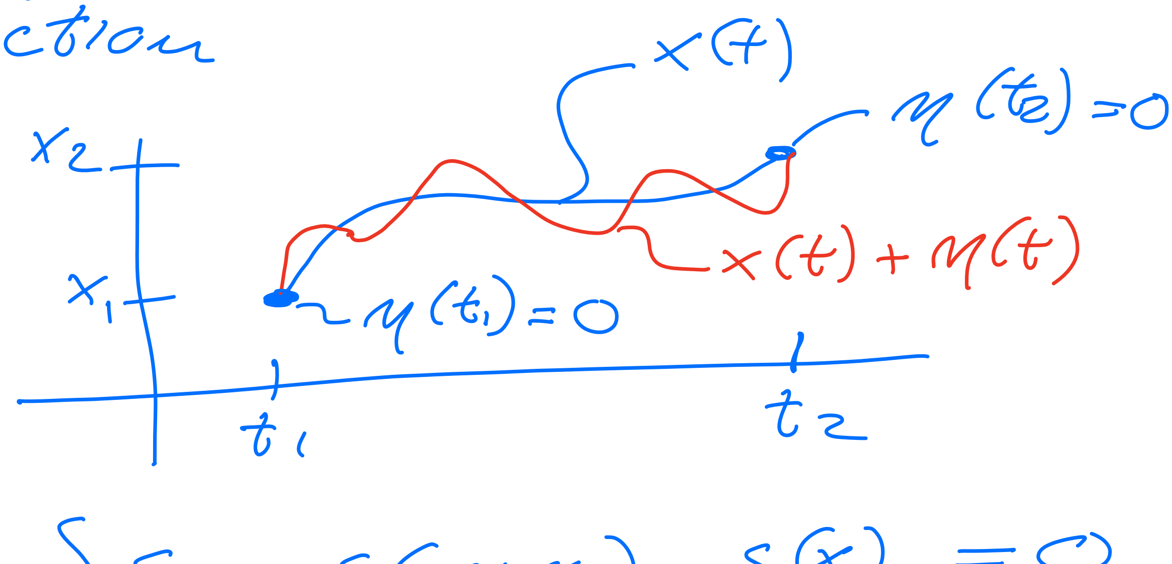
PHY 321, APRIL 18, 2022

- principle of least action
- variational calculus

↳ deals with finding min or max of a quantity which can be expressed as an integral

$$S = \int_{t_1}^{t_2} \mathcal{L}\left(x, \frac{dx}{dt}, t\right) dt$$

action



$$\delta S = S(x+\eta) - S(x) = 0$$

$$\mathcal{L}(x, v, t) = \frac{1}{2} m \left(\frac{dx}{dt}\right) - V(x)$$

$$S = \int_{t_1}^{t_2} \left[\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 - V(x) \right] dt$$

$$= S(x)$$

$$S(x+\eta) = \int_{t_1}^{t_2} \left\{ \frac{1}{2} m \left[\frac{d}{dt}(x+\eta) \right]^2 - V(x+\eta) \right\} dt$$

$$\left(\frac{dx}{dt} + \frac{d\eta}{dt} \right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + 2 \frac{dx}{dt} \frac{d\eta}{dt}$$

$$V(x+\eta) = V(x) + \eta \frac{dV}{dx} + \frac{\eta^2}{2!} V'' + O(\eta^3)$$

ignore

$$S(x+\eta) \approx \int^{t_2} \left[\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 \right.$$

$$S(x+\eta) = \int_{t_1} \Big[\frac{1}{2} m \left(\frac{dx}{dt}\right)$$

$$- V(x)\Big]\, dt$$

$$+ \int_{t_1}^{t_2} \left[\frac{1}{2} m \left(2\frac{dx}{dt}\frac{d\eta}{dt}\right) - \eta\frac{dV}{dx}\right] dt$$

(skip higher terms in η)

$$S(x+\eta) - S(x) = \delta S = 0$$

$$= \int_{t_1}^{t_2} \left[\frac{1}{2} m \left(2\frac{dx}{dt}\frac{d\eta}{dt}\right) - \eta\frac{dV}{dx}\right] dt$$

$x(t) = ?$ and we want to get rid of $\frac{d\eta}{dt}$

$$\frac{d}{dt}\left(\frac{dx}{dt}\eta\right) = \frac{dx}{dt}\frac{d\eta}{dt} + \frac{d^2x}{dt^2}\eta$$

$$\frac{dx}{dt}\frac{d\eta}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\eta\right) - \frac{d^2x}{dt^2}\eta$$

$$\int_{t_1}^{t_2} \frac{dx}{dt}\frac{d\eta}{dt}\,dt = \int_{t_1}^{t_2} \frac{d}{dt}\left[\frac{dx}{dt}\eta\right]dt$$

$$- \int_{t_1}^{t_2} \frac{d^2x}{dt^2}\eta\,dt$$

$$\underbrace{\frac{dx}{dt}\eta\Big|_{t_1}^{t_2}}_{=0} - \int_{t_1}^{t_2} \frac{d^2x}{dt^2}\eta\,dt$$

$\eta(t_1) = \eta(t_2) = 0$

$$\delta S = 0 = \int_{t_1}^{t_2} \left[-m\frac{d^2x}{dt^2}\eta - \frac{dV}{dx}\eta\right]dt$$

$(\eta(t) \neq 0)$

$$= \int_{t_1}^{t_2} \left[-m\frac{d^2x}{dt^2} - \frac{dV}{dx}\right]\eta\,dt$$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} = -\frac{dV}{dx} = F(x)$$

$dt^2 \qquad dx$

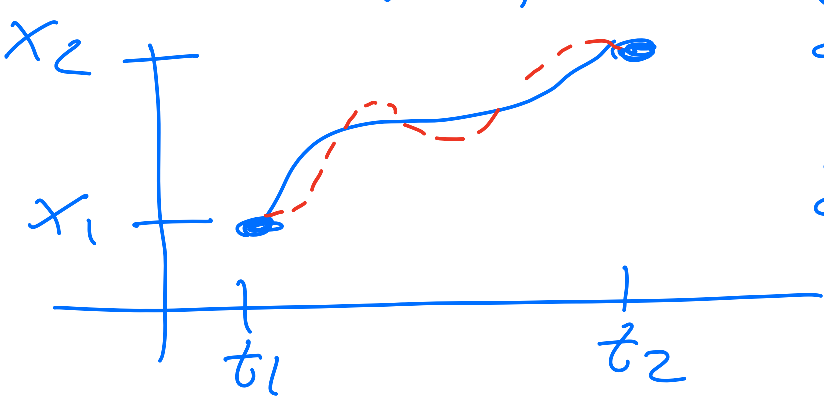
$=$ Newton's Law

Euler-Lagrange equations

$\mathcal{L} = \mathcal{L}(x, v, t)$

$x(t) + \delta x(t) \quad \wedge \quad v(t) + \delta v(t)$

$(\eta(t))$



$\delta x(t_1) =$
$\delta x(t_2) = 0$
$\delta v(t_1) =$
$\delta v(t_2) = 0$

$$\mathcal{L}(x + \delta x, v + \delta v, t) =$$

$$\mathcal{L}(x, v, t) + \frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial v}\delta v$$

$$+ O((\delta x)^2) + O((\delta v)^2)$$

$$+ O(\delta v \, \delta x)$$

$$\delta S = 0 = \int \mathcal{L}(x + \delta x, v + \delta v, t)\, dt$$

$$- \int \mathcal{L}(x, v, t)\, dt$$

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x}\,\delta x + \underbrace{\frac{\partial L}{\partial v}\,\delta v}_{\frac{\partial L}{\partial v}\frac{d}{dt}\delta x} \right) dt = 0$$

integrate by parts

$$\delta S = \frac{\partial L}{\partial v}\,\delta x \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial v} \right] \delta x \, dt$$

$$\delta x(t_1) = \delta x(t_2) = 0$$

$$\Rightarrow \quad \boxed{\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial v} = 0}$$

Euler-Lagrange eq.

Example: shortest

distance between
two points.