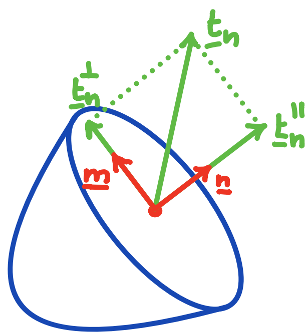
# Normal and Shear Stresses




Consider an arbitrary surface in B with normal $\underline{n}$. Then we have the two projection matrices

$$\underline{\underline{P}}^{\parallel} = \underline{n} \otimes \underline{n} \quad \text{and} \quad \underline{\underline{P}}^{\perp} = \underline{\underline{I}} - \underline{n} \otimes \underline{n} = \underline{m} \otimes \underline{m}$$

that define the

normal stress: $\underline{t}_n^{\parallel} = \underline{\underline{P}}^{\parallel} \underline{t}_n = (\underline{n} \cdot \underline{t}_n)\, \underline{n} = \sigma_n \underline{n}$

shear stress: $\underline{t}_n^{\perp} = \underline{\underline{P}}^{\perp} \underline{t}_n = (\underline{m} \cdot \underline{t}_n)\underline{m} = \tau\, \underline{m}$

The magnitudes of these stresses are

$$\sigma_n = \underline{n} \cdot \underline{t}_n = \underline{n} \cdot \underline{\underline{\sigma}}\, \underline{n} \quad \text{or} \quad \sigma_n = n_i\, \sigma_{ij}\, n_j$$

$$\tau = \underline{m} \cdot \underline{t}_n = \underline{m} \cdot \sigma\, \underline{n} \quad \text{or} \quad \tau = m_i\, \sigma_{ij}\, n_j$$

If $\sigma_n > 0$ the normal stresses are tensile if $\sigma_n < 0$ the normal stresses are compressive.

From geometry: $\underline{t}_n = \underline{t}_n^{\parallel} + \underline{t}_n^{\perp}$

$$|\underline{t}_n|^2 = |\sigma_n \underline{n}|^2 + |\tau\, \underline{n}|^2 = \sigma_n^2 + \tau_n^2$$

# Extremal Stress Values

## I) Maximum and Minimum Normal Stresses

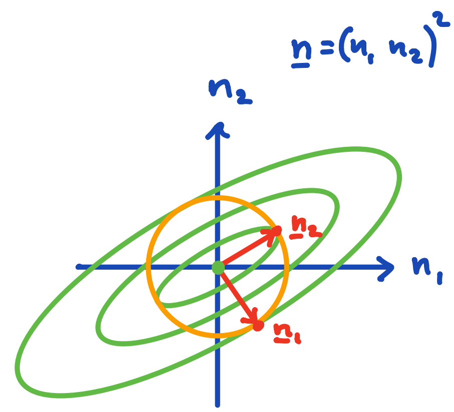
Given a state of stress $\underline{\underline{\sigma}}$ at point $\underline{x}$, what are the unit normals $\underline{n}$ corresponding to min. and max. normal stress $\sigma_n$.

This is a constrained optimization problem, because we want to find extrema of the function $\sigma_n = \sigma_n(\underline{n})$ with the constraint that $|\underline{n}| = 1$.

Lagrange multiplier method

$$\boxed{\begin{aligned} \mathcal{L}(\underline{n},\lambda) &= \underline{n}\cdot\underline{\underline{\sigma}}\,\underline{n} - \lambda(\underline{n}\cdot\underline{n}-1)\\ \mathcal{L}(n_i,\lambda) &= n_i\sigma_{ij}n_j - \lambda(n_in_i-1)\end{aligned}}$$

$$\underbrace{n_i\sigma_{ij}n_j}_{\text{function}} - \underbrace{\lambda}_{\text{Lagrange multiplier}}\underbrace{(n_in_i-1)}_{\text{constraint}}$$




$f(\underline{n}) = \underline{n}\cdot\underline{\underline{\sigma}}\,\underline{n}$ is quadratic

Function $f(\underline{n}) = \underline{n}\cdot\underline{\underline{\sigma}}\,\underline{n}$ is quadratic in components of $\underline{n}$.

If eigenvalues of $\underline{\underline{\sigma}}$ are positive then the level sets of $f(\underline{n})$ are ellipsoids as shown.

The extremal values are the stationary points of $\mathcal{L}(\underline{n},\lambda)$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = n_i n_i - 1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial n_k} = \sigma_{ij}\left(n_{i,k}\, n_j + n_i\, n_{j,k}\right) - \lambda\left(2\, n_i\, n_{i,k}\right) = 0$$

where $n_{i,k} = \delta_{i,k}$ $\quad n_{j,k} = \delta_{j,k}$

$$= \sigma_{ij}\left(\delta_{ik}\, n_j + \delta_{jk}\, n_i\right) - \lambda\left(2\, n_i\, \delta_{ik}\right)$$

$$= \sigma_{kj}\, n_j + \sigma_{ik}\, n_i - 2\lambda\, n_k$$

$$= 2\left(\sigma_{ik}\, n_k - \lambda n_k\right) = 0$$

In symbolic notation: $\boxed{(\underline{\underline{\sigma}} - \lambda \underline{\underline{I}})\underline{n} = 0 \quad \text{and} \quad |\underline{n}| = 1}$

The Lagrange multiplier method leads to an eigen problem, where the Lagrange multiplier, $\lambda$, is the eigenvalue and the normal, $\underline{n}$, the eigenvalue.

We can see that $\lambda$ is the magnitude of the normal stress by taking the dot product of eigenproblem with $\underline{n}$.

$$\underline{n} \cdot (\underline{\underline{\sigma}} - \lambda \underline{\underline{I}})\underline{n} = 0 \;\Rightarrow\; \underline{n} \cdot \underline{\underline{\sigma}}\, \underline{n} = \lambda\, \underline{n} \cdot \underline{n} \;\Rightarrow\; \boxed{\sigma_N = \lambda}$$

Hence to find the extremal stress values we must find the eigenvalues $\lambda_i$ and eigenvectors $\underline{n}_i$.

$\lambda_i$'s are the principal normal stresses $\Rightarrow \lambda_i = \sigma_i$

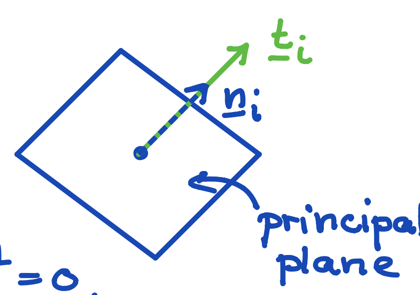
$\underline{n}$'s are the principal directions of $\underline{\underline{\sigma}}$

Since $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$ all $\lambda_i$ are real and the set $\{\underline{n}_i\}$ form a mutually orthogonal basis, so that $\underline{\underline{\sigma}}$ can be represented as

$$\underline{\underline{\sigma}} = \sum_{i=1}^{3} \sigma_i \, \underline{n}_i \otimes \underline{n}_i$$

The tractions in the principal directions are

$$\underline{t}_{\underline{n}_i} = \underline{\underline{\sigma}}\,\underline{n}_i = \sigma_i \underline{n}_i$$




since $\underline{t}_i \parallel \underline{n}_i$ there is no shear stress on the principal planes, $\underline{t}_i^{\perp} = 0$.

If the $\sigma_i$'s are distinct and ordered $\sigma_1 > \sigma_2 > \sigma_3$ then $\sigma_1$ and $\sigma_3$ are the max. and min. normal stresses.

## II. Maximum and minimum shear stresses

Given the principal directions $\underline{n}_1$, $\underline{n}_2$ and $\underline{n}_3$ at $\underline{x}$ what is the unit vector $\underline{s} = [s_1 \; s_2 \; s_3]$ that gives the max. and min. values of the shear stresses $\tau$?

In the frame of the principal directions $\{\underline{n}_i\}$ the traction vector associated with $\underline{s}$ is

$$\underline{t}_s = \sigma_1 s_1 \underline{n}_1 + \sigma_2 s_2 \underline{n}_2 + \sigma_3 s_3 \underline{n}_3$$

The magnitudes of normal, $\sigma_N$, and shear stress, $\tau$, are

$$\sigma_n = \underline{s} \cdot \underline{t}_s = \sigma_1 s_1^2 + \sigma_2 s_2^2 + \sigma_3 s_3^2$$

$$\tau^2 = |\underline{t}_s|^2 - \sigma_n^2 = \sigma_1^2 s_1^2 + \sigma_2^2 s_2^2 + \sigma_3^2 s_3^2 - (\sigma_1 s_1^2 + \sigma_2 s_2^2 + \sigma_3 s_3^2)^2$$

Hence we have the following expression for the shear stress $\tau^2 = \sum_{i=1}^{3} \sigma_i^2 s_i^2 - \left(\sum_{i=1}^{3} \sigma_i s_i^2\right)^2$ we are looking for the extremal values of $\tau^2$ under the constraint $|\underline{s}|^2 - 1 = 0$.

$\Rightarrow$ Solve using Lagrange mult. or direct elimination.

I) Eliminate $n_3 = 1 - n_1 - n_2 \Rightarrow \tau^2 = \tau^2(s_1, s_2)$.

We just need to find $\frac{\partial \tau^2}{\partial s_1} = \frac{\partial \tau^2}{\partial s_2} = 0$.

$$\frac{\partial \tau^2}{\partial s_1} = 2 s_1 (\sigma_1 - \sigma_3)\left\{\sigma_1 - \sigma_3 - 2\left[(\sigma_1 - \sigma_3)s_1^2 + (\sigma_2 - \sigma_3)s_2^2\right]\right\} = 0$$

$$\frac{\partial \tau^2}{\partial s_2} = 2 s_2 (\sigma_2 - \sigma_3)\left\{\sigma_2 - \sigma_3 - 2\left[(\sigma_1 - \sigma_2)s_1^2 + (\sigma_2 - \sigma_3)s_2^2\right]\right\} = 0$$

First solution: $s_1 = s_2 = 0 \Rightarrow s_3 = 1 \quad \underline{s} = \pm \underline{n}_3$

$$\tau^2 = \sigma_3^2 \cdot 1 - (\sigma_3 \cdot 1)^2 = 0$$

$\Rightarrow$ minimum in the shear stress

which vanishes on principal plane

Second solution: $s_1 = 0$

$$\frac{\partial \tau^2}{\partial n_2} = \sigma_2 - \sigma_3 - 2\left[(\sigma_2 - \sigma_3)s_2^2\right] = 0$$

$$(\sigma_2 - \sigma_3)(1 - 2 s_2^2) = 0 \quad \Rightarrow \quad s_2 = \pm \frac{1}{\sqrt{2}}$$

from $\quad s_2^2 + s_3^2 = 1 \quad \Rightarrow \quad s_3 = \pm \frac{1}{\sqrt{2}}$

$$\Rightarrow \quad \underline{s} = \pm \frac{1}{\sqrt{2}} \underline{n}_2 \pm \frac{1}{\sqrt{2}} \underline{n}_3$$

$$\tau^2 = \frac{\sigma_2^2}{2} + \frac{\sigma_2^2}{2} - \left(\frac{\sigma_2}{2} + \frac{\sigma_3}{2}\right)^2$$

$$= \frac{\sigma_2^2}{2} + \frac{\sigma_2^2}{2} - \left(\frac{\sigma_2^2}{4} + 2\frac{\sigma_2}{2}\frac{\sigma_3}{2} + \frac{\sigma_3^2}{4}\right)$$

$$\tau^2 = \left(\frac{\sigma_2}{2}\right)^2 - 2\frac{\sigma_2}{2}\frac{\sigma_3}{2} + \left(\frac{\sigma_3}{2}\right)^2 = \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2$$

We have the following two solutions:

$$\boxed{\begin{array}{lll} \text{min.} & \tau = 0 & \text{for } \underline{s} = \pm \underline{n}_3 \\ \text{max.} & \tau = \frac{1}{2}(\sigma_2 - \sigma_3) & \text{for } \underline{s} = \pm \frac{\underline{n}_2}{\sqrt{2}} \pm \frac{\underline{n}_3}{\sqrt{2}} \end{array}}$$

Two additional pairs of solutions can be obtained by eliminating $\underline{n}_1$ or $\underline{n}_2$ from $\tau^2$ and folling similar steps. So that we have

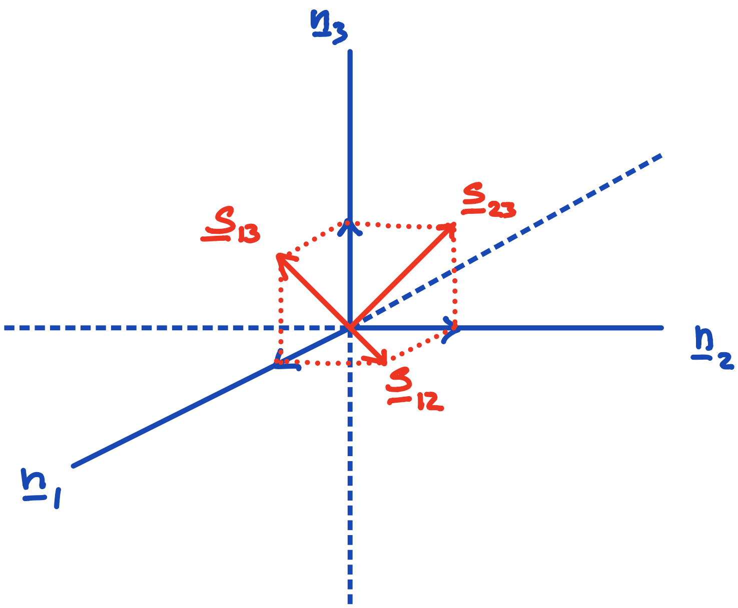
Minimum shear stresses:

$$\boxed{\tau = 0 \quad \text{on} \quad \underline{s} = \pm \underline{n}_1 \quad \underline{s} = \pm \underline{n}_2 \quad \underline{s} = \pm \underline{n}_3}$$

Maximum shear stresses:

$$\boxed{\begin{array}{lll} \tau = \frac{1}{2}(\sigma_2 - \sigma_3) & \text{on} & \underline{s}_{23} = \frac{1}{\sqrt{2}}(\pm \underline{n}_2 \pm \underline{n}_3) \\ \tau = \frac{1}{2}(\sigma_1 - \sigma_3) & \text{on} & \underline{s}_{13} = \frac{1}{\sqrt{2}}(\pm \underline{n}_1 \pm \underline{n}_3) \\ \tau = \frac{1}{2}(\sigma_1 - \sigma_2) & \text{on} & \underline{s}_{12} = \frac{1}{\sqrt{2}}(\pm \underline{n}_1 \pm \underline{n}_2) \end{array}}$$

where we assume $\sigma_1 \geq \sigma_2 \geq \sigma_3$




Note: G&S do this with Lagrange multipliers but it leads to odd expressions in index notation, such as

$$4\left(\sum_{j=1}^{3} n_j^2 \sigma_j\right) n_i \sigma_i = 2\lambda n_i \qquad ?$$

where 'i' seems to be a dummy on the l.h.s. but a free index on the r.h.s.

$\Rightarrow$ we did it the pedestrian way.