PHY321: Two-body problems, gravitational forces, scattering and begin Lagrangian formalism

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Mar 31, 2021

Aims and Overarching Motivation

Monday.

1. Physical interpretation of various orbit types and summary gravitational forces, examples on whiteboard and handwritten notes
2. Start discussion two-body scattering

Reading suggestion: Taylor chapter 8 and sections 14.1-14.2 and Lecture notes

Wednesday.

1. Two-body scattering

Reading suggestion: Taylor and sections 14.3-14.6

Friday.

1. Lagrangian formalism

Reading suggestion: Taylor Sections 6.1-6.2

Code example with gravitional force. The code example here is meant to illustrate how we can make a plot of the final orbit. We solve the equations in polar coordinates (the example here uses the minimum of the potential as initial value) and then we transform back to cartesian coordinates and plot x versus y. We see that we get a perfect circle when we place ourselves at the minimum of the potential energy, as expected.

Plotting the potential first

The code here plots the effective potential

The following code plots this effective potential for a simple choice of parameters, with a standard gravitational potential $-\alpha/r$. Here we have chosen $L = m = \alpha = 1$.

```
# Common imports
import numpy as np
from math import *
import matplotlib.pyplot as plt

Deltax = 0.01
#set up arrays
xinitial = 0.3
xfinal = 5.0
alpha = 1.0   # spring constant
m = 1.0   # mass, you can change these
AngMom = 1.0  #  The angular momentum
n = ceil((xfinal-xinitial)/Deltax)
x = np.zeros(n)
for i in range(n):
    x[i] = xinitial+i*Deltax
V = np.zeros(n)
V = -alpha/x+0.5*AngMom*AngMom/(m*x*x)
# Plot potential
fig, ax = plt.subplots()
ax.set_xlabel('r[m]')
ax.set_ylabel('V[J]')
ax.plot(x, V)
fig.tight_layout()
plt.show()
```

Solving the differential equations, Radial Degrees of Freedom only

This code and the subsequent one are relevant for homework 9 (the numerical exercise).

```
# Simple Gravitational Force   -alpha/r
# Common imports
import numpy as np
import pandas as pd
from math import *
import matplotlib.pyplot as plt
import os

# Where to save the figures and data files
PROJECT_ROOT_DIR = "Results"
FIGURE_ID = "Results/FigureFiles"
DATA_ID = "DataFiles/"

if not os.path.exists(PROJECT_ROOT_DIR):
    os.mkdir(PROJECT_ROOT_DIR)

if not os.path.exists(FIGURE_ID):
    os.makedirs(FIGURE_ID)
```

```
if not os.path.exists(DATA_ID):
    os.makedirs(DATA_ID)

def image_path(fig_id):
    return os.path.join(FIGURE_ID, fig_id)

def data_path(dat_id):
    return os.path.join(DATA_ID, dat_id)

def save_fig(fig_id):
    plt.savefig(image_path(fig_id) + ".png", format='png')



DeltaT = 0.01
#set up arrays
tfinal = 8.0
n = ceil(tfinal/DeltaT)
# set up arrays for t, v and r
t = np.zeros(n)
v = np.zeros(n)
r = np.zeros(n)
phi = np.zeros(n)
x = np.zeros(n)
y = np.zeros(n)
# Constants of the model, setting all variables to one for simplicity
alpha = 1.0
AngMom = 1.0  #  The angular momentum
m = 1.0  # scale mass to one
c1 = AngMom*AngMom/(m*m)
c2 = AngMom*AngMom/m
rmin = (AngMom*AngMom/m/alpha)
# Initial conditions, place yourself at the potential min
r0 = rmin
v0 = 0.0  # starts at rest
r[0] = r0
v[0] = v0
phi[0] = 0.0
# Start integrating using the Velocity-Verlet  method
for i in range(n-1):
    # Set up acceleration
    a = -alpha/(r[i]**2)+c1/(r[i]**3)
    # update velocity, time and position using the Velocity-Verlet method
    r[i+1] = r[i] + DeltaT*v[i]+0.5*(DeltaT**2)*a
    anew = -alpha/(r[i+1]**2)+c1/(r[i+1]**3)
    v[i+1] = v[i] + 0.5*DeltaT*(a+anew)
    t[i+1] = t[i] + DeltaT
    phi[i+1] = t[i+1]*c2/(r0**2)
# Find cartesian coordinates for easy plot
x = r*np.cos(phi)
y = r*np.sin(phi)
fig, ax = plt.subplots(3,1)
ax[0].set_xlabel('time')
ax[0].set_ylabel('radius')
ax[0].plot(t,r)
ax[1].set_xlabel('time')
ax[1].set_ylabel('Angle $\cos{\phi}$')
ax[1].plot(t,np.cos(phi))
ax[2].set_ylabel('y')
ax[2].set_xlabel('x')
ax[2].plot(x,y)
```

```
save_fig("Phasespace")
plt.show()
```

Changing initial conditions

Try to change the initial value for r and see what kind of orbits you get. In order to test different energies, it can be useful to look at the plot of the effective potential discussed above.

However, for orbits different from a circle the above code would need modifications in order to allow us to display say an ellipse. For the latter, it is much easier to run our code in cartesian coordinates, as done here. In this code we test also energy conservation and see that it is conserved to numerical precision. The code here is a simple extension of the code we developed for homework 4.

```
# Common imports
import numpy as np
import pandas as pd
from math import *
import matplotlib.pyplot as plt

DeltaT = 0.01
#set up arrays
tfinal = 10.0
n = ceil(tfinal/DeltaT)
# set up arrays
t = np.zeros(n)
v = np.zeros((n,2))
r = np.zeros((n,2))
E = np.zeros(n)
# Constants of the model
m = 1.0   # mass, you can change these
alpha = 1.0
# Initial conditions as compact 2-dimensional arrays
x0 = 0.5; y0= 0.
r0 = np.array([x0,y0])
v0 = np.array([0.0,1.0])
r[0] = r0
v[0] = v0
rabs = sqrt(sum(r[0]*r[0]))
E[0] = 0.5*m*(v[0,0]**2+v[0,1]**2)-alpha/rabs
# Start integrating using the Velocity-Verlet  method
for i in range(n-1):
    # Set up the acceleration
    rabs = sqrt(sum(r[i]*r[i]))
    a =  -alpha*r[i]/(rabs**3)
    # update velocity, time and position using the Velocity-Verlet method
    r[i+1] = r[i] + DeltaT*v[i]+0.5*(DeltaT**2)*a
    rabs = sqrt(sum(r[i+1]*r[i+1]))
    anew = -alpha*r[i+1]/(rabs**3)
    v[i+1] = v[i] + 0.5*DeltaT*(a+anew)
    E[i+1] = 0.5*m*(v[i+1,0]**2+v[i+1,1]**2)-alpha/rabs
    t[i+1] = t[i] + DeltaT
# Plot position as function of time
fig, ax = plt.subplots(3,1)
ax[0].set_ylabel('y')
ax[0].set_xlabel('x')
```

```
ax[0].plot(r[:,0],r[:,1])
ax[1].set_xlabel('time')
ax[1].set_ylabel('y position')
ax[1].plot(t,r[:,0])
ax[2].set_xlabel('time')
ax[2].set_ylabel('y position')
ax[2].plot(t,r[:,1])

fig.tight_layout()
save_fig("2DimGravity")
plt.show()
print(E)
```

Different Potential

Let us now try another potential, given by

$$V(r) = \beta r,$$

where β is constant we assume is larger than zero. This type of potential has played an importan role in modeling confinement of quarks in non-relativistic models for the interactions among quarks, see for example https://journals.aps.org/prl/pdf/10.1103/PhysRevLett.44.1369

Adding the angular momentum part, we obtain the effective potential

$$V_{\mathrm{eff}}(r) = \beta r + \frac{L^2}{2\mu r^2},$$

and taking the derivative with respect to r, we get the radial force

$$F_r = -\frac{dV_{\mathrm{eff}}(r)}{dr} = -\beta + \frac{L^2}{\mu r^3}.$$

It gives us in turn a radial acceleration a_r

$$a_r = -\frac{\beta}{\mu} + \frac{L^2}{\mu^2 r^3}.$$

This is the equation we need to include in our code. I have not been able to find out if there is an analytical solution to the above equation. If you can find one, there is a reward of 50 USD to the first one who finds. Numerically life is very easy, we just define a new acceleration, as seen below.

Plotting the Effective Potential

First however, we plot the effective potential in order to get a feeling of what we may expect.

The following code plots this effective potential for a simple choice of parameters, with a potential βr. Here we have chosen $L = m = \beta = 1$.

```
# Common imports
import numpy as np
from math import *
import matplotlib.pyplot as plt

Deltax = 0.01
#set up arrays
xinitial = 0.3
xfinal = 5.0
beta = 1.0   # spring constant
m = 1.0   # mass, you can change these
AngMom = 1.0  #  The angular momentum
n = ceil((xfinal-xinitial)/Deltax)
x = np.zeros(n)
for i in range(n):
    x[i] = xinitial+i*Deltax
V = np.zeros(n)
V = beta*x+0.5*AngMom*AngMom/(m*x*x)
# Plot potential
fig, ax = plt.subplots()
ax.set_xlabel('r[m]')
ax.set_ylabel('V[J]')
ax.plot(x, V)
fig.tight_layout()
plt.show()
```

Finding the Minimum for Circular Orbits

We take now the derivative of the effective potential in order to find its minimum, that is

$$\frac{dV_{\mathrm{eff}}(r)}{dr} = \beta - \frac{L^2}{\mu r^3} = 0,$$

which gives us r_{min}

$$r_{\mathrm{min}} = \left[\frac{L^2}{\beta\mu}\right]^{1/3}.$$

With the above choice of parameters this gives $r_{\mathrm{min}} = 1$.

In the code here we solve the equations of motion and find the time-evolution of the radius r.

Numerical Studies

```
DeltaT = 0.01
#set up arrays
tfinal = 8.0
n = ceil(tfinal/DeltaT)
# set up arrays for t, v and r
t = np.zeros(n)
v = np.zeros(n)
r = np.zeros(n)
# Constants of the model, setting all variables to one for simplicity
beta = 1.0
AngMom = 1.0  #  The angular momentum
m = 1.0  # scale mass to one
```

```
c1 = AngMom*AngMom/(m*m)
c2 = AngMom*AngMom/m
rmin = (AngMom*AngMom/m/beta)**(1./3.)
# Initial conditions, place yourself at the potential min
r0 = rmin
v0 = 0.0  # starts at rest
r[0] = r0
v[0] = v0
# Start integrating using the Velocity-Verlet  method
for i in range(n-1):
    # Set up acceleration
    a = -beta+c1/(r[i]**3)
    # update velocity, time and position using the Velocity-Verlet method
    r[i+1] = r[i] + DeltaT*v[i]+0.5*(DeltaT**2)*a
    anew = -beta+c1/(r[i+1]**3)
    v[i+1] = v[i] + 0.5*DeltaT*(a+anew)
    t[i+1] = t[i] + DeltaT
#plotting
plt.xlabel('time')
plt.ylabel('radius')
plt.plot(t,r)
save_fig("LinearPotential")
plt.show()
```

We see that if we run with the initial condition corresponding to a circular orbit, our radius stays constant as function of time.

Scattering and Cross Sections

Scattering experiments don't measure entire trajectories. For elastic collisions, they measure the distribution of final scattering angles at best. Most experiments use targets thin enough so that the number of scatterings is typically zero or one. The cross section, σ, describes the cross-sectional area for particles to scatter with an individual target atom or nucleus. Cross section measurements form the basis for MANY fields of physics. BThe cross section, and the differential cross section, encapsulates everything measurable for a collision where all that is measured is the final state, e.g. the outgoing particle had momentum $\boldsymbol{p}_f$. y studying cross sections, one can infer information about the potential interaction between the two particles. Inferring, or constraining, the potential from the cross section is a classic *inverse* problem. Collisions are either elastic or inelastic. Elastic collisions are those for which the two bodies are in the same internal state before and after the collision. If the collision excites one of the participants into a higher state, or transforms the particles into different species, or creates additional particles, the collision is inelastic. Here, we consider only elastic collisions.

Scattering: Coulomb forces

For Coulomb forces, the cross section is infinite because the range of the Coulomb force is infinite, but for interactions such as the strong interaction in nuclear or particle physics, there is no long-range force and cross-sections are finite. Even for Coulomb forces, the part of the cross section that corresponds to a specific

scattering angle, $d\sigma/d\Omega$, which is a function of the scattering angle θ_s is still finite.

If a particle travels through a thin target, the chance the particle scatters is $P_{\text{scatt}} = \sigma dN/dA$, where dN/dA is the number of scattering centers per area the particle encounters. If the density of the target is ρ particles per volume, and if the thickness of the target is t, the areal density (number of target scatterers per area) is $dN/dA = \rho t$. Because one wishes to quantify the collisions independently of the target, experimentalists measure scattering probabilities, then divide by the areal density to obtain cross-sections,

$$\sigma = \frac{P_{\text{scatt}}}{dN/dA}. \tag{1}$$

Scattering, more details

Instead of merely stating that a particle collided, one can measure the probability the particle scattered by a given angle. The scattering angle θ_s is defined so that at zero the particle is unscattered and at $\theta_s = \pi$ the particle is scattered directly backward. Scattering angles are often described in the center-of-mass frame, but that is a detail we will neglect for this first discussion, where we will consider the scattering of particles moving classically under the influence of fixed potentials $U(\boldsymbol{r})$. Because the distribution of scattering angles can be measured, one expresses the differential cross section,

$$\frac{d^2\sigma}{d\cos\theta_s \; d\phi}. \tag{2}$$

Usually, the literature expresses differential cross sections as

$$d\sigma/d\Omega = \frac{d\sigma}{d\cos\theta d\phi} = \frac{1}{2\pi}\frac{d\sigma}{d\cos\theta}, \tag{3}$$

where the last equivalency is true when the scattering does not depend on the azimuthal angle ϕ, as is the case for spherically symmetric potentials.

The differential solid angle $d\Omega$ can be thought of as the area subtended by a measurement, dA_d, divided by r^2, where r is the distance to the detector,

$$dA_d = r^2 d\Omega. \tag{4}$$

With this definition $d\sigma/d\Omega$ is independent of the distance from which one places the detector, or the size of the detector (as long as it is small).

Differential scattering cross sections

Differential scattering cross sections are calculated by assuming a random distribution of impact parameters b. These represent the distance in the xy plane for particles moving in the z direction relative to the scattering center. An impact

parameter $b = 0$ refers to being aimed directly at the target's center. The impact parameter describes the transverse distance from the $z = 0$ axis for the trajectory when it is still far away from the scattering center and has not yet passed it. The differential cross section can be expressed in terms of the impact parameter,

$$d\sigma = 2\pi b db, \tag{5}$$

which is the area of a thin ring of radius b and thickness db. In classical physics, one can calculate the trajectory given the incoming kinetic energy E and the impact parameter if one knows the mass and potential.

More on Differential Cross Sections

From the trajectory, one then finds the scattering angle $\theta_s(b)$. The differential cross section is then

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi}\frac{d\sigma}{d\cos\theta_s} = b\frac{db}{d\cos\theta_s} = \frac{b}{(d/db)\cos\theta_s(b)}. \tag{6}$$

Typically, one would calculate $\cos\theta_s$ and $(d/db)\cos\theta_s$ as functions of b. This is sufficient to plot the differential cross section as a function of θ_s.

The total cross section is

$$\sigma_{\rm tot} = \int d\Omega \frac{d\sigma}{d\Omega} = 2\pi \int d\cos\theta_s \ \frac{d\sigma}{d\Omega}. \tag{7}$$

Even if the total cross section is infinite, e.g. Coulomb forces, one can still have a finite differential cross section as we will see later on.

Rutherford Scattering

This refers to the calculation of $d\sigma/d\Omega$ due to an inverse square force, $F_{12} = \pm\alpha/r^2$ for repulsive/attractive interaction. Rutherford compared the scattering of α particles (^{4}He nuclei) off of a nucleus and found the scattering angle at which the formula began to fail. This corresponded to the impact parameter for which the trajectories would strike the nucleus. This provided the first measure of the size of the atomic nucleus. At the time, the distribution of the positive charge (the protons) was considered to be just as spread out amongst the atomic volume as the electrons. After Rutherford's experiment, it was clear that the radius of the nucleus tended to be roughly 4 orders of magnitude smaller than that of the atom, which is less than the size of a football relative to Spartan Stadium.

Rutherford Scattering, more details

In order to calculate differential cross section, we must find how the impact parameter is related to the scattering angle. This requires analysis of the trajectory. We consider our previous expression for the trajectory where we

derived the elliptic form for the trajectory, For that case we considered an attractive force with the particle's energy being negative, i.e. it was bound. However, the same form will work for positive energy, and repulsive forces can be considered by simple flipping the sign of α. For positive energies, the trajectories will be hyperbolas, rather than ellipses, with the asymptotes of the trajectories representing the directions of the incoming and outgoing tracks.

Rutherford Scattering, final trajectories

We have

$$r = \frac{1}{\frac{m\alpha}{L^2} + A\cos\theta}. \tag{8}$$

Once A is large enough, which will happen when the energy is positive, the denominator will become negative for a range of θ. This is because the scattered particle will never reach certain angles. The asymptotic angles θ' are those for which the denominator goes to zero,

$$\cos\theta' = -\frac{m\alpha}{AL^2}. \tag{9}$$

Rutherford Scattering, Closest Approach

The trajectory's point of closest approach is at $\theta = 0$ and the two angles θ', which have this value of $\cos\theta'$, are the angles of the incoming and outgoing particles. From Fig (**to come**), one can see that the scattering angle θ_s is given by,

$$\begin{aligned} 2\theta' - \pi &= \theta_s, \quad \theta' = \frac{\pi}{2} + \frac{\theta_s}{2}, \\ \sin(\theta_s/2) &= -\cos\theta' \\ &= \frac{m\alpha}{AL^2}. \end{aligned} \tag{10}$$

Now that we have θ_s in terms of m, α, L and A, we wish to re-express L and A in terms of the impact parameter b and the energy E. This will set us up to calculate the differential cross section, which requires knowing $db/d\theta_s$. It is easy to write the angular momentum as

$$L^2 = p_0^2 b^2 = 2mEb^2. \tag{11}$$

Rutherford Scattering, getting there

Finding A is more complicated. To accomplish this we realize that the point of closest approach occurs at $\theta = 0$, so from Eq. (8)

$$\begin{aligned} \frac{1}{r_{\min}} &= \frac{m\alpha}{L^2} + A, \\ A &= \frac{1}{r_{\min}} - \frac{m\alpha}{L^2}. \end{aligned} \tag{12}$$

Next, $r_{\min}$ can be found in terms of the energy because at the point of closest approach the kinetic energy is due purely to the motion perpendicular to $\hat{r}$ and

$$E = -\frac{\alpha}{r_{\min}} + \frac{L^2}{2mr_{\min}^2}. \tag{13}$$

Rutherford Scattering, More Manipulations

One can solve the quadratic equation for $1/r_{\min}$,

$$\frac{1}{r_{\min}} = \frac{m\alpha}{L^2} + \sqrt{(m\alpha/L^2)^2 + 2mE/L^2}. \tag{14}$$

We can plug the expression for $r_{\min}$ into the expression for A, Eq. (12),

$$A = \sqrt{(m\alpha/L^2)^2 + 2mE/L^2} = \sqrt{(\alpha^2/(4E^2b^4) + 1/b^2} \tag{15}$$

Rutherford Scattering, final expression

Finally, we insert the expression for A into that for the scattering angle, Eq. (10),

$$\begin{aligned} \sin(\theta_s/2) &= \frac{m\alpha}{AL^2} \\ &= \frac{a}{\sqrt{a^2+b^2}}, \quad a \equiv \frac{\alpha}{2E} \end{aligned} \tag{16}$$

Rutherford Scattering, the Differential Cross Section

The differential cross section can now be found by differentiating the expression for θ_s with b,

$$
\begin{aligned}
\frac{1}{2}\cos(\theta_s/2)d\theta_s &= \frac{ab\ db}{(a^2+b^2)^{3/2}} = \frac{bdb}{a^2}\sin^3(\theta_s/2), \qquad (17)\\
d\sigma &= 2\pi b db = \frac{\pi a^2}{\sin^3(\theta_s/2)}\cos(\theta_s/2)d\theta_s\\
&= \frac{\pi a^2}{2\sin^4(\theta_s/2)}\sin\theta_s d\theta_s\\
\frac{d\sigma}{d\cos\theta_s} &= \frac{\pi a^2}{2\sin^4(\theta_s/2)},\\
\frac{d\sigma}{d\Omega} &= \frac{a^2}{4\sin^4(\theta_s/2)}.
\end{aligned}
$$

where $a = \alpha/2E$. This the Rutherford formula for the differential cross section. It diverges as $\theta_s \to 0$ because scatterings with arbitrarily large impact parameters still scatter to arbitrarily small scattering angles. The expression for $d\sigma/d\Omega$ is the same whether the interaction is positive or negative.

Rutherford Scattering, Example

Consider a particle of mass m and charge z with kinetic energy E (Let it be the center-of-mass energy) incident on a heavy nucleus of mass M and charge Z and radius R. We want to find the angle at which the Rutherford scattering formula breaks down.

Let $\alpha = Zze^2/(4\pi\epsilon_0)$. The scattering angle in Eq. (16) is

$$
\sin(\theta_s/2) = \frac{a}{\sqrt{a^2+b^2}}, \quad a \equiv \frac{\alpha}{2E}.
$$

The impact parameter b for which the point of closest approach equals R can be found by using angular momentum conservation,

$$
\begin{aligned}
p_0 b &= b\sqrt{2mE} = Rp_f = R\sqrt{2m(E-\alpha/R)},\\
b &= R\frac{\sqrt{2m(E-\alpha/R)}}{\sqrt{2mE}}\\
&= R\sqrt{1-\frac{\alpha}{ER}}.
\end{aligned}
$$

Rutherford Scattering, Example, wrapping up

Putting these together

$$
\theta_s = 2\sin^{-1}\left\{\frac{a}{\sqrt{a^2+R^2(1-\alpha/(RE))}}\right\}, \quad a = \frac{\alpha}{2E}.
$$

It was from this departure of the experimentally measured $d\sigma/d\Omega$ from the Rutherford formula that allowed Rutherford to infer the radius of the gold nucleus, R.

Variational Calculus

The calculus of variations involves problems where the quantity to be minimized or maximized is an integral.

The usual minimization problem one faces involves taking a function $\mathcal{L}(x)$, then finding the single value x for which $\mathcal{L}$ is either a maximum or minimum. In multivariate calculus one also learns to solve problems where you minimize for multiple variables, $\mathcal{L}(x_1, x_2, \cdots x_n)$, and finding the points $(x_1 \cdots y_n)$ in an n-dimensional space that maximize or minimize the function. Here, we consider what seems to be a much more ambitious problem. Imagine you have a function $\mathcal{L}(x(t), \dot{x}(t), t)$, and you wish to find the extrema for an infinite number of values of x, i.e. x at each point t. The function $\mathcal{L}$ will not only depend on x at each point t, but also on the slope at each point, plus an additional dependence on t. Note we are NOT finding an optimum value of t, we are finding the set of optimum values of x at each point t, or equivalently, finding the function $x(t)$.

Variational Calculus, introducing the action

One treats the function $x(t)$ as being unknown while minimizing the action

$$S = \int_{t_1}^{t_2} dt\ \mathcal{L}(x(t), \dot{x}(t), t).$$

Thus, we are minimizing S with respect to an infinite number of values of $x(t_i)$ at points t_i. As an additional criteria, we will assume that $x(t_1)$ and $x(t_2)$ are fixed, and that that we will only consider variations of x between the boundaries. The dependence on the derivative, $\dot{x} = dx/dt$, is crucial because otherwise the solution would involve simply finding the one value of x that minimized $\mathcal{L}$, and $x(t)$ would equal a constant if there were no explicit t dependence. Furthermore, x wouldn't need to be continuous at the boundary.

Variational Calculus, general Action

In the general case we have an integral of the type

$$S[q] = \int_{t_1}^{t_2} \mathcal{L}(q(t), \dot{q}(t), t)dt,$$

where S is the quantity which is sought minimized or maximized. The problem is that although $\mathcal{L}$ is a function of the general variables $q(t), \dot{q}(t), t$ (note our change of variables), the exact dependence of q on t is not known. This means again that even though the integral has fixed limits t_1 and t_2, the path of integration is not known. In our case the unknown quantities are the positions

and general velocities of a given number of objects and we wish to choose an integration path which makes the functional $S[q]$ stationary. This means that we want to find minima, or maxima or saddle points. In physics we search normally for minima. Our task is therefore to find the minimum of $S[q]$ so that its variation δS is zero subject to specific constraints. The constraints can be treated via the technique of Lagrangian multipliers as we will see below.

Variational Calculus, Optimal Path

We assume the existence of an optimum path, that is a path for which $S[q]$ is stationary. There are infinitely many such paths. The difference between two paths δq is called the variation of q.

We call the variation $\eta(t)$ and it is scaled by a factor α. The function $\eta(t)$ is arbitrary except for

$$\eta(t_1) = \eta(t_2) = 0,$$

and we assume that we can model the change in q as

$$q(t, \alpha) = q(t) + \alpha\eta(t),$$

and

$$\delta q = q(t, \alpha) - q(t, 0) = \alpha\eta(t).$$

Variational Calculus, Condition for an Extreme Value

We choose $q(t, \alpha = 0)$ as the unkonwn path that will minimize S. The value $q(t, \alpha \neq 0)$ describes a neighbouring path.

We have

$$S[q(\alpha)] = \int_{t_1}^{t_2} \mathcal{L}(q(t, \alpha), \dot{q}(t, \alpha), t)dt.$$

The condition for an extreme of

$$S[q(\alpha)] = \int_{t_1}^{t_2} \mathcal{L}(q(t, \alpha), \dot{q}(t, \alpha), t)dt,$$

is

$$\left[\frac{\partial S[q(\alpha)]}{\partial t}\right]_{\alpha=0} = 0.$$

Variational Calculus. α Dependence

The α dependence is contained in $q(t,\alpha)$ and $\dot{q}(t,\alpha)$ meaning that

$$\left[\frac{\partial E[q(\alpha)]}{\partial \alpha}\right] = \int_{t_1}^{t_2} \left(\frac{\partial}{\partial q}\frac{\partial q}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial \alpha}\right) dt.$$

We have defined

$$\frac{\partial q(x,\alpha)}{\partial \alpha} = \eta(x)$$

and thereby

$$\frac{\partial \dot{q}(t,\alpha)}{\partial \alpha} = \frac{d(\eta(t))}{dt}.$$

INtegrating by Parts

Using

$$\frac{\partial q(t,\alpha)}{\partial \alpha} = \eta(t),$$

and

$$\frac{\partial \dot{q}(t,\alpha)}{\partial \alpha} = \frac{d(\eta(t))}{dt},$$

in the integral gives

$$\left[\frac{\partial S[q(\alpha)]}{\partial \alpha}\right] = \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q}\eta(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{d(\eta(t))}{dt}\right) dt.$$

Integrating the second term by parts

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{d(\eta(t))}{dt}dt = \eta(t)\frac{\partial \mathcal{L}}{\partial \dot{q}}|_{t_1}^{t_2} - \int_a^b \eta(t)\frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \dot{q}}dt,$$

and since the first term dissappears due to $\eta(a) = \eta(b) = 0$, we obtain

$$\left[\frac{\partial S[q(\alpha)]}{\partial \alpha}\right] = \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\eta(t)dt = 0.$$

Euler-Lagrange Equations

The latter can be written as

$$\left[\frac{\partial S[q(\alpha)]}{\partial \alpha}\right]_{\alpha=0} = \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\delta q(t)dt = \delta S = 0.$$

The condition for a stationary value is thus a partial differential equation

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0,$$

known as the **Euler-Lagrange** equation.