frac{1}{2}\sinh 2t)$.
4. $e^{-t}(3\sinh 2t - 5\cosh 2t)$.
5. $e^{-\frac{3}{2}t}\sin 3t$. [May 2008]
6. $\sinh 3t \cos t$. [Nov 2009]
7. $e^{-7t}t^{-\frac{1}{2}}$. [Jan 2009]
8. $e^{-3t}\cos^2 t$. [Dec 2007]
9. $e^{2t}\sin^3 t$. [May 2003]
10. $e^{4t}\sin 2t \sin t$. [May 2006]
11. $e^{-3t}\cos 4t \cos t$.
12. $e^{2t}\cos 3t \sin t$.
13. $e^t\cos^3 t$.
14. $e^t \sin 4t \cos 2t$.
15. $(t+1)^3 e^{-t}$.

—

Theorem. If $L[f(t)] = F(s)$, then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$.

Proof. We have $L[f(t)] = \int_0^\infty e^{-st} f(t)dt$

$$F(s) = \int_0^\infty e^{-st} f(t)dt.$$

Differentiating w.r.t s we get,

$$\frac{d}{ds}[F(s)] = \int_0^\infty \frac{d}{ds}[e^{-st} f(t)]dt$$

$$= \int_0^\infty e^{-st}(-t) f(t)dt$$

$$-\frac{d}{ds}[F(s)] = \int_0^\infty e^{-st} t f(t)dt = L[t f(t)].$$

Now $L[t^2 f(t)] = L[t\ t f(t)] = -\frac{d}{ds} L[t f(t)]$

$$= -\frac{d}{ds}\Big(-\frac{d}{ds}(F(s))\Big) = (-1)^2 \frac{d^2}{ds^2} F(s)$$

In general, $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\big[F(s)\big]$.

Worked Examples

Example 5.27. Find $L[t\ \cosh 3t]$. [May 1995]

Solution. $L[t\ \cosh 3t] = -\frac{d}{ds}\big\{L[\cosh 3t]\big\} = -\frac{d}{ds}\Big[\frac{s}{s^2-9}\Big] = -\Big[\frac{s^2+9-s\times 2s}{(s^2-9)^2}\Big]$

$$= -\frac{9-s^2}{(s^2-9)^2} = \frac{s^2-9}{(s^2-9)^2} = \frac{1}{(s^2-9)}.$$

Example 5.28. Find $L[t^2 \sin at]$. [May 2001]

Solution. $L[t^2 \sin at] = (-1)^2 \frac{d^2}{ds^2}\big[L[\sin at]\big] = \frac{d^2}{ds^2}\Big[\frac{a}{s^2+a^2}\Big]$

$$= a\frac{d^2}{ds^2}(s^2+a^2)^{-1} = a\frac{d}{ds}[(-1)(s^2+a^2)^{-2}2s] = -2a\frac{d}{ds}\Big[\frac{s}{(s^2+a^2)^2}\Big]$$

$$= -2a\left(\frac{(s^2+a^2)^2.1 - s.2.(s^2+a^2)2s}{(s^2+a^2)^4}\right)$$

$$= -2a\frac{(s^2+a^2)\big[s^2+a^2-4s^2\big]}{(s^2+a^2)^4} = -2a\frac{[-3s^2+a^2]}{(s^2+a^2)^3} = 2a\frac{[3s^2-a^2]}{(s^2+a^2)^3}.$$

—

Example 5.29. Find $L[t\ \cos 3t]$.

Solution. $L[t\ \cos 3t] = -\frac{d}{ds}\{L[\cos 3t]\} = -\frac{d}{ds}\Big[\frac{s}{s^2+9}\Big]$

$$= -\Big[\frac{s^2+9-s2s}{(s^2+9)^2}\Big] = -\frac{9-s^2}{(s^2+9)^2} = \frac{s^2-9}{(s^2+9)^2}.$$

Example 5.30. Find Laplace transform of $t\sin 2t$. [Dec 2010]

Solution. $L[t\sin 2t] = -\frac{d}{ds}\{L[\sin 2t]\}$

$$= -\frac{d}{ds}\left[\frac{2}{s^2+4}\right]$$

$$= -\frac{d}{ds}\left[2\cdot(s^2+4)^{-1}\right]$$

$$= -2\cdot(-1)\cdot(s^2+4)^{-2}\cdot 2s$$

$$= \frac{4s}{(s^2+4)^2}.$$

Example 5.31. Find $L[t\sin at]$.

Solution. $L[t\sin at] = -\frac{d}{ds}\{L[\sin at]\} = -\frac{d}{ds}\Big(\frac{a}{s^2+a^2}\Big) = -a\frac{d}{ds}(s^2+a^2)^{-1}$

$$= -a(-1)(s^2+a^2)^{-2}.2s = \frac{2as}{(s^2+a^2)^2}.$$

Example 5.32. Find $L[t\cos at]$.

Solution. $L[t\cos at] = -\frac{d}{ds}\Big[L[\cos at]\Big] = -\frac{d}{ds}\Big[\frac{s}{s^2+a^2}\Big] = -\frac{s^2+a^2-s2s}{(s^2+a^2)^2}$

$$= -\Big[\frac{a^2-s^2}{(s^2+a^2)^2}\Big] = \frac{s^2-a^2}{(s^2+a^2)^2}.$$

Example 5.33. Find $L[\sin at - at\cos at]$. [May 2003]

Solution. $L[\sin at - at\cos at] = L[\sin at] - aL[t\cos at]$

$$= \frac{a}{s^2+a^2} - a\Big(-\frac{d}{ds}L[\cos at]\Big)$$

$$= \frac{a}{s^2+a^2} + a\frac{d}{ds}\Big(\frac{s}{s^2+a^2}\Big)$$

$$= \frac{a}{s^2+a^2} + a\frac{(s^2+a^2).1 - s2s}{(s^2+a^2)^2}$$

$$= \frac{a}{s^2+a^2} + a\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}$$

$$= \frac{a}{s^2+a^2} + \frac{a(a^2-s^2)}{(s^2+a^2)^2} = a\left[\frac{s^2+a^2+a^2-s^2}{(s^2+a^2)^2}\right] = \frac{2a^3}{(s^2+a^2)^2}.$$

Example 5.34. Find $L[te^{-t}\sin t]$.

Solution. $L[te^{-t}\sin t] = -\frac{d}{ds}\{L[e^{-t}\sin t\} = -\frac{d}{ds}\big[L[\sin t]_{s\to s+1}\big]$

$$= -\frac{d}{ds}\Big[\frac{1}{s^2+1}\Big]_{s\to s+1} = -\frac{d}{ds}\Big[\frac{1}{(s+1)^2+1}\Big]$$

$$= -\frac{d}{ds}\Big[\frac{1}{s^2+2s+2}\Big] = -(-1)(s^2+2s+2)^{-2}(2s+2)$$

$$= \frac{2s+2}{(s^2+2s+2)^2}.$$

Example 5.35. Find $L[t\sin 3t\cos 2t]$. [May 2008]

Solution. $L[t\sin 3t\cos 2t] = L\left[t\frac{[\sin 5t+\sin t]}{2}\right]$

$$= \frac{1}{2}\Big(-\frac{d}{ds}\Big)\Big(L[\sin 5t] + L[\sin t]\Big)$$

$$= -\frac{1}{2}\frac{d}{ds}\Big[\frac{5}{s^2+25} + \frac{1}{s^2+1}\Big]$$

$$= -\frac{1}{2}\frac{d}{ds}\Big[5(s^2+25)^{-1} + (s^2+1)^{-}1\Big]$$

$$= -\frac{1}{2}\Big[\frac{-5.\,2s}{(s^2+25)^2} - \frac{2s}{(s^2+1)^2}\Big]$$

$$= -\frac{1}{2}(-2)\Big[\frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}\Big]$$

$$= \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}.$$

Example 5.36. Find $L[t^2e^{-3t}\sin 2t]$. [Jun 2013, May 2000]

Solution. $L[t^2e^{-3t}\sin 2t] = \frac{d^2}{ds^2}\Big[L[e^{-3t}\sin 2t]\Big] = \frac{d^2}{ds^2}\Big[L[\sin 2t]_{s\to s+3}\Big]$

$$= \frac{d^2}{ds^2}\Big[\frac{2}{(s+3)^2+4}\Big]$$

$$= \frac{d^2}{ds^2}\Big[\frac{2}{s^2+6s+13}\Big]$$

$$= 2\frac{d^2}{ds^2}(s^2+6s+13)^{-1}$$

$$= 2\frac{d}{ds}\Big[(-1)(s^2+6s+13)^{-2}(2s+6)\Big]$$

$$= -2\frac{d}{ds}\Big[\frac{2s+6}{(s^2+6s+13)^2}\Big]$$
$$= -4\frac{d}{ds}\Big[\frac{s+3}{(s^2+6s+13)^2}\Big]$$
$$= -4\frac{(s^2+6s+13)^2-(s+3)2(s^2+6s+13)(2s+6)}{(s^2+6s+13)^4}$$
$$= -4\frac{s^2+6s+13-4(s^2+6s+9)}{(s^2+6s+13)^3}$$
$$= -4\frac{s^2+6s+13-4s^2-36-24s}{(s^2+6s+13)^3}$$
$$= -4\frac{-3s^2-18s-23}{(s^2+6s+13)^3} = 4\frac{3s^2+18s+23}{(s^2+6s+13)^3}.$$

Exercise 5 C

Find the Laplace transform of the following functions.

1. $t\cos^3 t$.
2. $te^{-4t}\sin 3t$.
3. $t^2e^{-t}\cos t$.
4. $te^t\cos t$. [May 2005]
5. $t^2\sin 2t$. [Dec 2004]
6. $t^2\cosh 2t$. [Jan 2005]
7. $te^{2t}\sin 3t$. [Dec 2006]
8. $t\cosh 5t$. [Dec 2003]
9. t^2e^{-3t}. [Dec 2009]
10. $t^2e^t\sin t$. [May 2007]

Theorem. If $L[f(t)] = F[s]$ and if $\lim_{t\to 0}\frac{f(t)}{t}$ exists, then $L\Big[\frac{f(t)}{t}\Big] = \int_s^\infty F(s)ds$.

Proof. We have, $F(s) = L[f(t)] = \int_0^\infty e^{-st}f(t)dt$

Integrating both sides w.r.t s from s to ∞

$$\int_s^\infty F(s)ds = \int_s^\infty\int_0^\infty e^{-st}f(t)dtds = \int_0^\infty f(t)\Big[\int_s^\infty e^{-st}ds\Big]dt$$
$$= \int_0^\infty f(t)\Big(\frac{e^{-st}}{-t}\Big)_s^\infty dt = -\int_0^\infty \frac{f(t)}{t}(e^{-\infty}-e^{-st})dt = \int_0^\infty e^{-st}\frac{f(t)}{t}dt = L\Big[\frac{f(t)}{t}\Big].$$

Extension

$$L\Big[\frac{f(t)}{t^n}\Big] = \int_s^\infty \int_s^\infty \cdots \int_s^\infty F(s)(ds)^n.$$

Worked Examples

Example 5.37. Find $L\Big[\frac{1-e^t}{t}\Big]$. [Dec 2008]

Solution.
$$L\Big[\frac{1-e^t}{t}\Big] = \int_s^\infty L[1-e^t]ds = \int_s^\infty \Big(\frac{1}{s} - \frac{1}{s-1}\Big)ds$$
$$= \Big[\log s - \log(s-1)\Big]_s^\infty = \log\Big(\frac{s}{s-1}\Big)_s^\infty$$
$$= \log\Big(\frac{1}{1-\frac{1}{s}}\Big)_s^\infty = \log 1 - \log(\frac{s}{s-1})$$
$$= \log\Big(\frac{s-1}{s}\Big).$$

Example 5.38. Find the Laplace transform of $\frac{1-e^{-t}}{t}$. [Dec 2013]

Solution.
$$L\left[\frac{1-e^{-t}}{t}\right] = \int_s^\infty L[1-e^{-t}]ds = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+1}\right)ds$$
$$= [\log s - \log(s+1)]_s^\infty = \left[\log\frac{s}{s+1}\right]_s^\infty$$
$$= \left[\log\left(\frac{1}{1+\frac{1}{s}}\right)\right]_s^\infty = \log 1 - \log\left(\frac{s}{s+1}\right)$$
$$= \log\left(\frac{s+1}{s}\right).$$

Example 5.39. Find $L\Big[\frac{\sin 2t}{t}\Big]$. [Jan 2006]

Solution.
$$L\Big[\frac{\sin 2t}{t}\Big] = \int_s^\infty L[\sin 2t]ds = \int_s^\infty \frac{2}{s^2+4}ds$$
$$= 2\frac{1}{2}\Big(\tan^{-1}\frac{s}{2}\Big)_s^\infty = \tan^{-1}\infty - \tan^{-1}\Big(\frac{s}{2}\Big)$$
$$= \frac{\pi}{2} - \tan^{-1}\Big(\frac{s}{2}\Big)$$
$$= \cot^{-1}\Big(\frac{s}{2}\Big).$$

—

Example 5.40. Find $L\Big[\frac{1-\cos 2t}{t}\Big]$. [Apr 2004]

Solution. $$L\Big[\frac{1-\cos 2t}{t}\Big] = \int_s^\infty L\big[1-\cos 2t\big]ds = \int_s^\infty \Big(\frac{1}{s} - \frac{s}{s^2+4}\Big)ds$$
$$= \log s - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+4}ds$$
$$= \Big[\log s - \frac{1}{2}\log(s^2+4)\Big]_s^\infty$$
$$= \Big[\log\frac{s}{\sqrt{s^2+4}}\Big]_s^\infty$$
$$= \log\frac{\sqrt{s^2+4}}{s}.$$

Example 5.41. Find $L\Big[\frac{\sin at}{t}\Big]$. [Dec 2009]

Solution. $$L\Big[\frac{\sin at}{t}\Big] = \int_s^\infty L[\sin at]ds = \int_s^\infty \frac{a}{s^2+a^2}ds$$
$$= a\frac{1}{a}\Big(\tan^{-1}\frac{s}{a}\Big)_s^\infty = \tan^{-1}\infty - \tan^{-1}\Big(\frac{s}{a}\Big)$$
$$= \frac{\pi}{2} - \tan^{-1}\Big(\frac{s}{a}\Big) = \cot^{-1}\Big(\frac{s}{a}\Big).$$

Example 5.42. Find $L\Big[\frac{\cos at - \cos bt}{t}\Big]$. [Dec 2012, May 2011, Dec 2007]

Solution. $$L\Big[\frac{\cos at-\cos bt}{t}\Big] = \int_s^\infty L\big[\cos at - \cos bt\big]ds$$
$$= \int_s^\infty \Big(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\Big)ds$$
$$= \frac{1}{2}\Big[\log(s^2+a^2) - \log(s^2+b^2)\Big]_s^\infty$$
$$= \frac{1}{2}\Big[\log\frac{s^2+a^2}{s^2+b^2}\Big]_s^\infty = \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}.$$

Example 5.43. Find $L\Big[\frac{e^{-at}-e^{-bt}}{t}\Big]$. [May 2006]

Solution. $$L\Big[\frac{e^{-at}-e^{-bt}}{t}\Big] = \int_s^\infty L\big[e^{-at}-e^{-bt}\big]ds = \int_s^\infty \Big(\frac{1}{s+a} - \frac{1}{s+b}\Big)ds$$
$$= \Big[\log(s+a) - \log(s+b)\Big]_s^\infty = \Big[\log\frac{s+a}{s+b}\Big]_s^\infty$$
$$= \log\frac{s+b}{s+a}.$$

—

Example 5.44. Find the Laplace transform of $\frac{e^{at}-e^{-bt}}{t}$. [Jun 2012]

Solution. $$L\left[\frac{e^{at}-e^{-bt}}{t}\right] = \int_s^\infty L\left[e^{at}-e^{-bt}\right]ds = \int_s^\infty \left(\frac{1}{s-a}-\frac{1}{s+b}\right)ds$$
$$= [\log(s-a)-\log(s+b)]_s^\infty = \left[\log\frac{s-a}{s+b}\right]_s^\infty$$
$$= \log\frac{s+b}{s-a}.$$

Example 5.45. Find $L\left[\frac{e^{-3t}\sin 2t}{t}\right]$. [May 2007]

Solution. $$L\left[\frac{e^{-3t}\sin 2t}{t}\right] = \int_s^\infty L\left[e^{-3t}\sin 2t\right]ds = \int_s^\infty L\left[\sin 2t\right]_{s\to s+3}ds$$
$$= \int_s^\infty \frac{2}{(s+3)^2+2^2}ds$$
$$= 2\frac{1}{2}\left(\tan^{-1}\frac{s+3}{2}\right)_s^\infty$$
$$= \frac{\pi}{2}-\tan^{-1}\left(\frac{s+3}{2}\right)$$
$$= \cot^{-1}\left(\frac{s+3}{2}\right).$$

Exercise 5 D

Find the Laplace transform of the following functions.

1. $\frac{\cos t-\cos 2t}{t}$.
2. $\frac{1-e^{-t}}{t}$.
3. $\frac{e^{t}-e^{2t}}{t}$.
4. $\frac{e^{at}-\cos bt}{t}$.
5. $\frac{1-\cos t}{t}$. [Jun 2009]
6. $\frac{\sin^2 t}{t}$. [Dec 2007]
7. $\frac{\sin 3t\cos t}{t}$. [Nov 2009]
8. $\frac{\cos 4t\sin 2t}{t}$. [Dec 2009]
9. $\frac{e^{-3t}-e^{-4t}}{t}$. [Jan 2009]
10. $\frac{\cos 2t-\cos 3t}{t}$.

—

Laplace transform of periodic functions

A function $f(t)$ is said to be periodic if there exists a positive constant T such that $f(t+T) = f(t)$ for all t.The smallest of such T is called the period of the function.

Example. $\sin(t+2\pi) = \sin t$. Sine function is a periodic function with period 2π.

Theorem. If $f(t)$ is a periodic function with period T, then

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt.$$

Proof. By the definition of Laplace transforms, we have

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)dt = \int_0^{T} e^{-st} f(t)dt + \int_T^{2T} e^{-st} f(t)dt + \int_{2T}^{3T} e^{-st} f(t)dt + \cdots \qquad (1)$$

Consider $\displaystyle\int_T^{2T} e^{-st} f(t)dt.$

Let $t = u + T$.

$dt = du$.

When $t = T, u = 0$.

When $t = 2T, u = T$.

Now $$\int_T^{2T} e^{-st} f(t)dt = \int_0^{T} e^{-s(u+T)} f(u+T)du = \int_0^{T} e^{-su}e^{-sT} f(u+T)du = e^{-sT}\int_0^{T} e^{-su} f(u+T)du$$

$$= e^{-sT}\int_0^{T} e^{-su} f(u)du [\because\ f(u) \text{ is periodic in the period } T]$$

$$= e^{-sT}\int_0^{T} e^{-st} f(t)dt.$$

In a similar way in the integral $\displaystyle\int_{2T}^{3T} e^{-st} f(t)dt$ if we assume $t = u + 2T$ we arrive at

$$\int_{2T}^{3T} e^{-st} f(t)dt = e^{-s2T}\int_0^{T} e^{-st} f(t)dt.$$

—

If we replace t by $u + 3T$ in the next integral we obtain

$$\int_{3T}^{4T} e^{-st} f(t)dt = e^{-s3T} \int_{0}^{T} e^{-st} f(t)dt.$$

In the same way the subsequent integrals can be replaced by equivalent quantities.

Substituting everything in (1) we obtain

$$\begin{aligned} L[f(t)] &= \int_{0}^{T} e^{-st} f(t)dt + e^{-sT} \int_{0}^{T} e^{-st} f(t)dt + e^{-s2T} \int_{0}^{T} e^{-st} f(t)dt + \cdots \\ &= \left(1 + e^{-sT} + e^{-s2T} + e^{-s3T} + \cdots\right) \int_{0}^{T} e^{-st} f(t)dt \\ &= \left(1 + e^{-sT} + (e^{-sT})^2 + (e^{-sT})^3 + \cdots\right) \int_{0}^{T} e^{-st} f(t)dt \\ &= \frac{1}{1 - e^{-sT}} \int_{0}^{T} e^{-st} f(t)dt. \end{aligned}$$

Worked Examples

Example 5.46. Find the Laplace transform of the rectangular wave function given by $f(t) = \begin{cases} 1 & 0 < t < b \\ -1 & b < t < 2b \end{cases}$ with $f(t + 2b) = f(t)$. [May 2009]

Solution. Since $f(t + 2b) = f(t)$, it is a periodic function with period $2b$.

$$\begin{aligned} \therefore L[f(t)] &= \frac{1}{1 - e^{-2bs}} \int_{0}^{2b} e^{-st} f(t)dt = \frac{1}{1 - e^{-2bs}} \left[\int_{0}^{b} e^{-st} f(t)dt + \int_{b}^{2b} e^{-st} f(t)dt\right] \\ &= \frac{1}{1 - e^{-2bs}} \left[\int_{0}^{b} e^{-st}dt - \int_{b}^{2b} e^{-st}dt\right] \\ &= \frac{1}{1 - e^{-2bs}} \left[\left(\frac{e^{-st}}{-s}\right)_{0}^{b} - \left(\frac{e^{-st}}{-s}\right)_{b}^{2b}\right] \end{aligned}$$

$$= \frac{1}{1-e^{-2bs}}\Big(-\frac{1}{s}(e^{-bs}-1)+\frac{1}{s}(e^{-2bs}-e^{-bs})\Big)$$
$$= \frac{1}{s(1-e^{-2bs})}\Big(-e^{-bs}+1+e^{-2bs}-e^{-bs}\Big)$$
$$= \frac{1}{s(1-e^{-2bs})}\Big(1-2e^{-bs}+e^{-2bs}\Big)$$
$$= \frac{1}{s(1-(e^{-bs})^2)}\Big(1-2e^{-bs}+(e^{-bs})^2\Big)$$
$$= \frac{(1-e^{-bs})^2}{s(1-e^{-bs})(1+e^{-bs})} = \frac{1-e^{-bs}}{s(1+e^{-bs})} = \frac{e^{\frac{bs}{2}}-e^{-\frac{bs}{2}}}{s\left(e^{\frac{bs}{2}}+e^{-\frac{bs}{2}}\right)} = \frac{1}{s}\tan h\left(\frac{bs}{2}\right).$$

Example 5.47. Find the Laplace transform of the square-wave function (or Meoander function) of period a defined as $f(t) = \begin{cases} 1 & \text{when } 0 < t < \frac{a}{2} \\ -1 & \text{when } \frac{a}{2} < t < a. \end{cases}$ [Jun 2013]

Solution. Since $f(t)$ is a periodic function of period a, we have

$$L[f(t)] = \frac{1}{1-e^{-as}}\int_0^a e^{-st}f(t)dt$$
$$= \frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}} e^{-st}f(t)dt + \int_{\frac{a}{2}}^a e^{-st}f(t)dt\right]$$
$$= \frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}} e^{-st}dt + \int_{\frac{a}{2}}^a e^{-st}(-1)dt\right]$$
$$= \frac{1}{1-e^{-as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{\frac{a}{2}} - \left(\frac{e^{-st}}{-s}\right)_{\frac{a}{2}}^a\right]$$
$$= \frac{1}{s(1-e^{-as})}\left[-\left(e^{-\frac{a}{2}s}-1\right)+e^{-as}-e^{-\frac{a}{2}s}\right]$$
$$= \frac{1}{s(1-e^{-as})}\left[1-2e^{-\frac{a}{2}s}+e^{-as}\right]$$
$$= \frac{\left(1-e^{-\frac{a}{2}s}\right)^{\cancel{2}}}{s\cancel{\left(1-e^{-\frac{a}{2}s}\right)}\left(1+e^{-\frac{a}{2}s}\right)}$$
$$= \frac{1-e^{-\frac{a}{2}s}}{s\left(1+e^{-\frac{a}{2}s}\right)}$$

—

$$= \frac{1}{s} \cdot \frac{e^{\frac{as}{4}} - e^{-\frac{as}{4}}}{e^{\frac{as}{4}} + e^{-\frac{as}{4}}}$$
$$= \frac{1}{s} \tan h\left(\frac{as}{4}\right).$$

Example 5.48. Find the Laplace transform of a square wave function given by
$$f(t) = \begin{cases} \epsilon & \text{for } 0 \le t \le \frac{a}{2} \\ -\epsilon & \text{for } \frac{a}{2} \le t \le a \end{cases}$$
and $f(t+a) = f(t)$. [Dec 2011]

Solution. Since $f(t+a) = f(t)$, $f(t)$ is a periodic function of period a.

$$\therefore L[f(t)] = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)dt$$
$$= \frac{1}{1-e^{-as}} \left[\int_0^{\frac{a}{2}} e^{-st} f(t)dt + \int_{\frac{a}{2}}^{a} e^{-st} f(t)dt \right]$$
$$= \frac{1}{1-e^{-as}} \left[\int_0^{\frac{a}{2}} e^{-st} \epsilon dt + \int_{\frac{a}{2}}^{a} e^{-st} (-\epsilon)dt \right]$$
$$= \frac{\epsilon}{1-e^{-as}} \left[\left(\frac{e^{-st}}{-s}\right)_0^{\frac{a}{2}} - \left(\frac{e^{-st}}{-s}\right)_{\frac{a}{2}}^{a} \right]$$
$$= \frac{\epsilon}{s(1-e^{-as})} \left[-\left(e^{-\frac{sa}{2}} - 1\right) + e^{-as} - e^{-\frac{as}{2}} \right]$$
$$= \frac{\epsilon}{s(1-e^{-as})} \left[1 - 2e^{-\frac{as}{2}} + e^{-as} \right]$$
$$= \frac{\epsilon \left(1 - e^{-\frac{as}{2}}\right)^{\cancel{2}}}{s\left(1 + e^{-\frac{as}{2}}\right)\cancel{\left(1 - e^{-\frac{as}{2}}\right)}}$$
$$= \frac{\epsilon}{s} \frac{\left(1 - e^{-\frac{as}{2}}\right)}{\left(1 + e^{-\frac{as}{2}}\right)}$$
$$= \frac{\epsilon}{s} \cdot \frac{e^{\frac{as}{4}} - e^{-\frac{as}{4}}}{e^{\frac{as}{4}} + e^{-\frac{as}{4}}}$$
$$= \frac{\epsilon}{s} \tan h\left(\frac{as}{4}\right).$$

Example 5.49. Find the Laplace transform of $f(t) = \begin{cases} \epsilon & 0 \le t \le a \\ -\epsilon & a \le t \le 2a \end{cases}$ and $f(t+2a) = f(t)$ for all t. [Dec 2010]

Solution. Since $f(t+2a) = f(t)$ for all t, $f(t)$ is a periodic function with period $2a$.

$$\therefore L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} f(t)dt + \int_a^{2a} e^{-st} f(t)dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} \epsilon dt + \int_a^{2a} e^{-st} (-\epsilon)dt \right]$$

$$= \frac{\epsilon}{1-e^{-2as}} \left[\left(\frac{e^{-st}}{-s} \right)_0^a - \left(\frac{e^{-st}}{-s} \right)_a^{2a} \right]$$

$$= \frac{\epsilon}{s(1-e^{-2as})} \left[-(e^{-as} - 1) + e^{-2as} - e^{-as} \right]$$

$$= \frac{\epsilon \left(1 - 2e^{-as} + e^{-2as}\right)}{s\,(1-e^{-as})\,(1+e^{-as})}$$

$$= \frac{\epsilon\,(1-e^{-as})^{\cancel{2}}}{s\,\cancel{(1-e^{-as})}\,(1+e^{-as})}$$

$$= \frac{\epsilon}{s} \frac{(1-e^{-as})}{(1+e^{-as})} = \frac{\epsilon}{s} \frac{\left(e^{\frac{as}{2}} - e^{-\frac{as}{2}}\right)}{\left(e^{\frac{as}{2}} + e^{-\frac{as}{2}}\right)}$$

$$= \frac{\epsilon}{s} \tan h \left(\frac{as}{2} \right).$$

Example 5.50. Find the Laplace transform of the triangular wave function defined by $f(t) = \begin{cases} t & 0 \le t \le a \\ 2a - t & a < t \le 2a \end{cases}$ and $f(t)$ is of period $2a$. [May 2015,May 2011]

Solution. Since $f(t)$ is of period $2a$,we have

$$L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)dt$$

—

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st} f(t)dt + \int_a^{2a} e^{-st} f(t)dt\right]$$
$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st} tdt + \int_a^{2a} e^{-st}(2a-t)dt\right]$$
$$= \frac{1}{1-e^{-2as}}\left[\int_0^a td(\frac{e^{-st}}{-s}) + \int_a^{2a}(2a-t)d(\frac{e^{-st}}{-s})\right]$$
$$= \frac{1}{1-e^{-2as}}\left[\left(t\frac{e^{-st}}{-s}\right)_0^a - \int_0^a \frac{e^{-st}}{-s}dt + \left((2a-t)\frac{e^{-st}}{-s}\right)_a^{2a} - \int_a^{2a}\frac{e^{-st}}{-s}(-dt)\right]$$
$$= \frac{1}{1-e^{-2as}}\left[a\frac{e^{-as}}{-s} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_0^a + 0 + \frac{a}{s}e^{-as} - \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_a^{2a}\right]$$
$$= \frac{1}{1-e^{-2as}}\left[-a\frac{e^{-as}}{s} - \frac{1}{s^2}(e^{-as}-1) + \frac{a}{s}e^{-as} + \frac{1}{s^2}(e^{-2as}-e^{-as})\right]$$
$$= \frac{1}{1-e^{-2as}}\left[-a\frac{e^{-as}}{s} - \frac{1}{s^2}e^{-as} + \frac{1}{s^2} + a\frac{e^{-as}}{s} + \frac{1}{s^2}e^{-2as} - \frac{1}{s^2}e^{-as}\right]$$
$$= \frac{1}{1-e^{-2as}}\left[\frac{e^{-2as}-e^{-as}-e^{-as}+1}{s^2}\right] = \frac{1}{1-e^{-as}}\left[\frac{1-2e^{-2as}+e^{-2as}}{s^2}\right]$$
$$= \frac{1}{(1-e^{-as})(1+e^{-as})}\cdot\frac{(1-e^{-as})^2}{s^2} = \frac{(1-e^{-as})}{s^2.(1+e^{-as})}$$
$$= \frac{e^{\frac{as}{2}} - e^{-\frac{as}{2}}}{s^2\left(e^{\frac{as}{2}} + e^{-\frac{as}{2}}\right)} = \frac{1}{s^2}\tan h\left(\frac{as}{2}\right).$$

Example 5.51. Find the Laplace transform of the triangular wave function given by $f(t) = \begin{cases} t & 0 \le t \le \pi \\ 2\pi - t & \pi \le t \le 2\pi \end{cases}$
and $f(t+2\pi) = f(t)$. [Jun 2012, Jun 2010]

Solution. Since $f(t+2\pi) = f(t)$ for all t, $f(t)$ is a periodic function with period 2π.

In the previous Example, we get the solution by replacing $2a$ by 2π

i.e., $a = \pi$.

$$\therefore L[f(t)] = \frac{1}{s^2}\tan h\left(\frac{\pi}{2}s\right).$$

Example 5.52. Find the Laplace transform of the half-sine wave rectifier function defined by $f(t) = \begin{cases} \sin\omega t & 0 < t < \frac{\pi}{\omega} \\ 0 & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}. \end{cases}$ [Jun 2014, Dec 2012]

Solution. $f(t)$ is defined in the interval $(0, \frac{2\pi}{\omega})$

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$$f(t+\frac{2\pi}{\omega}) = \sin\omega(t+\frac{2\pi}{\omega}) = \sin(\omega t+2\pi) = \sin\omega t = f(t)$$

$\therefore\ f(t)$ is periodic with period $T = \frac{2\pi}{\omega}$.

We know that, $L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt = \frac{1}{1-e^{-\frac{s2\pi}{\omega}}}\int_0^{\frac{2\pi}{\omega}} e^{-st}\sin\omega t dt$

$$= \frac{1}{1-e^{-\frac{s2\pi}{\omega}}}\Big[\frac{e^{-st}}{s^2+w^2}[-s\sin\omega t-\omega\cos\omega t]\Big]_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{-\frac{s2\pi}{\omega}}}\Big[\frac{e^{\frac{-s\pi}{\omega}}}{s^2+w^2}[-s.0-\omega(-1)] - \frac{-\omega}{\omega^2+s^2}\Big]$$

$$= \frac{1}{(1-e^{-\frac{s2\pi}{\omega}})(s^2+\omega^2)}\Big\{\omega e^{\frac{-s\pi}{\omega}}+\omega\Big\}$$

$$= \frac{\omega(1+e^{-\frac{s\pi}{\omega}})}{(s^2+\omega^2)(1-e^{\frac{-s\pi}{\omega}})(1+e^{\frac{-s\pi}{\omega}})} = \frac{\omega}{(s^2+\omega^2)(1-e^{\frac{-s\pi}{\omega}})}.$$

Example 5.53. Find the Laplace transform of the saw-toothed wave function of period T given by $f(t) = \frac{t}{T}, 0 < t < T$.

Solution. Since $f(t)$is a periodic function of period T, we have

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-st}\frac{t}{T}dt = \frac{1}{T(1-e^{-sT})}\int_0^T td\Big(\frac{e^{-st}}{-s}\Big)$$

$$= \frac{1}{T(1-e^{-sT})}\Big[\Big(\frac{te^{-st}}{-s}\Big)_0^T - \int_0^T\frac{e^{-st}}{-s}dt\Big]$$

$$= \frac{1}{T(1-e^{-sT})}\Big[\frac{Te^{-sT}}{-s}+\frac{1}{s}\Big(\frac{e^{-st}}{-s}\Big)_0^T\Big]$$

$$= \frac{1}{sT(1-e^{-sT})}\Big[-Te^{-sT}-\frac{1}{s}(e^{-sT}-1)\Big]$$

$$= \frac{1}{sT(1-e^{-sT})}\Big[\frac{1}{s}-\frac{1}{s}e^{-sT}-Te^{-sT}\Big]$$

$$= \frac{1}{sT(1-e^{-sT})}\Big[\frac{1-e^{-sT}}{s}-Te^{-sT}\Big]$$

$$= \frac{1}{Ts^2}-\frac{e^{-sT}}{s(1-e^{-sT})}.$$

Exercise 5 E

1. Find the Laplace transform of the periodic function $f(t) = t$ for $0 < t < 4$ and $f(t+4) = f(t)$. [Dec 2009]

2. Find the Laplace transform of the periodic function $f(t) = \begin{cases} t & 0 < t < 1 \\ 2-t & 1 < t < 2. \end{cases}$ [Oct 2001]

3. Find the Laplace transform of the function $f(t) = \begin{cases} t & 0 < t < \frac{\pi}{2} \\ \pi - t & \frac{\pi}{2} < t < \pi \end{cases}$ and $f(\pi + t) = f(t)$. [May 2009]

4. Find the Laplace transform of the function $f(t) = \begin{cases} t & 0 \le t \le 3 \\ 0 & 3 \le t \le 6 \end{cases}$ and $f(t+6) = f(t)$. [May 2006]

5. Find the Laplace transform of the function $f(t) = \begin{cases} k & 0 \le t \le a \\ -k & 3 \le t \le 2a \end{cases}$ and $f(2a+t) = f(t)$. [Dec 2008]

6. Find the Laplace transform of the periodic saw - tooth wave function given by $f(t) = \dfrac{\alpha t}{\omega}, 0 < t < \omega$ and $f(t) = f(t+\omega)$. [May 2007]

7. Find the Laplace transform of the periodic function $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$ and $f(t+2\pi) = f(t)$. [May 2004]

8. Find the Laplace transform of $f(t) = \begin{cases} 0 & 0 < t < \frac{\pi}{\omega} \\ -\sin \omega t & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$ and $f(t+2\pi) = f(t)$. [May 2003]

9. Find the Laplace transform of the periodic function defined by $f(t) = \begin{cases} \frac{t}{a} & 0 \le t \le a \\ \frac{2a-t}{a} & a \le t \le 2a \end{cases}$ and $f(t+2a) = f(t)$. [May 2007]

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5.3 Laplace transform of derivatives

Theorem 1. Let $f(t)$ be a continuous function for all $t \geq 0$ and $f'(t)$ is piecewise continuous on every finite interval $0 \leq t \leq a$ in $[0, \infty)$. If $f(t)$ and $f'(t)$ are of exponential order as $t \to \infty$, then $L[f'(t)] = sL[f(t)] - f(0)$.

Solution.
$$\begin{aligned} L\Big[f'(t)\Big] &= \int_0^\infty e^{-st} f'(t)dt = \int_0^\infty e^{-st}\frac{d}{dt}(f(t))dt \\ &= \int_0^\infty e^{-st} d(f(t)) = \Big(e^{-st} f(t)\Big)_0^\infty - \int_0^\infty f(t)e^{-st}(-s)dt \\ &= 0 - f(0) + s\int_0^\infty e^{-st} f(t)dt = sL[f(t)] - f(0). \end{aligned}$$
$$\therefore L\Big[\frac{dy}{dt}\Big] = sL[y] - y(0).$$

Theorem 2. Find $L[f''(t)]$.

Solution.
$$\begin{aligned} L\Big[f''(t)\Big] &= \int_0^\infty e^{-st} f''(t)dt = \int_0^\infty e^{-st}\frac{d}{dt}\Big(f'(t)\Big)dt = \int_0^\infty e^{-st} d\Big(f'(t)\Big) \\ &= \Big(e^{-st} f'(t)\Big)_0^\infty - \int_0^\infty f'(t)e^{-st}(-s)dt \\ &= 0 - f'(0) + s\int_0^\infty e^{-st} f'(t)dt \\ &= -f'(0) + sL[f'(t)] = -f'(0) - s[sL[f(t)] - f(0)] \\ &= -f'(0) + s^2 L[f(t)] - sf(0) \end{aligned}$$
$$\therefore L\Big[f''(t)\Big] = s^2 L[f(t)] - sf(0) - f'(0).$$

Extension

$$L[f'''(0)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0).$$

In general $L[f^{(n)}] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$.

Note. The above results will be used later in solving differential equations using Laplace transforms.

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5.3.1 Laplace transform of an integral

If $f(t)$ is piecewise continuous in every finite interval $0 \le t \le a$ in $[0, \infty)$ and $f(t)$ is of exponential order $\alpha > 0$ and if $L[f(t)] = F(s)$ then

$$L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s} = \frac{L[f(t)]}{s} \text{ for } s > \alpha.$$

Proof. Let $g(t) = \int\limits_0^t f(u)du.$

Then $g'(t) = f(t)$ and $g(0) = \int\limits_0^0 f(u)du = 0.$

Now, $L[f(t)] = L[g'(t)] = sL[g(t)] - g(0)$

$$\frac{L[f(t)]}{s} = L[g(t)].$$

i.e., $L\left[\int\limits_0^t f(u)du\right] = \dfrac{L[f(t)]}{s}$

i.e., $L\left[\int\limits_0^t f(t)dt\right] = \dfrac{L[f(t)]}{s}.$

Result. $L\left[\int\limits_0^t \int\limits_0^t f(t)dtdt\right] = \dfrac{L\left[\int\limits_0^t f(t)dt\right]}{s} = \dfrac{L[f(t)]}{s^2}.$

Worked Examples

Example 5.54. Find $L\left[e^{-t}\int_0^t t\cos t dt\right]$. [May 2005]

Solution. $L\left[e^{-t}\int_0^t t\cos t dt\right] = L\left[\int_0^t t\cos t dt\right]_{s\to s+1} = \left[\dfrac{L[t\cos t]}{s}\right]_{s\to s+1}$

$$= \frac{1}{s+1}\left(-\frac{d}{ds}\{L(\cos t)\}\right)_{s\to s+1}$$

$$= -\frac{1}{s+1}\frac{d}{ds}\left\{\frac{s+1}{s^2+2s+2}\right\}$$

$$= -\frac{1}{s+1}\left\{\frac{(s^2+2s+2)-(s+1)(2s+2)}{(s^2+2s+2)^2}\right\}$$

—

$$= -\frac{1}{s+1}\left\{\frac{(s^2+2s+2)-2(s+1)^2}{(s^2+2s+2)^2}\right\}$$
$$= -\frac{1}{s+1}\left\{\frac{s^2+2s+2-2s^2-4s-2)}{(s^2+2s+2)^2}\right\}$$
$$= -\frac{1}{s+1}\left\{\frac{-s^2-2s}{(s^2+2s+2)^2}\right\}$$
$$= \frac{s(s+2)}{(s+1)(s^2+2s+2)^2}.$$

Example 5.55. Find $L\Big[\int_0^t \frac{e^{-t}\sin t}{t}dt\Big]$. [May 2008]

Solution. $L\Big[\int_0^t \frac{e^{-t}\sin t}{t}dt\Big] = \frac{L\left[\frac{e^{-t}\sin t}{t}\right]}{s} = \frac{1}{s}\int_s^\infty L[e^{-t}\sin t]ds$

$$= \frac{1}{s}\int_s^\infty L[\sin t]_{s\to s+1}ds$$
$$= \frac{1}{s}\int_s^\infty \frac{1}{(s+1)^2+1}ds$$
$$= \frac{1}{s}\Big[\tan^{-1}(s+1)\Big]_s^\infty$$
$$= \frac{1}{s}\Big[\frac{\pi}{2}-\tan^{-1}(s+1)\Big]$$
$$= \frac{1}{s}\cot^{-1}(s+1).$$

Example 5.56. Find $L\Big[\int_0^t te^{-t}\sin t dt\Big]$. [Jun 2005]

Solution. $L\Big[\int_0^t te^{-t}\sin t dt\Big] = \frac{L[te^{-t}\sin t]}{s} = \frac{1}{s}\Big(-\frac{d}{ds}\Big)L[e^{-t}\sin t]$

$$= -\frac{1}{s}\frac{d}{ds}\Big\{L[\sin t]_{s\to s+1}\Big\}$$
$$= -\frac{1}{s}\frac{d}{ds}\Big[\frac{1}{(s+1)^2+1}\Big]$$
$$= -\frac{1}{s}\frac{d}{ds}\Big\{(s^2+2s+2)^{-1}\Big\}$$
$$= -\frac{1}{s}(-1)(s^2+2s+2)^{-2}(2s+2)$$
$$= \frac{2(s+1)}{s(s^2+2s+2)^2}.$$

Example 5.57. Find $L\left[e^{-t}\int_0^t \frac{\sin t}{t}dt\right]$.

Solution.
$$\begin{aligned}L\left[e^{-t}\int_0^t \frac{\sin t}{t}dt\right] &= L\left[\int_0^t \frac{\sin t}{t}dt\right]_{s\to s+1} = \left[\frac{1}{s}L\left[\frac{\sin t}{t}\right]\right]_{s\to s+1}\\ &= \frac{1}{s+1}\left[\int_s^\infty L[\sin t]ds\right]_{s\to s+1}\\ &= \frac{1}{s+1}\left[\int_s^\infty \frac{1}{s^2+1}ds\right]_{s\to s+1}\\ &= \frac{1}{s+1}\left[\tan^{-1}(s)_s^\infty\right]_{s\to s+1} = \frac{1}{s+1}\cot^{-1}(s+1).\end{aligned}$$

5.3.2 Evaluation of integrals

Certain real integrals can be evaluated by comparing the definition of the Laplace transform. The following examples illustrate this.

Worked Examples

Example 5.58. Evaluate $\int_0^\infty e^{-t}\cos 2t dt$.

Solution. We know that $L[\cos 2t] = \int_0^\infty e^{-st}\cos 2t dt$.

Comparing the definition of the Laplace transform with the given integral we observe that

$$\int_0^\infty e^{-t}\cos 2t dt = L[\cos 2t]_{s=1} = \left[\frac{s}{s^2+4}\right]_{s=1} = \frac{1}{1+4} = \frac{1}{5}.$$

Example 5.59. Evaluate $\int_0^\infty e^{-2t}\sin 3t dt$.

Solution. We have $L[\sin 3t] = \int_0^\infty e^{-st}\sin 3t dt$.

Comparing the definition of the Laplace transform with the given integral, we observe that

$$\int_0^\infty e^{-2t}\sin 3t dt = L[\sin 3t]_{s=2} = \left[\frac{3}{s^2+9}\right]_{s=2} = \frac{3}{4+9} = \frac{3}{13}.$$

Example 5.60. Find $\int\limits_0^\infty te^{-2t}\cos t dt$ using Laplace transform [Jun 2012, Dec 2011]

Solution. Comparing the definition of the Laplace transform and the given

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integral, we observe that

$$\int_0^\infty te^{-2t}\cos t dt = L[t\cos t]_{s=2} = -\frac{d}{ds}[L(\cos t)]_{s=2}$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]_{s=2} = -\left[\frac{s^2+1-s\cdot 2s}{(s^2+1)^2}\right]_{s=2} = -\left[\frac{1-s^2}{(s^2+1)^2}\right]_{s=2}$$

$$= \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s=2} = \frac{4-1}{(4+1)^2} = \frac{3}{25}.$$

Example 5.61. Evaluate $\int_0^\infty e^{-2t} t\sin 3t dt.$ [May 2004]

Solution. Comparing the definition of the Laplace transform with the given integral we observe that

$$\int_0^\infty e^{-2t} t\sin 3t dt = L[t\sin 3t]_{s=2} = \Big[-\frac{d}{ds}L[\sin 3t]\Big]_{s=2}$$

$$= -\frac{d}{ds}\Big(\frac{3}{s^2+9}\Big)_{s=2} = \Big[-3(-1)(s^2+9)^{-2}2s\Big]_{s=2}$$

$$= \Big[\frac{6s}{(s^2+9)^2}\Big]_{s=2} = \frac{12}{13^2} = \frac{12}{169}.$$

Example 5.62. Evaluate $\int_0^\infty e^{-3t} t\sin t dt.$

Solution. Comparing the definition of the Laplace transform with the given integral we observe that

$$\int_0^\infty e^{-3t} t\sin t dt = L[t\sin t]_{s=3} = \Big[-\frac{d}{ds}L(\sin t)\Big]_{s=3} = -\frac{d}{ds}\Big[\frac{1}{s^2+1}\Big]_{s=3}$$

$$= \Big[-(-1)(s^2+1)^{-2}2s\Big]_{s=3} = \Big[\frac{2s}{(s^2+1)^2}\Big]_{s=3} = \frac{6}{100} = \frac{3}{50}.$$

Example 5.63. Evaluate $\int_0^\infty \frac{\cos at - \cos bt}{t} dt.$ [May 2015]

Solution. Comparing the definition of the Laplace transform with the given integral we observe that

$$\int_0^\infty \frac{\cos at - \cos bt}{t} dt = L\Big[\frac{\cos at - \cos bt}{t}\Big]_{s=0} = \Big[\int_s^\infty L[\cos at - \cos bt] ds\Big]_{s=0}$$

$$= \Big[\int_s^\infty \Big(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\Big) ds\Big]_{s=0} = \Big[\frac{1}{2}\Big[\log\frac{s^2+a^2}{s^2+b^2}\Big]_s^\infty\Big]_{s=0}$$

$$= \frac{1}{2}\Big[\log\frac{s^2+b^2}{s^2+a^2}\Big]_{s=0} = \frac{1}{2}\log\frac{b^2}{a^2} = \log\frac{b}{a}.$$

Example 5.64. Evaluate $\int_0^\infty \frac{e^{-3t}-e^{-6t}}{t}dt.$

Solution. Comparing the given integral and the definition of the Laplace transform we get

$$\int_0^\infty \frac{e^{-3t}-e^{-6t}}{t}dt = L[\frac{e^{-3t}-e^{-6t}}{t}]_{s=0} = \Big[\int_s^\infty L(e^{-3t}-e^{-6t})ds\Big]_{s=0}$$

$$= \Big[\int_s^\infty \Big(\frac{1}{s+3}-\frac{1}{s+6}\Big)ds\Big]_{s=0}$$

$$= \Big[\Big(\log\frac{s+3}{s+6}\Big)_s^\infty\Big]_{s=0}$$

$$= \Big[\log\frac{s+6}{s+3}\Big]_{s=0} = \log 2.$$

Example 5.65. Evaluate $\int_0^\infty e^{-t}\frac{\sin^2 t}{t}dt.$

Solution. we notice that

$$\int_0^\infty e^{-t}\frac{\sin^2 t}{t}dt = L\Big[\frac{\sin^2 t}{t}\Big]_{s=1} = \frac{1}{2}L\Big[\frac{1-\cos 2t}{t}\Big]_{s=1} = \Big[\frac{1}{2}\int_s^\infty L[1-\cos 2t]ds\Big]_{s=1}$$

$$= \Big[\frac{1}{2}\int_s^\infty \Big(\frac{1}{s}-\frac{s}{s^2+4}\Big)ds\Big]_{s=1} = \Big[\frac{1}{2}\Big[\log s-\log\sqrt{s^2+4}\Big]_s^\infty\Big]_{s=1}$$

$$= \Big[\frac{1}{2}\Big[\log\frac{s}{\sqrt{s^2+4}}\Big]_s^\infty\Big]_{s=1} = \frac{1}{2}\Big[\log\frac{\sqrt{s^2+4}}{s}\Big]_{s=1}$$

$$= \frac{1}{2}\log\sqrt{5} = \frac{1}{4}\log 5.$$

Example 5.66. Evaluate $\int_0^\infty e^{-t}\frac{1-\cos t}{t}dt.$

Solution. We have

$$\int_0^\infty e^{-t}\frac{1-\cos t}{t}dt = L\Big[\frac{1-\cos t}{t}\Big]_{s=1} = \Big[\int_s^\infty L[1-\cos t]ds\Big]_{s=1}$$

$$= \Big[\int_s^\infty L[1]-L[\cos t]ds\Big]_{s=1} = \int_s^\infty \Big[L[1]-L[\cos t]ds\Big]_{s=1}$$

$$= \int_s^\infty \Big[\frac{1}{s}-\frac{s}{s^2+1}ds\Big]_{s=1} = \Big[\Big(\log s-\log\sqrt{s^2+1}\Big)\Big]_{s=1}$$

$$= \Big[\Big(\log\frac{s}{\sqrt{s^2+1}}\Big)_s^\infty\Big]_{s=1} = \Big[\log\frac{\sqrt{s^2+1}}{s}\Big]_{s=1}$$

$$= \log\sqrt{2} = \frac{1}{2}\log 2.$$

Example 5.67. Evaluate $\int_0^\infty e^{-t}\frac{\sin\sqrt{3}t}{t}dt.$

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Solution. We have $\int_0^\infty e^{-t}\frac{\sin\sqrt{3}t}{t}dt = \left[L\left[\frac{\sin\sqrt{3}t}{t}\right]\right]_{s=1} = \left[\int_s^\infty L\left[\sin\sqrt{3}t\right]ds\right]_{s=1}$

$$= \left[\int_s^\infty \frac{\sqrt{3}}{s^2+3}ds\right]_{s=1} = \left[\sqrt{3}\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{s}{\sqrt{3}}\right)_s^\infty\right]_{s=1}$$

$$= \left[\frac{\pi}{2} - \tan^{-1}\frac{s}{\sqrt{3}}\right]_{s=1} = \frac{\pi}{2} - \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{2} - \frac{\pi}{6}$$

$$= \frac{3\pi - \pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}.$$

Initial value theorem. If the Laplace transform of $f(t)$ and $f'(t)$ exist and $L[f(t)] = F(s)$, then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$.

Proof. By Laplace transform of derivative of $f(t)$ we have

$$L[f'(t)] = sL[f(t)] - f(0).$$

$$\int_0^\infty e^{-st}f'(t)dt = sF(s) - f(0).$$

Taking limit as $s \to \infty$ we obtain.

$$\lim_{s\to\infty}\int_0^\infty e^{-st}f'(t)dt = \lim_{s\to\infty} sF(s) - f(0).$$

$$\int_0^\infty \lim_{s\to\infty} e^{-st}f(t)dt = \lim_{s\to\infty} sF(s) - f(0).$$

$$0 = \lim_{s\to\infty} sF(s) - f(0).$$

$$f(0) = \lim_{s\to\infty} sF(s).$$

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$$

Final value theorem. If the Laplace transform of $f(t)$ and $f'(t)$ exist and $F(s) = L[f(t)]$, then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$. [Dec 2014]

Proof. We have $L[f'(t)] = sL[f(t)] - f(0)$.

ie., $\int_0^\infty e^{-st}f'(t)dt = sF(s) - f(0).$

Taking limit as $s \to 0$ we obtain.

$$\lim_{s\to 0}\int_0^\infty e^{-st}f'(t)dt = \lim_{s\to 0} sF(s) - f(0).$$

$$\int_0^\infty \lim_{s\to 0} e^{-st} f'(t)dt = \lim_{s\to 0} sF(s) - f(0).$$

$$\int_0^\infty f'(t)dt = \lim_{s\to 0} sF(s) - f(0).$$

$$[f(t)]_0^\infty = \lim_{s\to 0} sF(s) - f(0).$$

$$f(\infty) - f(0) = \lim_{s\to 0} sF(s) - f(0)$$

$$f(\infty) = \lim_{s\to 0} sF(s)$$

$$\therefore \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

Worked Examples

Example 5.68. Verify the initial value theorem for the function $2+3\cos t$.

Solution. $f(t) = 2+3\cos t$

$$L[f(t)] = L[2+3\cos t] = 2L[1]+3L[\cos t]$$

$$= 2\frac{1}{s} + 3\frac{s}{s^2+1}$$

$$= \frac{2}{s} + \frac{3s}{s^2+1} = F(s)$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}(2+3\cos t) = 2+3\cos 0 = 2+3 = 5.$$

$$\therefore \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left(\frac{2}{s} + \frac{3s}{s^2+1}\right) = \lim_{s\to\infty}\left(2+\frac{3s^2}{s^2+1}\right)$$

$$= 2+3\lim_{s\to\infty}\left(\frac{s^2}{s^2\left(1+\frac{1}{s^2}\right)}\right)$$

$$= 2+3\lim_{s\to\infty}\left(\frac{1}{1+\frac{1}{s^2}}\right) = 2+3 = 5.$$

$$\therefore \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$$

Example 5.69. Verify the initial and final value theorems for the function $f(t) = ae^{-bt}$. [Jun 2013, May 1997]

Solution. Let $f(t) = ae^{-bt}$.

$$L[f(t)] = \frac{a}{s+b} = F(s)$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} ae^{-bt} = a$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{as}{s+b} = \lim_{s\to\infty} \frac{as}{s(1+\frac{b}{s})} = a$$

$$\therefore \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$\lim_{s\to\infty} f(t) = \lim_{t\to 0} ae^{-bt} = 0$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} ae^{-st} = 0$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{as}{s^2+a^2} = 0$$

$$\therefore \quad \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

Example 5.70. Verify the initial value theorem for the function $f(t) = 1 + e^{-t}(\sin t + \cos t)$. [Jun 2012, Dec 2010, Jun 2010]

Solution. Let $f(t) = 1 + e^{-t}(\sin t + \cos t)$.

$$\begin{aligned} L[f(t)] &= L[1 + e^{-t}(\sin t + \cos t)] \\ &= L[1] + L[e^{-t}(\sin t + \cos t)] \\ &= \frac{1}{s} + L[\sin t + \cos t]_{s\to s+1} \\ &= \frac{1}{s} + L\left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]_{s\to s+1} \\ &= \frac{1}{s} + \left[\frac{s+1}{s^2+1}\right]_{s\to s+1} \\ F(s) &= \frac{1}{s} + \frac{s+2}{(s+1)^2+1} \end{aligned}$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}[1 + e^{-t}(\sin t + \cos t)] = 1 + 1 = 2.$$

$$\begin{aligned} \lim_{s\to\infty} sF(s) &= \lim_{s\to\infty}\left[1 + \frac{s(s+2)}{(s+1)^2+1}\right] \\ &= 1 + \lim_{s\to\infty} \frac{s^2(1+\frac{2}{s})}{s^2(1+\frac{1}{s})^2 + \frac{1}{s^2}} = 1 + \frac{1}{1} = 2. \end{aligned}$$

$$\therefore \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$$

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Example 5.71. Verify the final value theorem for the function $1 - e^{-3t}$.

[May 2006]

Solution. Let $f(t) = 1 - e^{-3t}$.

$$L[f(t)] = L[1 - e^{-3t}] = L[1] - L[e^{-3t}] = \frac{1}{s} - \frac{1}{s+3}.$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}[1 - e^{-3t}] = 1 - 0 = 1.$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} s\left[\frac{1}{s} - \frac{1}{s+3}\right] = \lim_{s\to 0}\left(1 + \frac{s}{s+3}\right) = 1 - \lim_{s\to 0}\frac{1}{1+\frac{3}{s}} = 1 - 0 = 1.$$

$\therefore \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$.

Example 5.72. Verify the initial and final value theorems for the function $f(t) = e^{-t}\sin t$.

Solution. $f(t) = e^{-t}\sin t$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} e^{-t}\sin t = 0 \tag{1}$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} e^{-t}\sin t = 0 \tag{2}$$

$$L[f(t)] = \frac{1}{(s+1)^2+1} = F(s)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\frac{s}{s^2+2as+2} = \lim_{s\to\infty}\frac{s}{s^2(1+\frac{2as}{s}+\frac{2}{s^2})} = 0 \tag{3}$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\frac{s}{s^2+2as+2} = 0 \tag{4}$$

(1) = (3) $\Longrightarrow$ Initial value theorem is verified.

(2) = (4) $\Longrightarrow$ Final value theorem is verified.

Exercise 5 F

I. Evaluate the following integrals using Laplace transforms

1. $\displaystyle\int_0^\infty te^{-t}\sin t\,dt.$
2. $\displaystyle\int_0^\infty e^{-2t}t\sin 3t\,dt.$
3. $\displaystyle\int_0^\infty \frac{\cos 6t - \cos 4t}{t}dt.$

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4. $\int_0^\infty t^3 e^{-t} \sin t dt.$

5. $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt.$

6. $\int_0^\infty \frac{e^{-t} \sin \sqrt{3}t}{t} dt.$

II. Prove the initial and final value theorems for the following functions.

1. $3e^{-t}$. [Dec 2011]
2. $(2t-3)^2$.
3. $1 - e^{-at}$.
4. $t^2 e^{-3t}$.
5. $e^{-t}(t+2)^2$.
6. If $F(s) = \frac{3s^2 + 5s + 2}{s^3 + 4s^2 + 2s}$ find $f(0)$ and $f(\infty)$.

Unit Step Function. The unit step function u is defined as

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0. \end{cases}$$

$u(t)$ has jump discontinuity at $t = 0$.

More generally we define $u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a. \end{cases}$

Laplace transform of unit step function [Jun 2010]

$$\begin{aligned} L[u(t-a)] &= \int_0^\infty e^{-st} u(t-a) dt = \int_0^a e^{-st} u(t-a) dt + \int_a^\infty e^{-st} u(t-a) dt \\ &= \int_0^a e^{-st} .0 dt + \int_a^\infty e^{-st} dt = 0 + \left(\frac{e^{-st}}{-s}\right)_a^\infty \\ &= \frac{-1}{s}[0 - e^{-as}] = \frac{e^{-as}}{s} \text{ if } s > 0. \end{aligned}$$

Second Shifting property

If $L[f(t)] = F(s)$, then $L[f(t-a)u(t-a)] = e^{-as}F(s) = e^{-as}L[f(t)]$.

Proof. We have $u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a. \end{cases}$

—

$$\text{Now } f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t \geq a. \end{cases}$$

$$\therefore L[f(t-a)u(t-a)] = \int_0^\infty e^{-st} f(t-a)u(t-a)dt$$

$$= \int_0^a e^{-st} f(t-a)u(t-a)dt + \int_a^\infty e^{-st} f(t-a)u(t-a)dt$$

$$= \int_0^a e^{-st} 0 dt + \int_a^\infty e^{-st} f(t-a)dt$$

Let $t - a = x$ When $t = a, x = 0$.

$dt = dx$ When $t = \infty, x = \infty$.

$$\therefore L[f(t-a)u(t-a)] = \int_0^\infty e^{-s(a+x)} f(x)dx = \int_0^\infty e^{-as-sx} f(x)dx = \int_0^\infty e^{-as} e^{-sx} f(x)dx$$

$$= e^{-as} \int_0^\infty e^{-sx} f(x)dx = e^{-as} \int_0^\infty e^{-st} f(t)dt = e^{-as} F(s).$$

Note. The second shifting property can be stated as follows.
If $L[f(t)] = F(s)$, then $L[f(t-a)u(t-a)] = e^{-as} L[f(t)]$ where $u(t-a)$ is the unit step function.

The unit impulse function. For any positive ϵ, the impulse function δ_ϵ is defined as $\delta_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}(u(t) - u(t-\epsilon)), & 0 \leq t < \epsilon \\ 0, & \text{Otherwise} \end{cases}$

Dirac delta function. $\lim\limits_{\epsilon \to 0} \delta_\epsilon(t)$ is called the Dirac delta function, denoted by $\delta(t)$.

$$\therefore \delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

In general $\delta(t-a) = \begin{cases} 0, & t \neq a \\ \infty, & t = a. \end{cases}$

—

Results. 1. Find $L[\delta_\epsilon(t-a)]$.

Solution

By the definition, $\delta_\epsilon(t-a) = \frac{1}{\epsilon}(u(t-a) - u(t-a-\epsilon))$ if $a \le t \le a+\epsilon$ $[0 \le t-a < \epsilon]$.

$$\text{Now, } L[\delta_\epsilon(t-a)] = \frac{1}{\epsilon}\Big[L[u(t-a) - u(t-a-\epsilon)]\Big] = \frac{1}{\epsilon}\Big[L[u(t-a)] - L[u(t-a)-\epsilon]\Big]$$

$$= \frac{1}{\epsilon}\Big[L[u(t-a)] - L[u(t-(a+\epsilon)]\Big] = \frac{1}{\epsilon}\Big[\frac{e^{-as}}{s} - \frac{e^{-(a+\epsilon)s}}{s}\Big]$$

$$= \frac{1}{\epsilon}\Big[\frac{e^{-as}}{s} - \frac{e^{-as}.e^{-\epsilon s}}{s}\Big]$$

$$L[\delta_\epsilon(t-a)] = \frac{1}{\epsilon s}e^{-as}(1-e^{-\epsilon s})$$

Taking limit as $\epsilon \to 0$ we obtain

$$L[\lim_{\epsilon\to 0}\delta_\epsilon(t-a)] = e^{-as}\lim_{\epsilon s\to 0}\frac{1-e^{-\epsilon s}}{\epsilon s}$$

$$L[\delta(t-a)] = e^{-as}.1 \qquad [\lim_{x\to 0}\frac{1-e^{x}}{x} = 1]$$

$$= e^{-as}.$$

2. When $a = 0, L[\delta(t)] = 1$.

3. One important property of Dirac Delta funtion is $\int_0^\infty f(t)\delta(t-a)dt = f(a)$.

Worked Examples

Example 5.73. If $f(t) = \begin{cases} 0, & \text{When } 0 < t < 2 \\ 3, & \text{When } t > 2 \end{cases}$ find $L[f(t)]$. [May 2007]

Solution. $L[f(t)] = \int_0^\infty e^{-st}f(t)dt = \int_0^2 e^{-st}f(t)dt + \int_2^\infty e^{-st}f(t)dt = \int_2^\infty e^{-st}3dt$

$$= 3\left(\frac{e^{-st}}{-s}\right)_2^\infty = -\frac{3}{s}[0 - e^{-2s}] = \frac{3e^{-2s}}{s}.$$

Example 5.74. If $f(t) = \begin{cases} e^t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$ find $L[f(t)]$. [Dec 2009]

Solution. $L[f(t)] = \int_0^\infty e^{-st}f(t)dt = \int_0^1 e^{-st}f(t)dt + \int_1^\infty e^{-st}f(t)dt = \int_0^1 e^{-st}e^t dt$

$$= \int_0^1 e^{-st+t}dt = \int_0^1 e^{t(1-s)}dt = \left(\frac{e^{t(1-s)}}{1-s}\right)_0^1$$
$$= \frac{1}{1-s}[e^{1-s} - 1] = \frac{e^{1-s}-1}{1-s}.$$

Example 5.75. Find $L[f(t)]$ if $f(t) = \begin{cases} \sin 2t, & 0 < t < \pi \\ 0, & t > \pi. \end{cases}$ [Apr 2010]

Solution. $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = \int_0^\pi e^{-st} f(t)dt + \int_\pi^\infty e^{-st} f(t)dt.$

$$= \int_0^\pi e^{-st} \sin 2t dt = \left(e^{-st}\left(\frac{-s\sin 2t - 2\cos 2t}{s^2+4}\right)\right)_0^\pi$$
$$= \frac{e^{-s\pi}}{s^2+4}(-2) + \frac{2}{s^2+4}$$
$$= \frac{2}{s^2+4}(1 - e^{-\pi s}).$$

Example 5.76. If $f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$ find $L[f(t)]$. [Apr 2010]

Solution. Let $g\left(t - \dfrac{2\pi}{3}\right) = f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}.$

$$\text{Now } L[f(t)] = L\left[g\left(t - \frac{2\pi}{3}\right)\right] = e^{\frac{-2\pi s}{3}} L[g(t)] = e^{\frac{-2\pi s}{3}} L[\cos t] = e^{\frac{-2\pi}{3}s} \frac{s}{s^2+1}.$$

Example 5.77. Find $L[(t-1)^2 u(t-1)]$.

Solution. $L[(t-1)^2 u(t-1)] = e^{-s}L[t^2] = e^{-s}\dfrac{2}{s^3} = \dfrac{2e^{-s}}{s^3}.$

Example 5.78. Find $L[e^{-4t}u(t-1)]$.

Solution. $L[e^{-4t}u(t-1)] = L[e^{-4(t-1+1)}u(t-1)] = L[e^{-4(t-1)-4)}u(t-1)]$

$$= e^{-4}e^{-s}L[e^{-4t}] = e^{-4}e^{-s}\frac{1}{s+4} = \frac{e^{-(4+s)}}{s+4}.$$

—

Example 5.79. Find $L[\sin t u(t-\frac{\pi}{2})]$.

Solution. $L[\sin t u(t-\frac{\pi}{2})] = L[\sin(t-\frac{\pi}{2}+\frac{\pi}{2})u(t-\frac{\pi}{2})] = L\Big[\cos(t-\frac{\pi}{2})u(t-\frac{\pi}{2})\Big]$

$$= e^{\frac{-\pi s}{2}} L[\cos t] = e^{\frac{-\pi s}{2}} \frac{s}{s^2+1} = \frac{se^{\frac{-\pi s}{2}}}{s^2+1}.$$

Exercise 5 G

Find the Laplace transform of the following functions.

1. $f(t) = \begin{cases} t+1, & 0 \le t \le 2 \\ 3, & t > 2. \end{cases}$

2. $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi. \end{cases}$

3. $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \sin t, & t > \pi. \end{cases}$

4. $f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3. \end{cases}$

5. $f(t) = \begin{cases} (t-1)^2, & t > 1 \\ 0, & 0 < t < 1. \end{cases}$

6. $f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1. \end{cases}$

7. $f(t) = \begin{cases} t, & 0 < t < \frac{1}{2} \\ t-1, & \frac{1}{2} < t < 1. \end{cases}$

8. $f(t) = \begin{cases} t, & 1 < t < 2 \\ 0, & \text{otherwise}. \end{cases}$

9. $f(t) = \begin{cases} 0, & 0 < t < 1 \\ 1+5t, & t \ge 1. \end{cases}$

10. $f(t) = \begin{cases} \frac{t}{T}, & 0 < t < T \\ 1, & t > T. \end{cases}$

5.4 Inverse Laplace Transforms

Definition. If the Laplace transform of a function $f(t)$ is $F(s)$, then $f(t)$ is called the inverse Laplace transform of $F(s)$.

i.e., If $L[f(t)] = F(s)$ then $L^{-1}[F(s)] = f(t)$.

Standard Results

1. $L[1] = \dfrac{1}{s} \Rightarrow L^{-1}\left[\dfrac{1}{s}\right] = 1.$

2. $L[e^{at}] = \dfrac{1}{s-a} \Rightarrow L^{-1}\left[\dfrac{1}{s-a}\right] = e^{at}.$

3. $L[e^{-at}] = \dfrac{1}{s+a} \Rightarrow L^{-1}\left[\dfrac{1}{s+a}\right] = e^{-at}.$

4. $L[t] = \dfrac{1}{s^2} \Rightarrow L^{-1}\left[\dfrac{1}{s^2}\right] = t.$

5. $L[t^n] = \dfrac{n!}{s^{n+1}} \Rightarrow L^{-1}\left[\dfrac{n!}{s^{n+1}}\right] = t^n.$

6. $L[\cosh at] = \dfrac{s}{s^2-a^2} \Rightarrow L^{-1}\left[\dfrac{s}{s^2-a^2}\right] = \cosh at.$

7. $L[\sinh at] = \dfrac{a}{s^2-a^2} \Rightarrow L^{-1}\left[\dfrac{a}{s^2-a^2}\right] = \sinh at.$

8. $L[\cos at] = \dfrac{s}{s^2+a^2} \Rightarrow L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at.$

9. $L[\sin at] = \dfrac{a}{s^2+a^2} \Rightarrow L^{-1}\left[\dfrac{a}{s^2+a^2}\right] = \sin at.$

10. $L[t\sin at] = \dfrac{2as}{(s^2+a^2)^2} \Rightarrow L^{-1}\left[\dfrac{2as}{(s^2+a^2)^2}\right] = t\sin at.$

11. $L[t\cos at] = \dfrac{s^2-a^2}{(s^2+a^2)^2} \Rightarrow L^{-1}\left[\dfrac{s^2-a^2}{(s^2+a^2)^2}\right] = t\cos at.$

Basic theorems

1. $L^{-1}[aF(s)+bG(s)] = aL^{-1}[F(s)] + bL^{-1}[G(s)]$ where a and b are constants.

Shifting theorems

2. (i) If $L[f(t)] = F(s)$, then

$$L[e^{-at}f(t)] = F(s+a)$$

$$\Longrightarrow L^{-1}[F(s+a)] = e^{-at}f(t) = e^{-at}L^{-1}[F(s)]$$

(ii) $L[e^{at}f(t)] = F(s-a)$

$$\Longrightarrow L^{-1}[F(s-a)] = e^{at}f(t) = e^{at}L^{-1}[F(s)].$$

—

Worked Examples

Example 5.80. Find $L^{-1}\left[\frac{1}{s^2}+\frac{1}{s+4}+\frac{1}{s^2+4}+\frac{s}{s^2-9}\right]$. [Jan 2008]

Solution. $L^{-1}\left[\frac{1}{s^2}+\frac{1}{s+4}+\frac{1}{s^2+4}+\frac{s}{s^2-9}\right]$

$$= L^{-1}\left[\frac{1}{s^2}\right]+L^{-1}\left[\frac{1}{s+4}\right]+L^{-1}\left[\frac{1}{s^2+4}\right]+L^{-1}\left[\frac{s}{s^2-9}\right]$$

$$= t+e^{-4t}+\frac{1}{2}L^{-1}\left[\frac{2}{s^2+4}\right]+\cosh 3t$$

$$= t+e^{-4t}+\frac{1}{2}\sin 2t+\cosh 3t.$$

Example 5.81. Find $L^{-1}\left[\frac{s^2-3s+4}{s^3}\right]$.

Solution. $L^{-1}\left[\frac{s^2-3s+4}{s^3}\right] = L^{-1}\left[\frac{1}{s}-\frac{3}{s^2}+\frac{4}{s^3}\right] = L^{-1}\left[\frac{1}{s}\right]-3L^{-1}\left[\frac{1}{s^2}\right]+4L^{-1}\left[\frac{1}{s^3}\right]$

$$= 1-3t+\frac{4}{2}L^{-1}\left[\frac{2}{s^3}\right] = 1-3t+2t^2.$$

Example 5.82. Find $L^{-1}\left[\frac{3s+5}{s^2+9}\right]$.

Solution. $L^{-1}\left[\frac{3s+5}{s^2+9}\right] = 3L^{-1}\left[\frac{s}{s^2+9}\right]+5L^{-1}\left[\frac{1}{s^2+9}\right] = 3\cos 3t+\frac{5}{3}\sin 3t$

$$= \frac{9\cos 3t+5\sin 3t}{3}.$$

Example 5.83. Find $L^{-1}\left[\frac{3s+2}{s^2-4}\right]$.

Solution. $L^{-1}\left[\frac{3s+2}{s^2-4}\right] = 3L^{-1}\left[\frac{s}{s^2-4}\right]+L^{-1}\left[\frac{2}{s^2-4}\right] = 3\cosh 2t+\sinh 2t.$

Example 5.84. Find $L^{-1}\left[\frac{s}{a^2s^2+b^2}\right]$.

Solution. $L^{-1}\left[\frac{s}{a^2s^2+b^2}\right] = L^{-1}\left[\frac{s}{a^2(s^2+\frac{b^2}{a^2})}\right] = \frac{1}{a^2}L^{-1}\left[\frac{s}{s^2+\frac{b^2}{a^2}}\right] = \frac{1}{a^2}\cos\left(\frac{b}{a}t\right).$

Example 5.85. Find $L^{-1}\left[\frac{1}{(s-3)^5}\right]$.

Solution. $L^{-1}\left[\frac{1}{(s-3)^5}\right] = e^{3t}L^{-1}\left[\frac{1}{s^5}\right] = \frac{e^{3t}}{4!}t^4 = \frac{e^{3t}t^4}{24}.$

—

Example 5.86. Find $L^{-1}\Big[\dfrac{s}{(s+6)^3}\Big]$.

Solution. $$L^{-1}\Big[\frac{s}{(s+6)^3}\Big] = L^{-1}\Big[\frac{s+6-6}{(s+6)^3}\Big] = e^{-6t}L^{-1}\Big[\frac{s-6}{s^3}\Big] = e^{-6t}\Big[L^{-1}\Big[\frac{1}{s^2}\Big] - 6L^{-1}\Big[\frac{1}{s^3}\Big]\Big]$$
$$= e^{-6t}\Big[t - \frac{6}{2}t^2\Big] = e^{-6t}[t - 3t^2].$$

Example 5.87. Find $L^{-1}\Big[\dfrac{s+3}{s^2-4s+13}\Big]$.

Solution. $$L^{-1}\Big[\frac{s+3}{s^2-4s+13}\Big] = L^{-1}\Big[\frac{s+3}{(s-2)^2+13-4}\Big] = L^{-1}\Big[\frac{s+3}{(s-2)^2+9}\Big]$$
$$= L^{-1}\Big[\frac{s-2+2+3}{(s-2)^2+9}\Big]$$
$$= e^{2t}\Big\{L^{-1}\Big[\frac{s}{s^2+9}\Big] + L^{-1}\Big[\frac{5}{s^2+9}\Big]\Big\}$$
$$= e^{2t}\Big[\cos 3t + \frac{5}{3}\sin 3t\Big].$$

Example 5.88. Find $L^{-1}\Big[\dfrac{s}{s^2-4s+5}\Big]$.

Solution. $$L^{-1}\Big[\frac{s}{s^2-4s+5}\Big] = L^{-1}\Big[\frac{s}{(s-2)^2+5-4}\Big] = L^{-1}\Big[\frac{s-2+2}{(s-2)^2+1}\Big]$$
$$= e^{2t}L^{-1}\Big[\frac{s+2}{s^2+1}\Big]$$
$$= e^{2t}\Big[L^{-1}\Big[\frac{s}{s^2+1}\Big] + 2L^{-1}\Big[\frac{1}{s^2+1}\Big]\Big]$$
$$= e^{2t}\Big[\cos t + 2\sin t\Big].$$

Example 5.89. Find $L^{-1}\left[\dfrac{1}{s^2+4s+2}\right]$. [Dec 2010]

Solution. $$L^{-1}\left[\frac{1}{s^2+4s+2}\right] = L^{-1}\left[\frac{1}{(s+2)^2+2-4}\right] = L^{-1}\left[\frac{1}{(s+2)^2-2}\right]$$
$$= e^{-2t}L^{-1}\left[\frac{1}{s^2-(\sqrt{2})^2}\right]$$
$$= \frac{e^{-2t}}{\sqrt{2}}L^{-1}\left[\frac{\sqrt{2}}{s^2-(\sqrt{2})^2}\right]$$
$$= \frac{e^{-2t}}{\sqrt{2}}\sin(\sqrt{2}t).$$

Example 5.90. Find $L^{-1}\left[\frac{2s-3}{s^2+4s+13}\right]$.

Solution. $L^{-1}\left[\frac{2s-3}{s^2+4s+13}\right] = L^{-1}\left[\frac{2s-3}{(s+2)^2+13-4}\right] = L^{-1}\left[\frac{2(s+2)-7}{(s+2)^2+9}\right]$

$$= e^{-2t}L^{-1}\left[\frac{2s-7}{s^2+9}\right] = e^{-2t}\left[2L^{-1}\left[\frac{s}{s^2+9}\right] - 7L^{-1}\left[\frac{1}{s^2+9}\right]\right]$$

$$= e^{-2t}\left[2\cos 3t - \frac{7}{3}\sin 3t\right] = \frac{e^{-2t}}{3}\left[6\cos 3t - 7\sin 3t\right].$$

Example 5.91. Find $L^{-1}\left[\frac{s}{(s+2)^2+1}\right]$. [May 2007]

Solution. $L^{-1}\left[\frac{s}{(s+2)^2+1}\right] = L^{-1}\left[\frac{s+2-2}{(s+2)^2+1}\right] = e^{-2t}L^{-1}\left[\frac{s-2}{s^2+1}\right]$

$$= e^{-2t}\left[L^{-1}\left[\frac{s}{s^2+1}\right] - 2L^{-1}\left[\frac{1}{s^2+1}\right]\right]$$

$$= e^{-2t}[\cos t - 2\sin t].$$

Example 5.92. Find $L^{-1}\left[\frac{s}{(s+3)^2}\right]$. [Dec 2009]

Solution. $L^{-1}\left[\frac{s}{(s+3)^2}\right] = L^{-1}\left[\frac{s+3-3}{(s+3)^2}\right] = e^{-3t}L^{-1}\left[\frac{s-3}{s^2}\right]$

$$= e^{-3t}\left[L^{-1}\left[\frac{1}{s} - \frac{3}{s^2}\right]\right] = e^{-3t}\left[L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{3}{s^2}\right]\right]$$

$$= e^{-3t}[1-3t].$$

Example 5.93. Find $L^{-1}\left[\frac{1}{(s-4)^5} + \frac{5}{(s-2)^2+25} + \frac{s+3}{(s+3)^2+36}\right]$. [Dec 2007]

Solution. $L^{-1}\left[\frac{1}{(s-4)^5} + \frac{5}{(s-2)^2+25} + \frac{s+3}{(s+3)^2+36}\right]$

$$= L^{-1}\left[\frac{1}{(s-4)^5}\right] + L^{-1}\left[\frac{5}{(s-2)^2+25}\right] + L^{-1}\left[\frac{s+3}{(s+3)^2+36}\right]$$

$$= e^{4t}L^{-1}\left[\frac{1}{s^5}\right] + e^{2t}L^{-1}\left[\frac{5}{s^2+25}\right] + e^{-3t}L^{-1}\left[\frac{s}{s^2+36}\right]$$

$$= e^{4t}\frac{t}{4!} + e^{2t}\sin 5t + e^{-3t}\cos 6t$$

$$= e^{4t}\frac{t}{24} + e^{2t}\sin 5t + e^{-3t}\cos 6t.$$

Example 5.94. Find $L^{-1}\left[\frac{3}{(s-3)^2+25}\right]$. [Dec 2009]

Solution. $L^{-1}\left[\frac{3}{(s-3)^2+25}\right] = e^{3t}L^{-1}\left[\frac{3}{s^2+25}\right] = \frac{3}{5}e^{3t}L^{-1}\left[\frac{5}{s^2+25}\right] = \frac{3}{5}e^{3t}\sin 5t.$

Exercise 5 H

Find the inverse Laplace transform of the following functions.

1. $\dfrac{1}{s-3}+\dfrac{1}{s}+\dfrac{s}{s^2-4}$.
2. $\dfrac{2}{2s-3}+\dfrac{s}{s^2-1}$.
3. $\dfrac{s}{s^2+49}+\dfrac{1}{s+\sqrt{7}}$.
4. $\dfrac{2s}{s^2-25}+\dfrac{1}{s^3}$.
5. $\dfrac{s}{s^2-225}+\dfrac{1}{s-\sqrt{7}}+\dfrac{1}{s^2+16}$.
6. $\dfrac{1}{(s+1)^2+1}$.
7. $\dfrac{s-3}{(s-3)^2+4}$.
8. $\dfrac{s}{(s-6)^2+a^2}$.
9. $\dfrac{1}{(s-3)^5}+\dfrac{2}{(s+1)^2+4}$.
10. $\dfrac{s+6}{(s+6)^2+9}$.
11. $\dfrac{s-a}{(s-a)^2-b^2}$.
12. $\dfrac{s+7}{(s+7)^2+49}$.
13. $\dfrac{5s+3}{s^2+2s+5}$.
14. $\dfrac{3s-2}{s^2-4s+20}$.

5.4.1 Inverse Laplace transform by the method of partial fractions

Worked Examples

Example 5.95. Find the inverse Laplace transform of $\dfrac{1}{(s+1)(s+2)}$ [Dec 2012]

Solution. Let $\dfrac{1}{(s+1)(s+2)}=\dfrac{A}{s+1}+\dfrac{B}{s+2}=\dfrac{A(s+2)+B(s+1)}{(s+1)(s+2)}$

$\therefore 1=A(s+2)+B(s+1)$.

when $s=-1, A=1$.

when $s=-2, -B=1 \Rightarrow B=-1$.

$$\therefore \frac{1}{(s+1)(s+2)}=\frac{1}{s+1}-\frac{1}{s+2}$$

$$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right]=L^{-1}\left[\frac{1}{s+1}-\frac{1}{s+2}\right]$$

$$= L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] = e^{-t} - e^{-2t}.$$

Example 5.96. Find $L^{-1}\left[\frac{1}{(s+1)(s+3)}\right]$. [May 2008]

Solution. Let $\frac{1}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$

$$1 = A(s+3) + B(s+1)$$

Put $s = -1, 1 = 2A \Rightarrow A = \frac{1}{2}$.

Put $s = -3, 1 = -2B \Rightarrow B = \frac{-1}{2}$.

$$\therefore \frac{1}{(s+1)(s+3)} = \frac{\frac{1}{2}}{s+1} - \frac{\frac{1}{2}}{s+3}$$

$$\begin{aligned} L^{-1}\left[\frac{1}{(s+1)(s+3)}\right] &= L^{-1}\left[\frac{\frac{1}{2}}{s+1} - \frac{\frac{1}{2}}{s+3}\right] \\ &= \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s+3}\right] \\ &= \frac{1}{2}(e^{-t} - e^{-3t}). \end{aligned}$$

Example 5.97. Find $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$.

Solution. Let $\frac{1}{(s+a)(s+b)} = \frac{A}{s+a} + \frac{B}{s+b}$

$$1 = A(s+b) + B(s+a)$$

Put $s = -a \Rightarrow A = \frac{1}{b-a}$

Put $s = -b \Rightarrow B = \frac{1}{a-b}$

$$\begin{aligned} L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] &= L^{-1}\left[\frac{\frac{1}{b-a}}{s+a} + \frac{\frac{1}{a-b}}{s+b}\right] \\ &= \frac{1}{b-a}e^{-at} + \frac{1}{a-b}e^{-bt} \\ &= \frac{1}{a-b}(e^{-bt} - e^{-at}). \end{aligned}$$

Example 5.98. Find $L^{-1}\Big[\dfrac{1}{s(s+3)^3}\Big]$. [Nov 2005]

Solution. Let $\dfrac{1}{s(s+3)^3} = \dfrac{A}{s} + \dfrac{B}{s+3} + \dfrac{C}{(s+3)^2} + \dfrac{D}{(s+3)^3}$.

$$1 = A(s+3)^3 + Bs(s+3)^2 + Cs(s+3) + Ds.$$

When $s = 0, 1 = 27A \Rightarrow A = \dfrac{1}{27}$.

When $s = -3, 1 = -3D \Rightarrow B = \dfrac{-1}{3}$.

Equating the coeff. of s^3 we get

$A + B = 0 \Rightarrow B = -A = \dfrac{-1}{27}$.

Equating the coeff. of s^2 we get

$$9A + 6B + C = 0$$
$$9 \times \frac{1}{27} + 6\Big(\frac{-1}{27}\Big) + C = 0$$
$$\frac{9}{27} - \Big(\frac{6}{27}\Big) + C = 0$$
$$\frac{3}{27} + C = 0$$
$$C = -\frac{1}{9}.$$

$$\therefore \frac{1}{s(s+3)^3} = \frac{\frac{1}{27}}{s} - \frac{\frac{1}{27}}{s+3} - \frac{\frac{1}{9}}{(s+3)^2} + \frac{\frac{1}{3}}{(s+3)^3}$$

$$L^{-1}\Big[\frac{1}{s(s+3)^3}\Big] = \frac{1}{27}L^{-1}\Big[\frac{1}{s}\Big] - \frac{1}{27}\Big[\frac{1}{s+3}\Big] - \frac{1}{9}\Big[\frac{1}{(s+3)^2}\Big] - \frac{1}{3}\Big[\frac{1}{(s+3)^3}\Big]$$
$$= \frac{1}{27} \times 1 - \frac{1}{27}e^{-3t} - \frac{1}{9}e^{-3t}t - \frac{1}{3}e^{-3t}\frac{t^2}{2!}$$
$$= \frac{1}{27} - \frac{e^{-3t}}{27} - \frac{e^{-3t}t}{9} - \frac{t^2e^{-3t}}{6}.$$

Example 5.99. Find $L^{-1}\Big[\dfrac{s}{(s-4)(s+4)}\Big]$.

Solution. Let $\dfrac{s}{(s-4)(s+4)} = \dfrac{A}{s-4} + \dfrac{B}{s+4}$

$s = A(s+4) + B(s-4)$

Put $s = 4 \Rightarrow A = \frac{1}{2}$

Put $s = -4 \Rightarrow B = \frac{1}{2}$

$$L^{-1}\Big[\frac{s}{(s-4)(s+4)}\Big] = L^{-1}\Big[\frac{\frac{1}{2}}{s-4} + \frac{\frac{1}{2}}{s+4}\Big]$$
$$= \frac{1}{2}e^{4t} + \frac{1}{2}e^{-4t} = \cosh 4t.$$

Example 5.100. Find $L^{-1}\Big[\frac{4s+5}{(s-1)^2(s+2)}\Big]$.

Solution. Let $\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$

On computation, we get $A = \frac{1}{3}, B = 3, C = \frac{1}{3}$.

$$\therefore \frac{4s+5}{(s-1)^2(s+2)} = \frac{\frac{1}{3}}{s-1} + \frac{3}{(s-1)^2} + \frac{\frac{1}{3}}{s+2}.$$
$$\therefore L^{-1}\Big[\frac{4s+5}{(s-1)^2(s+2)}\Big] = \frac{1}{3}e^t + 3e^t t + \frac{1}{3}e^{-2t} = \frac{1}{3}(e^t + 9te^t + e^{-2t}).$$

Example 5.101. Find $L^{-1}\Big[\frac{2s+1}{(s+2)^2(s-1)^2}\Big]$.

Solution. Let $\frac{2s+1}{(s+2)^2(s-1)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$

$$= \frac{A(s+2)(s-1)^2 + B(s-1)^2 + C(s-1)(s+2)^2 + D(s+2)^2}{(s+2)^2(s-1)^2}$$

$\therefore 2s+1 = A(s+2)(s-1)^2 + B(s-1)^2 + C(s-1)(s+2)^2 + D(s+2)^2.$

When $s = -2$ When $s = 1$

$9B = -3 \Rightarrow B = -\frac{1}{3}.$ $9D = 3 \Rightarrow D = \frac{1}{3}.$

Equating the coefficients of s^3 we get

$A + C = 0.$ (1)

Equating the constants we get

$$2A + B - 4C + 4D = 1$$
$$2A - \frac{1}{3} - 4C + \frac{4}{3} = 1$$
$$2A - 4C + 1 = 1$$
$$2A - 4C = 0$$

$$A - 2C = 0. \tag{2}$$

$$(1) - (2) \Rightarrow 3C = 0 \Rightarrow C = 0.$$

$$(1) \Rightarrow A = 0.$$

$$\therefore \frac{2s+1}{(s+2)^2(s-1)^2} = \frac{-\frac{1}{3}}{(s+2)^2} + \frac{\frac{1}{3}}{(s-1)^2}$$

$$\text{Now, } L^{-1}\Big[\frac{2s+1}{(s+2)^2(s-1)^2}\Big] = -\frac{1}{3}L^{-1}\Big[\frac{1}{(s+2)^2}\Big] + \frac{1}{3}L^{-1}\Big[\frac{1}{(s-1)^2}\Big]$$

$$= -\frac{1}{3}e^{-2t}L^{-1}\Big[\frac{1}{s^2}\Big] + \frac{1}{3}e^{t}L^{-1}\Big[\frac{1}{s^2}\Big]$$

$$= -\frac{1}{3}e^{-2t}t + \frac{1}{3}e^{t}t = \frac{t}{3}(e^{t} - e^{-2t}).$$

Example 5.102. Find $L^{-1}\Big[\dfrac{5s+3}{(s-1)(s^2+2s+5)}\Big]$.

Solution. Let $\dfrac{5s+3}{(s-1)(s^2+2s+5)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+2s+5}$

$5s+3 = A(s^2+2s+5) + (Bs+C)(s-1)$

Put $s = 1 \Rightarrow 8A = 8 \Rightarrow A = 1$.

Equating the coeff. $s^2 \Rightarrow A + B = 0 \Rightarrow B = -1$.

$s = 0 \Rightarrow 5A - C = 3 \Rightarrow C = 5 - 3 = 2$.

$$L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = L^{-1}\left[\frac{1}{s-1} + \frac{-s+2}{s^2+2s+5}\right]$$

$$= e^{t} - L^{-1}\left[\frac{s-2}{(s+1)^2+5-1}\right]$$

$$= e^{t} - L^{-1}\left[\frac{s+1-3}{(s+1)^2+4}\right]$$

$$= e^{t} - e^{-t}L^{-1}\left[\frac{s-3}{s^2+4}\right]$$

$$= e^{t} - e^{-t}\left[\cos 2t - \frac{3}{2}\sin 2t\right]$$

$$= e^{t} - \frac{e^{-t}}{2}[2\cos 2t - 3\sin 2t].$$

Example 5.103. Find $L^{-1}\Big[\dfrac{s+9}{(s+2)(s^2+3)}\Big]$.

Solution. Let $\dfrac{s+9}{(s+2)(s^2+3)} = \dfrac{A}{s+2} + \dfrac{Bs+C}{s^2+3}$

$$= \frac{A(s^2+3)+(Bs+c)(s+2)}{(s+2)(s^2+3)}.$$

$\therefore\ s+9 = A(s^2+3)+(Bs+c)(s+2).$

When $s=-2$, $7A = 7 \Rightarrow A = 1$.

Equating the coefficient of s^2 we get

$A+B = 0 \Rightarrow B = -A = -1$.

Equating the constants we get

$$3A + 2c = 9$$
$$3 + 2c = 9$$
$$2c = 6 \Rightarrow c = 3.$$

$$\therefore \frac{s+9}{(s+2)(s^2+3)} = \frac{1}{s+2} + \frac{-s+3}{s^2+3} = \frac{1}{s+2} - \frac{s-3}{s^2+3}$$

$$\text{Now, } L^{-1}\Big[\frac{s+9}{(s+2)(s^2+3)}\Big] = L^{-1}\Big[\frac{1}{s+2}\Big] - L^{-1}\Big[\frac{s}{s^2+3}\Big] - 3L^{-1}\Big[\frac{1}{s^2+3}\Big]$$
$$= e^{-2t} - \cos\sqrt{3}t - \frac{3}{\sqrt{3}}\sin\sqrt{3}t.$$

Example 5.104. Find $L^{-1}\Big[\dfrac{s^2+16}{(s^2+1)(s^2+4)}\Big]$.

Solution. Let $\dfrac{s^2+16}{(s^2+1)(s^2+4)} = \dfrac{A}{s^2+1} + \dfrac{B}{s^2+4} = \dfrac{5}{s^2+1} - \dfrac{4}{s^2+4}$

$$L^{-1}\Big[\frac{s^2+16}{(s^2+1)(s^2+4)}\Big] = 5\sin t - 2\sin 2t.$$

Example 5.105. Find $L^{-1}\Big[\dfrac{s}{(s^2+a^2)(s^2+b^2)}\Big]$.

Solution. Let $\dfrac{1}{(s^2+a^2)(s^2+b^2)} = \dfrac{A}{s^2+a^2} + \dfrac{B}{s^2+b^2}$

$$= \frac{1}{b^2-a^2}\frac{1}{s^2+a^2} - \frac{1}{b^2-a^2}\frac{1}{s^2+b^2}$$

$$\therefore \frac{s}{(s^2+a^2)(s^2+b^2)} = \frac{1}{b^2-a^2}\Big[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\Big]$$

$$\text{Hence, } L^{-1}\Big[\frac{s}{(s^2+a^2)(s^2+b^2)}\Big] = \frac{1}{b^2-a^2}[\cos at - \cos bt].$$

—

Example 5.106. Find $L^{-1}\left[\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}\right]$.

Solution. Let $L^{-1}\left[\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}\right] = L^{-1}\left[\dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right]$

$$\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$2s^2-6s+5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

$s = 1 \Rightarrow 2A = 1 \Rightarrow A = \dfrac{1}{2}$.

$s = 2 \Rightarrow -B = 1 \Rightarrow B = -1$.

$s = 3 \Rightarrow 2C = 5 \Rightarrow C = \dfrac{5}{2}$.

$$\therefore L^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right] = L^{-1}\left[\frac{\frac{1}{2}}{s-1} - \frac{1}{s-2} + \frac{5}{2}\frac{1}{s-3}\right]$$
$$= \frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}.$$

Example 5.107. Find $L^{-1}\left[\dfrac{3s^2+16s+26}{s(s^2+4s+13)}\right]$. [Dec 2013]

Solution. Let $\dfrac{3s^2+16s+26}{s(s^2+4s+13)} = \dfrac{A}{s} + \dfrac{Bs+c}{s^2+4s+13}$

$3s^2+16s+26 = A(s^2+4s+13) + (Bs+c)s$.

When $s = 0, 13A = 26 \Rightarrow A = 2$.

Equating the coefficients of s^2

$A + B = 3$

$2 + B = 3$

$B = 1$.

Equating the coefficients of s

$4A + c = 16$

$8 + c = 16$

$c = 8$.

$$\therefore \frac{3s^2+16s+26}{s(s^2+4s+13)} = \frac{2}{s} + \frac{s+8}{s^2+4s+13}.$$

—

$$L^{-1}\left[\frac{3s^2+16s+26}{s(s^2+4s+13)}\right] = L^{-1}\left[\frac{2}{s}+\frac{s+8}{s^2+4s+13}\right]$$
$$= 2L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{s+8}{s^2+4s+13}\right]$$
$$= 2\cdot 1 + L^{-1}\left[\frac{s+2+6}{(s+2)^2+13-4}\right]$$
$$= 2 + L^{-1}\left[\frac{s+2+6}{(s+2)^2+9}\right]$$
$$= 2 + e^{-2t}L^{-1}\left[\frac{s+6}{s^2+9}\right]$$
$$= 2 + e^{-2t}\left[L^{-1}\left(\frac{s}{s^2+9}\right) + 2L^{-1}\left(\frac{3}{s^2+9}\right)\right]$$
$$= 2 + e^{-2t}[\cos 3t + 2\sin 3t].$$

Example 5.108. Find $L^{-1}\left[\dfrac{s^2}{s^4+4a^4}\right]$. [Dec 2011]

Solution. We have $\dfrac{s^2}{s^4+4a^4} = \dfrac{s^2}{(s^2-2as+2a^2)(s^2+2as+2a^2)}$

Let $\dfrac{s^2}{(s^2-2as+2a^2)(s^2+2as+2a^2)} = \dfrac{As+B}{s^2-2as+2a^2} + \dfrac{Cs+D}{s^2+2as+2a^2}$

$\therefore\ s^2 = (As+B)(s^2+2as+2a^2) + (Cs+D)(s^2-2as+2a^2).$

Equating the coefficients of s^3 we get

$$A + C = 0 \tag{1}$$

Equating the coefficient of s^2 we get

$$2aA + B - 2aC + D = 1. \tag{2}$$

Equating the coefficient of s we get

$$2a^2A + 2aB + 2a^2C - 2aD = 0 \tag{3}$$

Equating the constants we get

$$2a^2B + 2a^2D = 0$$

$$2a^2(B+D) = 0$$

$$B + D = 0. \tag{4}$$

From (1) and (4) we get $C = -A$ and $D = -B$.

(2) $\Rightarrow 2aA + B + 2aA - B = 1$

$$4aA = 1$$

$$A = \frac{1}{4a}.$$

$$\Rightarrow C = -\frac{1}{4a}.$$

$$(3) \Rightarrow 2a^2A + 2aB - 2a^2A + 2aB = 0$$

$$4aB = 0$$

$$B = 0.$$

$$\Rightarrow D = 0.$$

$$\therefore \frac{s^2}{s^4 + 4a^4} = \frac{1}{4a}\frac{s}{s^2 - 2as + 2a^2} - \frac{1}{4a}\frac{s}{s^2 + 2as + 2a^2}$$

$$\begin{aligned}
L^{-1}\left[\frac{s^2}{s^4 + 4a^4}\right] &= \frac{1}{4a}L^{-1}\left[\frac{s}{s^2 - 2as + 2a^2}\right] - \frac{1}{4a}L^{-1}\left[\frac{s}{s^2 + 2as + 2a^2}\right] \\
&= \frac{1}{4a}L^{-1}\left[\frac{s - a + a}{(s-a)^2 + a^2}\right] - \frac{1}{4a}L^{-1}\left[\frac{s + a - a}{(s+a)^2 + a^2}\right] \\
&= \frac{1}{4a}e^{at}L^{-1}\left[\frac{s + a}{s^2 + a^2}\right] - \frac{1}{4a}e^{-at}L^{-1}\left[\frac{s - a}{s^2 + a^2}\right] \\
&= \frac{1}{4a}e^{at}\left\{L^{-1}\left[\frac{s}{s^2 + a^2}\right] + L^{-1}\left[\frac{a}{s^2 + a^2}\right]\right\} \\
&\qquad - \frac{1}{4a}e^{-at}\left\{L^{-1}\left[\frac{s}{s^2 + a^2}\right] - L^{-1}\left[\frac{a}{s^2 + a^2}\right]\right\} \\
&= \frac{1}{4a}e^{at}\left[\cos at + \sin at\right] - \frac{1}{4a}e^{-at}\left[\cos at - \sin at\right] \\
&= \frac{1}{4a}\left[e^{at}\cos at + e^{at}\sin at - e^{-at}\cos at + e^{-at}\sin at\right] \\
&= \frac{1}{4a}\left[\cos at\left(e^{at} - e^{-at}\right) + \sin at\left(e^{at} + e^{-at}\right)\right] \\
&= \frac{1}{4a}\left[\cos at \cdot 2\sin hat + \sin at \cdot 2\cos hat\right] \\
&= \frac{1}{2a}\left[\cos at \sin hat + \sin at \cos hat\right].
\end{aligned}$$

Example 5.109. Find $L^{-1}\left[\frac{s}{s^4 + 4a^4}\right]$ [Dec 2008]

Solution. Let $\dfrac{s}{s^4 + 4a^4} = \dfrac{s}{(s^2 - 2as + 2a^2)(s^2 + 2as + 2a^2)}$

Let $\dfrac{s}{(s^2 - 2as + 2a^2)(s^2 + 2as + 2a^2)} = \dfrac{As + B}{(s^2 - 2as + 2a^2)} + \dfrac{Cs + D}{(s^2 + 2as + 2a^2)}$

—

$\therefore\ s = (As + B)(s^2 + 2as + 2a^2) + (Cs + D)(s^2 - 2as + 2a^2).$

Equating the coefficients of s^3, we get

$A + C = 0.$ (1)

Equating the coefficients of s^2, we get

$2aA + B - 2aC + D = 0.$ (2)

Equating the coefficients of s, we get

$2a^2A + 2aB + 2a^2C - 2aD = 1.$ (3)

Equating the constants we get

$2a^2B + 2a^2D = 0$

$2a^2(B + D) = 0$

$B + D = 0.$ (4)

From (1) and (4) we get

$C = -A$ and $D = -B$.

$$\therefore (2) \Rightarrow 2aA + 2aA = 0$$

$$4aA = 0$$

$$\Rightarrow A = 0.$$

$$\therefore C = 0.$$

$$(3) \Rightarrow 2a^2(A + C) + 2a(B - D) = 1$$

$$2a^2 \times 0 + 2a(B + B) = 1$$

$$4aB = 1 \Rightarrow B = \frac{1}{4a}.$$

$$\Rightarrow D = -\frac{1}{4a}.$$

$$\therefore \frac{s}{s^4 + 4a^4} = \frac{1}{4a} \cdot \frac{1}{s^2 - 2as + 2a^2} - \frac{1}{4a}\frac{1}{s^2 + 2as + 2a^2}$$

$$= \frac{1}{4a}\frac{1}{(s-a)^2 + a^2} - \frac{1}{4a}\frac{1}{(s+a)^2 + a^2}$$

$$L^{-1}\left[\frac{s}{s^4 + 4a^4}\right] = \frac{1}{4a}\left[L^{-1}\left(\frac{1}{(s-a)^2 + a^2}\right) - L^{-1}\left(\frac{1}{(s+a)^2 + a^2}\right)\right]$$

$$= \frac{1}{4a}\left[e^{at}L^{-1}\left(\frac{1}{s^2+a^2}\right) - e^{-at}L^{-1}\left(\frac{1}{s^2+a^2}\right)\right]$$
$$= \frac{1}{4a}\left[e^{at}\frac{\sin at}{a} - e^{-at}\frac{\sin at}{a}\right]$$
$$= \frac{1}{4a^2}\sin at[e^{at} - e^{-at}]$$
$$= \frac{1}{4a^2}\sin at \cdot 2\sin hat$$
$$= \frac{\sin at \sin hat}{2a^2}.$$

Exercise 5 I

Find the Laplace inverse of the following.

1. $\dfrac{s+2}{s(s+4)(s+9)}$.
2. $\dfrac{1}{s(s+1)(s+2)}$.
3. $\dfrac{1-s}{(s+1)(s^2+4s+13)}$. [Apr 2009]
4. $\dfrac{4s^2-3s+5}{(s+1)(s^2-3s+2)}$. [Dec 2007]
5. $\dfrac{1}{(s^2+a^2)(s^2+b^2)}$.
6. $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$. [Nov 2008]
7. $\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}$. [Apr 2009]
8. $\dfrac{1}{s^4-1}$. [Nov 2009]
9. $\dfrac{s}{(s+1)^2(s^2+1)}$.
10. $\dfrac{1}{s^2(s^2+81)}$. [Jun 2008]
11. $\dfrac{3s^2+2s-1}{(s^2+4)(s^2-2s+1)}$.
12. $\dfrac{2s-1}{s^2(s-1)^2}$.
13. $\dfrac{1}{s^3(s^2+1)}$.

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Results

1. If $L^{-1}\left[F(s)\right] = f(t)$ and $f(0) = 0$ then
$L^{-1}\left[sF(s)\right] = f'(t) = \frac{d}{dt}[f(t)] = \frac{d}{dt}\left[L^{-1}[F(s)]\right]$.
In general $L^{-1}\left[s^n F(s)\right] = f^{(n)}(t)$ if $f(0) = 0 = f'(0) = \cdots = f^{(n-1)}(0)$.
i.e., $L^{-1}\left[s^n F(s)\right] = \frac{d^n}{dt^n}[L^{-1}[F(s)]]$.

2. If $L^{-1}\left[F(s)\right] = f(t)$ then $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)dt = \int_0^t L^{-1}[F(s)]dt$
Similarly, $L^{-1}\left[\frac{F(s)}{s^2}\right] = \int_0^t \int_0^t L^{-1}[F(s)]dtdt$.

3. We know that $L[tf(t)] = -\frac{d}{ds}[f(s)] = -F'(s)$.
$$L^{-1}\left[F'(s)\right] = -tf(t) = -tL^{-1}\left[F(s)\right].$$

Worked Examples

Example 5.110. Find $L^{-1}\left[\frac{s}{(s+2)^2+4}\right]$. [May 2008]

Solution. $L^{-1}\left[\frac{s}{(s+2)^2+4}\right] = L^{-1}\left[s\frac{1}{(s+2)^2+4}\right] = \frac{d}{dt}\left(L^{-1}\left[\frac{1}{(s+2)^2+4}\right]\right)$

$$= \frac{d}{dt}\left(e^{-2t}L^{-1}\left[\frac{1}{s^2+4}\right]\right)$$
$$= \frac{d}{dt}\left(\frac{e^{-2t}}{2}L^{-1}\left[\frac{2}{s^2+4}\right]\right)$$
$$= \frac{d}{dt}\left(\frac{e^{-2t}}{2}\sin 2t\right)$$
$$= \frac{1}{2}\left(e^{-2t}2\cos 2t - 2e^{-2t}\sin 2t\right)$$
$$= e^{-2t}(\cos 2t - \sin 2t).$$

Example 5.111. Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$.

—

Solution. $L^{-1}\Big[\dfrac{s^2}{(s^2+a^2)^2}\Big] = L^{-1}\Big[s\dfrac{s}{(s^2+a^2)^2}\Big] = \dfrac{d}{dt}\Big\{L^{-1}\Big[\dfrac{s}{(s^2+a^2)^2}\Big]\Big\}$

$$= \frac{d}{dt}\Big\{\frac{t\sin at}{2a}\Big\} = \frac{1}{2a}\Big\{t\cos at.a + \sin at\Big\}$$

$$= \frac{1}{2a}\Big\{\sin at + at\cos at\Big\}.$$

Example 5.112. Find $L^{-1}\Big[\dfrac{s+2}{(s^2+4s+5)^2}\Big]$. [May 2006]

Solution. $L^{-1}\Big[\dfrac{s+2}{(s^2+4s+5)^2}\Big] = L^{-1}\Big[\dfrac{s+2}{((s+2)^2+1)^2}\Big] = e^{-2t}L^{-1}\Big[\dfrac{s}{(s^2+1)^2}\Big] = e^{-2t}\Big[\dfrac{t\sin t}{2}\Big].$

Example 5.113. Find $L^{-1}\Big[\dfrac{s^2}{(s-1)^4}\Big]$. [Dec 2008]

Solution. $L^{-1}\Big[\dfrac{s^2}{(s-1)^4}\Big] = L^{-1}\Big[s\dfrac{s}{(s-1)^4}\Big] = \dfrac{d}{dt}\left(L^{-1}\Big[\dfrac{s}{(s-1)^4}\Big]\right)$

$$= \frac{d}{dt}\left(L^{-1}\Big[\frac{s+1-1}{(s-1)^4}\Big]\right) = \frac{d}{dt}\left(e^tL^{-1}\Big[\frac{s+1}{s^4}\Big]\right)$$

$$= \frac{d}{dt}\left(e^tL^{-1}\Big[\frac{1}{s^3}+\frac{1}{s^4}\Big]\right) = \frac{d}{dt}\left(e^tL^{-1}\Big[\frac{1}{s^3}\Big] + L^{-1}\Big[\frac{1}{s^4}\Big]\right)$$

$$= \frac{d}{dt}\left(e^tL^{-1}\Big[\frac{t^2}{2}+\frac{t^3}{6}\Big]\right) = \frac{1}{6}\frac{d}{dt}(e^t(3t^2+t^3))$$

$$= \frac{1}{6}\Big(e^t(6t+3t^2)+e^t(3t^2+t^3)\Big) = \frac{e^t}{6}\Big(6t+6t^2+t^3\Big).$$

Example 5.114. Find $L^{-1}\Big[\dfrac{s}{(s^2+a^2)^2}\Big]$. [Dec 2013, May 1991]

Solution. From the standard results $L^{-1}\Big[\dfrac{s}{(s^2+a^2)^2}\Big] = \dfrac{t\sin at}{2a}$.

Example 5.115. Find $L^{-1}\Big[\dfrac{1}{s(s^2+a^2)}\Big]$. [Jun 2007]

Solution. $L^{-1}\Big[\dfrac{1}{s(s^2+a^2)}\Big] = L^{-1}\Big[\dfrac{F(s)}{s}\Big]$ where $F(s) = \dfrac{1}{s^2+a^2}$

$$= \int_0^t L^{-1}[F(s)]dt = \int_0^t L^{-1}\Big[\frac{1}{s^2+a^2}\Big]dt = \int_0^t \frac{\sin at}{a}dt$$

$$= \frac{1}{a}\left(-\frac{\cos at}{a}\right)_0^t = -\frac{1}{a^2}(\cos at - 1) = \frac{1-\cos at}{a^2}.$$

—

Example 5.116. Find $L^{-1}\left[\frac{1}{s(s+2)^3}\right]$. [May 2007]

Solution. $L^{-1}\left[\frac{1}{s(s+2)^3}\right] = L^{-1}\left[\frac{F(s)}{s}\right]$ where $F(s) = \frac{1}{(s+2)^3}$

$$= \int_0^t L^{-1}\left[F(s)\right]dt = \int_0^t L^{-1}\left[\frac{1}{(s+2)^3}\right]dt = \int_0^t e^{-2t}L^{-1}\left[\frac{1}{s^3}\right]dt$$

$$= \int_0^t e^{-2t}\frac{t^2}{2}dt = \frac{1}{2}\int_0^t t^2 d\left(\frac{e^{-2t}}{-2}\right) = \frac{1}{2}\left[\left(\frac{t^2e^{-2t}}{-2}\right)_0^t - \int_0^t \left(\frac{e^{-2t}}{-2}\right)2tdt\right]$$

$$= \frac{1}{2}\left[\frac{t^2e^{-2t}}{-2} + \int_0^t td\left(\frac{e^{-2t}}{-2}\right)\right] = \frac{1}{2}\left[\frac{t^2e^{-2t}}{-2} + \left(\frac{te^{-2t}}{-2}\right)_0^t - \int_0^t \left(\frac{e^{-2t}}{-2}\right)dt\right]$$

$$= \frac{1}{2}\left[\frac{t^2e^{-2t}}{-2} - \frac{te^{-2t}}{2} + \frac{1}{2}\left(\frac{e^{-2t}}{-2}\right)_0^t\right] = \frac{1}{2}\left[\frac{t^2e^{-2t}}{-2} - \frac{te^{-2t}}{2} - \frac{1}{4}\left(e^{-2t}-1\right)\right]$$

$$= \frac{1}{2}\left[\frac{t^2e^{-2t}}{-2} - \frac{te^{-2t}}{2} - \frac{1}{4}e^{-2t} + \frac{1}{4}\right] = \frac{1}{8}\left[1 - e^{-2t}\left(2t^2+2t+1\right)\right].$$

Example 5.117. Find $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$. [Dec 2013]

Solution. $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{s}\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{F(s)}{s}\right]$ where $F(s) = \frac{s}{(s^2+a^2)^2}$

$$= \int_0^t L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]dt = \int_0^t \frac{t\sin at}{2a}dt = \frac{1}{2a}\int_0^t td\left(\frac{-\cos at}{a}\right)$$

$$= \frac{1}{2a}\left[\left(\frac{-t\cos at}{a}\right)_0^t - \int_0^t \left(\frac{-\cos at}{a}\right)dt\right]$$

$$= \frac{1}{2a}\left[\frac{-t\cos at}{a} + \frac{1}{a}\left(\frac{\sin at}{a}\right)_0^t\right] = \frac{1}{2a}\left[\frac{-t\cos at}{a} + \frac{1}{a^2}\sin at\right]$$

$$= \frac{1}{2a^3}\left[\sin at - at\cos at\right].$$

Example 5.118. Find $L^{-1}\left[\frac{1}{(s^2+9s+13)^2}\right]$. [Dec 2013]

Solution. $L^{-1}\left[\frac{1}{(s^2+9s+13)^2}\right] = L^{-1}\left[\frac{1}{[(s+3)^2+13-9]^2}\right]$

$$= e^{-3t}L^{-1}\left[\frac{1}{(s^2+4)^2}\right], \text{which is of the form } \frac{1}{(s^2+a^2)^2} \text{ with } a = 2.$$

$$= e^{-3t}\left[\frac{1}{2\times 2^3}(\sin 2t - 2t\cos 2t)\right]$$

$$= \frac{e^{-3t}}{16}[\sin 2t - 2t\cos 2t].$$

Example 5.119. Find $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right]$ and find $L^{-1}\left[\dfrac{1}{(s^2+a^2)^2}\right]$ and hence find $L^{-1}\left[\dfrac{1}{(s^2+9s+13)^2}\right]$. [Dec 2013]

Solution. Solve examples 3.114, 3.117&3.118.

Example 5.120. Find $L^{-1}\left[\dfrac{1}{s^2(s+a)}\right]$. [Nov 2009]

Solution. $L^{-1}\left[\dfrac{1}{s^2(s+a)}\right] = L^{-1}\left[\dfrac{F(s)}{s^2}\right]$ where $F(s) = \dfrac{1}{s+a}, L^{-1}[F(s)] = e^{-at}$

$$
\begin{aligned}
&= \int_0^t \int_0^t L^{-1}[F(s)]dtdt \\
&= \int_0^t \int_0^t e^{-at}dtdt = \int_0^t \left(\frac{e^{-at}}{-a}\right)_0^t dt \\
&= \frac{-1}{a}\int_0^t ((e^{-at}-1))dt = \frac{-1}{a}\left[\left(\frac{e^{-at}}{-a}\right)_0^t - (t)_0^t\right] \\
&= \frac{-1}{a}\left[\frac{e^{-at}}{-a} + \frac{1}{a} - t\right] \\
&= \frac{1}{a^2}e^{-at} - \frac{1}{a^2} + \frac{1}{a}t \\
&= \frac{1}{a^2}\left(e^{-at} - 1 + at\right).
\end{aligned}
$$

Example 5.121. Find $L^{-1}\left[\dfrac{s}{(s^2-a^2)^2}\right]$. [Dec 2007]

Solution. Let $F'(s) = \dfrac{s}{(s^2-a^2)^2}$

$$
\begin{aligned}
L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right] &= L^{-1}\left[F'(s)\right] \\
&= -tL^{-1}[F(s)] \\
&= -tL^{-1}\left[-\frac{1}{2(s^2-a^2)}\right] \\
&= \frac{t}{2}\frac{\sinh at}{a} \\
&= \frac{t}{2a}\sinh at.
\end{aligned}
$$

$$
\begin{aligned}
F(s) &= \int \frac{s}{(s^2-a^2)^2}ds \\
&= \frac{1}{2}\int \frac{2s}{(s^2-a^2)^2}ds \\
&= \frac{1}{2}\int \frac{d(s^2-a^2)}{(s^2-a^2)^2}ds \\
&= \frac{1}{2}\frac{(s^2-a^2)^{-1}}{-1} \\
&= \frac{-1}{2}\frac{1}{s^2-a^2}.
\end{aligned}
$$

—

Example 5.122. Find $L^{-1}\left[\dfrac{2(s+1)}{(s^2+2s+2)^2}\right]$. [Dec 2009]

Solution. $L^{-1}\left[\dfrac{2(s+1)}{(s^2+2s+2)^2}\right] = L^{-1}\left[\dfrac{2(s+1)}{((s+1)^2+2-1)^2}\right] = L^{-1}\left[\dfrac{2(s+1)}{((s+1)^2+1)^2}\right]$

$$= e^{-t}L^{-1}\left[\frac{2s}{(s^2+1)^2}\right] = e^{-t}L^{-1}\left[F'(s)\right].$$

Where $F'(s) = \dfrac{2s}{(s^2+1)^2}$.

$$F(s) = \int \frac{2s}{(s^2+1)^2}ds = \int \frac{d(s^2+1)}{(s^2+1)^2} = -\frac{1}{s^2+1}.$$

$$L^{-1}\left[\frac{2(s+1)}{(s^2+2s+2)^2}\right] = e^{-t}\left\{-tL^{-1}\left[F(s)\right]\right\} = -te^{-t}L^{-1}\left[\frac{-1}{s^2+1}\right] = te^{-t}\sin t.$$

Example 5.123. Find $L^{-1}\left[\dfrac{s+3}{(s^2+6s+13)^2}\right]$. [Jun 2005]

Solution. $L^{-1}\left[\dfrac{s+3}{(s^2+6s+13)^2}\right] = L^{-1}\left[\dfrac{s+3}{((s+3)^2+13-9)^2}\right] = L^{-1}\left[\dfrac{s+3}{((s+3)^2+4)^2}\right]$

$$= e^{-3t}L^{-1}\left[\frac{s}{(s^2+4)^2}\right] = e^{-3t}L^{-1}\left[F'(s)\right].$$

where $F'(s) = \dfrac{s}{(s^2+4)^2}$.

$$\therefore F(s) = \int \frac{s}{(s^2+4)^2}ds = \frac{1}{2}\int \frac{2s}{(s^2+4)^2}ds = \frac{1}{2}\int \frac{d(s^2+4)}{(s^2+4)^2} = \frac{1}{2}\left(-\frac{1}{s^2+4}\right) = -\frac{1}{2}\frac{1}{s^2+4}.$$

$$\therefore L^{-1}\left[\frac{s+3}{(s^2+6s+13)^2}\right] = e^{-3t}\left\{-tL^{-1}\left[F(s)\right]\right\} = -te^{-3t}L^{-1}\left[\frac{-1}{2}\frac{1}{s^2+4}\right]$$

$$= \frac{te^{-3t}}{2}\frac{\sin 2t}{2} = \frac{te^{-3t}\sin 2t}{4}.$$

Exercise 5 J

Find the Inverse Laplace transform of the following functions.

1. $\dfrac{s}{(s+2)^2}$. [Nov 2008]
2. $\dfrac{s^2}{(s-2)^3}$.
3. $\dfrac{s-3}{s^2+4s+13}$. [Decc 2009]
4. $\dfrac{s}{(s-a)^5}$. [May 2009]

—

5. $\dfrac{1}{(s-2)^2+13^2}$.

6. $\dfrac{1}{s(s+3)}$.

7. $\dfrac{1}{s(s^2-2s+5)}$.

8. $\dfrac{1}{(s^2+a^2)^2}$.

9. $\dfrac{s}{(s^2+a^2)^2}$.

10. $\dfrac{1}{s^2(s^2+a^2)^2}$.

5.4.2 Inverse Laplace Transform of Logarithmic Functions

Worked Examples

Example 5.124. Find $L^{-1}\left[\log\dfrac{1+s}{s^2}\right]$. [Dec 2009]

Solution. Let $F(s)=\log\dfrac{s+1}{s^2}=\log(s+1)-\log s^2=\log(s+1)-2\log s$

$$F'(S)=\frac{1}{s+1}-\frac{2}{s}$$

we know that $L[tf(t)]=-F'(s)$

$$=-\left[\frac{1}{s+1}-\frac{2}{s}\right]=\frac{2}{s}-\frac{1}{s+1}$$

$$tf(t)=L^{-1}\left[\frac{2}{s}-\frac{1}{s+1}\right]=2L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{1}{s+1}\right]$$

$$\therefore tf(t)=2-e^{-t}$$

$$f(t)=\frac{2-e^{-t}}{t}.$$

Example 5.125. Find $L^{-1}\left[\log\dfrac{s(s+1)}{s^2+1}\right]$. [May 2009]

Solution. Let $F(s)=\log\dfrac{s(s+1)}{s^2+1}=\log s+\log(s+1)-\log(s^2+1)$

$$F'(S)=\frac{1}{s}+\frac{1}{s+1}-\frac{2s}{s^2+1}.$$

we have

$$L[tf(t)]=-F'(s)=\frac{2s}{s^2+1}-\frac{1}{s+1}-\frac{1}{s}$$

$$tf(t)=L^{-1}\left[\frac{2s}{s^2+1}-\frac{1}{s+1}-\frac{1}{s}\right]=2L^{-1}\left[\frac{s}{s^2+1}\right]-L^{-1}\left[\frac{1}{s+1}\right]-L^{-1}\left[\frac{1}{s}\right]$$

$$t f(t) = 2\cos t - e^{-t} - 1$$
$$f(t) = \frac{2\cos t - e^{-t} - 1}{t}.$$

Example 5.126. Find $L^{-1}\Big[\log \frac{s^2 + a^2}{s^2 - b^2}\Big]$. [Jun 2002]

Solution. Let $F(s) = \log \frac{s^2 + a^2}{s^2 - b^2} = \log(s^2 + a^2) - \log(s^2 - b^2)$

$$F'(S) = \frac{2s}{s^2 + a^2} - \frac{2s}{s^2 - b^2}.$$

$$F'(s) = \frac{2s}{s^2 + a^2} - \frac{2s}{s^2 - b^2}$$
$$L^{-1}[F'(s)] = 2L^{-1}\left[\frac{s}{s^2 + a^2}\right] - 2L^{-1}\left[\frac{s}{s^2 - b^2}\right]$$
$$-t f(t) = 2\cos at - 2\cosh bt$$
$$f(t) = \frac{2}{t}(\cosh bt - \cos at).$$

Example 5.127. Find $L^{-1}\Big[\log \frac{s + 1}{s - 1}\Big]$. [Dec 2013]

Solution. Let $F(s) = \log \frac{s + 1}{s - 1} = \log(s + 1) - \log(s - 1)$

$$F'(S) = \frac{1}{s + 1} - \frac{1}{s - 1}$$

We know that

$$L[t f(t)] = -F'(s) = -\frac{1}{s + 1} + \frac{1}{s - 1}$$
$$\therefore\ t f(t) = L^{-1}\Big[\frac{1}{s - 1} - \frac{1}{s + 1}\Big] = L^{-1}\Big[\frac{1}{s - 1}\Big] - L^{-1}\Big[\frac{1}{s + 1}\Big] = e^t - e^{-t}$$
$$\therefore\ f(t) = \frac{e^t - e^{-t}}{t}.$$

Example 5.128. Find $L^{-1}\Big[s \log \frac{s + 1}{s - 1} + 2\Big]$. [Apr 1995]

Solution. Let $F(s) = s \log \frac{s + 1}{s - 1} + 2 = s[\log(s + 1) - \log(s - 1)] + 2$

$$F'(s) = s\Big[\frac{1}{s + 1} - \frac{1}{s - 1}\Big] + \log(s + 1) - \log(s - 1)$$
$$= s\Big[\frac{s - 1 - s - 1}{s^2 - 1}\Big] + \log \frac{s + 1}{s - 1}$$

$$= \frac{-2s}{s^2-1} + \log\frac{s+1}{s-1}$$

$$tf(t) = 2L^{-1}\Big[\frac{s}{s^2-1}\Big] - L^{-1}\Big[\log\frac{s+1}{s-1}\Big]$$

$$= 2\cos ht - \frac{e^t - e^{-t}}{t} = 2\cos ht - \frac{2}{t}\cos ht.$$

$$f(t) = \frac{2\cos ht}{t} - \frac{2\cos ht}{t^2} = \frac{2\cos ht}{t^2}(t-1).$$

5.4.3 Inverse Laplace Transform of inverse Trignometric Functions

Example 5.129. Find $L^{-1}\Big[\tan^{-1}\frac{a}{s}\Big]$.

Solution. Let $F(s) = \tan^{-1}\left(\frac{a}{s}\right)$

$$F'(s) = \frac{1}{1+\frac{a^2}{s^2}}\Big(\frac{-a}{s^2}\Big) = \frac{s^2}{s^2+a^2}\Big(\frac{-a}{s^2}\Big) = -\frac{a}{s^2+a^2}.$$

We have $L[tf(t)] = -F'(s) = \dfrac{a^2}{s^2+a^2}$

$$tf(t) = L^{-1}\Big[\frac{a}{s^2+a^2}\Big] = \sin at$$

$$f(t) = \frac{\sin at}{t}.$$

Example 5.130. Find $L^{-1}\left[\cot^{-1}(s)\right]$. [May 2011, Jun 2010]

Solution. Let $F(s) = \cot^{-1}(s)$

$$F'(s) = -\frac{1}{1+s^2}$$

we have $L[tf(t)] = -F'(s) = \dfrac{1}{s^2+1}$

$$\therefore\ t[f(t)] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t.$$

$$\therefore\ f(t) = \frac{\sin t}{t}.$$

Example 5.131. Find $L^{-1}\Big[\tan^{-1}\Big(\frac{a}{s}\Big) + \cot^{-1}\Big(\frac{s}{b}\Big)\Big]$.

Solution. Let $F(s) = \tan^{-1}\left(\frac{a}{s}\right) + \cot^{-1}\left(\frac{s}{b}\right)$

$$F'(s) = \frac{1}{1+\frac{a^2}{s^2}}\Big(-\frac{a}{s^2}\Big) - \frac{1}{1+\frac{s^2}{b^2}}\Big(\frac{1}{b}\Big) = \frac{-a}{s^2+a^2} - \frac{b}{s^2+b^2}.$$

—

$$\text{We know that } L[tf(t)] = -F'(s) = \frac{a}{s^2+a^2} + \frac{b}{s^2+b^2}$$

$$tf(t) = \sin at + \sin bt$$

$$f(t) = \frac{\sin at + \sin bt}{t}.$$

Example 5.132. Find $L^{-1}\left[\cot^{-1}\left(\frac{2}{s+1}\right)\right]$. [May 2008]

Solution. Let $F(s) = \cot^{-1}\left(\frac{2}{s+1}\right)$.

$$F'(s) = -\frac{1}{1+\frac{4}{(s+1)^2}}\left(\frac{-2}{(s+1)^2}\right) = \frac{2}{(s+1)^2+4}.$$

$$L[tf(t)] = -F'(s) = \frac{-2}{(s+1)^2+4}.$$

$$tf(t) = L^{-1}\left[-\frac{2}{(s+1)^2+4}\right] = -e^{-t}L^{-1}\left[\frac{2}{s^2+4}\right] = -e^{-t}\sin 2t.$$

$$f(t) = -e^{-t}\frac{\sin 2t}{t}.$$

Example 5.133. Find $L^{-1}\left[s\log\frac{s}{\sqrt{s^2+1}} + \cot^{-1}s\right]$.

Solution. Let $F(s) = s\log\frac{s}{\sqrt{s^2+1}} + \cot^{-1}s$

$$= s\left[\log s - \frac{1}{2}\log(s^2+1)\right] + \cot^{-1}(s)$$

$$F'(s) = s\left[\frac{1}{s} - \frac{1}{2}\frac{2s}{s^2+1}\right] + \log s - \frac{1}{2}\log(s^2+1) - \frac{1}{1+s^2}$$

$$= 1 - \frac{s^2}{s^2+1} - \frac{1}{s^2+1} + \log s - \frac{1}{2}\log(s^2+1)$$

$$= 1 - \frac{s^2+1}{s^2+1} + \log s - \frac{1}{2}(\log s^2+1).$$

$$F'(s) = \log s - \frac{1}{2}\log(s^2+1)$$

$$L[tf(t)] = -F'(s) = \frac{1}{2}\log(s^2+1) - \log s$$

$$tf(t) = L^{-1}\left[\frac{1}{2}\log(s^2+1) - \log s\right]. \tag{1}$$

$$\text{Let } G(s) = \frac{1}{2}\log(s^2+1) - \log s$$

$$G'(s) = \frac{1}{2}\frac{2s}{s^2+1} - \frac{1}{s} = \frac{s}{s^2+1} - \frac{1}{s}$$

$$
\begin{aligned}
L[tg(t)] &= -G'(s) = \frac{1}{s} - \frac{s}{s^2+1}\\
tg(t) &= 1 - \cos t\\
g(t) &= \frac{1-\cos t}{t}\\
(1) \Rightarrow \quad tf(t) &= \frac{1-\cos t}{t}\\
f(t) &= \frac{1-\cos t}{t^2}.
\end{aligned}
$$

Example 5.134. Find $L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right]$.

Solution. Let $F(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$

$$
\begin{aligned}
F'(s) &= \frac{1}{1+\frac{4}{s^4}} \cdot 2(-2)(s^{-3})\\
&= \frac{s^4}{s^4+4}\left(\frac{-4}{s^3}\right) = -\frac{4s}{s^4+4}.
\end{aligned}
$$

$$
\begin{aligned}
\text{We know that } L[f(t)] = -F'(s) &= \frac{4s}{s^4+4}\\
tf(t) &= L^{-1}\left[\frac{4s}{s^4+4}\right]\\
\therefore\ f(t) &= \frac{4}{t}L^{-1}\left[\frac{s}{s^4+4}\right]\\
\text{Now, } \frac{s}{s^4+4} &= \frac{s}{(s^2+2s+2)(s^2-2s+2)}\\
&= \frac{As+B}{(s^2+2s+2)} + \frac{Cs+D}{(s^2-2s+2)}\\
s &= (As+B)(s^2-2s+2) + (Cs+D)(s^2+2s+2).
\end{aligned}
$$

Equating the coefficients of s^3, we get

$$A + C = 0. \qquad (1)$$

Equating the coefficients of s^2, we get

$$-2A + B + 2C + D = 0. \qquad (2)$$

Equating the coefficients of s, we get

—

$$2A - 2B + 2C + 2D = 1. \tag{3}$$

Equating the constants we get

$$2B + 2D = 0$$

$$B + D = 0. \tag{4}$$

From (1) and (4) we get

$$C = -A, \quad D = -B.$$

$$\begin{aligned}
(2) \Rightarrow -2A - 2A &= 0 \\
-4A &= 0 \\
A &= 0. \\
\Rightarrow C &= 0. \\
-2B - 2B &= 1 \\
-4B &= 1 \\
B &= -\frac{1}{4} \\
D &= \frac{1}{4}.
\end{aligned}$$

$$\begin{aligned}
\therefore \frac{s}{s^2+4} &= \frac{1}{4} \cdot \frac{1}{s^2-2s+2} - \frac{1}{4} \cdot \frac{1}{s^2+2s+2} \\
L^{-1}\left[\frac{s}{s^2+4}\right] &= \frac{1}{4} L^{-1}\left[\frac{1}{s^2-2s+2}\right] - \frac{1}{4} L^{-1}\left[\frac{1}{s^2+2s+2}\right] \\
&= \frac{1}{4}\left[L^{-1}\left(\frac{1}{(s-1)^2+1}\right) - L^{-1}\left(\frac{1}{(s+1)^2+1}\right)\right] \\
&= \frac{1}{4}\left[e^t L^{-1}\left(\frac{1}{s^2+1}\right) - e^{-t} L^{-1}\left(\frac{1}{s^2+1}\right)\right] \\
&= \frac{1}{4}\left[e^t \sin t - e^{-t} \sin t\right] \\
&= \frac{1}{4} \sin t (e^t - e^{-t}) \\
&= \frac{\sin t}{4} \cdot 2 \sin ht \\
&= \frac{\sin t \sin ht}{2}
\end{aligned}$$

—

$$\therefore\ f(t) = \frac{4}{t}\cdot\frac{\sin t \sin ht}{2}$$
$$= \frac{2\sin t \sin ht}{t}.$$

Exercise 5 K

Find the inverse Laplace transform of the following functions.

1. $\log\left(1+\frac{\omega^2}{s^2}\right)$. [May 2009]
2. $\log\left(\frac{s-a}{s^2+a^2}\right)$. [Apr 2007]
3. $\log\left(\frac{s+2}{s+4}\right)$.
4. $\cot^{-1}(s+1)$.
5. $\log\left(\frac{s^2(s+3)}{s-5}\right)$. [May 2006]
6. $\log\left(\frac{(s+3)(s+7)}{s-1}\right)$. [Apr 2005]
7. $\log\left(\frac{s^2+4}{s^2+1}\right)$. [Dec 2004]
8. $\tan^{-1}\left(\frac{s}{a}\right)$. [Apr 2007]
9. $\log\left(\frac{s^2+9}{s^2+16}\right)$.
10. $\cot^{-1}\left(\frac{s}{k}\right)$.

5.4.4 Inverse Laplace transform using contour integration

Let $L[f(t)] = F(s)$. If $F(s) \to 0$ as $s \to \infty$ then $f(t) =$ sum of the residues of $e^{st}F(s)$ at the poles of $F(s)$.

Worked Examples

Example 5.135. Find $L^{-1}\left[\frac{2s+3}{(s-2)(s+1)^2}\right]$ using the method of residues.

Solution. Let $F(s) = \frac{2s+3}{(s-2)(s+1)^2}$.

Clearly $F(s) \to 0$ as $s \to \infty$.

$\therefore\ f(t) =$ sum of the residues of $e^{st}F(s)$ at the poles of $F(s)$.

The poles of $F(s)$ are at $s = 2$ and at $s = -1$.

—

$s=2$ is a simple pole and $s=-1$ is a pole of order 2.

$$R(2)=\lim_{s\to 2}(s-2)e^{st}F(s)=\lim_{s\to 2}(s-2)e^{st}\frac{2s+3}{(s-2)(s+1)^2}=\frac{7e^{2t}}{9}.$$

$$R(-1)=\frac{1}{(2-1)!}\lim_{s\to -1}\frac{d}{ds}(s+1)^2e^{st}F(s)=\lim_{s\to -1}\frac{d}{ds}(s+1)^2e^{st}\frac{2s+3}{(s-2)(s+1)^2}$$

$$=\lim_{s\to -1}\frac{d}{ds}\left[\frac{e^{st}(2s+3)}{s-2}\right]=\lim_{s\to -1}\frac{(s-2)\{e^{st}2+(2s+3)te^{st}\}-e^{st}(2s+3)}{(s-2)^2}$$

$$=\frac{(-3)(2e^{-t}+te^{-t})-e^{-t}}{9}=\frac{-6e^{-t}-3te^{-t}-e^{-t}}{9}$$

$$=\frac{-7e^{-t}-3te^{-t}}{9}=\frac{-e^{-t}}{9}(7+3t).$$

$$f(t)=\text{ sum of the residues }=\frac{7}{9}e^{2t}-\frac{e^{-t}}{9}(7+3t).$$

Example 5.136. Evaluate the inverse Laplace transform of $\frac{1}{(s+1)(s-2)^2}$ by the method of residues.

Solution. Let $F(s)=\frac{1}{(s+1)(s-2)^2}$

By the method of residues,

$f(t)$ =sum of the residues of $e^{st}F(s)$ at the poles of F(s).

Poles of F(s) are $-1$and 2

-1 is a simple pole and 2 is a pole of order 2.

$$\therefore R(-1)=\lim_{s\to -1}(s+1)e^{st}\frac{1}{(s+1)(s-2)^2}=\frac{e^{-t}}{9}.$$

$$R(2)=\lim_{s\to 2}\frac{d}{ds}\left\{(s-2)^2e^{st}\frac{1}{(s+1)(s-2)^2}\right\}$$

$$=\lim_{s\to 2}\frac{(s+1)te^{st}-e^{st}}{(s+1)^2}=\frac{3te^{2t}-e^{2t}}{9}$$

$$=\frac{e^{2t}}{9}(3t-1)$$

$$\therefore\ f(t)=\text{ sum of the residues }=\frac{e^{-t}}{9}+\frac{e^{2t}}{9}(3t-1).$$

Example 5.137. Find the inverse Laplace transform of $\frac{2s-2}{(s+1)(s^2+2s+5)}$ using contour integral.

—

Solution. Let $F(s) = \dfrac{2s-2}{(s+1)(s^2+2s+5)}$

Poles are given by $s+1=0 \Rightarrow s=-1$ and $s^2+2s+5=0 \Rightarrow (s+1)^2+4=0$

$\Rightarrow (s+1+2i)(s+1-2i)=0 \Rightarrow s=-(1+2i)$ and $s=-(1-2i)$.

All are simple poles.

$$R(-1) = \lim_{s\to -1}(s+1)\Big[\frac{e^{st}(2s-2)}{(s+1)(s^2+2s+5)}\Big]$$
$$= \frac{e^{-t}(-4)}{1-2+5} = \frac{-4}{4}e^{-t} = -e^{-t}$$
$$R(-(1+2i)) = \lim_{s\to-(1+2i)}\Big[\frac{(s+1+2i)e^{st}(2s-2)}{(s+1)(s+1+2i)(s+1-2i)}\Big]$$
$$= \frac{e^{-(1+2i)t}(-2(1+2i)-2)}{(-1-2i+1)(-1-2i+1-2i)}$$
$$= \frac{(-4-4i)e^{-t}e^{-2it}}{-8} = \frac{e^{-t}e^{-2it}(1+i)}{2}.$$

changing i to -i

$$R(-(1-2i)) = \frac{e^{-t}e^{2it}(1-i)}{2}$$

By the method of residues

$$f(t) = -e^{-t} + \frac{e^{-t}e^{-2it}}{2}(1+i) + \frac{e^{-t}e^{2it}}{2}(1-i)$$
$$= e^{-t}\Big[-1 + \frac{e^{-2it}+ie^{-2it}}{2} + \frac{e^{2it}-ie^{2it}}{2}\Big]$$
$$= e^{-t}\Big[-1 + \frac{e^{-2it}+e^{-2it}}{2} - i\frac{e^{2it}-e^{2it}}{2}\Big]$$
$$= e^{-t}\Big[-1 + 2\cos 2t - i2i\sin 2t\Big]$$
$$= e^{-t}\Big[-1 + \cos 2t + 2\sin 2t\Big]$$
$$= e^{-t}\Big[\cos 2t + \sin 2t - 1\Big].$$

Example 5.138. Find $L^{-1}\Big[\dfrac{1}{(s-1)(s^2+1)}\Big]$ by the method of residues.

Solution. Let $F(s) = \dfrac{1}{(s-1)(s^2+1)} = \dfrac{1}{(s-1)(s+i)(s-i)}$.

Clearly $F(s) \to 0$ as $s \to \infty$.

—

$\therefore\quad f(t) =$ sum of the residues of $e^{st}F(s)$ at the poles of $F(s)$.

The poles of $F(s)$ are $1, i, -i$ and all are simple poles.

$$R(1) = \lim_{s\to 1}(s-1)e^{st}F(s) = \lim_{s\to 1}(s-1)e^{st}\frac{1}{(s-1)(s^2+1)} = \frac{e^t}{2}.$$

$$R(i) = \lim_{s\to i}(s-i)e^{st}\frac{1}{(s-1)(s+i)(s-i)} = \frac{e^{it}}{(i-1)2i} = \frac{-ie^{it}}{(i-1)2}$$

$$= \frac{i}{(1-i)2}e^{it} = \frac{ie^{it}}{4}(1+i).$$

$$R(-i) = \frac{-i}{4}e^{-it}(1-i).$$

$$f(t) = \text{sum of the residues} = R(1)+R(i)+R(-i)$$

$$= \frac{e^t}{2} + \frac{i}{4}e^{it}(1+i) - \frac{i}{4}e^{-it}(1-i)$$

$$= \frac{e^t}{2} + \frac{i}{4}\left(e^{it} + ie^{it} - e^{-it} + ie^{-it}\right)$$

$$= \frac{e^t}{2} + \frac{i}{4}\left(e^{it} - e^{-it} + i(e^{it} + e^{-it})\right)$$

$$= \frac{e^t}{2} + \frac{i}{4}2i\sin t - \frac{1}{4}2\cos t$$

$$= \frac{e^t}{2} - \frac{1}{2}\sin t - \frac{\cos t}{2} = \frac{1}{2}(e^t - \sin t - \cos t).$$

Example 5.139. Find the inverse Laplace transform of $\frac{1}{(s^2+1)^2}$ by the method of residues.

Solution. Let $F(s) = \frac{1}{(s^2+1)^2}$

Poles of F(s) are $s = i$ and $-i$

i and $-i$ are poles of order 2.

$$R(i) = \lim_{s\to i}\frac{d}{ds}\left[(s-i)^2e^{st}\frac{1}{(s^2+1)^2}\right] = \lim_{s\to i}\frac{d}{ds}\left[(s-i)^2e^{st}\frac{1}{(s+i)^2(s-i)^2}\right]$$

$$= \lim_{s\to i}\frac{(s+i)^2te^{st} - e^{st}2(s+i)}{(s+i)^4} = \frac{-4te^{it} - 4ie^{it}}{16} = -\frac{1}{4}\left[te^{it} + ie^{it}\right]$$

Replacing i by -i, we get

$$R(-i) = -\frac{1}{4}\left[te^{-it} - ie^{-it}\right].$$

By the method of residues,

—

$$
\begin{aligned}
f(t) &= \text{sum of the residues} \\
&= -\frac{1}{4}\left[te^{it} + te^{-it} + i(e^{it} - e^{-it})\right] \\
&= -\frac{1}{4}\left[2t\cos t - 2\sin t\right] \\
&= -\frac{1}{2}\left(t\cos t - \sin t\right).
\end{aligned}
$$

Example 5.140. Evaluate the inverse Laplace transform of $\frac{1}{s^2(s^2-a^2)}$ by the method of residues.

Solution. Let $F(s) = \frac{1}{s^2(s^2-a^2)} = \frac{1}{s^2(s-a)(s+a)}$

Poles of F(s) are $s = 0, a, -a$.

$s = 0$ is a pole of order 2.

$$
\begin{aligned}
\therefore R(0) &= \lim_{s\to 0}\frac{d}{ds}\left[s^2e^{st}F[s]\right] = \lim_{s\to 0}\frac{d}{ds}\left[s^2e^{st}\frac{1}{s^2(s^2-a^2)}\right] \\
&= \lim_{s\to 0}\frac{(s^2-a^2)te^{st} - e^{st}2s}{(s^2-a^2)^2} = \frac{-a^2t}{a^4} = \frac{-t}{a^2}.
\end{aligned}
$$

$s = a$ is a simple pole.

$$
\begin{aligned}
R(a) &= \lim_{s\to a}(s-a)e^{st}\frac{1}{s^2(s-a)(s+a)} \\
&= \frac{e^{at}}{a^2 2a} = \frac{e^{at}}{2a^3}
\end{aligned}
$$

$s = -a$ is a simple pole.

$$
\begin{aligned}
R(-a) &= \lim_{s\to -a}(s+a)e^{st}\frac{1}{s^2(s-a)(s+a)} \\
&= \frac{e^{at}}{a^2(-2a)} = -\frac{e^{-at}}{2a^3}.
\end{aligned}
$$

By the method of residues

$$f(t) = \text{sum of the residues of } e^{st}F(s)$$

$$
\begin{aligned}
f(t) &= R(0) + R(a) + R(-a) \\
&= -\frac{t}{a^2} + \frac{e^{at}}{2a^3} - \frac{e^{-at}}{2a^3} \\
&= -\frac{t}{a^2} + \frac{1}{a^3}\sinh at.
\end{aligned}
$$

—

5.5 Inverse Laplace transform using Second Shifting theorem

Result. If $L[f(t)] = F(s)$ then, $L[f(t-a)u(t-a)] = e^{-as}L[f(t)] = e^{-as}F(s)$.

$$\begin{aligned} \therefore L^{-1}\left[e^{-as}F(s)\right] &= f(t-a)u(t-a) \\ &= f(t-a), \quad \text{when } t > a. \\ &= [f(t)]_{t\to t-a} = [L^{-1}\{F(s)\}]_{t\to t-a}. \end{aligned}$$

Example 5.141. Find $L^{-1}\left[\dfrac{e^{-as}}{s}\right]$. [Dec 2011]

Solution. $F(s) = \dfrac{1}{s}$

$L^{-1}[F(s)] = 1.$

Now by Second Shifting theorem,

$$\begin{aligned} L^{-1}[e^{-as}F(s)] &= \left[L^{-1}F(s)\right]_{t\to t-a} u(t-a) = [1]_{t\to t-a}u(t-a) \\ &= 1 \cdot u(t-a) = u(t-a). \end{aligned}$$

Example 5.142. Find $L^{-1}\left[\dfrac{e^{-3s}}{s^3}\right]$.

Solution. $F(s) = \dfrac{1}{s^3} \cdot L^{-1}[F(s)] = L^{-1}\left[\dfrac{1}{s^3}\right] = \dfrac{t^2}{2!}$.

$$\begin{aligned} L^{-1}\left[e^{-3s} \cdot \frac{1}{s^3}\right] &= L^{-1}\left[\frac{1}{s^3}\right]_{t\to t-3}, \quad t > 3 \\ &= \left[\frac{t^2}{2!}\right]_{t\to t-3}, \quad t > 3 \\ &= \frac{(t-3)^2}{2}, \quad t > 3. \end{aligned}$$

Example 5.143. Find $L^{-1}\left[\dfrac{e^{-\pi s}}{s^2+1}\right]$.

Solution. $F(s) = \dfrac{1}{s^2+1}$.

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t.$$

—

By Second Shifting theorem,

$$L^{-1}\left[e^{-\pi s}\frac{1}{s^2+1}\right] = L^{-1}\left[\frac{1}{s^2+1}\right]_{t\to t-\pi}, \quad t>\pi$$
$$= [\sin t]_{t\to t-\pi}, \quad t>\pi$$
$$= \sin(t-\pi), \quad t>\pi.$$

Example 5.144. Find $L^{-1}\left[\frac{e^{-\pi s}}{s^2}\right]$.

Solution. $F(s) = \frac{1}{s^2}$.

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s^2}\right] = t.$$

By Second Shifting theorem,

$$L^{-1}\left[\frac{e^{-\pi s}}{s^2}\right] = L^{-1}\left[\frac{1}{s^2}\right]_{t\to t-\pi}, \quad t>\pi$$
$$= [t]_{t\to t-\pi}, \quad t>\pi$$
$$= t-\pi, \quad t>\pi.$$

Example 5.145. Find $L^{-1}\left[\frac{se^{-2s}}{s^2-1}\right]$.

Solution. $F(s) = \frac{s}{s^2-1}$.

$$L^{-1}[F(s)] = L^{-1}\left[\frac{s}{s^2-1}\right] = \cos ht.$$

By Second Shifting theorem,

$$L^{-1}\left[\frac{se^{-2s}}{s^2-1}\right] = L^{-1}\left[\frac{s}{s^2-1}\right]_{t\to t-2}, \quad t>2$$
$$= [\cos ht]_{t\to t-2}, \quad t>2$$
$$= \cos h(t-2), \quad t>2.$$

Example 5.146. Find $L^{-1}\left[\frac{e^{-2s}}{s-3}\right]$.

Solution. $F(s) = \frac{1}{s-3}$.

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s-3}\right] = e^{3t}.$$

By Second Shifting theorem,

$$L^{-1}\left[\frac{e^{-2s}}{s-3}\right] = L^{-1}\left[\frac{1}{s-3}\right]_{t\to t-2}, \quad t>2$$
$$= \left[e^{3t}\right]_{t\to t-2}, \quad t>2$$
$$= e^{3(t-2)}, \quad t>2.$$

Example 5.147. Find $L^{-1}\left[\dfrac{e^{-s}}{(s-1)(s-2)}\right]$.

Solution. $F(s) = \dfrac{1}{(s-1)(s-2)} = \dfrac{1}{s-2} - \dfrac{1}{s-1}$

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s-2} - \frac{1}{s-1}\right] = e^{2t} - e^{t}.$$

By Second Shifting theorem,

$$L^{-1}\left[\frac{e^{-s}}{(s-1)(s-2)}\right] = L^{-1}\left[\frac{1}{(s-1)(s-2)}\right]_{t\to t-1}, \quad t>1$$
$$= \left(e^{2t} - e^{t}\right)_{t\to t-1}, \quad t>1$$
$$= e^{2(t-1)} - e^{(t-1)}, \quad t>1.$$

Example 5.148. Find $L^{-1}\left[\dfrac{se^{-\frac{s}{2}} + \pi e^{-s}}{s^2+\pi^2}\right]$.

Solution. By Second Shifting theorem we have

$$L^{-1}\left[\frac{se^{-\frac{s}{2}} + \pi e^{-s}}{s^2+\pi^2}\right] = L^{-1}\left[e^{-\frac{s}{2}}\cdot\frac{s}{s^2+\pi^2} + e^{-s}\frac{\pi}{s^2+\pi^2}\right]$$
$$= L^{-1}\left[\frac{s}{s^2+\pi^2}\right]_{t\to t-\frac{1}{2}} + L^{-1}\left[\frac{\pi}{s^2+\pi^2}\right]_{t\to t-1}$$
$$= [\cos\pi t]_{t\to t-\frac{1}{2}} + [\sin\pi t]_{t\to t-1}$$
$$= \cos\pi\left(t-\frac{1}{2}\right) + \sin\pi(t-1), \qquad t>1.$$

5.6 Inverse Laplace transform by the method of convolution

Convolution. Let $f(t)$ and $g(t)$ be two functions defined for all $t \geq 0$. The convolution of $f(t)$ and $g(t)$ is defined as

$$(f*g)(t) = f(t)*g(t) = \int_0^t f(u)g(t-u)du$$

—

(or)

$$(f * g)(t) = \int_0^t g(u) f(t-u)du.$$

Convolution theorem. If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$ then $L[f(t) * g(t)] = F(s).G(s)$. Also $L^{-1}[F(s)G(s)] = f(t) * g(t) = L^{-1}[F(s)] * L^{-1}[G(s)]$.

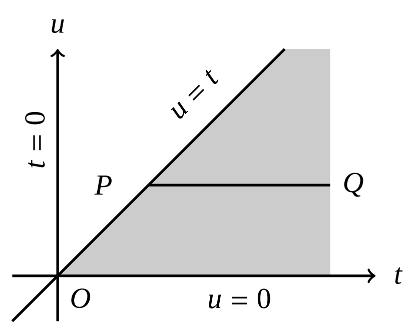
Proof. By definition $(f * g)(t) = \int\limits_0^t f(u)g(t-u)du$

$$L[f(t) * g(t)] = \int\limits_0^\infty e^{-st}(f(t) * g(t))dt = \int\limits_0^\infty e^{-st}\left[\int\limits_0^t f(u)g(t-u)du\right]dt$$

$$= \int\limits_0^\infty \int\limits_0^t e^{-st} f(u)g(t-u)dudt. \tag{1}$$

In the above double integration, the region of integration is given by $u = 0, u = t$, $t = 0$ and $t = \infty$.

We shall evaluate the double integral by changing the order of integration. Divide the region in to strips parallel to the t-axis. PQ is one such strip. Along this strip t varies from u to ∞ and as this strip traverses the entire region, u varies from 0 to ∞.



$$\therefore L[f(t) * g(t)] = \int\limits_{u=0}^\infty \int\limits_{t=u}^\infty e^{-st} f(u)g(t-u)dtdu.$$

Let $v = t - u$. When $t = u, v = 0$.

$dv = dt$ When $t = \infty, v = \infty$.

$$= \int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-s(u+v)} f(u)g(v)dvdu = \int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-su}e^{-sv} f(u)g(v)dvdu$$

$$= \int_{u=0}^{\infty} e^{-su} f(u)du \int_{v=0}^{\infty} e^{-sv}g(v)dv = L[f(u)]L[g(v)]$$

$$= L[f(t)]L[g(t)] = F(s)G(s).$$

$$\therefore L^{-1}[f(t)g(t)] = f(t) * g(t).$$

Worked Examples

Example 5.149. Using convolution theorem,evaluate $L^{-1}[\frac{1}{(s+1)(s+2)}]$.

[May 2007]

Solution. $\frac{1}{(s+1)(s+2)} = \frac{1}{s+1}.\frac{1}{s+2}$

Let $F(s) = \frac{1}{s+1}$ *and* $G(s) = \frac{1}{s+2}$.

$$L^{-1}[\frac{1}{(s+1)(s+2)}] = L^{-1}\Big[F(s)G(s)\Big] = L^{-1}\Big[F(s)\Big] * L^{-1}\Big[G(s)\Big]$$

$$= L^{-1}[\frac{1}{s+1}] * L^{-1}[\frac{1}{s+2}] = e^{-t} * e^{-2t}$$

$$= \int_0^t e^{-u}e^{-2(t-u)}du = \int_0^t e^{-u}e^{-2t+2u}du$$

$$= \int_0^t e^{-2t+u}du = \int_0^t e^{-2t}e^{u}du$$

$$= e^{-2t}.(e^u)_0^t = e^{-2t}(e^t - 1) = e^{-t} - e^{-2t}.$$

Example 5.150. Using Convolution theorem find $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$ [May 2011]

Solution. $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\frac{1}{(s+a)} \cdot \frac{1}{(s+b)}\right]$

$$= L^{-1}\left[\frac{1}{(s+a)}\right] * L^{-1}\left[\frac{1}{(s+b)}\right]$$

$$= e^{-at} * e^{-bt} = \int_0^t e^{-au} \cdot e^{-b(t-u)}du$$

$$= \int_0^t e^{-au} \cdot e^{-bt+bu} du = \int_0^t e^{-au} \cdot e^{-bt} \cdot e^{bu} du$$

$$= e^{-bt} \int_0^t e^{(b-a)u} du = e^{-bt} \left[\frac{e^{(b-a)u}}{b-a} \right]_0^t$$

$$= \frac{e^{-bt}}{b-a} \left(e^{(b-a)t} - 1 \right)$$

$$= \frac{1}{b-a} \left[e^{-bt+bt-at} - e^{-bt} \right]$$

$$= \frac{1}{b-a} \left[e^{-at} - e^{-bt} \right].$$

Example 5.151. Find $L^{-1}\left[\frac{1}{s(s^2-a^2)}\right]$ using convolution theorem. [Jun 2008]

Solution. $\frac{1}{s(s^2-a^2)} = \frac{1}{s} \cdot \frac{1}{s^2-a^2}$

Let $F(s) = \frac{1}{s}$ *and* $G(s) = \frac{1}{s^2-a^2}$

$$L^{-1}\left[\frac{1}{s(s^2-a^2)}\right] = L^{-1}\left[F(s)G(s)\right] = L^{-1}\left[F(s)\right] * L^{-1}\left[G(s)\right]$$

$$= L^{-1}\left[\frac{1}{s}\right] * L^{-1}\left[\frac{1}{s^2-a^2}\right] = 1 * \frac{1}{a}\sinh at$$

$$= \frac{1}{a}\int_0^t \sinh au.1du = \frac{1}{a}\left(\frac{\cosh au}{a}\right)_0^t = \frac{1}{a^2}\left(\cosh at - 1\right).$$

Example 5.152. Find $L^{-1}\left[\frac{1}{s(s^2+4)}\right]$ using Convolution theorem. [Dec 2011]

Solution.

$$L^{-1}\left[\frac{1}{s(s^2+4)}\right] = L^{-1}\left[\frac{1}{s} \cdot \frac{1}{s^2+4}\right]$$

$$= L^{-1}\left[\frac{1}{s}\right] * L^{-1}\left[\frac{1}{s^2+4}\right] = 1 * \frac{\sin 2t}{2}$$

$$= \frac{1}{2}\int_0^t \sin 2u \cdot du$$

$$= \frac{1}{2}\left[-\frac{\cos 2u}{2}\right]_0^t$$

$$= -\frac{1}{4}(\cos 2t - 1) = \frac{1 - \cos 2t}{4}.$$

—

Example 5.153. Using convolution theorem,evaluate $L^{-1}\Big[\dfrac{1}{(s+1)(s^2+1)}\Big]$.

[Nov 2004]

Solution.
$$L^{-1}\Big[\frac{1}{(s+1)(s^2+1)}\Big] = L^{-1}\Big[\frac{1}{s+1}.\frac{1}{s^2+1}\Big] = L^{-1}\Big[\frac{1}{s+1}\Big] * L^{-1}\Big[\frac{1}{s^2+1}\Big]$$
$$= e^{-t} * \sin t = \int_0^t \sin u e^{-(t-u)} du$$
$$= \int_0^t \sin u e^{-t+u} du = \int_0^t \sin u e^{-t} e^u du$$
$$= e^{-t}\int_0^t e^u \sin u du = e^{-t}\Big[\frac{e^u}{2}(\sin u - \cos u)\Big]_0^t$$
$$= \frac{e^{-t}}{2}\Big[e^t(\sin t - \cos t) + 1\Big] = \frac{1}{2}\Big[\sin t - \cos t + e^{-t}\Big].$$

Example 5.154. Find $L^{-1}\Big[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\Big]$ using convolution theorem.

[Dec 2015, Dec 2014, Jun 2014, Dec 2010, May 2003]

Solution.
$$L^{-1}\Big[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\Big] = L^{-1}\Big[\frac{s}{s^2+a^2}.\frac{s}{s^2+b^2}\Big]$$
$$= L^{-1}\Big[\frac{s}{s^2+a^2}\Big] * L^{-1}\Big[\frac{s}{s^2+b^2}\Big] = \cos at * \cos bt$$
$$= \int_0^t \cos au \cos b(t-u) du = \int_0^t \cos au \cos(bt - bu) du$$
$$= \frac{1}{2}\int_0^t \{\cos(au + bt - bu) + \cos(au - bt + bu)\} du$$
$$= \frac{1}{2}\Big[\frac{\sin(au + bt - bu)}{a-b} + \frac{\sin(au - bt + bu)}{a+b}\Big]_0^t$$
$$= \frac{1}{2}\Big[\frac{\sin at}{a-b} - \frac{\sin bt}{a-b} + \frac{\sin at}{a+b} + \frac{\sin bt}{a+b}\Big]$$
$$= \frac{1}{2}\Big[\frac{a+b+a-b}{a^2-b^2}\sin at + \frac{a-b-a-b}{a^2-b^2}\sin bt\Big]$$
$$= \frac{1}{2(a^2-b^2)}\Big[2a\sin at - 2b\sin bt\Big] = \frac{a\sin at - b\sin bt}{a^2-b^2}.$$

Example 5.155. Find $L^{-1}\Big[\dfrac{s}{(s^2+a^2)^2}\Big]$ using convolution theorem.

[Jun 2012, Jun 2010, May 2008]

Solution. Let $\dfrac{s}{(s^2+a^2)^2} = \dfrac{s}{s^2+a^2}.\dfrac{1}{s^2+a^2}$.
$$\therefore F(s) = \frac{s}{s^2+a^2}, G(s) = \frac{1}{s^2+a^2}.$$
$$L^{-1}\Big[\frac{s}{(s^2+a^2)^2}\Big] = L^{-1}\Big[F(s)G(s)\Big] = L^{-1}F(s) * L^{-1}G(s)$$

$$= L^{-1}\Big[\frac{s}{(s^2+a^2)}\Big] * \Big[\frac{1}{(s^2+a^2)}\Big] = \cos at * \frac{\sin at}{a}$$
$$= \frac{1}{a}\int_0^t \cos au \sin a(t-u)du$$
$$= \frac{1}{2a}\int_0^t 2\sin(at-au)\cos au du$$
$$= \frac{1}{2a}\int_0^t \Big\{\sin(at-au+au)+\sin(at-au-au)\Big\}du$$
$$= \frac{1}{2a}\Big\{\int_0^t \sin at du + \int_0^t \sin(at-2au)du\Big\}$$
$$= \frac{1}{2a}\Big[\sin at(u)_0^t - [\frac{\cos(at-2au)}{-2a}]_0^t\Big]$$
$$= \frac{1}{2a}\Big[t\sin at + \frac{1}{2a}(\cos(-at)-\cos at)\Big]$$
$$= \frac{1}{2a}\Big[t\sin at - \frac{1}{2a}0\Big] = \frac{t\sin at}{2a}.$$

Example 5.156. Find $L^{-1}\left[\frac{s^2}{(s^2+4)^2}\right]$ using convolution theorem. [Dec 2012]

Solution. $L^{-1}\left[\frac{s^2}{(s^2+4)^2}\right] = L^{-1}\left[\frac{s}{s^2+4}\cdot\frac{s}{s^2+4}\right]$

$$= L^{-1}\left[\frac{s}{s^2+4}\right] * L^{-1}\left[\frac{s}{s^2+4}\right]$$
$$= \cos 2t * \cos 2t$$
$$= \int_0^t \cos 2u \cdot \cos 2(t-u)du$$
$$= \int_0^t \cos 2u \cos(2t-2u)du$$
$$= \int_0^t \left(\frac{\cos(2u+2t-2u)+\cos(2u-2t+2u)}{2}\right)du$$
$$= \frac{1}{2}\int_0^t (\cos 2t + \cos[4u-2t])du$$

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$$= \frac{1}{2}\left[\cos 2t \int_0^t du + \int_0^t \cos(4u-2t)du\right]$$

$$= \frac{1}{2}\left[\cos 2t[u]_0^t + \left[\frac{\sin(4u-2t)}{4}\right]_0^t\right]$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{1}{4}\{\sin 2t - \sin(-2t)\}\right]$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{1}{4}2\sin 2t\right]$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{\sin 2t}{2}\right] = \frac{1}{4}(\sin 2t + 2t\cos 2t).$$

Example 5.157. Find $L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right]$ using convolution theorem.

[May 2007]

Solution. $L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right] = L^{-1}\left[\frac{1}{(s+1)((s+1)^2+1)}\right] = e^{-t}L^{-1}\left[\frac{1}{s(s^2+1)}\right]$

$$= e^{-t}\left[L^{-1}\left(\frac{1}{s}.\frac{1}{s^2+1}\right)\right] = e^{-t}\left[L^{-1}\left(\frac{1}{s}\right) * L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$

$$= e^{-t}\left[1 * \sin t\right] = e^{-t}\int_0^t \sin u.1du$$

$$= e^{-t}(-\cos u)_0^t = -e^{-t}(\cos t - 1) = e^{-t}(1-\cos t).$$

Example 5.158. Using convolution theorem, find $L^{-1}\left[\frac{4}{(s^2+2s+5)^2}\right]$.

[Jun 2013, May 2006]

Solution. $L^{-1}\left[\frac{4}{(s^2++2s+5)^2}\right] = L^{-1}\left[\frac{4}{(s+1)^2+4)^2}\right] = e^{-t}L^{-1}\left[\frac{4}{(s^2+4)^2}\right]$ (1)

Now, $L^{-1}\left[\frac{4}{(s^2+4)^2}\right] = L^{-1}\left[\frac{2}{s^2+4}\frac{2}{s^2+4}\right] = L^{-1}\left[\frac{2}{s^2+4}\right] * L^{-1}\left[\frac{2}{s^2+4}\right]$

$$= \sin 2t * \sin 2t = \int_0^t \sin 2u \sin 2(t-u)du$$

$$= \frac{1}{2}\int_0^t \Big(\cos\{2u-2(t-u)\} - \cos\{2u+2t-2u\}\Big)du$$

$$= \frac{1}{2}\left[\int_0^t \cos(4u-2t)du - \int_0^t \cos 2tdu\right]$$

$$= \frac{1}{2}\left[\left(\frac{\sin(4u-2t)}{4}\right)_0^t - \cos 2t(u)_0^t\right]$$

$$= \frac{1}{2}\left[\frac{\sin 2t}{4} + \frac{\sin 2t}{4} - t\cos 2t\right] = \frac{1}{2}\left[\frac{\sin 2t}{2} - t\cos 2t\right].$$

—

$$From(1) \quad L^{-1}\Big[\frac{1}{(s^2++2s+5)^2}\Big] = \frac{1}{4}\Big[\sin 2t - 2t\cos 2t\Big]e^{-t}.$$

Exercise 5 L

Find the Laplace inverse of the following using convolution theorem.

1. $\dfrac{1}{(s^2+1)^2}$.
2. $\dfrac{s}{(s^2+4)^2}$. [May 2005]
3. $\dfrac{s^2}{(s^2+a^2)^2}$. [Dec 2006]
4. $\dfrac{s^2}{(s^4-a^4)}$.
5. $\dfrac{1}{s(s^2+1)}$. [May 2008]
6. $\dfrac{1}{(s^2+4)^2}$. [May 2008]
7. $\dfrac{2}{s^3(s^2+5)}$.
8. $\dfrac{1}{(s^2+4)(s+2)}$.
9. $\dfrac{s^2}{(s-3)(s^2+5)}$.

Find the inverse Laplace transform of the following functions using the method of residues.

10. $\dfrac{2}{(s+1)(s^2+1)}$.
11. $\dfrac{2s}{2s^2+1}$.
12. $\dfrac{1}{(s-1)^2(s^2+1)}$.
13. $\dfrac{1}{(s+1)^3(s-2)^2}$.

5.7 Solution of linear second order differential equations

Worked Examples

Example 5.159. Solve $y'' + 5y' + 6y = 2$ given $y'(0) = 0$ and $y(0) = 0$ using Laplace transform method. [Jun 2013]

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Solution. $y'' + 5y' + 6y = 2.$

Taking Laplace transform both sides we get

$$L[y''] + 5L[y'] + 6L[y] = L[2]$$

$$s^2L[y] - sy(0) - y'(0) + 5[sL[y] - y(0)] + 6L[y] = 2 \cdot L[1]$$

$$L[y][s^2 + 5s + 6] = \frac{2}{s}$$

$$L[y] = \frac{2}{s(s^2 + 5s + 6)}$$

$$= \frac{2}{s(s+2)(s+3)}$$

$$y = L^{-1}\left[\frac{2}{s(s+2)(s+3)}\right]$$

$$\text{Let } \frac{2}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$= \frac{A(s+2)(s+3) + Bs(s+3) + Cs(s+2)}{s(s+2)(s+3)}$$

$$\therefore 2 = A(s+2)(s+3) + Bs(s+3) + Cs(s+2).$$

When $s = 0, 6A = 2 \Rightarrow A = \frac{1}{3}$.

When $s = -2, -2B = 2 \Rightarrow B = -1$.

When $s = -3, 3C = 2 \Rightarrow C = \frac{2}{3}$.

$$\therefore \frac{2}{s(s+2)(s+3)} = \frac{1}{3} \cdot \frac{1}{s} - \frac{1}{s+2} + \frac{2}{3} \cdot \frac{1}{s+3}$$

$$\text{Now } y = L^{-1}\left[\frac{2}{s(s+2)(s+3)}\right]$$

$$= L^{-1}\left[\frac{1}{3} \cdot \frac{1}{s} - \frac{1}{s+2} + \frac{2}{3} \cdot \frac{1}{s+3}\right]$$

$$= \frac{1}{3}L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+2}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{1}{3} \times 1 - e^{-2t} + \frac{2}{3}e^{-3t}$$

$$y = \frac{1}{3} - e^{-2t} + \frac{2}{3}e^{-3t}.$$

—

Example 5.160. Solve $\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} + 2x = 2$ given $x = 0$ and $\dfrac{dx}{dt} = 5$ for $t = 0$ using Laplace transform method. [Dec 2012, May 2011]

Solution. $\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} + 2x = 2.$

Taking Laplace transform both sides we get

$$L\left[\frac{d^2x}{dt^2}\right] - 3L\left[\frac{dx}{dt}\right] + 2L[x] = L[2]$$

$$s^2L[x] - s\cancel{x(0)} - x'(0) - 3[sL[x] - \cancel{x(0)}] + 2L[x] = \frac{2}{s}$$

$$s^2L[x] - 5 - 3sL[x] + 2L[x] = \frac{2}{s}$$

$$L[x][s^2 - 3s + 2] = \frac{2}{s} + 5 = \frac{5s+2}{s}$$

$$L[x] = \frac{5s+2}{s(s^2-3s+2)} = \frac{5s+2}{s(s-1)(s-2)}$$

$$x = L^{-1}\left[\frac{5s+2}{s(s-1)(s-2)}\right].$$

$$\text{Let } \frac{5s+2}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$= \frac{A(s-1)(s-2) + Bs(s-2) + Cs(s-1)}{s(s-1)(s-2)}$$

$$5s + 2 = A(s-1)(s-2) + Bs(s-2) + Cs(s-1).$$

When $s = 0, 2A = 2 \Rightarrow A = 1.$

When $s = 1, -B = 7 \Rightarrow B = -7.$

When $s = 2, 2C = 12 \Rightarrow C = 6.$

$$\therefore \frac{5s+2}{s(s-1)(s-2)} = \frac{1}{s} - \frac{7}{s-1} + \frac{6}{s-2}.$$

$$\text{Now, } x = L^{-1}\left[\frac{5s+2}{s(s-1)(s-2)}\right]$$

$$= L^{-1}\left[\frac{1}{s} - \frac{7}{s-1} + \frac{6}{s-2}\right]$$

$$= L^{-1}\left[\frac{1}{s}\right] - 7L^{-1}\left[\frac{1}{s-1}\right] + 6L^{-1}\left[\frac{1}{s-2}\right]$$

$$x = 1 - 7e^t + 6e^{2t}.$$

—

Example 5.161. Solve using Laplace transforms $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 8y = 0$ given $y(0) = 3, y'(0) = 6$. [May 2008]

Solution. The given differential equation is $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 8y = 0$.

Taking Laplace transform on both sides, we get

$$s^2L[y] - sy(0) - y'(0) - 2[sL(y) - y(0)] - 8L[y] = 0$$

$$L[y][s^2 - 2s - 8] - 3s - 6 + 6 = 0$$

$$L[y](s-4)(s+2) = 3s$$

$$L[y] = \frac{3s}{(s-4)(s-2)}$$

$$y = L^{-1}\Big[\frac{3s}{(s-4)(s-2)}\Big].$$

Let $\frac{3s}{(s-4)(s-2)} = \frac{A}{s-4} + \frac{B}{s-2}$

$$3s = A(s+2) + B(s-4).$$

When $s = 4, 6A = 12 \Rightarrow A = 2.$

When $s = -2, -6B = 6 \Rightarrow B = -1.$

$$y = L^{-1}\Big[\frac{2}{s-4} - \frac{1}{s-2}\Big] = 2e^{4t} - e^{2t}.$$

Example 5.162. Solve $(D^2+5D+6)y = e^{-t}$ given that $y(0) = 0, y'(0) = 0$ using Laplace transforms. [Nov 2003]

Solution. Given $(D^2 + 5D + 6)y = e^{-t}$

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = e^{-t}.$$

Taking Laplace transform on both sides we get

$$L\left[\frac{d^2y}{dt^2}\right] + 5L\left[\frac{dy}{dt}\right] + 6L[y] = L[e^{-t}]$$

$$s^2L[y] - sy(0) - y'(0) + 5[sL(y) - y(0)] + 6L[y] = \frac{1}{s+1}$$

$$s^2L[y] + 5sL(y) + 6L[y] = \frac{1}{s+1}$$

$$L[y](s^2 + 5s + 6) = \frac{1}{s+1}$$

$$L[y] = \frac{1}{(s+1)(s^2+5s+6)}$$

—

$$y = L^{-1}\left[\frac{1}{(s+1)(s^2+5s+6)}\right] = L^{-1}\left[\frac{1}{(s+1)(s+2)(s+3)}\right] \quad (1)$$

Let $\dfrac{1}{(s+1)(s+2)(s+3)} = \dfrac{A}{s+1} + \dfrac{B}{s+2} + \dfrac{C}{s+3}$.

$$1 = A(s+2)(s+3)+B(s+1)(s+3) + C(s+1)(s+2).$$

When $s = -1$,

$1 = A(1)(2)$

$A = \dfrac{1}{2}$.

When $s = -2$

$1 = B(-1)(1)$

$B = -1$.

When $s = -3$

$1 = C(-2)(-1)$

$C = \dfrac{1}{2}$.

$$\therefore \frac{1}{(s+1)(s+2)(s+3)} = \frac{\frac{1}{2}}{s+1} - \frac{1}{s+2} + \frac{\frac{1}{2}}{s+3}$$

$$L^{-1}\left[\frac{1}{(s+1)(s+2)(s+3)}\right] = \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+3}\right]$$

$$y = \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}.$$

Example 5.163. Solve using Laplace transforms $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 3y = e^{-t}$ given $y(0) = 1, y'(0) = 0$. [May 2004]

Solution. The given differential equation is $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 3y = e^{-t}$.

Taking Laplace on both sides, we get

$$L[\frac{d^2y}{dt^2}] - 4L[\frac{dy}{dt}] + 3L[y] = L[e^{-t}]$$

$$s^2L[y] - sy(0) - y'(0) - 4[sL(y) - y(0)] + 3L[y] = \frac{1}{s+1}$$

$$L[y][s^2 - 4s + 3] - s + 4 = \frac{1}{s+1}$$

$$L[y][s^2 - 4s + 3] = s - 4 + \frac{1}{s+1}$$

$$L[y][(s-3)(s-1)] = \frac{(s-4)(s+1)+1}{s+1}$$

$$L[y] = \frac{s^2 - 3s - 4 + 1}{(s+1)(s-1)(s-3)} = \frac{s^2 - 3s - 3}{(s-3)(s-1)(s+1)}.$$

$$y = L^{-1}\Big[\frac{s^2-3s-3}{(s-3)(s-1)(s+1)}\Big].$$

$$\text{Let } \frac{s^2-3s-3}{(s-3)(s-1)(s+1)} = \frac{A}{s-3}+\frac{B}{s-1}+\frac{C}{s+1}$$

$$s^2-3s-3 = A(s-1)(s+1)+B(s-3)(s+1)+C(s-3)(s-1).$$

When $s=3, 8A=-3 \Rightarrow A=-\frac{3}{8}$.

When $s=1, -4B=-5 \Rightarrow B=\frac{5}{4}$.

When $s=-1, 8C=1 \Rightarrow C=\frac{1}{8}$.

$$y = L^{-1}\Big[\frac{\frac{-3}{8}}{s-3}+\frac{\frac{5}{4}}{s-1}+\frac{\frac{1}{8}}{s+1}\Big] = -\frac{3}{8}e^{3t}+\frac{5}{4}e^{t}+\frac{1}{8}e^{-t}.$$

Example 5.164. Solve the differential equation $\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=e^{-t}$ with $y(0)=1$, and $y'(0)=0$ using Laplace transforms. [Jun 2012]

Solution. $\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=e^{-t}$.

Taking Laplace transform both sides we get

$$L\left[\frac{d^2y}{dt^2}\right]-3L\left[\frac{dy}{dt}\right]+2L[y]=L[e^{-t}]$$

$$s^2L[y]-sy(0)-\cancel{y'(0)}-3[sL[y]-y(0)]+2L[y]=\frac{1}{s+1}$$

$$s^2L[y]-s-3sL[y]+3+2L[y]=\frac{1}{s+1}$$

$$L[y][s^2-3s+2]=\frac{1}{s+1}+s-3$$

$$L[y](s-1)(s-2)=\frac{1+(s-3)(s+1)}{s+1}$$

$$=\frac{1+s^2-2s-3}{s+1}$$

$$=\frac{s^2-2s-2}{s+1}$$

$$L[y]=\frac{s^2-2s-2}{(s+1)(s-1)(s-2)}.$$

$$y=L^{-1}\left[\frac{s^2-2s-2}{(s+1)(s-1)(s-2)}\right].$$

$$\text{Let } \frac{s^2-2s-2}{(s+1)(s-1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s-2}$$
$$= \frac{A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1)}{(s+1)(s-1)(s-2)}.$$

$$\therefore \quad s^2-2s-2 = A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1).$$

When $s = 1, -2B = -3 \Rightarrow B = \frac{3}{2}$.

When $s = -1, 6A = 1 \Rightarrow A = \frac{1}{6}$.

When $s = 2, 3C = -2 \Rightarrow C = -\frac{2}{3}$.

$$\therefore \frac{s^2-2s-2}{(s+1)(s-1)(s-2)} = \frac{1}{6}\cdot\frac{1}{s+1} + \frac{3}{2}\cdot\frac{1}{s-1} - \frac{2}{3}\cdot\frac{1}{s-2}$$

$$\text{Now}, y = L^{-1}\left[\frac{s^2-2s-2}{(s+1)(s-1)(s-2)}\right]$$
$$= L^{-1}\left[\frac{1}{6}\cdot\frac{1}{s+1} + \frac{3}{2}\cdot\frac{1}{s-1} - \frac{2}{3}\cdot\frac{1}{s-2}\right]$$
$$y = \frac{1}{6}e^{-t} + \frac{3}{2}e^{t} - \frac{2}{3}e^{2t}.$$

Example 5.165. Using Laplace transform, solve the differential equation $y'' - 3y' - 4y = 2e^{-t}$ with $y(0) = 1 = y'(0)$. [Dec 2010, Jun 2010]

Solution. $y'' - 3y' - 4y = 2e^{-t}$.

Taking Laplace transform both sides we get

$$L[y''] - 3L[y'] - 4L[y] = 2L[e^{-t}]$$
$$s^2L[y] - sy(0) - y'(0) - 3[sL[y] - y(0)] - 4L[y] = \frac{2}{s+1}$$
$$s^2L[y] - s - 1 - 3[sL[y] - 1] - 4L[y] = \frac{2}{s+1}$$
$$L[y][s^2 - 3s - 4] - s - 1 + 3 = \frac{2}{s+1}$$
$$L[y](s-4)(s+1) - s + 2 = \frac{2}{s+1}$$
$$L[y](s-4)(s+1) = \frac{2}{s+1} + s - 2$$
$$= \frac{2 + (s+1)(s-2)}{s+1} = \frac{2 + s^2 - s - 2}{s+1} = \frac{s^2 - s}{s+1}$$

$$L[y] = \frac{s^2 - s}{(s+1)(s-4)(s+1)}$$
$$= \frac{s^2 - s}{(s-4)(s+1)^2}.$$
$$y = L^{-1}\left[\frac{s^2 - s}{(s-4)(s+1)^2}\right].$$
$$\text{Let } \frac{s^2 - s}{(s-4)(s+1)^2} = \frac{A}{s-4} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$
$$= \frac{A(s+1)^2 + B(s-4)(s+1) + C(s-4)}{(s-4)(s+1)^2}$$
$$s^2 - s = A(s+1)^2 + B(s-4)(s+1) + C(s-4).$$

When $s = 4,\ 25A = 12 \Rightarrow A = \frac{12}{25}$.
When $s = -1, -5C = 2 \Rightarrow C = -\frac{2}{5}$.
Equating the coefficient of s^2 we get
$A + B = 1 \ \Rightarrow \frac{12}{25} + B = 1 \Rightarrow B = 1 - \frac{12}{25} = \frac{13}{25}$.

$$\therefore \frac{s^2 - s}{(s-4)(s+1)^2} = \frac{12}{25}\cdot\frac{1}{s-4} + \frac{13}{25}\cdot\frac{1}{s+1} - \frac{2}{5}\cdot\frac{1}{(s+1)^2}$$
$$\text{Now}, y = L^{-1}\left[\frac{s^2 - s}{(s-4)(s+1)^2}\right]$$
$$= L^{-1}\left[\frac{12}{25}\cdot\frac{1}{s-4} + \frac{13}{25}\cdot\frac{1}{s+1} - \frac{2}{5}\cdot\frac{1}{(s+1)^2}\right]$$
$$= \frac{12}{25}e^{4t} + \frac{13}{25}e^{-t} - \frac{2}{5}e^{-t}L^{-1}\left[\frac{1}{s^2}\right]$$
$$y = \frac{12}{25}e^{4t} + \frac{13}{25}e^{-t} - \frac{2}{5}te^{-t}.$$

Example 5.166. Solve using Laplace transforms, $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t^2 + 2t$, given that $y = 4, y' = -2$ when $t = 0$. [Dec 2013, May 2007]

Solution. The given differential equation is $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t^2 + 2t$.

Taking Laplace transform on both sides, we get

$$s^2L[y] - sy(0) - y'(0) + sL(y) - y(0) - y(0) = \frac{2}{s^3} + \frac{2}{s^2}$$

—

$$(s^2+s)L[y]-4s+2-4=\frac{2(s+1)}{s^3}$$

$$s(s+1)L(y)=4s+2+\frac{2(s+1)}{s^3}=2s+2+2s+\frac{2(s+1)}{s^3}$$

$$=2(s+1)+2s+\frac{2(s+1)}{s^3}$$

$$L(y)=\frac{2}{s}+\frac{2}{s+1}+\frac{2}{s^4}$$

$$y=L^{-1}\Big[\frac{2}{s}+\frac{2}{s+1}+\frac{2}{s^4}\Big]$$

$$=2+2e^{-t}+\frac{2}{3!}t^3$$

$$y=2+2e^{-t}+\frac{1}{3}t^3.$$

Example 5.167. Solve $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=\sin t$, if $\frac{dy}{dt}=0$ and $y=2$ when $t=0$, using Laplace transforms. [Dec 2011]

Solution. $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=\sin t.$

Taking Laplace transform both sides we get

$$L\left[\frac{d^2y}{dt^2}\right]+4L\left[\frac{dy}{dt}\right]+4L[y]=L[\sin t]$$

$$s^2L[y]-sy(0)-\cancel{y'(0)}+4[sL[y]-y(0)]+4L[y]=\frac{1}{s^2+1}$$

$$s^2L[y]-2s+4[sL[y]-2]+4L[y]=\frac{1}{s^2+1}$$

$$L[y][s^2+4s+4]-2s-8=\frac{1}{s^2+1}$$

$$L[y](s+2)^2=\frac{1}{s^2+1}+2s+8$$

$$=\frac{1+(2s+8)(s^2+1)}{s^2+1}$$

$$=\frac{1+2s^3+2s+8s^2+8}{s^2+1}$$

$$=\frac{2s^3+8s^2+2s+9}{s^2+1}$$

$$L[y]=\frac{2s^3+8s^2+2s+9}{(s+2)^2(s^2+1)}$$

$$y=L^{-1}\left[\frac{2s^3+8s^2+2s+9}{(s+2)^2(s^2+1)}\right].$$

—

$$\text{Let } \frac{2s^3+8s^2+2s+9}{(s+2)^2(s^2+1)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{Cs+D}{s^2+1}$$
$$= \frac{A(s+2)(s^2+1)+B(s^2+1)+(Cs+D)(s+2)^2}{(s+2)^2(s^2+1)}$$
$$\therefore 2s^3+8s^2+2s+9 = A(s+2)(s^2+1)+B(s^2+1)+(Cs+D)(s+2)^2.$$

When $s=-2$, $5B=21 \Rightarrow B=\frac{21}{5}$.

Equating the coefficients of s^3, we get

$$A+C=2. \tag{1}$$

Equating the constants we get

$$2A+B+4D=9.$$
$$2A+\frac{21}{5}+4D=9$$
$$2A+4D=9-\frac{21}{5}=\frac{24}{5} \tag{2}$$

Equating the coefficients of s^2 we get

$$2A+B+4C+D=8.$$
$$2A+\frac{21}{5}+4C+D=8$$
$$2A+4C+D=8-\frac{21}{5}$$
$$2A+4C+D=\frac{19}{5}. \tag{3}$$

From (1), $C=2-A$.

$$(3) \Rightarrow 2A+4(2-A)+D=\frac{19}{5}$$
$$2A+8-4A+D=\frac{19}{5}$$
$$-2A+D=\frac{19}{5}-8=-\frac{21}{5}$$
$$2A-D=\frac{21}{5}. \tag{4}$$
$$(2)-(4) \Rightarrow 5D=\frac{24}{5}-\frac{21}{5}=\frac{3}{5}.$$
$$D=\frac{3}{25}.$$

$$(4) \Rightarrow 2A - \frac{3}{5} = \frac{21}{5}$$
$$2A = \frac{21}{5} + \frac{3}{5} = \frac{24}{5}.$$
$$A = \frac{12}{5}.$$
$$C = 2 - A = 2 - \frac{12}{5} = -\frac{2}{5}.$$

$$\therefore \frac{2s^3 + 8s^2 + 2s + 9}{(s+2)^2(s^2+1)} = \frac{12}{5}\frac{1}{s+2} + \frac{21}{5} \cdot \frac{1}{(s+2)^2} - \frac{2}{5}\frac{s}{s^2+1} + \frac{3}{5}\frac{1}{s^2+1}$$

$$\text{Now}, y = L^{-1}\left[\frac{2s^3 + 8s^2 + 2s + 9}{(s+2)^2(s^2+1)}\right]$$
$$= L^{-1}\left[\frac{12}{5}\frac{1}{s+2} + \frac{21}{5} \cdot \frac{1}{(s+2)^2} - \frac{2}{5}\frac{s}{s^2+1} + \frac{3}{5}\frac{1}{s^2+1}\right]$$
$$= \frac{12}{5}L^{-1}\left[\frac{1}{s+2}\right] + \frac{21}{5}L^{-1}\left[\frac{1}{(s+2)^2}\right] - \frac{2}{5}L^{-1}\left[\frac{s}{s^2+1}\right] + \frac{3}{5}L^{-1}\left[\frac{1}{s^2+1}\right]$$
$$= \frac{12}{5}e^{-2t} + \frac{21}{5}e^{-2t}L^{-1}\left[\frac{1}{s^2}\right] - \frac{2}{5}\cos t + \frac{3}{5}\sin t$$
$$= \frac{12}{5}e^{-2t} + \frac{21}{5}e^{-2t}t - \frac{2}{5}\cos t + \frac{3}{5}\sin t.$$

Example 5.168. Solve $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8y = \cos 2t, y = 2$ and $\frac{dy}{dt} = 1$ where $t = 0$.

[May 2002]

Solution. The given differential equation is $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8y = \cos 2t$.

Taking Laplace transform on both sides, we get

$$L\left[\frac{d^2y}{dt^2}\right] + 4L\left[\frac{dy}{dt}\right] + 8L[y] = L[\cos 2t]$$
$$s^2L[y] - sy(0) - y'(0) + 4[sL(y) - y(0)] + 8L[y] = \frac{s}{s^2+4}$$
$$s^2L[y] - 2s - 1 + 4[sL(y) - 2] + 8L[y] = \frac{s}{s^2+4}$$
$$L[y][s^2 + 4s + 8] - 2s - 1 - 8 = \frac{s}{s^2+4}$$
$$L[y][s^2 + 4s + 8] = 2s + 9 + \frac{s}{s^2+4}$$
$$= \frac{(2s+9)(s^2+4) + s}{s^2+4}$$

$$= \frac{2s^3 + 8s + 9s^2 + 36 + s}{s^2 + 4} = \frac{2s^3 + 9s^2 + 9s + 36}{s^2 + 4}$$

$$L[y] = \frac{2s^3 + 9s^2 + 9s + 36}{(s^2 + 4)(s^2 + 4s + 8)}$$

$$y = L^{-1}\Big[\frac{2s^3 + 9s^2 + 9s + 36}{(s^2 + 4)(s^2 + 4s + 8)}\Big].$$

Let $\frac{2s^3 + 9s^2 + 9s + 36}{(s^2 + 4)(s^2 + 4s + 8)} = \frac{As + B}{s^2 + 4} + \frac{Cs + D}{s^2 + 4s + 8}$

$2s^3 + 9s^2 + 9s + 36 = (As + B)(s^2 + 4s + 8) + (Cs + D)(s^2 + 4)$

Equating the coefficients of s^3 we get

$$A + C = 2. \tag{1}$$

Equating the coefficients of s^2 we get

$$4A + B + D = 9. \tag{2}$$

Equating the coefficients of s we get

$$8A + 4B + 4C = 9. \tag{3}$$

Equating the constants we get, $8B + 4D = 36$

$$2B + D = 9. \tag{4}$$

$$(3) \Longrightarrow 8A + 4B + 4(2 - A) = 9 \qquad \text{[using (1)]}$$

$$8A + 4B + 8 - 4A = 9$$

$$4A + 4B = 1. \tag{5}$$

$$(2) \Longrightarrow 4A + B + 9 - 2B = 9 \qquad \text{[from (4)]}$$

$$4A - B = 0$$

$$B = 4A \tag{6}$$

$$(5) \Longrightarrow 4A + 4(4A) = 1 \Longrightarrow 20A = 1 \Longrightarrow A = \frac{1}{20}.$$

—

$$(6) \Longrightarrow B = 4\frac{1}{20} = \frac{1}{5}.$$

$$(1) \Longrightarrow C = 2 - A = 2 - \frac{1}{20} = \frac{39}{20}.$$

$$(4) \Longrightarrow D = 9 - 2B = 9 - \frac{2}{5} = \frac{43}{5}.$$

$$\therefore \frac{2s^3 + 9s^2 + 9s + 36}{(s^2+4)(s^2+4s+8)} = \frac{\frac{1}{20}s + \frac{1}{5}}{s^2+4} + \frac{\frac{39}{20}s + \frac{43}{5}}{s^2+4s+8}$$

$$\text{Now, } y = L^{-1}\left[\frac{2s^3 + 9s^2 + 9s + 36}{(s^2+4)(s^2+4s+8)}\right] = L^{-1}\left[\frac{\frac{1}{20}s + \frac{1}{5}}{s^2+4}\right] + L^{-1}\left[\frac{\frac{39}{20}s + \frac{43}{5}}{s^2+4s+8}\right]$$

$$\begin{aligned}
&= \frac{1}{20}L^{-1}\left[\frac{s}{s^2+4}\right] + \frac{1}{5}L^{-1}\left[\frac{1}{s^2+4}\right] + \frac{39}{20}L^{-1}\left[\frac{s}{s^2+4s+8}\right] + \frac{43}{5}L^{-1}\left[\frac{1}{s^2+4s+8}\right]\\
&= \frac{1}{20}\cos 2t + \frac{1}{10}\sin 2t + \frac{39}{20}L^{-1}\left[\frac{s}{(s+2)^2+4}\right] + \frac{43}{5}L^{-1}\left[\frac{1}{(s+2)^2+4}\right]\\
&= \frac{1}{20}\cos 2t + \frac{1}{10}\sin 2t + \frac{39}{20}e^{-2t}L^{-1}\left[\frac{s-2}{s^2+4}\right] + \frac{43}{5}e^{-2t}L^{-1}\left[\frac{1}{s^2+4}\right]\\
&= \frac{1}{20}\cos 2t + \frac{1}{10}\sin 2t + \frac{39}{20}e^{-2t}\cos 2t - \frac{39}{20}e^{-2t}\sin 2t + \frac{43}{5}e^{-2t}\sin 2t\\
&= \frac{1}{20}\cos 2t + \frac{1}{10}\sin 2t + \frac{39}{20}e^{-2t}\cos 2t + \frac{47}{20}e^{-2t}\sin 2t.
\end{aligned}$$

Example 5.169. Solve $y''(t) + 9y(t) = 18t$ given that $y(0) = 0, y\left(\frac{\pi}{2}\right) = 0$. [May 2005]

Solution. The given initial conditions are $y(0) = 0, y\left(\frac{\pi}{2}\right) = 0$.

Since $y'(0)$ is not given, let us assume that $y'(0) = k$.

The given Differential equation is $y'' + 9y = 18t$.

—

Taking Laplace transform both sides we get

$$L[y''] + 9L[y] = 18L[t]$$

$$s^2L[y] - sy(0) - y'(0) + 9L[y] = 18\frac{1}{s^2}$$

$$s^2L[y] - k + 9L[y] = \frac{18}{s^2}$$

$$L[y](s^2+9) = k + \frac{18}{s^2} = \frac{ks^2+18}{s^2}$$

$$L[y] = \frac{ks^2+18}{s^2(s^2+9)}$$

$$y = L^{-1}\left[\frac{ks^2+18}{s^2(s^2+9)}\right] \tag{1}$$

Let $\frac{ks^2+18}{s^2(s^2+9)} = \frac{A}{s^2} + \frac{B}{s^2+9}$.
$ks^2 + 18 = A(s^2+9) + Bs^2$.

When $s = 0$,

$9A = 18$

$A = 2.$

Equating the Coeff. of s^2 we get

$A + B = k$

$B = k - A = k - 2.$

$$\therefore \frac{ks^2+18}{s^2(s^2+9)} = \frac{2}{s^2} + \frac{k-2}{s^2+9}.$$

Now, $y = L^{-1}\left[\frac{ks^2+18}{s^2(s^2+9)}\right] = 2L^{-1}\left[\frac{1}{s^2}\right] + (k-2)L^{-1}\left[\frac{1}{s^2+9}\right]$.

$$y = 2t + \frac{k-2}{3}\sin 3t.$$

Now $y\left(\frac{\pi}{2}\right) = 0$

$$\therefore 0 = 2\frac{\pi}{2} + \frac{k-2}{3}\sin\frac{3\pi}{2} = \pi + \frac{k-2}{3}(-1).$$

$$\frac{k-2}{3} = \pi$$

$$k - 2 = 3\pi$$

$$k = 3\pi + 2.$$

$\therefore$ The solution is $y = 2t + \frac{3\pi+2-2}{3}\sin 3t.$

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$y = 2t + \pi \sin 3t.$

Exercise 5 M

Solve the following differential equations using Laplace transforms

1. $(D^2 + 4D + 4)y = e^{-t}$ given that $y(0) = 0, y'(0) = 0.$ [May 2006]
2. $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 3\cos 3t - 11\sin 3t$ with $y(0) = 0, y'(0) = 6.$ [May 2008]
3. $(D^2 + 9)y = \cos 2t$ given that $y(0) = 1, y(\frac{\pi}{2}) = -1.$
4. $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$, given that $x = 2, \frac{dx}{dt} = -1$ when $t = 0.$ [Dec 2008]
5. $y'' + 5y' + 6y = 2$ given that $y(0) = 1, y'(0) = 0.$ [May 2006]
6. $y'' - 3y' + 2y = 4$ given that $y(0) = 2, y'(0) = 3.$ [May 2005]
7. $y'' + 2y' - 3y = 3$, given $y(0) = 4, y'(0) = -7.$ [May 2007]
8. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = e^{-t}\sin t$, given that $x(0) = 0, x'(0) = 1.$ [May 1998]
9. $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 10\sin t$, given $y = 0 = \frac{dy}{dt}$ when $t = 0.$ [May 2004]
10. $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 2e^{-3t}, y(0) = 1, y'(0) = -2.$ [May 2002]
11. $2\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 2y = 3\sin t$, given that $y(0) = y'(0) = 0.$ [Apr 1996]
12. $y'' - 3y' + 2y = e^{-t}$, given $y(0) = 1, y'(0) = 0.$ [May 2000]
13. $y'' - 2y' + y = e^t$, given $y(0) = 2, y'(0) = 1.$ [May 2005]
14. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = te^{-t}$, given that $x(0) = 0, \frac{dx}{dt} = 2$ for $t = 0.$ [Apr 2005]
15. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t$, given, $y = 0 = \frac{dy}{dt}$ when $t = 0.$ [May 2005]

—

Also $|f(z)| = |\frac{1}{z^2+a^2}| \to 0$ as $|z| \to \infty$.

$\therefore$ By Jordon's lemma, $\int_s e^{imz} f(z)dz \to 0$ as $R \to \infty$.

Now (1)$\Rightarrow \frac{\pi}{a}e^{-am} = \int\limits_{-R}^{R} e^{imx} f(x)dx + \int_s e^{imz} f(z)dz.$

Taking limit as $R \to \infty$ we get

$$\frac{\pi}{a}e^{-am} = \lim_{R\to\infty}\int\limits_{-R}^{R} e^{imx} f(x)dx + \lim_{R\to\infty}\int_s e^{imz} f(z)dz$$

$$\frac{\pi}{a}e^{-am} = \int\limits_{-\infty}^{\infty} \frac{e^{imx}}{x^2+a^2}dx + 0$$

$$= \int\limits_{-\infty}^{\infty} \frac{\cos mx + i\sin mx}{x^2+a^2}dx.$$

Equating the real parts we get

$$\int\limits_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2}dx = \frac{\pi e^{-am}}{a}$$

$$2\int\limits_{0}^{\infty} \frac{\cos mx}{x^2+a^2}dx = \frac{\pi e^{-am}}{a}$$

$$\int\limits_{0}^{\infty} \frac{e^{mx}}{x^2+a^2}dx = \frac{\pi e^{-am}}{2a}.$$

2.5 Unit V Laplace Transforms.

2.5.1 May/June 2016 (R 2013)

Part A

5. State convolution theorem on laplace transforms.

   Solution. Refer page 289 of the main text book.

6. Find $L^{-1}\left(\dfrac{s}{s^2+4s+5}\right)$.

Solution. $L^{-1}\left(\dfrac{s}{s^2+4s+5}\right) = L^{-1}\left(\dfrac{s}{s^2+4s+4+1}\right)$

$$= L^{-1}\left(\frac{s+2-2}{(s+2)^2+1}\right)$$

$$= L^{-1}\left(\frac{s+2}{(s+2)^2+1}\right) - 2L^{-1}\left(\frac{1}{(s+2)^2+1}\right)$$

$$= e^{-2t}L^{-1}\left(\frac{s}{s^2+1}\right) - 2e^{-2t}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= e^{-2t}\cos t - 2e^{-2t}\sin t$$

Part B

13. (a) (i) Evaluate:

(1) $L\left(t^2e^{-t}\cos t\right)$

(2) $L^{-1}\left(e^{-2s}\dfrac{1}{(s^2+s+1)^2}\right)$.

Solution.

$$(1) L\left(t^2e^{-t}\cos t\right) = \frac{d^2}{ds^2}\left\{L\left[e^{-t}\cos t\right]\right\}$$

$$= \frac{d^2}{ds^2}\left\{L\left[\cos t\right]_{s\to s+1}\right\}$$

$$= \frac{d^2}{ds^2}\left[\frac{s}{s^2+1}\right]_{s\to s+1}$$

$$= \left[\frac{d^2}{ds^2}\frac{s}{s^2+1}\right]_{s\to s+1}$$

$$= \left[\frac{d}{ds}\frac{s^2+1-s(2s)}{(s^2+1)^2}\right]_{s\to s+1}$$

$$= \left[\frac{d}{ds}\frac{s^2+1-2s^2}{(s^2+1)^2}\right]_{s\to s+1}$$

$$= \left[\frac{d}{ds}\frac{1-s^2}{(s^2+1)^2}\right]_{s\to s+1}$$

$$= \left[\frac{(s^2+1)^2(-2s)-(1-s^2)(2s)}{(s^2+1)^4}\right]_{s\to s+1}$$

$$= \left[\frac{(s^2+1)^2(-2s)-(1-s^2)2(s^2+1)(2s)}{(s^2+1)^4}\right]_{s\to}{}^{s+1}$$

$$= \left[\frac{(s^2+1)(-2s)-(1-s^2)2(2s)}{(s^2+1)^3}\right]_{s\to s+1}$$

$$= \left[\frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}\right]_{s\to s+1}$$

$$= \left[\frac{(2s^3-6s}{(s^2+1)^3}\right]_{s\to s+1}$$

$$= \left[\frac{2s(s^2-3)}{(s^2+1)^3}\right]_{s\to s+1}$$

$$= \left[\frac{2(s+1)(s^2+2s-2)}{(s^2+2s+2)^3}\right].$$

$$\text{(2)Let} F(s) = \frac{1}{(s^2+s+1)^2}$$

$$= \frac{1}{\left(\left(s^2+\frac{1}{2}\right)^2+1-\frac{1}{4}\right)^2} = \frac{1}{\left(\left(s^2+\frac{1}{2}\right)^2+\frac{3}{4}\right)^2}$$

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{\left(\left(s^2+\frac{1}{2}\right)^2+\frac{3}{4}\right)^2}\right] = e^{-\frac{t}{2}}L^{-1}\left[\frac{1}{\left(s^2+\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$= e^{-\frac{t}{2}}\frac{1}{2\left(\frac{\sqrt{3}}{2}\right)^3}\left[\sin\left(\frac{\sqrt{3}}{2}t\right)-\left(\frac{\sqrt{3}}{2}t\right)\cos\left(\frac{\sqrt{3}}{2}t\right)\right]$$

$$L^{-1}[F(s)] = e^{-\frac{t}{2}}\frac{4}{\sqrt{3}}\left[\sin\left(\frac{\sqrt{3}}{2}t\right)-\left(\frac{\sqrt{3}}{2}t\right)\cos\left(\frac{\sqrt{3}}{2}t\right)\right].$$

$$\text{Now } L^{-1}\left[e^{-2s}\frac{1}{(s^2+s+1)^2}\right] = L^{-1}\left[e^{-2s}F(s)\right]$$

$$= L^{-1}\left[F(s)\right]_{t\to t-2} \text{ (by second shifting theorem.)}$$

$$= \left\{e^{-\frac{t}{2}}\frac{4}{\sqrt{3}}\left[\sin\left(\frac{\sqrt{3}}{2}t\right)-\left(\frac{\sqrt{3}}{2}t\right)\cos\left(\frac{\sqrt{3}}{2}t\right)\right]\right\}_{t\to t-2}$$

$$= e^{-\frac{(t-2)}{2}}\frac{4}{\sqrt{3}}\left[\sin\left(\frac{\sqrt{3}(t-2)}{2}\right)-\left(\frac{\sqrt{3}(t-2)}{2}\right)\cos\left(\frac{\sqrt{3}(t-2)}{2}\right)\right].$$

13. (a) (ii) Find the inverse Laplace transform of $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$ using convolution theorem.

Solution. Refer Example 3.154 on page 292 of the main text book.

13. (b) (i) Find the Laplace transform of $f(t) = \begin{cases} E & \text{if } 0 < t < a/2 \\ -E & \text{if } a/2 < t < a \end{cases}$

where $f(t+a) = f(t)$

Solution. Refer Example 3.48 on page 235 of the main text book.

13. (b) (ii) Using Laplace transform technique solve $y'' + y' = t^2 + 2t$, given $y = 4$, $y' = -2$ when $t = 0$.

Solution. Refer Example 3.166 on page 302 of the main text book.

2.5.2 Dec.2015/Jan. 2016 (R 2013)

Part A

5. Prove that $L\left(\int_0^t f(t)\,dt\right) = \frac{F(s)}{s}$, where $L(f(t)) = F(s)$.

Solution. Refer Example 3.3.1 on page 241 of the main text book.

6. Find $L^{-1}\left(\log \frac{s}{s-a}\right)$.

Solution. Let $F(s) = \log\left(\frac{s}{s-a}\right) = \log s - \log(s-a)$.

$$F'(s) = \frac{1}{s} - \frac{1}{s-a}.$$

We know that $L[tf(t)] = -F'(s) = -\left[\frac{1}{s} - \frac{1}{s-a}\right] = \frac{1}{s-a} - \frac{1}{s}$.

$$\therefore tf(t) = L^{-1}\left[\frac{1}{s-a} - \frac{1}{s}\right] = L^{-1}\left[\frac{1}{s-a}\right] - L^{-1}\left[\frac{1}{s}\right] = e^{at} - 1$$

$$\therefore f(t) = \frac{e^{at}-1}{t}.$$

Part B

13. (a) (i) Find $L\left(e^{-t}\sin^2 3t\right)$ and $L\left(\dfrac{e^{-t}-\cos t}{t}\right)$.

Solution. we have $\cos 2\theta = 1 - 2\sin^2\theta$.

$$\Rightarrow 2\sin^2\theta = 1 - \cos 2\theta$$
$$\Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\therefore \sin^2 3t = \frac{1-\cos 6t}{2}.$$

$$\begin{aligned} L\left(e^{-t}\sin^2 3t\right) &= L\left[e^{-t}\frac{1-\cos 6t}{2}\right] \\ &= \frac{1}{2}L\left[e^{-t} - e^{-t}\cos 6t\right] \\ &= \frac{1}{2}L\left[e^{-t}\right] - \frac{1}{2}L\left[e^{-t}\cos 6t\right] \\ &= \frac{1}{2}\left[\frac{1}{s+1}\right] - \frac{1}{2}L\left[\cos 6t\right]_{s\to s+1} \\ &= \frac{1}{2}\left[\frac{1}{s+1}\right] - \frac{1}{2}\left[\frac{s}{s^2+36}\right]_{s\to s+1} \\ &= \frac{1}{2(s+1)} - \frac{1}{2}\left[\frac{s+1}{(s+1)^2+36}\right]. \end{aligned}$$

13. (a) (ii) Solve $x'' + 2x' + 5x = e^{-t}\sin t$; $x(0) = 0$ and $x'(0) = 1$ using Laplace transform.

Solution. Given, $x'' + 2x' + 5x = e^{-t}\sin t$, $x(0) = 0$ and $x'(0) = 1$.

Taking Laplace transform on both sides we get

$$L[x''] + 2L[x'] + 5L[x] = L[e^{-t}\sin t]$$
$$s^2L[x] - sx(0) - x'(0) + 2\left[sL[x] - x(0)\right] + 5L[x] = L[\sin t]_{s\to s+1}$$
$$s^2L[x] - s\times 0 - 1 + 2sL[x] - 2\times 0 + 5L[x] = \left[\frac{1}{s^2+1}\right]_{s\to s+1}$$
$$s^2L[x] - 1 + 2sL[x] + 5L[x] = \frac{1}{(s+1)^2+1}$$

$$L[x](s^2+2s+5) = \frac{1}{s^2+2s+1+1} + 1$$

$$= \frac{1}{s^2+2s+2} + 1$$
$$= \frac{1+s^2+2s+2}{s^2+2s+2} = \frac{s^2+2s+3}{s^2+2s+2}.$$
$$L[x] = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$
$$L[x] = \frac{(s+1)^2+2}{((s+1)^2+1)((s+1)^2+4)}.$$
$$x = L^{-1}\left[\frac{(s+1)^2+2}{((s+1)^2+1)((s+1)^2+4)}\right]$$
$$x = e^{-t}L^{-1}\left[\frac{s^2+2}{(s^2+1)(s^2+4)}\right]. \tag{1}$$

Let $\frac{s^2+2}{(s^2+1)(s^2+4)} = \frac{A}{(s^2+1)} + \frac{B}{(s^2+4)}$.

$$\therefore s^2+2 = A(s^2+4) + B(s^2+1)$$

Put $s^2 = -1 \quad \Rightarrow 1 = 3A \quad \Rightarrow A = \frac{1}{3}$.

Put $s^2 = -4 \quad \Rightarrow -2 = -3B \quad \Rightarrow B = \frac{2}{3}$.

$$\therefore \frac{s^2+2}{(s^2+1)(s^2+4)} = \frac{\frac{1}{3}}{(s^2+1)} + \frac{\frac{2}{3}}{(s^2+4)}.$$
$$L^{-1}\left[\frac{s^2+2}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}L^{-1}\left[\frac{1}{(s^2+1)}\right] + \frac{1}{3}L^{-1}\left[\frac{2}{(s^2+4)}\right].$$
$$= \frac{1}{3}\sin t + \frac{1}{3}\sin 2t.$$
$$(1) \Rightarrow x = e^{-t}\left[\frac{1}{3}\sin t + \frac{1}{3}\sin 2t\right]$$
$$= \frac{e^{-t}}{3}\left[\sin t + \sin 2t\right].$$

13. (b) (i) State second shifting theorem and also find $L^{-1}\left(\frac{e^{-s}}{\sqrt{s+1}}\right)$.

Solution. $L^{-1}\left(\frac{e^{-s}}{\sqrt{s+1}}\right) = L^{-1}\left(\frac{e^{-s-1+1}}{\sqrt{s+1}}\right)$

$$= L^{-1}\left(\frac{e^{-(s+1)+1}}{\sqrt{s+1}}\right)$$

$$= L^{-1}\left(\frac{e^{-(s+1)}e^{1}}{\sqrt{s+1}}\right)$$
$$= eL^{-1}\left(\frac{e^{-(s+1)}}{\sqrt{s+1}}\right)$$
$$= ee^{-t}L^{-1}\left(\frac{e^{-s}}{\sqrt{s}}\right)$$
$$= e^{1-t}L^{-1}\left[\frac{1}{\sqrt{s}}\right]_{t\to t-1} \qquad \left[\text{since, } L\left[\frac{1}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{\sqrt{s}}\right]$$
$$= e^{1-t}\left[\frac{1}{\sqrt{\pi t}}\right]_{t\to t-1}$$
$$= e^{1-t}\left[\frac{1}{\sqrt{\pi(t-1)}}\right]$$

13. (b) (ii) Find $L^{-1}\left(\frac{3s+1}{(s+1)^4}\right)$.

Solution. $L^{-1}\left(\frac{3s+1}{(s+1)^4}\right) = L^{-1}\left(\frac{3(s+1-1)+1}{(s+1)^4}\right)$

$$= L^{-1}\left(\frac{3(s+1)-3+1}{(s+1)^4}\right)$$
$$= L^{-1}\left(\frac{3(s+1)-2}{(s+1)^4}\right)$$
$$= L^{-1}\left(\frac{3(s+1)}{(s+1)^4} - \frac{2}{(s+1)^4}\right)$$
$$= 3L^{-1}\left(\frac{1}{(s+1)^3}\right) - 2L^{-1}\left(\frac{1}{(s+1)^4}\right)$$
$$= 3e^{-t}L^{-1}\left(\frac{1}{s^3}\right) - 2e^{-t}L^{-1}\left(\frac{1}{s^4}\right)$$
$$= \frac{3e^{-t}}{2!}L^{-1}\left(\frac{2}{s^3}\right) - \frac{2e^{-t}}{3!}L^{-1}\left(\frac{3!}{s^4}\right)$$
$$= \frac{3e^{-t}}{2}t^2 - \frac{2e^{-t}}{6}t^3 = \frac{3e^{-t}}{2}t^2 - \frac{e^{-t}}{3}t^3$$
$$= e^{-t}t^2\left(\frac{3}{2} - \frac{t}{3}\right) = \frac{e^{-t}t^2}{6}(9-2t).$$

13. (b) (iii) Find the Laplace transform for $f(t) = \sin\frac{\pi t}{a}$, such that $f(t+a) = f(t)$.

Solution. Refer Example 3.52 on page 237 of the main text book. To get the correct solution replace ω by $\frac{\pi}{a}$.

$$\therefore L\left[\sin\left(\frac{\pi t}{a}\right)\right] = \frac{\frac{\pi}{a}}{\left(s^2 + \frac{\pi^2}{a^2}\right)(1 - e^{-as\pi})} = \frac{\pi a}{(a^2 s^2 + \pi^2)(1 - e^{-as\pi})}.$$

2.5.3 April/May 2015 (R 2013)

Part A

5. State the sufficiency condition for the existence of Laplace transform.

Solution. Let $f(t)$ be defined for all $t \geq 0$ such that (i) $f(t)$ is piecewise continuous in the interval $[0, \infty)$ and (ii) $f(t)$ is of exponential order $\alpha > 0$, then the Laplace transform of $f(t)$ exists for $s > \alpha$.

6. Evaluate using Laplace transform.

Solution. By definition
$$\int_0^\infty te^{-2t} \sin t \, dt = [L[t \sin t]]_{s=2}$$
$$= \left[-\frac{d}{ds}\{L[\sin t]\}\right]_{s=2}$$
$$= \left[-\frac{d}{ds}\left[\frac{1}{s^2+1}\right]\right]_{s=2}$$
$$= \left[-\frac{d}{ds}\left[\cdot(s^2+1)^{-1}\right]\right]_{s=2}$$
$$= -\left[(-1)\cdot(s^2+1)^{-2}\cdot 2s\right]_{s=2}$$
$$= \left[\frac{2s}{(s^2+1)^2}\right]_{s=2} = \frac{2\times 2}{(4+1)^2} = \frac{4}{25}.$$

Part B

13. (a) (i) Find the Laplace transform of the triangular wave function $f(t)$ defined by $f(t) = \begin{cases} t & \text{, in } 0 < t \leq c \\ 2c - t & \text{, in } c < t < 2c, \end{cases}$ and

$f(t+2c) = f(t)$ for all t.

Solution. Since $f(t)$ is of period $2c$,we have

$$L[f(t)] = \frac{1}{1-e^{-2cs}} \int_0^{2c} e^{-st} f(t)dt$$
$$= \frac{1}{1-e^{-2cs}} \left[\int_0^{c} e^{-st} f(t)dt + \int_c^{2c} e^{-st} f(t)dt\right]$$
$$= \frac{1}{1-e^{-2cs}} \left[\int_0^{c} e^{-st} tdt + \int_c^{2c} e^{-st}(2c-t)dt\right]$$
$$= \frac{1}{1-e^{-2cs}} \left[\int_0^{c} td\left(\frac{e^{-st}}{-s}\right) + \int_c^{2c} (2c-t)d\left(\frac{e^{-st}}{-s}\right)\right]$$
$$= \frac{1}{1-e^{-2cs}} \left[\left(t\frac{e^{-st}}{-s}\right)_0^c - \int_0^{c} \frac{e^{-st}}{-s}dt + \left((2c-t)\frac{e^{-st}}{-s}\right)_c^{2c} - \int_c^{2c} \frac{e^{-st}}{-s}(-dt)\right]$$
$$= \frac{1}{1-e^{-2cs}} \left[c\frac{e^{-cs}}{-s} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_0^c + 0 + \frac{c}{s}e^{-cs} - \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_c^{2c}\right]$$
$$= \frac{1}{1-e^{-2cs}} \left[-c\frac{e^{-cs}}{s} - \frac{1}{s^2}(e^{-cs}-1) + \frac{c}{s}e^{-cs} + \frac{1}{s^2}(e^{-2cs}-e^{-cs})\right]$$
$$= \frac{1}{1-e^{-2cs}} \left[-c\frac{e^{-cs}}{s} - \frac{1}{s^2}e^{-cs} + \frac{1}{s^2} + c\frac{e^{-cs}}{s} + \frac{1}{s^2}e^{-2cs} - \frac{1}{s^2}e^{-cs}\right]$$
$$= \frac{1}{1-e^{-2cs}} \left[\frac{e^{-2cs}-e^{-cs}-e^{-cs}+1}{s^2}\right] = \frac{1}{1-e^{-cs}}\left[\frac{1-2e^{-2cs}+e^{-2cs}}{s^2}\right]$$
$$= \frac{1}{(1-e^{-cs})(1+e^{-cs})}.\frac{(1-e^{-cs})^2}{s^2} = \frac{(1-e^{-cs})}{s^2.(1+e^{-cs})}$$
$$= \frac{e^{\frac{cs}{2}} - e^{-\frac{cs}{2}}}{s^2\left(e^{\frac{cs}{2}} + e^{-\frac{cs}{2}}\right)} = \frac{1}{s^2}\tan h\left(\frac{cs}{2}\right).$$

13. (a) (ii) Find $L^{-1}\left\{\frac{s^2}{(s^2+1)(s^2+4)}\right\}$.

Solution. $L^{-1}\left[\frac{s^2}{(s^2+1^2)(s^2+2^2)}\right]$

$$= L^{-1}\left[\frac{s}{s^2+1^2}.\frac{s}{s^2+2^2}\right]$$
$$= L^{-1}\left[\frac{s}{s^2+1^2}\right] * L^{-1}\left[\frac{s}{s^2+2^2}\right]$$
$$= \cos t * \cos 2t$$
$$= \int_0^t \cos u \cos 2(t-u)du$$

$$= \int_0^t \cos u \cos(2t - 2u) du$$
$$= \frac{1}{2} \int_0^t \{\cos(u + 2t - 2u) + \cos(u - 2t + 2u)\} du$$
$$= \frac{1}{2} \int_0^t \{\cos(2t - u) + \cos(3u - 2t)\} du$$
$$= \frac{1}{2} \left[\frac{\sin(2t - u)}{-1} + \frac{\sin(3u - 2t)}{3} \right]_0^t$$
$$= \frac{1}{2} \left[\frac{\sin t}{-1} - \frac{\sin 2t}{-1} + \frac{\sin t}{3} + \frac{\sin 2t}{3} \right]$$
$$= \frac{1}{2} \left[\frac{2}{-3} \sin t + \frac{-4}{-3} \sin 2t \right]$$
$$= \frac{1}{2(-3)} [2 \sin t - 4 \sin 2t]$$
$$= \frac{\sin t - 2 \sin 2t}{-3} = \frac{2 \sin 2t - \sin t}{3}.$$

13. (b) (i) Solve the differential equation $y'' - 3y' + 2y = 4t + e^{3t}$, where $y(0) = 1$ and $y'(0) = -1$ using Laplace transforms.

Solution. $y'' - 3y' + 2y = 4t + e^{3t}$.

Taking Laplace transform both sides we get

$$L[y''] - 3L[y'] + 2L[y] = 4L[t] + L[e^{3t}]$$
$$s^2L[y] - sy(0) - y'(0) - 3[sL[y] - y(0)] + 2L[y] = \frac{4}{s^2} + \frac{1}{s-3}$$
$$s^2L[y] - s + 1 - 3sL[y] + 3 + 2L[y] = \frac{4(s-3) + s^2}{s^2(s-3)}$$
$$L[y][s^2 - 3s + 2] - s + 4 = \frac{4s - 12 + s^2}{s^2(s-3)}$$
$$L[y][s^2 - 3s + 2] = \frac{4s - 12 + s^2}{s^2(s-3)} + s - 4$$
$$L[y] = \frac{s^2 + 4s - 12}{s^2(s-3)(s^2 - 3s + 2)} + \frac{s-4}{s^2 - 3s + 2}$$
$$L[y] = \frac{(s+6)\cancel{(s-2)}}{s^2(s-3)\cancel{(s-2)}(s-1)} + \frac{s-4}{(s-2)(s-1)}$$
$$L[y] = \frac{(s+6)}{s^2(s-3)(s-1)} + \frac{s-4}{(s-1)(s-2)}$$

$$L[y] = \frac{(s+6)(s-2)+s^2(s-3)(s-4)}{s^2(s-1)(s-2)(s-3)}$$

$$L[y] = \frac{(s+6)(s-2)+s^2(s-3)(s-4)}{s^2(s-1)(s-2)(s-3)} \quad (1)$$

Let $\dfrac{(s+6)(s-2)+s^2(s-3)(s-4)}{s^2(s-1)(s-2)(s-3)} = \dfrac{A}{s}+\dfrac{B}{s^2}+\dfrac{C}{s-1}+\dfrac{D}{s-2}+\dfrac{E}{s-3}$

$$\begin{aligned}(s+6)(s-2)+s^2(s-3)(s-4) &= As(s-1)(s-2)(s-3)\\ &\quad + B(s-1)(s-2)(s-3)+Cs^2(s-2)(s-3)\\ &\quad + Ds^2(s-1)(s-3)+Es^2(s-1)(s-2).\end{aligned}$$

Put $s=0$, $B(0-1)(0-2) = (0+6)(0-2) \Rightarrow 2B = -12 \Rightarrow B = -6.$

Put $s=1$, $\quad C\cdot 1\cdot(0-1)(0-2) = 7(-1)+(-3)(0-2)$

$$\Rightarrow 2C = -7+6 = -1 \Rightarrow C = \frac{-1}{2}.$$

Put $s=2, (-2)4(-1) = D(4)(1)(-4) \Rightarrow 8 = -4D \Rightarrow D = -2.$

Put $s=3, 9\times 1 = E(9)(2)(1) \Rightarrow 2E = 1 \Rightarrow E = \dfrac{1}{2}.$

Equating the coefficient of s^4 we get

$$\begin{aligned}1 &= A+C+D+E\\ 1 &= A-\frac{1}{2}-2+\frac{1}{2}\\ 1 &= A-2 \quad \Rightarrow A = 3\end{aligned}$$

$$y = L^{-1}\left[\frac{3}{s}-\frac{6}{s^2}-\frac{\frac{1}{2}}{s-1}-\frac{2}{s-2}+\frac{\frac{1}{2}}{s-3}\right]$$

$$= 3L^{-1}\left[\frac{1}{s}\right]-6L^{-1}\left[\frac{1}{s^2}\right]-\frac{1}{2}L^{-1}\left[\frac{1}{s-1}\right]-2L^{-1}\left[\frac{1}{s-2}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s-3}\right]$$

$$y = 3\cdot 1-6t-\frac{1}{2}e^{t}-2e^{2t}+\frac{1}{2}e^{3t}.$$

13. (b) (ii) Find $L\left\{\dfrac{\cos at-\cos bt}{t}\right\}$.

Solution. $L\left[\dfrac{\cos at-\cos bt}{t}\right] = \displaystyle\int_s^\infty L[\cos at-\cos bt]ds$

$$= \int_s^\infty \left(\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right) ds$$
$$= \frac{1}{2} \left[\log \frac{s^2 + a^2}{s^2 + b^2} \right]_s^\infty$$
$$= \frac{1}{2} \left[\log \frac{s^2 + b^2}{s^2 + a^2} \right]$$

2.5.4 November/December 2014 (R 2013)

Part A

5. Find the Laplace transform of $e^{-t} \sin 2t$.

Solution. $L[e^{-t} \sin 2t] = [L[\sin 2t]]_{s=s+1}$

$$= \left[\frac{2}{s^2 + 4} \right]_{s=s+1}$$
$$= \frac{2}{(s+1)^2 + 4}$$
$$= \frac{2}{s^2 + 2s + 1 + 4} = \frac{2}{s^2 + 2s + 5}.$$

6. Find $f(t)$ if the Laplace transform of $f(t)$ is $\dfrac{s}{(s+1)^2}$.

Solution. $f(t) = L^{-1} \left[\dfrac{s}{(s+1)^2} \right]$

$$= L^{-1} \left[\frac{s+1-1}{(s+1)^2} \right]$$
$$= e^{-t} L^{-1} \left[\frac{s-1}{s^2} \right]$$
$$= e^{-t} L^{-1} \left[\frac{1}{s} - \frac{1}{s^2} \right]$$
$$= e^{-t} \left\{ L^{-1} \left[\frac{1}{s} \right] - L^{-1} \left[\frac{1}{s^2} \right] \right\} = e^{-t} [1 - t].$$

Part B

13. (a) (i) Find the Laplace transform of the following functions

(1) $e^{-t}t\cos t$, (2) $\dfrac{1-\cos t}{t}$.

Solution. $L\left[e^{-t}t\cos t\right] = L\,[t\cos t]_{s\to s+1}$

$$= \left\{-\frac{d}{ds}L[\cos t]\right\}_{s\to s+1}$$

$$= \left\{-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right\}_{s\to s+1}$$

$$= -\left\{\frac{s^2+1-s\times 2s}{(s^2+1)^2}\right\}_{s\to s+1}$$

$$= -\left\{\frac{1-s^2}{(s^2+1)^2}\right\}_{s\to s+1}$$

$$= \frac{(s+1)^2-1}{((s+1)^2+1)^2}$$

$$= \frac{s^2+2s+1-1}{(s^2+2s+1+1)^2} = \frac{s^2+2s}{(s^2+2s+2)^2}.$$

$$L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty L\,[1-\cos t]\,ds$$

$$= \int_s^\infty \left(\frac{1}{s}-\frac{s}{s^2+1}\right)ds$$

$$= \int_s^\infty \left(\frac{1}{s}-\frac{1}{2}\frac{2s}{s^2+1}\right)ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left[\log\frac{s}{\sqrt{s^2+1}}\right]_s^\infty$$

$$= \left[\log\frac{s}{s\sqrt{1+\frac{1}{s^2}}}\right]_s^\infty$$

$$= \left[\log\frac{1}{\sqrt{1+\frac{1}{s^2}}}\right]_s^\infty$$

$$= \log 1 - \log \frac{1}{\sqrt{1+\frac{1}{s^2}}}$$

$$= -\left[\log 1 - \log \sqrt{1+\frac{1}{s^2}}\right]$$

$$= \log \sqrt{\frac{s^2+1}{s^2}} = \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right).$$

13. (a) (ii) Apply convolution theorem to evaluate $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$.

Solution.

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= \cos at * \cos bt$$

$$= \int_0^t \cos au \cos b(t-u)du$$

$$= \int_0^t \cos au \cos(bt-bu)du$$

$$= \frac{1}{2}\int_0^t \{\cos(au+bt-bu)+\cos(au-bt+bu)\}du$$

$$= \frac{1}{2}\left[\frac{\sin(au+bt-bu)}{a-b}+\frac{\sin(au-bt+bu)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b}-\frac{\sin bt}{a-b}+\frac{\sin at}{a+b}+\frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{a+b+a-b}{a^2-b^2}\sin at+\frac{a-b-a-b}{a^2-b^2}\sin bt\right]$$

$$= \frac{1}{2(a^2-b^2)}[2a\sin at - 2b\sin bt] = \frac{a\sin at - b\sin bt}{a^2-b^2}.$$

13. (b) (i) Find the Laplace transform of $f(t) = \begin{cases} t & ,\text{for } 0<t<a \\ 2a-t & ,\text{for } a<t<2a, \end{cases}$ $f(t=2a) = f(t)$.

Solution. Since $f(t)$ is of period $2a$,we have

$$L[f(t)] = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)dt$$
$$= \frac{1}{1-e^{-2as}}\left[\int_0^{a} e^{-st} f(t)dt + \int_a^{2a} e^{-st} f(t)dt\right]$$
$$= \frac{1}{1-e^{-2as}}\left[\int_0^{a} e^{-st} t dt + \int_a^{2a} e^{-st}(2a-t)dt\right]$$
$$= \frac{1}{1-e^{-2as}}\left[\int_0^{a} t d\left(\frac{e^{-st}}{-s}\right) + \int_a^{2a} (2a-t)d\left(\frac{e^{-st}}{-s}\right)\right]$$
$$= \frac{1}{1-e^{-2as}}\left[\left(t\frac{e^{-st}}{-s}\right)_0^a - \int_0^{a} \frac{e^{-st}}{-s}dt + \left((2a-t)\frac{e^{-st}}{-s}\right)_a^{2a} - \int_a^{2a} \frac{e^{-st}}{-s}(-dt)\right]$$
$$= \frac{1}{1-e^{-2as}}\left[a\frac{e^{-as}}{-s} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_0^a + 0 + \frac{a}{s}e^{-as} - \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_a^{2a}\right]$$
$$= \frac{1}{1-e^{-2as}}\left[-a\frac{e^{-as}}{s} - \frac{1}{s^2}(e^{-as}-1) + \frac{a}{s}e^{-as} + \frac{1}{s^2}(e^{-2as}-e^{-as})\right]$$
$$= \frac{1}{1-e^{-2as}}\left[-a\frac{e^{-as}}{s} - \frac{1}{s^2}e^{-as} + \frac{1}{s^2} + a\frac{e^{-as}}{s} + \frac{1}{s^2}e^{-2as} - \frac{1}{s^2}e^{-as}\right]$$
$$= \frac{1}{1-e^{-2as}}\left[\frac{e^{-2as}-e^{-as}-e^{-as}+1}{s^2}\right] = \frac{1}{1-e^{-as}}\left[\frac{1-2e^{-2as}+e^{-2as}}{s^2}\right]$$
$$= \frac{1}{(1-e^{-as})(1+e^{-as})}\cdot\frac{(1-e^{-as})^2}{s^2} = \frac{(1-e^{-as})}{s^2.(1+e^{-as})}$$
$$= \frac{e^{\frac{as}{2}}-e^{-\frac{as}{2}}}{s^2\left(e^{\frac{as}{2}}+e^{-\frac{as}{2}}\right)} = \frac{1}{s^2}\tan h\left(\frac{as}{2}\right).$$

13. (b) (ii) Use Laplace transform to solve $(D^2-3D+2)y = e^{3t}$, with $y(0) = 1$ and $y'(0) = 0$.

Solution. $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}$.

Taking Laplace transform both sides we get

$$L\left[\frac{d^2y}{dt^2}\right] - 3L\left[\frac{dy}{dt}\right] + 2L[y] = L[e^{3t}]$$
$$s^2L[y] - sy(0) - \cancel{y'(0)} - 3[sL[y] - y(0)] + 2L[y] = \frac{1}{s-3}$$
$$s^2L[y] - s - 3sL[y] + 3 + 2L[y] = \frac{1}{s-3}$$

$$L[y][s^2-3s+2]=\frac{1}{s-3}+s-3$$
$$L[y](s-1)(s-2)=\frac{1+(s-3)^2}{s-3}=\frac{1+s^2-6s+9}{s-3}=\frac{s^2-6s+10}{s-3}$$
$$L[y]=\frac{s^2-6s+10}{(s-1)(s-2)(s-3)}.$$
$$y=L^{-1}\left[\frac{s^2-6s+10}{(s-1)(s-2)(s-3)}\right].$$

Let $\dfrac{s^2-6s+10}{(s-1)(s-2)(s-3)}=\dfrac{A}{s-1}+\dfrac{B}{s-2}+\dfrac{C}{s-3}$
$$=\frac{A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2)}{(s-1)(s-2)(s-3)}.$$

$\therefore\ s^2-6s+10=A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2).$

Put $s=1, 2A=5\Rightarrow A=\dfrac{5}{2}$.

Put $s=2, -B=2\Rightarrow B=-2$.

Put $s=3, 2C=1\Rightarrow C=\dfrac{1}{2}$.

$$\therefore\frac{s^2-6s+10}{(s-1)(s-2)(s-3)}=\frac{5}{2}\cdot\frac{1}{s-1}+-2\cdot\frac{1}{s-2}+\frac{1}{2}\cdot\frac{1}{s-3}$$

Now, $y=L^{-1}\left[\dfrac{s^2-6s+10}{(s-1)(s-2)(s-3)}\right]$
$$=L^{-1}\left[\frac{5}{2}\cdot\frac{1}{s-1}-2\cdot\frac{1}{s-2}+\frac{1}{2}\cdot\frac{1}{s-3}\right]$$
$$y=\frac{5}{2}e^t-2e^{2t}+\frac{1}{2}e^{3t}.$$

2.5.5 May/June 2014 (R 2013)

Part A

5. Find $L\left[\dfrac{\sin t}{t}\right]$.

Solution. $L\left[\dfrac{\sin t}{t}\right]=\displaystyle\int_s^\infty L[\sin t]ds=\int_s^\infty \frac{1}{s^2+1}]ds=\left[\tan^{-1}s\right]_s^\infty$

$$= \tan^{-1}(\infty) - \tan^{-1}(s) = \frac{\pi}{2} - \tan^{-1}(s) = \cot^{-1}(s).$$

6. Evaluate $L^{-1}\left[\dfrac{1}{s^2+6s+13}\right]$.

Solution. $$L^{-1}\left[\frac{1}{s^2+6s+13}\right] = L^{-1}\left[\frac{1}{(s+3)^2+13-9}\right]$$
$$= L^{-1}\left[\frac{1}{(s+3)^2+4}\right]$$
$$= L^{-1}\left[\frac{1}{(s+3)^2+2^2}\right]$$
$$= e^{-3t}L^{-1}\left[\frac{1}{s^2+2^2}\right] = \frac{e^{-3t}\sin 2t}{2}.$$

Part B

13. (a) (i) Find the Laplace transform of $f(t)$, where

$$f(t) = \begin{cases} \sin\omega t & ,\text{for } 0 < t < \frac{\pi}{\omega} \\ 0 & ,\text{for } \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases} \quad \text{and } f(t+\tfrac{2\pi}{\omega}) = f(t).$$

Solution. $f(t)$ is defined in the interval $(0, \frac{2\pi}{\omega})$

$$f(t+\frac{2\pi}{\omega}) = \sin\omega(t+\frac{2\pi}{\omega}) = \sin(\omega t + 2\pi) = \sin\omega t = f(t)$$

$$\therefore\ f(t) \text{ is periodic with period } T = \frac{2\pi}{\omega}.$$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$$
$$= \frac{1}{1-e^{-\frac{s2\pi}{\omega}}}\int_0^{\frac{2\pi}{\omega}} e^{-st}f(t)dt$$
$$= \frac{1}{1-e^{-\frac{s2\pi}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st}\sin\omega t dt$$
$$= \frac{1}{1-e^{-\frac{s2\pi}{\omega}}}\left[\frac{e^{-st}}{s^2+w^2}[-s\sin\omega t - \omega\cos\omega t]\right]_0^{\frac{\pi}{\omega}}$$
$$= \frac{1}{1-e^{-\frac{s2\pi}{\omega}}}\left[\frac{e^{\frac{-s\pi}{\omega}}}{s^2+w^2}[-s.0-\omega(-1)] - \frac{-\omega}{\omega^2+s^2}\right]$$
$$= \frac{1}{\left(1-e^{-\frac{s2\pi}{\omega}}\right)(s^2+\omega^2)}\left\{\omega e^{\frac{-s\pi}{\omega}}+\omega\right\}$$

$$= \frac{\omega(1+e^{\frac{s\pi}{\omega}})}{(s^2+\omega^2)(1-e^{\frac{-s\pi}{\omega}})(1+e^{\frac{-s\pi}{\omega}})}$$
$$= \frac{\omega}{(s^2+\omega^2)(1-e^{\frac{-s\pi}{\omega}})}.$$

13. (a) (ii) Using convolution theorem find the inverse Laplace transform of $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$.

Solution. Refer question 13. (a) (ii) of Nov./Dec.2014.

13. (b) (i) Find the Laplace transform of $f(t) = te^{-3t}\cos 2t$.

Solution. $L\left[te^{-3t}\cos 2t\right] = -\frac{d}{ds}L[e^{-3t}\cos 2t]$

$$= -\frac{d}{ds}\{L[\cos 2t]\}_{s\to s+3} = -\frac{d}{ds}\left\{\frac{s}{s^2+4}\right\}_{s\to s+3} = -\frac{d}{ds}\left(\frac{s+3}{(s+3)^2+4}\right)$$
$$= -\frac{d}{ds}\left(\frac{s+3}{s^2+6s+9+4}\right) = -\frac{d}{ds}\left(\frac{s+3}{s^2+6s+13}\right)$$
$$= -\frac{(s^2+6s+13)\cdot 1-(s+3)(2s+6)}{(s^2+6s+13)^2}$$
$$= -\frac{s^2+6s+13-(2s^2+6s+6s+18)}{(s^2+6s+13)^2} = -\frac{-s^2-6s-5}{(s^2+6s+13)^2}$$
$$= \frac{s^2+6s+5}{(s^2+6s+13)^2}.$$

13. (b) (ii) Using Laplace transform, solve $\frac{d^2y}{dt^2}+4y = \sin 2t$, given $y(0) = 3$ and $y'(0) = 4$.

Solution. The given differential equation is $\frac{d^2y}{dt^2}+4y = \sin 2t$.

Taking Laplace transform on both sides we get

$$L\left[\frac{d^2y}{dt^2}\right]+4L[y] = L[\sin 2t]$$
$$s^2L[y]-sy(0)-y'(0)+4L[y] = \frac{2}{s^2+4}$$
$$s^2L[y]-3s-4+4L[y] = \frac{2}{s^2+4}$$
$$L[y][s^2+4] = \frac{2}{s^2+4}+3s+4$$

$$L[y] = \frac{2 + (3s+4)(s^2+4)}{(s^2+4)^2}$$

$$= \frac{2 + 3s^3 + 12s + 4s^2 + 16}{(s^2+4)^2}$$

$$= \frac{3s^3 + 4s^2 + 12s + 18}{(s^2+4)^2}.$$

$$y = L^{-1}\left[\frac{3s^3 + 4s^2 + 12s + 18}{(s^2+4)^2}\right].$$

Let $\dfrac{3s^3 + 4s^2 + 12s + 18}{(s^2+4)^2} = \dfrac{As+B}{s^2+4} + \dfrac{Cs+D}{(s^2+4)^2}$

$$\therefore 3s^3 + 4s^2 + 12s + 18 = (As+B)(s^2+4) + (Cs+D).$$

Equating the coefficients of s^3 we get

$A = 3.$

Equating the coefficients of s^2 we get

$B = 4.$

Equating the coefficients of s we get

$4A + C = 12 \quad \Rightarrow 12 + C = 12 \quad \Rightarrow C = 0.$

Equating the constants we get

$4B + D = 18 \quad \Rightarrow 16 + D = 18 \quad \Rightarrow D = 2.$

$$\therefore \frac{3s^3 + 4s^2 + 12s + 18}{(s^2+4)^2} = \frac{3s+4}{s^2+4} + \frac{2}{(s^2+4)^2}$$

$$\therefore y = L^{-1}\left[\frac{3s+4}{s^2+4}\right] + 2L^{-1}\left[\frac{1}{(s^2+4)^2}\right]$$

$$= 3L^{-1}\left[\frac{s}{s^2+4}\right] + 4L^{-1}\left[\frac{1}{s^2+4}\right] + 2L^{-1}\left[\frac{1}{(s^2+4)^2}\right]$$

$$= 3\cos 2t + 4\frac{\sin 2t}{2} + 2 \cdot \frac{1}{2 \times 2^3}[\sin 2t - 2t\cos 2t]$$

$$\left[\because L^{-1}\left[\frac{2a^3}{(s^2+a^2)^2}\right] = \sin at - at\cos at\right]$$

$$y = 3\cos 2t + 2\sin 2t + \frac{1}{8}[\sin 2t - 2t\cos 2t].$$

2.5.6 November/December 2013(R 2008)

Part A

5. Find the Laplace transform of the function $f(f) = \dfrac{1-e^t}{t}$.

Solution. $L\left[\dfrac{1-e^t}{t}\right] = \displaystyle\int_s^\infty L[1-e^t]ds = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right)ds$

$$= [\log s - \log(s-1)]_s^\infty = \log\left(\frac{s}{s-1}\right)_s^\infty$$

$$= \log\left(\frac{1}{1-\frac{1}{s}}\right)_s^\infty = \log 1 - \log(\frac{s}{s-1})$$

$$= \log\left(\frac{s-1}{s}\right).$$

6. Find the Laplace transform of the function $f(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$

Solution. $L[1] = \displaystyle\int_0^\infty e^{-st} \times 1dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{-1}{s}[0-1] = \frac{1}{s}.$

Part B

13. (a) (i) Find $L^{-1}\left[\dfrac{3s^2+16s+26}{s(s^2+4s+13)}\right]$.

Solution. Let $\dfrac{3s^2+16s+26}{s(s^2+4s+13)} = \dfrac{A}{s} + \dfrac{Bs+c}{s^2+4s+13}$

$3s^2+16s+26 = A(s^2+4s+13) + (Bs+c)s.$

When $s = 0, 13A = 26 \Rightarrow A = 2.$

Equating the coefficients of s^2

$A + B = 3$

$2 + B = 3$

$B = 1.$

Equating the coefficients of s

$$4A + c = 16$$

$$8 + c = 16$$

$$c = 8.$$

$$\therefore \frac{3s^2 + 16s + 26}{s(s^2 + 4s + 13)} = \frac{2}{s} + \frac{s+8}{s^2 + 4s + 13}.$$

$$\begin{aligned}
L^{-1}\left[\frac{3s^2 + 16s + 26}{s(s^2 + 4s + 13)}\right] &= L^{-1}\left[\frac{2}{s} + \frac{s+8}{s^2 + 4s + 13}\right] \\
&= 2L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{s+8}{s^2 + 4s + 13}\right] \\
&= 2 \cdot 1 + L^{-1}\left[\frac{s+2+6}{(s+2)^2 + 13 - 4}\right] \\
&= 2 + L^{-1}\left[\frac{s+2+6}{(s+2)^2 + 9}\right] \\
&= 2 + e^{-2t} L^{-1}\left[\frac{s+6}{s^2 + 9}\right] \\
&= 2 + e^{-2t}\left[L^{-1}\left(\frac{s}{s^2+9}\right) + 2L^{-1}\left(\frac{3}{s^2+9}\right)\right] \\
&= 2 + e^{-2t}[\cos 3t + 2 \sin 3t].
\end{aligned}$$

13. (a) (ii) Find $L^{-1}\left[\log \frac{s+1}{s-1}\right]$.

Solution. Let $F(s) = \log \frac{s+1}{s-1} = \log(s+1) - \log(s-1)$

$$F'(S) = \frac{1}{s+1} - \frac{1}{s-1}$$

$$L[tf(t)] = -F'(s) = -\frac{1}{s+1} + \frac{1}{s-1}$$

$$\therefore tf(t) = L^{-1}\left[\frac{1}{s-1} - \frac{1}{s+1}\right] = L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s+1}\right] = e^t - e^{-t}$$

$$\therefore f(t) = \frac{e^t - e^{-t}}{t}.$$

13. (b) (i) Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ and find $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$ and hence find $L^{-1}\left[\frac{1}{(s^2+9s+13)^2}\right]$.

Solution.

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{2a}$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{s}\frac{s}{(s^2+a^2)^2}\right]$$

$$= L^{-1}\left[\frac{F(s)}{s}\right] \text{ where } F(s) = \frac{s}{(s^2+a^2)^2}$$

$$= \int_0^t L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]dt$$

$$= \int_0^t \frac{t\sin at}{2a}dt = \frac{1}{2a}\int_0^t td\left(\frac{-\cos at}{a}\right)$$

$$= \frac{1}{2a}\left[\left(\frac{-t\cos at}{a}\right)_0^t - \int_0^t\left(\frac{-\cos at}{a}\right)dt\right]$$

$$= \frac{1}{2a}\left[\frac{-t\cos at}{a} + \frac{1}{a}\left(\frac{\sin at}{a}\right)_0^t\right] = \frac{1}{2a}\left[\frac{-t\cos at}{a} + \frac{1}{a^2}\sin at\right]$$

$$= \frac{1}{2a^3}\left[\sin at - at\cos at\right].$$

$$L^{-1}\left[\frac{1}{(s^2+9s+13)^2}\right] = L^{-1}\left[\frac{1}{[(s+3)^2+13-9]^2}\right]$$

$$= e^{-3t}L^{-1}\left[\frac{1}{(s^2+4)^2}\right], \text{which is of the form } \frac{1}{(s^2+a^2)^2} \text{ with } a = 2.$$

$$= e^{-3t}\left[\frac{1}{2\times 2^3}(\sin 2t - 2t\cos 2t)\right] = \frac{e^{-3t}}{16}[\sin 2t - 2t\cos 2t].$$

13. (b) (ii) Solve using Laplace transforms, $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t^2 + 2t$, given that $y = 4, y' = -2$ when $t = 0$.

Solution. The given differential equation is $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t^2 + 2t$.

Taking Laplace transform on both sides, we get

$$s^2L[y] - sy(0) - y'(0) + sL(y) - y(0) - y(0) = \frac{2}{s^3} + \frac{2}{s^2}$$

$$(s^2+s)L[y] - 4s + 2 - 4 = \frac{2(s+1)}{s^3}$$

$$s(s+1)L(y) = 4s + 2 + \frac{2(s+1)}{s^3} = 2s + 2 + 2s + \frac{2(s+1)}{s^3}$$

$$= 2(s+1) + 2s + \frac{2(s+1)}{s^3}$$

$$L(y) = \frac{2}{s} + \frac{2}{s+1} + \frac{2}{s^4}$$

$$y = L^{-1}\left[\frac{2}{s} + \frac{2}{s+1} + \frac{2}{s^4}\right] = 2 + 2e^{-t} + \frac{2}{3!}t^3$$

$$y = 2 + 2e^{-t} + \frac{1}{3}t^3.$$

2.5.7 May/June 2013 (R 2008)

Part A

5. Find the Laplace transform of $\frac{t}{e^t}$.

   Solution. $L\left[\frac{t}{e^t}\right] = L[e^{-t}t] = [L(t)]_{s\to s+1} = \frac{1}{(s+1)^2}$.

6. Verify the initial and final value theorems for the function $f(t) = ae^{-bt}$.

   Solution. Let $f(t) = ae^{-bt}$.

$$L[f(t)] = \frac{a}{s+b} = F(s).$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} ae^{-bt} = a.$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{as}{s+b} = \lim_{s\to\infty} \frac{as}{s(1+\frac{b}{s})} = a.$$

$$\therefore \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$$

   Hence, the initial value theorem is verified.

Part B

13. (a) (i) Find $L[te^{-3t}\sin 2t]$.

Solution. $L[te^{-3t}\sin 2t] = \frac{-d}{ds}\left[L[e^{-3t}\sin 2t]\right]$

$$= \frac{-d}{ds}[L[\sin 2t]_{s\to s+3}] = \frac{-d}{ds}\left[\frac{2}{s^2+4}\right]_{s\to s+3}$$

$$= \frac{-d}{ds}\left[\frac{2}{(s+3)^2+4}\right]$$

$$= \frac{-d}{ds}\left[\frac{2}{s^2+6s+9+4}\right]$$

$$= \frac{-d}{ds}\left[\frac{2}{s^2+6s+13}\right]$$

$$= -2\frac{d}{ds}(s^2+6s+13)^{-1}$$

$$= -2\left[(-1)(s^2+6s+13)^{-2}(2s+6)\right]$$

$$= 2\left[\frac{2s+6}{(s^2+6s+13)^2}\right] = 4\left[\frac{s+3}{(s^2+6s+13)^2}\right].$$

13. (a) (ii) Find the Laplace transform of the square-wave function (or Meoander function) of period a defined as

$$f(t) = \begin{cases} 1 & \text{when } 0 < t < \dfrac{a}{2} \\ -1 & \text{when } \dfrac{a}{2} < t < a. \end{cases}$$

Solution. Since $f(t)$ is a periodic function of period a, we have

$$L[f(t)] = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}} e^{-st} f(t)dt + \int_{\frac{a}{2}}^{a} e^{-st} f(t)dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}} e^{-st}dt + \int_{\frac{a}{2}}^{a} e^{-st}(-1)dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{\frac{a}{2}} - \left(\frac{e^{-st}}{-s}\right)_{\frac{a}{2}}^{a}\right]$$

$$= \frac{1}{s(1-e^{-as})}\left[-\left(e^{-\frac{a}{2}s}-1\right) + e^{-as} - e^{-\frac{a}{2}s}\right]$$

$$= \frac{1}{s(1-e^{-as})}\left[1 - 2e^{-\frac{a}{2}s} + e^{-as}\right]$$

$$= \frac{\left(1-e^{-\frac{a}{2}s}\right)^{\cancel{2}\,1}}{s\cancel{\left(1-e^{-\frac{a}{2}s}\right)}\left(1+e^{-\frac{a}{2}s}\right)}$$

$$= \frac{1 - e^{-\frac{a}{2}s}}{s\left(1 + e^{-\frac{a}{2}s}\right)} = \frac{1}{s} \cdot \frac{e^{\frac{as}{4}} - e^{-\frac{as}{4}}}{e^{\frac{as}{4}} + e^{-\frac{as}{4}}} = \frac{1}{s} \tan h\left(\frac{as}{4}\right).$$

13. (b) (i) Using convolution theorem, find $L^{-1}\left[\frac{4}{(s^2+2s+5)^2}\right]$.

Solution.

$$L^{-1}\left[\frac{4}{(s^2+2s+5)^2}\right] = L^{-1}\left[\frac{4}{((s+1)^2+4)^2}\right] = e^{-t}L^{-1}\left[\frac{4}{(s^2+4)^2}\right] \qquad (1)$$

Now, $L^{-1}\left[\frac{4}{(s^2+4)^2}\right] = L^{-1}\left[\frac{2}{s^2+4}\,\frac{2}{s^2+4}\right]$

$$= L^{-1}\left[\frac{2}{s^2+4}\right] * L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$= \sin 2t * \sin 2t = \int_0^t \sin 2u \sin 2(t-u)du$$

$$= \int_0^t (\cos\{2u - 2(t-u)\} - \cos\{2u + 2t - 2u\})\,du$$

$$= \left[\int_0^t \cos(4u-2t)du - \int_0^t \cos 2t du\right]$$

$$= \left[\left(\frac{\sin(4u-2t)}{4}\right)_0^t - \cos 2t(u)_0^t\right]$$

$$= \left[\frac{\sin 2t}{4} + \frac{\sin 2t}{4} - t\cos 2t\right]$$

$$= \left[\frac{\sin 2t}{2} - t\cos 2t\right].$$

$$(1) \Rightarrow L^{-1}\left[\frac{1}{(s^2++2s+5)^2}\right] = \frac{1}{2}\left[\sin 2t - 2t\cos 2t\right]e^{-t}.$$

13. (b) (ii) Solve $y'' + 5y' + 6y = 2$ given $y'(0) = 0$ and $y(0) = 0$ using Laplace transform method.

Solution. $y'' + 5y' + 6y = 2$.

Taking Laplace transform both sides we get

$$L[y''] + 5L[y'] + 6L[y] = L[2]$$

$$s^2L[y] - sy(0) - y'(0) + 5[sL[y] - y(0)] + 6L[y] = 2 \cdot L[1]$$

$$L[y][s^2 + 5s + 6] = \frac{2}{s}$$

$$L[y] = \frac{2}{s(s^2+5s+6)}$$
$$= \frac{2}{s(s+2)(s+3)}$$
$$y = L^{-1}\left[\frac{2}{s(s+2)(s+3)}\right]$$

Let $$\frac{2}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$
$$= \frac{A(s+2)(s+3) + Bs(s+3) + Cs(s+2)}{s(s+2)(s+3)}$$
$$\therefore 2 = A(s+2)(s+3) + Bs(s+3) + Cs(s+2).$$

When $s = 0, 6A = 2 \Rightarrow A = \frac{1}{3}$.

When $s = -2, -2B = 2 \Rightarrow B = -1$.

When $s = -3, 3C = 2 \Rightarrow C = \frac{2}{3}$.

$$\therefore \frac{2}{s(s+2)(s+3)} = \frac{1}{3}\cdot\frac{1}{s} - \frac{1}{s+2} + \frac{2}{3}\cdot\frac{1}{s+3}$$

Now $$y = L^{-1}\left[\frac{2}{s(s+2)(s+3)}\right]$$
$$= L^{-1}\left[\frac{1}{3}\cdot\frac{1}{s} - \frac{1}{s+2} + \frac{2}{3}\cdot\frac{1}{s+3}\right]$$
$$= \frac{1}{3}L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+2}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s+3}\right]$$
$$= \frac{1}{3}\times 1 - e^{-2t} + \frac{2}{3}e^{-3t}$$
$$y = \frac{1}{3} - e^{-2t} + \frac{2}{3}e^{-3t}.$$

2.5.8 November/December 2012 (R 2008)

Part A

5. Is the linear property applicable to $L\left[\frac{1-\cos t}{t}\right]$?

   Solution. Linear property is applicable for continuous functions only. Here $\frac{1-\cos t}{t}$ is not even defined at $t = 0$.

Hence, for this function, linearity property is not applicable.

6. Find the inverse Laplace transform of $\dfrac{1}{(s+1)(s+2)}$.

Solution.
$$
\begin{aligned}
L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] &= L^{-1}\left[\frac{s+2-(s+1)}{(s+1)(s+2)}\right]\\
&= L^{-1}\left[\frac{s+2}{(s+1)(s+2)} - \frac{s+1}{(s+1)(s+2)}\right]\\
&= L^{-1}\left[\frac{1}{s+1} - \frac{1}{s+2}\right]\\
&= L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right]\\
&= e^{-t} - e^{-2t}.
\end{aligned}
$$

Part B

13. (a) (i) Find $L^{-1}\left[\dfrac{s^2}{(s^2+4)^2}\right]$ using Convolution theorem.

Solution. $L^{-1}\left[\dfrac{s^2}{(s^2+4)^2}\right] = L^{-1}\left[\dfrac{s}{s^2+4}\cdot\dfrac{s}{s^2+4}\right]$

$$
\begin{aligned}
&= L^{-1}\left[\frac{s}{s^2+4}\right] * L^{-1}\left[\frac{s}{s^2+4}\right]\\
&= \cos 2t * \cos 2t\\
&= \int_0^t \cos 2u \cdot \cos 2(t-u)du\\
&= \int_0^t \cos 2u \cos(2t-2u)du\\
&= \int_0^t \left(\frac{\cos(2u+2t-2u)+\cos(2u-2t+2u)}{2}\right)du\\
&= \frac{1}{2}\int_0^t (\cos 2t + \cos[4u-2t])du\\
&= \frac{1}{2}\left[\cos 2t \int_0^t du + \int_0^t \cos(4u-2t)du\right]
\end{aligned}
$$

$$= \frac{1}{2}\left[\cos 2t[u]_0^t + \left[\frac{\sin(4u-2t)}{4}\right]_0^t\right]$$
$$= \frac{1}{2}\left[t\cos 2t + \frac{1}{4}\{\sin 2t - \sin(-2t)\}\right]$$
$$= \frac{1}{2}\left[t\cos 2t + \frac{1}{4}2\sin 2t\right]$$
$$= \frac{1}{2}\left[t\cos 2t + \frac{\sin 2t}{2}\right]$$
$$= \frac{1}{4}(\sin 2t + 2t\cos 2t).$$

13. (a) (ii) Find the Laplace transform of the half-sine wave rectifier function defined by $f(t) = \begin{cases} \sin\omega t & 0 < t < \frac{\pi}{\omega} \\ 0 & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}. \end{cases}$

Solution. Refer question 13. (a) (i) on of May/June 2014.

13. (b) (i) Find $L\left[\frac{\cos at - \cos bt}{t}\right]$.

Solution. $L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty L\,[\cos at - \cos bt]\,ds$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$$
$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$
$$= \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty = \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}.$$

13. (b) (ii) Solve $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2$ given $x = 0$ and $\frac{dx}{dt} = 5$ for $t = 0$ using Laplace transform method.

Solution. $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2.$

Taking Laplace transform both sides we get

$$L\left[\frac{d^2x}{dt^2}\right] - 3L\left[\frac{dx}{dt}\right] + 2L[x] = L[2]$$
$$s^2L[x] - s\cancel{x(0)} - x'(0) - 3[sL[x] - \cancel{x(0)}] + 2L[x] = \frac{2}{s}$$
$$s^2L[x] - 5 - 3sL[x] + 2L[x] = \frac{2}{s}$$

$$L[x][s^2-3s+2] = \frac{2}{s}+5 = \frac{5s+2}{s}$$

$$L[x] = \frac{5s+2}{s(s^2-3s+2)} = \frac{5s+2}{s(s-1)(s-2)}$$

$$x = L^{-1}\left[\frac{5s+2}{s(s-1)(s-2)}\right].$$

Let $\dfrac{5s+2}{s(s-1)(s-2)} = \dfrac{A}{s}+\dfrac{B}{s-1}+\dfrac{C}{s-2}$

$$= \frac{A(s-1)(s-2)+Bs(s-2)+Cs(s-1)}{s(s-1)(s-2)}$$

$$5s+2 = A(s-1)(s-2)+Bs(s-2)+Cs(s-1).$$

When $s=0, 2A=2 \Rightarrow A=1.$

When $s=1, -B=7 \Rightarrow B=-7.$

When $s=2, 2C=12 \Rightarrow C=6.$

$$\therefore \frac{5s+2}{s(s-1)(s-2)} = \frac{1}{s}-\frac{7}{s-1}+\frac{6}{s-2}.$$

Now, $x = L^{-1}\left[\dfrac{5s+2}{s(s-1)(s-2)}\right]$

$$= L^{-1}\left[\frac{1}{s}-\frac{7}{s-1}+\frac{6}{s-2}\right]$$

$$= L^{-1}\left[\frac{1}{s}\right]-7L^{-1}\left[\frac{1}{s-1}\right]+6L^{-1}\left[\frac{1}{s-2}\right]$$

$$x = 1-7e^{t}+6e^{2t}.$$

2.5.9 May/June 2012 (R 2008)

Part A

5. State the first shifting theorem on Laplace transforms.

   Statement: If $L[f(t)] = F(s)$, then $L[e^{-at}f(t)] = F(s+a)$.

6. Verify the initial value theorem for the function

   $f(t) = 1+e^{-t}(\sin t+\cos t)$.

Solution.

$$\text{Let } f(t) = 1 + e^{-t}(\sin t + \cos t).$$

$$L[f(t)] = L[1 + e^{-t}(\sin t + \cos t)]$$

$$= L[1] + L[e^{-t}(\sin t + \cos t)]$$

$$= \frac{1}{s} + L[\sin t + \cos t]_{s\to s+1}$$

$$= \frac{1}{s} + L\left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]_{s\to s+1}$$

$$= \frac{1}{s} + \left[\frac{s+1}{s^2+1}\right]_{s\to s+1}$$

$$F(s) = \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}[1 + e^{-t}(\sin t + \cos t)]$$

$$= 1 + 1 = 2.$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[1 + \frac{s(s+2)}{(s+1)^2+1}\right]$$

$$= 1 + \lim_{s\to\infty}\frac{s^2(1+\frac{2}{s})}{s^2(1+\frac{1}{s})^2 + \frac{1}{s^2}} = 1 + \frac{1}{1} = 2.$$

$$\therefore \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$$

Part B

13. (a) (i) Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ using convolution theorem.

Solution.

$$\text{Let } \frac{s}{(s^2+a^2)^2} = \frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}.$$

$$\therefore F(s) = \frac{s}{s^2+a^2}, G(s) = \frac{1}{s^2+a^2}.$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}[F(s)G(s)] = L^{-1}F(s) * L^{-1}G(s)$$

$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * \left[\frac{1}{(s^2+a^2)}\right] = \cos at * \frac{\sin at}{a}$$

$$= \frac{1}{a}\int_0^t \cos au \sin a(t-u)du$$
$$= \frac{1}{2a}\int_0^t 2\sin(at-au)\cos au du$$
$$= \frac{1}{2a}\int_0^t \left\{\sin(at-au+au)+\sin(at-au-au)\right\}du$$
$$= \frac{1}{2a}\left\{\int_0^t \sin at du + \int_0^t \sin(at-2au)du\right\}$$
$$= \frac{1}{2a}\left[\sin at(u)_0^t - \left[\frac{\cos(at-2au)}{-2a}\right]_0^t\right]$$
$$= \frac{1}{2a}\left[t\sin at + \frac{1}{2a}(\cos(-at)-\cos at)\right]$$
$$= \frac{1}{2a}\left[t\sin at - \frac{1}{2a}0\right] = \frac{t\sin at}{2a}.$$

13. (a) (ii) Find the Laplace transform of the triangular wave function given by $f(t) = \begin{cases} t & 0 \le t \le \pi \\ 2\pi - t & \pi \le t \le 2\pi \end{cases}$ and $f(t+2\pi) = f(t)$.

Solution. Since $f(t)$ is of period 2π,we have

$$L[f(t)] = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} f(t)dt$$
$$= \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} e^{-st} f(t)dt + \int_{\pi}^{2\pi} e^{-st} f(t)dt\right]$$
$$= \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} e^{-st} tdt + \int_{\pi}^{2\pi} e^{-st}(2\pi-t)dt\right]$$
$$= \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} td(\frac{e^{-st}}{-s}) + \int_{\pi}^{2\pi} (2\pi-t)d(\frac{e^{-st}}{-s})\right]$$
$$= \frac{1}{1-e^{-2\pi s}}\left[\left(t\frac{e^{-st}}{-s}\right)_0^{\pi} - \int_0^{\pi}\frac{e^{-st}}{-s}dt + \left((2\pi-t)\frac{e^{-st}}{-s}\right)_{\pi}^{2\pi} - \int_{\pi}^{2\pi}\frac{e^{-st}}{-s}(-dt)\right]$$
$$= \frac{1}{1-e^{-2\pi s}}\left[\pi\frac{e^{-\pi s}}{-s} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_0^{\pi} + 0 + \frac{\pi}{s}e^{-\pi s} - \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_{\pi}^{2\pi}\right]$$
$$= \frac{1}{1-e^{-2\pi s}}\left[-\pi\frac{e^{-\pi s}}{s} - \frac{1}{s^2}(e^{-\pi s}-1) + \frac{\pi}{s}e^{-\pi s} + \frac{1}{s^2}(e^{-2\pi s}-e^{-\pi s})\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[-\pi\frac{e^{-\pi s}}{s} - \frac{1}{s^2}e^{-\pi s} + \frac{1}{s^2} + \pi\frac{e^{-\pi s}}{s} + \frac{1}{s^2}e^{-2\pi s} - \frac{1}{s^2}e^{-\pi s}\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-2\pi s}-e^{-\pi s}-e^{-\pi s}+1}{s^2}\right] = \frac{1}{1-e^{-2\pi s}}\left[\frac{1-2e^{-2\pi s}+e^{-2\pi s}}{s^2}\right]$$

$$= \frac{1}{(1-e^{-\pi s})(1+e^{-\pi s})}\cdot\frac{(1-e^{-\pi s})^2}{s^2} = \frac{(1-e^{-\pi s})}{s^2.(1+e^{-\pi s})}$$

$$= \frac{e^{\frac{\pi s}{2}}-e^{-\frac{\pi s}{2}}}{s^2\left(e^{\frac{\pi s}{2}}+e^{-\frac{\pi s}{2}}\right)} = \frac{1}{s^2}\tan h\left(\frac{\pi s}{2}\right).$$

13. (b) (i) Find the Laplace transform of $\dfrac{e^{at}-e^{-bt}}{t}$.

Solution. $L\left[\dfrac{e^{at}-e^{-bt}}{t}\right] = \displaystyle\int_s^\infty L\left[e^{at}-e^{-bt}\right]ds = \int_s^\infty\left(\frac{1}{s-a}-\frac{1}{s+b}\right)ds$

$$= [\log(s-a)-\log(s+b)]_s^\infty = \left[\log\frac{s-a}{s+b}\right]_s^\infty$$

$$= \log\frac{s+b}{s-a}.$$

13. (b) (ii) Find $\displaystyle\int_0^\infty te^{-2t}\cos t dt$ using Laplace transform .

Solution. Comparing the definition of the Laplace transform and the given integral, we observe that

$$\int_0^\infty te^{-2t}\cos t dt = L[t\cos t]_{s=2} = -\frac{d}{ds}[L(\cos t)]_{s=2}$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]_{s=2} = -\left[\frac{s^2+1-s\cdot 2s}{(s^2+1)^2}\right]_{s=2}$$

$$= -\left[\frac{1-s^2}{(s^2+1)^2}\right]_{s=2} = \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s=2} = \frac{4-1}{(4+1)^2} = \frac{3}{25}.$$

13. (b) (iii) Solve the differential equation $\dfrac{d^2y}{dt^2}-3\dfrac{dy}{dt}+2y = e^{-t}$ with $y(0)=1$, and $y'(0)=0$ using Laplace transforms.

Solution. $\dfrac{d^2y}{dt^2}-3\dfrac{dy}{dt}+2y = e^{-t}$.

Taking Laplace transform both sides we get

$$L\left[\frac{d^2y}{dt^2}\right] - 3L\left[\frac{dy}{dt}\right] + 2L[y] = L[e^{-t}]$$

$$s^2L[y] - sy(0) - \cancel{y'(0)} - 3[sL[y] - y(0)] + 2L[y] = \frac{1}{s+1}$$

$$s^2L[y] - s - 3sL[y] + 3 + 2L[y] = \frac{1}{s+1}$$

$$L[y][s^2 - 3s + 2] = \frac{1}{s+1} + s - 3$$

$$L[y](s-1)(s-2) = \frac{1 + (s-3)(s+1)}{s+1}$$

$$= \frac{1 + s^2 - 2s - 3}{s+1} = \frac{s^2 - 2s - 2}{s+1}$$

$$L[y] = \frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)}.$$

$$y = L^{-1}\left[\frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)}\right].$$

Let $\dfrac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)} = \dfrac{A}{s+1} + \dfrac{B}{s-1} + \dfrac{C}{s-2}$

$$= \frac{A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1)}{(s+1)(s-1)(s-2)}.$$

$$\therefore \quad s^2 - 2s - 2 = A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1).$$

When $s = 1, -2B = -3 \Rightarrow B = \frac{3}{2}$.

When $s = -1, 6A = 1 \Rightarrow A = \frac{1}{6}$.

When $s = 2, 3C = -2 \Rightarrow C = -\frac{2}{3}$.

$$\therefore \frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)} = \frac{1}{6}\cdot\frac{1}{s+1} + \frac{3}{2}\cdot\frac{1}{s-1} - \frac{2}{3}\cdot\frac{1}{s-2}$$

$$\text{Now}, y = L^{-1}\left[\frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)}\right]$$

$$= L^{-1}\left[\frac{1}{6}\cdot\frac{1}{s+1} + \frac{3}{2}\cdot\frac{1}{s-1} - \frac{2}{3}\cdot\frac{1}{s-2}\right]$$

$$y = \frac{1}{6}e^{-t} + \frac{3}{2}e^{t} - \frac{2}{3}e^{2t}.$$