

# Review of Vectors

Def: Vector is a quantity with a magnitude & direction.

$$\underline{v} = |\underline{v}| \hat{v}$$

$$|\underline{v}| = \text{magnitude} \quad (|\underline{v}| \geq 0)$$

$$\hat{v} = \frac{\underline{v}}{|\underline{v}|} \text{ direction } (|\hat{v}| = 1) \quad \text{unit vector}$$

Examples: force, velocities, displacements, ...

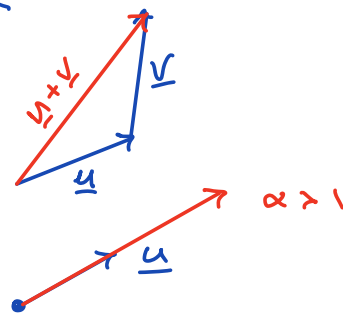
Q: Is it possible to have vector without direction?

Def: Vector space,  $\mathcal{V}$ , is a collection of objects that is closed under addition and scalar multiplication.

$$\underline{u} \in \mathcal{V} \quad \underline{v} \in \mathcal{V} \quad \alpha \in \mathbb{R}$$

$$1) \quad \underline{u} + \underline{v} \in \mathcal{V}$$

$$2) \quad \alpha \underline{u} \in \mathcal{V}$$

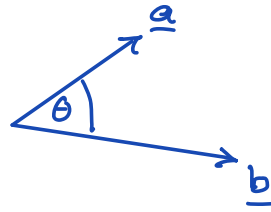


Q1: Do vectors in  $\mathbb{R}^3$  form vector space?

Q2: Do vectors in  $\mathbb{R}^1$  form vector space?

Scalar product:  $\underline{a}, \underline{b} \in \mathcal{V}$

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos \theta \quad \theta \in [0, \pi]$$



$$\underline{a} \cdot \underline{b} = 0 \quad \underline{a} \perp \underline{b}$$

$$\underline{a} \cdot \underline{a} = |\underline{a}|^2$$

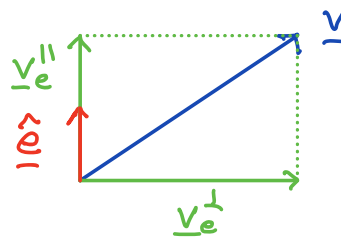
$$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \quad \text{commutative}$$

Projection:  $\underline{\hat{e}}$  unit vector &  $\underline{v} \in \mathcal{V}$

$$\underline{v} = \underline{v}^{\parallel} + \underline{v}^{\perp}$$

$$\underline{v}^{\parallel} = \underline{v} \cdot \underline{\hat{e}}$$

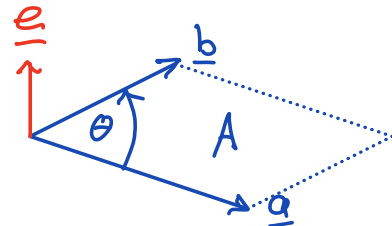
$$\underline{v}^{\perp} = \underline{v} - \underline{v}^{\parallel}$$



Vector product:  $\underline{a}, \underline{b} \in \mathcal{V}$

$$\underline{a} \times \underline{b} = |\underline{a}| |\underline{b}| \sin \theta \underline{\hat{e}} \quad \theta \in [0, \pi]$$

$\underline{\hat{e}}$  unit vector  $\perp$  to  $\underline{a}$  &  $\underline{b}$   
direction right-hand rule



$|\underline{a} \times \underline{b}| = \text{Area of parallelogram spanned by } \underline{a} \text{ \& \; } \underline{b}$

Note:  $\underline{a} \times \underline{b} = -(\underline{b} \times \underline{a})$  not commutative

Q: What does it mean when  $\underline{a} \times \underline{b} = \underline{0}$ ?  
( $\underline{a} \neq \underline{0}$ ,  $\underline{b} \neq \underline{0}$ )

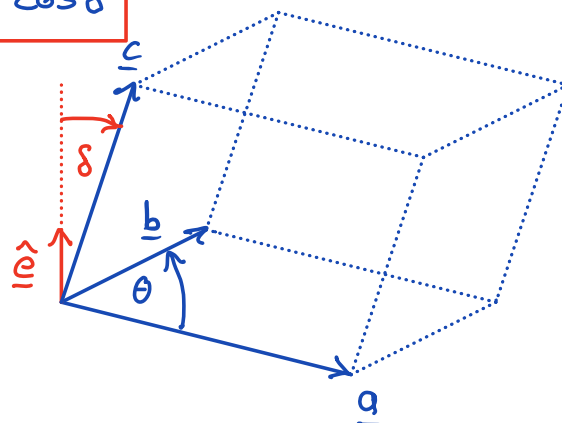
## Triple scalar product $\underline{a}, \underline{b}, \underline{c} \in V^3$

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = |\underline{a}| |\underline{b}| |\underline{c}| \sin \theta \cos \delta$$

$\theta$  angle from  $\underline{a}$  to  $\underline{b}$

$\hat{\underline{e}}$  right handed normal  
to  $\underline{a}$  and  $\underline{b}$

$\delta$  angle from  $\hat{\underline{e}}$  to  $\underline{c}$



$(\underline{a} \times \underline{b}) \cdot \underline{c} = 0 \Rightarrow \underline{a}, \underline{b}, \underline{c}$  linearly dependent

$(\underline{a} \times \underline{b}) \cdot \underline{c} > 0 \Rightarrow \underline{a}, \underline{b}, \underline{c}$  form right handed system

$(\underline{a} \times \underline{b}) \cdot \underline{c} < 0 \Rightarrow \underline{a}, \underline{b}, \underline{c}$  form left handed system

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = (\underline{b} \times \underline{c}) \cdot \underline{a} = (\underline{c} \times \underline{a}) \cdot \underline{b}$$

$\Rightarrow$  Volume of parallelepiped spanned by  $\underline{a}, \underline{b}, \underline{c}$

$$Q: (\underline{a} \times \underline{b}) \cdot \underline{c} \stackrel{?}{=} (\underline{b} \times \underline{a}) \cdot \underline{c}$$

## Triple vector product

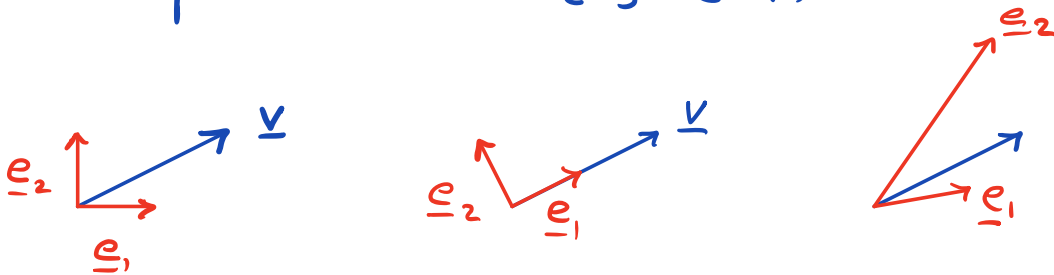
This may be new - we'll talk more about it later

$$\begin{aligned} (\underline{a} \times \underline{b}) \times \underline{c} &= (\underline{a} \cdot \underline{c}) \underline{b} - (\underline{b} \cdot \underline{c}) \underline{a} \\ \underline{a} \times (\underline{b} \times \underline{c}) &= (\underline{a} \cdot \underline{c}) \underline{b} - (\underline{a} \cdot \underline{b}) \underline{c} \end{aligned}$$

## Basis for a vector space

Def.: Basis for  $\mathcal{V}$  is a set of linearly independent vectors  $\{\underline{e}\}$  that span the space.

Examples in 2D:  $\{\underline{e}\} = \{\underline{e}_1, \underline{e}_2\}$



many choices  $\Rightarrow$  not unique

In this class we will always choose a right-handed orthonormal basis  $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

ortho-normal:  $\underline{e}_1 \times \underline{e}_2 = \underline{e}_3$ ,  $\underline{e}_2 \times \underline{e}_3 = \underline{e}_1$ ,  $\underline{e}_3 \times \underline{e}_1 = \underline{e}_2$

right handed:  $(\underline{e}_1 \times \underline{e}_2) \cdot \underline{e}_3 = 1$

$\Rightarrow$  called Cartesian reference frame

## Components of a vector in a basis

Project  $\underline{v}$  onto basis vectors to get components.

$$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3$$

$$v_1 = \underline{v} \cdot \underline{e}_1$$

$$v_2 = \underline{v} \cdot \underline{e}_2$$

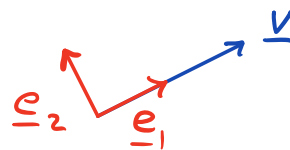
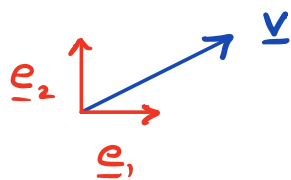
$$v_3 = \underline{v} \cdot \underline{e}_3$$

$$[\underline{v}] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Here  $[\underline{v}]$  is the representation of  $\underline{v}$  in  $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

The distinction between a vector and its representation is important for this class.

Example:



$$[\underline{v}] = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$[\underline{v}] = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}$$

$$|\underline{v}| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$|\underline{v}| = \sqrt{5^2 + 0^2} = \sqrt{5}$$

The vector is the same but representation is not.