

Balance of energy and entropy in local Eulerian form

Before deriving local forms of First and Second laws, we derive a relation between the rate of change of Kinetic energy and the power of external and internal forces.

Net working in Eulerian form

Power: $P = \underline{f} \cdot \underline{v}$

Newton's 2nd law: $\underline{f} = m \underline{a} \rightarrow \underline{f} = \frac{d}{dt}(m \underline{v}) = m \dot{\underline{v}}$

Start by taking dot product of \underline{v} and lin. mom. balance

$$\rho \dot{\underline{v}} \cdot \underline{v} = \rho \underline{v} \cdot \dot{\underline{v}} = (\nabla_x \cdot \underline{\underline{s}}) \cdot \underline{v} + \rho \underline{b} \cdot \underline{v}$$

integrating over an arbitrary $\Omega_t \subseteq B_t$

$$\int_{\Omega_t} \rho \underline{v} \cdot \dot{\underline{v}} dV_x = \int_{\Omega_t} (\nabla_x \cdot \underline{\underline{s}}) \cdot \underline{v} + \rho \underline{b} \cdot \underline{v} dV_x$$

use identity $\nabla \cdot (\underline{A}^T \underline{b}) = (\nabla \cdot \underline{A}) \cdot \underline{b} + \underline{A} : \nabla \underline{b}$ (Lecture 4)

$$\int_{\Omega_t} \rho \underline{v} \cdot \dot{\underline{v}} dV_x = \int_{\Omega_t} -\underline{\underline{s}} : \nabla_x \underline{v} + \nabla \cdot (\underline{\underline{s}}^T \underline{v}) + \rho \underline{b} \cdot \underline{v} dV_x$$

Using property $\underline{\underline{s}} : \underline{\underline{D}} = \underline{\underline{s}} : \text{sym}(\underline{\underline{D}})$ if $\underline{\underline{s}} = \underline{\underline{s}}^T$ we can introduce the rate of strain tensor $\underline{\underline{d}} = \text{sym}(\nabla_x \underline{v}) = \frac{1}{2}(\nabla_x \underline{v} + \nabla_x \underline{v}^T)$.

$$\int_{\Omega_t} \rho \underline{v} \cdot \dot{\underline{v}} \, dV_x = \int_{\Omega_t} \underline{\underline{\underline{\sigma}}} : \underline{\underline{\underline{d}}} + \rho \underline{b} \cdot \underline{v} \, dV_x + \int_{\partial\Omega_t} \underline{\underline{\underline{\sigma}}} \underline{v} \cdot \underline{n} \, dA_x$$

where we have used tensor divergence thm.

Using the definition of transpose $\underline{\underline{\underline{\sigma}}} \underline{v} \cdot \underline{n} = \underline{v} \cdot \underline{\underline{\underline{\sigma}}}^T \underline{n} = \underline{v} \cdot \underline{t}$

$$\int_{\Omega_t} \rho \underline{v} \cdot \dot{\underline{v}} \, dV_x = \int_{\Omega_t} \underline{\underline{\underline{\sigma}}} : \underline{\underline{\underline{d}}} \, dV_x + \underbrace{\int_{\Omega_t} \rho \underline{b} \cdot \underline{v} \, dV_x + \int_{\partial\Omega_t} \underline{t} \cdot \underline{v} \, dA_x}_{\mathcal{P}[\Omega_t]}$$

Now we can identify the left hand side as

$$\frac{d}{dt} K[\Omega_t] = \frac{d}{dt} \int_{\Omega_t} \frac{1}{2} \rho \underline{v} \cdot \underline{v} \, dV_x = \frac{1}{2} \int_{\Omega_t} \rho \frac{d}{dt} (\underline{v} \cdot \underline{v}) \, dV_x$$

$$\frac{d}{dt} (v_i v_i) = \dot{v}_i v_i + v_i \dot{v}_i = 2 (v_i \dot{v}_i)$$

$$\frac{d}{dt} K[\Omega_t] = \int_{\Omega_t} \rho \underline{v} \cdot \dot{\underline{v}} \, dV_x$$

so that we have the result

$$\frac{d}{dt} K[\Omega_t] + \int_{\Omega_t} \underline{\underline{\underline{\sigma}}} : \underline{\underline{\underline{d}}} \, dV_x = \mathcal{P}[\Omega_t]$$

by comparison with $W[\Omega_t] = \mathcal{P}[\Omega_t] - \frac{d}{dt} K[\Omega_t]$

$$\Rightarrow W[\Omega_t] = \int_{\Omega_t} \underline{\underline{\underline{\sigma}}} : \underline{\underline{\underline{d}}} \, dV_x$$

The quantity $\underline{\underline{\underline{\sigma}}} : \underline{\underline{\underline{d}}}$ is called the stress power associated with a motion. It corresponds to the rate of work done by internal forces (stresses) in a continuum body.

Local Eulerian form of First Law

The integral form of First Law

$$\frac{d}{dt} U[\Omega_t] = Q[\Omega_t] + W[\Omega_t]$$

where $U[\Omega_t] = \int_{\Omega_t} \rho \phi \, dV_x$

$$Q[\Omega_t] = \int_{\Omega_t} \rho r \, dV_x - \int_{\partial\Omega_t} \mathbf{q} \cdot \mathbf{n} \, dA_x$$

$$W[\Omega_t] = \int_{\Omega_t} \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV_x$$

Hence we have

$$\frac{d}{dt} \int_{\Omega_t} \rho \phi \, dV_x = \int_{\Omega_t} \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV_x - \int_{\partial\Omega_t} \mathbf{q} \cdot \mathbf{n} \, dA_x + \int_{\Omega_t} \rho r \, dV_x$$

using derivative relative to mass and divergence Thm

$$\int_{\Omega} (\rho \dot{\phi} - \underline{\underline{\sigma}} : \underline{\underline{d}} + \nabla_x \cdot \mathbf{q} + \rho r) \, dV_x$$

by the arbitrary ness of Ω_t we have

$\rho \dot{\phi} = \underline{\underline{\sigma}} : \underline{\underline{d}} - \nabla_x \cdot \mathbf{q} + \rho r$

 local Eulerian form

To write it in conservative form we expand
expand the material time derivative and use
the balance of mass

$$\begin{aligned} \rho \dot{\phi} &= \rho \left(\frac{\partial \phi}{\partial t} + \nabla_x \phi \cdot \underline{u} \right) = \frac{\partial}{\partial t} (\rho \phi) - \phi \frac{\partial \rho}{\partial t} + \rho \nabla_x \phi \cdot \underline{u} \\ &= \frac{\partial}{\partial t} (\rho \phi) + \phi \nabla_x \cdot (\rho \underline{u}) + \nabla_x \phi \cdot (\rho \underline{u}) \\ &= \frac{\partial}{\partial t} (\rho \phi) + \nabla_x \cdot (\underline{u} \rho \phi) \end{aligned}$$

Substituting into the local form and collecting the flux terms we have

$$\frac{\partial}{\partial t}(\rho\phi) + \nabla_x \cdot [\underline{v} \rho\phi + \underline{q}] = \underline{\underline{\sigma}} : \underline{\underline{\underline{\epsilon}}} + \rho r$$

conservative local Eulerian form

Local Eulerian Form of the Second Law

The integral form of the Clausius-Duhem form of the Second Law is

$$\frac{d}{dt} \int_{\Omega_t} \rho s \, dV_x \geq \int_{\Omega_t} \frac{\rho r}{\theta} \, dV_x - \int_{\partial\Omega_t} \frac{\mathbf{q} \cdot \mathbf{n}}{\theta} \, dA_x$$

After applying the Divergence Thm and invoking the arbitrariness of Ω_t we have

$$\boxed{\rho \dot{s} \geq \rho r / \theta - \nabla_x \cdot (\mathbf{q} / \theta)} \quad \text{Clausius-Duhem inequality in local Eulerian form}$$

After multiplying by θ and expanding the divergence

$$\theta \rho \dot{s} \geq \rho r - \nabla_x \cdot \mathbf{q} + \theta^{-1} \mathbf{q} \cdot \nabla_x \theta$$

Which can be written as

$$\boxed{S - \theta^{-1} \mathbf{q} \cdot \nabla_x \theta \geq 0}$$

where $S = \theta \rho \dot{s} - (\rho r - \nabla_x \cdot \mathbf{q})$ is the internal dissipation density per unit volume. Difference between local entropy increase and the local heating.

Note:

I) Any point where $\nabla_x \theta = 0$ the dissipation is non-negative, $S \geq 0$. \Rightarrow bodies with homogeneous θ have non-neg. dissipation.

II) If $S = 0$, i.e. a reversible process, then $\mathbf{q} \cdot \nabla_x \theta \leq 0$.

 Thus \mathbf{q} is at an angle > 90 from $\nabla_x \theta$.

\Rightarrow heat flows down the temperature gradient.

To study the consequences of Clausius-Duhem inequality for constitutive laws we introduce the field

$$\psi(\underline{x}, t) = \phi(\underline{x}, t) - \theta(\underline{x}, t) s(\underline{x}, t)$$

Helmholtz free energy density. This is the portion of the free energy available for performing work at const. θ .

\Rightarrow Reformulate Clausius-Duhem in terms of ψ

Material derivative of free energy

$$\begin{aligned}
 \frac{d}{dt}(\theta s) &= \frac{\partial}{\partial t}(\theta s) + \nabla_x(\theta s) \cdot \underline{v} = \theta \frac{\partial s}{\partial t} + s \frac{\partial \theta}{\partial t} + \theta \nabla_x s \cdot \underline{v} \\
 &\quad + s \nabla_x \theta \cdot \underline{v} \\
 &= \theta \left(\frac{\partial s}{\partial t} + \nabla_x s \cdot \underline{v} \right) + s \left(\frac{\partial \theta}{\partial t} + \nabla_x \theta \cdot \underline{v} \right) \\
 &= \theta \dot{s} + s \dot{\theta}
 \end{aligned}$$

from definition of $\dot{\psi}$

$$\dot{\psi} = \dot{\phi} - \theta \dot{s} - s \dot{\theta} \quad \Rightarrow \quad \dot{\phi} = \dot{\psi} + \theta \dot{s} + s \dot{\theta}$$

substituting into local form of 1st law

$$\rho \dot{\phi} = \underline{\underline{\sigma}} : \underline{\underline{L}} - \nabla_x \cdot \underline{q} + \rho r$$

$$\rho \dot{\psi} + \rho \theta \dot{s} + \rho s \dot{\theta} = \underline{\underline{\sigma}} : \underline{\underline{L}} - \nabla_x \cdot \underline{q} + \rho r$$

$$\rho \theta \dot{s} = \underline{\underline{\sigma}} : \underline{\underline{L}} - \nabla_x \cdot \underline{q} + \rho r - \rho \dot{\psi} - \rho s \dot{\theta}$$

substituting into 2nd law

$$\theta \rho \dot{s} \geq \rho r - \nabla_x \cdot \underline{q} + \theta^{-1} \underline{q} \cdot \nabla_x \theta$$

$$\underline{\underline{\sigma}} : \underline{\underline{L}} - \cancel{\nabla_x \cdot \underline{q}} + \cancel{\rho r} - \rho \dot{\psi} - \rho s \dot{\theta} \geq \cancel{\rho r} - \cancel{\nabla_x \cdot \underline{q}} + \theta^{-1} \underline{q} \cdot \nabla_x \theta$$

solve for $\rho \dot{\psi}$

$$\rho \dot{\psi} \leq \underline{\underline{\sigma}} : \underline{\underline{L}} - \rho s \dot{\theta} - \theta^{-1} \underline{q} \cdot \nabla_x \theta$$

This is called the reduced Clausius-Duhem inequality, because it is independent of local heat supply r and heat flux, q , if $\nabla_x \theta = 0$. \Rightarrow homogeneous bodies of classical thermo.

Note:

In a homogeneous body, $\nabla \theta = 0$, we have that

$$\rho \dot{\phi} \leq \underline{\underline{\underline{\sigma}}} : \underline{\underline{\underline{L}}}$$

for a reversible process this becomes an equality.

\Rightarrow rate of change of free energy is equal to the stress power.