Constitutive Theory

Common constitutive laws:

Newtonian fluid: $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \eta\,(\nabla\underline{v} + \nabla\underline{v}^T)$

$p = -\frac{1}{3}\,\mathrm{tr}(\underline{\underline{\sigma}})$ η = viscosity $\underline{v}$ = velocity

Linear elastic solid: $\underline{\underline{\sigma}} = \lambda\,\nabla\cdot\underline{u}\,\underline{\underline{I}} + \mu\,(\nabla\underline{u} + \nabla\underline{u}^T)$

λ, μ = lame parameters $\underline{u}$ = displacement

Both derive from the functional form

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \lambda\,\mathrm{tr}(\underline{\underline{A}}) + 2\mu\,\mathrm{sym}(\underline{\underline{A}})}$$

Newtonian fluid: $\underline{\underline{A}} = \nabla\underline{v}$

Linear elastic solid: $\underline{\underline{A}} = \nabla\underline{u}$

remember $\nabla\cdot\underline{a} = \mathrm{tr}(\nabla\underline{a})$

$\Rightarrow$ direct for lin. elastic solid

for fluid there is a complication due to incompressibility !

Why do const. relations have this form ?

Change of observer

In Lecture 6 we discussed Change in basis

$$\underline{v} = \underline{\underline{Q}}\,\underline{v}' \quad \text{and} \quad \underline{\underline{S}} = \underline{\underline{Q}}\,\underline{\underline{S}}'\,\underline{\underline{Q}}^T$$

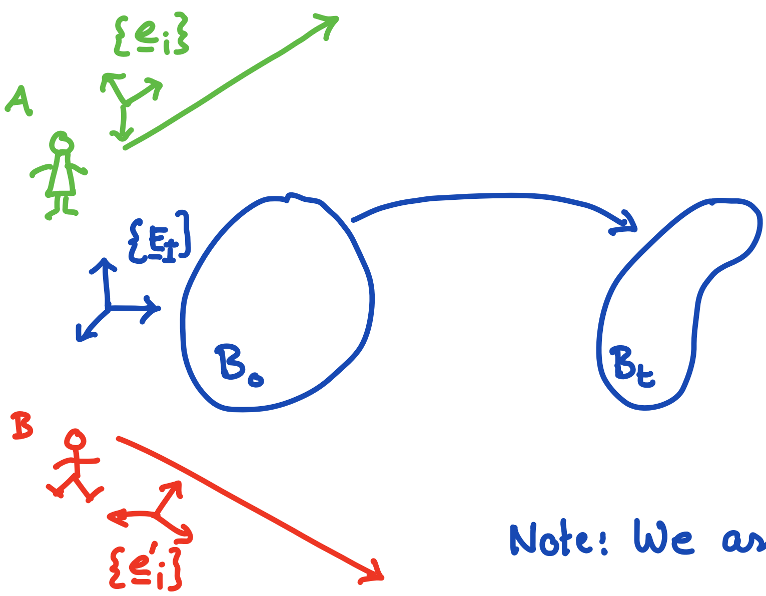
where $\underline{\underline{Q}}$ is change in basis tensor.

Q is a rotation: 1) orthonormal $\underline{\underline{Q}}\,\underline{\underline{Q}}^T = \underline{\underline{Q}}^T\underline{\underline{Q}} = \underline{\underline{I}}$

2) $\det(\underline{\underline{Q}}) = 1$

Change in basis is passive change of frame.

Active change in frame $\rightarrow$ change in observer



$$\underline{x} = \underline{\varphi}(\underline{X}, t)$$

$$\underline{x}' = \underline{\varphi}'(\underline{X}, t)$$

Note: Material ref. frame is common

Note: We assume both observers use same clock.

Since change in observer cannot induce a deformation. Two ref. frames must be related by a rigid body motion.

$$\boxed{\underline{x}' = Q(t)\,\varphi(\underline{X},t) + \underline{c}(t)} \qquad \text{Eulerian transformation}$$

$\underline{\underline{Q}}$ = rotation $\qquad$ $\underline{c}$ = translation

Our description of forces and deformations cannot depend on the observer (objective).

Effect on kinematic quantities

$$\underline{x} = \varphi(\underline{X},t) \qquad \nabla\varphi = \underline{\underline{F}}$$

$$\underline{x}' = \varphi'(\underline{X},t) = Q\,\varphi(\underline{X},t) + c \qquad \nabla\varphi' = \underline{\underline{Q}}\,\underline{\underline{F}} = \underline{\underline{F}}'$$

Right Cauchy-Green Strain tensor

$$\underline{\underline{C}}' = \underline{\underline{F}}'^T\,\underline{\underline{F}}' = (\underline{\underline{Q}}\,\underline{\underline{F}})^T(\underline{\underline{Q}}\,\underline{\underline{F}}) = \underline{\underline{F}}^T\underline{\underline{Q}}^T\underline{\underline{Q}}\,\underline{\underline{F}} = \underline{\underline{F}}^T\underline{\underline{F}} = \underline{\underline{C}}$$

$\Rightarrow$ not affected by rigid body motion because it is a material tensor C_{IJ} (naturally objective)

What about <u>spatial</u> tensors?

Axiom of frame indifference

Fields ϕ, $\underline{w}$ and $\underline{\underline{S}}$ are called frame indifferent or objective if for all superposed rigid body motions $\underline{x}' = \underline{\underline{Q}}\,\underline{x} + \underline{c}$ we have for all spatial fields

$$\boxed{\begin{aligned} \phi'(\underline{x}',t) &= \phi(\underline{x},t) \\ \underline{w}'(\underline{x}',t) &= \underline{\underline{Q}}\;\underline{w}(\underline{x},t) \\ \underline{\underline{S}}'(\underline{x}',t) &= \underline{\underline{Q}}\;\underline{\underline{S}}(\underline{x},t)\underline{\underline{Q}}^T \end{aligned}} \quad \begin{aligned} &\text{scalar field} \\ &\text{vector field} \\ &\text{tensor field} \end{aligned}$$

$\Rightarrow$ from Lecture 6.

Is spatial velocity gradient objective?

From lecture 16: $\underline{\underline{\ell}} = \nabla_x \underline{v} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$

$\underline{\underline{F}}' = \underline{\underline{Q}}\,\underline{\underline{F}}$ $\qquad\qquad$ $\underline{\underline{\ell}}' = \nabla_{x'}\underline{v}' = \dot{\underline{\underline{F}}}'\,\underline{\underline{F}}'^{-1}$

$$\dot{\underline{\underline{F}}}' = \frac{d}{dt}(\underline{\underline{Q}}\,\underline{\underline{F}}) = \underline{\underline{Q}}\,\dot{\underline{\underline{F}}} + \dot{\underline{\underline{Q}}}\,\underline{\underline{F}}$$

$$\underline{\underline{F}}'^{-1} = (\underline{\underline{Q}}\,\underline{\underline{F}})^{-1} = \underline{\underline{F}}^{-1}\underline{\underline{Q}}^{-1} = \underline{\underline{F}}^{-1}\underline{\underline{Q}}^T$$

$$\underline{\underline{\ell}}' = \dot{\underline{\underline{F}}}'\,\underline{\underline{F}}'^{-1} = (\underline{\underline{Q}}\,\dot{\underline{\underline{F}}} + \dot{\underline{\underline{Q}}}\,\underline{\underline{F}})\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T$$

$$= \underline{\underline{Q}}\,\dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underline{\underline{F}}\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T = \underline{\underline{Q}}\,\underline{\underline{\ell}}\,\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T$$

$\Rightarrow \underline{\underline{\ell}}' = \underline{\underline{Q}}\,\underline{\underline{\ell}}\,\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T$ <u>not</u> objective!

that is why $\nabla_x \underline{v}$ is not used in constitutive laws

The "non-objective" term is $\underline{\underline{\Omega}} = \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T$

it represents rigid body angular velocity between observers. see HW9

Show $\underline{\underline{\Omega}} = -\underline{\underline{\Omega}}^T$ skew-symmetric

Non-objetive part of $\underline{\underline{\ell}} = \nabla_x \underline{v}$ is skew-sym.

$\Rightarrow$ simply take symmetric part of $\underline{\underline{\ell}}$!

$\underline{\underline{d}} = \text{sym}(\underline{\underline{\ell}}) = \frac{1}{2}(\nabla_x \underline{v} + \nabla_x \underline{v}^T)$

rate of strain tensor is objective

$\Rightarrow$ used in constitutive laws

Note that velocity itself

Material frame indifferent functions

Fields: $\phi(\underline{x},t)$ scalar

$\underline{w}(\underline{x},t)$ vector

$\underline{\underline{\sigma}}(\underline{x},t)$ tensor

fields because they depend on $\underline{x}$.

Constitutive functions are not fields but they depend on fields as input.

internal energy: $u(\underline{x},t) = \hat{u}\big(\underbrace{\rho(\underline{x},t),\ \theta(\underline{x},t)}_{\text{input fields}}\big)$

output field

constitutive function

heat flow: $\underline{q}(\underline{x},t) = \hat{\underline{q}}\big(\theta(\underline{x},t)\big)$

Cauchy stress: $\underline{\underline{\sigma}}(\underline{x},t) = \hat{\underline{\underline{\sigma}}}\big(\rho(\underline{x},t),\ \theta(\underline{x},t),\ \underline{\underline{d}}(\underline{x},t)\big)$

Constitutive functions: $\hat{u}(\rho,\theta)$, $\hat{\underline{q}}(\theta)$, $\hat{\underline{\underline{\sigma}}}(\rho,\theta,\underline{\underline{d}})$

As such constitutive functions are not directly dependent on frame but their input fields are.

Consider frames $\{\underline{e}_i\}$ and $\{\underline{e}_i'\}$ then to be frame indifference requires

$$\hat{\underline{\underline{\sigma}}}(\rho', \theta', \underline{\underline{d}}') = \underline{\underline{Q}}\ \hat{\underline{\underline{\sigma}}}(\rho, \theta, \underline{\underline{d}})\ \underline{\underline{Q}}^T$$

substituting $\underline{\underline{d}}' = \underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T$

$$\boxed{\hat{\underline{\underline{\sigma}}}(\rho', \underline{\theta}', \underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\ \hat{\underline{\underline{\sigma}}}(\rho, \theta, \underline{\underline{d}})\ \underline{\underline{Q}}^T}$$

$\Rightarrow$ both input & output of constitutive function $\hat{\underline{\underline{\sigma}}}$ must be frame invariant

Isotropic functions

Functions that are frame invariant are called isotropic. Consider the following

$\hat{\phi}$ = scalar fun. $\hat{\underline{w}}$ = vector fun. $\hat{\underline{\underline{s}}}$ = tensor fun.

θ = scalar $\underline{v}$ = vector $\underline{\underline{S}}$ = tensor

Then for two frames related by rigid body rotation $\underline{\underline{Q}}$ we have following isotropic functions:

$$\boxed{\begin{array}{lll}
\hat{\phi}(\theta) = \hat{\phi}(\theta) & \hat{\phi}(\underline{\underline{Q}}\,\underline{v}) = \hat{\phi}(\underline{v}) & \hat{\phi}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \hat{\phi}(\underline{\underline{S}}) \\
\hat{\underline{w}}(\theta) = \underline{\underline{Q}}\,\hat{\underline{w}}(\theta) & \hat{\underline{w}}(\underline{\underline{Q}}\,\underline{v}) = \underline{\underline{Q}}\,\hat{\underline{w}}(\underline{v}) & \hat{\underline{w}}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\hat{\underline{w}}(\underline{\underline{S}}) \\
\hat{\underline{\underline{s}}}(\theta) = \underline{\underline{Q}}\,\hat{\underline{\underline{s}}}(\theta)\,\underline{\underline{Q}}^T & \hat{\underline{\underline{s}}}(\underline{\underline{Q}}\,\underline{v}) = \underline{\underline{Q}}\,\hat{\underline{\underline{s}}}(\underline{v})\,\underline{\underline{Q}}^T & \hat{\underline{\underline{s}}}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\hat{\underline{\underline{s}}}(\underline{\underline{S}})\,\underline{\underline{Q}}^T
\end{array}}$$

Examples:

1) $\hat{\phi}(\underline{\underline{S}}) = \det(\underline{\underline{S}})$

$$\hat{\phi}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \det(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \det(\underline{\underline{Q}})\det(\underline{\underline{S}})\det(\underline{\underline{Q}}^T) = \det(\underline{\underline{S}}) \checkmark$$

2) $\hat{\underline{u}}(\underline{v}, \underline{\underline{A}}) = \underline{\underline{A}}\,\underline{v}$

$$\hat{\underline{u}}(\underline{\underline{Q}}\,\underline{v}, \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T\,\underline{\underline{Q}}\,\underline{v} = \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{v} = \underline{\underline{Q}}\,\hat{\underline{u}}(\underline{v}, \underline{\underline{A}}) \checkmark$$

Isotropic material: stress/strain principal directions

Objectivity $\Rightarrow$ isotropic function

fluids: $\underline{Q}\, \hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\, \underline{Q}^T = \hat{\underline{\underline{\sigma}}}(\underline{Q}\, \underline{\underline{d}}\, \underline{Q}^T)$ rate of strain

solids: $\underline{Q}\, \hat{\underline{\underline{\sigma}}}(\underline{\underline{\varepsilon}})\, \underline{Q}^T = \hat{\underline{\underline{\sigma}}}(\underline{Q}\, \underline{\underline{\varepsilon}}\, \underline{Q}^T)$ strain

Since $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$ and $\underline{\underline{d}} = \underline{\underline{d}}^T$ $(\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}^T)$ they can all be written in spectral decomposition.

$\underline{\underline{S}} = \underline{\underline{S}}^T \quad \Rightarrow \quad \underline{\underline{S}} = \sum_{i=1}^{3} \lambda_i\, \underline{v}_i \otimes \underline{v}_i$

where $\quad \underline{\underline{S}}\, \underline{v}_i = \lambda_i \underline{v}_i$

Q: How are $\underline{v}_i$'s of $\underline{\underline{\sigma}}$ and $\underline{\underline{d}} / \underline{\underline{\varepsilon}}$ related?

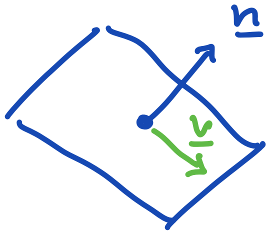
$\Rightarrow$ same eigenvectors!

$\underline{\underline{d}}\, \lambda_i = \lambda_i \underline{v}_i \quad \Rightarrow \quad \hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\, \omega_i = \omega_i \underline{v}_i$

Can be shown with reflections & projections

Projection: $\underline{\underline{P}}_n = \underline{n} \otimes \underline{n}$

Reflection: $\underline{\underline{R}}_n = \underline{\underline{I}} - 2\, \underline{n} \otimes \underline{n}$



Note: $\underline{\underline{R}}_n \underline{n} = -\underline{n}$

$\underline{\underline{R}}_n \underline{a} = \underline{a}$ if $\underline{a} \cdot \underline{n} = 0$ $\underline{a} \perp \underline{n}$

$\Rightarrow$ reflections help to detect colinear vectors.

If $\underline{n} = \underline{v}_1$ one eigenvector

$\Rightarrow \underline{\underline{R}}_{v_1} \underline{v}_1 = -v_1$ but $\underline{\underline{R}}_{v_1} \underline{v}_2 = \underline{v}_2$ & $\underline{\underline{R}}_{v_1} \underline{v}_3 = \underline{v}_3$

Step 1: $\boxed{\underline{\underline{R}}_{v_1} \underline{\underline{S}} \underline{\underline{R}}_{v_1}^T = \underline{\underline{S}}}$ $\underline{\underline{S}} = \underline{\underline{S}}^T$

$$= \underline{\underline{R}}_{v_1} \left(\sum_{i=1}^{3} \lambda_i (\underline{v}_i \otimes \underline{v}_i) \right) \underline{\underline{R}}_{v_1}^T$$

$$= \sum_{i=1}^{3} \lambda_i \underline{\underline{R}}_{v_1} (\underline{v}_i \otimes \underline{v}_i) \underline{\underline{R}}_{v_1}^T$$

use identities: $\underline{\underline{A}}(\underline{a} \otimes \underline{b}) = (\underline{\underline{A}}\underline{a}) \otimes \underline{b}$

$$(\underline{a} \otimes \underline{b}) \underline{\underline{B}} = \underline{a} \otimes (\underline{\underline{B}}^T \underline{b})$$

$$\Rightarrow \underline{\underline{R}}_{v_1} \underline{\underline{S}} \underline{\underline{R}}_{v_1}^T = \sum_{i=1}^{3} \lambda_i (\underline{\underline{R}}_{v_1} \underline{v}_i) \otimes (\underline{\underline{R}}_{v_1} \underline{v}_i)$$

$$= \lambda_1 (-\underline{v}_1) \otimes (-\underline{v}_1) + \lambda_2 \underline{v}_2 \otimes \underline{v}_2 + \lambda_3 \underline{v}_3 \otimes \underline{v}_3$$

$$= \sum_{i=1}^{3} \lambda_i \underline{v}_i \otimes \underline{v}_i = \underline{\underline{S}}$$

$$\Rightarrow \underline{\underline{R}}_{v_1} \underline{\underline{d}} \underline{\underline{R}}_{v_1}^T = \underline{\underline{d}}$$

$$\underline{\underline{R}}_{v_1} \underline{\underline{\varepsilon}} \underline{\underline{R}}_{v_1}^T = \underline{\underline{\varepsilon}}$$

Step 2: $\boxed{\underline{\underline{R}}_{v_i}\, \underline{\underline{\hat{\sigma}}}(\underline{\underline{d}}) = \underline{\underline{\hat{\sigma}}}(\underline{\underline{d}})\, \underline{\underline{R}}_{v_i}}$ commute

isotropic material: $\underline{\underline{Q}}\, \underline{\underline{\hat{\sigma}}}(\underline{\underline{d}})\underline{\underline{Q}}^T = \underline{\underline{\hat{\sigma}}}(\underline{\underline{Q}}\, \underline{\underline{d}}\, \underline{\underline{Q}}^T)$

$\underline{\underline{Q}}$ = orthogonal (rotation or reflection)

$Q = \underline{\underline{R}}_{v_1}$: $\underline{\underline{R}}_{v_1}\underline{\underline{\hat{\sigma}}}(\underline{\underline{d}})\underline{\underline{R}}_{v_1}^T = \underline{\underline{\hat{\sigma}}}(\underline{\underline{R}}_{v_1}\underline{\underline{d}}\, \underline{\underline{R}}_{v_1}^T)$

$$= \underline{\underline{\hat{\sigma}}}(\underline{\underline{d}})$$

$$\underline{\underline{R}}_{v_1}\underline{\underline{\hat{\sigma}}}(\underline{\underline{d}})\cancel{\underline{\underline{R}}_{v_1}^T \underline{\underline{R}}_{v_1}}^{\underline{\underline{I}}} = \underline{\underline{\hat{\sigma}}}(\underline{\underline{d}})\, \underline{\underline{R}}_{v_1}$$

$$\Rightarrow \quad \underline{\underline{R}}_{v_1}\underline{\underline{\hat{\sigma}}}(\underline{\underline{d}}) = \underline{\underline{\hat{\sigma}}}(\underline{\underline{d}})\, \underline{\underline{R}}_{v_1}$$

Step 3: $\underline{\underline{\hat{\sigma}}}(\underline{\underline{d}})\, \underline{v}_i \parallel \underline{v}_i$ where $\underline{\underline{d}}\, \underline{v}_i = \lambda_i \underline{v}_i$

$$\underline{\underline{R}}_{v_i}\underline{\underline{\hat{\sigma}}}(\underline{\underline{d}})\, \underline{v}_i = \underline{\underline{\hat{\sigma}}}(\underline{\underline{d}})\, \underbrace{\underline{\underline{R}}_{v_i}\underline{v}_i}_{-\underline{v}_i}$$

$$\underline{\underline{R}}_{v_i}\, \underline{\underline{\hat{\sigma}}}(\underline{\underline{d}})\, \underline{v}_i = -\underline{v}_i$$

$$\Rightarrow \quad \underline{\underline{\hat{\sigma}}}(\underline{\underline{d}})\, \underline{v}_i \parallel \underline{v}_i \quad \Rightarrow \quad \underline{\underline{\hat{\sigma}}}(\underline{\underline{d}})\, \underline{v}_i = \omega_i \underline{v}_i$$

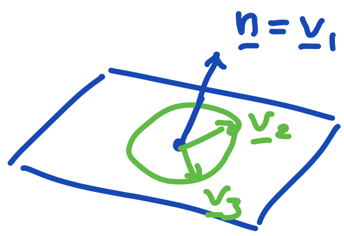
principal directions of stress and strain are same (if material is isotropic)

Representation theorem

Note: $\underline{\underline{S}} = \sum_{i=1}^{3} \lambda_i \underbrace{\underline{v}_i \otimes \underline{v}_i}_{\underline{\underline{P}}_{v_i}} = \sum_{i=1}^{3} \lambda_i \underline{\underline{P}}_{v_i}$

where $\underline{\underline{P}}_{v_i} = \underline{v}_i \otimes \underline{v}_i$ projection tensor

Eigenproblem: $\underline{\underline{P}}_n \lambda_i = \lambda_i v_i$



$\lambda_1 = 1 \quad \lambda_2 = \lambda_3 = 0$

$\Rightarrow \underline{v}_1 = \underline{n}$

$\underline{v}_2$ & $\underline{v}_3$ are any two perpendicular vectors in plane

For any $\underline{\underline{E}}$ with $\underline{v}_1\ \underline{v}_2\ \underline{v}_3$ indep. λ_1 & $\lambda = \lambda_2 = \lambda_3$

$\underline{\underline{E}} = \lambda \underline{\underline{I}} + (\lambda_1 - \lambda) \underline{\underline{P}}_{v_1}$

Consider $\underline{\underline{P}}_{v_i}$ where $\underline{\underline{d}}\, \underline{v}_i = \lambda_i \underline{v}_i$

$\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_{v_i})\, \underline{v}_i = \omega_i \underline{v}_i \qquad 3\ \underline{v}_i\text{'s}, \quad \omega_1\ \&\ \omega_2 = \omega_3 = \omega \neq 0$

$\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_{v_i}) = \omega \underline{\underline{I}} + (\omega_1 - \omega) \underline{\underline{P}}_{v_1}$

$= \lambda(\underline{v}_1) \underline{\underline{I}} + 2\mu(\underline{v}_1) \underline{\underline{P}}_{v_1}$

Show λ & μ are independent of $\underline{v}_i$:

$|\underline{e}| = |\underline{f}| = 1 \qquad \underline{\underline{R}}\,\underline{e} = \underline{f} \qquad \underline{\underline{R}}\,\underline{\underline{R}}^T = \underline{\underline{I}} \qquad \det(\underline{\underline{R}}) = 1$

$$\Rightarrow \underline{\underline{P}}_f = \underline{f} \otimes \underline{f} = (\underline{\underline{R}}\,\underline{e}) \otimes (\underline{\underline{R}}\,\underline{e}) = \underline{\underline{R}}(\underline{e} \otimes \underline{e})\underline{\underline{R}}^T =$$

$$= \underline{\underline{R}}\,\underline{\underline{P}}_e\,\underline{\underline{R}}^T$$

isotropic: $\quad \hat{\underline{\underline{\sigma}}}(\underline{\underline{R}}\,\underline{\underline{P}}_e\,\underline{\underline{R}}^T) = \underline{\underline{R}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_e)\,\underline{\underline{R}}^T$

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_f) = \underline{\underline{R}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_f)\,\underline{\underline{R}}^T$$

$$\Rightarrow \underline{\underline{R}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_f)\,\underline{\underline{R}}^T - \hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_f) = \underline{\underline{0}}$$

substituting: $\quad \hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_f) = \lambda(\underline{f})\underline{\underline{I}} + 2\mu(\underline{f})\,\underline{\underline{P}}_f$

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_e) = \lambda(\underline{e})\underline{\underline{I}} + 2\mu(\underline{e})\,\underline{\underline{P}}_e$$

$$[\lambda(\underline{e}) - \lambda(\underline{f})]\,\underline{\underline{I}} + 2[\mu(\underline{e}) - \mu(\underline{f})]\,\underline{\underline{P}}_f = \underline{\underline{0}}$$

since $\underline{\underline{I}}$ and $\underline{\underline{P}}_f$ are linearly independent

$$\Rightarrow \lambda(\underline{e}) = \lambda(\underline{f}) = \lambda \qquad \mu(\underline{e}) = \mu(\underline{f}) = \mu$$

λ & μ are constants

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_{v_i}) = \lambda\,\underline{\underline{I}} + 2\mu\,\underline{\underline{P}}_{v_i}$$

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{d}}) = \hat{\underline{\underline{\sigma}}}\left(\sum_{i=1}^{3} \omega_i \underline{\underline{P}}_{v_i}\right) = \sum_{i=1}^{3} \omega_i \, \hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_{v_i})$$

$$= \sum_{i=1}^{3} \omega_i \left(\lambda \underline{\underline{I}} + 2\mu \underline{\underline{P}}_{v_i}\right)$$

$$= \lambda \underbrace{(\omega_1 + \omega_2 + \omega_3)}_{tr(\underline{\underline{d}})} \underline{\underline{I}} + 2\mu \underbrace{\left(\omega_1 \underline{\underline{P}}_{v_1} + \omega_2 \underline{\underline{P}}_{v_2} + \omega_3 \underline{\underline{P}}_{v_3}\right)}_{\underline{\underline{d}}}$$

Representation Thm for linear isotropic functions

$$\boxed{\hat{\underline{\underline{\sigma}}} = \lambda \, tr(\underline{\underline{d}}) \, \underline{\underline{I}} + 2\mu \, \underline{\underline{d}}}$$

Representation for linear isotropic Tensor function

An linear isotropic function $\underline{\underline{G}}(\underline{\underline{E}})$ that maps symmetric tensors $\underline{\underline{E}}$ into symmetric tensors $\underline{\underline{G}}(\underline{\underline{E}})$ must have following form

$$\boxed{\underline{\underline{G}}(\underline{\underline{E}}) = \lambda \, tr(\underline{\underline{E}}) \, \underline{\underline{I}} + 2\mu \underline{\underline{E}}}$$

where $\lambda, \mu \in \mathbb{R}$ are scalars

if we substitute $\underline{\underline{E}} = sym(\underline{\underline{A}})$ and $tr(sym(\underline{\underline{A}})) = tr(\underline{\underline{A}})$

$$\Rightarrow \quad \underline{\underline{G}}(A) = \lambda \, tr(\underline{\underline{A}}) \, \underline{\underline{I}} + 2\mu \, sym(\underline{\underline{A}})$$

Representation of isotropic tensor functions

An isotropic function $\underline{\underline{G}}(\underline{\underline{A}}) : \mathcal{V}^2 \to \mathcal{V}^2$ that maps symmetric tensors to symmetric tensors must have the following form

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \alpha_0(I_A)\,\underline{\underline{I}} + \alpha_1(I_A)\,\underline{\underline{A}} + \alpha_2(I_A)\,A^2}$$

Rivlin - Ericksen representation Thm

where α_0, α_1 and α_2 are functions of the set of pricipal invariants of $\underline{\underline{A}}$, $I_A = \{I_1(\underline{\underline{A}}), I_2(\underline{\underline{A}}), I_3(\underline{\underline{A}})\}$

- $\underline{\underline{G}}$ is clearly sym. if $\underline{\underline{A}}$ is sym.
- To see $\underline{\underline{G}}$ is isotropic assume α_0, α_1, α_2 = const

$$\underline{\underline{G}}(\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T) = \alpha_0 \underline{\underline{I}} + \alpha_1 \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T + \alpha_2 \underline{\underline{Q}}\,\underline{\underline{A}}\,\underbrace{\underline{\underline{Q}}^T\underline{\underline{Q}}}_{\underline{\underline{I}}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T$$

$$= \alpha_0 \underline{\underline{I}} + \alpha_1 \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T + \alpha_2 \underline{\underline{Q}}\,\underline{\underline{A}}^2\underline{\underline{Q}}^T$$

$$\underline{\underline{Q}}\,\underline{\underline{G}}(\underline{\underline{A}})\,\underline{\underline{Q}}^T = \alpha_0 \underline{\underline{Q}}\,\underline{\underline{Q}}^T + \alpha\, \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T + \alpha\, \underline{\underline{Q}}\,\underline{\underline{A}}^2\underline{\underline{Q}}^T$$

$$\Rightarrow \quad \underline{\underline{G}}(\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\underline{\underline{G}}(\underline{\underline{A}})\,\underline{\underline{Q}}^T$$

isotropic for constant coefficients.

If coefficents α_0, α_1, α_2 only depend on the invariants of $\underline{\underline{A}}$, then $\underline{\underline{G}}$ remains isotropic.

This is the most general form of a constitutive eqn for an isotropic material.

Linear Isotropic Function

Most standard constitutive laws are linear

If in addition we require:

1) $\mathbb{C}\underline{A} \in \mathcal{V}^2$ is symmetric for every symmetric $\underline{A} \in \mathcal{V}^2$

2) $\mathbb{C}\,\underline{\underline{W}} = \underline{\underline{0}}$ for every skew-symmetric $\underline{\underline{W}} \in \mathcal{V}^2$

Then there are scalars μ and λ such that

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\,\underline{\underline{A}} = \lambda\,\mathrm{tr}(\underline{\underline{A}})\,\underline{\underline{I}} + 2\mu\,\mathrm{sym}(\underline{\underline{A}})} \quad \text{for all } \underline{\underline{A}} \in \mathcal{V}^2$$

This follows from the representation Thm

$$\underline{\underline{G}}(\underline{\underline{A}}) = \alpha_0(\mathcal{I}_A)\,\underline{\underline{I}} + \alpha_1(\mathcal{I}_A)\,\underline{\underline{A}} + \alpha_2(\mathcal{I}_A)\,\underline{\underline{A}}^2$$

where the set of invariants of $\underline{\underline{A}}$ is

$$\mathcal{I}_A = \left\{ \mathrm{tr}\,\underline{\underline{A}},\ \tfrac{1}{2}\left[(\mathrm{tr}\,\underline{\underline{A}})^2 - \mathrm{tr}(\underline{\underline{A}}^2)\right],\ \det \underline{\underline{A}} \right\}$$ (Lecture 6)

Note that only $\mathrm{tr}(\underline{\underline{A}})$ is linear function!

since $\underline{\underline{G}}(\underline{\underline{A}})$ is linear in $\underline{\underline{A}}$ the only possibilities are

$\alpha_0(I_A) = c_0 \,\mathrm{tr}\,\underline{\underline{A}} + c_1$, $\alpha_1(I_A) = c_2$ and $\alpha_2(I_A) = 0$

where c_0, c_1 and c_2 are scalar constants.

Since $\underline{\underline{G}}(\underline{\underline{0}}) = \underline{\underline{0}} \Rightarrow c_1 = 0$

Hence setting $c_0 = \lambda$ and $c_2 = 2\lambda$

$$\underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\,\underline{\underline{A}} = \lambda\,\mathrm{tr}(\underline{\underline{A}}) + 2\mu\,\underline{\underline{A}}$$

since $\underline{\underline{G}}(\underline{\underline{W}}) = 0$ and $\mathrm{tr}(\underline{\underline{W}}) = 0$

$$\Rightarrow \underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\,\underline{\underline{A}} = \lambda\,\mathrm{tr}\,\underline{\underline{A}} + 2\mu\,\mathrm{sym}\,\underline{\underline{A}} \checkmark$$