COMP20003 Algorithms and Data Structures Recurrences

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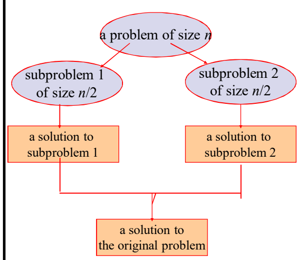
Divide and Conquer Algorithms

Mergesort and quicksort are instances of divide-and-conquer algorithms:

- Solve the problem by continually dividing into smaller problems

Other examples?

Split-solve-join approach:



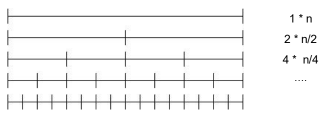
For problems where the output is a transformation of the input, need to:

- process both sub-problems, and
- join the sub-solutions after processing

Recurrence for divide and conquer sorting algorithms

One pass through the data reduces problem size by half. Process both halves:

- Operation (process) takes constant time c
- Base case takes time d



Recurrence for divide and conquer sorting algorithms

One pass through the data reduces problem size by half. Process both halves

- Operation takes constant time c
- Base case takes time d

$T(1) = d$

$T(n) = 2T(n/2) + nc$

$= nc + 2cn/2 + 4cn/4 \ldots + n/2 * 2c + nd$

$= cn \log_2 n + nd$

Divide and Conquer: Recurrences to Master Theorem

- Most common case:

  $T(n) = 2T(n/2) + n$

- General case:

  $T(n) = aT(n/b) + f(n)$

  $f(n) \in \Theta(n^d)$

- Most common case:

  $T(n) = 2T(n/2) + n$

  $a=2,\ b=2,\ d=1$

Divide and Conquer: Recurrences to Master Theorem

- Familiar examples?

$T(n) = T(n/2) + f(1)$ $\quad a=1,\ b=2,\ d=0$

?

$T(n) = 2T(n/2) + f(1)$ $\quad a=2,\ b=2,\ d=0$

?

Master Theorem for Divide and Conquer

- $T(n) = aT(n/b) + f(n)$

  $f(n) \in \Theta(n^d)$

- *T(n) closed form varies, depending on whether:*
  - $d > \log_b a$ $\quad T(n) \in \Theta(n^d)$
  - $d = \log_b a$ $\quad T(n) \in \Theta(n^d \log n)$
  - $d < \log_b a$ $\quad T(n) \in \Theta(n^{\log_b a})$

Master Theorem for Divide and Conquer

- $T(n) = aT(n/b) + f(n)$, where
  $a \geq 1$, $b > 1$, n^d asymptotically positive
- $T(n)$ closed form varies, depending on whether:
  - $d > \log_b a$ $\quad T(n) \in \Theta(n^d)$
  - $d = \log_b a$ $\quad T(n) \in \Theta(n^d \log n)$
  - $d < \log_b a$ $\quad T(n) \in \Theta(n^{\log_b a})$

Where do Θ() solutions to the Master Theorem come from?

$T(n) = aT(n/b) + f(n),\ f(n) \in \Theta(n^d)$

Size of subproblems decreases by b
- So base case reached after $\log_b n$ levels
- Recursion tree $\log_b n$ levels

Branch factor is a
- At kth level, have a^k subproblems

At level k, total work is then
- a^k * $O(n/b^k)^d$
- (#subproblems * cost of solving one)

Where do Θ() solutions to the Master Theorem come from?

$T(n) = aT(n/b) + f(n),\ f(n) \in \Theta(n^d)$

- At level k, total work is then
  - a^k * $O(n/b^k)^d$ = $O(n^d)$ * $(a/b^d)^k$
- As k (levels) goes from 0 to $\log_b n$, this is a geometric series, with ratio a/b^d

  $\Sigma\ O(n^d)$* $(a/b^d)^k$

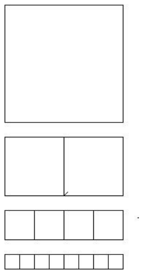
Where do Θ() solutions to the Master Theorem come from?

$T(n) = aT(n/b) + f(n),\ f(n) \in \Theta(n^d)$

- Geometric series: $O(n^d)$ * $(a/b^d)^k$
  - as k goes from $0 \rightarrow \log_b n$
- Case 1: ratio $a/b^d < 1$ or $d > \log_b a$
  - $(a/b^d)^k$ gets smaller as k goes from $1 \rightarrow \log n$
  - a/b^d First term is the largest, and is <1
  - $O(n^d)$

Example for $a/b^d < 1$

$T(n) = 2T(n/2) + n^2$



Where do the solutions to the Master Theorem come from?

$T(n) = aT(n/b) + f(n),\ f(n) \in \Theta(n^d)$

- Geometric series: $O(n^d) * (a/b^d)^k$
  - as k goes from $0 \rightarrow \log_b n$
- Case 2: ratio $a/b^d = 1$ or $d = \log_b a$
  - Series is $O(n^d) + O(n^d) + \ldots$
    - For $\log_b n$ levels
  - Sum $= O(n^d \log_b n)$

Example for most common case $a/b^d = 1$

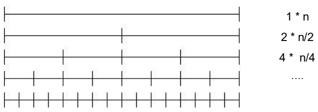
$T(n) = 2T(n/2) + n$

$T(n) = 2(2T(n/4) + n/2) + n$

$\quad = 4T(n/4) + n + n$

$\quad = 8T(n/8) + n + n + n$

….



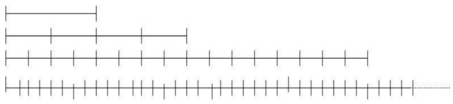
Where do $\Theta()$ solutions to the Master Theorem come from?

$T(n) = aT(n/b) + f(n),\ f(n) \in \Theta(n^d)$

- Geometric series: $O(n^d) * (a/b^d)^k$
  - as k goes from $0 \rightarrow \log_b n$
- Case 3: ratio $a/b^d > 1$ or $d < \log_b a$
  - $a/b^d > 1 \rightarrow$ series is *increasing*
  - Sum dominated by last term:
    - $O(n^d)(a/b^d)^{\log(b)n} = n^{\log(b)a}$

Example for $a/b^d > 1$

$T(n) = 4T(n/2) + n$



- For more on geometric series, and calculation of closed form, see: http://www.youtube.com/watch?v=JJZ-shHiayU
- 4 minute tutorial from Rose-Hulman Institute of Technology