جامعة الملك عبدالله
للعلوم والتقنية
King Abdullah University of
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A package for symbolic PDE analysis

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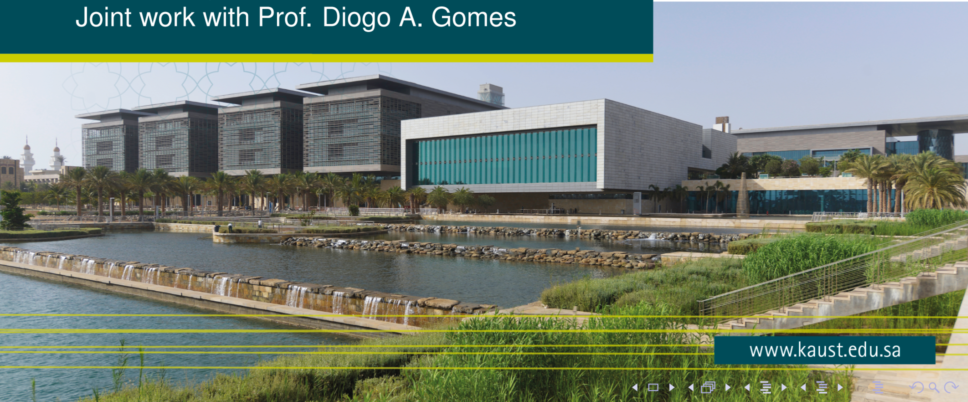
Joint work with Prof. Diogo A. Gomes



Simple symbolic analysis

Let us analyze the 1D heat equation

$$u_t(t,x) = u_{xx}(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{T}.$$

Take the expression $u(t,x)$ and compute time derivative:

$$\frac{d}{dt}\int_{\mathbb{T}} u(t,x)\,dx = \int_{\mathbb{T}} \underbrace{\frac{\delta}{\delta u}u(t,x)}_{\text{Variational Derivative}} u_t(t,x)\,dx = \int_{\mathbb{T}} \underbrace{u_{xx}(t,x)}_{\substack{\text{Null-Lagrangian,} \\ \frac{\delta}{\delta u}u_{xx}(t,x)=0}} dx = 0.$$

$$\frac{d}{dt}\int_{\mathbb{T}} u(t,x)\,dx = 0 \quad - \quad \text{Conservation of Heat}$$

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$$= \int_{\mathbb{T}} 2u(t,x)u_{xx}(t,x)\,dx$$

$$\text{(Integration by Parts)} = \int_{\mathbb{T}} \underbrace{-2u_x^2(t,x)}_{\leq 0}\,dx \leq 0.$$

$$\frac{d}{dt}\int_{\mathbb{T}} \frac{1}{2}u^2(t,x)\,dx \leq 0 \quad - \quad \text{Dissipation of Energy}$$

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