Index notation

1) Dummy Indices

Given basis $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

$$\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3 = \sum_{i=1}^{3} a_i \underline{e}_i \equiv a_i \underline{e}_i$$

If an index is repeated twice in a term, summation is implied.
The repeated index is called a dummy index.

$\Rightarrow$ Einstein summation convention

$$\sum_{i=1}^{N} a_i b_i = a_i b_i$$

Note: Symbol for index does not matter

$$\underline{a} = a_i \underline{e}_i = a_k \underline{e}_k = a_q \underline{e}_q$$

$\Rightarrow$ we can rename dummy indices

2) Free indices

A free index occurs only once in a term.

Example: $a_i = c_j b_j b_i$ i = free index
j = dummy index

Short hand for the set of equations:

$$a_1 = \left(\sum_{j=1}^{3} c_j b_j\right) b_1, \quad a_2 = \left(\sum_{j=1}^{3} c_j b_j\right) b_2, \quad a_3 = \left(\sum_{j=1}^{3} c_j b_j\right) b_3$$

Basis: $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\} = \{\underline{e}_i\}$

Note:
- all terms must have same free indices
- there can be more than one free index
- same symbol cannot be used for dummy & free ind.
- dummy's can only be repeated twice

Why are these expressions meaning less?

1) $a_i = b_j$
2) $a_i b_j = c_i d_j d_j$
3) $a_i b_j = c_i c_k d_k d_j + d_p c_e c_e d_q$
4) $a_i = b_k c_k d_k e_i$

To express standard vector operations in index notation we need to introduce new symbols.

Kronecker delta

For any frame $\{\underline{e}_i\}$ we associate

$$\boxed{\delta_{ij} = \underline{e}_i \cdot \underline{e}_j = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i \neq j \end{cases}}$$

result of ortho normal basis

$$\delta_{ij} = \delta_{ji} \qquad \text{symmetry}$$

$$\underline{e}_i = \delta_{ij}\,\underline{e}_j \qquad \text{transfer property}$$

Example: Projection onto basis

$$\underline{u} \cdot \underline{e}_j = (u_i \underline{e}_i) \cdot \underline{e}_j = u_i\,(\underline{e}_i \cdot \underline{e}_j) = u_i\,\delta_{ij} = u_j$$

Example: Scalar Product

$$\underline{a} \cdot \underline{b} = (a_i \underline{e}_i) \cdot (b_j \underline{e}_j) = a_i\, b_j (\underline{e}_i \cdot \underline{e}_j)$$

$$= a_i\, b_j\, \delta_{ij} = a_i\, b_i = a_j\, b_j = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Kronecker delta expresses scalar product in index notation.

Permutation symbol (Levi-Civita)

To express the vector product we introduce

$$\boxed{\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \in \{123, 231, 312\} \text{ (even)} \\ -1 & \text{if } ijk \in \{321, 213, 132\} \text{ (odd)} \\ 0 & \text{repeated index} \end{cases}}$$

Flipping any two indices changes sign

$$\epsilon_{ijk} = -\epsilon_{kji} = -\epsilon_{ikj} = -\epsilon_{jik}$$

Invariant under cyclic permutation

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$$

Alternative definitions

$$\boxed{\begin{aligned} \epsilon_{ijk} &= (\underline{e}_i \times \underline{e}_j) \cdot \underline{e}_k \\ \epsilon_{ijk} &= \det([\underline{e}_i, \underline{e}_j, \underline{e}_k]) \end{aligned}}$$

For a orthonormal frame we have

$$\boxed{\underline{e}_i \times \underline{e}_j = \epsilon_{ijk} \underline{e}_k}$$

Vector product : $\underline{a} \times \underline{b} = \underline{c}$

$\underline{a} = a_i \underline{e}_i \quad \underline{b} = b_j \underline{e}_j \quad \underline{c} = c_k \underline{e}_k$

$$\underline{a} \times \underline{b} = (a_i \underline{e}_i) \times (b_j \underline{e}_j) = a_i b_j (\underline{e}_i \times \underline{e}_j)$$
$$= \underbrace{a_i b_j \epsilon_{ijk}}\, \underline{e}_k$$
$$= c_k \underline{e}_k$$

$$\boxed{c_k = a_i b_j \epsilon_{ijk}}$$

To express $(\underline{a} \times \underline{b}) \cdot \underline{c}$ in index notation

$$\left((a_i \underline{e}_i) \times (b_j \underline{e}_j)\right) \cdot (c_m \underline{e}_m) = a_i b_j c_m (\underline{e}_i \times \underline{e}_j) \cdot \underline{e}_m$$
$$= a_i b_j c_m \epsilon_{ijk} \underline{e}_k \cdot \underline{e}_m$$
$$= a_i b_j c_m \epsilon_{ijk} \delta_{km}$$
$$= a_i b_j c_k \epsilon_{ijk}$$

Frame identities

Summariz relations between basis vectors

$$\boxed{\underline{e}_i = \delta_{ij} \underline{e}_j} \quad \text{and} \quad \boxed{\underline{e}_i \times \underline{e}_j = \underline{e}_k}$$

consequence of orthonormal frame

Epsilon-delta identities

In a right-handed frame we have

$$\boxed{\begin{aligned}\epsilon_{pqs}\epsilon_{nrs} &= \delta_{pn}\delta_{qr} - \delta_{pr}\delta_{qn} \\ \epsilon_{pqs}\epsilon_{rqs} &= 2\,\delta_{pr}\end{aligned}}$$

Very helpful in establishing vector identities.

Example: $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c} \equiv \underline{d}$

$\underline{a} = a_q \underline{e}_q \qquad \underline{b} = b_i \underline{e}_i \qquad \underline{c} = c_j \underline{e}_j \qquad \underline{d} = d_p \underline{e}_p$

$\underline{b} \times \underline{c} = \epsilon_{ijk} b_i c_j \underline{e}_k$

$$a_q \underline{e}_q \times (\epsilon_{ijk} b_i c_j \underline{e}_k) = \epsilon_{ijk}\, a_q b_i c_j \,(\underline{e}_q \times \underline{e}_k)$$

$\underline{e}_q \times \underline{e}_k = \epsilon_{qkp} \underline{e}_p$ (frame identity)

$$= \epsilon_{ijk}\epsilon_{qkp}\, a_q b_i c_j \,\underline{e}_p \qquad \epsilon_{pqk} = \epsilon_{qkp}$$

$$= \epsilon_{ijk}\epsilon_{pqk}\, a_q b_i c_j \,\underline{e}_p$$

index notation for triple vector prod.

use $\epsilon\delta$ identity

$$= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\, a_q b_i c_j \underline{e}_p$$

First term: $\delta_{ip}\delta_{jq}\, a_q b_i c_j \underline{e}_p = a_q b_p c_q \underline{e}_p =$

$$= (a_q c_q)\, b_p \underline{e}_p$$

$$= (\underline{a} \cdot \underline{c})\, \underline{b}$$

Second term: $\delta_{iq}\delta_{jp}\, a_q b_i c_j \underline{e}_p = a_q b_q c_p \underline{e}_p$

$$= (\underline{a} \cdot \underline{b})\, \underline{c}$$

$$\Rightarrow \underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\, \underline{b} - (\underline{a} \cdot \underline{b})\, \underline{c}$$