COMP20003 Algorithms and Data Structures Greedy Algorithms and the MST

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Semester 2

Greedy Algorithms

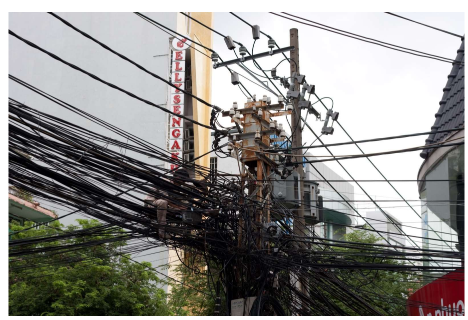
Greedy algorithms are used in optimization problems

Greedy algorithms keep taking the next best step repeatedly, until the best solution is reached

- Dijkstra's algorithm is greedy: takes the next best edge to add to the path tree

Minimum Spanning Tree

- Undirected weighted graphs
- Minimum spanning tree = subgraph that is:
  - A tree (no cycles)
  - Contains every vertex (spans)
  - Minimum sum of edge weights
- Also called:
  - Minimum weight spanning tree (sum of weights)
  - Minimal spanning tree (might be more than one)



Saigon X, Health and Safety? What's That? Reena Mahtani
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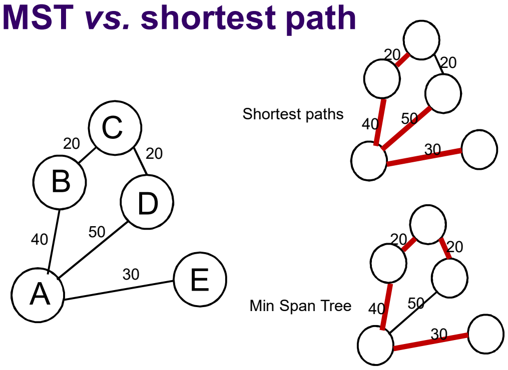
MST *vs.* shortest path



MST and Graph characteristics

- Graph must be connected
- MST *must* have exactly *V-1* edges
- No cycles in MST

Building a MST: General approach

- **Start** with isolated vertices (all), **no edges**
- **Begin** with **any vertex** (Prim's) **or** the **least cost edge** (Kruskal's)
  - This is a MST subtree

Keep adding vertices/edges to **extend** this MST **subtree**

- Shortest connections
- No cycles

Famous MST algorithms

- Prim's
  - **Shortest connection networks and some generalizations**. R.C. Prim, *Bell System Technical Journal* ***36***(6), 1389-1401, 1957.
- Kruskal's
  - **On the shortest spanning subtree of a graph and the traveling salesman problem**. J.B. Kruskal, *Proceedings of the American Mathematical Society* **7**, 48-50, 1956.
- Borůvka's (1926, published in Czech)
  - **Otakar Borůvka on minimum spanning tree problem: translation of both the 1926 papers, comments, history**. Nešetřil, Jaroslav; Milková, Eva; Nešetřilová, Helena (2001). *Discrete Mathematics* **233** (1–3): 3–36

Prim’s MST algorithm

- Preferred method for dense graphs
- Easiest with matrix representation
- Prim’s algorithm relies on picking the next best edge that joins two set of vertices:
  - Vertices already in the tree (S)
  - Vertices not yet in the tree (V-S)

These two sets form a “cut”

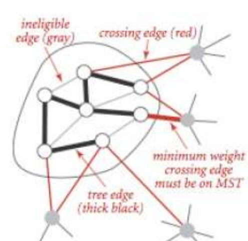
Definitions

- A Cut *(V,V-S)* of *G* is a partition of *V*
- Cross: an edge *(u,v)* in *E* with one endpoint in *S* and the other in *V-S*
- Light edge: the minimum weight edge crossing the cut
- Respect: a cut respects a set *A* of edges if no edge in *A* crosses the cut

Cut during MST construction

Cut:

- *S*: set of vertices already **in** the MST
- *V-S*: **not** yet in the MST
  - Fringe: part of *V-S* **one step away** from the MST
  - **Vertices in *V-S*** have a cost (**distance**) from the MST subtree so far constructed
  - **Distances** between non-MST vertices and MST vertices **are updated** as vertices are added to MST



Prim’s MST algorithm

From R. Sedgewick, Algorithms 4th edition

Prim's MST construction

Start:

- *S* = {any vertex}
- *S-V* = {all the others}
- The **cut** *S/V-S* **respects** edges **in** the **MST** as it is being constructed
- The cut itself changes

Prim's MST construction

Respect:

- The cut S/V-S respects edges in the MST being constructed
  - Fringe: vertices in V-S one step away from the MST
  - Vertices in V-S have a cost(distance) from the MST subtree so far constructed (some may be ∞)

Prim's MST construction

- Pick **lightest** edge **crossing** the cut:
  - Crossing edge *(u,v)* has *u* in *S* and *v* in *V-S*
    - Add v to S
    - Keep track of path (pred[ ])
    - Update distances between non-MST vertices and MST vertices (could be closer now) (w[ ])
- Repeat until V-S = `{0}`
- Reconstruct connections and distances from `pred`[ ] and `wt`[ ]

Prim's: Pseudocode

```
void prim(G,wt,root)
{
   for every u in V { dist[u] = ∞;  inmst[u] = FALSE;}
   dist[root] = 0; pred[root] = NULL;
   PQ = makePQ(V); /* all vertices in PQ */
   while(!empty PQ){
     u = deletemin(PQ);
     for every (v adjacent to u){
         if ((inmst[v]==FALSE)&& (w[u][v]< dist[v])){
            dist[v] = w[u][v]; /* update distance */
            decreasewt(PQ,v,dist[v]);/* update PQ dist*/
            pred[v]=u; /* update path information */
         }
     }
     inmst[u] = TRUE;
   }
 } /* at end: MST = {{v,pred[v]}: v in V - {root}} */
```

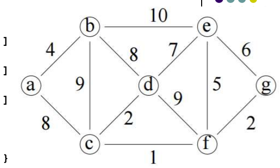
Fringe vertices are in a priority queue

- This is a Priority-First Search

Prim's example

	a	b	c	d	e	f	g	
Dist = [								]
pred = [								]
inMST =[								]

PQ = { }

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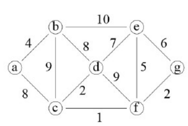
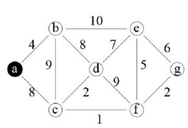
Prim's example

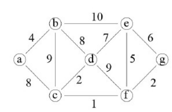


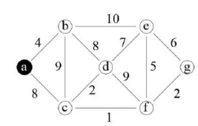
Connected graph



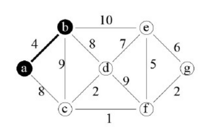
Step 0
S={a}
V \ S = {b,c,d,e,f,g}
lightest edge = {a,b}

Example from Mordechai Golin, Hong Kong University of Science and Technology
http://www.cse.ust.hk/faculty/golin/COMP271Sp03/Notes/MyL10.pdf

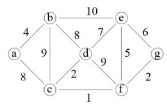


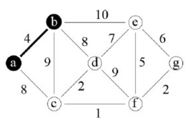


Step 1.1 before
S={a}
V \ S = {b,c,d,e,f,g}
A={}
lightest edge = {a,b}

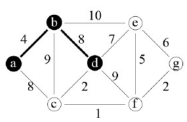


Step 1.1 after
S={a,b}
V \ S = {c,d,e,f,g}
A={{a,b}}
lightest edge = {b,d}, {a,c}

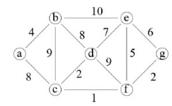


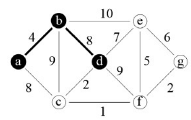


Step 1.2 before
S={a,b}
V \ S = {c,d,e,f,g}
A={{a,b}}
lightest edge = {b,d}, {a,c}

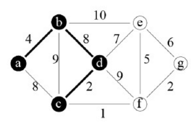


Step 1.2 after
S={a,b,d}
V \ S = {c,e,f,g}
A={{a,b},{b,d}}
lightest edge = {d,c}

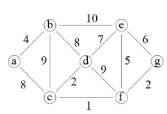


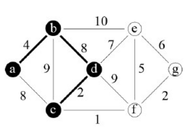


Step 1.3 before
S={a,b,d}
V \ S = {c,e,f,g}
A={{a,b},{b,d}}
lightest edge = {d,c}

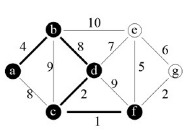


Step 1.3 after
S={a,b,c,d}
V \ S = {e,f,g}
A={{a,b},{b,d},{c,d}}
lightest edge = {c,f}

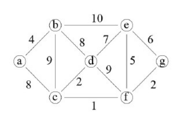


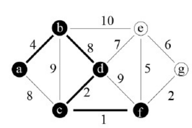


Step 1.4 before
S={a,b,c,d}
V \ S = {e,f,g}
A={{a,b},{b,d},{c,d}}
lightest edge = {c,f}

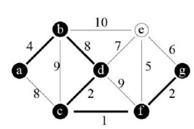


Step 1.4 after
S={a,b,c,d,f}
V \ S = {e,g}
A={{a,b},{b,d},{c,d},{c,f}}
lightest edge = {f,g}

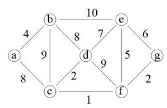


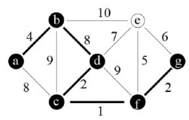


Step 1.5 before
S={a,b,c,d,f}
V \ S = {e,g}
A={{a,b},{b,d},{c,d},{c,f}}
lightest edge = {f,g}

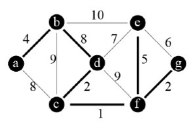


Step 1.5 after
S={a,b,c,d,f,g}
V \ S = {e}
A={{a,b},{b,d},{c,d},{c,f},
{f,g}}
lightest edge = {f,e}





Step 1.6 before
S={a,b,c,d,f,g}
V \ S = {e}
A={{a,b},{b,d},{c,d},{c,f},
{f,g}}
lightest edge = {f,e}

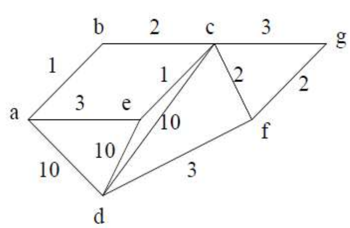


Step 1.6 after
S={a,b,c,d,e,f,g}
V \ S = {}
A={{a,b},{b,d},{c,d},{c,f},
{f,g},{f,e}}
MST completed

Prim's: Analysis

Initialize arrays pred[] dist[]: O(v)
Make PQ of vertices: if heap O(v)
Loop while PQ not empty V*
  Deletemin (if heap) O(log v)
  Update adjacent weights,
    adjust wt in PQ O(degree of u * log v)
    = O(V* (log V + deg(u) * logV))
 = O(V log V + V deg(u) * log V)
Noting that V deg(u) = E,
 = O(V log V + E log V)
 = O((V+E)logV)
 = O(E log V) for dense graphs with heap PQ

Prim's *vs.* Dijkstra's



Kruskal's MST algorithm

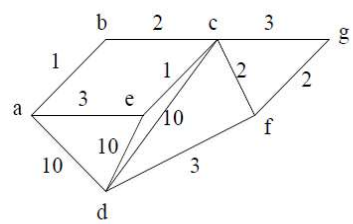
- **Prim's** algorithm adds the next closest vertex.
- **Kruskal's** algorithm adds the next lowest weight edge *that **doesn't form a cycle***.

Kruskal's Algorithm for MST

```
E1: edges in MST so far
E2: remaining edges

E1=EMPTYSET,E2=E
Sort edges in E2 by weight
while |E1| < |V|-1 edges and E2 not EMPTYSET
      Pick min cost edge e(i,j) from E2
      E2 = E2 \ e(I,j)
      if V(i),V(j) are not in same MST-so far, then
            E1 = E1 Union e(I,j)
            unite MSTs with V(i) and V(j)
```


Kruskal's




```
if V(i),V(j) are not in same MST-so far, then
            unite MSTs with V(i) and V(j)
```

- Prevents cycles (not in same MST-so far)
- Unites MSTs (new edge in MST-so far)

Kruskal's algorithm

Sort edges:
- E log E

E* get next edge and check for cycle

E * merge subsets

Kruskal's Algorithm for MST

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??


```
if V(i),V(j) are not in same MST-so far, then
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```

- Prevents cycles (not in same MST-so far)
- Unites MSTs (new edge in MST-so far)

Sounds easy, but…

…. **requires new data structure and algorithm**

- Disjoint-set data structure
- Union-find algorithm

Union-find

- Have disjoint (non-overlapping) subsets
  - Find: Which subset is an element in?
  - Union: Join two subsets into a single subset
- For Kruskal's algorithm:
  - **Find**: Is the new edge in an existing subset?
    - If yes, this is a cycle! – don't use!
  - **Union**: Does the new edge join two subjects?
    - If yes, join the two subsets

Union-find

- Have disjoint (non-overlapping) subsets
  - Find: Which subset is an element in?
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Union-find

- Have disjoint (non-overlapping) subsets
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- Naïve union-find: array

Union-find: array

- Have disjoint (non-overlapping) subsets
  - Find: Which subset is an element in?
  - Union: Join two subsets into a single subset
- Naïve union-find: array
- id[]

0	1	2	3	4	5	6	7

Union-find: array

- Start: Singleton Sets
- id[]

0	1	2	3	4	5	6	7
0	1	2	3	4	5	6	7

Example: Put (2, 3) in same set, and (4,6):

- change entry in id[]: choose representatives

0	1	2	3	4	5	6	7
0	1	2	2	4	5	4	7

Union-find: array

0	1	2	3	4	5	6	7
0	1	2	2	4	5	4	7
4	2	2	2	4	4	4	2
2	2	2	2	2	2	2	2

Union-find: array

- Naïve algorithm, using array:
- Find:
  - id[p] == id[q]
  - O(1)
- Union:
  - id[p and all in same subset] = id[q]
  - O(n)

Speeding up the Union in Union-Find

- Speed up union: tree-based approach
  - id[] is a parent array
  - **Root** is the **representative of the subset**

To union two subsets – make the root of one the parent of the root of the other

- O(1)

0	1	2	3	4	5	6	7

Find in Tree-based Union-Find

- Find:
  - Traverse back through parent array to root
  - Nodes are in the same subset if they have the same root
  - O(n)

Time for trace depends on depth of tree

0	1	2	3	4	5	6	7

Improvements in Union-Find

- Find:
  - **Time** for trace depends on **depth of tree**
  - Weighted: merge smaller tree into larger
    - **keeps tree broader**
  - Path compression
- Analysis: E union-finds on V vertices
  - Naïve: O(EV)
  - Weighted or path compress: O(V + E log V)
  - Weighted AND path compress: O(E+V) α(V) ≈ O(E+V)

Union-Find Analysis

- Analysis: E union-finds on V vertices
  - Naïve: O(EV)
    - Array: O(1) find; O(n) union
    - Tree: O(1) union; O(n) find
  - Weighted OR path compress: O(V + E log V)
  - Weighted AND path compression:
    - $O(E^*\alpha(E,V) + V)$
    - $\alpha(n)$: inverse Ackermann function, small constant
    - $\approx O(E+V)$

Kruskal's: Analysis with best union-find

- Sort edges:
  - E log E
- E finds and E unions:
  - E+V
- O(E log E + E + V) = O(E log E)
- Time is dominated by sorting the edges!

Kruskal's: Analysis with best union-find

- Time is dominated by sorting the edges!

Any ideas for what we might do?

Improvement to Kruskals: Partial sort

When sorting dominates performance, partial sorting can help…

… only need the smallest V-1 edges

e.g. quicksort-like partition, but

- Works if graph is connected
- Doesn't work if longest edge needs to be in MST
  - *e.g.* tight clusters connected by one or more long edges

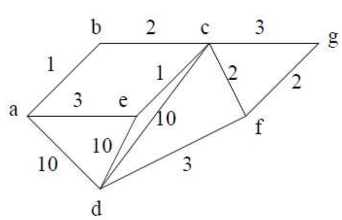
	Prim	Kruskal
General	(E+V) log V	E log E
Dense Graph	E log V	E log E
V<<E, Prim's is faster		
Sparse Graph	V log V	V log V
Kruskal's is faster because of the data structures		

Kruskal's algorithm: an overview (Skiena)

A large-scale view of Kruskal's algorithm:

http://www.cs.sunysb.edu/~skiena/combinatorica/animations/mst.html

Kruskal's *vs.* Prim's



More advanced MSTs

- Euclidean MSTs:
  - Given points on a plane, build MST
  - Could construct complete graph, then use Prim's. – Slow!

Other more clever algorithms exist

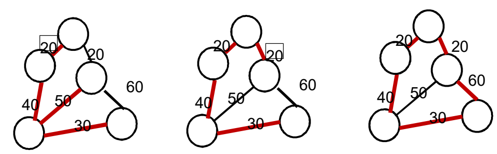
More advanced MSTs

- Randomized MST algorithm
  - Random partition of the graph
  - Expected time linear, but bad worst case
  - Karger, David R.; Klein, Philip N.; Tarjan, Robert E. (1995). "A randomized linear-time algorithm to find minimum spanning trees". *JACM* **42** (2): 321–328.
  - Linear MST algorithms exist for restricted types of graphs

The general solution for linear time MST creation is an open research problem

MST and the Travelling Salesperson Problem

- Travelling salesperson problem (TSP):
  - Given a list of cities and the distances between each pair of cities, find:
    - shortest possible route that
    - visits each city exactly once
    - and returns to the origin city



MST and the Travelling Salesperson Problem

- Travelling salesperson problem (TSP):
  - Given a list of cities and the distances between each pair of cities, find:
    - shortest possible route that
    - visits each city exactly once
    - and returns to the origin city
- Much harder than MST!
- Greedy (nearest neighbor) doesn't work!

Graph algorithms

- Graph search:
  - Depth-first
  - Breadth-first
  - Priority-first
- Undirected graphs
- Directed graphs

Graph algorithms

- Graph search
- Algorithms on undirected graphs
- Algorithms on directed graphs

Graph algorithms

- Graph search
  - Depth-first search
  - Breadth-first search
  - Priority-first search
  - (Connected components)
- Algorithms on undirected graphs
- Algorithms on directed graphs

Graph algorithms

- Graph search
- Algorithms on undirected graphs
- Algorithms on directed graphs
  - Single source shortest path (Dijkstra's)
  - Transitive closure (Warshall)
  - All pairs shortest path (Floyd-Warshall)

Graph algorithms

- Graph search
- Algorithms on undirected graphs
  - Minimum spanning tree
    - Prim's
    - Kruskal's
  - Travelling salesperson
- Algorithms on directed graphs

Graphs in the real world

- Many real-world problems can be modelled as graphs
- Many specialized types of graphs allow modelling of complex problems
- People have been working on graph algorithms for a long time, so
- Huge library of algorithms available

Take away lesson

If you can model a problem as a graph, there is a very good chance that there is already an algorithm to solve the problem…

… or evidence that the problem is intractible