Easily Made and Implemented C1 Continuous Acceleration Profile

Miles Higman

February 2024

1 Introduction

In control engineering, discontinuities in jerk are seen as a hindrance to accurate, smooth motion. To try and overcome this hindrance, I have devised an easy way to partition the sections of a modified trapezoidal acceleration profile such that the net change in velocity is zero while the resulting displacement, maximum acceleration, and maximum jerk are any constants specified.

2 Preconditions

To build the acceleration profile, it is best to consider the preconditions that the motion profile must satisfy.

First, the first integral of the acceleration profile must be zero.

$$\int_0^{t_{end}} a(t)dt = 0 \tag{1}$$

Then, the double integral of the acceleration profile must be some constant L.

$$\int_0^{t_{end}} \int a(t)dt^2 = L \tag{2}$$

Also, $\frac{\mathrm{d}(a(t))}{\mathrm{d}t}$ should be continuous and have a maximum of j.
Furthermore, $a(t)$ should have a maximum of a.
Finally, $a(0)$ and $a(t_{end})$ should be equal to 0.

3 The Tunable Function

A modified trapezoidal profile that is C1 continuous is as follows:

$$a(t) = \begin{cases} a\sin\left(\frac{jt}{a}\right) & 0 \le t < \frac{\pi a}{2j} \\ a & \frac{\pi a}{2j} \le t < \frac{\pi a}{2j} + t_1 \\ a\cos\left(\frac{j}{a}\left(t - \left(t_1 + \frac{\pi a}{2j}\right)\right)\right) & \frac{\pi a}{2j} + t_1 \le t < \frac{3\pi a}{2j} + t_1 \\ -a & \frac{3\pi a}{2j} + t_1 \le t < \frac{3\pi a}{2j} + 2t_1 \\ a\sin\left(\frac{j}{a}\left(t - \left(2t_1 - \frac{2\pi a}{j}\right)\right)\right) & \frac{3\pi a}{2j} + 2t_1 \le t \le \frac{2\pi a}{j} + 2t_1 \end{cases} \tag{3}$$

This equation has some variables that need clarifying...

a is the maximum acceleration the profile allows.

j is the maximum jerk the profile allows.

t_1 is the duration of the maximum acceleration of the profile. This quantity is automatically calculated given the distance (L) the profile needs to achieve, a, and finally j.

4 Finding t_1

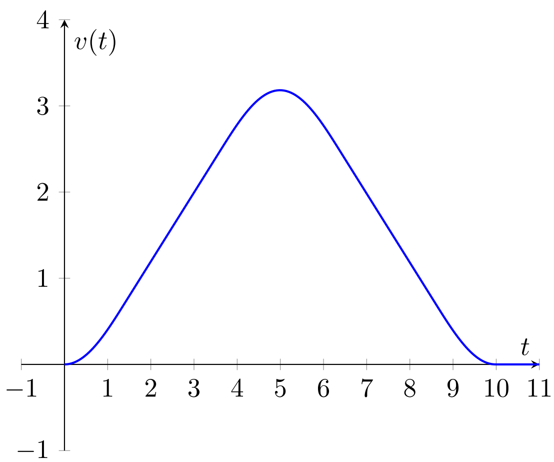
One of the more challenging parts of creating this motion profile is finding the duration of the extrema of the profile. It is best to extend the maximums of the profile over the changes (sine waves), as this distinction allows for faster motion.

The most pertinent precondition in Section 2 that dictates t_1 is the precondition outlined in Equation 2. To find t_1, it's imperative to know what $v(t)$ is to better understand the integrand of the final integral in Equation 2. $v(t)$ is as follows...

$$v(t) = \begin{cases} \frac{a^2}{j}\left(1 - \cos\left(\frac{j}{a}t\right)\right) & 0 \le t < \frac{\pi a}{2j} \\ at + \left(1 - \frac{\pi}{2}\right)\cdot\frac{a^2}{j} & \frac{\pi a}{2j} \le t < \frac{\pi a}{2j} + t_1 \\ \frac{a^2}{j}\sin\left(\frac{j}{a}t - \frac{jt_1}{a} - \frac{\pi}{2}\right) + at_1 + \frac{a^2}{j} & \frac{\pi a}{2j} + t_1 \le t < \frac{3\pi a}{2j} + t_1 \\ -at \ + \ 2at_1 + \left(1 + \frac{3\pi}{2}\right)\cdot\frac{a^2}{j} & \frac{3\pi a}{2j} + t_1 \le t < \frac{3\pi a}{2j} + 2t_1 \\ \frac{a^2}{j}\left(1 - \cos\left(\frac{j}{a}t - \frac{2jt_1}{a}\right)\right) & \frac{3\pi a}{2j} + 2t_1 \le t \le \frac{2\pi a}{j} + 2t_1 \end{cases} \tag{4}$$

This is a complicated function, to say the least. However, most of its complexity derives from the constants that allow the piecewise function to be continuous from the start to the end of the motion, which have been put in terms of a, j, and t_1.

A good way to find t_1 is to look at the problem geometrically. Below is the graph of $v(t)$ where $a = 0.8$, $j = 0.9$, and $L = 8\pi\frac{a^3}{j^2}$:



To find the position, we will eventually take this function's integral. However, since the function is symmetric over $t = \frac{t_{end}}{2}$, we only need to consider half of it. So:

$$\int_0^{\frac{t_{end}}{2}} v(t)dt = \frac{L}{2} \tag{5}$$

Now, we need to find this integral. I find the most intuitive method to find this integral is via geometry.

To make this integral easier, let us consider two integrals that will not change due to t_1.

$$\int_0^{\frac{\pi a}{2j}} v(t)dt \tag{6}$$

$$\int_{\frac{\pi a}{2j}+t_1}^{\frac{\pi a}{j}+t_1} \left(v(t) - at_1 - \frac{a^2}{j} \right) dt \tag{7}$$

Now, through simple integration, we get...

$$\int_0^{\frac{\pi a}{2j}} v(t)dt + \int_{\frac{\pi a}{2j}+t_1}^{\frac{\pi a}{j}+t_1} \left(v(t) - at_1 - \frac{a^2}{j} \right) dt = \frac{\pi a^3}{2j^2} \tag{8}$$

Now, we evaluate the part of Equation 5 using geometry (two squares and a triangle) depending on t_1.

$$\int_{\frac{\pi a}{2j}}^{\frac{\pi a}{2j}+t_1} v(t)dt + \int_{\frac{\pi a}{2j}+t_1}^{\frac{\pi a}{j}+t_1} \left(at_1 + \frac{a^2}{j} \right) dt = \frac{a}{2}t_1^2 + \left(1 + \frac{\pi}{2}\right) \frac{t_1 a^2}{j} + \frac{\pi a^3}{2j^2} \tag{9}$$

Notice the relationship between the integrals.

$$\int_0^{\frac{\pi a}{2j}} v(t)dt + \int_{\frac{\pi a}{2j}}^{\frac{\pi a}{2j}+t_1} v(t)dt + \int_{\frac{\pi a}{2j}+t_1}^{\frac{\pi a}{j}+t_1} v(t)dt = \int_0^{\frac{t_{end}}{2}} v(t)dt \tag{10}$$

Substituting and simplifying, we get:

$$\frac{a}{2}t_1^2 + \left(1 + \frac{\pi}{2}\right)\frac{t_1 a^2}{j} + \frac{\pi a^3}{j^2} = \frac{L}{2} \tag{11}$$

Notice that Equation 11 is a quadratic equation with t_1 being the independent variable. First, let's define the a, b, and c of a conventional quadratic as a_q, b_q, and c_q

$$\begin{cases} a_q = \frac{a}{2} \\ b_q = \left(1 + \frac{\pi}{2}\right)\frac{a^2}{j} \\ c_q = \frac{\pi a^3}{j^2} - \frac{L}{2} \end{cases} \tag{12}$$

Substituting the variables defined in Equation 12 into the quadratic formula and simplifying, we get:

$$t_1 = -\left(1 + \frac{\pi}{2}\right) \cdot \frac{a}{j} + \sqrt{\left(1 - \pi + \frac{\pi^2}{4}\right) \cdot \frac{a^2}{j^2} + \frac{L}{a}} \tag{13}$$

This is about as simple as I will go, but feel free to simplify further.

5 Putting It All Together: Finding $s(t)$

It is best to clarify what $s(t)$ is to ensure this profile fully applies to any path function. By integrating $v(t)$, we get...

$$\int v(t)dt = s(t) = \begin{cases} \frac{a^2\left(jt - a\sin\left(\frac{j}{a}t\right)\right)}{j^2} & 0 \le t < \frac{\pi a}{2j} \\ \left(1 - \frac{\pi}{2}\right)\frac{a^2 x}{j} + \frac{a x^2}{2} + \frac{a^3}{j^2}\left(\frac{\pi^2}{8} - 1\right) & \frac{\pi a}{2j} \le t < \frac{\pi a}{2j} + t_1 \\ \frac{a^3 \sin\left(\frac{j(t_1 - t)}{a}\right) + \left(a^2 j + a j^2 t_1\right)t}{j^2} - \frac{\pi a^2 t_1}{2j} - \frac{a t_1^2}{2} & \frac{\pi a}{2j} + t_1 \le t < \frac{3\pi a}{2j} + t_1 \\ -\frac{a t^2}{2} + \left(2 a t_1 + \left(1 + \frac{3\pi}{2}\right)\frac{a^2}{j}\right)t + \frac{\left(8 - 9\pi^2\right)a^3}{8j^2} - \frac{2\pi a^2 t_1}{j} - a t_1^2 & \frac{3\pi a}{2j} + t_1 \le t < \frac{3\pi a}{2j} + 2t_1 \\ \frac{a^2 t}{j} - \frac{a^3}{j^2}\sin\left(\frac{j}{a}t - \frac{2 j t_1}{a}\right) + a t_1^2 + \frac{\pi a^2 t_1}{j} & \frac{3\pi a}{2j} + 2t_1 \le t < \frac{2\pi a}{j} + 2t_1 \end{cases} \tag{14}$$

6 Conclusion

This precise profile has many applications. It is a very convenient profile as the percentage of each section of the modified trapezoidal profile is automatically optimized for the arc length. As long as a robot needs to follow a path, this C1 continuous profile can significantly improve accuracy and precision. However, more practical testing is needed to test the profile's general optimization and fault tolerance.