Continuum Mass and Force Concepts

Introduce the notion fo a continuum body and the various types of forces acting on it. A continuum is is infinitely divisible, and hence we ignore the atomic nature of materials. At length scales much larger than atomic spacing this leads to effective models.

The discussion of the internal forces leads to notion of stress tensor field. We introduce mechanical equilibrium and corresponding differential equations.

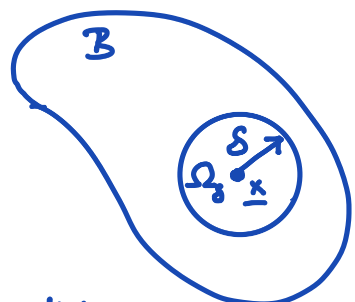
Important ideas:

1) Notion of mass density field
2) Notion of body and surface forces
3) Cauchy - stress field
4) Equations of equilibrium

Mass Density

Mass is a physical property of matter that quantifies its resistance to acceleration when a force is applied. In the continuum assumption we assume that mass is continuously distributed throughout the volume of a body, B.

We assume that any subset Ω of B with positive volume has a positive mass.



$$v_\Omega = \int_\Omega dV \qquad m_\Omega = \int_\Omega \rho(\underline{x})\, dV$$

where $\rho(\underline{x})$ is the mass density field, which can be defined at any point $\underline{x}$ as

$$\boxed{\rho(\underline{x}) = \lim_{\delta \to 0} m_{\Omega_\delta} / v_{\Omega_\delta}}$$

The center of volume (centroid) and mass are

$$\underline{x}_v = \frac{1}{v_\Omega} \int_\Omega \underline{x}\, dV \quad \text{and} \quad \underline{x}_m = \frac{1}{m_\Omega} \int_\Omega \underline{x}\, \rho\, dV$$

Body Forces

The interactions between parts of a body or a body and its environment are described by forces. Any force that not due to physical contact is a body force and acts on the entire body. Common body forces originate from gravitational and electromagnetic field.

If $\underline{b}$ is a body force field with units $\frac{\text{force}}{\text{volume}}$

Units: $\frac{F}{V} = \frac{a\,m}{V}$ $\left[\frac{1}{L^3}\frac{L}{T^2}M = \frac{M}{L^2T^2}\right]$

the resultant force on a body is

$$\underline{F}_b = \int_\Omega \underline{b}(\underline{x})\,dV$$

and the torque on a body about a point $\underline{z}$ is given by $\underline{T}_b = \int_\Omega (\underline{x}-\underline{z}) \times \underline{b}(\underline{x})\,dV$

Example: gravitational body force

$$\boxed{\underline{b}_g = \rho g} \qquad \left[\frac{M}{L^3}\frac{L}{T^2} = \frac{M}{L^2T^2}\right]$$

Inertial/fictitious forces in rotating frames such as the centrifugal and the Coriolis force act similar to body forces.

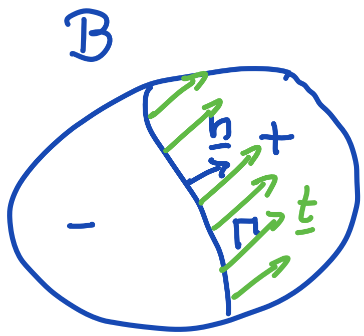
Surface/Contact Forces

arise due to the physical contact between bodies. Forces along imaginary surfaces within a body are called internal forces while forces along the bounding surface of a body are external.

Internal surface forces hold a body together.

External surface forces describe the interaction with the environment.

Traction Field (Euler-Cauchy cut principle)



Consider an arbitrary surface Γ in $\mathcal{B}$ with unit normal $\underline{n}(\underline{x})$ that defines the positive and negative sides of $\mathcal{B}$.

The force per unit area exerted by material on the pos. side upon material on the neg. side is given by the traction field $\underline{t}_n$ for Γ.

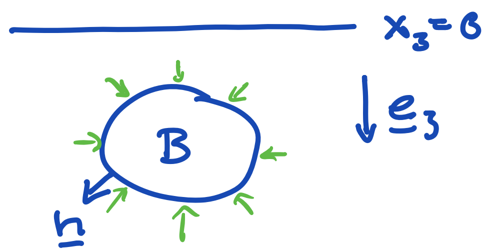
The resultant force due to a traction field on Γ is

$$\boxed{\underline{r}_s[\Gamma] = \int_\Gamma \underline{t}_n(\underline{x})\, dA}$$

The resultant torque about point $\underline{z}$ due to a traction field on Γ is

$$\boxed{\underline{\tau}_s[\Gamma] = \int_\Gamma (\underline{x} - \underline{z}) \times \underline{t}_n(\underline{x})\, dA}$$

Example: Archimedes' principle



"Any object, wholly or partially submerged in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object."

$$\Rightarrow \quad \boxed{\underline{r}_s[\partial B] = -W\underline{e}_3 = -\rho g V_B \underline{e}_3}$$

The hydrostatic force field on the bounding surface $\underline{t} = -p\underline{n}$ where $p = \rho g x_3$.

Substituting into the definition of resulting force $\underline{r}_s[\partial B] = \int_{\partial B} \underline{t}\, dA = \int_{\partial B} -p\underline{n}\, dA =$

Multiply by arbitrary constant $\underline{c} \in \mathcal{V}$ to apply the divergence Thm

$$\underline{c} \cdot \underline{r}_s[\partial B] = \int_{\partial B} -p\underline{c} \cdot \underline{n}\, dA = \int_B -\nabla \cdot (p\underline{c})\, dV$$

because $\underline{c}$ is constant we have

$$\underline{c} \cdot \underline{r}_s[\partial B] = \int_B -(\underline{c} \cdot \nabla p + p \cancel{\nabla \cdot \underline{c}}) \, dV$$

$$= \underline{c} \cdot \int_B -\nabla p \, dV$$

By the arbitraryness of $\underline{c}$ we have

$$\underline{r}_s[\partial B] = -\int_B \nabla p \, dV = -\int_B \rho g \underline{e}_3 \, dV = -\rho g V_B \underline{e}_3$$

Hence we have $\underline{r}_s[\partial B] = -\rho g V_B \underline{e}_3$ ✓

Note: According to "Mechanics in the Earth and Environmental Sciences" by Middelton & Wilcock there is confusion in Geology as to whether the "buoyant force" is a surface or a body force.

$\Rightarrow$ "hydrostatic surface force"

Show that the hydrostatic surface force acts at the center of volume $\bar{x}$.

$$\underline{\tau}_s[\partial B] = \int_{\partial B} (\underline{x} - \bar{\underline{x}}) \times \underline{t} \, dA = \int_{\partial B} -(x - \bar{\underline{x}}) \times p\underline{n} \, dA$$

set $\underline{x} - \bar{\underline{x}} = \underline{\delta}$

$$= \int_{\partial B} -\underline{\delta} \times p\,\underline{n} \, dA$$

multiply by constant vector $\underline{c} \in V$

$$\underline{c} \cdot \underline{\tau}_s[\partial B] = \int_{\partial B} -p \, \underline{c} \cdot \underline{\delta} \times \underline{n} \, dA = \int_{\partial B} -p \, \underline{\delta} \times \underline{n} \cdot \underline{c} \, dA$$

triple scalar product is invariant under cyclic perm.

$$= \int_{\partial B} -p \, \underline{c} \times \underline{\delta} \cdot \underline{n} \, dA$$

apply divergence thm

$$= \int_B - \nabla \cdot (p\underline{c} \times \underline{\delta}) \, dV$$

use identity $\nabla \cdot (\underline{a} \times \underline{b}) = \underline{b} \cdot \nabla \times a - \underline{a} \cdot \nabla b$

$$= -\int_B \underline{\delta} \cdot \nabla \times (p\underline{c}) - p\underline{c} \cdot \cancelto{0}{\nabla \times \underline{\delta}} \, dV$$

$$\nabla \times \delta = \nabla \times (\underline{x} - \bar{\underline{x}}) = \nabla \times \underline{x} - \cancelto{0}{\nabla \times \bar{\underline{x}}}$$

$$\nabla \times \underline{x} = \epsilon_{ijk} \, x_{i,k} \, \underline{e}_j = 0$$

$i = k$ $\quad \epsilon_{ijk} = 0$

$i \neq k$ $\quad x_{i,k} = 0$

$$= -\int_B \underline{\delta} \cdot \nabla \times (p\,\underline{c}) \, dV$$

$$\nabla \times p\underline{c} = \epsilon_{ijk}(p c_i)_{,k}\,\underline{e}_j$$

$$= \epsilon_{ijk}\left(p_{,k} c_i + p\, \cancelto{0}{c_{i,k}}\right)\underline{e}_j$$

$$= \epsilon_{ijk}\, p_{,k} c_i\, \underline{e}_j \overset{cyc}{=} \epsilon_{kij}\, p_{,k} c_i\, \underline{e}_j$$

$$= \nabla p \times \underline{c}$$

$$\underline{c} \cdot \underline{T}_S[\partial B] = -\int_B \underline{\delta} \cdot \nabla p \times \underline{c}\; dV$$

$$= -\int_B \underline{c} \cdot \underline{\delta} \times \nabla p\; dV$$

from arbitraryness of $\underline{c}$

$$\underline{T}_S[\partial B] = -\int_B (\underline{x} - \overline{\underline{x}}) \times \nabla p\; dV$$

since $\nabla p = \rho g = \text{const.}$

$$\underline{T}_S[\partial B] = \left(-\int_B \underline{x}\, dV + \overline{\underline{x}} \int_B dV\right) \times \nabla p$$

from def $\overline{\underline{x}} = \frac{1}{V_B} \int_B \underline{x}\, dV$

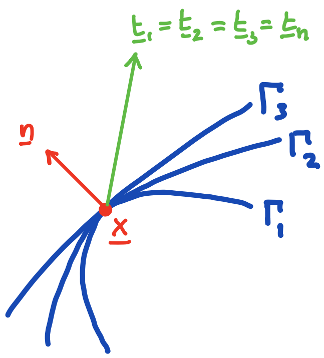
$$= \left(- V_B\, \overline{\underline{x}} + \cancelto{0}{\overline{\underline{x}}\; V_B}\right) \times \nabla p = \underline{0}$$

Hydrostatic surface force applies no torque to the center of volume $\overline{\underline{x}}$ $\quad \underline{T}[\partial B] = \underline{0}$

$\Rightarrow$ we can think of it acting at $\overline{\underline{x}}$

Cauchy's postulate

The traction field $\underline{t}_n$ on a surface Γ in B depends only pointwise on the unit normal field $\underline{n}$. In particular, there is a traction function such that $\boxed{\underline{t}_n = \underline{t}_n(\underline{n}(\underline{x}), \underline{x})}$.



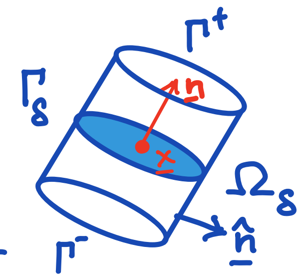
This assumes that the traction field is independent of $\nabla \underline{n}$ and hence the curvature of the surface. Therefore the traction $\underline{t}_i$ on the set of surfaces Γ_i that are tangent at $\underline{x}$ is the same, $\underline{t}_i = \underline{t}_n$.

Law of Action and Reaction

If the traction field, $\underline{t}(\underline{n},\underline{x})$, is continuous and bounded, then

$$\boxed{\underline{t}(-\underline{n},\underline{x}) = -\underline{t}(\underline{n},x)}$$

for all $\underline{n}$ and $\underline{x} \in B$.



To show this consider a disk D with arbitrary fixed radius around $\underline{x}$. Let Ω_δ be the cylinder with center $\underline{x}$, axis $\underline{n}$ and height $\delta > 0$.

We refer to the end-faces of the cylinder as as Γ^+ and Γ^- and the mantle as Γ_δ. Let $\hat{\underline{n}}$ be the outward normal on $\partial\Omega_\delta$.

Note: on Γ^+ $\hat{\underline{n}} = \underline{n}$ and on Γ^- $\hat{\underline{n}} = -\underline{n}$

Also as $\delta \to 0$ the area of the end faces approach the disk, $\Gamma^\pm \to D$ but $\Gamma_\delta \to 0$.

Since $\partial\Omega_\delta = \Gamma_\delta \cup \Gamma^+ \cup \Gamma^-$ we have

$$\lim_{\delta\to 0}\left[\int_{\Gamma_\delta} \underline{t}(\underline{\hat{n}}(y), y)\, dA + \int_{\Gamma^+} \underline{t}(\underline{n}, \underline{y})\, dA + \int_{\Gamma^-} \underline{t}(-n, \underline{y})\, dA\right] = 0$$

the first term vanishes because $\underline{t}$ is bounded and $\Gamma_\delta \to 0$. Using the fact that $\Gamma^\pm \to D$ we have in the limit

$$\int_D \underline{t}(\underline{n}, \underline{y}) + t(-\underline{n}, y)\, dA = 0$$

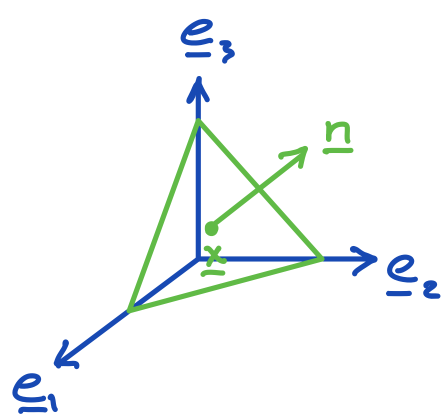
since the radius of D is arbitrary the integrand must vanish so that $\underline{t}(\underline{n}, \underline{x}) + \underline{t}(-\underline{n}, \underline{x}) = 0$ ✓

The Stress tensor

Cauchy's Theorem

Let $\underline{t}(\underline{n}, \underline{x})$ be the traction field for body B that satisfies Cauchy's postulate. Then $\underline{t}(\underline{n}, \underline{x})$ is linear in $\underline{n}$, that is, for each $\underline{x} \in B$ there is a second-order tensor field $\underline{\underline{\sigma}}(\underline{x}) \in \mathcal{V}^2$ such that $\boxed{\underline{t}(\underline{n}, \underline{x}) = \underline{\underline{\sigma}}(\underline{x})\, \underline{n}}$

called the Cauchy stress field for B.



To estabish this consider a frame $\{\underline{e}_i\}$, a point $\underline{x} \in B$ and a normal $\underline{n}$ s.t. $\underline{n} \cdot \underline{e}_i > 0$.

For $\delta > 0$, let Γ_δ denote a triangular region with center $\underline{x}$, normal $\underline{n}$ and maximum edge length δ.

Let Ω_δ be the tetrahedron bounded by Γ_δ and the three coordinate planes. These planes form three faces Γ_j with outward normals $\underline{n}_j = -\underline{e}_j$. The volume of Ω_δ goes to zero as δ becomes small.

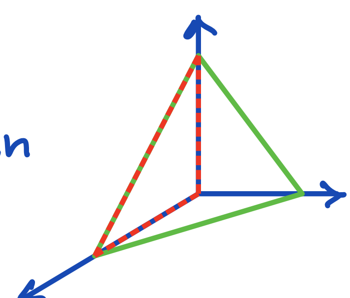
$$\lim_{\delta \to 0} \frac{1}{A_{\partial\Omega_\delta}} \int_{\partial\Omega_\delta} \underline{t}(\underline{n}(\underline{y}), \underline{y})\, dA = 0$$

where $A_{\partial\Omega_\delta}$ is the surface area of Ω_δ

Since $\partial\Omega_\delta = \Gamma_\delta \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ we have

$$\lim_{\delta \to 0} \frac{1}{A_{\partial\Omega_\delta}} \left[\int_{\Gamma_\delta} \underline{t}(\underline{n}, \underline{y})\, dA + \sum_{j=1}^{3} \int_{\Gamma_j} \underline{t}(-\underline{e}_j, \underline{y})\, dA \right] = 0$$

Since each face Γ_j can be linearly mapped onto Γ_δ with constant Jacobian $n_j = \underline{n} \cdot \underline{e}_j > 0$ so that $A_{\Gamma_j} = n_j A_{\Gamma_\delta}$



$$\Rightarrow A_{\partial\Omega_\delta} = A_{\Gamma_\delta} + \sum_{j=1}^{3} \Gamma_j = \lambda A_{\Gamma_\delta} \qquad \lambda = 1 + \sum_{j=1}^{3} n_j$$

substituting we obtain

$$\lim_{\delta\to 0} \frac{1}{A_{\partial\Omega_\delta}}\left[\int_{\Gamma_\delta} \underline{t}(\underline{n},\underline{y})\,dA + \sum_{j=1}^{3}\int_{\delta} t_n(-\underline{e}_j,\underline{y})\,n_j\,dA\right] = 0$$

$$\lim_{\delta\to 0} \frac{1}{\lambda A_{\Gamma_\delta}} \int_{\Gamma_\delta} \underline{t}(\underline{n},\underline{y}) + \sum_{j=1}^{3} t(-\underline{e}_j,\underline{y})\,n_j\,dA = 0$$

As $\delta \to 0$ the area Γ_δ shrinks to $\underline{x}$ so that by the mean value Thm for integrals the limit is given by the integrand. Hence

$$t(\underline{n},\underline{x}) + \sum_{j=1}^{3} \underline{t}(-\underline{e}_j,\underline{x})\,n_j = 0$$

Using the Law of Action and Reaction

$$\underline{t}(\underline{n},\underline{x}) = -\sum_{j=1}^{3} \underline{t}(-\underline{e}_j,\underline{x})\,n_j = \sum_{j=1}^{3} \underline{t}(\underline{e}_j,\underline{x})\,n_j$$

or with summation convention

$$\underline{t}(\underline{n},\underline{x}) = \underline{t}(\underline{e}_j,\underline{x})\,n_j$$

using the definition of dyadic product

$$(\underline{t}(\underline{e}_j, \underline{x}) \otimes \underline{e}_j)\underline{n} = \underbrace{(\underline{e}_j \cdot \underline{n})}_{n_i \, \underline{e}_j \cdot \underline{e}_i \, = \, n_i \, \delta_{ij} \, = \, n_j} \underline{t}(\underline{e}_j, \underline{x})$$

So that we have

$$\underline{t}(\underline{n}, \underline{x}) = (\underline{t}(\underline{e}_j, \underline{x}) \otimes \underline{e}_j)\underline{n} = \underline{\underline{\sigma}}\, \underline{n}$$

$$\underline{\underline{\sigma}} = \underline{t}(\underline{e}_j, \underline{x}) \otimes \underline{e}_j$$

substituting $\underline{t}(\underline{e}_j, \underline{x}) = t_i(\underline{e}_j, \underline{x})\, \underline{e}_i$ we obtain the definition of the Cauchy stress tensor

$$\boxed{\underline{\underline{\sigma}} = \sigma_{ij}\, \underline{e}_i \otimes \underline{e}_j \quad \text{with} \quad \sigma_{ij} = t_i(\underline{e}_j, \underline{x})}$$

Hence σ_{ij} is the i-th component of the traction on the j-th coordinate plane.

The traction vectors on the coor. planes at $\underline{x}$ are

$$\underline{t}(\underline{e}_1, \underline{x}) = t_i(\underline{e}_1, \underline{x})\, \underline{e}_i = S_{i1}(\underline{x})\, \underline{e}_i$$

$$\underline{t}(\underline{e}_2, \underline{x}) = t_i(\underline{e}_2, \underline{x})\, \underline{e}_i = S_{i2}(\underline{x})\, \underline{e}_i$$

$$\underline{t}(\underline{e}_3, \underline{x}) = t_i(\underline{e}_3, \underline{x})\, \underline{e}_i = S_{i3}(\underline{x})\, \underline{e}_i$$