Second-order tensors

Here we are interested in second-order tensors

Linear operators : $\underline{v} = \underline{\underline{A}}\,\underline{u}$

maps vector $\underline{u} \in \mathcal{V}$ into vector $\underline{v} \in \mathcal{V}$

Linearity requires that

1) $\underline{\underline{A}}(\underline{u} + \underline{v}) = \underline{\underline{A}}\,\underline{u} + \underline{\underline{A}}\,\underline{v}$ for all $\underline{u}, \underline{v} \in \mathcal{V}$

2) $\underline{\underline{A}}(\alpha \underline{v}) = \alpha \underline{\underline{A}}\,\underline{v}$ for all $\alpha \in \mathbb{R}$ and $\underline{v} \in \mathcal{V}$

Example: $\underline{\underline{A}}$ maps every $\underline{v} \in \mathcal{V}$ into $\underline{n} \neq \underline{0} \in \mathcal{V}$.

Is $\underline{\underline{A}}$ a tensor?

Consider $\underline{u}, \underline{v}, \underline{w} \in \mathcal{V}$

$\underline{w} = \underline{u} + \underline{v}$

$$\underline{\underline{A}}(\underline{u} + \underline{v}) \overset{?}{=} \underline{\underline{A}}\,\underline{u} + \underline{\underline{A}}\,\underline{v}$$

$$\underline{\underline{A}}\,\underline{w} \overset{?}{=} \underline{\underline{A}}\,\underline{u} + \underline{\underline{A}}\,\underline{v}$$

$$\underline{n} \neq \underline{n} + \underline{n}$$

$\Rightarrow$ $\underline{\underline{A}}$ is <u>not</u> a tensor, because it is not linear

Tensor algebra

For all $\underline{v} \in \mathcal{V}$ we define

1) $(\alpha \underline{\underline{A}}) \underline{v} = \underline{\underline{A}} (\alpha \underline{v})$ scalar multiplication
2) $(\underline{\underline{A}} + \underline{\underline{B}}) \underline{v} = \underline{\underline{A}} \underline{v} + \underline{\underline{B}} \underline{v}$ tensor sum
3) $(\underline{\underline{A}} \underline{\underline{B}}) \underline{v} = \underline{\underline{A}} (\underline{\underline{B}} \underline{v})$ tensor product

Note there is also a scalar product introduced later.

The set of all 2^{nd}-order tensors $\mathcal{V}^2$ is a vector space

1) $\alpha \underline{\underline{A}} \in \mathcal{V}^2$ for all $\underline{\underline{A}} \in \mathcal{V}$ and $\alpha \in \mathbb{R}$
2) $\underline{\underline{A}} + \underline{\underline{B}} \in \mathcal{V}^2$ for all $\underline{\underline{A}}, \underline{\underline{B}} \in \mathcal{V}^2$
3) $\underline{\underline{A}} \underline{\underline{B}} \in \mathcal{V}^2$ for all $\underline{\underline{A}}, \underline{\underline{B}} \in \mathcal{V}^2$

Any of these operations will produce another second-order tensor.

Q: What is a basis for $\mathcal{V}^2$?

Two tensors $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are equal if

$$\underline{\underline{A}}\,\underline{v} = \underline{\underline{B}}\,\underline{v} \quad \text{for all } \underline{v} \in \mathcal{V}$$

Zero tensor: $\underline{\underline{0}}\,\underline{v} = \underline{0}$ for all $\underline{v} \in \mathcal{V}$

Identity tensor: $\underline{\underline{I}}\,\underline{v} = \underline{v}$ for all $\underline{v} \in \mathcal{V}$

Representation of a tensor

In a frame $\{\underline{e}_i\}$ a second order tensor $\underline{\underline{S}}$ is represented by nine numbers

$$\boxed{S_{ij} = \underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j}$$

Consider $\underline{v} = \underline{\underline{S}}\,\underline{u}$ where $\underline{v} = v_i \underline{e}_i$, $\underline{u} = u_j \underline{e}_j$

$$v_i \underline{e}_i = \underline{\underline{S}}(u_j \underline{e}_j) = \underline{\underline{S}}\,\underline{e}_j\, u_j$$

multiply by $\underline{e}_k$ from left

$$v_i\, \underline{e}_k \cdot \underline{e}_i = \underline{e}_k \cdot \underline{\underline{S}}\,\underline{e}_j\, u_j$$

$$v_i\, \delta_{ki} = \underline{e}_k \cdot \underline{\underline{S}}\,\underline{e}_j\, u_j$$

$$v_i = (\underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j)\, u_j \qquad \boxed{v_i = S_{ij}\, u_j}$$

Matrix representation of tensor in $\{\underline{e}_i\}$

$$[\underline{\underline{S}}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \in \mathbb{R}^3 \times \mathbb{R}^3$$

Note that $[\underline{\underline{S}}]_{ij} = S_{ij}$

Dyadic Product

The dyadic product of two vectors $\underline{a}$ and $\underline{b}$ is the 2^{nd}-order tensor $\underline{a} \otimes \underline{b}$ defined by

$$\boxed{(\underline{a} \otimes \underline{b})\underline{v} = (\underline{b} \cdot \underline{v})\,\underline{a}} \qquad \text{for all } \underline{v} \in \mathcal{V}$$

This has the form: $\underline{\underline{A}}\,\underline{v} = \alpha\,\underline{a}$

in components: $A_{ij} v_j = \alpha\, a_i$

$\alpha = \underline{b} \cdot \underline{v} = b_j v_j$

$A_{ij} = [\underline{a} \otimes \underline{b}]_{ij}$

$\Rightarrow \quad [\underline{a} \otimes \underline{b}]_{ij}\, v_j = b_j v_j\, a_i$

$$[\underline{a}\otimes\underline{b}]_{ij}\, v_j = (a_i\, b_j)\, v_j$$

$$\Rightarrow \quad [\underline{a}\otimes\underline{b}]_{ij} = a_i b_j$$

So that

$$\boxed{[\underline{a}\otimes\underline{b}] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} = \underline{a}\,\underline{b}^T}$$

Linearity of dyadic product:

for scalars $\alpha, \beta \in \mathbb{R}$ and vectors $\underline{a}, \underline{b}, \underline{v}, \underline{w} \in \mathcal{V}$

$$\boxed{(\underline{a}\otimes\underline{b})(\alpha\,\underline{v} + \beta\,\underline{w}) = \alpha(\underline{a}\otimes\underline{b})\underline{v} + \beta(\underline{a}\otimes\underline{b})\underline{w}}$$

The product of two dyadic products

$$\boxed{(\underline{a}\otimes\underline{b})(\underline{c}\otimes\underline{d}) = (\underline{b}\cdot\underline{c})\,\underline{a}\otimes\underline{d}} \Rightarrow \text{HW2}$$

needed for tensor product.

Basis for $\mathcal{V}^2$

Given any frame $\{\underline{e}_i\}$ the nine dyadic products $\{\underline{e}_i \otimes \underline{e}_j\}$ form a basis for $\mathcal{V}^2$.

Any second-order tensor $\underline{\underline{S}}$ can be written as linear combination

$$\boxed{\underline{\underline{S}} = S_{ij}\, \underline{e}_i \otimes \underline{e}_j}$$

where $\quad S_{ij} = \underline{e}_i \cdot \underline{\underline{S}}\, \underline{e}_j$

Consider $\quad \underline{v} = \underline{\underline{S}}\,\underline{u} \quad$ with $\quad \underline{v} = v_i\, \underline{e}_i, \; \underline{u} = u_k \underline{e}_k$

$$
\begin{aligned}
v_i \underline{e}_i &= S_{ij} (\underline{e}_i \otimes \underline{e}_j)(u_k \underline{e}_k) \\
&= S_{ij}\, u_k\, (\underline{e}_i \otimes \underline{e}_j) \cdot \underline{e}_k \qquad \text{apply def. of dyadic} \\
&= S_{ij}\, u_k (\underline{e}_j \cdot \underline{e}_k)\, \underline{e}_i \; = \; S_{ij}\, u_k\, \delta_{kj}\, \underline{e}_i
\end{aligned}
$$

$$\boxed{\begin{aligned} v_i\, \underline{e}_i &= S_{ij}\, u_j\, \underline{e}_i \\ v_i &= S_{ij}\, u_j \end{aligned}} \qquad \text{used often}$$

Tensor algebra in components

Addition : $\underline{\underline{H}} = \underline{\underline{S}} + \underline{\underline{T}}$

$$H_{ij}\, \underline{e}_i \otimes \underline{e}_j = S_{ij}\, \underline{e}_i \otimes \underline{e}_j + T_{ij}\, \underline{e}_i \otimes \underline{e}_j$$
$$= (S_{ij} + T_{ij})\, \underline{e}_i \otimes \underline{e}_j$$
$$\boxed{H_{ij} = S_{ij} + T_{ij}}$$

Scalar multiplication : $\underline{\underline{H}} = \alpha \underline{\underline{S}} \Rightarrow \boxed{H_{ij} = \alpha S_{ij}}$

Product : $\underline{\underline{H}} = \underline{\underline{S}}\, \underline{\underline{T}}$

$$\underline{\underline{H}} = S_{ij} (\underline{e}_i \otimes \underline{e}_j)\, T_{kl} (\underline{e}_k \otimes \underline{e}_l)$$
$$= S_{ij} T_{kl} \underbrace{(\underline{e}_i \otimes \underline{e}_j)(\underline{e}_k \otimes \underline{e}_l)}_{\text{product of two dyads}}$$
$$= S_{ij} T_{kl} \underbrace{(\underline{e}_j \cdot \underline{e}_k)}_{\delta_{jk}} \underline{e}_i \otimes \underline{e}_l$$
$$= S_{ij} T_{jl}\, \underline{e}_i \otimes \underline{e}_l$$
$$H_{il}\, \underline{e}_i \otimes \underline{e}_l = S_{ij} T_{jl}\, \underline{e}_i \otimes \underline{e}_l$$

$$\Rightarrow \quad \boxed{H_{il} = S_{ij} T_{jl}}$$

note the dummy j!

Transpose of a tensor

To any $\underline{\underline{S}} \in \mathcal{V}^2$ we associate a transpose $\underline{\underline{S}}^T \in \mathcal{V}^2$ the unique tensor such that

$$\boxed{\underline{\underline{S}}\,\underline{u} \cdot \underline{v} = \underline{u} \cdot \underline{\underline{S}}^T \underline{v}} \quad \text{for all } \underline{u}, \underline{v} \in \mathcal{V}$$

This implies that $S^T_{ij} = S_{ji}$ as follows

$$(S_{ij} u_j \underline{e}_i) \cdot (v_\ell \underline{e}_\ell) = (u_k \underline{e}_k) \cdot (S^T_{ij} v_j \underline{e}_i)$$

$$S_{ij} u_j v_\ell (\underline{e}_i \cdot \underline{e}_\ell) = S^T_{ij} v_j u_k (\underline{e}_k \cdot \underline{e}_i)$$

$$S_{ij} u_j v_\ell \, \delta_{i\ell} = S^T_{ij} v_j u_k \, \delta_{ki}$$

$$S_{ij} u_j v_i = S^T_{ij} v_j u_i \qquad \text{rename indices } i \leftrightarrow j \text{ on rhs}$$

$$S_{ij} u_j v_i = S^T_{ji} u_j v_i$$

$$\Rightarrow \quad S_{ij} = S^T_{ji} \quad \checkmark$$

Properties of transpose:

$$(\underline{\underline{A}}^T)^T = \underline{\underline{A}}$$

$$(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$$

$$(\underline{u} \otimes \underline{v})^T = \underline{v} \otimes \underline{u}$$

$\underline{\underline{S}}$ is symmetric if $\underline{\underline{S}} = \underline{\underline{S}}^T$ $\quad S_{ij} = S_{ji}$

$\underline{\underline{S}}$ is skew-symmetric if $\underline{\underline{S}} = -\underline{\underline{S}}^T$ $\quad S_{ij} = -S_{ji}$

Symmetric-Skew decomposition:

Any tensor $\underline{\underline{S}} \in \mathcal{V}^2$ can be written as

$$\boxed{\begin{aligned} \underline{\underline{S}} &= \underline{\underline{E}} + \underline{\underline{W}} \\ \underline{\underline{E}} &= \tfrac{1}{2}(\underline{\underline{S}} + \underline{\underline{S}}^T) \\ \underline{\underline{W}} &= \tfrac{1}{2}(\underline{\underline{S}} - \underline{\underline{S}}^T) \end{aligned}}$$

$\underline{\underline{E}} = \underline{\underline{E}}^T$

$\underline{\underline{W}} = -\underline{\underline{W}}^T$

Note: $\underline{\underline{W}} = \begin{bmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & W_{23} \\ -W_{13} & -W_{23} & 0 \end{bmatrix}$ only 3 indep. comp.

$\Rightarrow$ can be related to an axial vector $\underline{w}$

$$\boxed{\underline{\underline{W}}\,\underline{v} = \underline{w} \times \underline{v}}$$ for all $\underline{v} \in \mathcal{V}$

Relation :

$$W_{ij} = -e_{ijk}\, w_k$$

$$w_k = -\tfrac{1}{2} e_{ijk}\, W_{ij}$$

$$\underline{\underline{W}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Trace of a tensor

We define the trace of a dyad as

$$\mathrm{tr}(\underline{a} \otimes \underline{b}) = \underline{a} \cdot \underline{b} = a_i b_i$$

this implies that

$$\boxed{\mathrm{tr}(\underline{\underline{A}}) = A_{ii} = A_{11} + A_{22} + A_{33}}$$

as follows $\quad \mathrm{tr}(A_{ij}\, \underline{e}_i \otimes \underline{e}_j) = A_{ij}\, \mathrm{tr}(\underline{e}_i \otimes \underline{e}_j)$

$$= A_{ij}\, \delta_{ij} = A_{ii}$$

Properties: $\mathrm{tr}(\underline{\underline{A}}^T) = \mathrm{tr}(\underline{\underline{A}})$

$$\mathrm{tr}(\underline{\underline{A}}\,\underline{\underline{B}}) = \mathrm{tr}(\underline{\underline{B}}\,\underline{\underline{A}})$$

$$\mathrm{tr}(\underline{\underline{A}} + \underline{\underline{B}}) = \mathrm{tr}(\underline{\underline{A}}) + \mathrm{tr}(\underline{\underline{B}})$$

$$\mathrm{tr}(\alpha \underline{\underline{A}}) = \alpha\, \mathrm{tr}(\underline{\underline{A}})$$

Decomposition: $\boxed{\underline{\underline{A}} = \alpha \underline{\underline{I}} + \mathrm{dev}\, \underline{\underline{A}}}$

Sperical tensor: $\alpha \underline{\underline{I}}$ where $\alpha = \frac{1}{3}\mathrm{tr}(\underline{\underline{A}})$

Deviatoric tensor: $\mathrm{dev}\, \underline{\underline{A}} = \underline{\underline{A}} - \alpha \underline{\underline{I}}$

$$\mathrm{tr}(\mathrm{dev}\, \underline{\underline{A}}) = 0$$

Tensor scalar product (Contraction)

analogous to scalar product of vectors

$$\boxed{\underline{\underline{A}} : \underline{\underline{B}} = tr(\underline{\underline{A}}^T \underline{\underline{B}}) = A_{ij} B_{ij}}$$

scalar ✓

explicitly:

$$\underline{\underline{A}} : \underline{\underline{B}} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} B_{ij} = A_{11} B_{11} + A_{12} B_{12} + A_{13} B_{13} + \dots$$
$$A_{21} B_{21} + A_{22} B_{22} + A_{23} B_{23} + \dots$$
$$A_{31} B_{31} + A_{32} B_{32} + A_{33} B_{33}$$

The index expression is derived as follows

$$\underline{\underline{A}} : \underline{\underline{B}} = tr(\underline{\underline{A}}^T \underline{\underline{B}}) =$$

$$\underline{\underline{A}}^T \underline{\underline{B}} = (A_{ji}\, \underline{e}_i \otimes \underline{e}_j)(B_{kl}\, \underline{e}_k \otimes \underline{e}_l)$$
$$= A_{ji} B_{kl} (\underline{e}_i \otimes \underline{e}_j)(\underline{e}_k \otimes \underline{e}_l)$$
$$= A_{ji} B_{kl} (\underline{e}_j \cdot \underline{e}_k)(\underline{e}_i \otimes \underline{e}_l) = A_{ji} B_{kl}\, \delta_{jk}\, \underline{e}_i \otimes \underline{e}_l$$
$$= A_{ji} B_{jl}\, \underline{e}_i \otimes \underline{e}_l$$

$$tr(\underline{\underline{A}}^T \underline{\underline{B}}) = A_{ji} B_{jl}\, \delta_{il} = A_{ji} B_{ji} = A_{ij} B_{ij} \quad ✓$$

Properties: 1) $\underline{\underline{A}} : \underline{\underline{B}} = \underline{\underline{B}} : \underline{\underline{A}}$

2) $(\underline{a} \otimes \underline{b}) : (\underline{c} \otimes \underline{d}) = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d})$

First follows from prop. of trace

$$\underline{\underline{A}} : \underline{\underline{B}} = \mathrm{tr}(\underline{\underline{A}}^T \underline{\underline{B}}) = \mathrm{tr}((\underline{\underline{A}}^T \underline{\underline{B}})^T) = \mathrm{tr}(\underline{\underline{B}}^T \underline{\underline{A}}) = \underline{\underline{B}} : \underline{\underline{A}}$$

Second property $[\underline{a} \otimes \underline{b}]_{ij} = a_i b_j$

$$(\underline{a} \otimes b) : (\underline{c} \otimes \underline{d}) = [\underline{a} \otimes \underline{b}]_{ij} [\underline{c} \otimes \underline{d}]_{ij} = a_i b_j c_i d_j$$
$$= a_i c_i b_j d_j$$
$$= (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d})$$

A common norm for tensors is

$$\boxed{|\underline{\underline{A}}| = \sqrt{\underline{\underline{A}} : \underline{\underline{A}}} = \sqrt{A_{ij} A_{ij}} \geq 0}$$

Note: Tensor scalar product will be important to express the work done during deformation. For example the shear heating in glaciology.

Determinant and Inverse

The determinat of $\underline{\underline{A}} \in \mathcal{V}^2$ is the salar

$$\boxed{\det(\underline{\underline{A}}) = \det \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \epsilon_{ijk} \, [\underline{\underline{A}}]_{i1} \, [\underline{\underline{A}}]_{j2} \, [\underline{\underline{A}}]_{k3}}$$

where $[\underline{\underline{A}}]_{i1}$, $[\underline{\underline{A}}]_{j2}$, $[\underline{\underline{A}}]_{k3}$ are the columns of $[\underline{\underline{A}}]$

Properties: $\det(\underline{\underline{A}}\,\underline{\underline{B}}) = \det(\underline{\underline{A}}) \det(\underline{\underline{B}})$

$$\det(\underline{\underline{A}}^T) = \det(\underline{\underline{A}})$$

$$\det(\alpha \underline{\underline{A}}) = \alpha^n \det(\underline{\underline{A}}) \qquad (\underline{\underline{A}} \text{ is } n \times n)$$

$\underline{\underline{A}}$ is singular if $\det \underline{\underline{A}} = 0$.

If $\det \underline{\underline{A}} \neq 0$ then the inverse $\underline{\underline{A}}^{-1}$ exists

$$\boxed{\underline{\underline{A}}^{-1} \underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}}}$$

Properties: $(\underline{\underline{A}}\,\underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}$

$$(\underline{\underline{A}}^{-1})^{-1} = \underline{\underline{A}}$$

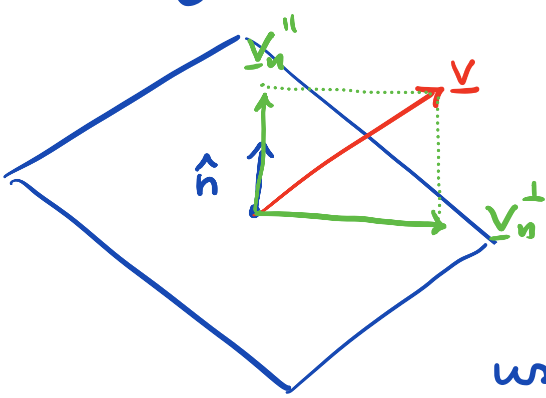
$$(\underline{\underline{A}}^{-1})^{T} = (\underline{\underline{A}}^{T})^{-1}$$

$$(\alpha A)^{-1} = \frac{1}{\alpha}\underline{\underline{A}}^{-1}$$

$$\det(\underline{\underline{A}}^{-1}) = \det(\underline{\underline{A}})^{-1} = \frac{1}{\det(\underline{\underline{A}})}$$

Projection & Reflection tensors

commonly used to partition forces on a surface.



$$\underline{v} = \underline{v}_n^{\parallel} + \underline{v}_n^{\perp}$$

$$\underline{v}_n^{\parallel} = (\underline{v}\cdot\underline{\hat{n}})\,\underline{\hat{n}}$$

$$\underline{v}_n^{\perp} = \underline{v} - \underline{v}_n^{\parallel}$$

use dyadic property

$$\underline{v}_n^{\parallel} = (\underline{v}\cdot\underline{\hat{n}})\underline{\hat{n}} = (\underline{\hat{n}}\otimes\underline{\hat{n}})\,\underline{v} = \underline{\underline{P}}_n^{\parallel}\,\underline{v}$$

$$\underline{v}_n^{\perp} = \underline{v} - (\underline{\hat{n}}\otimes\underline{\hat{n}})\underline{v} = (\underline{\underline{I}} - \underline{\hat{n}}\otimes\underline{\hat{n}})\,\underline{v} = \underline{\underline{P}}_n^{\perp}\,\underline{v}$$

$$\boxed{\underline{\underline{P}}_n^{\parallel} = \underline{\hat{n}}\otimes\underline{\hat{n}}} \qquad \boxed{\underline{\underline{P}}_n^{\perp} = \underline{\underline{I}} - \underline{\hat{n}}\otimes\underline{\hat{n}}}$$

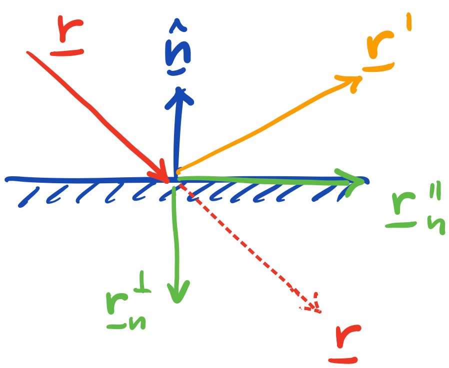
Properties: $\underline{\underline{P}} = \underline{\underline{P}}^T$ symmetric

$$\underline{\underline{P}}^2 = \underline{\underline{P}}$$

$$\underline{\underline{P}}^{\parallel} + \underline{\underline{P}}^{\perp} = \underline{\underline{I}}$$

$$\underline{\underline{P}}^{\parallel} \underline{\underline{P}}^{\perp} = \underline{\underline{0}}$$

Reflections



incoming: $\underline{r} = \underline{r}_n^{\parallel} + \underline{r}_n^{\perp}$

reflected: $\underline{r}' = \underline{r}_n^{\parallel} - \underline{r}_n^{\perp}$

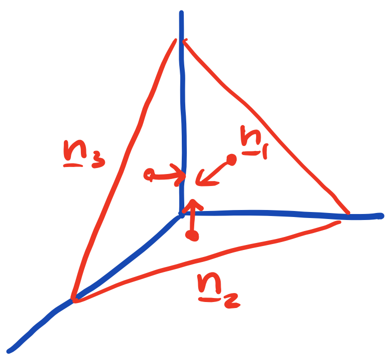
$$\underline{r}' = (\underline{\underline{P}}_n^{\parallel} - \underline{\underline{P}}^{\perp})\, \underline{r}$$

$$\underline{r}' = (\underline{\underline{I}} - 2\, \hat{\underline{n}} \otimes \hat{\underline{n}})\, \underline{r}$$

$$\underline{r}' = \underline{\underline{R}}_n\, \underline{r}$$

Reflection tensor: $\boxed{\underline{\underline{R}}_n = \underline{\underline{I}} - 2\, \hat{\underline{n}} \otimes \hat{\underline{n}}}$

Example: Corner reflector



Inverts the direction of any ray that reflects off all three surfaces.

Direction of triply reflected ray:

$$\underline{r}''' = \underline{\underline{R}}_{n_1} \underline{\underline{R}}_{n_2} \underline{\underline{R}}_{n_3} \underline{r}$$

Show that $\underline{r}''' = -\underline{r}$!