PHY321: Classical Mechanics 1

Second midterm project, due Friday April 16

Apr 1, 2021

Practicalities about homeworks and projects.

1. You can work in groups (optimal groups are often 2-3 people) or by yourself. If you work as a group you can hand in one answer only if you wish. **Remember to write your name(s)**!

2. How do I(we) hand in? Due to the extraordinary situation we are in now, the midterm should be handed in fully via D2L. You can scan your handwritten notes and upload to D2L or you can hand in everyhting (if you are ok with typing mathematical formulae using say Latex) as a jupyter notebook at D2L. The numerical part should always be handed in as a jupyter notebook.

Introduction to the second midterm project, total score 100 points. In this midterm we will attempt at writing a program that simulates the solar system. We start with the Earth-Sun system we studied in homework 5 and 6 and study elliptical orbits and their properties. We test also elliptical orbits and their dependence on powers β of r^{β}. We will test other aspects of the Earth-Sun system and link these to the theoretical discussion on two-body problems with central forces.

Thereafter, based on the three-body problem studied in homework 9, we attempt at making a code which simulates the solar system.

The relevant reading background is

1. chapter 8 of Taylor.

2. Lecture notes on central forces and two-body problems

3. Homeworks 5-9

Part 1 (50pt), the inverse-square law and the stability of planetary orbits. In homework 9 we studied an attractive potential

$$V(r) = -\alpha/r,$$

where the quantity r is the absolute value of the relative position and α is a constant.

When we rewrote the equations of motion in polar coordinates, the analytical solution to the radial equation of motion was

$$r(\phi) = \frac{c}{1 + \epsilon \cos(\phi)},$$

where $c = L^2/\mu\alpha$, with the reduced mass μ and the angular momentum L, as discussed during the lectures. With the transformation of a two-body problem to the center-of-mass frame, the actual equations look like an *effective* one-body problem.

The quantity ϵ is what we called the eccentricity. Since we will mainly study bounded orbits, we have $0 \le \epsilon < 1$. For the Earth, the orbit is indeed close to circular and at perihelion (the closest distance to the Sun), the Earth's center is about 0.98329 astronomical units (AU) or 147,098,070 km from the Sun's center. For Earth, the orbital eccentricity is $\epsilon \approx 0.0167$. The outer planets have more elliptical orbits. For example, Mars has its perihelion at 206,655,215 km and its apehelion at 249,232,432 km.

In this part we will limit ourselves to the Earth-Sun system we studied in homeworks 5 and 6. You can reuse your code with either the Velocity-Verlet or the Euler-Cromer algorithms from homework 5 or 6.

This means also that $\alpha = GM_{\odot}M_{\mathrm{Earth}}$. We will use α as a shorthand in the equations here. Keep in mind that in homework 5 you scaled $GM_{\odot} = 4\pi^2$ in your code.

The exercises here are all based on you analyzing the results from your code from homeworks 5, 6, 7 and 8.

- 1a (10pt) Use now your code from homework 5 (in cartesian coordinates). Start with a circular orbit setting $\epsilon = 0$ and plot x versus y. How would you choose the initial conditions to obtain a circular orbit?

- 1b (10pt) Check that for the case of a circular orbit that both the kinetic and the potential energies are conserved. Why do we expect such a result if we have a circular orbit?

- 1c (10pt) With the same initial conditions (circular orbit) Use Kepler's second law (see Taylor section 3.4) to show that angular momentum is conserved. Compare the value you get with the angular momentum you get from a circular orbit.

- 1d (10pt) Till now we have assumed that we have an inverse-square force $F(r) = -\alpha/r^2$. Let us rewrite this force as $F(r) = -\alpha/r^{\beta}$ with

$\beta = [2, 2.01, 2.10, 2.5, 3.0, 3.5]$. Run your Sun-Earth code with these values of β and plot x versus y (you can use the same initial conditions or switch to eliptical orbits). Discuss your results. Can you use the observations of planetary motion to determine by what amount Nature deviates from a perfect inverse-square law?

- 1e (10pt) Consider now an elliptical orbit with an initial position 1 AU from the Sun and an initial velocity of 5 AU/yr. Show that the total energy is a constant (the kinetic and potential energies will vary). Show also that the angular momentum is a constant. If you change the parameter β in $F(r) = -\alpha/r^\beta$ from $\beta = 2$ to $\beta = 3$, are these quantities conserved? Discuss your results. (Hint: relate your results to Kepler's laws).

Part 2 (50pt), making a program for the solar system. Our final aim is to write a code which includes the known planets of the solar system.

We will, as before, use so-called astronomical units when rewriting our equations. Using astronomical units (AU as abbreviation)it means that one astronomical unit of length, known as 1 AU, is the average distance between the Sun and Earth, that is 1 AU $= 1.5 \times 10^{11}$ m. It can also be convenient to use years instead of seconds since years match better the time evolution of the solar system. The mass of the Sun is $M_{\mathrm{sun}} = M_\odot = 2 \times 10^{30}$ kg. The masses of all relevant planets and their distances from the sun are listed in the table here in kg and AU.

Planet	Mass in kg	Distance to sun in AU
Earth	$M_{\mathrm{Earth}} = 6 \times 10^{24}$ kg	1AU
Jupiter	$M_{\mathrm{Jupiter}} = 1.9 \times 10^{27}$ kg	5.20 AU
Mars	$M_{\mathrm{Mars}} = 6.6 \times 10^{23}$ kg	1.52 AU
Venus	$M_{\mathrm{Venus}} = 4.9 \times 10^{24}$ kg	0.72 AU
Saturn	$M_{\mathrm{Saturn}} = 5.5 \times 10^{26}$ kg	9.54 AU
Mercury	$M_{\mathrm{Mercury}} = 3.3 \times 10^{23}$ kg	0.39 AU
Uranus	$M_{\mathrm{Uranus}} = 8.8 \times 10^{25}$ kg	19.19 AU
Neptun	$M_{\mathrm{Neptun}} = 1.03 \times 10^{26}$ kg	30.06 AU
Pluto	$M_{\mathrm{Pluto}} = 1.31 \times 10^{22}$ kg	39.53 AU

Pluto is no longer considered a planet, but we add it here for historical reasons. It is optional in this midterm project to include Pluto and eventual moons.

In setting up the equations we can limit ourselves to a co-planar motion and use only the x and y coordinates. But you should feel free to extend your equations to three dimensions, it is not very difficult and the data from NASA are all in three dimensions.

NASA has an excellent site at `http://ssd.jpl.nasa.gov/horizons.cgi#top`. From there you can extract initial conditions in order to start your differential equation solver. At the above website you need to change from **OBSERVER** to **VECTOR** and then write in the planet you are interested in. The generated data contain the x, y and z values as well as their corresponding velocities. The velocities are in units of AU per day. Alternatively they can be obtained in terms of km and km/s.

- 2a (50pt) Since the Sun is much more massive than all the other planets, we will define the Sun as our center of mass and set its velocity and position to zero. You can use your code from homework 9 and add gradually one planet at the time. Develop a code which simulates the solar system with the above planets and plot their orbits. Discuss your results.

In homework 4 we limited ourselves (in order to test the algorithm) to a hypothetical solar system with the Earth only orbiting around the Sun. We assumed that the only force in the problem is gravity. Newton's law of gravitation is given by a force F_G (we assume this is the force acting on Earth from the Sun)

$$F_G = -\frac{GM_{\odot}M_{\mathrm{Earth}}}{r^2},$$

where $M_{\odot}$ is the mass of the Sun and M_{Earth} is the mass of the Earth. The gravitational constant is G and r is the distance between the Earth and the Sun. We assumed that the Sun has a mass which is much larger than that of the Earth. We could therefore safely neglect the motion of the Sun.

In homework 4 assumed that the orbit of the Earth around the Sun was co-planar, and we took this to be the xy-plane. Using Newton's second law of motion we got the following equations

$$\frac{d^2x}{dt^2} = -\frac{F_{G,x}}{M_{\mathrm{Earth}}},$$

and

$$\frac{d^2y}{dt^2} = -\frac{F_{G,y}}{M_{\mathrm{Earth}}},$$

where $F_{G,x}$ and $F_{G,y}$ are the x and y components of the gravitational force.

We will again use so-called astronomical units when rewriting our equations. Using astronomical units (AU as abbreviation)it means that one astronomical unit of length, known as 1 AU, is the average distance between the Sun and Earth, that is 1 AU $= 1.5 \times 10^{11}$ m. It can also be convenient to use years instead of seconds since years match better the time evolution of the solar system. The mass of the Sun is $M_{\mathrm{sun}} = M_{\odot} = 2 \times 10^{30}$ kg. The masses of all relevant planets and their distances from the sun are listed in the table here in kg and AU.

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Using our code from homework 4, we will now add Jupyter and play around with different masses for this planet and study numerically a three-body problem. This is a well-studied problem in classical mechanics, with many interesting results, from stable orbits to chaotic motion.

Exercise 1 The three-body problem (100pt). We will now study the three-body problem, still with the Sun kept fixed as the center of mass of the system but including Jupiter (the most massive planet in the solar system, having a mass that is approximately 1000 times smaller than that of the Sun) together with the Earth. This leads to a three-body problem. Without Jupiter, the Earth's motion is stable and unchanging with time. The aim here is to find out how much Jupiter alters the Earth's motion.

The program you have developed in homework 4 can easily be modified by simply adding the magnitude of the force betweem the Earth and Jupiter.

This force is given again by

$$F_{\rm Earth-Jupiter} = -\frac{GM_{\rm Jupiter}M_{\rm Earth}}{r^2_{\rm Earth-Jupiter}},$$

where $M_{\rm Jupiter}$ is the mass of Jupyter and $M_{\rm Earth}$ is the mass of the Earth. The gravitational constant is G and $r_{\rm Earth-Jupiter}$ is the distance between Earth and Jupiter.

We assume again that the orbits of the two planets are co-planar, and we take this to be the xy-plane (you can easily extend the equations to three dimensions).

- 1a (20pt) Modify your coupled first-order differential equations from homework 4 in order to accomodate both the motion of the Earth and Jupiter by taking into account the distance in x and y between the Earth and Jupiter. Write out the differential equations for Earth and Jupyter, keeping the Sun at rest (mass center of the system).

- 1b (10pt) Scale these equations in terms of Astronomical Units.

- 1c (30pt) Set up the algorithm and plot the positions of the Earth and Jupiter using the Velocity Verlet algorithm. Discuss the stability of the solutions using your Velocity Verlet solver.

- 1c (40pt) Repeat the calculations by increasing the mass of Jupiter by a factor of 10, 100 and 1000 and plot the position of the Earth. Discuss your results and study again the stability of the Velocity Verlet solver. Is energy conserved?

Exercise 2, the bonus part (50pt). The perihelion precession of Mercury. This is the bonus exercise and gives an additional score of 50 points. It is fully optional. **I would grade this as a more difficult exercise compared to previous ones**. It requires also that you read some background literature about the perihelion of Mercury. You don't need to derive the relativistic correction here. This is something you will meet in a graduate course on General Relativity. The bonus here is that it allows you explore physics you could not have done without a numerical code.

An important test of the general theory of relativity was to compare its prediction for the perihelion precession of Mercury to the observed value. The observed value of the perihelion precession, when all classical effects (such as the perturbation of the orbit due to gravitational attraction from the other planets) are subtracted, is $43''$ (43 arc seconds) per century.

Closed elliptical orbits are a special feature of the Newtonian $1/r^2$ force. In general, any correction to the pure $1/r^2$ behaviour will lead to an orbit which is not closed, i.e. after one complete orbit around the Sun, the planet will not be at exactly the same position as it started. If the correction is small, then each orbit around the Sun will be almost the same as the classical ellipse, and the orbit can be thought of as an ellipse whose orientation in space slowly rotates. In other words, the perihelion of the ellipse slowly precesses around the Sun.

You will now study the orbit of Mercury around the Sun, adding a general relativistic correction to the Newtonian gravitational force, so that the force becomes

$$F = -\frac{GM_{\mathrm{Sun}}M_{\mathrm{Mercury}}}{r^2}\left[1 + \frac{3l^2}{r^2c^2}\right]$$

where M_{Mercury} is the mass of Mercury, r is the distance between Mercury and the Sun, $l = |\vec{r} \times \vec{v}|$ is the magnitude of Mercury's orbital angular momentum per unit mass, and c is the speed of light in vacuum. Run a simulation over one century of Mercury's orbit around the Sun with no other planets present, starting with Mercury at the perihelion on the x axis. Check then the value of the perihelion angle θ_{p}, using

$$\tan\theta_{\mathrm{p}} = \frac{y_{\mathrm{p}}}{x_{\mathrm{p}}}$$

where x_{p} (y_{p}) is the x (y) position of Mercury at perihelion, i.e. at the point where Mercury is at its closest to the Sun. You may use that the speed of Mercury at perihelion is 12.44 AU/yr, and that the distance to the Sun at perihelion is 0.3075 AU.

You need to make sure that the time resolution used in your simulation is sufficient, for example by checking that the perihelion precession you get with a pure Newtonian force is at least a few orders of magnitude smaller than the observed perihelion precession of Mercury. Can the observed perihelion precession of Mercury be explained by the general theory of relativity?