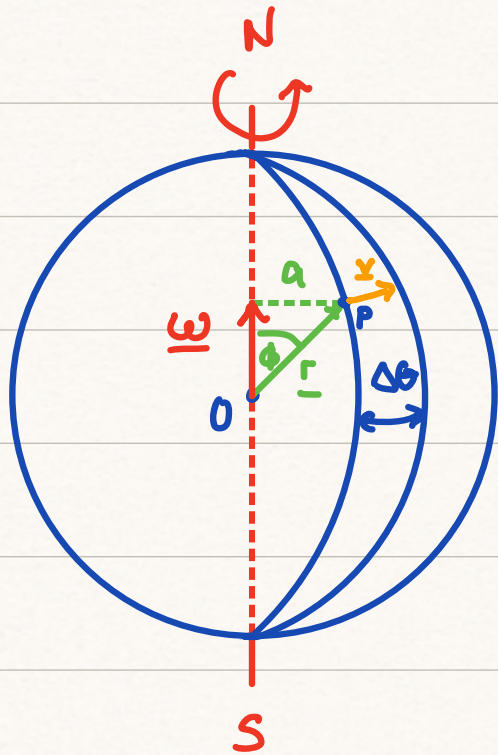# Angular Momentum and Torque

Rotational motion:

Angular velocity: $\underline{\omega} = |\underline{\omega}| \, \hat{\underline{e}}_\omega$

$$|\omega| = \frac{d\theta}{dt}$$

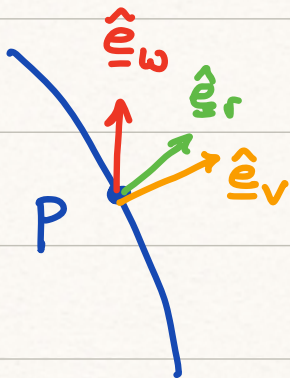Position vector: $\underline{r} = |\underline{r}| \, \hat{\underline{e}}_r$

$$|r| = \overline{OP}$$

Velocity: $\underline{v} = |\underline{v}| \, \hat{\underline{e}}_v$

$$|\underline{v}| = |\underline{\omega}| \, a = |\underline{\omega}||\underline{r}| \sin\phi$$

$$\Rightarrow \underline{v} = |\underline{\omega}||\underline{r}| \sin\phi \, \underline{e}_v$$

$\hat{\underline{e}}_v \perp$ plane of $\hat{\underline{e}}_\omega$ and $\hat{\underline{e}}_r$

$$\hat{e}_\omega \times \hat{\underline{e}}_r = \sin\phi \, \hat{\underline{e}}_v$$

substitute

$$\underline{v} = |\underline{\omega}| \, |\underline{r}| \, \hat{\underline{e}}_\omega \times \hat{\underline{e}}_r$$

$$\underline{v} = (|\underline{\omega}| \, \hat{\underline{e}}_\omega) \times (|\underline{r}| \, \hat{\underline{e}}_r)$$

$$\boxed{\underline{v} = \underline{\omega} \times \underline{r}}$$

local coord

Example: Your current velocity in ATX

$$\phi \approx 30° = \frac{\pi}{6} \approx 0.52 \qquad |r| = 6.37 \cdot 10^6 \, m$$

$$|\omega| = 7.3 \cdot 10^{-5} \, \frac{rad}{sec} \qquad |v| = |\omega||r| \sin\phi \approx 232 \, \frac{m}{s}$$
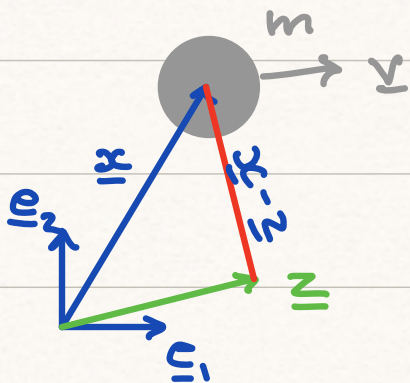
Angular momentum & torque

linear momentum: $\boxed{\underline{l} = m\underline{v}}$

angular momentum: $\boxed{\underline{j} = (\underline{x} - \underline{z}) \times \underline{l}}$

relative to $\underline{z}$

force $\rightarrow$ linear momentum

torque $\rightarrow$ angular momentum

torque: $\underline{\tau} = \dfrac{d\underline{j}}{dt} = \dfrac{d}{dt}\left[(\underline{x} - \underline{z}) \times m\underline{v}\right]$

$\qquad\qquad = m\dfrac{d}{dt}\left(\underline{x} \times \underline{v} - \underline{z} \times \underline{v}\right)$

$\dot{\underline{x}} = \underline{v}$

$\qquad\qquad = m\left(\dot{\underline{x}} \times \underline{v} + \underline{x} \times \dot{\underline{v}} - \dot{\underline{z}} \times \underline{v} - \underline{z} \times \dot{\underline{v}}\right)$

$\underline{a} = \dot{\underline{v}}$

$\qquad\qquad = m\left(\underline{v} \times \underline{v} + \underline{x} \times \underline{a} - \underline{z} \times \underline{a}\right)$

$\qquad\quad \underline{\tau} = (\underline{x} - \underline{z}) \times m\underline{a}$

$\boxed{\underline{\tau} = (\underline{x} - \underline{z}) \times \underline{f}}$

units of torque. $\left[FL = \dfrac{ML^2}{T^2}\right]$ or Nm

torque = moment of force or moment

Two basic relations:

1) ang. & lin. mom..:

2) torque & force :

$$\underline{j} = (\underline{x} - \underline{z}) \times \underline{L}$$

$$\underline{T} = (\underline{x} - \underline{z}) \times \underline{f}$$

Resultant torque due to

1) body force :

$$\underline{T}_b = \int_{\Omega} (\underline{x} - \underline{z}) \times \underline{b} \, dV$$

2) surface force :

$$\underline{T}_s = \int_{\Gamma} (\underline{x} - \underline{z}) \times \underline{t}_n \, dA$$

In rotation we have two important geometric locations in a body:

Center of volume :

$$\underline{x}_v = \frac{1}{V_B} \int_B \underline{x} \, dV$$

Center of mass :

$$\underline{x}_m = \frac{1}{m_B} \int_B \rho(x) \underline{x} \, dV$$

Note: $\rho = const$

$$\underline{x}_m = \frac{1}{m_\Omega} \int_\Omega \rho \, \underline{x} \, dV = \frac{\rho}{\rho V_\Omega} \int \underline{x} \, dV = \frac{1}{V_\Omega} \int_\Omega \underline{x} \, dV = \underline{x}_v$$

Gravitational body force

resultant around center of mass $\quad \underline{x}_m = $ const.

$$\underline{I}_b = \int_B (\underline{x} - \underline{x}_m) \times \rho \underline{g} \, dV \qquad\qquad \underline{g} = \text{const.}$$
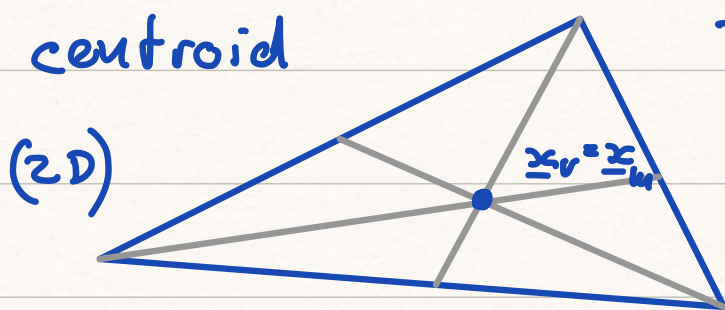
$$= \int_B \underline{x} \times \rho \underline{g} - \underline{x}_m \times \rho \underline{g} \, dV$$

$$= \underbrace{\int_B \underline{x} \rho \, dV}_{m \underline{x}_m} \times \underline{g} - \underline{x}_m \times \underline{g} \underbrace{\int_B \rho \, dV}_{m}$$

$$= m \underline{x}_m \times \underline{g} - \underline{x}_m \times m \underline{g}$$

$$= m \left(\underline{x}_m \times \underline{g} - \underline{x}_m \times \underline{g} \right) = \underline{0}$$

$\Rightarrow$ gravitational torque around
center of mass vanishes

centroid

(2D)

$\underline{x}_v = \underline{x}_m$

Triangle can be suspended
from its centroid in
any orientation without
inducing rotation.

Moment of gravity:

torque due to gravity around origin $\underline{z} = \underline{0}$

$$\boxed{\underline{T}_G = \int_B \underline{x} \times \rho_b \underline{g} \, dV} \qquad \underline{g} = -g\underline{e}_3$$

Simplify "moment of gravity"

$$\underline{T}_G = \int_B \underline{x} \times \rho_b \underline{g} \, dV = \int_B (\underline{x} - \underline{x}_m + \underline{x}_m) \times \rho_b \underline{g} \, dV$$

$$= \int_B (\underline{x} - \underline{x}_m) \times \rho_b \underline{g} \, dV \overset{\circ}{+} \int_B \underline{x}_m \times \rho_b \underline{g} \, dV$$

$$= \underline{x}_m \times \underline{g} \underbrace{\int_B \rho_b \, dV}_{m} = \underline{x}_m \times m_b \underline{g}$$

$$\boxed{\underline{T}_G = \underline{x}_m \times m_b \underline{g}} \qquad \text{Moment of Gravity}$$

center of mass $\Rightarrow$ center of gravity

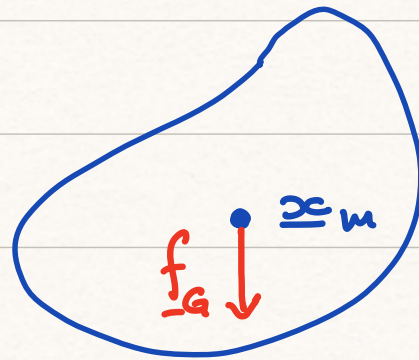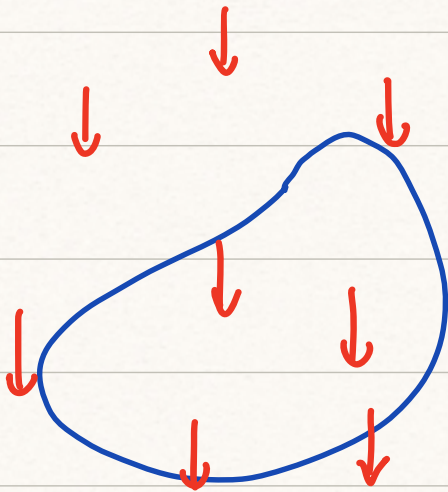$$\underline{x}_m \equiv \underline{x}_G$$

$\Rightarrow$ Gravity acts on the center of mass
  (Center of mass theorem)

Continuum $\longrightarrow$ Discrete

$$\underline{b}_G(\underline{x}) = \rho_b(\underline{x})\underline{g}$$

force field can be represented as acting on a
point where it does not induce a torque.

Hydrostatic surface force

Moment of buoyancy

Torque due to hydrostatic surface force around origin

$$\underline{T}_B = \oint_{\partial B} \underline{x} \times (-p\,\hat{\underline{n}})\, dA \qquad \underline{t}_n = -p\,\hat{\underline{n}}$$

Show that buoyancy acts on center of mass
of displaced fluid $\qquad \underline{x}_B = \frac{1}{V} \int_B \rho_f(\underline{x})\, dV$

$$\underline{T}_B = \oint_{\partial B} \underline{x}_v \times (-p\,\hat{\underline{n}})\, dA = 0$$

(requires some vector calculus $\Rightarrow$ later)

Use to simplify the moment of buoyancy

$$\Rightarrow \qquad \underline{T}_B = -\underline{x}_v \times (m_f\, g) \qquad\qquad \Rightarrow HW2$$

assumes body is fully submerged and $\rho_f = $ const.

Center of volume $\Rightarrow$ center of buoyancy

$$\underline{x}_v = \underline{x}_B$$

Hydrostatic moment

force balance: $\underline{f}_H = \underline{f}_G + \underline{f}_B = (m_f - m_b) g \underline{e}_3$

similarly

Torque balance: $\underline{\tau}_H = \underline{\tau}_G + \underline{\tau}_B$

$$= \underline{x}_m \times m_b \, \underline{g} - \underline{x}_v \times m_f \, \underline{g}$$

Neutrally boyant / floating: $\underline{f}_H = \underline{0} \Rightarrow m_b = m_f \equiv m$

Hydrostatic moment: $\boxed{\underline{\tau}_H = m(\underline{x}_m - \underline{x}_v) \times \underline{g}}$

Stability of fully submerged body

$\underline{\tau} \ne \underline{0}$

unstable

$\underline{\tau} = \underline{0}$

metastable

$\underline{\tau} = \underline{0}$

stable