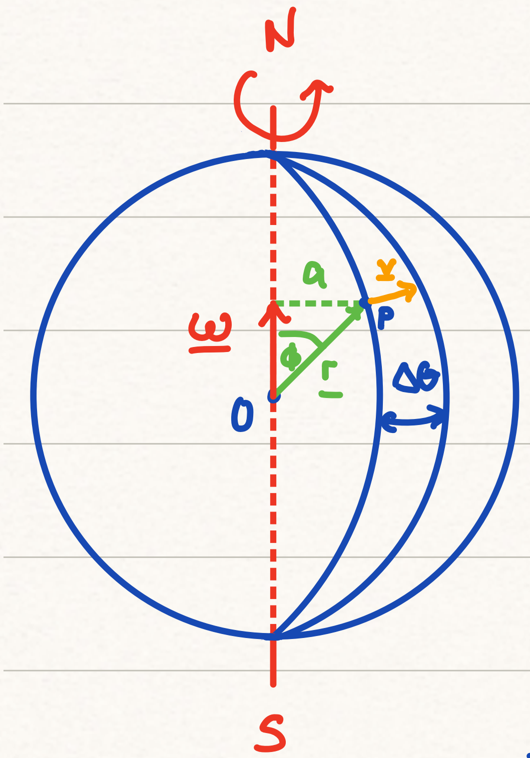
Angular Momentum and Torque

Rotational motion:



Angular velocticy: $\underline{\omega} = |\underline{\omega}|\,\hat{\underline{e}}_\omega$

$$|\omega| = \frac{d\theta}{dt}$$

Position vector: $\underline{r} = |\underline{r}|\,\hat{\underline{e}}_r$

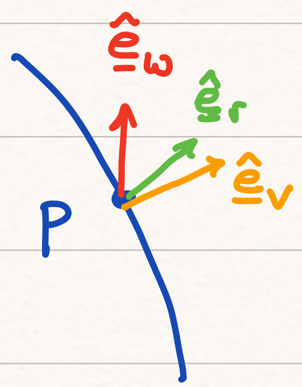
$$|r| = \overline{OP}$$

Velocity: $\underline{v} = |\underline{v}|\,\hat{\underline{e}}_v$

$$|\underline{v}| = |\underline{\omega}|\,a = |\underline{\omega}||\underline{r}|\sin\phi$$

$\Rightarrow\ \underline{v} = |\underline{\omega}||\underline{r}|\sin\phi\ \underline{e}_v$

local coord



$\hat{\underline{e}}_v \perp$ plane of $\hat{\underline{e}}_\omega$ and $\hat{\underline{e}}_r$

$\hat{e}_\omega \times \hat{\underline{e}}_r = \sin\phi\,\hat{\underline{e}}_v$

substitute

$$\underline{v} = |\underline{\omega}|\,|\underline{r}|\,\hat{\underline{e}}_\omega \times \hat{\underline{e}}_r$$

$$\underline{v} = (|\underline{\omega}|\,\hat{\underline{e}}_\omega) \times (|\underline{r}|\,\hat{\underline{e}}_r)$$

$$\boxed{\underline{v} = \underline{\omega} \times \underline{r}}$$

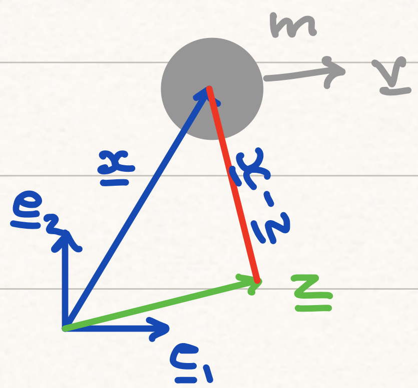
Example: Your current velocity in ATX (lat. ~30° N)

$\phi \approx 60° = \frac{\pi}{3} \approx 1.05$ $\qquad |r| = 6.37 \cdot 10^6\,\mathrm{m}$

$|\omega| = 7.3 \cdot 10^{-5}\,\frac{\mathrm{rad}}{\mathrm{sec}}$ $\qquad |v| = |\omega||r|\sin\phi \approx 403\,\frac{\mathrm{m}}{\mathrm{s}}$

Angular momentum & torque

$$\boxed{\begin{aligned} &\text{Linear momentum:} && \underline{l} = m\,\underline{v} \\ &\text{angular momentum:} && \underline{j} = (\underline{x} - \underline{z}) \times \underline{l} \end{aligned}}$$



relative to $\underline{z}$

force → linear momentum

torque → angular momentum

torque: $\underline{\tau} = \dfrac{d\underline{j}}{dt} = \dfrac{d}{dt}\left[(\underline{x} - \underline{z}) \times m\underline{v}\right]$

$$= m\frac{d}{dt}\left(\underline{x} \times \underline{v} - \underline{z} \times \underline{v}\right)$$

$\dot{\underline{x}} = \underline{v}$ $\qquad = m\left(\dot{\underline{x}} \times \underline{v} + \underline{x} \times \dot{\underline{v}} - \cancelto{0}{\dot{\underline{z}}} \times \underline{v} - \underline{z} \times \dot{\underline{v}}\right)$

$\underline{a} = \dot{\underline{v}}$ $\qquad = m\left(\cancelto{0}{\underline{v} \times \underline{v}} + \underline{x} \times \underline{a} - \underline{z} \times \underline{a}\right)$

$$\underline{\tau} = (\underline{x} - \underline{z}) \times m\underline{a}$$

$$\boxed{\underline{\tau} = (\underline{x} - \underline{z}) \times \underline{f}}$$

units of torque. $\left[FL = \dfrac{ML^2}{T^2}\right]$ or Nm

torque = moment of force or moment

Two basic relations:

1) ang. & lin. mom.: $\boxed{\underline{j} = (\underline{x} - \underline{z}) \times \underline{L}}$

2) torque & force: $\boxed{\underline{\tau} = (\underline{x} - \underline{z}) \times \underline{f}}$

Resultant torque due to

1) body force: $\underline{\tau}_b = \int_\Omega (\underline{x} - \underline{z}) \times \underline{b} \, dV$

2) surface force: $\underline{\tau}_s = \int_\Gamma (\underline{x} - \underline{z}) \times \underline{t}_n \, dA$

In rotation we have two important geometric locations in a body:

Center of volume: $\underline{x}_v = \frac{1}{V_B} \int_B \underline{x} \, dV$

Center of mass: $\underline{x}_m = \frac{1}{m_B} \int_B \rho(x) \underline{x} \, dV$

Note: $\rho = \text{const}$

$$\underline{x}_m = \frac{1}{m_\Omega} \int_\Omega \rho \underline{x} \, dV = \frac{\rho}{\rho V_\Omega} \int \underline{x} \, dV = \frac{1}{V_\Omega} \int_\Omega \underline{x} \, dV = \underline{x}_v$$

Gravitational body force

resultant around center of mass $\qquad \underline{x}_m = \text{const.}$

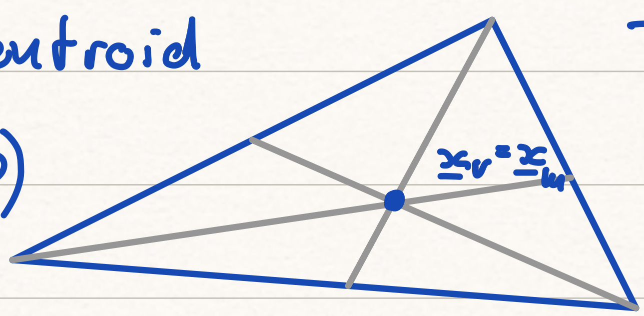
$\qquad \underline{g} = \text{const.}$

$$
\begin{aligned}
\underline{T}_b &= \int_B (\underline{x} - \underline{x}_m) \times \rho \underline{g} \, dV \\
&= \int_B \underline{x} \times \rho \underline{g} - \underline{x}_m \times \rho \underline{g} \, dV \\
&= \underbrace{\int_B \underline{x} \rho \, dV}_{m \underline{x}_m} \times \underline{g} - \underline{x}_m \times \underline{g} \underbrace{\int_B \rho \, dV}_{m} \\
&= m \underline{x}_m \times \underline{g} - \underline{x}_m \times m \underline{g} \\
&= m \left(\underline{x}_m \times \underline{g} - \underline{x}_m \times \underline{g} \right) = \underline{0}
\end{aligned}
$$

$\Rightarrow$ gravitational torque around center of mass vanishes

centroid

(2D)



Triangle can be suspended from its centroid in any orientation without inducing rotation.

Moment of gravity:

torque due to gravity around origin $\underline{z}=\underline{0}$

$$\boxed{\underline{\Gamma}_G = \int_B \underline{x} \times \rho_b \underline{g}\, dV} \qquad \underline{g} = -g\,\underline{e}_3$$

Simplify "moment of gravity"

$$\underline{\Gamma}_G = \int_B \underline{x} \times \rho_b \underline{g}\, dV = \int_B (\underline{x} - \underline{x}_m + \underline{x}_m) \times \rho_b \underline{g}\, dV$$

$$= \cancel{\int_B (\underline{x} - \underline{x}_m) \times \rho_b \underline{g}\, dV}^{\,\underline{0}} + \int_B \underline{x}_m \times \rho_b \underline{g}\, dV$$

$$= \underline{x}_m \times \underline{g} \underbrace{\int_B \rho_b\, dV}_{m} = \underline{x}_m \times m_b \underline{g}$$

$$\boxed{\underline{\Gamma}_G = \underline{x}_m \times m_b \underline{g}} \quad \text{Moment of Gravity}$$

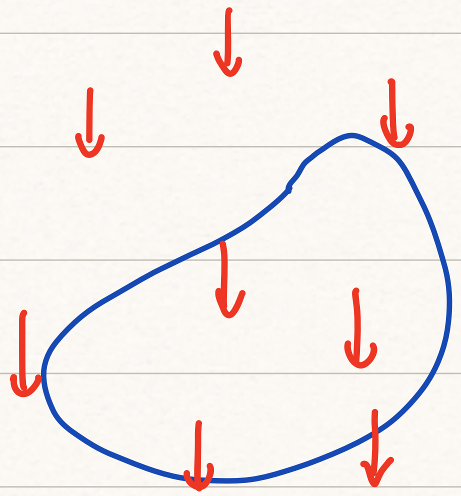
center of mass $\Rightarrow$ center of gravity

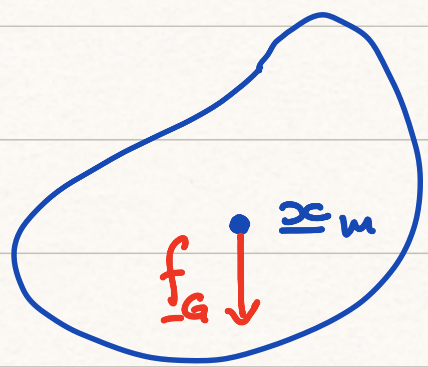
$$\underline{x}_m \equiv \underline{x}_G$$

⇒ Gravity acts on the center of mass (Center of mass theorem)

Continuum ⟶ Discrete

$\underline{b}_G(\underline{x}) = \rho_b(\underline{x})g$





force field can be represented as acting on a point where it does not induce a torque.

Hydrostatic surface force

Moment of buoyancy

Torque due to hydrostatic surface force around origin

$$\boxed{\underline{\tau}_B = \oint_{\partial B} \underline{x} \times (-p\hat{\underline{n}})\, dA} \qquad \underline{t}_n = -p\hat{\underline{n}}$$

Show that buoyancy acts on center of mass of displaced fluid $\quad \underline{x}_B = \frac{1}{V}\int_B \rho_f(\underline{x})\, dV$

$$\underline{\tau}_B = \oint_{\partial B} \underline{x}_V \times (-p\hat{\underline{n}})\, dA = 0$$

(requires some vector calculus $\Rightarrow$ later)

Use to simplify the moment of buoyancy

$$\Rightarrow \boxed{\underline{\tau}_B = -\underline{x}_V \times (m_f g)} \qquad \Rightarrow \text{HW 2}$$

assumes body is fully submerged and $\rho_f = \text{const.}$

Center of volume $\Rightarrow$ center of buoyancy

$$\underline{x}_V \equiv \underline{x}_B$$

Hydrostatic moment

force balance: $\underline{f}_H = \underline{f}_G + \underline{f}_B = (m_f - m_b)\, g\, \underline{e}_3$

similarly

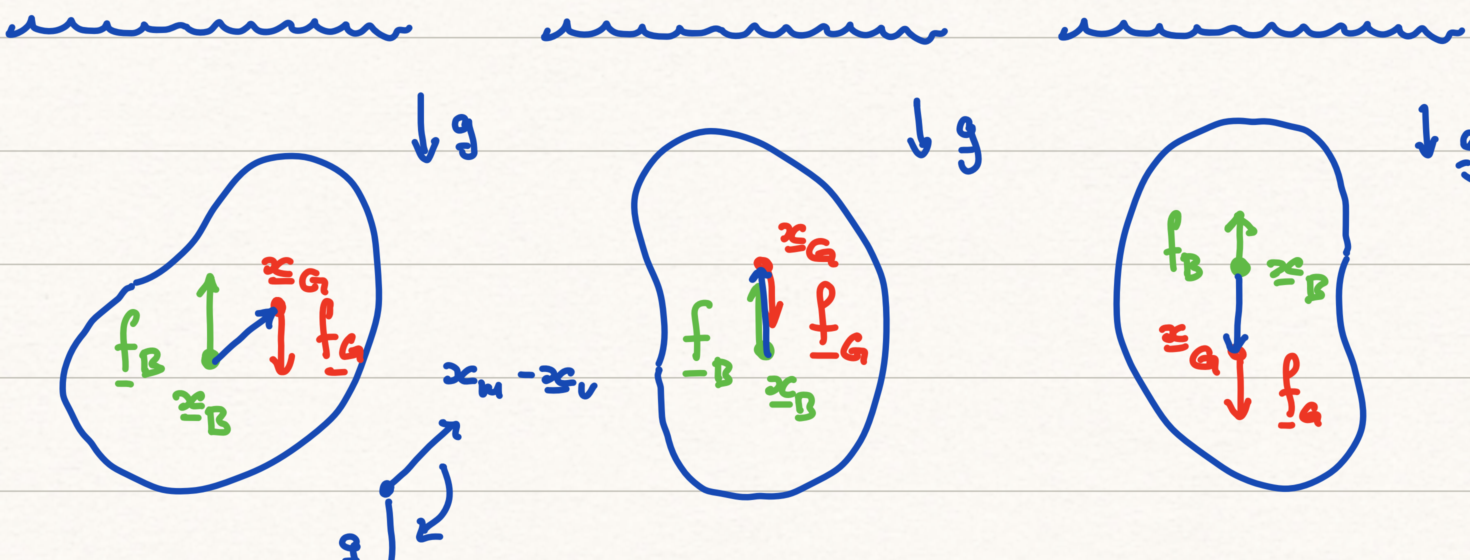
Torque balance: $\underline{\tau}_H = \underline{\tau}_G + \underline{\tau}_B$

$$= \underline{x}_G \times m_b \underline{g} - \underline{x}_B \times m_f \underline{g}$$

Neutrally boyant / floating: $\underline{f}_H = \underline{0} \Rightarrow m_b = m_f \equiv m$

Hydrostatic moment: $\boxed{\underline{\tau}_H = m\,(\underline{x}_G - \underline{x}_B) \times \underline{g}}$

Stability of fully submerged body



$\underline{\tau} \neq \underline{0}$	$\underline{\tau} = \underline{0}$	$\underline{\tau} = \underline{0}$
unstable	metastable	stable