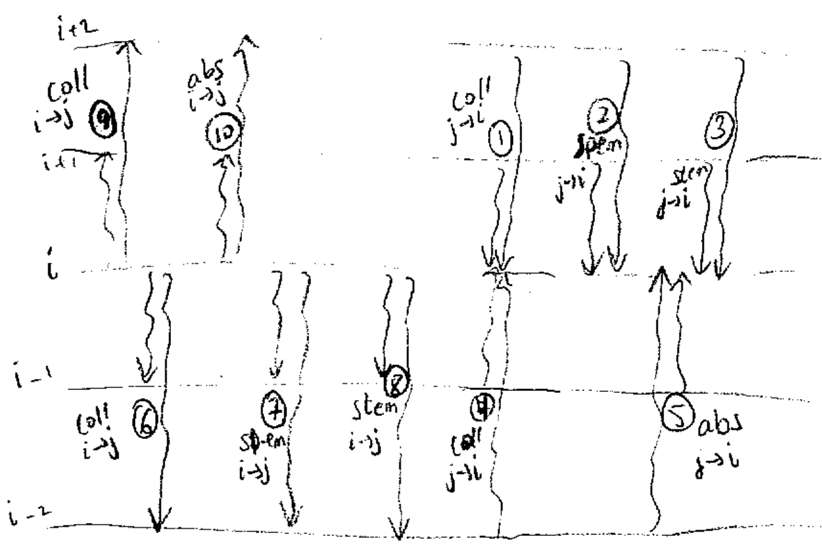
(1)

$$\frac{dn_i}{dt} = \sum_{j=i+1}^{N-1} n_j\left[\overset{①}{n_c K_{ji}} + (\overset{②}{1} + \overset{③}{\bar{n}_{\gamma_{ji}}})A_{ji}\right] + \sum_{j=0}^{i-1} \overset{④}{n_j n_c K_{ji}}$$

$$+ \sum_{j=0}^{i-1} \overset{⑤}{n_j \frac{g_i}{g_j}\bar{n}_{\gamma_{ij}} A_{ij}}$$

$$- n_i \sum_{j=0}^{i-1}\left[\overset{⑥}{n_c K_{ij}} + (\overset{⑦}{1} + \overset{⑧}{n_{\gamma_{ij}}})A_{ij}\right] - n_i \sum_{j=i+1}^{N-1} \overset{⑨}{n_c K_{ij}}$$

$$- n_i \sum_{j=i+1}^{N-1} \overset{⑩}{\frac{g_j}{g_i}\bar{n}_{\gamma_{ji}} A_{ji}}$$



In the lower triangular parts $i > j$, hence those correspond to transitions from an upper to a lower level

$$\circledast \quad A_{ij} = \begin{cases} \neq 0 & i > j \\ = 0 & \forall\ i \leq j \end{cases}$$

$$A = \begin{bmatrix} 0 & & 0 \\ & 0 & \\ A_{ij} & & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ A_{10} & 0 & 0 & 0 \\ A_{20} & A_{21} & 0 & 0 \\ A_{30} & A_{31} & A_{32} & 0 \end{bmatrix}$$

$$\begin{array}{c} \rightarrow j \qquad j > i \\ a_{ij} = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0\ldots} \\ a_{10} & a_{11} & & a_{\ldots} \\ \vdots & & & \\ a_{n0} & a_{n-1,1} & \cdots & a_{nn} \end{pmatrix} \\ i > j \end{array}$$

$$\circledast \quad \bar{n}_{\gamma_{ij}} = \frac{1}{e^{\Delta E_{ij}/k_B T} - 1},$$

$$n_{\gamma_{ij}} = n_{\gamma_{ji}}$$

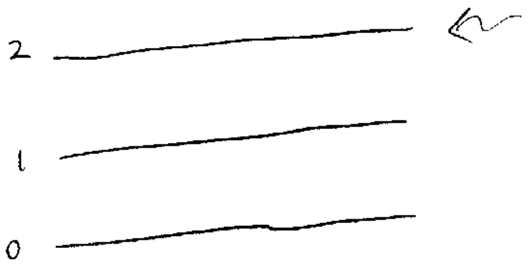
$$A' = \begin{bmatrix} 0 & & 0 \\ & 0 & \\ (1+n_{\gamma_{ij}})A_{ij} & & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ (1+n_{10})A_{10} & 0 & 0 & 0 \\ (1+n_{20})A_{20} & (1+n_{21})A_{21} & 0 & 0 \\ (1+n_{30})A_{30} & (1+n_{31})A_{31} & (1+n_{32})A_{32} & 0 \end{bmatrix}$$

$$A'_{abs} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \frac{g_i}{g_j} n_{\gamma_{ij}} A_{ij} & & 0 & 0 \\ & & & 0 \end{bmatrix}$$

$$\circledast \quad K_{ij} = \begin{cases} \neq 0 & i > j \\ 0 & i = j \\ \frac{g_j}{g_i} K_{ji} e^{-E_{ij}/k_B T_{kin}} & i < j \end{cases}$$

$$K = \begin{bmatrix} 0 & \frac{g_1}{g_0}K_{10}e^{-\frac{E_{10}}{k_B T_k}} & \frac{g_2}{g_0}K_{20}e^{-\frac{E_{20}}{kT}} & \frac{g_3}{g_0}K_{30}e^{-\frac{E_{30}}{kT}} \\ K_{10} & 0 & \frac{g_2}{g_1}K_{21}e^{-\frac{E_{21}}{kT}} & \frac{g_3}{g_1}K_{31}e^{-\frac{E_{31}}{kT}} \\ K_{20} & K_{21} & 0 & \frac{g_3}{g_2}K_{32}e^{-\frac{E_{32}}{kT}} \\ K_{30} & K_{31} & K_{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & K_{01} & K_{02} & K_{03} \\ K_{10} & 0 & K_{12} & K_{13} \\ K_{20} & K_{21} & 0 & K_{23} \\ K_{30} & K_{31} & K_{32} & 0 \end{bmatrix}$$

explicit example (back side) →



$$A = \begin{pmatrix} 0 & 0 & 0 \\ A_{10} & 0 & 0 \\ A_{20} & A_{21} & 0 \end{pmatrix}$$

$$A' = \begin{pmatrix} 0 & 0 & 0 \\ (1+n_{\gamma_{10}})A_{10} & 0 & 0 \\ (1+n_{\gamma_{20}})A_{20} & (1+n_{\gamma_{21}})A_{21} & 0 \end{pmatrix}$$

$$K = \begin{pmatrix} 0 & \frac{g_1}{g_0}K_{10}e^{-E_{10}/kT} & \frac{g_2}{g_0}K_{20}e^{-\frac{E_{20}}{kT}} \\ K_{10} & 0 & \frac{g_2}{g_1}K_{21}e^{-\frac{E_{21}}{kT}} \\ K_{20} & K_{21} & 0 \end{pmatrix}$$

$$\frac{dn_0}{dt} = n_1 n_c K_{10} + n_1(1+n_{\gamma_{10}})A_{10} + n_2 n_c K_{20} + n_2(1+n_{\gamma_{20}})A_{20} +$$
$$+ 0$$
$$- n_0 n_c K_{01} - 0 - 0$$
$$= \left[n_c K_{10} + (1+n_{\gamma_{10}})A_{10}\right]n_1 + \left[n_c K_{20} + (1+n_{\gamma_{20}})A_{20}\right]n_2 - \left[n_c K_{01} + n_c K_{02} + \frac{g_1}{g_0}n_{\gamma_{10}}A_{10} + \frac{g_2}{g_0}n_{\gamma_{20}}A_{20}\right]n_0$$

$$\frac{dn_1}{dt} = n_2 n_c K_{21} + n_2(1+n_{\gamma_{21}})A_{21} + n_0 n_c K_{01} + 0$$
$$+ n_0 \frac{g_1}{g_0} n_{\gamma_{10}} A_{10}$$
$$- n_1\left[n_c K_{10} + (1+n_{\gamma_{10}})A_{10}\right] - n_1 n_c K_{12}$$
$$- n_1 \frac{g_2}{g_1} n_{\gamma_{21}} A_{21}$$

$$A'_{abs} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{g_1}{g_0}A_{10}n_{\gamma_{10}} & 0 & 0 \\ \frac{g_2}{g_0}A_{20}n_{\gamma_{20}} & \frac{g_2}{g_1}A_{21}n_{\gamma_{21}} & 0 \end{pmatrix}$$

$$= \left[n_c K_{21} + (1+n_{\gamma_{21}})A_{21}\right]n_2 + \left[n_c K_{01} + \frac{g_1}{g_0}n_{\gamma_{10}}A_{10}\right]n_0 - \left[n_c K_{10} + (1+n_{\gamma_{10}})A_{10} + n_c K_{12} + \frac{g_2}{g_1}n_{\gamma_{21}}A_{21}\right]n_1$$

$$\frac{dn_2}{dt} = n_1 n_c K_{12} + 0 + n_0 n_c K_{02}$$
$$+ n_0 \frac{g_2}{g_0} n_{\gamma_{20}} A_{20} + n_1 \frac{g_1}{g_0} n_{\gamma_{21}} A_{21}$$
$$- n_2 n_c K_{20} + n_2(1+n_{\gamma_{20}})A_{20} - n_2 n_c K_{21} - n_2(1+n_{\gamma_{21}})A_{21}$$
$$- 0$$

$$= \left[n_c K_{12} + \frac{g_2}{g_1}n_{\gamma_{21}}A_{21}\right]n_1 + \left[n_c K_{02} + \frac{g_2}{g_0}n_{\gamma_{20}}A_{20}\right]n_0 - \left[n_c K_{20} + (1+n_{\gamma_{20}})A_{20} + n_c K_{21} + (1+n_{\gamma_{21}})A_{21}\right]n_2$$

(2)

$$\frac{dn_i}{dt} = \sum_{j=0}^{N-1}[n_c K_{ji}]n_j - \sum_{j=0}^{N-1}[n_c K_{ij}]n_i + \sum_{j=i+1}^{N-1}[(1+n_{\gamma_{ji}})A_{ji}]n_j - \sum_{j=0}^{i-1}[(1+n_{\gamma_{ij}})A_{ij}]n_i + \sum_{j=0}^{i-1}\left[\frac{g_i}{g_j} n_{\gamma_{ij}} A_{ij}\right]n_j - \sum_{j=i+1}^{N-1}\left[\frac{g_j}{g_i} n_{\gamma_{ji}} A_{ji}\right]n_i$$

$$\frac{d\bar{n}}{dt} = n_c \mathbf{K}^T \cdot \bar{n} + A'^T \bar{n} + A'_{abs} \cdot \bar{n} - D.E.(n_c K + A' + A'^T_{abs}) \cdot \bar{n}$$

$$= [F^T - D.E.F] \cdot \bar{n} \quad \text{where } F = n_c K + A' + A'^T_{abs}$$

, E is a matrix full of 1 $\quad E = \begin{pmatrix} 1 \end{pmatrix}$

D is an operator which takes the entries of the 1st col it operates on and put them on the diagonal of an diagonal matrix, ex

$$D.A = D.\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & g \end{pmatrix}$$

$$\bar{n} = \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_{N-1} \end{pmatrix}$$

in the lambda database, the transitions are listed as

transitionID upper lower — — —

upper is always > lower

So then fill the lower triangular part of A and K

⊛ not all entries in A' or A are filled, since most of them are forbidden transitions determined by selection rules.

$L_{ij} = n_i A_{ij} h\nu_{ij}$ (luminosity for line with freq ν_{ij})

$R_{ij} = A_{ij} h\nu_{ij}$

(see Robert's HW, he had solved it using least squares ...)

$$\frac{d\bar{n}}{dt} = [\underbrace{F^T - D.E.F}_{M}].\bar{n}$$

this can be solved numerically in a few ways

the most straight forward way is to solve it by
solving by inverting:

$$\frac{d\bar{n}}{dt} = M.n_{eq} = 0 \quad \text{(homogeneous linear system)}$$

by inverting M, but this yeilds the undesired trivial solution $\bar{n}_{eq} = 0$, since $\det(M) = 0$,

we replace one row of M by ones, representing the conservation equation $\sum_{i=0}^{N-1} \frac{n_i(t)}{n_{gas}} = 1$

which turns the system into the form

$$Ax = b \quad \left| \quad A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ & M_{row\,1} & & \\ & M_{row\,2} & & \\ & \vdots & & \\ & M_{row\,N-1} & & \end{pmatrix} \begin{matrix} \leftarrow \text{row } 0 \\ \\ \\ \\ \\ \end{matrix} \qquad b = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \} \text{ conservation eq} \\ \} \; dn_i/dt \end{matrix} \right.$$

now, $x = A^{-1}b$

Solving by minimization. In this case, we keep all the entries is M but append the conservation equation as an extra row and solve by a minimization procedure

$$A = \begin{pmatrix} \overset{N\times N}{M} \\ \overset{1\times N}{1} \end{pmatrix}, \quad b = \begin{pmatrix} \overset{N\times 1}{0} \\ \overset{1\times 1}{1} \end{pmatrix}, \quad x = \overset{N\times 1}{\bar{n}/n}$$

$$\overset{(N+1)\times N}{A}.\overset{N\times 1}{x} = \overset{(N+1)\times 1}{b}$$

the minimization procedure minimizes $\|b - A.x\|^2$, i.e it looks for the values x which minimize that.

Solving by evolving wrt time and looking for the evolution to stabilize

$$\frac{d\bar{n}(t)}{dt} = \underset{\substack{\uparrow \\ \text{independent} \\ \text{of time}}}{M}.n(t)$$

at LTE, the fractional abundances would be

$$n_{i_{LTE}} = n g_i \exp(-E_i/KT)$$