



A fractional SEIQR model on diphtheria disease

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Abstract

The purpose of this paper is to study a fractional mathematical model on diphtheria disease, considering the parameters of natural immunity, treatment, and vaccination. This model includes five compartments, namely, the susceptible, exposed, infected, quarantined, and recovered sub population. All compartments involve the memory effect and long-rate interactions that is modeled by a Caputo fractional derivative. This paper starts with the study of some analytical results. We first present the preliminary concepts of fractional calculus. The well-posedness of our fractional model is proved based on the boundedness, non-negativity, existence, and uniqueness. The existence of local, and global stability are also studied based on the Magniton's theorem and the appropriate Lyapunov function. We further apply the predictor–corrector scheme to establish the numerical simulations. The validation of our fractional model is based on the real data by using the least square technique. We also present the comparison results of root mean square error for varying fractional order $\alpha = 1$, $\alpha = 0.95$, $\alpha = 0.9$, $\alpha = 0.85$, and $\alpha = 0.8$.

Keywords Fractional-order differential equation · Stability analysis · Magniton's theorem · Lyapunov function · Least square technique · Predictor–corrector scheme · Vaccination · Disease transmission

Introduction

The proposed model of this paper is adapted from the former model in Izzati and Andriani (2021) under the case of integer-order derivative ($\alpha = 1$)

$$\begin{aligned} \frac{ds}{dt} &= (1-p)\mu N - \eta \frac{si}{N} - \delta s + \phi e, \\ \frac{de}{dt} &= \eta \frac{si}{N} - (\beta + \phi + \delta)e, \\ \frac{di}{dt} &= \beta e - (\gamma + \delta + \theta)i, \\ \frac{dq}{dt} &= \gamma i - (\varepsilon + \delta)q, \\ \frac{dr}{dt} &= p\mu N + \varepsilon q - \delta r, \end{aligned} \quad (1)$$

where the parameters N , p , μ , δ , η , β , γ , ε , ϕ , and θ represent the total population, the vaccination, the natural increase rate, the natural mortality rate, the probability of

interaction between susceptible and infected, the infection rate, the treatment rate (rate of quarantined individuals for each unit of time), the recovery rate, the exposed individuals with natural immunity, and the mortality due to diphtheria respectively. Moreover, we provide the following non-dimensional variables

$$s = \frac{S}{N}, \quad e = \frac{E}{N}, \quad i = \frac{I}{N}, \quad q = \frac{Q}{N}, \quad r = \frac{R}{N}, \quad (2)$$

where s , e , i , q , and r denote the proportion of number of individuals for each subpopulation. We substitute (2) into (1) to obtain

$$\begin{aligned} \frac{dS}{dt} &= (1-p)\mu - \eta SI - \delta S + \phi E, \\ \frac{dE}{dt} &= \eta SI - (\beta + \phi + \delta)E, \\ \frac{dI}{dt} &= \beta E - (\gamma + \delta + \theta)I, \\ \frac{dQ}{dt} &= \gamma I - (\varepsilon + \delta)Q, \\ \frac{dR}{dt} &= p\mu + \varepsilon Q - \delta R, \end{aligned} \quad (3)$$

The epidemiological's SEIQR model has been extensively studied in many cases as in Rahimi et al. (2021), Hussain

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et al. (2021), Youssef et al. (2021), Prabakaran et al. (2021), Youssef et al. (2022), Gerberry and Milner (2009), Bhadauria et al. (2022). We notice that the growth rates in the system (3) only capture on the current state. In fact, the growth rates of population capture not only the current state but also the memory effects and long-rate interactions which exist in the diphtheria model but are neglected in the case of integer-order derivative. The fractional derivatives are then applied to get fractional differentiation or to include such memory effects and long-rate interactions in a most biological system (Owolabi and Atangana 2017). We further notice that the fractional differential equations (FDEs) are generalized differential equations by employing the application of a fractional operator consisting of the non-integer order (Almeida et al. 2015; George and Chakrabarti 1995). In this paper, we consider the Caputo fractional derivative which is given by the following definition.

Definition 1 (See Podlubny 1999). Let $\alpha > 0$. Then the Caputo fractional derivative of order α of function $f \in C^n$ is given by

$$D_*^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{1+\alpha-n}} ds, \quad (4)$$

where $n = \lceil \alpha \rceil$. In particular, if $0 < \alpha \leq 1$, then (4) becomes

$$D_*^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds. \quad (5)$$

Since the Caputo fractional derivative of order α is the nonlocal operator (it includes the initial and current states), then the Caputo fractional derivative of order α is employed in the dynamical system model (3) to present the influence of the memory effects and long-rate interactions (see Hamdan and Kilicman 2018, 2019; Moustafa et al. 2018, 2019). Moreover, the growth rates in the system (3) provide the current condition and also the initial states to get the memory effects and long-rate interactions. The first derivatives of system (3) are then replaced by the Caputo fractional derivative of order α to obtain

$$\begin{aligned} D_*^\alpha S(t) &= (1-p)\mu - \eta SI - \delta S + \phi E, \\ D_*^\alpha E(t) &= \eta SI - (\beta + \phi + \delta)E, \\ D_*^\alpha I(t) &= \beta E - (\gamma + \delta + \theta)I, \\ D_*^\alpha Q(t) &= \gamma I - (\varepsilon + \delta)Q, \\ D_*^\alpha R(t) &= p\mu + \varepsilon Q - \delta R. \end{aligned} \quad (6)$$

where $S(0) = S_0$, $E(0) = E_0$, $I(0) = I_0$, $Q(0) = Q_0$, $R(0) = R_0$, and $0 < \alpha \leq 1$. Moreover, we consider $0 \leq p \leq 1$ for the proportion of vaccinated individuals in a population. We notice that the nonlinear fractional system (6) can be very

complicated to establish the analytical solution. In the case of such nonlinear dynamical systems, the Lie algebra can be employed to approach the exact solutions as studied in Shang (2012, 2013, 2017) and the fractional dynamical systems can be found in the study (Alshamrani et al. 2017; Gazizov and Kasatkin 2013). The Lie algebra constructs the symmetric properties of the system to find the solution. Unfortunately, this Lie algebra can not widely be used because the coupled differential equations (6) are complicated and have less symmetry. In this paper, we are not concerned with the analytical solution of fractional systems (6). The stability of system (3) has been studied in Izzati and Andriani (2020, 2021) for the case of integer-order derivative ($\alpha = 1$) with or without optimal control.

We further organize this paper as follows. In Section 2, we provide some basic definitions and the necessary results which are useful to prove our main results (boundedness, non-negativity, existence, uniqueness, local stability, and global stability). The boundedness, non-negativity, existence and uniqueness, basic reproduction number, local and global stability are studied in Section 3. The local stability is obtained through the theory of Routh–Hurwitz fractional-order criteria (for $0 < \alpha \leq 1$) (Ahmed et al. 2006) for both disease-free and endemic equilibrium points. Moreover, the global stability are proved based on basic definitions and necessary results in Section 2. The basic reproduction number is obtained by employing the next generation matrix. In Section 4, the numerical results consist of the discretization process of the model (6) by applying the predictor–corrector scheme. The simulation results with the different values of fractional-order α , vaccination p , natural immunity ϕ , and treatment γ are analyzed. Moreover, the validation of our fractional dynamical system (6) is also studied by fitting our simulation results of Runge–Kutta fourth order with the real data through the least square technique.

Preliminary concepts

In this section, we present some lemmas, theorems, and corollary which are useful to prove the main results of our paper. We first give the comparison theorem which is useful to establish the boundedness.

Theorem 1 (See Li et al. 2010, 2017) *Let $v(t) \in C([0, +\infty))$, $\alpha \in (0, 1]$, $\Phi, \psi \in \mathbb{R}$, and $\Phi \neq 0$. If $v(t)$ satisfies*

$$D_*^\alpha v(t) \leq -\Phi v(t) + \psi, \quad v(0) = v_0 \in \mathbb{R}, \quad (7)$$

then

$$v(t) \leq \left(v_0 - \frac{\psi}{\Phi} \right) \mathcal{M}_\alpha[-\Phi t^\alpha] + \frac{\psi}{\Phi}, \quad (8)$$

where $\mathcal{M}_\alpha(k)$ is said to be the Mittag-Leffler function of one parameter and is defined by

$$\mathcal{M}_\alpha(k) = \sum_{j=1}^{\infty} \frac{k^j}{\Gamma(\alpha j + 1)}. \quad (9)$$

For the special cases, if $\alpha = 1$ then (9) becomes the exponential function which has an important role in the classical calculus

$$e^k = \mathcal{M}_1(k) = \sum_{j=1}^{\infty} \frac{k^j}{\Gamma(j+1)}. \quad (10)$$

In Saleh and Kilicman (2019), the correlations between the Mittag-Leffler and Wright functions are studied. The further lemma and corollary are presented to show the non-negativity of the solution in the system (6).

Lemma 1 (See Odibat and Shawagfeh 2007). *Let $(v, D_*^\alpha)(t) \in C[0, \zeta]$, and $0 < \alpha \leq 1$. Then, one has*

$$v(t) = v(0) + \frac{1}{\Gamma(\alpha)} D_*^\alpha v(\xi) t^\alpha, \quad (11)$$

where $\xi \in [0, y]$, for all $y \in [0, \zeta]$.

Corollary 1 (See Odibat and Shawagfeh 2007) *($v, D_*^\alpha)(t) \in C[0, \zeta]$, and $0 < \alpha \leq 1$. The function $v(t)$ is increased for all $t \in (0, \zeta)$ if $D_*^\alpha v(t) \geq 0$ and is decreased for all $t \in (0, \zeta)$ if $D_*^\alpha v(t) \leq 0$.*

Lemma 2 (See Vargas-De-Leon 2015). *We consider $v(t) \in C(\mathbb{R}_+)$ and for any $0 < \alpha \leq 1$ the fractional system of order α exist. Then, one has*

$$D_*^\alpha \left[v(t) - v^* - v^* \ln \frac{v(t)}{v^*} \right] \leq \left(1 - \frac{v^*}{v(t)} \right) D_*^\alpha v(t), \quad (12)$$

$v^* \in \mathbb{R}_+$, for any $t > 0$.

Lemma 3 (See Huo et al. 2015). *Let Ω be a bounded and closed set and every solution of*

$$D_*^\alpha v(t) = f(v(t)), \quad (13)$$

start from and remain in Ω for all time. If there exists a continuous function $V(v) : \Omega \rightarrow \mathbb{R}$ satisfying

$$D_*^\alpha V(t)|_{D_*^\alpha v(t)=f(v(t))} \leq 0, \quad (14)$$

and \mathcal{K} be the largest invariant set of $E := \{v | D_^\alpha V(t)|_{D_*^\alpha v(t)=f(v(t))} = 0\}$. Then, for each solution $v(t)$ starting from Ω approaches to \mathcal{K} as $t \rightarrow \infty$.*

We notice that the Lyapunov function in Lemma 3 can be stated as the continuous function $V(v)$. It is necessary

to construct a suitable Lyapunov function first in order to satisfy Lemma 3. Moreover, the negativity of the Lyapunov function is usually shown through Lemma 2.

Mathematical analysis

This section provides the more detailed analysis of dynamical system (6).

Boundedness

Theorem 2 All state variables $S(t)$, $E(t)$, $I(t)$, $Q(t)$, and $R(t)$ of dynamical system (6) are bounded in a region $\Phi = \{w(t) \in \mathbb{R}^5 : w(t) \leq \frac{\mu}{\delta}\}$.

Proof Let $w(t)$ be the total population of dynamical system (6) defined as $w(t) = S(t) + E(t) + I(t) + Q(t) + R(t)$. Then, we employ the fractional derivative D_*^α to get

$$\begin{aligned} D_*^\alpha w(t) &= D_*^\alpha S(t) + D_*^\alpha E(t) + D_*^\alpha I(t) + D_*^\alpha Q(t) + D_*^\alpha R(t) \\ &= \mu - (S + E + I + Q + R)\delta - \theta I \\ &= \mu - w(t)\delta - \theta I(t) \\ &\leq \mu - w(t)\delta, \end{aligned} \quad (15)$$

which implies that $D_*^\alpha w(t) + w(t)\delta \leq \mu$.

By the standard comparison theorem for fractional derivative of order α in Theorem 1, (15) leads to

$$w(t) \leq w(0)\mathcal{M}_\alpha[-\delta t^\alpha] + \frac{\mu}{\delta}(1 - \mathcal{M}_\alpha[-\delta t^\alpha]), \quad (16)$$

where \mathcal{M}_α is the Mittag-Leffler function. Moreover, we apply Lemma 5 and Corollary 6 in Choi et al. (2014) to get

$$\mathcal{M}_\alpha(-\delta t^\alpha) \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (17)$$

Consequently, we obtain

$$\limsup_{t \rightarrow \infty} w(t) \leq \frac{\mu}{\delta}. \quad (18)$$

□

Non-negativity

Theorem 3 All state variables $S(t)$, $E(t)$, $I(t)$, $Q(t)$, and $R(t)$ of dynamical system (6) with non-negative initial conditions have the non-negative solution for all $t > 0$.

Proof It follows from (6)₁, one can derive

$$\begin{aligned} D_*^\alpha S(t) &= (1-p)\mu + \phi E - (\eta I + \delta)S \\ &\geq -(\eta I + \delta)S \geq -\left(\frac{\eta\mu}{\delta} + \delta\right)S, \end{aligned} \quad (19)$$

which implies that

$$D_*^\alpha S(t) + \left(\frac{\eta\mu}{\delta} + \delta\right)S \geq 0. \quad (20)$$

Applying Theorem 1 (the standard comparison theorem) gives

$$\begin{aligned} S(t) &\geq [S(0) - 0]\mathcal{M}_\alpha\left[-\left(\frac{\eta\mu}{\delta} + \delta\right)t^\alpha\right] + 0 \\ &= [S(0) - 0]\mathcal{M}_\alpha\left[-\left(\frac{\eta\mu}{\delta} + \delta\right)t^\alpha\right]. \end{aligned} \quad (21)$$

Since $\mathcal{M}_\alpha\left[-\left(\frac{\eta\mu}{\delta} + \delta\right)t^\alpha\right] \approx 0$ as $(t \rightarrow \infty)$, then we can conclude that $S(t) \geq 0$. Moreover, we can employ the similar ways to (6)₂, and one has

$$\begin{aligned} D_*^\alpha E(t) &= \eta SI - (\beta + \phi + \delta)E \\ &\geq -(\beta + \phi + \delta)E, \end{aligned} \quad (22)$$

which gives

$$D_*^\alpha E(t) + (\beta + \phi + \delta)E \geq 0. \quad (23)$$

Because we have $E(t) \geq E(0)\mathcal{M}_\alpha[-(\beta + \phi + \delta)t^\alpha]$ and $\mathcal{M}_\alpha[-(\beta + \phi + \delta)t^\alpha] \approx 0$ as $(t \rightarrow \infty)$, then the positivity of $E(t) \geq 0$ is achieved. Consequently, we have the same results for (6)₃, (6)₄, and (6)₅, by having $I(t) \geq I(0)\mathcal{M}_\alpha[-(\gamma + \delta + \theta)t^\alpha]Q(t) \geq Q(0)\mathcal{M}_\alpha[-(\varepsilon + \delta)t^\alpha]$, $R(t) \geq R(0)\mathcal{M}_\alpha[-\delta t^\alpha]$ and $\mathcal{M}_\alpha[-(\gamma + \delta + \theta)t^\alpha] \approx 0$, $\mathcal{M}_\alpha[-(\varepsilon + \delta)t^\alpha] \approx 0$, $\mathcal{M}_\alpha[-\delta t^\alpha] \approx 0$ as $(t \rightarrow \infty)$. Then, we can conclude that $I(t) \geq 0$, $Q(t) \geq 0$, $R(t) \geq 0$. \square

Existence and uniqueness

This section provides the existence and uniqueness of the fractional dynamical system (6) through the following maximality condition. Basically, the existence and uniqueness are the way to show the essence of first-order fractional dynamical system so that the given initial conditions can be satisfied.

Lemma 4 Consider the constant \mathcal{P}_k , $\bar{\mathcal{P}}_k$ such that

$$\begin{aligned} |\mathcal{G}_k(f_k, t) - \mathcal{G}_k(f_k^*, t)|^2 &\leq \mathcal{P}_k|f_k - f_k^*|^2 \\ \text{for all } (f, t) &\in (\mathbb{R}^5 \times (0, T)). \end{aligned} \quad (24)$$

$$\begin{aligned} |\mathcal{G}_k(f_k, t)|^2 &\leq \bar{\mathcal{P}}_k(1 + |f_k|^2) \\ \text{for all } (f, t) &\in (\mathbb{R}^5 \times (0, T)). \end{aligned} \quad (25)$$

Before we prove the existence and uniqueness of our dynamical system (6) through (24)–(25), then as the first step, we assume that

$$\begin{aligned} \mathcal{G}_1(t, S, E, I, Q, R) &:= D_*^\alpha S(t) = (1-p)\mu - \eta SI - \delta S + \phi E, \\ \mathcal{G}_2(t, S, E, I, Q, R) &:= D_*^\alpha E(t) = \eta SI - (\beta + \phi + \delta)E, \\ \mathcal{G}_3(t, S, E, I, Q, R) &:= D_*^\alpha I(t) = \beta E - (\gamma + \delta + \theta)I, \\ \mathcal{G}_4(t, S, E, I, Q, R) &:= D_*^\alpha Q(t) = \gamma I - (\varepsilon + \delta)Q, \\ \mathcal{G}_5(t, S, E, I, Q, R) &:= D_*^\alpha R(t) = p\mu + \varepsilon Q - \delta R, \\ (f_1, f_2, f_3, f_4, f_5) &:= (S, E, I, Q, R), \\ (f_1^*, f_2^*, f_3^*, f_4^*, f_5^*) &:= (S^*, E^*, I^*, Q^*, R^*). \end{aligned} \quad (26)$$

Then, we first establish the following inequality

$$|\mathcal{G}_1(S, t) - \mathcal{G}_1(S^*, t)|^2 \leq \mathcal{P}_1|S - S^*|^2. \quad (27)$$

It follows from (26), one has

$$\begin{aligned} |\mathcal{G}_1(S, t) - \mathcal{G}_1(S^*, t)|^2 &= |-\eta(S - S^*)I - \delta(S - S^*)|^2 \\ &\leq |(-\eta I - \delta)(S - S^*)|^2 \\ &\leq 2\left(\eta^2 \sup_{0 \leq t \leq \tau} |I|^2 + \delta^2\right)|S - S^*|^2 \\ &\leq 2(\eta^2 \|I\|_\infty^2 + \delta^2)|S - S^*|^2 \\ &\leq \mathcal{P}_1|S - S^*|^2, \end{aligned} \quad (28)$$

where $\mathcal{P}_1 = 2(\eta^2 \|I\|_\infty^2 + \delta^2)$. By the similar way for $|\mathcal{G}_2(E, t) - \mathcal{G}_2(E^*, t)|^2 \leq \mathcal{P}_2|E - E^*|^2$, $|\mathcal{G}_3(I, t) - \mathcal{G}_3(I^*, t)|^2 \leq \mathcal{P}_3|I - I^*|^2$, $|\mathcal{G}_4(Q, t) - \mathcal{G}_4(Q^*, t)|^2 \leq \mathcal{P}_4|Q - Q^*|^2$, and $|\mathcal{G}_5(R, t) - \mathcal{G}_5(R^*, t)|^2 \leq \mathcal{P}_5|R - R^*|^2$, we have

$$\begin{aligned} |\mathcal{G}_2(E, t) - \mathcal{G}_2(E^*, t)|^2 &= |-(\beta + \phi + \delta)(E - E^*)|^2 \\ &\leq 2(\beta^2 + \phi^2 + \delta^2)|E - E^*|^2 \\ &\leq \mathcal{P}_2|E - E^*|^2, \end{aligned} \quad (29)$$

$$\begin{aligned} |\mathcal{G}_3(I, t) - \mathcal{G}_3(I^*, t)|^2 &= |-(\gamma + \delta + \theta)(I - I^*)|^2 \\ &\leq 2(\gamma^2 + \delta^2 + \theta^2)|I - I^*|^2 \\ &\leq \mathcal{P}_3|I - I^*|^2, \end{aligned} \quad (30)$$

$$\begin{aligned} |\mathcal{G}_4(Q, t) - \mathcal{G}_4(Q^*, t)|^2 &= |-(\varepsilon + \delta)(Q - Q^*)|^2 \\ &\leq 2(\varepsilon^2 + \delta^2)|Q - Q^*|^2 \\ &\leq \mathcal{P}_4|Q - Q^*|^2, \end{aligned} \quad (31)$$

$$\begin{aligned} |\mathcal{G}_5(R, t) - \mathcal{G}_5(R^*, t)|^2 &= |-\delta(R - R^*)|^2 \\ &\leq 2\delta^2|R - R^*|^2 \\ &\leq \bar{\mathcal{P}}_5|R - R^*|^2, \end{aligned} \quad (32)$$

where $\mathcal{P}_2 = 2(\beta^2 + \phi^2 + \delta^2)$, $\mathcal{P}_3 = 2(\gamma^2 + \delta^2 + \theta^2)$, $\mathcal{P}_4 = 2(\varepsilon^2 + \delta^2)$, $\mathcal{P}_5 = 2\delta^2$ respectively. Based on the above calculations, we can conclude that the first inequality (24) in Lemma 4 is satisfied. We continue to prove the second inequality (25) that the following inequality is satisfied

$$|\mathcal{G}_1(S, t)|^2 \leq \bar{\mathcal{P}}_1(1 + |S|)^2. \quad (33)$$

Firstly, we have

$$\begin{aligned} |\mathcal{G}_1(S, t)|^2 &= |(1-p)\mu - \eta SI - \delta S + \phi E|^2 \\ &\leq |(1-p)\mu - (\eta I + \delta)S + \phi E|^2 \\ &\leq 2(1-p)^2\mu^2 + 2\left(\eta^2 \sup_{0 \leq t \leq \tau} |I|^2 + \delta^2\right) \\ &\quad |S|^2 + 2\phi^2 \sup_{0 \leq t \leq \tau} |E|^2 \\ &\leq 2((1-p)^2\mu^2 + \phi^2\|E\|_\infty^2) \\ &\quad + 2(\eta^2\|I\|_\infty^2 + \delta^2)|S|^2 \\ &\leq \bar{\mathcal{P}}_1(1 + |S|^2), \end{aligned} \quad (34)$$

which gives $\frac{\eta^2\|I\|_\infty^2 + \delta^2}{(1-p)^2\mu^2 + \phi^2\|E\|_\infty^2} < 1$, where $\bar{\mathcal{P}}_1 = (1-p)^2\mu^2 + \phi^2\|E\|_\infty^2$. Applying the same strategy for $|\mathcal{G}_2(E, t)|^2 \leq \bar{\mathcal{P}}_2(1 + |E|)^2$, $|\mathcal{G}_3(I, t)|^2 \leq \bar{\mathcal{P}}_3(1 + |I|)^2$, $|\mathcal{G}_4(Q, t)|^2 \leq \bar{\mathcal{P}}_4(1 + |Q|)^2$, and $|\mathcal{G}_5(R, t)|^2 \leq \bar{\mathcal{P}}_5(1 + |R|)^2$, one gets

$$\begin{aligned} |\mathcal{G}_2(E, t)|^2 &= |\eta SI - (\beta + \phi + \delta)E|^2 \\ &\leq 2\eta^2 \sup_{0 \leq t \leq \tau} |S|^2 \sup_{0 \leq t \leq \tau} |I|^2 + 2(\beta^2 + \phi^2 + \delta^2)|E|^2 \\ &\leq 2\eta^2\|S\|_\infty^2\|I\|_\infty^2 + 2(\beta^2 + \phi^2 + \delta^2)|E|^2 \\ &\leq \bar{\mathcal{P}}_2(1 + |E|^2), \end{aligned} \quad (35)$$

which gives $\frac{\beta^2 + \phi^2 + \delta^2}{\eta^2\|S\|_\infty^2\|I\|_\infty^2} < 1$, where $\bar{\mathcal{P}}_2 = \eta^2\|S\|_\infty^2\|I\|_\infty^2$.

$$\begin{aligned} |\mathcal{G}_3(I, t)|^2 &= |\beta E - (\gamma + \delta + \theta)I|^2 \\ &\leq 2\beta^2 \sup_{0 \leq t \leq \tau} |E|^2 + 2(\gamma^2 + \delta^2 + \theta^2)|I|^2 \\ &\leq 2\beta^2\|E\|_\infty^2 + 2(\gamma^2 + \delta^2 + \theta^2)|I|^2 \\ &\leq \bar{\mathcal{P}}_3(1 + |I|^2), \end{aligned} \quad (36)$$

which gives $\frac{\gamma^2 + \delta^2 + \theta^2}{\beta^2\|E\|_\infty^2} < 1$, where $\bar{\mathcal{P}}_3 = \beta^2\|E\|_\infty^2$.

$$\begin{aligned} |\mathcal{G}_4(Q, t)|^2 &= |\gamma I - (\varepsilon + \delta)Q|^2 \\ &\leq 2\gamma^2 \sup_{0 \leq t \leq \tau} |I|^2 + 2(\varepsilon^2 + \delta^2)|Q|^2 \\ &\leq 2\gamma^2\|I\|_\infty^2 + 2(\varepsilon^2 + \delta^2)|Q|^2 \\ &\leq \bar{\mathcal{P}}_4(1 + |Q|^2), \end{aligned} \quad (37)$$

which gives $\frac{\varepsilon^2 + \delta^2}{\gamma^2\|I\|_\infty^2} < 1$, where $\bar{\mathcal{P}}_4 = \gamma^2\|I\|_\infty^2$.

$$\begin{aligned} |\mathcal{G}_5(R, t)|^2 &= |p\mu + \varepsilon Q - \delta R|^2 \\ &\leq 2p^2\mu^2 + 2\varepsilon^2 \sup_{0 \leq t \leq \tau} |Q|^2 + 2\delta^2|R|^2 \\ &\leq 2p^2\mu^2 + 2\varepsilon^2\|Q\|_\infty^2 + 2\delta^2|R|^2 \\ &\leq \bar{\mathcal{P}}_5(1 + |R|^2), \end{aligned} \quad (38)$$

which gives $\frac{\delta^2}{p^2\mu^2 + \varepsilon^2\|Q\|_\infty^2} < 1$, where $\bar{\mathcal{P}}_5 = p^2\mu^2 + \varepsilon^2\|Q\|_\infty^2$.

Finally, the second inequality (25) can be established in the following maximality condition

$$\text{Max} \left\{ \frac{\eta^2\|I\|_\infty^2 + \delta^2}{(1-p)^2\mu^2 + \phi^2\|E\|_\infty^2}, \frac{\beta^2 + \phi^2 + \delta^2}{\eta^2\|S\|_\infty^2\|I\|_\infty^2}, \frac{\gamma^2 + \delta^2 + \theta^2}{\beta^2\|E\|_\infty^2}, \right. \\ \left. \frac{\varepsilon^2 + \delta^2}{\gamma^2\|I\|_\infty^2}, \frac{\delta^2}{p^2\mu^2 + \varepsilon^2\|Q\|_\infty^2} \right\} < 1. \quad (39)$$

Basic reproduction number

The transmission of a disease is indicated by the value of \mathcal{R}_0 which is called basic reproduction number. The basic reproduction number \mathcal{R}_0 is obtained from the derivative of the affected classes (6)₂ and (6)₃. By linearizing around the disease-free equilibrium (\mathcal{E}_0), one has

$$\begin{aligned} \begin{pmatrix} E \\ I \end{pmatrix}_t &= \begin{pmatrix} \eta SI - (\beta + \phi + \delta)E \\ \beta E - (\gamma + \delta + \theta)I \end{pmatrix} \\ &= \begin{pmatrix} \eta SI \\ 0 \end{pmatrix} - \begin{pmatrix} (\beta + \phi + \delta)E \\ -\beta E + (\gamma + \delta + \theta)I \end{pmatrix} \\ &= \begin{pmatrix} 0 & \eta S \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E \\ I \end{pmatrix} - \begin{pmatrix} \beta + \phi + \delta & 0 \\ -\beta & \gamma + \delta + \theta \end{pmatrix} \begin{pmatrix} E \\ I \end{pmatrix} \\ &= (\mathcal{F} - \mathcal{V}) \begin{pmatrix} E \\ I \end{pmatrix}, \end{aligned} \quad (40)$$

where \mathcal{F} is the transmission matrix of new infected individuals, and \mathcal{V} is the transition matrix of individual displacements between groups of individuals. Therefore, the basic reproduction number \mathcal{R}_0 can be established from the following dominant Eigen values of $\mathcal{F}\mathcal{V}^{-1}$

$$|\lambda I - \mathcal{F}\mathcal{V}^{-1}| = \begin{vmatrix} \lambda - \frac{\eta\beta(1-p)\mu}{\delta(\beta+\phi+\delta)(\gamma+\delta+\theta)} & -\frac{\eta(1-p)\mu}{\delta(\gamma+\delta+\theta)} \\ 0 & \lambda \end{vmatrix} \quad (41)$$

which gives $\lambda_1 = 0$ and $\lambda_2 = \frac{\eta\beta(1-p)\mu}{\delta(\beta+\phi+\delta)(\gamma+\delta+\theta)}$. Hence, the basic reproduction number of system (6) is

$$\mathcal{R}_0 = \frac{\eta\beta(1-p)\mu}{\delta(\beta+\phi+\delta)(\gamma+\delta+\theta)}. \quad (42)$$

Stability

Theorem 4 (See Matignon 1996; Petras 2011). *Given a fractional system of order α*

$$D_*^\alpha \vec{v} = \vec{f}(\vec{v}), \vec{v}(0) = \vec{v}_0, 0 < \alpha \leq 1. \quad (43)$$

If the condition $\vec{f}(\vec{v}^*) = 0$ can be satisfied then a point \vec{v}^* is known as equilibrium point. If the condition $|\arg(\lambda_j)| > \alpha\pi/2$ can be satisfied by all Eigen values λ_j obtained from the Jacobian matrix $J = df/dv$ at the point \vec{v}^* then this equilibrium point is asymptotic stable.

The dynamical system (43) has the similar equilibrium points as the integer-order system. By applying the right-hand side of (43) equal to zero, then the equilibrium point for both disease-free and endemic are given below.

Theorem 5 For all the values of parameters, if $\mathcal{R}_0 < 1$, the dynamical system (6) has disease-free equilibrium point $\mathcal{E}_0 = (S^0, E^0, I^0, Q^0, R^0) = ((1-p)\mu/\delta, 0, 0, 0, p\mu/\delta)$. If $\mathcal{R}_0 > 1$, the endemic equilibrium is $\mathcal{E}_1 = (S^*, E^*, I^*, Q^*, R^*)$ where $S^* = \frac{(\gamma+\delta+\theta)(\beta+\phi+\delta)}{\eta\beta}$, $E^* = \frac{(\gamma+\delta+\theta)(\beta+\phi+\delta)}{\eta\beta(\beta+\delta)}(\mathcal{R}_0 - 1)$, $I^* = \frac{\beta+\phi+\delta}{\eta(\beta+\delta)}(\mathcal{R}_0 - 1)$, $Q^* = \frac{\gamma(\beta+\phi+\delta)}{\eta(\epsilon+\delta)(\beta+\delta)}(\mathcal{R}_0 - 1)$, and $R^* = \frac{p\mu}{\delta} + \frac{\epsilon\gamma(\beta+\phi+\delta)}{\eta\delta(\epsilon+\delta)(\beta+\delta)}(\mathcal{R}_0 - 1)$.

Local stability

In the next analysis, the discussion of local stability for both disease-free and endemic equilibrium point are stated in the following theorems.

Theorem 6 Let $\mathcal{R}_0 < 1$. Then, the disease-free equilibrium point \mathcal{E}_0 is locally asymptotically stable.

Proof The Jacobian matrix of the dynamical system (6) at \mathcal{E}_0 is given as follows

$$J(\mathcal{E}_0) = \begin{pmatrix} -\delta & \phi & -\frac{\eta(1-p)\mu}{\delta} & 0 & 0 \\ 0 & -(\beta + \phi + \delta) & \frac{\eta(1-p)\mu}{\delta} & 0 & 0 \\ 0 & \beta & -(\gamma + \delta + \theta) & 0 & 0 \\ 0 & 0 & \gamma & -(\epsilon + \delta) & 0 \\ 0 & 0 & 0 & \epsilon & -\delta \end{pmatrix}. \quad (44)$$

The Eigen values of (44) are obtained by the solution of characteristic equation $|J(\mathcal{E}_0) - \lambda I| = 0$, then we obtain

$$\lambda_{1,2} = -\delta, \lambda_3 = \frac{-(\beta + \phi + \gamma + \theta + 2\delta) - \sqrt{\Delta}}{2}, \quad (45)$$

$$\lambda_4 = \frac{-(\beta + \phi + \gamma + \theta + 2\delta) + \sqrt{\Delta}}{2}$$

where

$$\Delta = (\beta + \phi + \gamma + \theta + 2\delta)^2 - 4(\beta + \phi + \delta)(\gamma + \delta + \theta) + \frac{4\beta\eta(1-p)\mu}{\delta}. \quad (46)$$

Therefore $\Delta > 0$. This condition implies that λ_3 is negative. If $\mathcal{R}_0 < 1$, one has $\frac{\eta\beta(1-p)\mu}{\delta} < (\beta + \phi + \delta)(\gamma + \delta + \theta)$, which implies that

$$\Delta < (\beta + \phi + \gamma + \theta + 2\delta)^2. \quad (47)$$

So that, $\lambda_4 = \frac{-(\beta + \phi + \gamma + \theta + 2\delta) + \sqrt{\Delta}}{2} < \frac{-(\beta + \phi + \gamma + \theta + 2\delta) + \sqrt{(\beta + \phi + \gamma + \theta + 2\delta)^2}}{2} = 0$, giving $\lambda_4 < 0$. Then all Eigen values λ_k , for $k = 1, 2, 3, 4$ satisfy the Theorem 4. Then we can conclude that all Eigen values have negative value if $\mathcal{R}_0 < 1$. \square

Theorem 7 Let $\mathcal{R}_0 > 1$. Then, the endemic equilibrium point \mathcal{E}_1 is locally asymptotically stable.

Proof Similarly, for the equilibrium point in endemic condition, the Jacobian matrix is given below

$$J(\mathcal{E}_1) = \begin{pmatrix} -(\eta I^* + \delta) & \phi & -\eta S^* & 0 & 0 \\ \eta I^* & -(\beta + \phi + \delta) & \eta S^* & 0 & 0 \\ 0 & \beta & -(\gamma + \delta + \theta) & 0 & 0 \\ 0 & 0 & \gamma & -(\epsilon + \delta) & 0 \\ 0 & 0 & 0 & \epsilon & -\delta \end{pmatrix}. \quad (48)$$

The Eigen values of (48) are obtained by the solution of characteristic equation $|J(\mathcal{E}_1) - \lambda I| = 0$. By the similar ways, we determine the minor values of matrix $J(\mathcal{E}_1)$ to get the cofactor expansion, then Eigen values satisfy the following equation

$$(-\delta - \lambda)(-\epsilon - \delta - \lambda)det(B) = 0, \quad (49)$$

where

$$B = \begin{pmatrix} -(\eta I^* + \delta) & \phi & -\eta S^* \\ \eta I^* & -(\beta + \phi + \delta) & \eta S^* \\ 0 & \beta & -(\gamma + \delta + \theta) \end{pmatrix}. \quad (50)$$

Obviously, the Eigen values $\lambda_1 = -\delta < 0$, $\lambda_2 = -(\varepsilon + \delta) < 0$. Moreover, the Eigen values $\lambda_{3,4,5}$ determine the stability criteria at the equilibrium point \mathcal{E}_1 , which can be derived from the following characteristic equation

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0, \quad (51)$$

where

$$\begin{aligned} b_1 &= \eta I^* + 2\delta + \beta + \phi + \gamma + \theta, \\ b_2 &= (\eta I^* + \delta)(\beta + \delta) + (\eta I^* + 2\delta + \beta + \phi) \\ &\quad (\gamma + \delta + \theta) - \eta \beta S^*, \\ b_3 &= (\eta I^* + \delta)(\beta + \delta)(\gamma + \delta + \theta) - \eta \beta S^* \delta. \end{aligned} \quad (52)$$

Based on the Routh–Hurwitz criterion for third order polynomial, the system with characteristic equation (51) is stable at the equilibrium point \mathcal{E}_1 if $b_1 > 0$, $b_2 > 0$, $b_3 > 0$ and $b_1 b_2 > b_3$. Therefore, the condition $b_3 > 0$ is equivalent to

$$\begin{aligned} &(\eta I^* + \delta)(\beta + \delta)(\gamma + \delta + \theta) - \eta \beta S^* \delta > 0, \\ &\eta I^* (\beta + \delta)(\gamma + \delta + \theta) + \delta(\beta + \delta)(\gamma + \delta + \theta) - \eta \beta S^* \delta > 0. \end{aligned} \quad (53)$$

By substituting the values of S^* and I^* in Theorem 5, one can get

$$\delta(\beta + \phi + \delta)(\gamma + \delta + \theta)(\mathcal{R}_0 - 1) > 0. \quad (54)$$

Hence, $b_3 > 0$ is satisfied only if $\mathcal{R}_0 > 1$. \square

Global stability

In this section, we prove the global stability for disease-free and endemic equilibrium points. The proofs are based on the Lemma 4 and Lemma 2.

Theorem 8 If $\mathcal{R}_0 < 1$, then the disease-free equilibrium point \mathcal{E}_0 is globally asymptotically stable.

Proof Consider the Lyapunov function

$$\begin{aligned} V(t) &= \left(S - S^0 - S^0 \ln \left(\frac{S}{S^0} \right) \right) + \frac{\beta E}{\beta + \phi + \delta} + I + E \\ &\quad + \left(R - R^0 - R^0 \ln \left(\frac{R}{R^0} \right) \right). \end{aligned} \quad (55)$$

By applying Lemma 4 into (55), then (55) becomes

$$\begin{aligned} D_*^\alpha V(t) &\leq \left(\frac{S - S^0}{S} \right) D_*^\alpha S(t) + \left(\frac{\beta}{\beta + \phi + \delta} \right) D_*^\alpha E(t) \\ &\quad + D_*^\alpha I(t) + D_*^\alpha E(t) + \left(\frac{R - R^0}{R} \right) D_*^\alpha R(t) \\ &\leq \left(\frac{S - S^0}{S} \right) ((1-p)\mu - \eta SI - \delta S + \phi E) \\ &\quad + \left(\frac{\beta}{\beta + \phi + \delta} \right) (\eta SI - (\beta + \phi + \delta)E) \\ &\quad + \beta E - (\gamma + \delta + \theta)I + (\eta SI - (\beta + \phi + \delta)E) \\ &\quad + \left(\frac{R - R^0}{R} \right) (p\mu + \varepsilon Q - \delta R). \end{aligned} \quad (56)$$

Let $(1-p)\mu = \delta S^0$ and $p\mu = \delta R^0$, then we have

$$\begin{aligned} D_*^\alpha V(t) &\leq \left(\frac{S - S^0}{S} \right) (-\delta(S - S^0) - \eta SI + \phi E) \\ &\quad + \left(\frac{\beta}{\beta + \phi + \delta} \right) (\eta SI - (\beta + \phi + \delta)E) \\ &\quad + \beta E - (\gamma + \delta + \theta)I + (\eta SI - (\beta + \phi + \delta)E) \\ &\quad + \left(\frac{R - R^0}{R} \right) (-\delta(R - R^0) + \varepsilon Q) \\ &\leq -\delta \left(\frac{(S - S^0)^2}{S} \right) - \left(\frac{S - S^0}{S} \right) \phi E \\ &\quad + \left(\frac{\eta \beta S}{\beta + \phi + \delta} - (\gamma + \delta + \theta) \right) I \\ &\quad + (\eta SI - (\beta + \phi + \delta)E) \\ &\quad - \delta \left(\frac{(R - R^0)^2}{R} \right) + \left(\frac{R - R^0}{R} \right) \varepsilon Q. \end{aligned} \quad (57)$$

It follows from $I = \frac{\beta}{\gamma + \delta + \theta}$, one can derive

$$\begin{aligned} D_*^\alpha V(t) &\leq -\delta \left(\frac{(S - S^0)^2}{S} \right) - \left(\frac{S - S^0}{S} \right) \phi E \\ &\quad + \left(\frac{\eta \beta S}{\beta + \phi + \delta} - (\gamma + \delta + \theta) \right) I \\ &\quad + \left(\frac{\eta \beta S}{\gamma + \delta + \theta} - (\beta + \phi + \delta) \right) E \\ &\quad - \delta \left(\frac{(R - R^0)^2}{R} \right) + \left(\frac{R - R^0}{R} \right) \varepsilon Q. \end{aligned} \quad (58)$$

Based on the Theorem 1, then we have

$$S(t) \leq \left(S(0) - \frac{(1-p)\mu}{\delta} \right) \mathcal{M}_\alpha[-\delta t^\alpha] + \frac{(1-p)\mu}{\delta}, \quad (59)$$

where \mathcal{M}_α is the Mittag-Leffler function. Since

$$\mathcal{M}_\alpha[-\delta t^\alpha] \rightarrow 0 \text{ as } t \rightarrow \infty, \quad (60)$$

then by Lemma 5 and Corollary 6 in Choi et al. (2014), one gets

$$S(t) \leq \frac{(1-p)\mu}{\delta} \text{ as } t \rightarrow \infty, \quad (61)$$

Substituting (61) into (58), we have

$$\begin{aligned} D_*^\alpha V(t) &\leq -\delta \left(\frac{(S-S^0)^2}{S} \right) - \left(\frac{\delta S - \delta S^0}{S} \right) \frac{\phi E}{\delta} \\ &+ (\mathcal{R}_0 - 1)((\gamma + \delta + \theta)I + (\beta + \phi + \delta)E) \\ &- \delta \left(\frac{(R-R^0)^2}{R} \right) + \left(\frac{\delta R - \delta R^0}{R} \right) \frac{\varepsilon Q}{\delta}. \end{aligned} \quad (62)$$

Similarly, by applying $\delta S^0 = (1-p)\mu$, $\delta R^0 = p\mu$, $\phi E = \eta SI - (\beta + \delta)E$, (61), $E = \frac{\gamma+\delta+\theta}{\beta}I$, and $I = \frac{\varepsilon+\delta}{\gamma}Q$, we obtain

$$\begin{aligned} D_*^\alpha V(t) &\leq -\delta \left(\frac{(S-S^0)^2}{S} \right) - \frac{(\beta + \delta)\phi}{(1-p)\mu} E^2 \\ &+ (\mathcal{R}_0 - 1)((\gamma + \delta + \theta)I + (\beta + \phi + \delta)E) \\ &- \delta \left(\frac{(R-R^0)^2}{R} \right) + \frac{\varepsilon^2}{\delta} Q^2 \\ &\leq -\delta \left(\frac{(S-S^0)^2}{S} \right) + \frac{\varepsilon^2(\beta + \phi + \delta)(\gamma + \delta + \theta)}{\eta\beta(1-p)\mu} \\ &(\mathcal{R}_0 - 1)Q^2 \\ &+ (\mathcal{R}_0 - 1)((\gamma + \delta + \theta)I + (\beta + \phi + \delta)E) \\ &- \delta \left(\frac{(R-R^0)^2}{R} \right). \end{aligned} \quad (63)$$

Then one has $D_*^\alpha V(t) < 0$ if $\mathcal{R}_0 < 1$ for all positive parameters. Moreover, one has $D_*^\alpha V(t) = 0$ if $\mathcal{R}_0 = 1$. Therefore, if $\mathcal{R}_0 < 1$ then the global asymptotic \mathcal{E}_0 is achieved. \square

Theorem 9 If $\mathcal{R}_0 > 1$, then the endemic equilibrium point \mathcal{E}_1 is globally asymptotically stable.

Proof Given a Lyapunov function as follows

$$V(t) = V_1(S(t)) + V_2(E(t)) + V_3(I(t)) + V_4(Q(t)) + V_5(R(t)), \quad (64)$$

where

$$\begin{aligned} V_1(S(t)) &= S - S^* - S^* \ln \left(\frac{S}{S^*} \right), \\ V_2(E(t)) &= E - E^* - E^* \ln \left(\frac{E}{E^*} \right), \\ V_3(I(t)) &= I - I^* - I^* \ln \left(\frac{I}{I^*} \right), \end{aligned} \quad (65)$$

$$\begin{aligned} V_4(Q(t)) &= Q - Q^* - Q^* \ln \left(\frac{Q}{Q^*} \right), \\ V_5(R(t)) &= R - R^* - R^* \ln \left(\frac{R}{R^*} \right), \end{aligned}$$

By applying the Lemma 2 and (6), we get

$$\begin{aligned} D_*^\alpha V &\leq \left(1 - \frac{S^*}{S} \right) D_*^\alpha S + \left(1 - \frac{E^*}{E} \right) D_*^\alpha E + \left(1 - \frac{I^*}{I} \right) D_*^\alpha I \\ &+ \left(1 - \frac{Q^*}{Q} \right) D_*^\alpha Q + \left(1 - \frac{R^*}{R} \right) D_*^\alpha R \\ &= A_1 + A_2 + A_3 + A_4 + A_5, \end{aligned} \quad (66)$$

where

$$\begin{aligned} A_1 &= \left(1 - \frac{S^*}{S} \right) ((1-p)\mu - \eta SI - \delta S + \phi E), \\ A_2 &= \left(1 - \frac{E^*}{E} \right) (\eta SI - \varphi_3 E), \\ A_3 &= \left(1 - \frac{I^*}{I} \right) (\beta E - \varphi_2 I), \quad A_4 = \left(1 - \frac{Q^*}{Q} \right) (\gamma I - \varphi_1 Q), \\ A_5 &= \left(1 - \frac{R^*}{R} \right) (p\mu + \varepsilon Q - \delta R). \end{aligned} \quad (67)$$

By assuming $S = S - S^*$, $E = E - E^*$, $I = I - I^*$, $Q = Q - Q^*$, and $R = R - R^*$ and substituting them in (67), then one has

$$\begin{aligned} D_*^\alpha V &\leq \left(\frac{S - S^*}{S} \right) ((1-p)\mu - \eta(S - S^*)(I - I^*) - \delta(S - S^*) \\ &+ \phi(E - E^*)) \\ &+ \left(\frac{E - E^*}{E} \right) (\eta(S - S^*)(I - I^*) - \varphi_3(E - E^*)) \\ &+ \left(\frac{I - I^*}{I} \right) (\beta(E - E^*) - \varphi_2(I - I^*)) \\ &+ \left(\frac{Q - Q^*}{Q} \right) (\gamma(I - I^*) - \varphi_1(Q - Q^*)) \\ &+ \left(\frac{R - R^*}{R} \right) (p\mu + \varepsilon(Q - Q^*) - \delta(R - R^*)), \end{aligned} \quad (68)$$

where

$$\varphi_3 = \beta + \phi + \delta, \quad \varphi_2 = \gamma + \delta + \theta, \quad \varphi_1 = \varepsilon + \delta. \quad (69)$$

Making simplification of this calculation, one has

$$D_*^\alpha V \leq \mathcal{H}_1 - \mathcal{H}_2, \quad (70)$$

where

$$\begin{aligned} \mathcal{H}_1 &= 1 + \eta \left(\frac{(S - S^*)^2}{S} I^* + SI + S^* I^* + \frac{E^*}{E} (SI^* + S^* I) \right) \\ &\quad + \phi \left(E + \frac{S^*}{S} E^* \right) \\ &\quad + \beta \left(E + \frac{I^*}{I} E^* \right) + \gamma \left(I + \frac{Q^*}{Q} I^* \right) + \varepsilon \left(Q + \frac{R^*}{R} Q^* \right), \\ \mathcal{H}_2 &= (1-p)\mu \frac{S^*}{S} \\ &\quad + \eta \left(\frac{(S - S^*)^2}{S} I + SI^* + S^* I + \frac{E^*}{E} (SI + S^* I^*) \right) \\ &\quad + \phi \left(E^* + \frac{S^*}{S} E \right) + \delta \left(\frac{(S - S^*)^2}{S} + \frac{(R - R^*)^2}{R} \right) \\ &\quad + \varphi_3 \frac{(E - E^*)^2}{E} \\ &\quad + \beta \left(\frac{I^*}{I} E + E^* \right) + \varphi_2 \frac{(I - I^*)^2}{I} + \gamma \left(\frac{Q^*}{Q} I + I^* \right) \\ &\quad + \varphi_1 \frac{(Q - Q^*)^2}{Q} \\ &\quad + p\mu \frac{R^*}{R} + \varepsilon \left(\frac{R^*}{R} Q + Q^* \right). \end{aligned} \quad (71)$$

Hence, $D_*^\alpha V < 0$ if $\mathcal{H}_1 - \mathcal{H}_2 < 0$. Moreover, if $S = S^*$, $E = E^*$, $I = I^*$, $Q = Q^*$, and $R = R^*$, then from (70), one has

$$\mathcal{H}_1 - \mathcal{H}_2 = 0 \Rightarrow D_*^\alpha V = 0. \quad (72)$$

Then the invariant set $\{(S^*, E^*, I^*, Q^*, R^*) \in \mathbb{R}_+^5 : D_*^\alpha V = 0\}$ at \mathcal{E}_1 is the endemic equilibrium and according to the Generalized Lasalle Invariance Principle in Lemma 3, \mathcal{E}_1 is globally asymptotically stable if $\mathcal{H}_1 - \mathcal{H}_2 < 0$. \square

Numerical results and discussion

In this section, before we provide the simulation results, we first discretize our dynamical system by the numerical technique predictor–corrector as introduced in Diethelm et al. (2002), Garrappa (2010, 2011). Then, the discretization results of our model (6) are shown in the following equation.

$$\begin{aligned} \begin{bmatrix} S_h(\tau_{n+1}) \\ E_h(\tau_{n+1}) \\ I_h(\tau_{n+1}) \\ Q_h(\tau_{n+1}) \\ R_h(\tau_{n+1}) \end{bmatrix} &= \begin{bmatrix} S(0) \\ E(0) \\ I(0) \\ Q(0) \\ R(0) \end{bmatrix} + \frac{h^\alpha}{\Gamma(\alpha + 2)} \\ &\quad \left[F_1(\tau_{n+1}, S_h^P(\tau_{n+1}), E_h^P(\tau_{n+1}), I_h^P(\tau_{n+1}), Q_h^P(\tau_{n+1}), R_h^P(\tau_{n+1})) \right. \\ &\quad \left. F_2(\tau_{n+1}, S_h^P(\tau_{n+1}), E_h^P(\tau_{n+1}), I_h^P(\tau_{n+1}), Q_h^P(\tau_{n+1}), R_h^P(\tau_{n+1})) \right. \\ &\quad \left. F_3(\tau_{n+1}, S_h^P(\tau_{n+1}), E_h^P(\tau_{n+1}), I_h^P(\tau_{n+1}), Q_h^P(\tau_{n+1}), R_h^P(\tau_{n+1})) \right. \\ &\quad \left. F_4(\tau_{n+1}, S_h^P(\tau_{n+1}), E_h^P(\tau_{n+1}), I_h^P(\tau_{n+1}), Q_h^P(\tau_{n+1}), R_h^P(\tau_{n+1})) \right. \\ &\quad \left. F_5(\tau_{n+1}, S_h^P(\tau_{n+1}), E_h^P(\tau_{n+1}), I_h^P(\tau_{n+1}), Q_h^P(\tau_{n+1}), R_h^P(\tau_{n+1})) \right] \\ &\quad + \frac{h^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^n a_{j,n+1} \\ &\quad \left[F_1(\tau_j, S_h^P(\tau_j), E_h^P(\tau_j), I_h^P(\tau_j), Q_h^P(\tau_j), R_h^P(\tau_j)) \right] \\ &\quad \left[F_2(\tau_j, S_h^P(\tau_j), E_h^P(\tau_j), I_h^P(\tau_j), Q_h^P(\tau_j), R_h^P(\tau_j)) \right] \\ &\quad \left[F_3(\tau_j, S_h^P(\tau_j), E_h^P(\tau_j), I_h^P(\tau_j), Q_h^P(\tau_j), R_h^P(\tau_j)) \right] \\ &\quad \left[F_4(\tau_j, S_h^P(\tau_j), E_h^P(\tau_j), I_h^P(\tau_j), Q_h^P(\tau_j), R_h^P(\tau_j)) \right] \\ &\quad \left[F_5(\tau_j, S_h^P(\tau_j), E_h^P(\tau_j), I_h^P(\tau_j), Q_h^P(\tau_j), R_h^P(\tau_j)) \right]. \end{aligned} \quad (73)$$

$$\begin{aligned} \begin{bmatrix} S_h^P(\tau_{n+1}) \\ E_h^P(\tau_{n+1}) \\ I_h^P(\tau_{n+1}) \\ Q_h^P(\tau_{n+1}) \\ R_h^P(\tau_{n+1}) \end{bmatrix} &= \begin{bmatrix} S(0) \\ E(0) \\ I(0) \\ Q(0) \\ R(0) \end{bmatrix} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1} \\ &\quad \left[F_1(\tau_j, S_h(\tau_j), E_h(\tau_j), I_h(\tau_j), Q_h(\tau_j), R_h(\tau_j)) \right] \\ &\quad \left[F_2(\tau_j, S_h(\tau_j), E_h(\tau_j), I_h(\tau_j), Q_h(\tau_j), R_h(\tau_j)) \right] \\ &\quad \left[F_3(\tau_j, S_h(\tau_j), E_h(\tau_j), I_h(\tau_j), Q_h(\tau_j), R_h(\tau_j)) \right] \\ &\quad \left[F_4(\tau_j, S_h(\tau_j), E_h(\tau_j), I_h(\tau_j), Q_h(\tau_j), R_h(\tau_j)) \right] \\ &\quad \left[F_5(\tau_j, S_h(\tau_j), E_h(\tau_j), I_h(\tau_j), Q_h(\tau_j), R_h(\tau_j)) \right]. \end{aligned} \quad (74)$$

where S_h , E_h , I_h , Q_h , and R_h represent the corrector schemes for susceptible, exposed, infected, quarantined, and recovered individuals respectively. Moreover, S_h^P , E_h^P , I_h^P , Q_h^P , and R_h^P are respectively the predictor schemes for susceptible, exposed, infected, quarantined, and recovered individuals. The parameter h is a step size, and

$$\begin{aligned} a_{j,n+1} &= \begin{cases} n^{\alpha+1} - (n - \alpha)(n + 1)^\alpha, & \text{if } j = 0, \\ (n - j + 2)^{\alpha+1} + (n - j)^{\alpha+1} - 2(n - j + 1)^{\alpha+1}, & \text{if } 1 \leq j \leq n, \\ 1, & \text{if } j = n + 1, \end{cases} \\ b_{j,n+1} &= \frac{h^\alpha}{\alpha} ((n + 1 - j)^\alpha - (n - j)^\alpha). \end{aligned} \quad (75)$$

Here, the initial values of states and parameters are based on the study in Izzati and Andriani (2021) and

references there in. The initial values of S, E, I, Q, R are given $S(0) = 0.95, E(0) = 0, I(0) = 0.05, Q(0) = 0, R(0) = 0$, indicating that the total susceptible and infected individuals are 95% and 5% respectively. Moreover, the parameter values are $\eta = 0.57, \beta = 0.23, \varepsilon = 0.5, \mu = 0.019, \delta = 0.006, \theta = 0.05$.

Table 1 represents all possible scenarios for both disease-free and endemic condition (i.e. $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$). The scenario A_1 shows that there are no vaccination (p), treatment (γ), and natural immunity for exposed individuals (ϕ), where in such case there is no quarantined individuals ($Q_f = 0$), recovered individuals ($R_f = 0$), and basic reproduction number $\mathcal{R}_0 = 31.4127 > 1$. Meanwhile, the scenario A_2 , the natural immunity for exposed individuals is not considered, and the basic reproduction number $\mathcal{R}_0 = 3.4589 > 1$ is lower than scenario A_1 . The scenarios B_1 and B_2 give same values of treatment and natural immunity but different values of vaccination, i.e. the vaccination in scenario B_2 higher than in scenario B_1 . Then, it causes the basic reproduction number \mathcal{R}_0 in scenario B_2 lower than in scenario B_1 . Moreover, the scenarios C_1 and C_2 only give same values of treatment, but the vaccination and natural immunity in scenario C_2 stronger than in scenario C_1 . In this case, it causes the basic reproduction number $\mathcal{R}_0 = 0$ in scenario C_2 lower than in scenario C_1 .

Figures 1 and 2 show the stability of endemic ($\mathcal{E}_1 = (S^*, E^*, I^*, Q^*, R^*)$) and disease-free ($\mathcal{E}_0 = (S^0, E^0, I^0, Q^0, R^0) = ((1-p)\mu/\delta, 0, 0, 0, p\mu/\delta)$) equilibrium points for different values of $\alpha = 0.5, \alpha = 0.75, \alpha = 0.8, \alpha = 0.9, \alpha = 0.95$. The results are obtained based on the basic reproduction number, i.e. if $\mathcal{R}_0 > 1$ then the existing infection can affect more new infection. Meanwhile, if $\mathcal{R}_0 < 1$ then the existing infection can affect less new infection. In Fig. 1, we use $p = \phi = \gamma = 0.3$ with $\mathcal{R}_0 = 1.5230 > 1$ and $p = \phi = 1, \gamma = 0.3$ with $\mathcal{R}_0 = 0 < 1$ in Fig. 2. Moreover, our results show that the higher the order of α , the less time it needs to converge. Figure 3a–f represent the SEIQR profiles for varying p (vaccination), ϕ (natural immunity), and γ (treatment) respectively. By referring to

the system (6), we can conclude that the more increased vaccination the more decreased susceptible individuals and the more increased recovered individuals as shown in Fig. 3a, b. Meanwhile, Fig. 3c, d imply that the natural immunity can reduce the number of exposed individuals and increase the number of susceptible individuals because the antibodies produced are able to protect the body from transmission of diphtheria disease. The treatment of infected individuals will cause the condition of individuals to improve, so that the rate of quarantined individuals can reduce as shown in Fig. 3e, f.

The effectiveness of vaccination (p), natural immunity (ϕ), and treatment (γ) toward the profiles of susceptible, exposed, infected, quarantined, and recovered individuals are represented in Figs. 4 and 5. For the purpose of analysis in those two figures (except in recovered individuals), we only focus on the stable condition from 400 days to 1000 days (there are no changes anymore for the profiles of susceptible, exposed, infected, and recovered individuals). Based on the Fig. 4, we can conclude that the vaccination ($p = 0.3, \gamma = 0, \phi = 0$) is at the first class (the most effective) than the natural immunity ($p = 0, \gamma = 0, \phi = 0.3$) at the second class, and the treatment ($p = 0, \gamma = 0.3, \phi = 0$) at the third class. In this case, the vaccination can reduce the susceptible individuals significantly, followed by the natural immunity and treatment. Similarly, it also happens to the exposed individuals, namely the vaccination is very effective in reducing the exposed individuals. The natural immunity and treatment are at the different position of effectiveness level than previous one, namely third and second classes respectively in reducing the exposed individuals.

Moreover, the results of infected individuals provide the different effectiveness level, the treatment, vaccination, and natural immunity are at the first, second, and third classes respectively in reducing the infected individuals. In other words, the treatment is the most effective to decrease the number of infected individuals. Meanwhile, the quarantined individuals only depend on the treatment. In this case, there are no results for the profile of quarantined individuals with $(p = \gamma = 0, \phi = 0.3)$ and $(p = 0.3, \gamma = \phi = 0)$. The recovered

Table 1 Numerical simulation results in the Caputo sense for $\alpha = 0.95$ and the final values of S_f, E_f, I_f, Q_f, R_f

Parameter	A1	A2	B1	B2	C1	C2
γ	0	0.3	0.3	0.3	0.3	0.3
ϕ	0	0	0.3	0.3	0.9	1
p	0	0.3	0.3	0.9	0.3	1
\mathcal{R}_0	31.4127	3.4589	1.5230	0.2176	0.7186	0
S_f	0.1009	0.6409	1.4533	0.3346	2.1758	2.1758
E_f	0.0782	0.0401	0.0192	0	0	0
I_f	0.3208	0.0259	0.0124	0	0	0
Q_f	0	0.0154	0.0074	0	0	0
R_f	0	2.1677	1.5088	2.7653	0.9240	0.9240

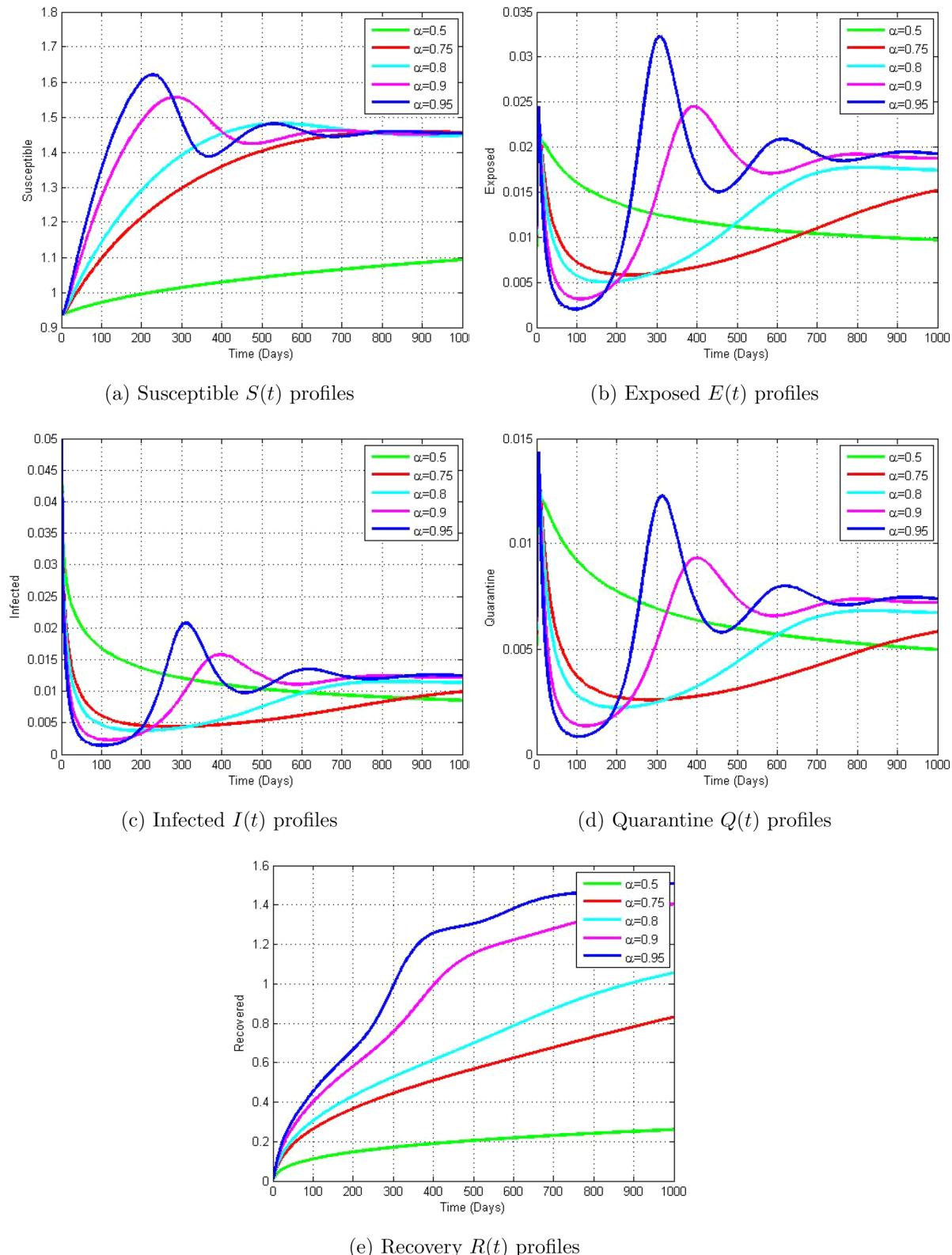


Fig. 1 Profiles of the endemic equilibrium \mathcal{E}_1 for $\alpha = 0.5, 0.75, 0.8, 0.9, 0.95$

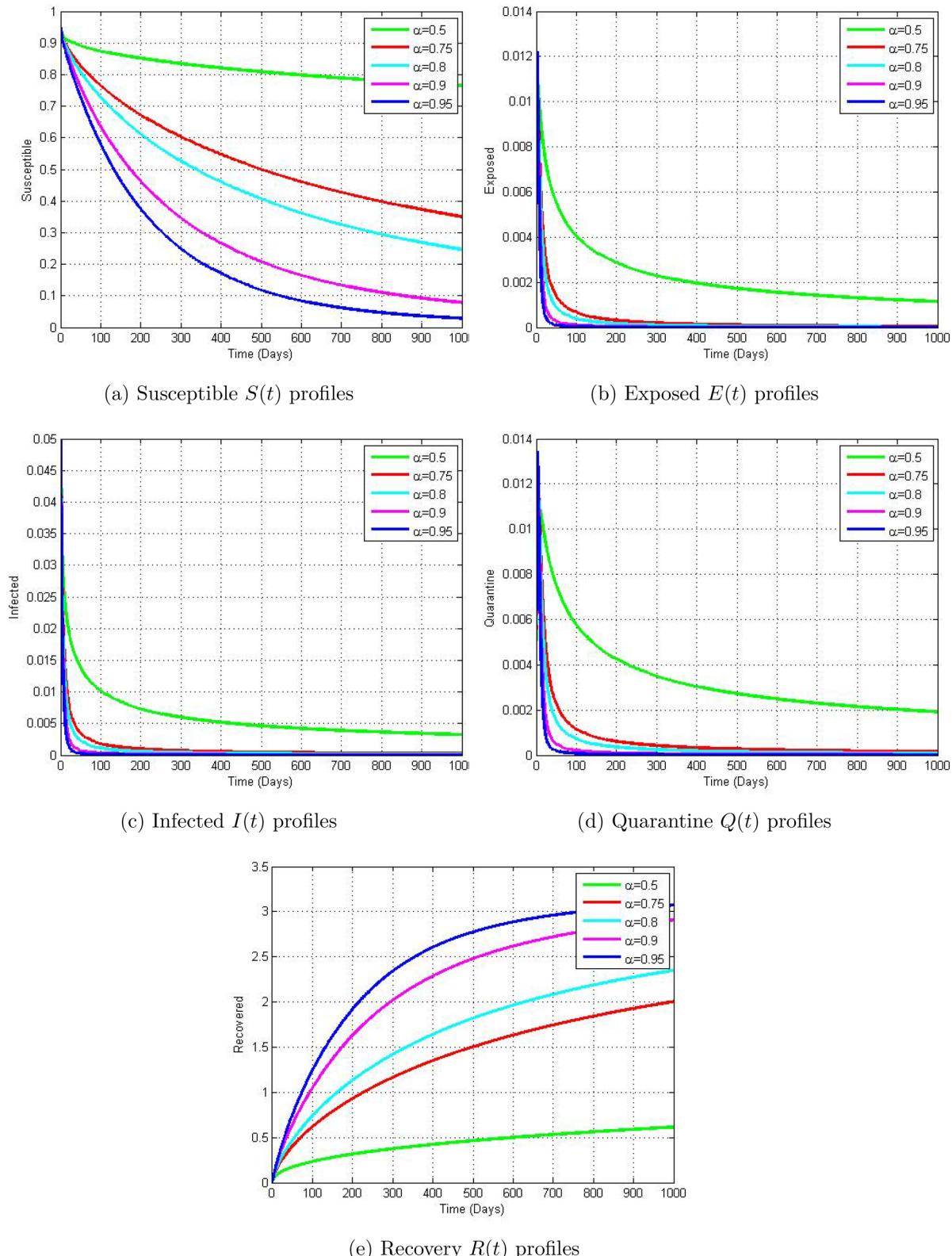


Fig. 2 Profiles of the disease-free equilibrium point E_0 for $\alpha = 0.5, 0.75, 0.8, 0.9, 0.95$

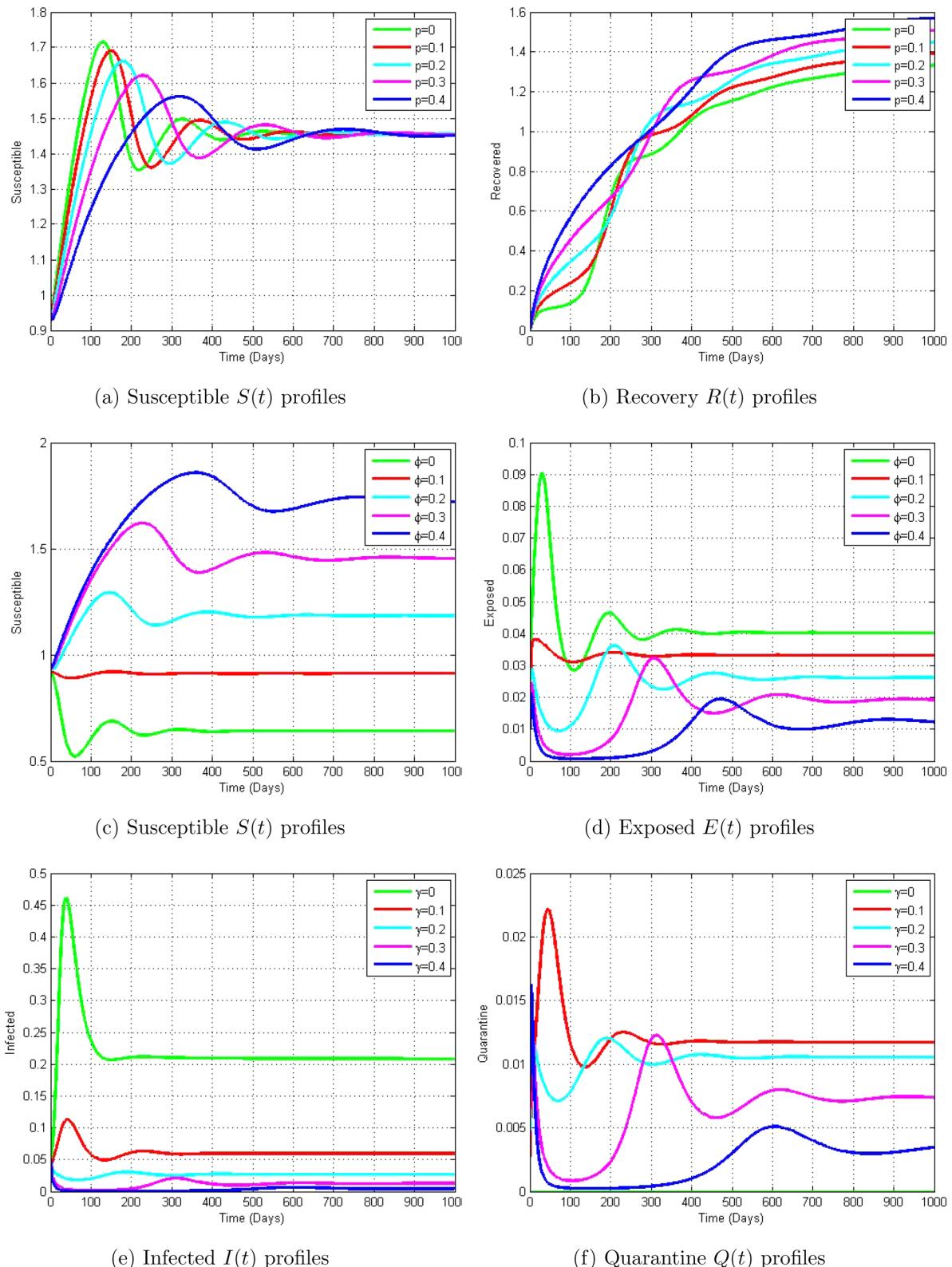


Fig. 3 $(S, E, I, Q, R)(t)$ for varying vaccination, natural immunity, and treatment

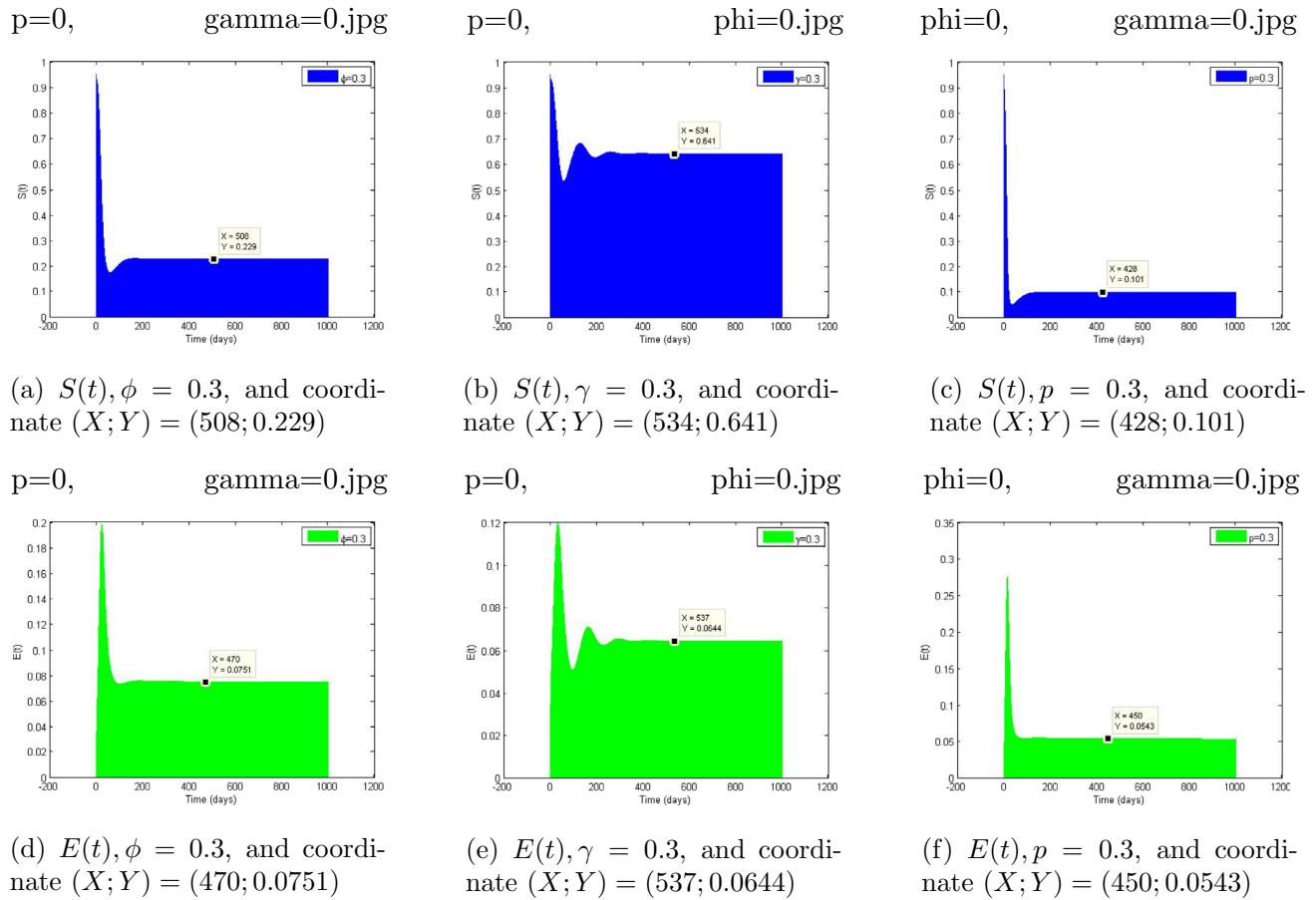


Fig. 4 The effectiveness of vaccination, natural immunity, and treatment toward the profiles of susceptible and exposed individuals with varying parameters $(p = 0, \phi = 0, \gamma = 0.3), (p = 0, \phi = 0.3, \gamma = 0), (p = 0.3, \phi = 0, \gamma = 0)$

individuals depend on treatment and vaccination but the natural immunity does not give the effect for recovered profile at all. The number of recovered individuals by treatment ($p = 0, \gamma = 0.3, \phi = 0$) is higher than the number of recovered individuals by vaccination ($p = 0.3, \gamma = 0, \phi = 0$). In this case, based on the previous results that the treatment is the most effective in reducing the infected and quarantined individuals from 400 to 1000 days.

We further validate our dynamical system (6) by fitting between the numerical results and the real data. In this paper, we apply least square technique to estimate the parameters (p, γ, ϕ) . We first consider the dynamical system as follows

$$D_*^\alpha X(t) = F(t, X, \mathcal{W}), \quad X(0) = X_0, \quad (76)$$

where $X = [S, E, I, Q, R]$, $\mathcal{W} = [p, \mu, \eta, \delta, \phi, \beta, \gamma, \theta, \varepsilon]$, and α are states of dynamical system, nine parameters, and fractional order respectively, in which these parameters are further to be estimated. The initial values X_0 are adapted from the study in Izzati and Andriani (2021) and the sum of squares error are stated as follows

$$E = \sum_{i=1}^N (Y(i) - X(i))^2, \quad (77)$$

where $X(i)$ is the numerical results obtained by Runge–Kutta fourth order, $Y(i)$ is the real data, and N is the amount of data. We further analyze the real data of Indonesia's diphtheria cases for year 2007–2017 adapted from the source (Sariadji

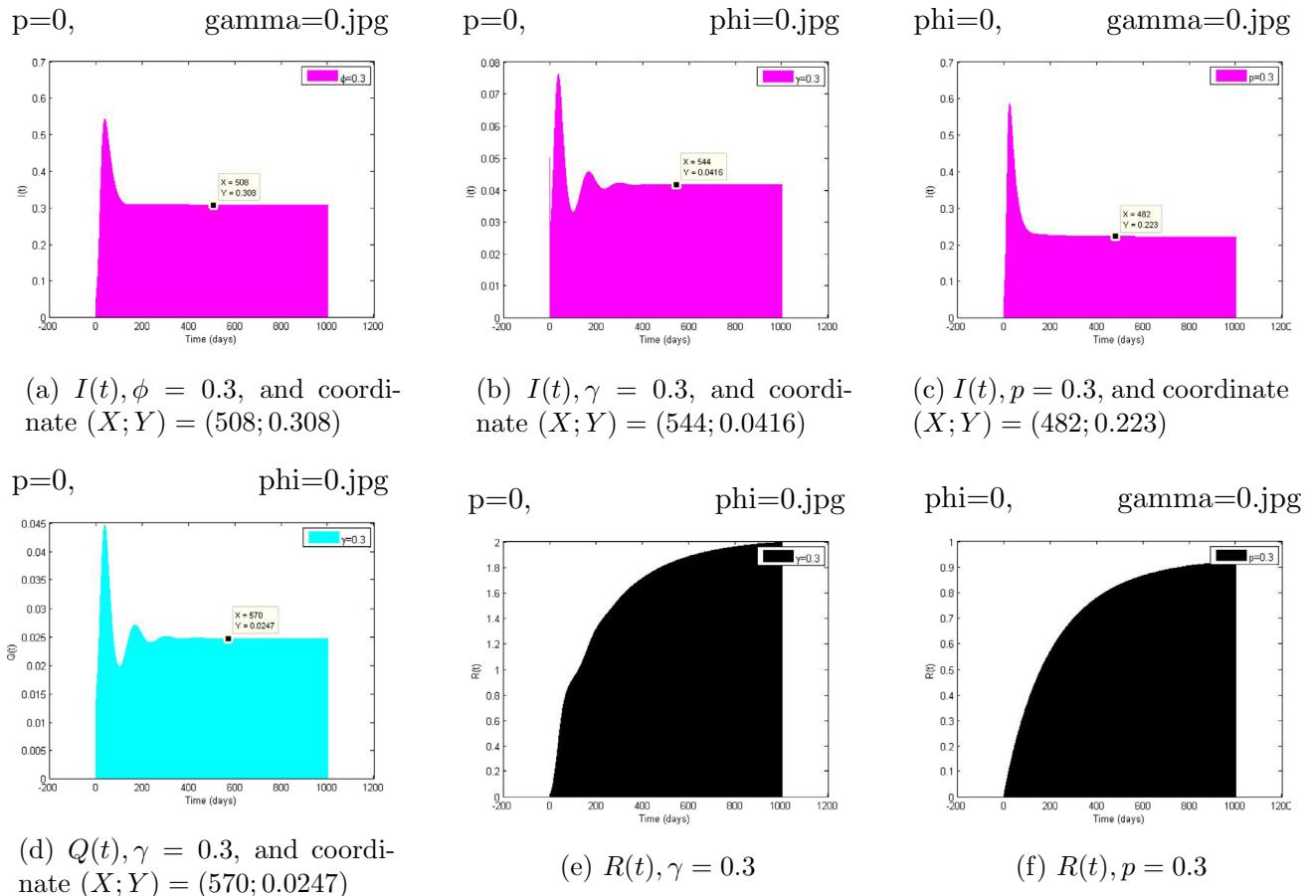
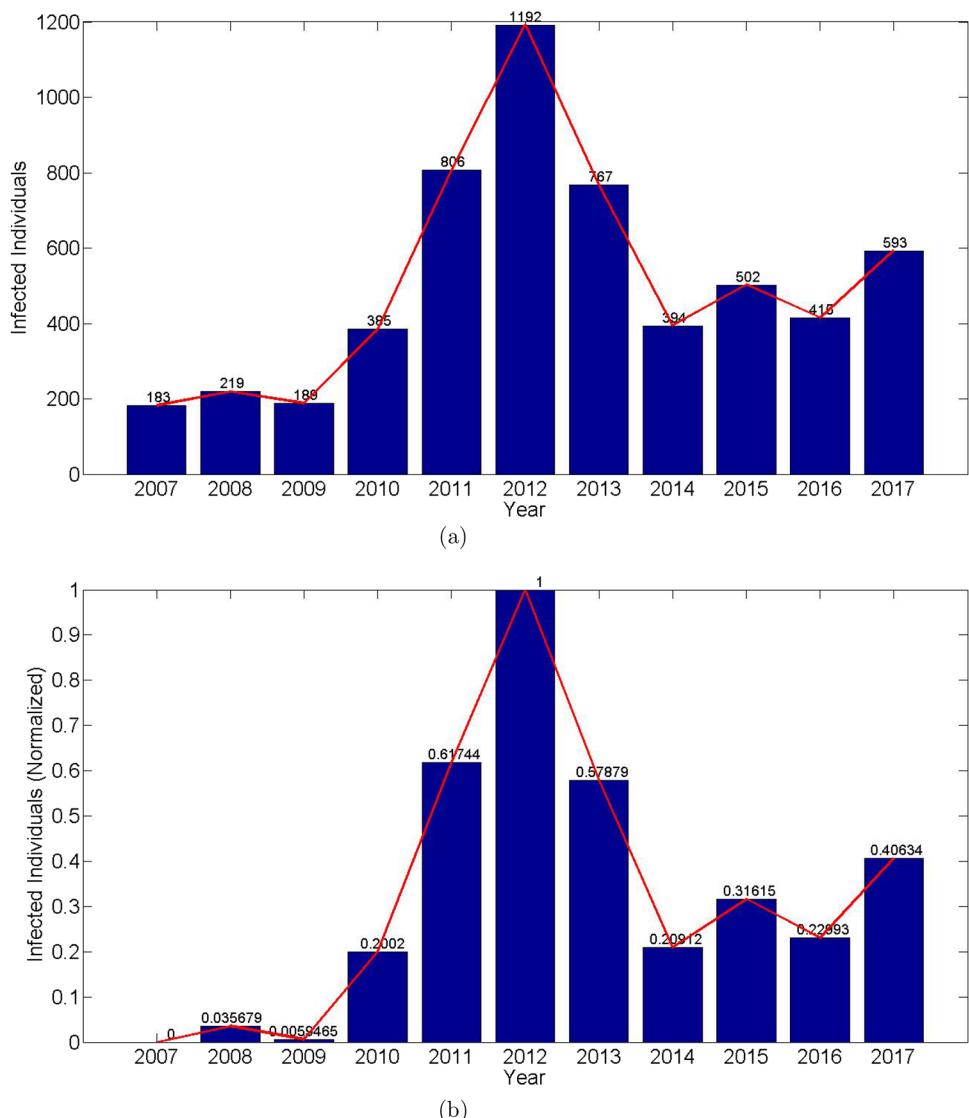


Fig. 5 The effectiveness of vaccination, natural immunity, and treatment toward the profiles of infected, quarantined, and recovered individuals with varying parameters ($p = 0, \phi = 0, \gamma = 0.3$), ($p = 0, \phi = 0.3, \gamma = 0$), ($p = 0.3, \phi = 0, \gamma = 0$)

2017), and the representative data is shown in Fig. 6a. We first normalize the data to get the results as in Fig. 6b. Based on those two data, the peak is achieved in year 2012 with the number of infected cases 1192 and 1 (normalized). Moreover, the least square technique is employed to fit our simulation obtained by Runge–Kutta fourth order (RK-4) with the normalized data. We can compare the results between our simulation of dynamical system (6) (with $\alpha = 0.95$) and real data as shown in Figs. 7a, 8a, 9a. Both results give the same trend and small values of root mean square error 2.63%, 2.87%, 2.66%, 5.85%, 2.18%, 5.25% (without/with) vaccination, (without/with) natural immunity, and (without/with) treatment respectively in Table 2.

The best-fit estimations of parameters are established by choosing the minimum RMSE through the least square technique. Moreover, the best-fit estimations of parameters are used to estimate the next 5 years (2018, 2019, 2020, 2021 and 2022) for the trend of infected individuals. In the green area, we can see that the Figs. 7b, 8b, 9b provide the decreasing trend of infected individuals in year 2018 and the trend is increased from year 2019 to year 2022 (without/with) vaccination, (without/with) natural immunity, and (without/with) treatment respectively. In the predication area, the peaks are achieved in year 2021 (with) natural immunity and treatment, and in year 2022 (without) natural immunity, treatment, and (with/without)

Fig. 6 **a** Original and **b** normalized data of Indonesia's diphtheria cases (Sariadji 2017)



vaccination. This peak in the number of infected individuals may be due to decreased natural immunity, vaccination, or treatment where these, among others, may be factors that caused the outbreak.

Moreover, we also present the comparison results for varying fractional order ($\alpha = 1, \alpha = 0.95, \alpha = 0.9, \alpha = 0.85, \alpha = 0.8$). Based on the those results, the root mean square error of dynamical system with fractional order is not always smaller than the root mean square error of dynamical system

with integer order (i.e., RMSE for $\alpha = 1$ and $\alpha = 0.9$), where this root mean square error is represented in the Table 3. Figure 10a shows the profiles of infected individuals for both numerical results solved by Runge–Kutta fourth order and real data of diphtheria case in Indonesia. Based on the results in Table 3 and Fig. 10b, we can conclude that the fractional order $\alpha = 0.8$ has the smallest error value than the other ones.

Fig. 7 **a** Fitting and **b** prediction results of diphtheria case with/without natural immunity

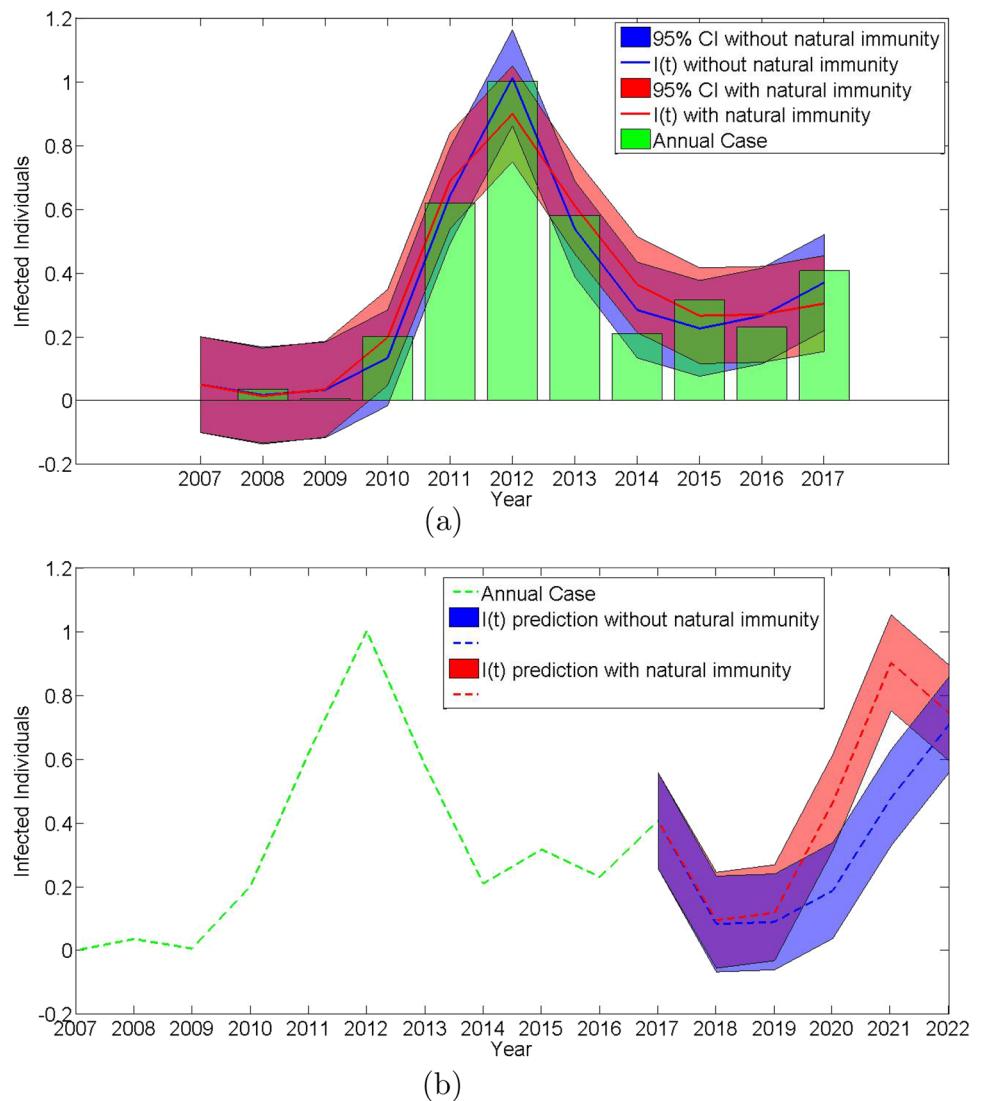
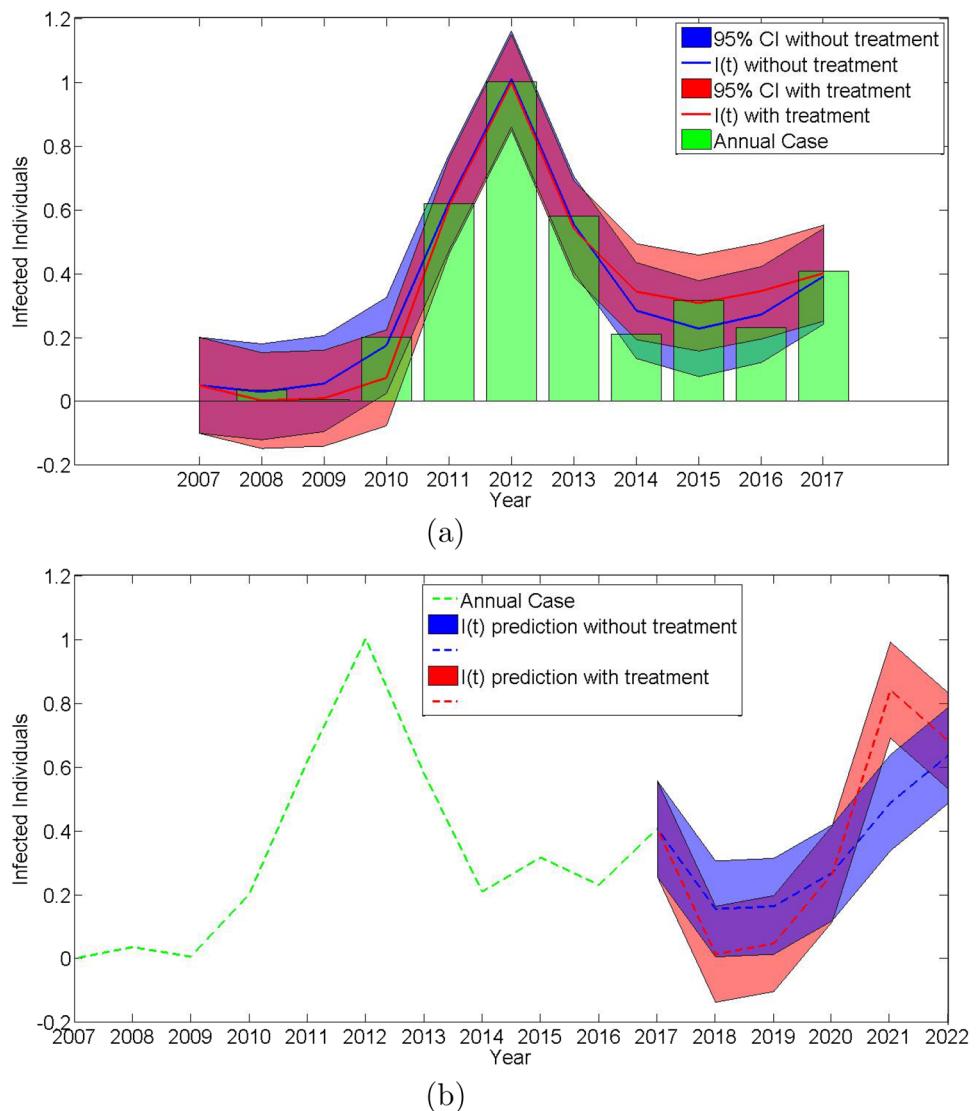


Table 2 Best-fit estimation of parameter for annual diphtheria case

Parameter	Vaccination		Natural Immunity		Treatment	
	Without	With	Without	With	Without	With
p	-2.3448	22.6133	-10.2996	-1.1539	1.7982	-8.0445
μ	0.2073	-0.0363	0.0836	-0.2771	-1.1833	0.5114
η	1.4564	1.4070	2.0195	1.8816	0.3911	1.7650
δ	-0.0780	-0.0943	-0.1190	0.0308	0.0447	0.0978
ϕ	38.8229	-0.6816	0.0765	-2.0850	-11.8333	-1.4857
β	145.6372	3.8853	2.4015	0.1313	15.5989	1.5421
γ	2.9544	2.7007	4.0233	1.5359	0.3044	10.7326
θ	-0.8293	-0.1464	-0.7414	0.0935	1.8333	0.0168
ϵ	1.5203	-8.4180	2.3787	3.5204	3.2747	11.3678
RMSE	2.63%	2.87%	2.66%	5.85%	2.18%	5.25%

Fig. 8 **a** Fitting and **b** prediction results of diphtheria case with/without treatment



Conclusions

We present the fractional SEIQR model on diphtheria disease by considering the effect of vaccination, treatment, and natural immunity. The basic analysis of boundedness, non-negativity, and existence and uniqueness are studied. The basic reproduction number (\mathcal{R}_0) is obtained based on the dominant Eigen values $\mathcal{F}\mathcal{V}^{-1}$, where this basic reproduction number provides the transmission of a disease ($\mathcal{R}_0 < 1$

indicates the disease-free condition and $\mathcal{R}_0 > 1$ indicates the endemic condition). The local stability for disease-free equilibrium point is proved based on the standard calculation. Moreover, the local stability for endemic equilibrium point was obtained by applying the Routh–Hurwitz criterion. The global stability for both the disease-free and endemic equilibrium points are established by constructing the appropriate Lyapunov function. The numerical simulations are obtained by using the predictor–corrector

Fig. 9 **a** Fitting and **b** prediction results of diphtheria case with/without vaccination

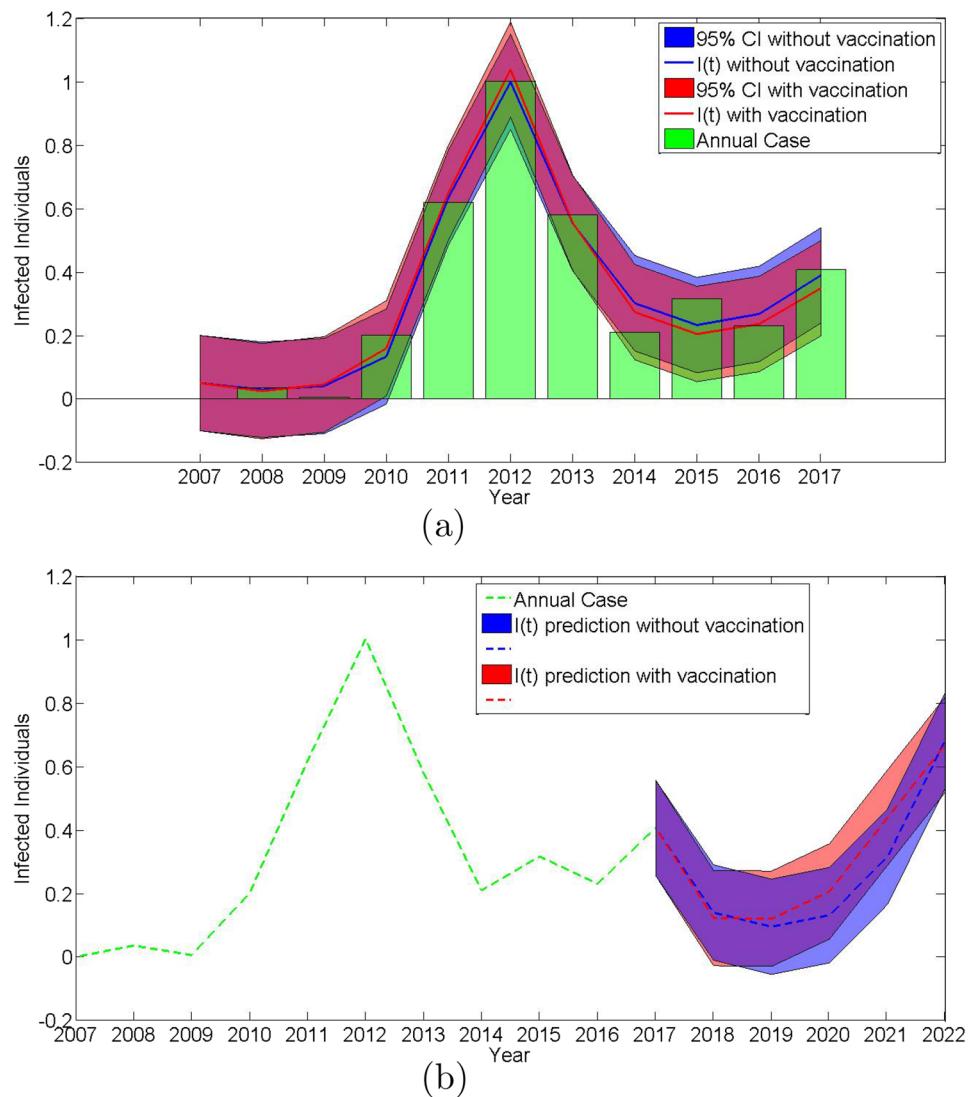


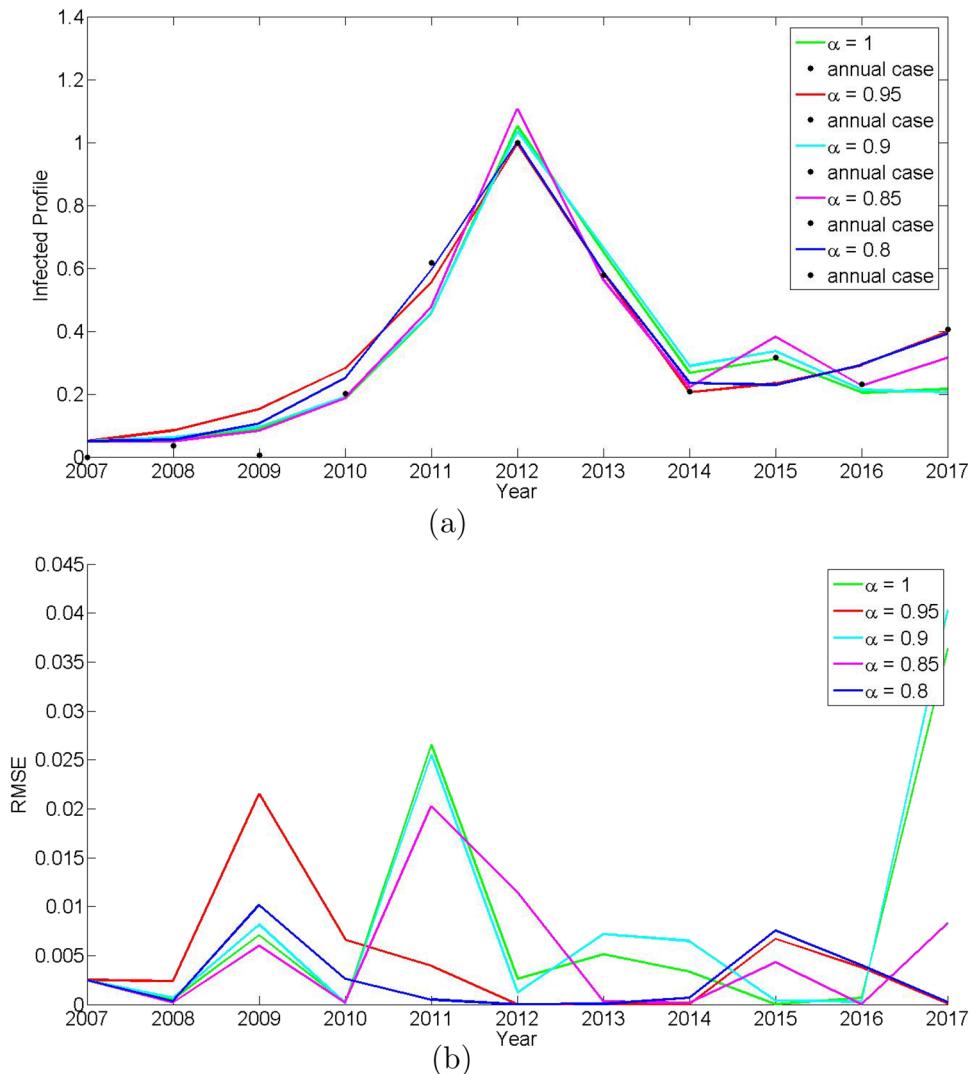
Table 3 Comparison results of root mean square error for varying fractional order

Error	Fractional order				
	$\alpha = 1$	$\alpha = 0.95$	$\alpha = 0.9$	$\alpha = 0.85$	$\alpha = 0.8$
RMSE	0.0077	0.0043	0.0084	0.0049	0.0026

technique. The aims of this numerical simulations are to provide the effect of fractional order. The different values of fractional order have no effect on the stability of our fractional order dynamical system. The most effectiveness is

achieved by the vaccination in reducing the susceptible and exposed individuals. Moreover, the treatment is the most effective in reducing the number of infected individuals and the treatment give the effect significantly for the quarantined individuals. We further apply the real data to validate our fractional dynamical system by fitting the numerical results by Runge–Kutta fourth order with the data of Indonesia's diphtheria cases. The results provide the small root mean square error (RMSE) for different values of fractional order ($\alpha = 1, \alpha = 0.95, \alpha = 0.9, \alpha = 0.85, \alpha = 0.8$). Based on those RMSE, we can conclude that the smaller fractional order does not always guarantee to give the smaller RMSE.

Fig. 10 **a** Infected profile and **b** error value for each varying fractional order ($\alpha = 1$, $\alpha = 0.95$, $\alpha = 0.9$, $\alpha = 0.85$, $\alpha = 0.8$)



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Data availability statement This study uses secondary data from the Ministry of Health of the Republic of Indonesia, which is published

online. The reference used as secondary data information for the purposes of this study refers to Sariadji (2017).

Declarations

Conflict of interest None of the authors of this paper has a financial or personal relationship with other people or organizations that could inappropriately influence or bias the content of the paper. It is to specifically state that “No Competing interests are at stake and there is No Conflict of Interest” with other people or organizations that could inappropriately influence or bias the content of the paper.

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