Homework 1 Solutions

Exercise 1 (10 pt)

1. (a) (2 pt)

$$\begin{aligned}
\cos\omega t &= \sum_{n=0}(-1)^n \frac{(\omega t)^{2n}}{(2n)!} \\
&= 1 - \frac{(\omega t)^2}{2!} + \frac{(\omega t)^4}{4!} - \cdots . \\
\sin\omega t &= \sum_{n=0}(-1)^n \frac{(\omega t)^{2n+1}}{(2n+1)!} \\
&= \omega t - \frac{(\omega t)^3}{3!} + \frac{(\omega t)^5}{5!} - \cdots .
\end{aligned}$$

(b) (2 pt)

$$\begin{aligned}
e^{i\omega t} &= \sum_{n=0} \frac{(i\omega t)^n}{(n)!} \\
&= 1 + i\omega t + \frac{(i\omega t)^2}{2!} + \frac{(i\omega t)^3}{3!} + \frac{(i\omega t)^4}{4!} + \frac{(i\omega t)^5}{5!} \cdots .
\end{aligned}$$

(c) (3 pt) Using $i^2 = -1$, we can rewrite the last result as

$$\begin{aligned}
e^{i\omega t} &= 1 + i\omega t - \frac{(\omega t)^2}{2!} - i\frac{(i\omega t)^3}{3!} + \frac{(i\omega t)^4}{4!} + i\frac{(i\omega t)^5}{5!} \cdots \\
&= \left(1 - \frac{(\omega t)^2}{2!} + \frac{(\omega t)^4}{4!} - \cdots\right) + i\left(\omega t - \frac{(\omega t)^3}{3!} + \frac{(\omega t)^5}{5!} - \cdots\right) \\
&= \cos\omega t + i\sin\omega t .
\end{aligned}$$

(d) (3 pt) Letting $\omega t = \pi$, we get

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 .$$

Exercise 2 (10 pt)

2. (10 pt) For a cube with sides of length 1, one vertex at the origin, and sides along the x, y, and z axes, the vector of the body diagonal from the origin can be written $\vec{b} = (1,1,1)$ and the vector of the face diagonal in the xy plane from the origin is $\vec{f} = (1,1,0)$. The length of the body diagonal is $b = \sqrt{1^2+1^2+1^2} = \sqrt{3}$ and the length of the face diagonal is $\sqrt{1^2+1^2} = \sqrt{2}$. From this we find

$$\begin{aligned} \vec{b}\cdot\vec{f} &= bf\cos\theta \\ 2 &= \sqrt{2}\sqrt{3}\cos\theta\,. \end{aligned}$$

So $\cos\theta = \sqrt{2/3}$ and $\theta = 35.3° = 0.615$ rad.

Exercise 3 (10 pt)

3. (a) (5 pt) Using the fact that multiplication of reals is distributive, we get

$$\begin{aligned} \vec{r}\cdot(\vec{u}+\vec{v}) &= r_1(u_1+v_1)+r_2(u_2+v_2)+r_3(u_3+v_3) \\ &= (r_1u_1+r_1v_1)+(r_2u_2+r_2v_2)+(r_3u_3+r_3v_3) \\ &= (r_1u_1+r_2u_2+r_3u_3)+(r_1v_1+r_2v_2+r_3v_3) \\ &= \vec{r}\cdot\vec{u}+\vec{r}\cdot\vec{v}\,. \end{aligned}$$

(b) (5 pt) Using the product rule for differentiating reals, we get

$$\begin{aligned} \frac{d}{dt}(\vec{r}\cdot\vec{s}) &= \frac{d}{dt}(r_1s_1+r_2s_2+r_3s_3) \\ &= \left(r_1\frac{ds_1}{dt}+\frac{dr_1}{dt}s_1\right)+\left(r_2\frac{ds_2}{dt}+\frac{dr_2}{dt}s_2\right)+\left(r_3\frac{ds_3}{dt}+\frac{dr_3}{dt}s_3\right) \\ &= \left(r_1\frac{ds_1}{dt}+r_2\frac{ds_2}{dt}+r_3\frac{ds_3}{dt}\right)+\left(\frac{dr_1}{dt}s_1+\frac{dr_2}{dt}s_2+\frac{dr_3}{dt}s_3\right) \\ &= \vec{r}\cdot\frac{d\vec{s}}{dt}+\frac{d\vec{r}}{dt}\cdot\vec{s}\,. \end{aligned}$$

Exercise 4 (10pt)

4a (5pt) $\vec{a} \times (\vec{b} + \vec{c}) =$

$$\vec{a} \times \left((b_x + c_x)\vec{e}_x + (b_y + c_y)\vec{e}_y + (c_z + b_z) e_z \right)$$

$$= \left(a_y (c_z + b_z) - (b_y + c_y) a_z \right) \vec{e}_x$$

$$+ \left(a_x (c_z + b_z) - (b_x + c_x) a_z \right) \vec{e}_y$$

$$+ \left(a_x (b_y + c_y) - a_y (b_x + c_x) \right) \vec{e}_z$$

$$= \left.\begin{aligned} &(a_y b_z - a_z b_y)\vec{e}_x + \\ &(a_x b_z - a_z b_y)\vec{e}_y + \\ &(a_x b_y - a_y b_x)\vec{e}_z + \end{aligned}\right\} \vec{a} \times \vec{b}$$

$$\left.\begin{aligned} &(a_y c_z - a_z c_y)\vec{e}_x + \\ &(a_x c_z - a_z c_x)\vec{e}_y + \\ &(a_x c_y - a_y c_x)\vec{e}_z \end{aligned}\right\} \vec{a} \times \vec{c}$$

$$= \underline{\vec{a} \times \vec{b} + \vec{a} \times \vec{c}}$$

4b (5pt) $\frac{d}{dt}(\vec{a} \times \vec{b}) =$

$$\frac{d}{dt}\Big[(a_y b_z - a_z b_y)\vec{e}_x + (a_x b_z - a_z b_y)\vec{e}_y + (a_x b_y - a_y b_x)\vec{e}_x\Big]$$

collecting all terms and using the chain rule we can easily see that it leads to

$$\vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b}$$

(note order of factors)

Note typo in original homework !

Sorry for this.

Exercise 5 (10pt)

(a) (5 pt) If we place vertex A at the origin and side $\mathbf{b}$ along the x axis, then the magnitude of the cross product of $\vec{b} \times \vec{c}$ gives

$$
\begin{aligned}
|\vec{b} \times \vec{c}| &= |b_x c_y - b_y c_x| = |b_x c_y - 0 \cdot c_x| \\
&= b_x c_y \\
&= (\text{base}) \cdot (\text{height}) \,,
\end{aligned}
$$

where we recognize that c_y is the height of the vertex B above the x axis. From this we get

$$
\begin{aligned}
\text{area} &= \frac{1}{2}(\text{base}) \cdot (\text{height}) \\
&= \frac{1}{2}|\vec{b} \times \vec{c}| \,.
\end{aligned}
$$

(Actually, the way the figure is drawn, we should use the vector $-\vec{b}$, but since we are taking the absolute value, the sign of the vector doesn't matter.) Results for the other two equations can be obtained analogously by putting C at the origin and $\mathbf{a}$ on the x axis, or by putting B at the origin and $\mathbf{c}$ on the x axis.

(b) (5 pt) From part (a) and using the formula for the magnitude of the cross product we have

$$
\text{area} = \frac{1}{2} ab \sin\gamma = \frac{1}{2} bc \sin\alpha = \frac{1}{2} ca \sin\beta \,.
$$

From this we can write

$$
\frac{abc}{2\,\text{area}} = \frac{c}{\sin\gamma} = \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \,.
$$

Exercise 6 (10pt)

6a (4pt)

$$\mathbb{R} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbb{R}^T = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6b (4pt)

$$\mathbb{R}^T\mathbb{R} = \begin{bmatrix} \cos^2\phi+\sin^2\phi & 0 & 0 \\ 0 & \cos^2\phi+\sin^2\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This means that for example the dot product of a vector is unchanged under such a rotation, To see this consider a rotation

$$\vec{b} = \mathbb{R}\vec{a}$$

we have $\vec{a}\cdot\vec{a} = a^T a$, the dot product.

The dot product of $\vec{b}$ is

$$b^T b = (\mathbb{R}a)^T(\mathbb{R}a) =$$

$$a^T \underbrace{\mathbb{R}^T\mathbb{R}}_{\mathbb{I}} a = a^T a$$

The dot product is conserved (it is a scalar).

6c (2pt) Rotation about the x-axis. It is then y and z which get changed.

$$\mathbb{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

$$\phi = 90 \qquad \cos\phi = 0$$

$$\sin\phi = 1$$

$$\Rightarrow$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$