Jacobian for transient unconfined flow

Consider the transient unconfined equation

$$\phi_0 h^m \frac{\partial h}{\partial t} - \nabla \cdot \left[\frac{K_0}{n+1} h^{n+1} \nabla h \right] = f_s$$

lets write this as a general non-linear diffusion eqn

$$\boxed{s(u) \frac{\partial u}{\partial t} - D_h \nabla \cdot \left[f(u) \nabla u \right] = f_s}$$

where $s(u)$ and $f(u)$ are arbitrary differentiable functions. In the case of unconfined flow $s = \phi_0 u^m$ and $f = \frac{K_0}{n+1} u^{n+1}$.

Discretization with Backward Euler

$$\{\underline{s}(\underline{u}^{n+1})\}_c (\underline{u}^{n+1} - \underline{u}^n) - \Delta t \, \underline{\underline{D}} \left[\{\underline{\underline{M}} \, \underline{f}(\underline{u}^{n+1})\}_f \, \underline{\underline{G}} \, \underline{u}^{n+1} \right] = \Delta t \, f_s$$

Discrete residual: $\quad \underline{u}^{n+1} \equiv \underline{u}$

$$\boxed{\underline{r}(\underline{u}) = \{\underline{s}(\underline{u})\}_c (\underline{u} - \underline{u}^n) - \Delta t \, D \left[\{\underline{\underline{M}} \, \underline{f}(\underline{u})\}_f \, \underline{\underline{G}} \, \underline{u} \right] - \Delta t \, f_s}$$

Now we have to solve the non-linear problem for $\underline{u} = \underline{u}^{n+1}$ using Newton-Raphson method. We have to be careful to distinguish the k superscripts for the iteration from the n superscripts for the timestep.

Directional derivative $D_{\underline{\hat u}}\, \underline{r}(\underline{\bar u})$

$$\frac{d}{d\epsilon}\, \underline{r}(\underline{\bar u}+\epsilon\underline{\hat u})\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\underbrace{\{s(\underline{\bar u}+\epsilon\underline{\hat u})\}_c(\underline{\bar u}+\epsilon\underline{\hat u}-\underline{u}^n)}_{acc} - \underbrace{\Delta t\, \underline{\underline{D}}\left[\{\underline{\underline{M}}\, f(\underline{\bar u}+\epsilon\underline{\hat u})\}_f\, \underline{\underline{G}}(\underline{\bar u}+\epsilon\underline{\hat u})\right]}_{flux} - \cancel{\Delta t\, \underline{f_s}}\Big|_{\epsilon=0}$$

acc:

$$\left\{\frac{ds}{du}(\underline{\bar u}+\cancelto{0}{\epsilon\underline{\hat u}})\, \underline{\hat u}\right\}_c(\underline{\bar u}+\cancelto{0}{\epsilon\underline{\hat u}}-\underline{u}^n) + \{\underline{s}(\underline{\bar u}+\cancelto{0}{\epsilon\underline{\hat u}})\}_c\, \underline{\hat u}\,\Big|_{\epsilon=0}$$

$$\left\{\frac{ds}{du}(\underline{\bar u})\, \underline{\hat u}\right\}_c(\underline{\bar u}-\underline{u}^n)+\{\underline{s}(\underline{\bar u})\}_c\, \underline{\hat u} = \underbrace{\left\{\frac{ds}{du}\right\}_c}_{\underline{\underline{dS}}}\{\underline{\bar u}-\underline{u}^n\}_c\, \underline{\hat u}+\underbrace{\{\underline{s}(\underline{\bar u})\}_c}_{\underline{\underline{S}}}\, \underline{\hat u}$$

$$acc = \left[\underline{\underline{dS}}\, \{\underline{\bar u}-\underline{u}^n\}_c+\underline{\underline{S}}\right]\underline{\hat u}$$

flux:

$$\Delta t\, \underline{\underline{D}}\left[\left\{\underline{\underline{M}}\frac{df}{du}(\underline{\bar u}+\cancelto{0}{\epsilon\underline{\hat u}})\, \underline{\hat u}\right\}_f \underline{\underline{G}}(\underline{\bar u}+\cancelto{0}{\epsilon\underline{\hat u}})+\{\underline{\underline{M}}\, \underline{f}(\underline{\bar u}+\cancelto{0}{\epsilon\underline{\hat u}})\}_f\, \underline{\underline{G}}\, \underline{\hat u}\right]\Big|_{\epsilon=0}$$

$$\Delta t\, \underline{\underline{D}}\left[\underbrace{\{\underline{\underline{G}}\, \underline{\bar u}\}_f}_{\underline{\underline{Gu}}}\underbrace{\left\{\underline{\underline{M}}\frac{df}{du}(\underline{\bar u})\, \underline{\hat u}\right\}_f}_{?}+\underbrace{\{\underline{\underline{M}}\, \underline{f}(\underline{\bar u})\}_f}_{\underline{\underline{F}}}\, \underline{\underline{G}}\, \underline{\hat u}\right]$$

$$\left\{\underline{\underline{M}}\frac{df}{du}(\underline{\bar u})\, \underline{\hat u}\right\}_f = \underline{\underline{M}}\left\{\frac{df}{du}(\underline{\bar u})\, \underline{\hat u}\right\}_c = \underline{\underline{M}}\left\{\frac{df}{du}(\underline{\bar u})\right\}_c \hat u = \underline{\underline{M}}\, \underline{\underline{dF}}\, \underline{u}$$

$$flux = \Delta t\, \underline{\underline{D}}\left[\underline{\underline{Gu}}\, \underline{\underline{M}}\, \underline{\underline{dF}} + \underline{\underline{F}}\, \underline{\underline{G}}\right]\underline{\hat u}$$

So that we have:

$$D_{\hat{\underline{u}}}\, \underline{r}(\bar{\underline{u}}) = \left\langle \underline{\underline{dS}} \left\{ \bar{\underline{u}} - \underline{u}^n \right\}_c + \underline{\underline{S}} - \Delta t\, \underline{\underline{D}} \left[\underline{\underline{GU}}\, \underline{\underline{M}}\, \underline{\underline{dF}} + \underline{\underline{F}}\, \underline{\underline{G}} \right] \right\rangle \hat{\underline{u}}$$

Hence the Jacobian for unconfined flow is

$$\underline{\underline{J}}(\bar{\underline{u}}, \underline{u}^n) = \underline{\underline{dS}} \left\{ \bar{\underline{u}} - \underline{u}^n \right\}_c + \underline{\underline{S}} - \Delta t\, \underline{\underline{D}} \left[\underline{\underline{dF}} \left\{ \underline{\underline{G}}\, \bar{\underline{u}} \right\}_f + \underline{\underline{F}}\, \underline{\underline{G}} \right]$$