Constitutive Theory

Common constitutive laws:

Newtonian fluid: $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \eta\,(\nabla\underline{v} + \nabla\underline{v}^T)$

$p = -\frac{1}{3}\,\mathrm{tr}(\underline{\underline{\sigma}})$ η = viscosity $\underline{v}$ = velocity

Linear elastic solid: $\underline{\underline{\sigma}} = \lambda\,\nabla\cdot\underline{u}\,\underline{\underline{I}} + \mu\,(\nabla\underline{u} + \nabla\underline{u}^T)$

λ, μ = lame parameters $\underline{u}$ = displacement

Both derive from the functional form

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\,\underline{\underline{A}} = \lambda\,\mathrm{tr}(\underline{\underline{A}}) + 2\mu\,\mathrm{sym}(\underline{\underline{A}})}$$

Newtonian fluid: $\underline{\underline{A}} = \nabla\underline{v}$

Linear elastic solid: $\underline{\underline{A}} = \nabla\underline{u}$

remember $\nabla\cdot\underline{a} = \mathrm{tr}(\nabla\underline{a})$

$\Rightarrow$ direct for lin. elastic solid

for fluid there is a complication due to incompressibility !

Why do const. relations have this form ?

Change of observer

In Lecture 6 we discussed Change in basis

$$\underline{v} = \underline{\underline{Q}}\,\underline{v}' \quad \text{and} \quad \underline{\underline{S}} = \underline{\underline{Q}}\,\underline{\underline{S}}'\,\underline{\underline{Q}}^T$$

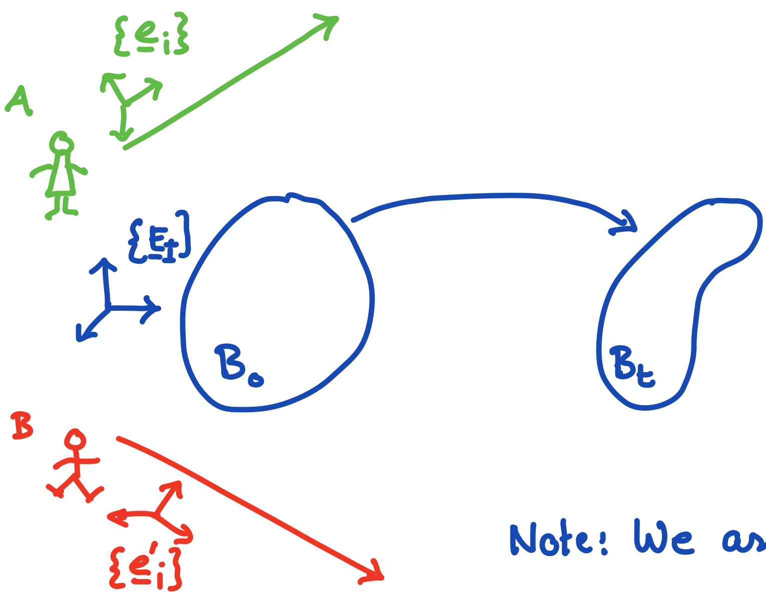
where $\underline{\underline{Q}}$ is change in basis tensor.

Q is a rotation : 1) orthonormal $\underline{\underline{Q}}\,\underline{\underline{Q}}^T = \underline{\underline{Q}}^T\underline{\underline{Q}} = \underline{\underline{I}}$

2) $\det(\underline{\underline{Q}}) = 1$

Change in basis is passive change of frame.

Active change in frame → change in observer



$$\underline{x} = \varphi(\underline{X}, t)$$

$$\underline{x}' = \varphi'(\underline{X}, t)$$

Note: Material ref. frame is common

Note: We assume both observers use same clock.

Since change in observer cannot induce a deformation. Two ref. frames must be related by a rigid body motion.

$$\boxed{\underline{x}' = Q(t)\,\varphi(\underline{X},t) + \underline{c}(t)} \quad \text{Eulerian transformation}$$

$\underline{\underline{Q}}$ = rotation $\qquad \underline{c}$ = translation

Our description of forces and deformations cannot depend on the observer (objective).

Effect on kinematic quantities

$\underline{x} = \varphi(\underline{X},t) \qquad \nabla\varphi = \underline{\underline{F}}$

$\underline{x}' = \varphi'(\underline{X},t) = Q\,\varphi(\underline{X},t) + c \qquad \nabla\varphi' = \underline{\underline{Q}}\,\underline{\underline{F}} = \underline{\underline{F}}'$

Right Cauchy-Green Strain tensor

$$\underline{\underline{C}}' = \underline{\underline{F}}'^{T}\,\underline{\underline{F}}' = (\underline{\underline{Q}}\,\underline{\underline{F}})^{T}(\underline{\underline{Q}}\,\underline{\underline{F}}) = \underline{\underline{F}}^{T}\underline{\underline{Q}}^{T}\underline{\underline{Q}}\,\underline{\underline{F}} = \underline{\underline{F}}^{T}\underline{\underline{F}} = \underline{\underline{C}}$$

$\Rightarrow$ not affected by rigid body motion because it is a material tensor C_{IJ} (naturally objective)

What about spatial tensors?

Axiom of frame indifference

Fields ϕ, $\underline{w}$ and $\underline{\underline{S}}$ are called frame indifferent or objective if for all superposed rigid body motions $\underline{x}' = \underline{\underline{Q}}\,\underline{x} + \underline{c}$ we have for all spatial fields

$$\boxed{\begin{aligned} \phi'(\underline{x}',t) &= \phi(\underline{x},t) \\ \underline{w}'(\underline{x}',t) &= \underline{\underline{Q}}\;\underline{w}(\underline{x},t) \\ \underline{\underline{S}}'(\underline{x}',t) &= \underline{\underline{Q}}\;\underline{\underline{S}}(\underline{x},t)\underline{\underline{Q}}^T \end{aligned}}
\begin{aligned} &\text{scalar field} \\ &\text{vector field} \\ &\text{tensor field} \end{aligned}$$

$\Rightarrow$ from Lecture 6.

Is spatial velocity gradient objective?

From lecture 16: $\underline{\underline{\ell}} = \nabla_x \underline{v} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$

$\underline{\underline{F}}' = \underline{\underline{Q}}\,\underline{\underline{F}}$ $\qquad\qquad$ $\underline{\underline{\ell}}' = \nabla_{x'}\underline{v}' = \dot{\underline{\underline{F}}}'\,\underline{\underline{F}}'^{-1}$

$$\dot{\underline{\underline{F}}}' = \frac{d}{dt}(\underline{\underline{Q}}\,\underline{\underline{F}}) = \underline{\underline{Q}}\,\dot{\underline{\underline{F}}} + \dot{\underline{\underline{Q}}}\,\underline{\underline{F}}$$

$$\underline{\underline{F}}'^{-1} = (\underline{\underline{Q}}\,\underline{\underline{F}})^{-1} = \underline{\underline{F}}^{-1}\underline{\underline{Q}}^{-1} = \underline{\underline{F}}^{-1}\underline{\underline{Q}}^T$$

$$\underline{\underline{\ell}}' = \dot{\underline{\underline{F}}}'\,\underline{\underline{F}}'^{-1} = (\underline{\underline{Q}}\,\dot{\underline{\underline{F}}} + \dot{\underline{\underline{Q}}}\,\underline{\underline{F}})\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T$$

$$= \underline{\underline{Q}}\,\dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underline{\underline{F}}\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T = \underline{\underline{Q}}\,\underline{\underline{\ell}}\,\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T$$

$\Rightarrow \underline{\underline{\ell}}' = \underline{\underline{Q}}\,\underline{\underline{\ell}}\,\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T$ <u>not</u> objective!

that is why $\nabla_x \underline{v}$ is not used in constitutive laws

The "non-objective" term is $\underline{\underline{\Omega}} = \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T$

it represents rigid body angular velocity between observers. see HW9

Show $\underline{\underline{\Omega}} = -\underline{\underline{\Omega}}^T$ skew-symmetric

Non-objetive part of $\underline{\underline{\ell}} = \nabla_x \underline{v}$ is skew-sym.

$\Rightarrow$ simply take symmetric part of $\underline{\underline{\ell}}$!

$\underline{\underline{d}} = \text{sym}(\underline{\underline{\ell}}) = \frac{1}{2}(\nabla_x \underline{v} + \nabla_x \underline{v}^T)$

rate of strain tensor is objective

$\Rightarrow$ used in constitutive laws

Note that velocity itself

Material frame indifferent functions

Fields: $\phi(\underline{x},t)$ scalar

$\underline{w}(\underline{x},t)$ vector

$\underline{\underline{\sigma}}(\underline{x},t)$ tensor

fields because they depend on $\underline{x}$.

Constitutive functions are not fields but they depend on fields as input.

internal energy: $u(\underline{x},t) = \hat{u}(\underbrace{\rho(\underline{x},t), \theta(\underline{x},t)}_{\text{input fields}})$

output field

constitutive function

heat flow: $\underline{q}(\underline{x},t) = \hat{\underline{q}}(\theta(\underline{x},t))$

Cauchy stress: $\underline{\underline{\sigma}}(\underline{x},t) = \hat{\underline{\underline{\sigma}}}(\rho(\underline{x},t), \theta(\underline{x},t), \underline{\underline{d}}(\underline{x},t))$

Constitutive functions: $\hat{u}(\rho,\theta)$, $\hat{\underline{q}}(\theta)$, $\hat{\underline{\underline{\sigma}}}(\rho,\theta,\underline{\underline{d}})$

As such constitutive functions are not directly dependent on frame but their input fields are.

Consider frames $\{\underline{e}_i\}$ and $\{\underline{e}_i'\}$ then to be frame indifference requires

$$\hat{\underline{\underline{\sigma}}}(\rho', \theta', \underline{\underline{d}}') = \underline{\underline{Q}} \; \hat{\underline{\underline{\sigma}}}(\rho, \theta, \underline{\underline{d}}) \; \underline{\underline{Q}}^T$$

substituting $\underline{\underline{d}}' = \underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T$

$$\boxed{\hat{\underline{\underline{\sigma}}}(\rho', \underline{\theta}', \underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}} \; \hat{\underline{\underline{\sigma}}}(\rho, \theta, \underline{\underline{d}}) \, \underline{\underline{Q}}^T}$$

$\Rightarrow$ both input & output of constitutive function $\hat{\underline{\underline{\sigma}}}$ must be frame invariant

Isotropic functions

Functions that are frame invariant are called isotropic. Consider the following

$\hat{\phi}$ = scalar fun. $\quad \hat{\underline{w}}$ = vector fun. $\quad \hat{\underline{\underline{S}}}$ = tensor fun.

θ = scalar $\quad \underline{v}$ = vector $\quad \underline{\underline{S}}$ = tensor

Then for two frames related by rigid body rotation $\underline{\underline{Q}}$ we have following isotropic functions:

$$\boxed{\begin{array}{lll}
\hat{\phi}(\theta) = \hat{\phi}(\theta) & \hat{\phi}(\underline{\underline{Q}}\,\underline{v}) = \hat{\phi}(\underline{v}) & \hat{\phi}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \hat{\phi}(\underline{\underline{S}}) \\
\hat{\underline{w}}(\theta) = \underline{\underline{Q}}\,\hat{\underline{w}}(\theta) & \hat{\underline{w}}(\underline{\underline{Q}}\,\underline{v}) = \underline{\underline{Q}}\,\hat{\underline{w}}(\underline{v}) & \hat{\underline{w}}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\hat{\underline{w}}(\underline{\underline{S}}) \\
\hat{\underline{\underline{S}}}(\theta) = \underline{\underline{Q}}\,\hat{\underline{\underline{S}}}(\theta)\,\underline{\underline{Q}}^T & \hat{\underline{\underline{S}}}(\underline{\underline{Q}}\,\underline{v}) = \underline{\underline{Q}}\,\hat{\underline{\underline{S}}}(\underline{v})\,\underline{\underline{Q}}^T & \hat{\underline{\underline{S}}}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\hat{\underline{\underline{S}}}(\underline{\underline{S}})\,\underline{\underline{Q}}^T
\end{array}}$$

Examples:

1) $\hat{\phi}(\underline{\underline{S}}) = \det(\underline{\underline{S}})$

$$\hat{\phi}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \det(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \det(\underline{\underline{Q}})\det(\underline{\underline{S}})\det(\underline{\underline{Q}}^T) = \det(\underline{\underline{S}}) \checkmark$$

2) $\hat{\underline{u}}(\underline{v}, \underline{\underline{A}}) = \underline{\underline{A}}\,\underline{v}$

$$\hat{\underline{u}}(\underline{\underline{Q}}\,\underline{v}, \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T\,\underline{\underline{Q}}\,\underline{v} = \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{v} = \underline{\underline{Q}}\,\hat{\underline{u}}(\underline{v}, \underline{\underline{A}}) \checkmark$$

Representation of isotropic tensor functions

An isotropic function $\underline{\underline{G}}(\underline{\underline{A}}) : \mathcal{V}^2 \to \mathcal{V}^2$ that maps symmetric tensors to symmetric tensors must have the following form

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \alpha_0(I_A)\,\underline{\underline{I}} + \alpha_1(I_A)\,\underline{\underline{A}} + \alpha_2(I_A)\,A^2}$$

Rivlin-Ericksen representation Thm

where α_0, α_1 and α_2 are functions of the set of pricipal invariants of $\underline{\underline{A}}$, $I_A = \{I_1(\underline{\underline{A}}), I_2(\underline{\underline{A}}), I_3(\underline{\underline{A}})\}$

- $\underline{\underline{G}}$ is clearly sym. if $\underline{\underline{A}}$ is sym.
- To see $\underline{\underline{G}}$ is isotropic assume $\alpha_0, \alpha_1, \alpha_2 = \text{const}$

$$\underline{\underline{G}}(\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T) = \alpha_0 \underline{\underline{I}} + \alpha_1 \underline{\underline{Q}}\,\underline{\underline{A}}\,Q^T + \alpha_2 \underline{\underline{Q}}\,\underline{\underline{A}}\,\underbrace{\underline{\underline{Q}}^T\underline{\underline{Q}}}_{\underline{\underline{I}}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T$$

$$= \alpha_0 \underline{\underline{I}} + \alpha_1 \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T + \alpha_2 \underline{\underline{Q}}\,\underline{\underline{A}}^2\underline{\underline{Q}}^T$$

$$\underline{\underline{Q}}\,\underline{\underline{G}}(\underline{\underline{A}})\underline{\underline{Q}}^T = \alpha_0 \underline{\underline{Q}}\,\underline{\underline{Q}}^T + \alpha\,\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T + \alpha\,\underline{\underline{Q}}\,\underline{\underline{A}}^2\underline{\underline{Q}}^T$$

$$\Rightarrow \quad \underline{\underline{G}}(\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\underline{\underline{G}}(\underline{\underline{A}})\underline{\underline{Q}}^T$$

isotropic for constant coefficients.

If coefficents $\alpha_0, \alpha_1, \alpha_2$ only depend on the invariants of $\underline{\underline{A}}$, then $\underline{\underline{G}}$ remains isotropic.

This is the most general form of a constitutive eqn for an isotropic material.

Isotropic Fourth-Order Tensors

If $\underline{\underline{G}}(\underline{\underline{A}})$ is a linear function then it can be written as

$$\underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\,\underline{\underline{A}}$$

where $\mathbb{C}$ is a fourth-order tensor.

If in addition we require:

1) $\mathbb{C}\underline{\underline{A}} \in \mathcal{V}^2$ is symmetric for every symmetric $\underline{\underline{A}} \in \mathcal{V}^2$

2) $\mathbb{C}\,\underline{\underline{W}} = \underline{\underline{0}}$ for every skew-symmetric $\underline{\underline{W}} \in \mathcal{V}^2$

Then there are scalars μ and λ such that

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\,\underline{\underline{A}} = \lambda\,\mathrm{tr}(\underline{\underline{A}})\,\underline{\underline{I}} + 2\mu\,\mathrm{sym}(\underline{\underline{A}})} \quad \text{for all } \underline{\underline{A}} \in \mathcal{V}^2$$

This follows from the representation Thm

$$\underline{\underline{G}}(\underline{\underline{A}}) = \alpha_0(\mathcal{I}_A)\,\underline{\underline{I}} + \alpha_1(\mathcal{I}_A)\,\underline{\underline{A}} + \alpha_2(\mathcal{I}_A)\,\underline{\underline{A}}^2$$

where the set of invariants of $\underline{\underline{A}}$ is

$$\mathcal{I}_A = \left\{ \mathrm{tr}\underline{\underline{A}},\ \tfrac{1}{2}\left[(\mathrm{tr}\underline{\underline{A}})^2 - \mathrm{tr}(\underline{\underline{A}}^2)\right],\ \det\underline{\underline{A}} \right\} \quad \text{(Lecture 6)}$$

Note that only $\mathrm{tr}(\underline{\underline{A}})$ is linear function!

since $\underline{\underline{G}}(\underline{\underline{A}})$ is linear in $\underline{\underline{A}}$ the only possibilities are

$\alpha_0(I_A) = c_0 \,\mathrm{tr}\,\underline{\underline{A}} + c_1$, $\alpha_1(I_A) = c_2$ and $\alpha_2(I_A) = 0$

where c_0, c_1 and c_2 are scalar constants.

Since $\underline{\underline{G}}(\underline{\underline{0}}) = \underline{\underline{0}} \Rightarrow c_1 = 0$

Hence setting $c_0 = \lambda$ and $c_2 = 2\lambda$

$$\underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\,\underline{\underline{A}} = \lambda\,\mathrm{tr}(\underline{\underline{A}}) + 2\mu\,\underline{\underline{A}}$$

since $\underline{\underline{G}}(\underline{\underline{W}}) = 0$ and $\mathrm{tr}(\underline{\underline{W}}) = 0$

$$\Rightarrow \underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\,\underline{\underline{A}} = \lambda\,\mathrm{tr}\,\underline{\underline{A}} + 2\mu\,\mathrm{sym}\,\underline{\underline{A}} \checkmark$$