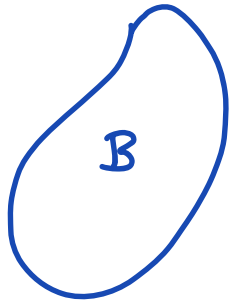


Continuum Mass and Force Concepts



Volume of a body B :

$$V_B = \int_B dV$$

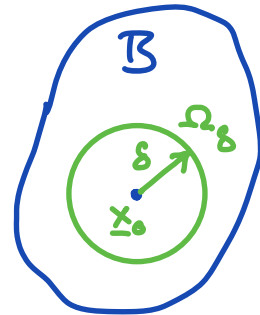
Mass of a body B :

$$m_B = \int_B \rho(\underline{x}) dV$$

$\rho(\underline{x})$ = mass density field

At any point \underline{x}_0 in B

$$\rho(\underline{x}_0) = \lim_{\delta \rightarrow 0} \frac{m_{\Omega_\delta}}{V_{\Omega_\delta}}$$



Important geometric quantities of a body are:

Center of volume :

$$\underline{x}_v = \frac{1}{V_B} \int_B \underline{x} dV$$

Center of mass :

$$\underline{x}_m = \frac{1}{m_B} \int_B \rho(\underline{x}) \underline{x} dV$$

Note: $\rho = \text{const}$

$$\underline{x}_m = \frac{1}{m_\Omega} \int_\Omega \rho \underline{x} dV = \frac{\rho}{\rho V_\Omega} \int_\Omega \underline{x} dV = \frac{1}{V_\Omega} \int_\Omega \underline{x} dV = \underline{x}_v$$

Important because resulting forces.

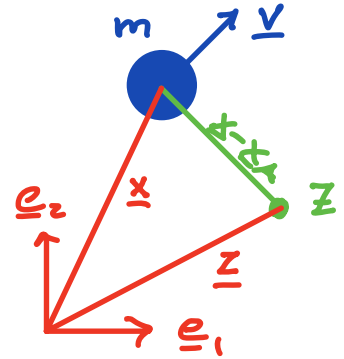
Short review of force and moment

Object with a mass m and velocity \underline{v} has a momentum:

Linear momentum: $\underline{L} = m \underline{v}$

angular momentum: $\underline{j} = (\underline{x} - \underline{z}) \times \underline{L}$

→ always relative to a point!



Newton's 1st law: "Principle of inertia"

In a fixed frame of reference every object preserves its state of motion unless it is acted upon by a force or torque.

Force: $\underline{f} = \frac{d\underline{L}}{dt} = \frac{d(m\underline{v})}{dt} = m \frac{d\underline{v}}{dt} = m \underline{a} \quad \frac{d\underline{v}}{dt} = \underline{\dot{v}}$

$$\left[\frac{ML}{T^2} \right] = N$$

$$\underline{f} = m \underline{a}$$

Newton's 2nd law

Torque: $\underline{\tau} = \frac{d\underline{j}}{dt} = m \frac{d}{dt} (\underline{x} \times \underline{v} - \underline{z} \times \underline{v}) =$

(moment of force) $= m (\underline{\dot{x}} \times \underline{v} + \underline{x} \times \underline{\dot{v}} - \underline{\dot{z}} \times \underline{v} - \underline{z} \times \underline{\dot{v}})$

$$\left[\frac{ML^2}{T^2} \right] = Nm$$

$$= m (\cancel{\underline{v} \times \underline{v}} + \underline{x} \times \underline{a} - \underline{z} \times \underline{a}) = m (\underline{x} - \underline{z}) \times \underline{a}$$

$$\underline{\tau} = (\underline{x} - \underline{z}) \times m \underline{a} = (\underline{x} - \underline{z}) \times \underline{f}$$

Body Forces

Any force that not due to physical contact is a body force and acts on the entire body.

Common body forces originate from gravitational and electromagnetic fields.

Example: gravitational body force

$$\underline{b}_g = \rho \underline{g} \quad \left[\frac{M}{L^3} \frac{L}{T^2} = \frac{M}{L^2 T^2} \right]$$

\Rightarrow body force field has units of $\frac{\text{force}}{\text{volume}}$

If a body force acts on a body B the net or resultant body force is:

$$\underline{F}_b[B] = \int_B \underline{b}(\underline{x}) dV \quad \text{units of force} \left[\frac{ML}{T^2} \right].$$

The net or resultant torque on a body about \underline{z}

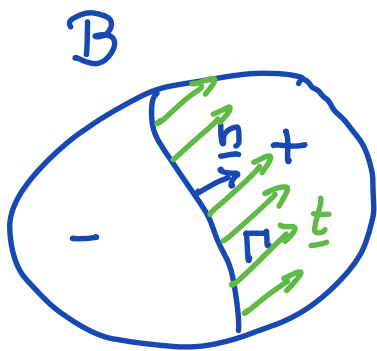
$$\underline{\tau}_b[B] = \int_B (\underline{x} - \underline{z}) \times \underline{b}(\underline{x}) dV$$

Surface/Contact Forces

arise due to the physical contact between bodies. Forces along imaginary surfaces within a body are called internal forces while forces along the bounding surface of a body are external.

Internal surface forces hold a body together. External surface forces describe the interaction with the environment.

Traction Field



Consider an arbitrary surface Γ in B with unit normal $\underline{n}(\underline{x})$ that defines the positive and negative sides of B .

The force per unit area exerted by material on the pos. side upon material on the neg. side is given by the traction field \underline{t}_n for Γ .

The resultant force due to a traction field on Γ is

$$\underline{F}_s[\Gamma] = \int_{\Gamma} \underline{t}_n(\underline{x}) dA$$

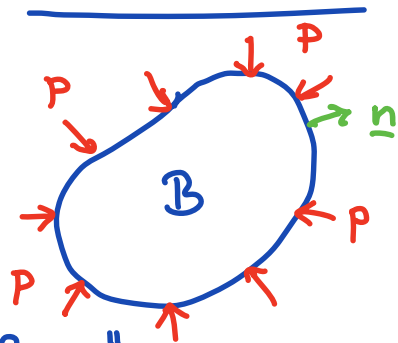
The resultant torque about point \underline{z} due to a traction field on Γ is

$$\underline{T}_s[\Gamma] = \int_{\Gamma} (\underline{x} - \underline{z}) \times \underline{t}_n(\underline{x}) dA$$

Example: Pressure, p , on submerged body

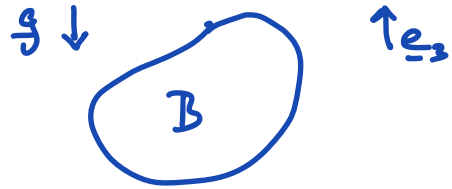
$$\underline{t} = -p \underline{n}$$

"Hydrostatic surface force"



Weight: Resultant gravitational body force

The weight of a body is the resultant force due to gravity.



$$\underline{f}_G = \underline{\tau}_b[B] = \int_B \rho_B \underline{g} dV$$

if $\rho_B = \text{const.}$ and $\underline{g} = \text{const.}$ $\underline{g} = -g \underline{e}_3$

$$\underline{f}_G = -\rho_B g \underline{e}_3 \int_B dV = -\rho_B g V_B \underline{e}_3 = -m_B g \underline{e}_3 = m_B \underline{g}$$

Acceleration of a free falling body in vacuum

$$\underline{f}_G = m_B \underline{a} \Rightarrow \underline{a} = \frac{1}{m_B} \underline{f}_G = -g \underline{e}_3$$

$$\underline{a} = -g \underline{e}_3$$

acceleration during free fall

is independent of mass (Galileo)

Q: Where on B does \underline{f}_G act?

Moment of Gravity

Resultant torque on body B about origin, $\underline{z} = \underline{0}$, due to gravitational body force:

$$\underline{\tau}_G = \underline{\tau}_b[B]_0 = \int_B \underline{x} \times \rho(\underline{x}) \underline{g} dV$$

$$\underline{g} = -g \underline{e}_3$$

Resultant torque around \underline{x}_m

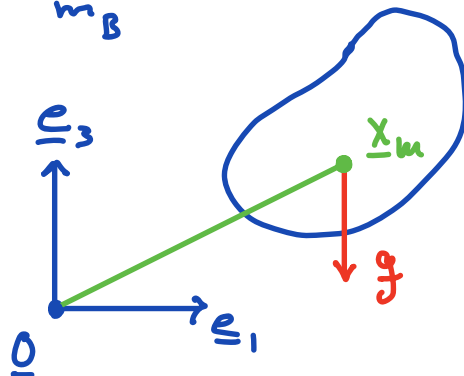
$$\begin{aligned}\underline{\tau}_b[B] &= \int_B (\underline{x} - \underline{x}_m) \times \rho \underline{g} dV \quad \text{note } \underline{x}_m = \text{const} \\ &= \int_B \underline{x} \times \rho \underline{g} - \underline{x}_m \times \rho \underline{g} dV \\ &= \underbrace{\int_B \underline{x} \rho dV}_{\underline{x}_m m_B} \times \underline{g} - \underline{x}_m \times \underline{g} \underbrace{\int_B \rho dV}_{m_B} \\ &= m_B \underline{x}_m \times \underline{g} - m_B \underline{x}_m \times \underline{g} = \underline{0}\end{aligned}$$

\Rightarrow gravitational torque around \underline{x}_m vanishes

Simplify "moment of gravity"

$$\begin{aligned}\underline{\tau}_G &= \int_B \underline{x} \times \rho \underline{g} dV = \int_B (\underline{x} - \underline{x}_m + \underline{x}_m) \times \rho \underline{g} dV \\ &= \int_B (\underline{x} - \underline{x}_m) \times \rho \underline{g} dV + \int_B \underline{x}_m \times \rho \underline{g} dV \\ &= \int_B \underline{x}_m \times \rho \underline{g} dV = \underline{x}_m \times \underline{g} \underbrace{\int_B \rho dV}_{m_B}\end{aligned}$$

\Rightarrow $\underline{\tau}_G = \underline{x}_m \times m_B \underline{g}$
moment of gravity
(torque about origin)



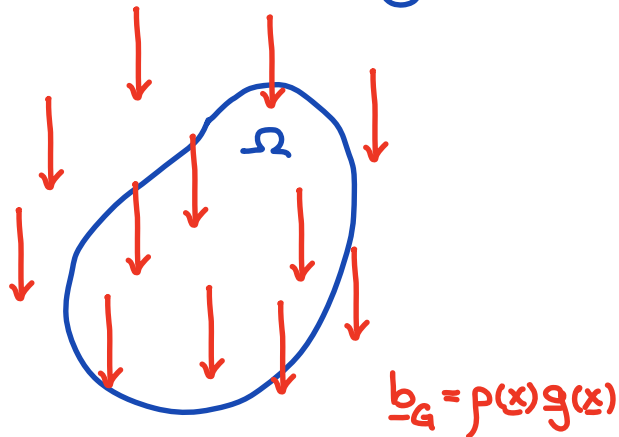
We say that: Gravity acts on the center of mass.

Because resultant torque about \underline{x}_m is zero.

\Rightarrow Center of Mass Theorem (prove it later)

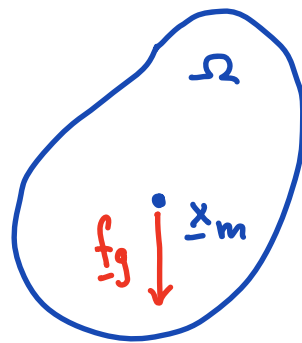
Provides the link between continuum & discrete!

Continuum system



gravitational body force field
acts everywhere

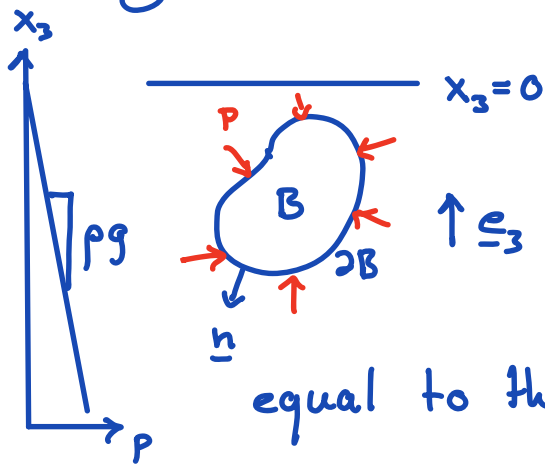
Discrete body



gravity vector \underline{f}_g
acts only on \underline{x}_m

\Rightarrow force field can be represented as acting on the point where it does not induce a torque.

Buoyancy: Resultant hydro static surface force



Any object, wholly or partially submerged in a fluid is bouyed up by a force

equal to the weight of the fluid displaced by the body (Archimedes principle).

Q: Is the buoyancy force a body or a surface force?

Hydrostatic pressure acts on the boundary of the object. \Rightarrow external surface force

Buoyancy force \rightarrow resultant surface force

$$\underline{f}_s[\partial B] = \underline{f}_B = W \underline{e}_3 = \rho_f g V_B \underline{e}_3 = -m_f g$$

ρ_f = fluid density

Hydrostatic pressure: $p = -\rho_f g x_3$

Hydrostatic traction on ∂B : $\underline{t} = -p \underline{n}$

Resulting surface force:

$$\underline{F}_s[\partial B] = \int_{\partial B} \underline{t} \, dA = - \int_{\partial B} p \, \underline{n} \, dA$$

need to convert this to volume integral

\Rightarrow Gradient theorem $\int_{\partial \Omega} \phi \, \underline{n} \, dA = \int_{\Omega} \nabla \phi \, dV \rightarrow \#W$

$$\Rightarrow \underline{F}_s[\partial B] = - \int_{\partial B} p \, \underline{n} \, dA = - \int_B \nabla p \, dV$$

where $\nabla p = \nabla(-\rho_f g x_3) = -\rho_f g \underline{e}_3$

$$\underline{F}_s[\partial B] = \int_B \rho_f g \underline{e}_3 \, dV = \rho_f g \underline{e}_3 \underbrace{\int_B dV}_{V_B} = \rho_f g V_B \underline{e}_3 \quad \checkmark$$

$$\underline{F}_s[\partial B] = m_f g \underline{e}_3 = - \underline{F}_b[B] \text{ of fluid}$$

Moment of Buoyancy

With arguments similar to those used for gravity, we can show that buoyancy force has zero resultant torque about

center of volume $\underline{x}_v = \frac{1}{V_B} \int_B \underline{x} dV$.

$$\underline{\tau}_B = -\underline{x}_v \times (m_f \underline{g}) = \underline{x}_v \times m_f g \underline{e}_3$$

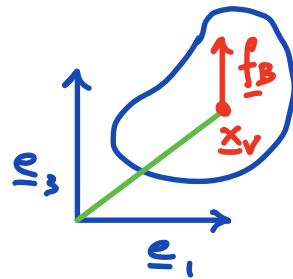
m_f = mass of displaced fluid

Buoyancy acts on center of volume.

\Rightarrow center of buoyancy

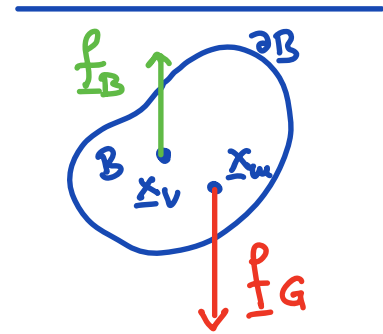
The implicit assumption is that $\rho_f = \text{const.}$

otherwise the buoyancy force acts on center of mass of displaced fluid (center of buoy.)



Hydrostatic force balance

Total resultant force \underline{f} on a submerged body in a gravitational field is the sum of weight and buoyancy.



$$\underline{f} = \underline{f}_G + \underline{f}_B = \underline{r}_b[B] + \underline{r}_s[\partial B]$$

$$= - \int_B \rho_b g \underline{e}_3 dV - \oint_{\partial B} p \underline{n} dS$$

substituting:

$$\underline{f} = \int_B (\rho_f - \rho_b) g \underline{e}_3 dV = (m_f - m_b) g \underline{e}_3$$

$\rho_f > \rho_b$: \underline{f} points up \rightarrow body rises (pos. buoyancy)

$\rho_f < \rho_b$: \underline{f} points down \rightarrow body sinks (neg. buoyancy)

$\rho_f = \rho_b$: $\underline{f} = \underline{0}$ \rightarrow body is neutrally buoyant

Note: The integrated expression assumes $g = \text{const.}$

Hydrostatic Moment of floating body

\underline{f}_G and \underline{f}_B act on different points

\Rightarrow induce a moment / net torque

$$\underline{\tau}_G = \underline{x}_m \wedge m_B \underline{g} \quad \text{and} \quad \underline{\tau}_B = -\underline{x}_v \wedge m_f \underline{g}$$

For floating body: $\underline{f} = (m_b - m_f) \underline{g} = 0$

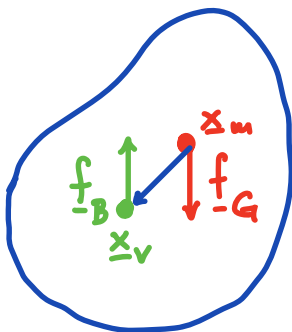
$$\Rightarrow m_b = m_f \equiv m$$

Total torque on body:

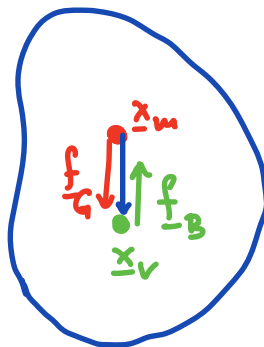
$$\underline{\tau} = \underline{\tau}_G + \underline{\tau}_B = \underline{x}_m \wedge m \underline{g} - \underline{x}_v \wedge m \underline{g}$$

$$\underline{\tau} = (\underline{x}_m - \underline{x}_v) \wedge m \underline{g}$$

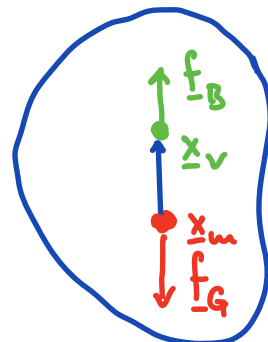
Stability of fully submerged body:



$\underline{\tau} \neq \underline{0}$
unstable



$\underline{\tau} = \underline{0}$
meta stable



$\underline{\tau} = \underline{0}$
stable