PHY 321, APRIL 4, 2022

$$\mu \ddot{r} = F(r) + \frac{L^2}{\mu r^3}$$

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{L}{\mu r^2}$$

$$x = r \cdot \cos\phi \quad \wedge \quad y = r \sin\phi$$

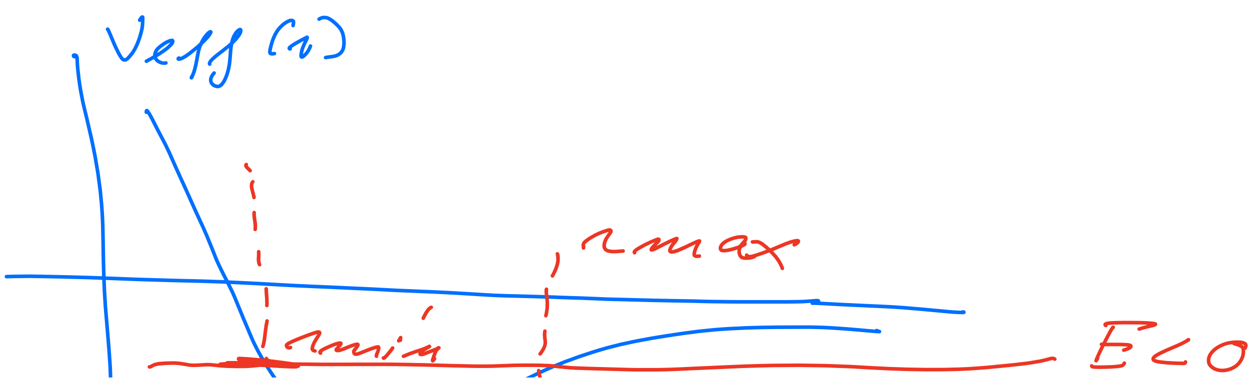
$$F(r) = -\alpha / r^2$$

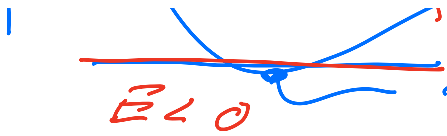
$$r(\phi) = \frac{c}{1 + \varepsilon \cos\phi}$$

$$c = \frac{L^2}{\mu \alpha}$$

(i) $\varepsilon = 0$, circular motion

(ii) $0 < \varepsilon < 1$, elliptical motion





$r'' = 0, \ \varepsilon = 0$

$0 < \varepsilon < 1$

$$r_{min} = \frac{c}{1+\varepsilon}$$

$$r_{max} = \frac{c}{1-\varepsilon}$$

(iii) $\boxed{\varepsilon = 1}$

$$r(\phi) = \frac{c}{1+\varepsilon\cos\phi}$$

$r(\phi)$ diverges if $\phi = \pm\pi$

$$r(1+\cos\phi) = c$$

$$r\cos\phi = x$$

$$r + x = c$$

$$r = c - x$$

square both sides

$$r^2 = x^2 + y^2 = (c-x)^2$$

$$= c^2 - 2cx + x^2$$

$$y^2 = c^2 - 2cx \implies$$

$$\boxed{x(y) = \frac{c}{2} - y^2/2c}$$ parabola

$$f(y) = a_0 + a_1 x + a_2 x^2$$
$$a_1 = 0$$

Link with energy (before analysis of $\varepsilon > 1$)

assume that at r_{min} we have $\frac{dr}{dt} = 0$

$$K = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r \dot{\phi}^2 \Rightarrow$$

$$V_{eff} = V(r) + \frac{l^2}{2\mu r^2}$$

$$E\Big|_{r=r_{min}} = -\frac{\alpha}{r_{min}} + \frac{l^2}{2\mu r_{min}^2}$$

$$\left(V(r) = -\alpha/r \right)$$

$$= \frac{1}{2 r_{min}} \left[\frac{l^2}{\mu r_{min}} - 2\alpha \right]$$

$$\left(r_{min} = \frac{c}{1+\varepsilon} = \frac{l^2}{\alpha \mu (1+\varepsilon)} \right)$$

$$= \frac{\alpha\mu(1+\varepsilon)}{2L^2}\left[\alpha(1+\varepsilon)-2\alpha\right]$$

$$= \frac{\alpha^2\mu}{2L^2}(1+\varepsilon)(\varepsilon-1)$$

$$= \frac{\alpha^2\mu}{2L^2}(\varepsilon^2-1)$$

$$\mu > 0 \qquad L^2 > 0 \qquad \alpha > 0$$

$$E(r=r_{min}) = \frac{\alpha^2\mu}{2L^2}(\varepsilon^2-1)$$

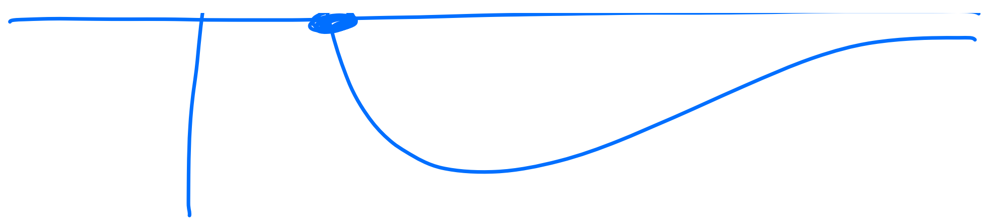
$$\varepsilon \geq 0$$

if $E < 0$, $0 \leq \varepsilon < 1$

$$\frac{\alpha^2\mu}{2L^2}(\varepsilon-1)(\varepsilon+1) < 0$$

then $\varepsilon < 1$

if $E = 0$



$$\varepsilon^2 - 1 = 0 \Rightarrow \varepsilon = \pm 1$$

$\Rightarrow$ parabolic motion

(iv) $E > 0$

$$\frac{\alpha^2 \mu}{2L^2}(\varepsilon^2 - 1) > 0 \Rightarrow$$

$$\boxed{\varepsilon > 1}$$

$$r(\phi) = \frac{c}{1 + \varepsilon \cos\phi}$$

Denominator vanishes at some value ϕ_{max}

$$\varepsilon \cos(\phi_{max}) = -1$$

$$\cos(\phi_{max}) = -1/\varepsilon$$

what orbit is this?

$$x = r\cos\phi$$

$$r(1 \; \varepsilon\cos\phi) = c$$

$r + \varepsilon x = c$

$r = c - \varepsilon x$ square

$r^2 = x^2 + y^2 = c^2 + \varepsilon^2 x^2 - 2c\varepsilon x$

$x^2(1-\varepsilon^2) + y^2 + 2c\varepsilon x = c^2$

$x^2(\varepsilon^2 - 1) - y^2 - 2c\varepsilon x = -c^2$

complete squares for x

$$(\varepsilon^2 - 1)(x - \delta)^2 - y^2 = -c^2 + \frac{\varepsilon^2 c^2}{\varepsilon^2 - 1}$$

$$(x-\delta)^2 = x^2 - 2x\delta + \delta^2$$

$\delta = \dfrac{c\varepsilon}{\varepsilon^2 - 1}$ $\qquad \delta = \varepsilon\alpha$

$\alpha = \dfrac{c}{\varepsilon^2 - 1}$ $\qquad \beta = \dfrac{c}{\sqrt{\varepsilon^2 - 1}}$

$$\frac{(x-\delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$$

equation for hyperbola

$$F(r) = -\alpha/r^2$$

$$V_{eff}(r) = \frac{L^2}{2\mu r^2} - \frac{\alpha}{r}$$

$$F(r) = -\frac{dV}{dr}$$

$$F(r) = \alpha/r^2$$

$\Rightarrow$ unbound

$$r(\phi) = \frac{c}{\varepsilon\cos\phi - 1}$$