PHY 321 Harmonic oscillator in 2-Dimensions

Of relevance for hw 8, exercises 2 and 3

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

(i) $\vec{R} = 0$ CoM-frame

$$\frac{d\vec{P}}{dt} = 0 \quad \vec{P} \text{ is conserved}$$

$$\vec{P} = M\frac{d\vec{R}}{dt}$$

(ii) $\vec{L} = \vec{r} \times \vec{p}$

$$\frac{d\vec{L}}{dt} = 0, \quad \vec{L} \text{ is conserved}$$

Two-dim problem

$$\vec{r} = x\vec{i} + y\vec{j}$$

$$r \in [0, \infty) \quad \phi \in [0, 2\pi]$$

$$x = r\cos\phi \quad \wedge \quad y = r\sin\phi$$

(iii) $K = \frac{1}{2}\mu(\dot{r}^2 + r^2 \cdot \dot{\phi}^2)$

$$\mu = \frac{m_1 m_2}{M}$$

$$r = \sqrt{x^2+y^2}$$

$$\dot{r} = \frac{dr}{dt} \qquad \dot{\phi} = \frac{d\phi}{dt}$$

(iv) $\dot{\phi} = \frac{L}{\mu r^2}$

$$\mu \ddot{r} = F(r) + \frac{L^2}{\mu r^3}$$

(v) $V_{eff}(r) = V(r) + \frac{L^2}{2\mu r^2}$

$$F(r) = -\frac{d V_{eff}(r)}{dr}$$

$$= -\frac{d V(r)}{dr} + \frac{L^2}{\mu r^3}$$

Harmonic oscillator

$$V(r) = \frac{1}{2} k r^2 = \frac{1}{2} k (x^2+y^2)$$

$$\boxed{\begin{aligned} \mu \ddot{r} &= -kr + \frac{L^2}{\mu r^3} \\ \dot{\phi} &= \frac{L}{\mu r^2} \end{aligned}}$$

x - y (cartesian coordinates)

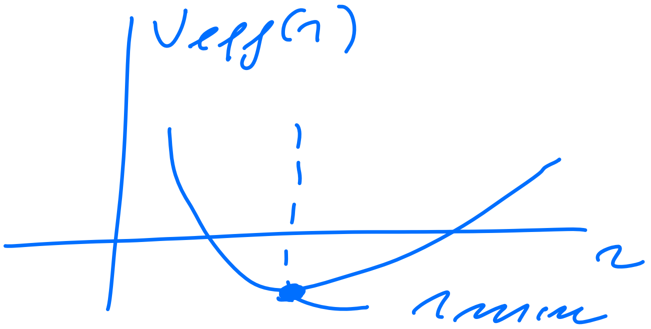
$$\boxed{\begin{aligned} m \frac{d^2x}{dt^2} &= -kx \\ m \frac{d^2y}{dt^2} &= -ky \end{aligned}}$$

x-y-equations are separable in x and y

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$
$$y(t) = C\cos(\omega_0 t) + D\sin(\omega_0 t)$$

HO in more detail

$$V_{eff}(r) = \frac{1}{2}kr^2 + \frac{L^2}{2\mu r^2}$$



→ circular motion

$$\frac{dV_{eff}}{dr} = 0 = kr - \frac{L^2}{\mu r^3}$$

$$r_{min} = \left(\frac{L^2}{k\mu}\right)^{1/4}$$

$$E = K + V$$

$$K = \frac{1}{2}\mu\left(v_x^2 + v_y^2\right)$$

$$= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right)$$

$$V = \frac{1}{2}k r^2 \qquad \dot{\phi} = \frac{L}{\mu r^2}$$

$$E = \frac{1}{2}\mu\dot{r}^2$$

radial velocity

$$+ \underbrace{V_{eff}(r)}_{\frac{1}{2}k r^2 + \frac{L^2}{\mu^2 r^2}\cdot\frac{1}{2}\mu}$$

$$= \frac{1}{2}\mu\dot{r}^2 + \boxed{\frac{1}{2}k r^2 + \frac{L^2}{2\mu r^2}}$$

comes from kinetic energy.

$$r_{min}^2 = \frac{L}{\sqrt{k\mu}}$$

$$V_{eff}(r_{min}) = L\cdot\omega_0$$

$$\omega_0 = \sqrt{k/\mu}$$

$$E(r_{min}) = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2\Big|_{r=r_{min}} + L\,\omega_0$$

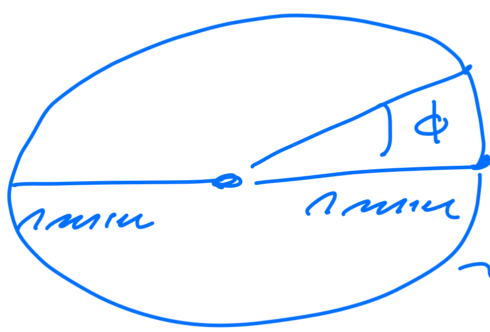
$$\mu\ddot{r} = F(r) + \frac{L^2}{\mu r^3}$$

$$= -kr + \frac{L^2}{\mu r^3}$$

$$\mu\left(\frac{d^2r}{dt^2}\right)\Big|_{r=r_{min}} = 0$$

zero acceleration, constant velocity?

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{L}{\mu r^2} = \omega_0$$



$\sim \frac{d\phi}{dt} = \omega_0$

$\sim$ motion in a circle with constant radial velocity $\frac{dr}{dt}$ and constant angular velocity $\frac{d\phi}{dt}$