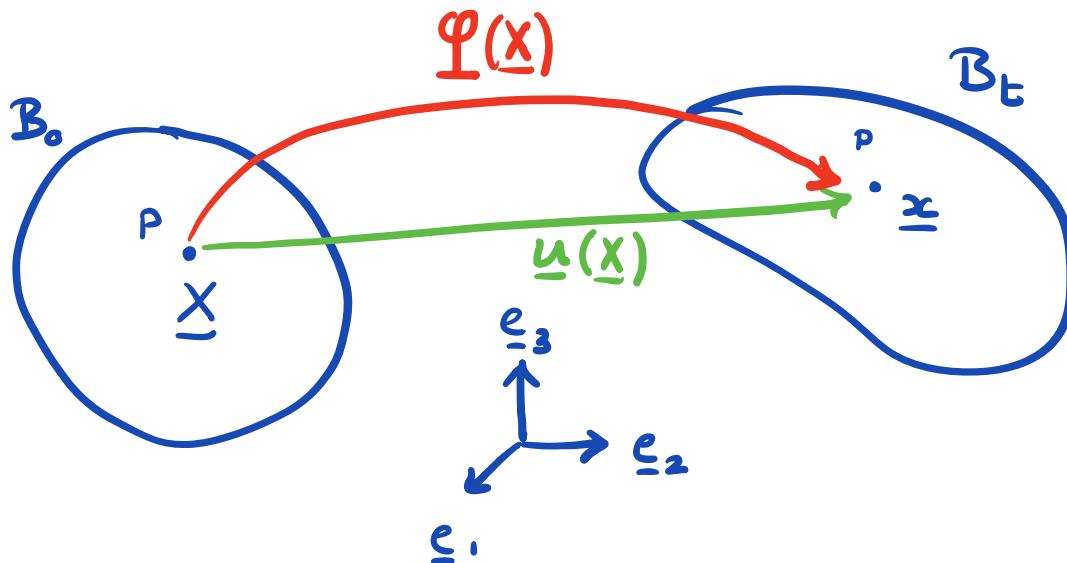


Kinematics

Study of geometry of motion without consideration of mass or stress.

⇒ Quantify the strain and rate of strain .

Deformation Mapping



B_0 = body in reference, initial , undeformed
or material configuration

B_t = body in current, spatial or deformed config.

P = material point in body

\underline{x} = location of P in B_0

\underline{x} = location of p in B_e

$\varphi(\underline{x})$ = deformation mapping

$\underline{u}(\underline{x})$ = displacement

$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ = frame

$\underline{x} = x_I \underline{e}_I$ x_I = components of \underline{x} in $\{\underline{e}_I\}$

$\underline{x} = x_i \underline{e}_i$ x_i = " " \underline{x} in $\{\underline{e}_i\}$

Convention:

Upper case quantities & indices \rightarrow reference. B_o

Lower case quantities & indices \rightarrow current. B_e

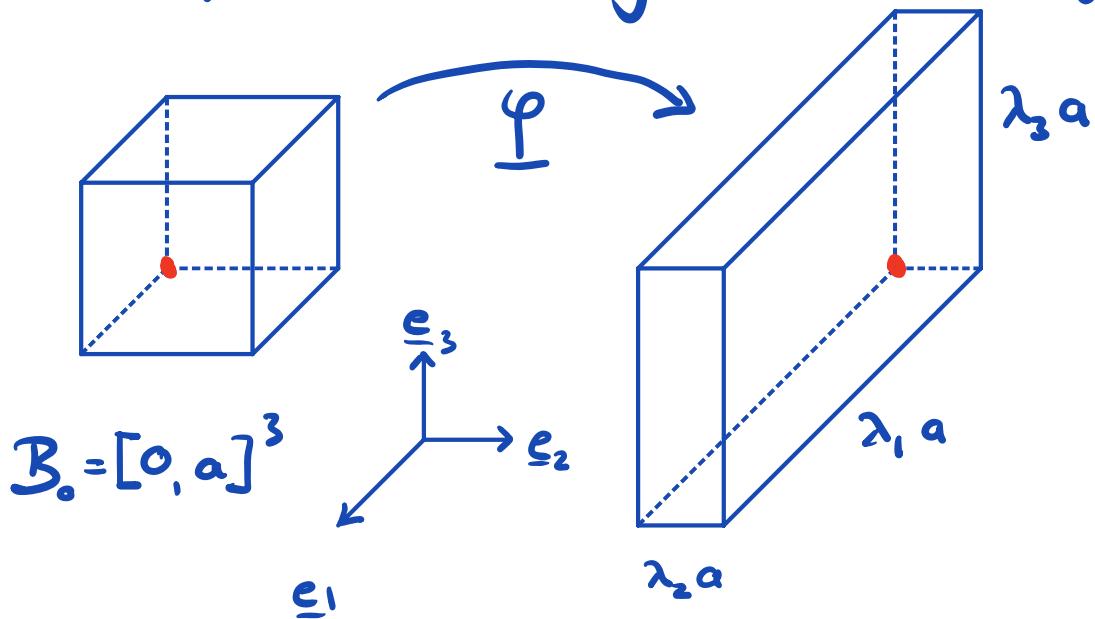
Definition of deformation mapping

$$\underline{x} = \varphi(\underline{x}) = \varphi_i(\underline{x}) \underline{e}_i$$

Displacement of a material particle

$$\underline{u}(\underline{x}) = \varphi(\underline{x}) - \underline{x}$$

Example: Stretching cube with edge length a



$$\text{deformation map: } \underline{x}_1 = \lambda_1 \underline{X}_1 + \underline{v}_1$$

$$\underline{x}_2 = \lambda_2 \underline{X}_2 + \underline{v}_2$$

$$\underline{x}_3 = \lambda_3 \underline{X}_3 + \underline{v}_3$$

λ = stretch ratio

\underline{v} = translation (only important in presence of body force)

$$(\underline{v} = 0)$$

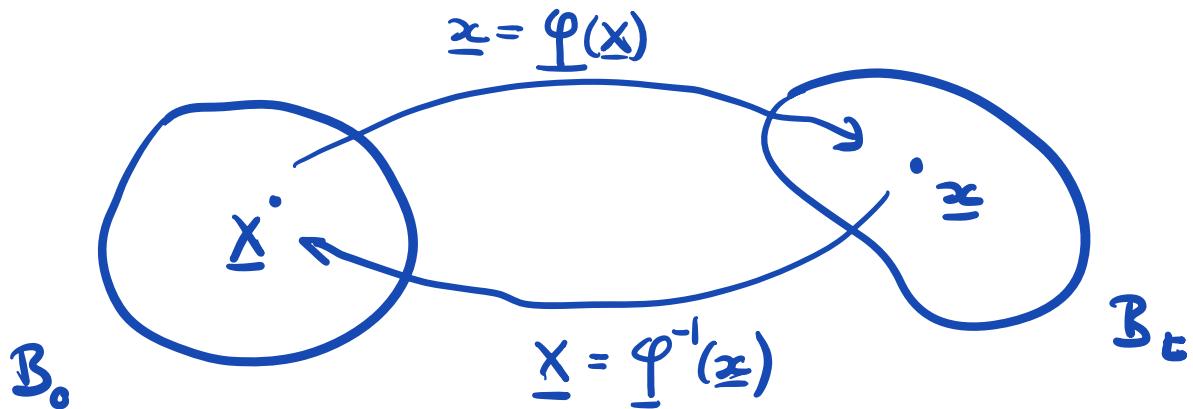
$$\underline{\underline{\epsilon}} = \underline{\varphi}(\underline{\underline{X}}) = \lambda_1 \underline{X}_1 \underline{e}_1 + \lambda_2 \underline{X}_2 \underline{e}_2 + \lambda_3 \underline{X}_3 \underline{e}_3 = \Lambda_{ij} \underline{X}_j \underline{e}_i$$

$$\underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\underline{\underline{\epsilon}} = \underline{\underline{\Lambda}} \underline{\underline{X}}$$

Inverse Mapping

If φ is admissible \Rightarrow well defined inverse φ^{-1}



inverse deformation map : $x = \underline{\varphi}^{-1}(z)$

Measures of Strain

In 1D we have simple measures

original:  ΔL

deformed: 

$$\Delta L = l - L$$

engineering strain: $e = \frac{\Delta L}{L} = \frac{l - L}{L}$

stretch ratio: $\lambda = \frac{l}{L} \Rightarrow e = \lambda - 1$

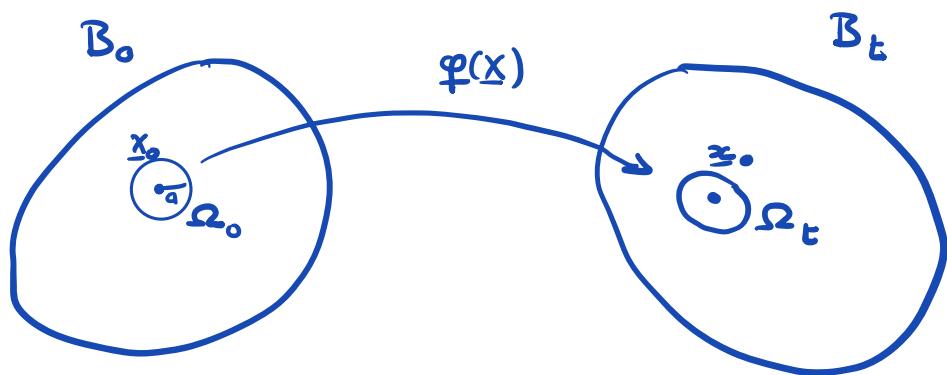
true or Hencky strain: $\varepsilon = \ln(\lambda)$

Graeu strain: $\varepsilon = \frac{1}{2}(\lambda^2 - 1)$

....

Description of strain is not unique !

Here we need to find a general 3D approach
that is not limited to small deformations.



Sphere Ω_0 of radius a around \underline{x}_0 .

Mapped to Ω_t around \underline{x}_t by $\varphi(\underline{x})$

$$\Omega_t = \{\underline{x} \in B_t \mid \underline{x} = \varphi(\underline{x}), \underline{x} \in \Omega_0\} \rightarrow \Omega_t = \varphi(\Omega_0)$$

Def: The strain at \underline{x}_0 is any relative difference
between Ω_0 and Ω_t in limit of $a \rightarrow 0$.

Deformation gradient

Natural way to quantify local strain

$$\underline{\underline{F}}(\underline{x}) = \nabla \varphi(\underline{x})$$

$$F_{ij} = \frac{\partial \varphi_i}{\partial x_j}$$

Expanding deformation in Taylor series around \underline{x}_0 we have

$$\begin{aligned}\varphi(\underline{x}) &= \varphi(\underline{x}_0) + \nabla \varphi(\underline{x}_0) (\underline{x} - \underline{x}_0) + \mathcal{O}(1|\underline{x} - \underline{x}_0|^2) \\ &= \underbrace{\varphi(\underline{x}_0)}_{\subseteq} - \underbrace{\nabla \varphi(\underline{x}_0) \underline{x}_0}_{\underline{\underline{F}}(\underline{x}_0)} + \underbrace{\nabla \varphi(\underline{x}_0) \underline{x}}_{\underline{\underline{F}}(\underline{x}_0) \underline{x}}\end{aligned}$$

locally we can approximate φ as

$$\varphi(\underline{x}) \approx \underline{\underline{c}} + \underline{\underline{F}}(\underline{x}_0) \underline{x} \quad (\text{affine deform.})$$

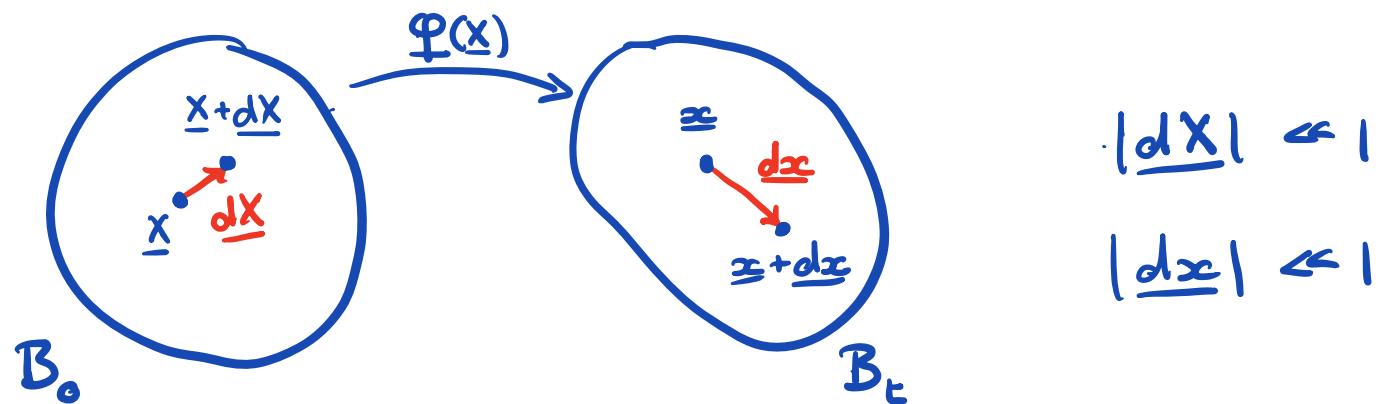
$\Rightarrow \underline{\underline{F}}(\underline{x}_0)$ characterizes local behavior of $\varphi(\underline{x})$

Homogeneous deformation

$\underline{\underline{F}}$ is constant

$$\Rightarrow \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}} = \varphi(\underline{x}) = \underline{\underline{c}} + \underline{\underline{F}} \underline{x}$$

Consider the mapping of line segment



$$|dx| \ll 1$$

$$|\bar{d}\bar{x}| \ll 1$$

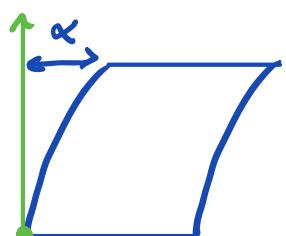
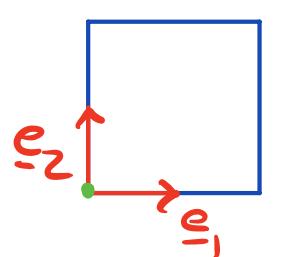
$$\bar{x} + d\bar{x} = \Phi(x + dx) \approx \Phi(x) + \nabla \Phi(x) dx = \bar{x} + \bar{\mathcal{F}}(x) dx$$

$$d\bar{x} = \bar{\mathcal{F}}(x) dx$$

$$d\bar{x}_i = \bar{\mathcal{F}}_{ij}(x) dx_j$$

$\bar{\mathcal{F}}$ maps material vectors into spatial vectors.

Example: Shear deformation



$$\Phi(\underline{x}) = [x_1 + \alpha x_2^2, x_2]$$

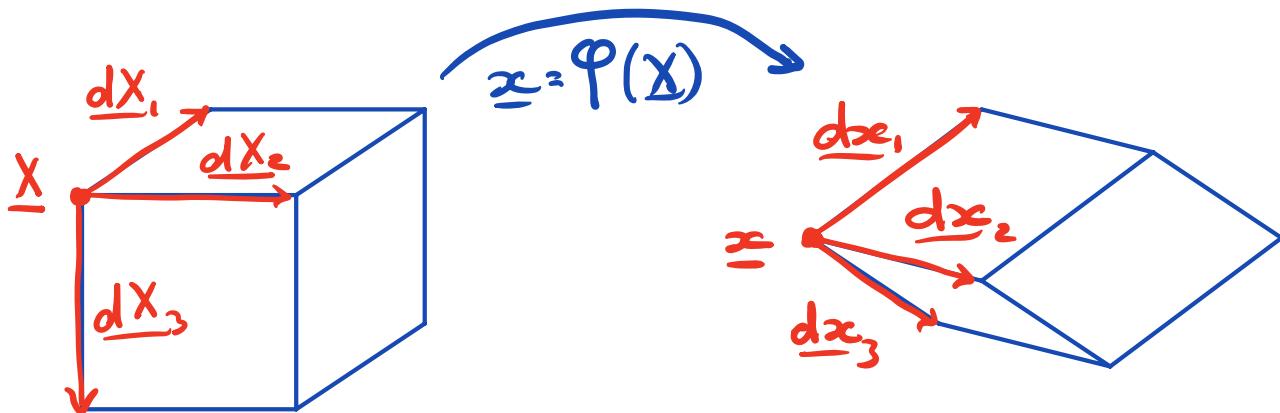
$$\nabla \Phi = \mathcal{F} = \begin{bmatrix} 1 & 2\alpha x_2 \\ 0 & 1 \end{bmatrix}$$

$$\bar{\mathcal{F}} e_1 = [1, 0]^T = e_1 \quad \text{unchanged}$$

$$\bar{\mathcal{F}} e_2 = [2\alpha x_2, 1]^T \quad \text{rotated and stretched}$$

Volume changes

Change in volume during deformation



$$\text{Volumes are: } dV_x = (\underline{dX}_1 \times \underline{dX}_2) \cdot \underline{dX}_3$$

$$dV_x = (\underline{dx}_1 \times \underline{dx}_2) \cdot \underline{dx}_3$$

$$= \det([\underline{dx}_1][\underline{dx}_2][\underline{dx}_3])$$

$$\text{substituting } \underline{dx} = \underline{F} \underline{dX}$$

$$dV_x = \det([\underline{F} \underline{dX}_1][\underline{F} \underline{dX}_2][\underline{F} \underline{dX}_3])$$

$$= \det(\underline{F}) \det(\underline{dX}) \quad \text{where } \underline{dX} = [\underline{dX}_1 \underline{dX}_2 \underline{dX}_3]$$

$$= \det(\underline{F}) \det(\underline{dX})$$

$$= \det(\underline{F}) (\underline{dX}_1 \times \underline{dX}_2) \cdot \underline{dX}_3$$

\Rightarrow

$$dV_x = \det(\underline{F}) dV_x$$

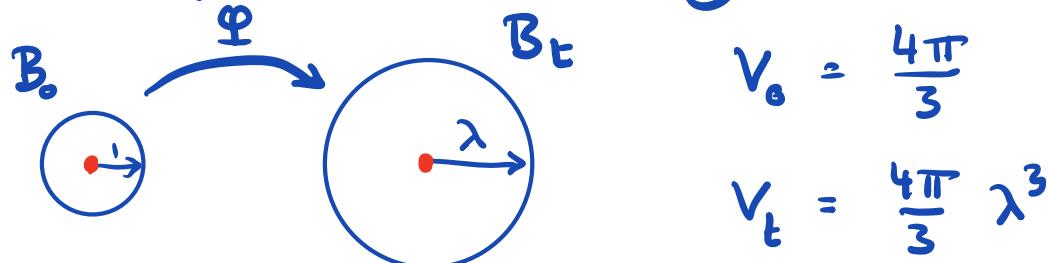
The field $J(\underline{x}) = \det(\underline{\underline{F}}) = \frac{dV_{\infty}}{dV_{\underline{x}}}$ is the Jacobian of φ and measures the volume strain.

$J(\underline{x}) > 1$: volume increase

$J(\underline{x}) < 1$: volume decrease

$J(\underline{x}) = 1$: no volume change

Example: Expanding sphere $V = \frac{4}{3}\pi R^3$



$$V_0 = \frac{4\pi}{3}$$

$$V_t = \frac{4\pi}{3} \lambda^3$$

Deformation map: $\underline{\underline{x}} = \varphi(\underline{x}) = \lambda \underline{x}$ $\lambda \approx 1$

$$\underline{\underline{F}} = \nabla \varphi = \lambda \underline{\underline{I}}$$

$J \neq J(\underline{x})$ because $\underline{\underline{F}}$ is const

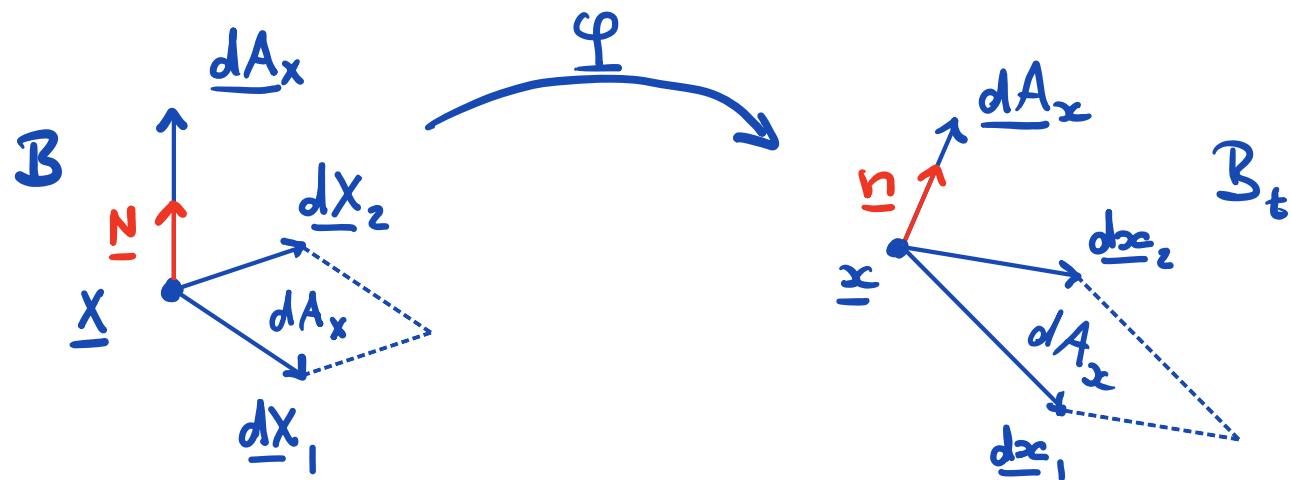
$$J = \det(\underline{\underline{F}}) = \det(\lambda \underline{\underline{I}}) = \lambda^3 \underbrace{\det(\underline{\underline{I}})}_1$$

$$J = \lambda^3$$

$$V_t = J V_0 = \frac{4\pi}{3} \lambda^3 \quad \checkmark$$

Surface area changes

How do surfaces change during deformation

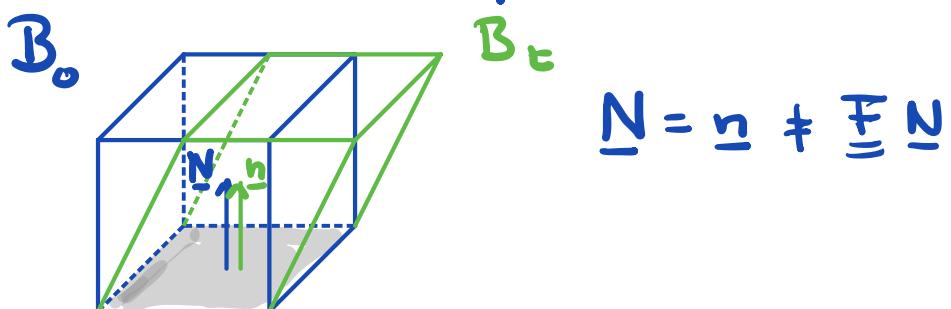


surface normals: $|\underline{N}| = |\underline{n}| = 1$

surface vector elements: $dA_x = \underline{N} dA_x = \underline{x} d\underline{z}_2$
 $dA_x' = \underline{n} dA_x' = \underline{z}_t d\underline{x}_2$

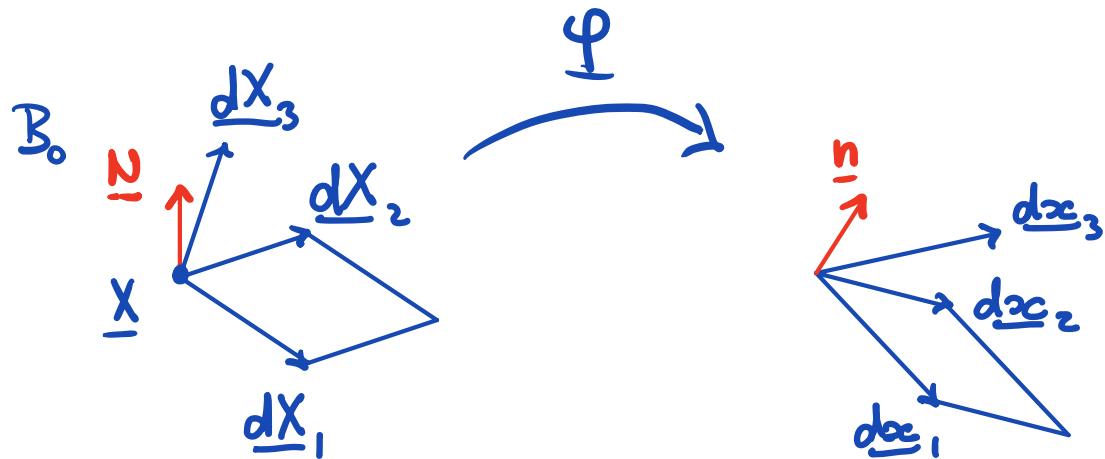
Important: $\underline{n} \neq \underline{\underline{E}} \underline{N}$!

Example: Simple shear



What is the relation between \underline{N} and \underline{n} ?

Consider \underline{dX}_3 so that $\underline{N} \cdot \underline{dX}_3 \neq 0$



$$\underline{dA}_x = \underline{dX}_1 \times \underline{dX}_2$$

$$dV_x = \underline{dA}_x \cdot \underline{dX}_3$$

$$\underline{dA}_x = \underline{dx}_1 \times \underline{dx}_2$$

$$dV_x = \underline{dA}_x \cdot \underline{dx}_3$$

Change in volume:

$$dV_x = \int dV_x$$

$$\underline{dA}_x \cdot \underline{dx}_1 = \int \underline{dA}_x \cdot \underline{dX}_1 \quad \text{with } \underline{dx}_1 = \underline{F} \underline{dX}_1,$$

$$\underline{dA}_x \cdot \underline{F} \underline{dX}_1 - \int \underline{dA}_x \cdot \underline{dX}_1 = 0 \quad \text{using transpose}$$

$$\underline{F}^T \underline{dA}_x \cdot \underline{dX}_1 - \int \underline{dA}_x \cdot \underline{dX}_1 = 0$$

$$(\underline{F}^T)^{-1} (\underline{F}^T \underline{dA}_x - \int \underline{dA}_x) \cdot \underline{dX}_1 = 0 \quad \text{since } \underline{dX}_1 \text{ is arbitrary}$$

\Rightarrow

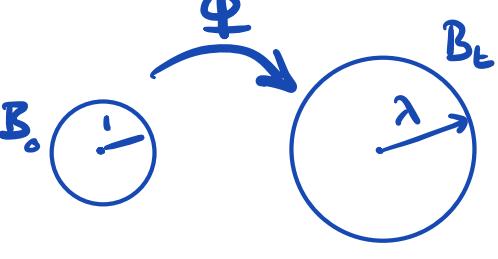
$$\underline{dA}_x = \int \underline{F}^{-T} \underline{dA}_x$$

$$\underline{n} \underline{dA}_x = \int \underline{F}^{-T} \underline{N} \underline{dA}_x$$

Nanson's formula
 $\underline{F}^{-T} = (\underline{F}^T)^{-1}$

so that $\underline{n} = \underbrace{\int dA_x}_{\underline{F}^{-T} \underline{N}}$
 normalization direction

Example : Expanding sphere

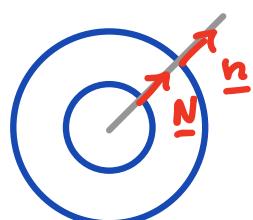


$$B_0 \xrightarrow{\varphi} B_t \quad A_0 = 4\pi \quad A_t = 4\pi \lambda^2$$

$$\Rightarrow \frac{A_t}{A_0} = \lambda^2$$

Get same result with Nausen's formula :

Both B_0 & B_t are spheres: $\underline{N} = \underline{n}$?

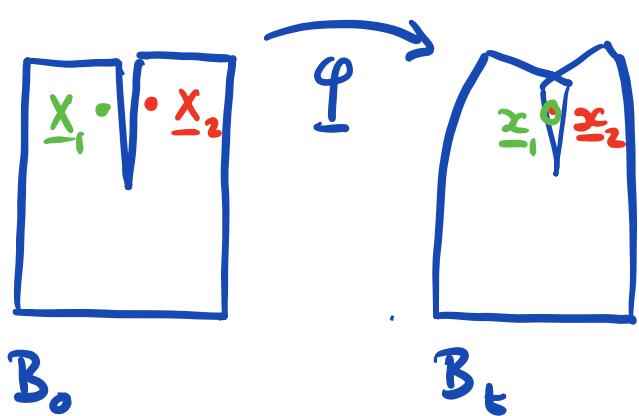


$\underline{n} dA_x = \int \underline{F}^{-T} \underline{N} dA_x$ $\underline{n} \frac{dA_x}{dA_x} = \int \underline{F}^{-T} \underline{N}$ substitute J & \underline{F}^{-T} $\underline{n} \frac{dA_x}{dA_x} = \lambda^3 \frac{1}{\lambda} \underline{I} \equiv \underline{n}$ $\underline{n} \frac{dA_x}{dA_x} = \lambda^2 \underline{N}$ since $\underline{n} = \underline{N}$ $\underline{n} \underline{F}' \frac{dA_x}{dA_x} = \lambda^2 \underline{n} \underline{F}'$ $\Rightarrow \frac{dA_x}{dA_x} = \lambda^2$ ✓	From volume change example $\varphi = \lambda \underline{x}$ & $\underline{F} = \lambda \underline{I}$ $J = \det(\underline{F}) = \lambda^3$ $\Rightarrow \underline{F}^{-T} = \underline{F}^{-1} = \frac{1}{\lambda} \underline{I}$
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Admissible deformations

For φ to represent the deformation of a body it must satisfy the following conditions:

1) $\varphi: B_0 \rightarrow B_t$ is one to one and onto

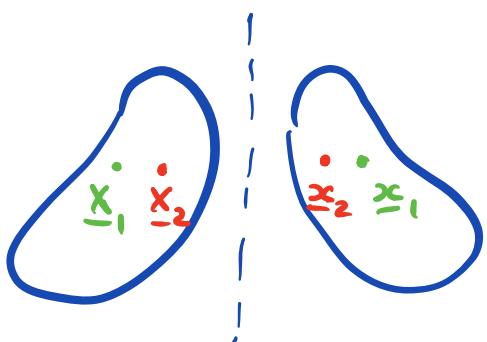


two separate points in B_0 cannot be mapped to same point in B_t .

one to one: for each \underline{x} in B_0 there is at most one \underline{x} in B_t s.t. $\underline{x} = \varphi(\underline{x})$

onto: for each \underline{x} in B_0 there is at least one \underline{x} in B_t s.t. $\underline{x} = \varphi(\underline{x})$

2) $\det(\nabla\varphi) > 0$



The orientation of a body is preserved, i.e., a body cannot be deformed into its mirror image.

Next time: Analysis of local deformation
series of decompositions

I) Translation - Fixed point decomposition

$\varphi(\underline{x}) \rightarrow$ translation & def. with fixed point

II) Polar decomposition

def with fixed point \rightarrow rotation & stretch

III) Spectral decomposition

stretch \rightarrow principal stretches

\Rightarrow allows us to formulate strain tensor