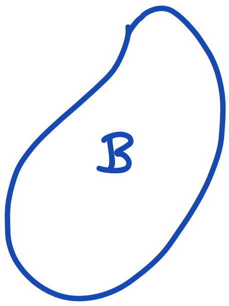
Mass and Density



Volume of a body B:

$$V_B = \int_B dV$$

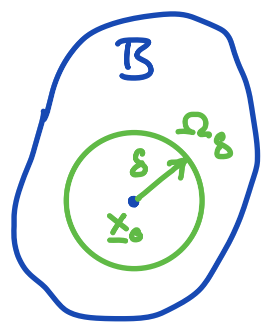
Mass of a body B:

$$m_B = \int_B \rho(\underline{x})\, dV$$

$\rho(\underline{x})$ = mass density field

At any point $\underline{x}_0$ in B

$$\boxed{\rho(\underline{x}_0) = \lim_{\delta \to 0} \frac{m_{\Omega_\delta}}{V_{\Omega_\delta}}}$$



Important geometric quantities of a body are:

Center of volume: $\underline{x}_V = \dfrac{1}{V_B} \displaystyle\int_B \underline{x}\, dV$

Center of mass: $\underline{x}_m = \dfrac{1}{m_B} \displaystyle\int_B \rho(\underline{x})\, \underline{x}\, dV$

Note: $\rho = \text{const}$

$$\underline{x}_m = \frac{1}{m_\Omega} \int_\Omega \rho\, \underline{x}\, dV = \frac{\rho}{\rho V_\Omega} \int \underline{x}\, dV = \frac{1}{V_\Omega} \int_\Omega \underline{x}\, dV = \underline{x}_V$$

Important because <u>resulting</u> forces.

Body Forces

Any force that not due to physical contact is a body force and acts on the entire body.

Example: gravitational body force

$$\boxed{\underline{b}_g = \rho \underline{g}} \qquad \left[\frac{M}{L^3}\frac{L}{T^2} = \frac{M}{L^2T^2}\right]$$

$\Rightarrow$ body force field has units of $\frac{\text{force}}{\text{volume}}$

If a body force acts on a body B the net or resultant body force is:

$$\boxed{\underline{r}_b[B] = \int_B \underline{b}(\underline{x})\, dV} \qquad \text{units of force} \left[\frac{ML}{T^2}\right]$$

Resultant force due to Gravity.

$$\underline{f}_G = \underline{r}_b[B] = \int_B \rho_b \underline{g}\, dV \qquad \text{if } \underline{g} \text{ is constant}$$

$$= \underline{g} \int_B \rho_b\, dV = m_b \underline{g}$$

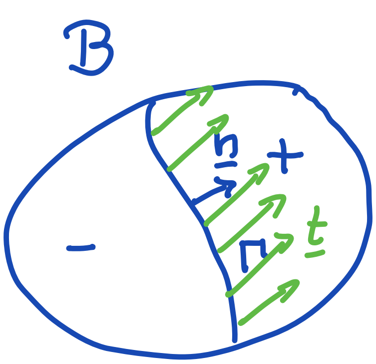
$\boxed{\underline{f}_G = m_b \underline{g}}$ $\longleftrightarrow$ Weight of body

Surface/Contact Forces

arise due to the physical contact between bodies. Forces along imaginary surfaces within a body are called internal forces while forces along the bounding surface of a body are external.

Internal surface forces hold a body together. External surface forces describe the interaction with the environment.

Traction Field



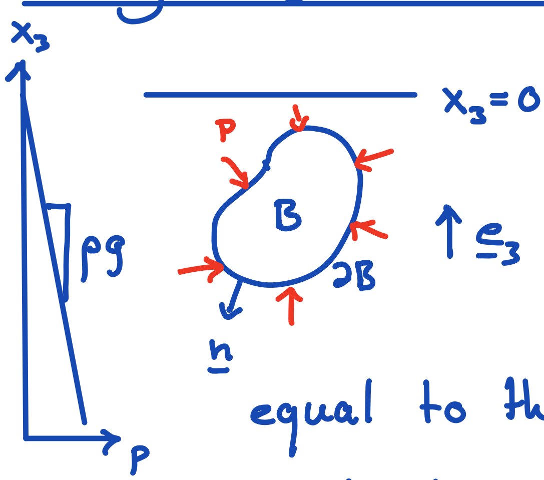
Consider an arbitrary surface Γ in B with unit normal $\underline{n}(\underline{x})$ that defines the positive and negative sides of B.

The force per unit area exerted by material on the pos. side upon material on the neg. side is given by the traction field $\underline{t}_n$ for Γ.

The resultant force due to a traction field on Γ is

$$\boxed{\underline{F}_s[\Gamma] = \int_\Gamma \underline{t}_n(\underline{x})\, dA}$$

Buoyancy: Resultant hydrostatic surface force



Any object, wholly or partially submerged in a fluid is bouyed up by a force equal to the weight of the fluid displaced by the body (Archimedes principle).

Hydrostatic pressure: $p = -\rho_f g x_3$

Hydrostatic traction on ∂B: $\underline{t} = -p\underline{n}$

Resulting surface force:

$$\underline{f}_B = \underline{\Gamma}_s[\partial B] = \int_{\partial B} \underline{t}\, dA = -\int_{\partial B} p\, \underline{n}\, dA$$

need to convert this to volume integral

$\Rightarrow$ Gradient theorem $\boxed{\int_{\partial\Omega} \phi\, \underline{n}\, dA = \int_{\Omega} \nabla\phi\, dV}$ $\rightarrow$ HW

$$\Rightarrow \quad \underline{f}_B = -\int_{\partial B} p\, \underline{n}\, dA = -\int_{B} \nabla p\, dV$$

where $\nabla p = \nabla(-\rho_f g x_3) = -\rho_f g\, \underline{e}_3 = \rho_f \underline{g}$

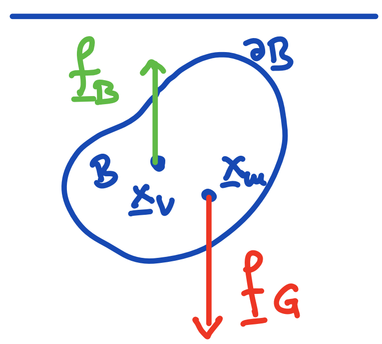
$$\underline{f}_B = -\int_B \rho_f \underline{g}\, dV = -\underline{g} \int_B \underbrace{\rho_f\, dV}_{V_B} = -m_f \underline{g}$$

$$\boxed{\underline{f}_B = -m_f \underline{g}}$$

Buoyancy force is minus the weight of the displaced fluid (Archimedes ✓)

Hydrostatic force balance

Total resultant force $\underline{f}$ on a submerged body in a gravitational field is the sum of weight and buoyancy.



$$\underline{f} = \underline{f}_G + \underline{f}_B = \underline{r}_b[B] + \underline{r}_s[\partial B]$$

$$= -\int_B \rho_b g \underline{e}_3 \, dV - \oint_{\partial B} p \underline{n} \, dS$$

substituting:

$$\boxed{\underline{f} = \int_B (\rho_f - \rho_b) \, g \, \underline{e}_3 \, dV = (m_f - m_b) \, g \, \underline{e}_3}$$

$\rho_f > \rho_b$: $\underline{f}$ points up $\rightarrow$ body rises (pos. buoyancy)

$\rho_f < \rho_b$: $\underline{f}$ points down $\rightarrow$ body sinks (neg. buoyancy)

$\rho_f = \rho_b$: $\underline{f} = \underline{0}$ $\rightarrow$ body is neutrally buoyant

Note: The integrated expression assumes $g = \text{const}$.