PHY 321, APRIL 17, 2023

- Principle of least action

$$S = \int_{x_1}^{x_2} \mathcal{L}(y, y', x)\, dx$$

- Lagrangian $\mathcal{L} = K - V$

(1d) $= \frac{1}{2} m v^2 - V(x)$

- variational calculus

Euler-Lagrange eqs

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v}$$

$$\mathcal{L} = \mathcal{L}(x, v, t)$$

Define general coordinates

$x \to q$

$v \to \dot{q}$ general velocity

Example

$x, y \to r\ \phi$

$$\mathcal{L}(x, \dot{x}, y, \dot{y}, t) = \mathcal{L}(r, \dot{r}, \phi, \dot{\phi}, t)$$

$$= \sum_{i=1}^{2} \mathcal{L}(q_i, \dot{q}_i, t)$$

$$q_1 = r \qquad \dot{q}_1 = \dot{r}$$

$$q_2 = \phi \qquad \dot{q}_2 = \dot{\phi}$$

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad i = 1, 2.$$

Variational calculus with constraints

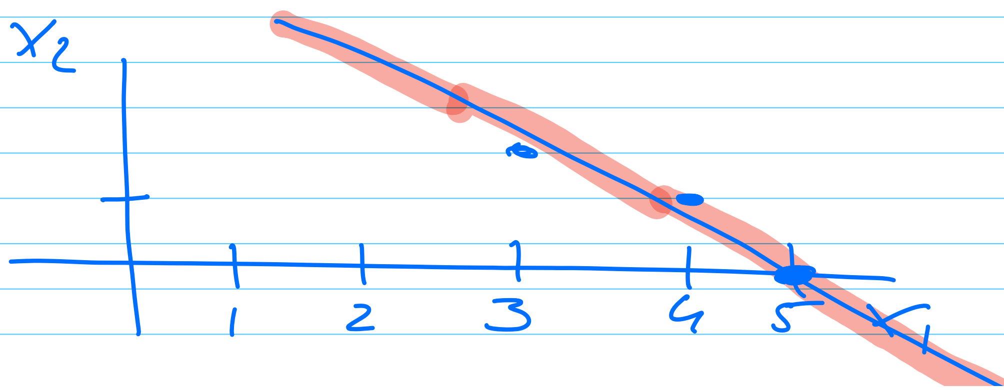
Example 2

minimize

$$f(x_1, x_2) = -3x_1^2 - 6x_1x_2 - 5x_2^2 + 7x_1 + 5x_2$$

subject to the constraint

$$x_1 + x_2 = 5$$



$x_2 = 5 - x_1$ in $f(x_1, x_2)$

$$f(x_1, x_2) \Rightarrow h(x_1) = -3x_1^2 - 6x_1(5 - x_1) - 5(5 - x_1)^2 + 7x_1 + 5(5 - x_1)$$

$$= -2x_1^2 + 22x_1 - 100$$

$$\frac{\partial h}{\partial x_1} = 0 = -4x_1 + 22 \Rightarrow$$

$$x_1 = 11/2$$

$$x_2 = 5 - 11/2 = -1/2$$

Top-down

$$\mathcal{L} = f(x_1, x_2)$$

$$\mathcal{L}' = \mathcal{L} + \lambda g(x_1 x_2)$$

$$g(x_1, x_2) = 0$$

Lagrangian multiplier

$$g(x_1, x_2) = 0 = x_1 + x_2 - 5$$

$$\left[\frac{\partial \mathcal{L}'}{\partial x_1} - \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{x}_1}\right] = 0$$

$$\left[\frac{\partial \mathcal{L}'}{\partial x_2} - \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{x}_2}\right] = 0$$

$$\frac{\partial \mathcal{L}'}{\partial x_1} = -6x_1 - 6x_2 + 7 + \lambda = 0$$

$$x_1 + x_2 = \frac{7+\lambda}{6}$$

$$x_1 + x_2 = 5$$

$$\frac{7+\lambda}{6} = 5 \Rightarrow \lambda = 23$$

$$\frac{\partial \mathcal{L}'}{\partial x_2} = -6x_1 - 10x_2 + 5 + \lambda = 0$$

$$x_1 = 5 - x_2$$

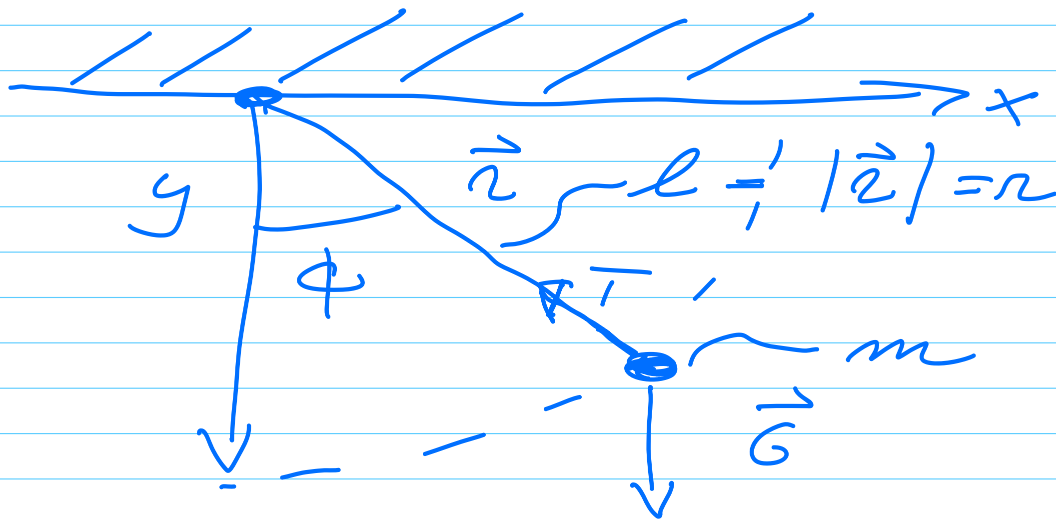
$$-6(5 - x_2) - 10x_2 + 5 + \lambda = 0$$

$$-4x_2 = 2 \Rightarrow x_2 = -1/2$$

$$x_1 = 5 + 1/2 = 11/2$$

Example 3

mathematical pendulum



$$\mathcal{L} = \frac{1}{2} m (v_x^2 + v_y^2) + m \cdot g \cdot y$$

$y = r \cdot \cos\phi \qquad x = r \cdot \sin\phi$

$x, y \Rightarrow r, \phi$

$$\mathcal{L} = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + m g r \cdot \cos\phi + \lambda \times \underbrace{\text{constraint}}_{(l - r) = 0}$$

$$\frac{\partial \mathcal{L}'}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{r}} = 0$$

$$mr\dot{\phi}^2 + mg\cos\phi - m\ddot{r} - \lambda = 0$$

$$mr\dot{\phi}^2 + mg\cos\phi - \lambda = m\ddot{r} = 0 \Rightarrow$$

$$\lambda = mr\dot{\phi}^2 + mg\cos\phi$$

(Tension force)

$$\frac{\partial \mathcal{L}'}{\partial \phi} - \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{\phi}}$$

$$\downarrow$$

$$-mgr\cdot\sin\phi - mr^2\ddot{\phi} = 0$$

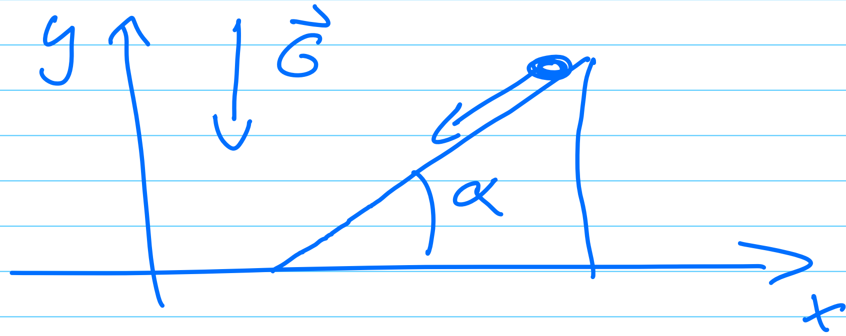
$$\ddot{\phi} = -\frac{g}{r}\sin\phi$$

$$r = l \quad \Rightarrow$$

$$\ddot{\phi} = -g/l\,\sin\phi = -w_0^2\sin\phi$$

$$\boxed{\ddot{\phi} = -\omega_0^2 \sin\phi} \qquad \omega_0^2 = g/l$$

Example 4



constraint

$$g(x,y) = y - x\tan\alpha = 0$$

$$y/x = \tan\alpha$$

$$\mathcal{L}' = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + ?$$
$$+ \lambda g(x,y)$$

$$= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + mgy$$
$$+ \lambda g(x,y)$$