PHY 321, MARCH 13, 2023

Formalism

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0\cos(\omega t)$$

$$\omega_0 = \sqrt{k/m} \qquad \gamma = \frac{b}{2m\omega_0}$$

$$\tilde{\omega} = \frac{\omega}{\omega_0}$$

$$\tau = \omega_0 \cdot t \qquad \tilde{F}_0 = \frac{F_0}{m\omega_0^2}$$

$$\frac{d^2x}{d\tau^2} + 2\gamma\frac{dx}{d\tau} + x = \tilde{F}_0\cos(\tilde{\omega}\cdot\tau)$$

$\tilde{F}_0 \neq 0$

$$x(\tau) = \underbrace{D\cos(\tilde{\omega}\tau - \delta)}_{x_p(\tau)} + C_1 e^{r_1\tau} + C_2 e^{r_2\tau}$$

Our guess on $x_p(\tau)$

$= D\cos(\tilde{\omega}\tau - \delta)$

Plug it into the diff. equation:

$$\frac{d^2x}{d\tilde{\tau}^2} + 2\gamma \frac{dx}{d\tilde{\tau}} + x = \tilde{F}_0 \cos(\tilde{\omega}\tilde{\tau})$$

$$D\left\{ -\tilde{\omega}^2 \cos(\tilde{\omega}\tilde{\tau} - \delta) - 2\gamma\tilde{\omega}\sin(\tilde{\omega}\tilde{\tau} - \delta) + \cos(\tilde{\omega}\tilde{\tau} - \delta) \right\} = \tilde{F}_0 \cos(\tilde{\omega}\tilde{\tau})$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\alpha = \tilde{\omega}\tilde{\tau} \quad \beta = \delta$$

$$D\left\{ \left(-\tilde{\omega}^2\cos\delta + 2\gamma\tilde{\omega}\sin\delta + \cos\delta\right)\cos(\tilde{\omega}\tilde{\tau}) + \left(-\tilde{\omega}^2\sin\delta - 2\gamma\tilde{\omega}\cos\delta + \sin\delta\right)\sin(\tilde{\omega}\tilde{\tau}) \right\} = \tilde{F}_0\cos(\tilde{\omega}\tilde{\tau})$$

In order for this to hold at all times, the sin and cos parts have to be equal

(i)

$$D\left\{ -\tilde{\omega}^2\cos\delta + 2\gamma\tilde{\omega}\sin\delta \right.$$

$$+\cos\delta\} = \tilde{F}_0$$

(ii)

$$-\tilde{\omega}^2\sin\delta - 2\gamma\tilde{\omega}\cos\delta + \sin\delta = 0$$

in (ii) divide by $\cos\delta$

$$-\tilde{\omega}^2\tan\delta - 2\gamma\tilde{\omega} = -\tan\delta \Rightarrow$$

$$\tan\delta = \frac{2\gamma\tilde{\omega}}{1-\tilde{\omega}^2}$$

$$\sin^2\delta + \cos^2\delta = 1$$

$$\tan^2\delta + 1 = \csc^2\delta$$

$$\sin\delta = \frac{\tan\delta}{\sqrt{\tan^2\delta - 1}}$$

$$\cos^2\delta = 1 - \sin^2\delta$$

$$\sin\delta = \frac{2\gamma\tilde{\omega}^2}{\sqrt{4\gamma^2\tilde{\omega}^2 + (1-\tilde{\omega}^2)^2}}$$

$$\cos\delta = \frac{(1-\tilde{\omega}^2)^2}{\sqrt{}}$$

insert i (i)

$$D = \frac{\tilde{F}_0}{\sqrt{(1-\tilde{\omega}^2)^2 + 4\tilde{\omega}^2\gamma^2}}$$

$$\delta = \tan^{-1}\left(\frac{2\gamma\tilde{\omega}}{1-\tilde{\omega}^2}\right)$$

$$x_p(\tau) = D \cdot \cos(\tilde{\omega}\tau - \delta)$$

Analysis

$$D^2 = \frac{\tilde{F}_0^2}{(1-\tilde{\omega}^2)^2 + 4\tilde{\omega}^2\gamma^2}$$

$$\gamma = \frac{b}{2m\,\omega_0^2}$$

$$\beta = \text{damping constant}$$

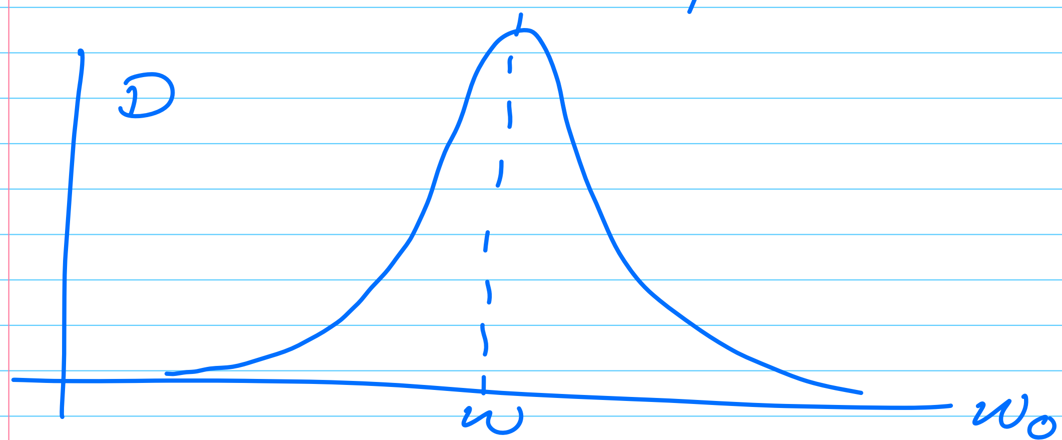
$$= \frac{b}{2m}$$

$$\tilde{F}_0^2 = F_0^2/m^2\omega_0^4 = f_0^2/\omega_0^4$$

$$D^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

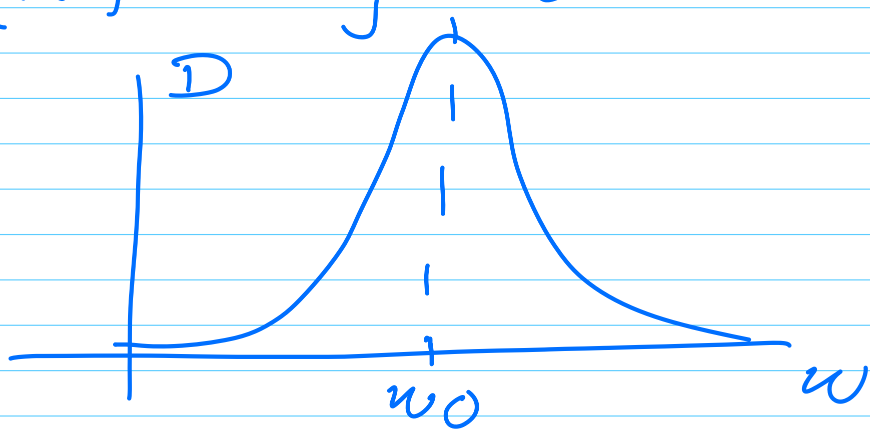
$$\tilde{\omega} = \frac{\omega}{\omega_0}$$

normal case : β is small



(i) vary ω_0

(ii) vary ω



when does the max D occur?

$$\frac{d}{d\omega}\left[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\right] = 0$$

$$= -4\omega(\omega_0^2 - \omega^2) + 8\beta^2\omega = 0$$

$\omega \neq 0$ Divide by ω

$$-\omega_0^2 + \omega^2 + 2\beta^2 = 0$$

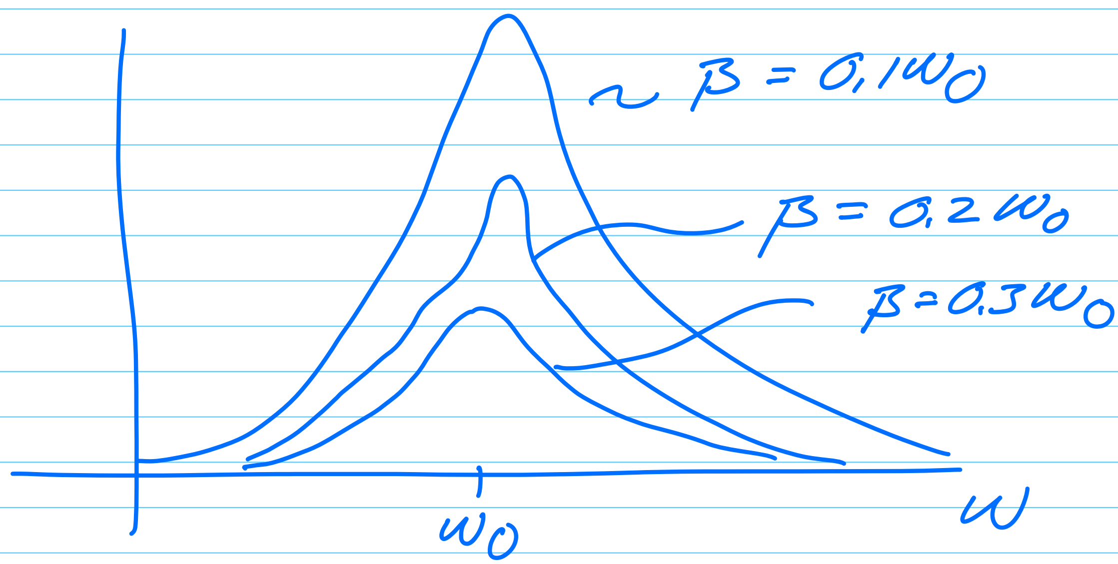
$$\omega^2 = \omega_0^2 - 2\beta^2$$

$$\omega = \sqrt{\omega_0^2 - 2\beta^2}$$

when $\beta \ll \omega_0$,

$$\omega \simeq \omega_0 \Rightarrow$$

$$D_{max} \simeq \frac{f_0}{2\beta\omega_0}$$



as β decreases the resonance peak gets higher and sharper.