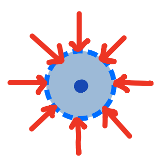
Simple states of stress

I) Hydrostatic stress

$\boxed{\underline{\underline{\sigma}} = -p\,\underline{\underline{I}}} \Rightarrow \underline{t}_n = \underline{\underline{\sigma}}\,\underline{n} = -p\,\underline{n}$ for all $\underline{n}$

$\underline{t}_n^{\parallel} = \underline{\underline{P}}_n^{\parallel}\,\underline{t} = (\underline{n}\otimes\underline{n})(-p\,\underline{n}) = -p\,(\underline{n}\cdot\underline{n})\,\underline{n} = -p\,\underline{n}$



$\Rightarrow \underline{t}_n = \underline{t}_n^{\parallel} \qquad \underline{t}_n^{\perp} = 0$

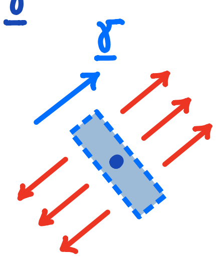
all normal stresses are $\sigma_1 = \sigma_2 = \sigma_3 = -p$

no shear stresses on any plane

II) Uniaxial stress

$\boxed{\underline{\underline{\sigma}} = \sigma\;\underline{\gamma}\otimes\underline{\gamma}} \Rightarrow \underline{t}_n = \underline{\underline{\sigma}}\,\underline{n} = \sigma\,(\underline{\gamma}\cdot\underline{n})\,\underline{\gamma}$

($\underline{\gamma}$ is unit vector) Traction is always parallel to $\underline{\gamma}$



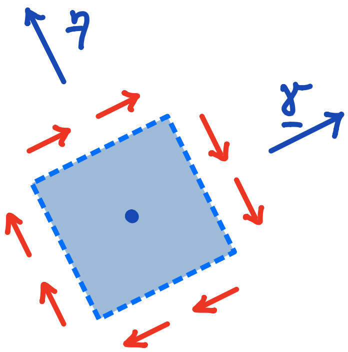
and vanishes on surfaces with normal perpendicular to $\underline{\gamma}$.

$\sigma > 0$: pure tension

$\sigma < 0$: pure compression

III, Pure shear stress

$$\boxed{\underline{\underline{\sigma}} = \tau\,(\underline{\gamma}\otimes\underline{\eta} + \underline{\eta}\otimes\underline{\gamma})} \Rightarrow \underline{t}_n = \underline{\underline{\sigma}}\,\underline{n} = \tau(\underline{\eta}\cdot\underline{n})\,\underline{\gamma} + \tau(\underline{\gamma}\cdot\underline{n})\,\underline{\eta}$$



$\underline{n} = \underline{\eta}: \quad \underline{t}_n = \tau\,\underline{\gamma}$

$\underline{n} = \underline{\gamma}: \quad \underline{t}_n = \tau\,\underline{\eta}$

IV, Plane stress

If there exists a pair of orthogonal vectors $\underline{\gamma}$ and $\underline{\eta}$ such that the matrix representation of $\underline{\underline{\sigma}}$ in the frame $\{\underline{\gamma}, \underline{\eta}, \underline{\gamma}\times\underline{\eta}\}$ is of the form

$$[\underline{\underline{\sigma}}] = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then a state of plane stress exists.

Spherical and deviatoric stress tensors

The Cauchy stress tensor can be decomposed as

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_S + \underline{\underline{\sigma}}_D$$

spherical stress tensor: $\underline{\underline{\sigma}}_S = -p\underline{\underline{I}}$ $\quad p = -\mathrm{tr}(\underline{\underline{\sigma}})$

deviatoric stress tensor: $\underline{\underline{\sigma}}_D = \underline{\underline{\sigma}} + p\underline{\underline{I}}$

The pressure $p = -\frac{1}{3}\mathrm{tr}(\underline{\underline{\sigma}}) = -\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ can be interpreted as the mean normal traction. The spherical stress is the part of $\underline{\underline{\sigma}}$ that changes the volume of the body. Note that $p > 0$ corresponds to compression.

The deviatoric stress is the part of $\underline{\underline{\sigma}}$ that changes the shape of a body without changing its volume. By definition $\mathrm{tr}\, \underline{\underline{\sigma}}_D = 0$.

Principal invariants of $\underline{\underline{\sigma}}_D$:

$$I_1(\underline{\underline{\sigma}}_D) = \mathrm{tr}\, \underline{\underline{\sigma}}_D = 0$$

$$J_2(\underline{\underline{\sigma}}) = -I_2(\underline{\underline{\sigma}}_D) = \frac{1}{2}\, \underline{\underline{\sigma}}_D : \underline{\underline{\sigma}}_D$$

$$J_3(\underline{\underline{\sigma}}) = I_3(\underline{\underline{\sigma}}_D) = \det \underline{\underline{\sigma}}_D$$

The invariants J_2 and J_3 of the deviatoric stress $\underline{\underline{\sigma}}_D$ are used to formulate yield functions in theory of plasticity.