PHY321: Two-body problems and Gravitational Forces

Morten Hjorth-Jensen[1,2]

[1]Department of Physics and Astronomy and Facility for Rare Ion Beams (FRIB), Michigan State University, USA
[2]Department of Physics, University of Oslo, Norway

Mar 21, 2022

Aims and Overarching Motivation

Monday March 21. Definition of the two-body problem, rewriting the equations in relative and center-of-mass coordinates
Reading suggestion: Taylor sections 8.2-8.3

Wednesday March 23. Preparing the ground for the gravitational force and its solution in two dimensions. Harmonic Oscillator example in two dimensions (if we get time).
Reading suggestion: Taylor chapter 8.4-8.5

Friday March 25. Discussion and work on homework 7

Videos of possible interest

- Video on solving differential equations numerically
- Video on Fourier aanalysis
- Handwritten notes for Fourier analysis

Two-body Problems

Two-body problems play a central role in physics. Some of these problems can, with appropriate transformations, be solved analytically. The gravitional force problem (as well as the Coulomb potential problem for two particles) is an example of this.

There are several small steps which we need to do in order to reach this solution. These are

© 1999-2022, "Morten Hjorth-Jensen":"http://mhjgit.github.io/info/doc/web/". Released under CC Attribution-NonCommercial 4.0 license

1. Rewriting the equations in terms of the degrees of freedom of the relative motion and the center of mass motion

2. Making a transformation either to polar (two dimensions) or to spherical coordinates (three dimensions)

3. This gives us uncoupled differential equations for each coordinate that we can solve analytically

The advantage in doing so is that we can extract a lot of interesting insights about the motion and the physics of these important physics problems. These insights can be transferred to other physics problems where the potentials are given by expressions proportional with the inverse relative distance.

The gravitational force

The gravitational potential energy and forces involving two masses a and b are

$$\begin{aligned} V_{ab} &= -\frac{Gm_a m_b}{|\boldsymbol{r}_a - \boldsymbol{r}_b|}, \\ F_{ba} &= -\frac{Gm_a m_b}{|\boldsymbol{r}_a - \boldsymbol{r}_b|^2}\hat{r}_{ab}, \\ \hat{r}_{ab} &= \frac{\boldsymbol{r}_b - \boldsymbol{r}_a}{|\boldsymbol{r}_a - \boldsymbol{r}_b|}. \end{aligned} \tag{1}$$

Here $G = 6.67 \times 10^{-11}$ $\mathrm{Nm}^2/\mathrm{kg}^2$, and F_{ba} is the force on b due to a. By inspection, one can see that the force on b due to a and the force on a due to b are equal and opposite. The net potential energy for a large number of masses would be

$$V = \sum_{a<b} U_{ab} = \frac{1}{2}\sum_{a\neq b} V_{ab}. \tag{2}$$

Relative and Center of Mass Motion

Thus far, we have considered the trajectory as if the force is centered around a fixed point. For two bodies interacting only with one another, both masses circulate around the center of mass. One might think that solutions would become more complex when both particles move, but we will see here that the problem can be reduced to one with a single body moving according to a fixed force by expressing the trajectories for $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ into the center-of-mass coordinate $\boldsymbol{R}_{\mathrm{cm}}$ and the relative coordinate $\boldsymbol{r}$,

$$\begin{aligned} \boldsymbol{R}_{\mathrm{cm}} &\equiv \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2}, \\ \boldsymbol{r} &\equiv \boldsymbol{r}_1 - \boldsymbol{r}_2. \end{aligned} \tag{3}$$

Relative and Center of Mass Motion, assumptions

Here, we assume the two particles interact only with one another, so $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$ (where $\boldsymbol{F}_{ij}$ is the force on i due to j. The equations of motion then become

$$\begin{aligned}
\ddot{\boldsymbol{R}}_{\rm cm} &= \frac{1}{m_1+m_2}\{m_1\ddot{\boldsymbol{r}}_1 + m_2\ddot{\boldsymbol{r}}_2\} \qquad (4)\\
&= \frac{1}{m_1+m_2}\{\boldsymbol{F}_{12} + \boldsymbol{F}_{21}\} = 0.\\
\ddot{\boldsymbol{r}} &= \ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2 = \left(\frac{\boldsymbol{F}_{12}}{m_1} - \frac{\boldsymbol{F}_{21}}{m_2}\right) \qquad (5)\\
&= \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\boldsymbol{F}_{12}.
\end{aligned}$$

The first expression simply states that the center of mass coordinate $\boldsymbol{R}_{\rm cm}$ moves at a fixed velocity. The second expression can be rewritten in terms of the reduced mass μ.

$$\begin{aligned}
\mu\ddot{\boldsymbol{r}} &= \boldsymbol{F}_{12}, \qquad (6)\\
\frac{1}{\mu} &= \frac{1}{m_1} + \frac{1}{m_2}, \quad \mu = \frac{m_1 m_2}{m_1+m_2}. \qquad (7)
\end{aligned}$$

Trajectory as a one-body problem

Thus, one can treat the trajectory as a one-body problem where the reduced mass is μ, and a second trivial problem for the center of mass. The reduced mass is especially convenient when one is considering gravitational problems because then

$$\begin{aligned}
\mu\ddot{r} &= -\frac{Gm_1m_2}{r^2}\hat{r} \qquad (8)\\
&= -\frac{GM\mu}{r^2}\hat{r}, \quad M \equiv m_1 + m_2.
\end{aligned}$$

For the gravitational problem, the reduced mass then falls out and the trajectory depends only on the total mass M.

The kinetic energy and momenta also have analogues in center-of-mass coordinates. The total and relative momenta are

$$\begin{aligned}
\boldsymbol{P} &\equiv \boldsymbol{p}_1 + \boldsymbol{p}_2 = M\dot{\boldsymbol{R}}_{\rm cm}, \qquad (9)\\
\boldsymbol{q} &\equiv \mu\dot{\boldsymbol{r}}.
\end{aligned}$$

Kinetic energy

With these definitions, a little algebra shows that the kinetic energy becomes

$$\begin{aligned} K &= \frac{1}{2}m_1|\boldsymbol{v}_1|^2 + \frac{1}{2}m_2|\boldsymbol{v}_2|^2 \\ &= \frac{1}{2}M|\dot{\boldsymbol{R}}_{\rm cm}|^2 + \frac{1}{2}\mu|\dot{\boldsymbol{r}}|^2 \\ &= \frac{P^2}{2M} + \frac{q^2}{2\mu}. \end{aligned} \tag{10}$$

The standard strategy is to transform into the center of mass frame, then treat the problem as one of a single particle of mass μ undergoing a force $\boldsymbol{F}_{12}$. Scattering angles can also be expressed in this frame, then transformed into the lab frame. In practice, one sees examples in the literature where $d\sigma/d\Omega$ expressed in both the "center-of-mass" and in the "laboratory" frame.

Deriving Elliptical Orbits

Kepler's laws state that a gravitational orbit should be an ellipse with the source of the gravitational field at one focus. Deriving this is surprisingly messy. To do this, we first use angular momentum conservation to transform the equations of motion so that it is in terms of r and θ instead of r and t. The overall strategy is to

1. Find equations of motion for r and t with no angle (θ) mentioned, i.e. $d^2r/dt^2 = \cdots$. Angular momentum conservation will be used, and the equation will involve the angular momentum L.

2. Use angular momentum conservation to find an expression for $\dot{\theta}$ in terms of r.

3. Use the chain rule to convert the equations of motions for r, an expression involving $r, \dot{r}$ and $\ddot{r}$, to one involving $r, dr/d\theta$ and $d^2r/d\theta^2$. This is quitecomplicated because the expressions will also involve a substitution $u = 1/r$ so that one finds an expression in terms of u and θ.

4. Once $u(\theta)$ is found, you need to show that this can be converted to the familiar form for an ellipse.

Euqations of motion

The equations of motion give

$$
\begin{aligned}
\frac{d}{dt}r^2 &= \frac{d}{dt}(x^2+y^2) = 2x\dot{x} + 2y\dot{y} = 2r\dot{r}, \qquad (11)\\
\dot{r} &= \frac{x}{r}\dot{x} + \frac{y}{r}\dot{y},\\
\ddot{r} &= \frac{x}{r}\ddot{x} + \frac{y}{r}\ddot{y} + \frac{\dot{x}^2+\dot{y}^2}{r} - \frac{\dot{r}^2}{r}.
\end{aligned}
$$

Reordering the equations

Recognizing that the numerator of the third term is the velocity squared, and that it can be written in polar coordinates,

$$
v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2, \qquad (12)
$$

one can write $\ddot{r}$ as

$$
\begin{aligned}
\ddot{r} &= \frac{F_x\cos\theta + F_y\sin\theta}{m} + \frac{\dot{r}^2 + r^2\dot{\theta}^2}{r} - \frac{\dot{r}^2}{r} \qquad (13)\\
&= \frac{F}{m} + \frac{r^2\dot{\theta}^2}{r}\\
m\ddot{r} &= F + \frac{L^2}{mr^3}.
\end{aligned}
$$

Force depends on r only

This derivation used the fact that the force was radial, $F = F_r = F_x\cos\theta + F_y\sin\theta$, and that angular momentum is $L = mrv_\theta = mr^2\dot{\theta}$. The term $L^2/mr^3 = mv^2/r$ behaves like an additional force. Sometimes this is referred to as a centrifugal force, but it is not a force. Instead, it is the consequence of considering the motion in a rotating (and therefore accelerating) frame.

Now, we switch to the particular case of an attractive inverse square force, $F = -\alpha/r^2$, and show that the trajectory, $r(\theta)$, is an ellipse. To do this we transform derivatives w.r.t. time to derivatives w.r.t. θ using the chain rule combined with angular momentum conservation, $\dot{\theta} = L/mr^2$.

$$
\begin{aligned}
\dot{r} &= \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{L}{mr^2}, \qquad (14)\\
\ddot{r} &= \frac{d^2r}{d\theta^2}\dot{\theta}^2 + \frac{dr}{d\theta}\left(\frac{d}{dr}\frac{L}{mr^2}\right)\dot{r}\\
&= \frac{d^2r}{d\theta^2}\left(\frac{L}{mr^2}\right)^2 - 2\frac{dr}{d\theta}\frac{L}{mr^3}\dot{r}\\
&= \frac{d^2r}{d\theta^2}\left(\frac{L}{mr^2}\right)^2 - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2\left(\frac{L}{mr^2}\right)^2
\end{aligned}
$$

Further manipulations

Equating the two expressions for $\ddot{r}$ in Eq.s (13) and (14) eliminates all the derivatives w.r.t. time, and provides a differential equation with only derivatives w.r.t. θ,

$$\frac{d^2r}{d\theta^2}\left(\frac{L}{mr^2}\right)^2 - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2\left(\frac{L}{mr^2}\right)^2 = \frac{F}{m} + \frac{L^2}{m^2r^3}, \tag{15}$$

that when solved yields the trajectory, i.e. $r(\theta)$. Up to this point the expressions work for any radial force, not just forces that fall as $1/r^2$.

Final manipulations, part 1

The trick to simplifying this differential equation for the inverse square problems is to make a substitution, $u \equiv 1/r$, and rewrite the differential equation for $u(\theta)$.

$$\begin{aligned} r &= 1/u, \qquad (16) \\ \frac{dr}{d\theta} &= -\frac{1}{u^2}\frac{du}{d\theta}, \\ \frac{d^2r}{d\theta^2} &= \frac{2}{u^3}\left(\frac{du}{d\theta}\right)^2 - \frac{1}{u^2}\frac{d^2u}{d\theta^2}. \end{aligned}$$

Plugging these expressions into Eq. (15) gives an expression in terms of u, $du/d\theta$, and $d^2u/d\theta^2$. After some tedious algebra,

$$\frac{d^2u}{d\theta^2} = -u - \frac{Fm}{L^2u^2}. \tag{17}$$

Final manipulations, part 2

For the attractive inverse square law force, $F = -\alpha u^2$,

$$\frac{d^2u}{d\theta^2} = -u + \frac{m\alpha}{L^2}. \tag{18}$$

The solution has two arbitrary constants, A and θ_0,

$$\begin{aligned} u &= \frac{m\alpha}{L^2} + A\cos(\theta-\theta_0), \qquad (19) \\ r &= \frac{1}{(m\alpha/L^2) + A\cos(\theta-\theta_0)}. \end{aligned}$$

The radius will be at a minimum when $\theta = \theta_0$ and at a maximum when $\theta = \theta_0 + \pi$. The constant A is related to the eccentricity of the orbit. When $A = 0$ the radius is a constant $r = L^2/(m\alpha)$, and the motion is circular. If one solved the expression $mv^2/r = -\alpha/r^2$ for a circular orbit, using the substitution $v = L/(mr)$, one would reproduce the expression $r = L^2/(m\alpha)$.

Final manipulations, part 3

The form describing the elliptical trajectory in Eq. (19) can be identified as an ellipse with one focus being the center of the ellipse by considering the definition of an ellipse as being the points such that the sum of the two distances between the two foci are a constant. Making that distance $2D$, the distance between the two foci as $2a$, and putting one focus at the origin,

$$\begin{aligned}
2D &= r + \sqrt{(r\cos\theta - 2a)^2 + r^2\sin^2\theta}, \\
4D^2 + r^2 - 4Dr &= r^2 + 4a^2 - 4ar\cos\theta, \\
r &= \frac{D^2 - a^2}{D + a\cos\theta} = \frac{1}{D/(D^2-a^2) - a\cos\theta/(D^2-a^2)}.
\end{aligned} \tag{20}$$

By inspection, this is the same form as Eq. (19) with $D/(D^2 - a^2) = m\alpha/L^2$ and $a/(D^2 - a^2) = A$.

Ellipse reminder

Let us remind ourselves about what an ellipse is before we proceed.

```
import numpy as np
from matplotlib import pyplot as plt
from math import pi

u=1.     #x-position of the center
v=0.5    #y-position of the center
a=2.     #radius on the x-axis
b=1.5    #radius on the y-axis

t = np.linspace(0, 2*pi, 100)
plt.plot( u+a*np.cos(t) , v+b*np.sin(t) )
plt.grid(color='lightgray',linestyle='--')
plt.show()
```

Effective or Centrifugal Potential

The total energy of a particle is

$$\begin{aligned}
E &= V(r) + \frac{1}{2}mv_\theta^2 + \frac{1}{2}m\dot{r}^2 \\
&= V(r) + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}m\dot{r}^2 \\
&= V(r) + \frac{L^2}{2mr^2} + \frac{1}{2}m\dot{r}^2.
\end{aligned} \tag{21}$$

The second term then contributes to the energy like an additional repulsive potential. The term is sometimes referred to as the "centrifugal" potential, even though it is actually the kinetic energy of the angular motion. Combined with $V(r)$, it is sometimes referred to as the "effective" potential,

$$V_{\text{eff}}(r) = V(r) + \frac{L^2}{2mr^2}. \tag{22}$$

Note that if one treats the effective potential like a real potential, one would expect to be able to generate an effective force,

$$\begin{aligned} F_{\text{eff}} &= -\frac{d}{dr}V(r) - \frac{d}{dr}\frac{L^2}{2mr^2} \\ &= F(r) + \frac{L^2}{mr^3} = F(r) + m\frac{v_\perp^2}{r}, \end{aligned} \tag{23}$$

which is indeed matches the form for $m\ddot{r}$ in Eq. (13), which included the **centrifugal** force.

Code example

The following code plots this effective potential for a simple choice of parameters, with a standard gravitational potential $-\alpha/r$. Here we have chosen $L = m = \alpha = 1$.

```
# Common imports
import numpy as np
from math import *
import matplotlib.pyplot as plt

Deltax = 0.01
#set up arrays
xinitial = 0.3
xfinal = 5.0
alpha = 1.0   # spring constant
m = 1.0   # mass, you can change these
AngMom = 1.0  #  The angular momentum
n = ceil((xfinal-xinitial)/Deltax)
x = np.zeros(n)
for i in range(n):
    x[i] = xinitial+i*Deltax
V = np.zeros(n)
V = -alpha/x+0.5*AngMom*AngMom/(m*x*x)
# Plot potential
fig, ax = plt.subplots()
ax.set_xlabel('r[m]')
ax.set_ylabel('V[J]')
ax.plot(x, V)
fig.tight_layout()
plt.show()
```

Gravitational force example

Using the above parameters, we can now study the evolution of the system using for example the velocity Verlet method. This is done in the code here for an initial radius equal to the minimum of the potential well. We seen then that the radius is always the same and corresponds to a circle (the radius is always constant).

```
# Common imports
import numpy as np
import pandas as pd
from math import *
import matplotlib.pyplot as plt
import os

# Where to save the figures and data files
PROJECT_ROOT_DIR = "Results"
FIGURE_ID = "Results/FigureFiles"
DATA_ID = "DataFiles/"

if not os.path.exists(PROJECT_ROOT_DIR):
    os.mkdir(PROJECT_ROOT_DIR)

if not os.path.exists(FIGURE_ID):
    os.makedirs(FIGURE_ID)

if not os.path.exists(DATA_ID):
    os.makedirs(DATA_ID)

def image_path(fig_id):
    return os.path.join(FIGURE_ID, fig_id)

def data_path(dat_id):
    return os.path.join(DATA_ID, dat_id)

def save_fig(fig_id):
    plt.savefig(image_path(fig_id) + ".png", format='png')


# Simple Gravitational Force   -alpha/r

DeltaT = 0.01
#set up arrays
tfinal = 100.0
n = ceil(tfinal/DeltaT)
# set up arrays for t, v and r
t = np.zeros(n)
v = np.zeros(n)
r = np.zeros(n)
# Constants of the model, setting all variables to one for simplicity
alpha = 1.0
AngMom = 1.0  #  The angular momentum
m = 1.0  # scale mass to one
c1 = AngMom*AngMom/(m*m)
c2 = AngMom*AngMom/m
rmin = (AngMom*AngMom/m/alpha)
# Initial conditions
r0 = rmin
v0 = 0.0
r[0] = r0
v[0] = v0
# Start integrating using the Velocity-Verlet  method
for i in range(n-1):
    # Set up acceleration
    a = -alpha/(r[i]**2)+c1/(r[i]**3)
    # update velocity, time and position using the Velocity-Verlet method
```

```
    r[i+1] = r[i] + DeltaT*v[i]+0.5*(DeltaT**2)*a
    anew = -alpha/(r[i+1]**2)+c1/(r[i+1]**3)
    v[i+1] = v[i] + 0.5*DeltaT*(a+anew)
    t[i+1] = t[i] + DeltaT
    # Plot position as function of time
fig, ax = plt.subplots(2,1)
ax[0].set_xlabel('time')
ax[0].set_ylabel('radius')
ax[0].plot(t,r)
ax[1].set_xlabel('time')
ax[1].set_ylabel('Velocity')
ax[1].plot(t,v)
save_fig("RadialGVV")
plt.show()
```

Changing the value of the initial position to a value where the energy is positive, leads to an increasing radius with time, a so-called unbound orbit. Choosing on the other hand an initial radius that corresponds to a negative energy and different from the minimum value leads to a radius that oscillates back and forth between two values.

Harmonic Oscillator in two dimensions

Consider a particle of mass m in a 2-dimensional harmonic oscillator with potential

$$V = \frac{1}{2}kr^2 = \frac{1}{2}k(x^2 + y^2).$$

If the orbit has angular momentum L, we can find the radius and angular velocity of the circular orbit as well as the b) the angular frequency of small radial perturbations.

We consider the effective potential. The radius of a circular orbit is at the minimum of the potential (where the effective force is zero). The potential is plotted here with the parameters $k = m = 0.1$ and $L = 1.0$.

```
# Common imports
import numpy as np
from math import *
import matplotlib.pyplot as plt

Deltax = 0.01
#set up arrays
xinitial = 0.5
xfinal = 3.0
k = 1.0   # spring constant
m = 1.0   # mass, you can change these
AngMom = 1.0  #  The angular momentum
n = ceil((xfinal-xinitial)/Deltax)
x = np.zeros(n)
for i in range(n):
    x[i] = xinitial+i*Deltax
V = np.zeros(n)
V = 0.5*k*x*x+0.5*AngMom*AngMom/(m*x*x)
# Plot potential
fig, ax = plt.subplots()
```

```
ax.set_xlabel('r[m]')
ax.set_ylabel('V[J]')
ax.plot(x, V)
fig.tight_layout()
plt.show()
```

$$V_{\text{eff}} = \frac{1}{2}kr^2 + \frac{L^2}{2mr^2}$$

Harmonic oscillator in two dimensions and effective potential

The effective potential looks like that of a harmonic oscillator for large r, but for small r, the centrifugal potential repels the particle from the origin. The combination of the two potentials has a minimum for at some radius r_{min}.

$$\begin{aligned}
0 &= kr_{\text{min}} - \frac{L^2}{mr_{\text{min}}^3}, \\
r_{\text{min}} &= \left(\frac{L^2}{mk}\right)^{1/4}, \\
\dot{\theta} &= \frac{L}{mr_{\text{min}}^2} = \sqrt{k/m}.
\end{aligned}$$

For particles at r_{min} with $\dot{r} = 0$, the particle does not accelerate and r stays constant, i.e. a circular orbit. The radius of the circular orbit can be adjusted by changing the angular momentum L.

For the above parameters this minimum is at $r_{\text{min}} = 1$.

Now consider small vibrations about r_{min}. The effective spring constant is the curvature of the effective potential.

$$\begin{aligned}
k_{\text{eff}} &= \left.\frac{d^2}{dr^2}V_{\text{eff}}(r)\right|_{r=r_{\text{min}}} = k + \frac{3L^2}{mr_{\text{min}}^4} \\
&= 4k, \\
\omega &= \sqrt{k_{\text{eff}}/m} = 2\sqrt{k/m} = 2\dot{\theta}.
\end{aligned}$$

Because the radius oscillates with twice the angular frequency, the orbit has two places where r reaches a minimum in one cycle. This differs from the inverse-square force where there is one minimum in an orbit. One can show that the orbit for the harmonic oscillator is also elliptical, but in this case the center of the potential is at the center of the ellipse, not at one of the foci.

The solution is also simple to write down exactly in Cartesian coordinates. The x and y equations of motion separate,

$$\begin{aligned}
\ddot{x} &= -kx, \\
\ddot{y} &= -ky.
\end{aligned}$$

The general solution can be expressed as

$$
\begin{aligned}
x &= A\cos\omega_0 t + B\sin\omega_0 t, \\
y &= C\cos\omega_0 t + D\sin\omega_0 t.
\end{aligned}
$$

The code here finds the solution for x and y using the code we developed in homework 5 and 6 and the midterm. Note that this code is tailored to run in Cartesian coordinates. There is thus no angular momentum dependent term.

Here we have chose initial conditions that correspond to the minimum of the effective potential r_{min}. We have chosen $x_0 = r_{\mathrm{min}}$ and $y_0 = 0$. Similarly, we use the centripetal acceleration to determine the initial velocity so that we have a circular motion (see back to the last question of the midterm). This means that we set the centripetal acceleration v^2/r equal to the force from the harmonic oscillator $-k\boldsymbol{r}$. Taking the magnitude of $\boldsymbol{r}$ we have then $v^2/r = k/mr$, which gives $v = \pm\omega_0 r$.

Since the code here solves the equations of motion in cartesian coordinates and the harmonic oscillator potential leads to forces in the x- and y-directions that are decoupled, we have to select the initial velocities and positions so that we don't get that for example $y(t) = 0$.

We set x_0 to be different from zero and v_{y0} to be different from zero.

```
DeltaT = 0.001
#set up arrays
tfinal = 10.0
n = ceil(tfinal/DeltaT)
# set up arrays
t = np.zeros(n)
v = np.zeros((n,2))
r = np.zeros((n,2))
radius = np.zeros(n)
# Constants of the model
k = 1.0   # spring constant
m = 1.0   # mass, you can change these
omega02 = k/m  # Frequency
AngMom = 1.0  #  The angular momentum
# Potential minimum
rmin = (AngMom*AngMom/k/m)**0.25
# Initial conditions as compact 2-dimensional arrays, x0=rmin and y0 = 0
x0 = rmin; y0= 0.0
r0 = np.array([x0,y0])
vy0 = sqrt(omega02)*rmin; vx0 = 0.0
v0 = np.array([vx0,vy0])
r[0] = r0
v[0] = v0
# Start integrating using the Velocity-Verlet  method
for i in range(n-1):
    # Set up the acceleration
    a =  -r[i]*omega02
    # update velocity, time and position using the Velocity-Verlet method
    r[i+1] = r[i] + DeltaT*v[i]+0.5*(DeltaT**2)*a
    anew = -r[i+1]*omega02
    v[i+1] = v[i] + 0.5*DeltaT*(a+anew)
    t[i+1] = t[i] + DeltaT
```

```python
# Plot position as function of time
radius = np.sqrt(r[:,0]**2+r[:,1]**2)
fig, ax = plt.subplots(3,1)
ax[0].set_xlabel('time')
ax[0].set_ylabel('radius squared')
ax[0].plot(t,r[:,0]**2+r[:,1]**2)
ax[1].set_xlabel('time')
ax[1].set_ylabel('x position')
ax[1].plot(t,r[:,0])
ax[2].set_xlabel('time')
ax[2].set_ylabel('y position')
ax[2].plot(t,r[:,1])

fig.tight_layout()
save_fig("2DimHOVV")
plt.show()
```

We see that the radius (to within a given error), we obtain a constant radius.

The following code shows first how we can solve this problem using the radial degrees of freedom only. Here we need to add the explicit centrifugal barrier. Note that the variable r depends only on time. There is no x and y directions since we have transformed the equations to polar coordinates.

```python
DeltaT = 0.01
#set up arrays
tfinal = 10.0
n = ceil(tfinal/DeltaT)
# set up arrays for t, v and r
t = np.zeros(n)
v = np.zeros(n)
r = np.zeros(n)
E = np.zeros(n)
# Constants of the model
AngMom = 1.0  #  The angular momentum
m = 1.0
k = 1.0
omega02 = k/m
c1 = AngMom*AngMom/(m*m)
c2 = AngMom*AngMom/m
rmin = (AngMom*AngMom/k/m)**0.25
# Initial conditions
r0 = rmin
v0 = 0.0
r[0] = r0
v[0] = v0
E[0] = 0.5*m*v0*v0+0.5*k*r0*r0+0.5*c2/(r0*r0)
# Start integrating using the Velocity-Verlet  method
for i in range(n-1):
    # Set up acceleration
    a = -r[i]*omega02+c1/(r[i]**3)
    # update velocity, time and position using the Velocity-Verlet method
    r[i+1] = r[i] + DeltaT*v[i]+0.5*(DeltaT**2)*a
    anew = -r[i+1]*omega02+c1/(r[i+1]**3)
    v[i+1] = v[i] + 0.5*DeltaT*(a+anew)
    t[i+1] = t[i] + DeltaT
    E[i+1] = 0.5*m*v[i+1]*v[i+1]+0.5*k*r[i+1]*r[i+1]+0.5*c2/(r[i+1]*r[i+1])
    # Plot position as function of time
fig, ax = plt.subplots(2,1)
ax[0].set_xlabel('time')
```

```
ax[0].set_ylabel('radius')
ax[0].plot(t,r)
ax[1].set_xlabel('time')
ax[1].set_ylabel('Energy')
ax[1].plot(t,E)
save_fig("RadialHOVV")
plt.show()
```

With some work using double angle formulas, one can calculate

$$
\begin{aligned}
r^2 &= x^2 + y^2 \\
&= (A^2 + C^2)\cos^2(\omega_0 t) + (B^2 + D^2)\sin^2\omega_0 t + (AB + CD)\cos(\omega_0 t)\sin(\omega_0 t) \\
&= \alpha + \beta\cos 2\omega_0 t + \gamma\sin 2\omega_0 t, \\
\alpha &= \frac{A^2 + B^2 + C^2 + D^2}{2}, \quad \beta = \frac{A^2 - B^2 + C^2 - D^2}{2}, \quad \gamma = AB + CD, \\
r^2 &= \alpha + (\beta^2 + \gamma^2)^{1/2}\cos(2\omega_0 t - \delta), \quad \delta = \arctan(\gamma/\beta),
\end{aligned}
$$

and see that radius oscillates with frequency $2\omega_0$. The factor of two comes because the oscillation $x = A\cos\omega_0 t$ has two maxima for x^2, one at $t = 0$ and one a half period later.

Stability of Orbits

The effective force can be extracted from the effective potential, $V_{\rm eff}$. Beginning from the equations of motion, Eq. (11), for r,

$$
\begin{aligned}
m\ddot{r} &= F + \frac{L^2}{mr^3} \qquad (24) \\
&= F_{\rm eff} \\
&= -\partial_r V_{\rm eff}, \\
F_{\rm eff} &= -\partial_r\left[V(r) + (L^2/2mr^2)\right].
\end{aligned}
$$

For a circular orbit, the radius must be fixed as a function of time, so one must be at a maximum or a minimum of the effective potential. However, if one is at a maximum of the effective potential the radius will be unstable. For the attractive Coulomb force the effective potential will be dominated by the $-\alpha/r$ term for large r because the centrifugal part falls off more quickly, $\sim 1/r^2$. At low r the centrifugal piece wins and the effective potential is repulsive. Thus, the potential must have a minimum somewhere with negative potential. The circular orbits are then stable to perturbation.

The effective potential is sketched for two cases, a $1/r$ attractive potential and a $1/r^3$ attractive potential. The $1/r$ case has a stable minimum, whereas the circular orbit in the $1/r^3$ case is unstable.

If one considers a potential that falls as $1/r^3$, the situation is reversed and the point where $\partial_r V$ disappears will be a local maximum rather than a local minimum. **Fig to come here with code**

The repulsive centrifugal piece dominates at large r and the attractive Coulomb piece wins out at small r. The circular orbit is then at a maximum of the effective potential and the orbits are unstable. It is the clear that for potentials that fall as r^n, that one must have $n > -2$ for the orbits to be stable.

Consider a potential $V(r) = \beta r$. For a particle of mass m with angular momentum L, find the angular frequency of a circular orbit. Then find the angular frequency for small radial perturbations.

For the circular orbit you search for the position $r_{\rm min}$ where the effective potential is minimized,

$$
\begin{aligned}
\partial_r \left\{ \beta r + \frac{L^2}{2mr^2} \right\} &= 0, \\
\beta &= \frac{L^2}{mr_{\rm min}^3}, \\
r_{\rm min} &= \left(\frac{L^2}{\beta m} \right)^{1/3}, \\
\dot{\theta} &= \frac{L}{mr_{\rm min}^2} = \frac{\beta^{2/3}}{(mL)^{1/3}}
\end{aligned}
$$

Now, we can find the angular frequency of small perturbations about the circular orbit. To do this we find the effective spring constant for the effective potential,

$$
\begin{aligned}
k_{\rm eff} &= \partial_r^2 \left. V_{\rm eff} \right|_{r_{\rm min}} \\
&= \frac{3L^2}{mr_{\rm min}^4}, \\
\omega &= \sqrt{\frac{k_{\rm eff}}{m}} \\
&= \frac{\beta^{2/3}}{(mL)^{1/3}} \sqrt{3}.
\end{aligned}
$$

If the two frequencies, $\dot{\theta}$ and ω, differ by an integer factor, the orbit's trajectory will repeat itself each time around. This is the case for the inverse-square force, $\omega = \dot{\theta}$, and for the harmonic oscillator, $\omega = 2\dot{\theta}$. In this case, $\omega = \sqrt{3}\dot{\theta}$, and the angles at which the maxima and minima occur change with each orbit.

Code example with gravitional force. The code example here is meant to illustrate how we can make a plot of the final orbit. We solve the equations in polar coordinates (the example here uses the minimum of the potential as initial value) and then we transform back to cartesian coordinates and plot x versus y. We see that we get a perfect circle when we place ourselves at the minimum of the potential energy, as expected.

```python
# Simple Gravitational Force   -alpha/r

DeltaT = 0.01
#set up arrays
tfinal = 8.0
n = ceil(tfinal/DeltaT)
# set up arrays for t, v and r
t = np.zeros(n)
v = np.zeros(n)
r = np.zeros(n)
phi = np.zeros(n)
x = np.zeros(n)
y = np.zeros(n)
# Constants of the model, setting all variables to one for simplicity
alpha = 1.0
AngMom = 1.0  #  The angular momentum
m = 1.0  # scale mass to one
c1 = AngMom*AngMom/(m*m)
c2 = AngMom*AngMom/m
rmin = (AngMom*AngMom/m/alpha)
# Initial conditions, place yourself at the potential min
r0 = rmin
v0 = 0.0  # starts at rest
r[0] = r0
v[0] = v0
phi[0] = 0.0
# Start integrating using the Velocity-Verlet  method
for i in range(n-1):
    # Set up acceleration
    a = -alpha/(r[i]**2)+c1/(r[i]**3)
    # update velocity, time and position using the Velocity-Verlet method
    r[i+1] = r[i] + DeltaT*v[i]+0.5*(DeltaT**2)*a
    anew = -alpha/(r[i+1]**2)+c1/(r[i+1]**3)
    v[i+1] = v[i] + 0.5*DeltaT*(a+anew)
    t[i+1] = t[i] + DeltaT
    phi[i+1] = t[i+1]*c2/(r0**2)
# Find cartesian coordinates for easy plot
x = r*np.cos(phi)
y = r*np.sin(phi)
fig, ax = plt.subplots(3,1)
ax[0].set_xlabel('time')
ax[0].set_ylabel('radius')
ax[0].plot(t,r)
ax[1].set_xlabel('time')
ax[1].set_ylabel('Angle $\cos{\phi}$')
ax[1].plot(t,np.cos(phi))
ax[2].set_ylabel('y')
ax[2].set_xlabel('x')
ax[2].plot(x,y)

save_fig("Phasespace")
plt.show()
```

Try to change the initial value for r and see what kind of orbits you get. In order to test different energies, it can be useful to look at the plot of the effective potential discussed above.

However, for orbits different from a circle the above code would need modifications in order to allow us to display say an ellipse. For the latter, it is much easier to run our code in cartesian coordinates, as done here. In this code we test also energy conservation and see that it is conserved to numerical precision. The code here is a simple extension of the code we developed for homework 4.

```
# Common imports
import numpy as np
import pandas as pd
from math import *
import matplotlib.pyplot as plt

DeltaT = 0.01
#set up arrays
tfinal = 10.0
n = ceil(tfinal/DeltaT)
# set up arrays
t = np.zeros(n)
v = np.zeros((n,2))
r = np.zeros((n,2))
E = np.zeros(n)
# Constants of the model
m = 1.0   # mass, you can change these
alpha = 1.0
# Initial conditions as compact 2-dimensional arrays
x0 = 0.5; y0= 0.
r0 = np.array([x0,y0])
v0 = np.array([0.0,1.0])
r[0] = r0
v[0] = v0
rabs = sqrt(sum(r[0]*r[0]))
E[0] = 0.5*m*(v[0,0]**2+v[0,1]**2)-alpha/rabs
# Start integrating using the Velocity-Verlet  method
for i in range(n-1):
    # Set up the acceleration
    rabs = sqrt(sum(r[i]*r[i]))
    a =  -alpha*r[i]/(rabs**3)
    # update velocity, time and position using the Velocity-Verlet method
    r[i+1] = r[i] + DeltaT*v[i]+0.5*(DeltaT**2)*a
    rabs = sqrt(sum(r[i+1]*r[i+1]))
    anew = -alpha*r[i+1]/(rabs**3)
    v[i+1] = v[i] + 0.5*DeltaT*(a+anew)
    E[i+1] = 0.5*m*(v[i+1,0]**2+v[i+1,1]**2)-alpha/rabs
    t[i+1] = t[i] + DeltaT
# Plot position as function of time
fig, ax = plt.subplots(3,1)
ax[0].set_ylabel('y')
ax[0].set_xlabel('x')
ax[0].plot(r[:,0],r[:,1])
ax[1].set_xlabel('time')
ax[1].set_ylabel('y position')
ax[1].plot(t,r[:,0])
ax[2].set_xlabel('time')
ax[2].set_ylabel('y position')
ax[2].plot(t,r[:,1])

fig.tight_layout()
save_fig("2DimGravity")
plt.show()
print(E)
```

Central forces are forces which are directed towards or away from a reference point. A familiar force is the gravitional force with the motion of our Earth around the Sun as a classic. The Sun, being approximately sixth order of magnitude heavier than the Earth serves as our origin. A force like the gravitational force is a function of the relative distance $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ only, where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the positions relative to a defined origin for object one and object two, respectively.

These forces depend on the spatial degrees of freedom only (the positions of the interacting objects/particles). As discussed earlier, from such forces we can infer that the total internal energy, the total linear momentum and total angular momentum are so-called constants of the motion, that is they stay constant over time. We say that energy, linear and anuglar momentum are conserved.

With a scalar potential $V(\boldsymbol{r})$ we define the force as the gradient of the potential

$$\boldsymbol{F}(\boldsymbol{r}) = -\boldsymbol{\nabla} V(\boldsymbol{r}).$$

In general these potentials depend only on the magnitude of the relative position and we will write the potential as $V(r)$ where r is defined as,

$$r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|.$$

In three dimensions our vectors are defined as (for a given object/particle i)

$$\boldsymbol{r}_i = x_i \boldsymbol{e}_1 + y_i \boldsymbol{e}_2 + z_i \boldsymbol{e}_3,$$

while in two dimensions we have

$$\boldsymbol{r}_i = x_i \boldsymbol{e}_1 + y_i \boldsymbol{e}_2.$$

In two dimensions the radius r is defined as

$$r = |\boldsymbol{r}_1 - \boldsymbol{r}_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

If we consider the gravitational potential involving two masses 1 and 2, we have

$$V_{12}(r) = V(r) = -\frac{Gm_1 m_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} = -\frac{Gm_1 m_2}{r}.$$

Calculating the gradient of this potential we obtain the force

$$\boldsymbol{F}(\boldsymbol{r}) = -\frac{Gm_1 m_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_1|^2} \hat{\boldsymbol{r}}_{12} = -\frac{Gm_a m_b}{r^2} \hat{\boldsymbol{r}},$$

where we have the unit vector

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{r}}_{12} = \frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}.$$

Here $G = 6.67 \times 10^{-11}$ $\mathrm{Nm}^2/\mathrm{kg}^2$, and $\boldsymbol{F}$ is the force on 2 due to 1. By inspection, one can see that the force on 2 due to 1 and the force on 1 due to 2 are equal and opposite. The net potential energy for a large number of masses would be

$$V = \sum_{i<j} V_{ij} = \frac{1}{2}\sum_{i\neq j} V_{ij}.$$

In general, the central forces that we will study can be written mathematically as

$$\boldsymbol{F}(\boldsymbol{r}) = f(r)\hat{r},$$

where $f(r)$ is a scalar function. For the above gravitational force this scalar term is $-Gm_1m_2/r^2$. In general we will simply write this scalar function $f(r) = \alpha/r^2$ where α is a constant that can be either negative or positive. We will also see examples of other types of potentials in the examples below.

Besides general expressions for the potentials/forces, we will discuss in detail different types of motion that arise, from circular to elliptical or hyperbolic or parabolic. By transforming to either polar coordinates or spherical coordinates, we will be able to obtain analytical solutions for the equations of motion and thereby obtain new insights about the properties of a system. Where possible, we will compare our analytical equations with numerical studies.

However, before we arrive at these lovely insights, we need to introduce some mathematical manipulations and definitions. We conclude this chapter with a discussion of two-body scattering.

Center of Mass and Relative Coordinates

Thus far, we have considered the trajectory as if the force is centered around a fixed point. For two bodies interacting only with one another, both masses circulate around the center of mass. One might think that solutions would become more complex when both particles move, but we will see here that the problem can be reduced to one with a single body moving according to a fixed force by expressing the trajectories for $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ into the center-of-mass coordinate $\boldsymbol{R}$ and the relative coordinate $\boldsymbol{r}$. We define the center-of-mass (CoM) coordinate as

$$\boldsymbol{R} \equiv \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1+m_2},$$

and the relative coordinate as

$$\boldsymbol{r} \equiv \boldsymbol{r}_1 - \boldsymbol{r_2}.$$

We can then rewrite $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in terms of the relative and CoM coordinates as

$$\boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{M}\boldsymbol{r},$$

and

$$\boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{M}\boldsymbol{r}.$$

Conservation of total Linear Momentum. In our discussions on conservative forces we defined the total linear momentum as

$$\boldsymbol{P} = \sum_{i=1}^{N} m_i \frac{d\boldsymbol{r}_i}{dt},$$

where $N = 2$ in our case. With the above definition of the center of mass position, we see that we can rewrite the total linear momentum as (multiplying the CoM coordinate with M)

$$\boldsymbol{P} = M\frac{d\boldsymbol{R}}{dt} = M\dot{\boldsymbol{R}}.$$

The net force acting on the system is given by the time derivative of the linear momentum (assuming mass is time independent) and we have

$$\boldsymbol{F}^{\mathrm{net}} = \dot{\boldsymbol{P}} = M\ddot{\boldsymbol{R}}.$$

The net force acting on the system is given by the sum of the forces acting on the two bodies, that is we have

$$\boldsymbol{F}^{\mathrm{net}} = \boldsymbol{F}_1 + \boldsymbol{F}_2 = \dot{\boldsymbol{P}} = M\ddot{\boldsymbol{R}}.$$

In our case the forces are given by the internal forces only. The force acting on object 1 is thus $\boldsymbol{F}_{12}$ and the one acting on object 2 is $\boldsymbol{F}_{12}$. We have also defined that $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$. This means thar we have

$$\boldsymbol{F}_1 + \boldsymbol{F}_2 = \boldsymbol{F}_{12} + \boldsymbol{F}_{21} = 0 = \dot{\boldsymbol{P}} = M\ddot{\boldsymbol{R}}.$$

We could alternatively had write this

$$\ddot{\boldsymbol{R}}_{\mathrm{cm}} = \frac{1}{m_1 + m_2}\left\{m_1\ddot{\boldsymbol{r}}_1 + m_2\ddot{\boldsymbol{r}}_2\right\} = \frac{1}{m_1 + m_2}\left\{\boldsymbol{F}_{12} + \boldsymbol{F}_{21}\right\} = 0.$$

This has the important consequence that the CoM velocity is a constant of the motion. And since the total linear momentum is given by the time-derivative of the CoM coordinate times the total mass $M = m_1 + m_2$, it means that linear momentum is also conserved. Stated differently, the center-of-mass coordinate $\boldsymbol{R}$ moves at a fixed velocity.

This has also another important consequence for our forces. If we assume that our force depends only on the relative coordinate, it means that the gradient of the potential with respect to the center of mass position is zero, that is

$$M d\ddot{\boldsymbol{R}} = -\boldsymbol{\nabla}_{\boldsymbol{R}} V = 0!$$

If we now switch to the equation of motion for the relative coordinate, we have

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2 = \left(\frac{\boldsymbol{F}_{12}}{m_1} - \frac{\boldsymbol{F}_{21}}{m_2}\right) = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\boldsymbol{F}_{12},$$

which we can rewrite in terms of the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2},$$

as

$$\mu \ddot{\boldsymbol{r}} = \boldsymbol{F}_{12}.$$

This has a very important consequence for our coming analysis of the equations of motion for the two-body problem. Since the acceleration for the CoM coordinate is zero, we can now treat the trajectory as a one-body problem where the mass is given by the reduced mass μ plus a second trivial problem for the center of mass. The reduced mass is especially convenient when one is considering forces that depend only on the relative coordinate (like the Gravitational force or the electrostatic force between two charges) because then for say the gravitational force we have

$$\mu \ddot{\boldsymbol{r}} = -\frac{G m_1 m_2}{r^2} \hat{\boldsymbol{r}} = -\frac{G M \mu}{r^2} \hat{\boldsymbol{r}},$$

where we have defined $M = m_1 + m_2$. It means that the acceleration of the relative coordinate is

$$\ddot{\boldsymbol{r}} = -\frac{GM}{r^2} \hat{\boldsymbol{r}},$$

and we have that for the gravitational problem, the reduced mass then falls out and the trajectory depends only on the total mass M.

The standard strategy is to transform into the center of mass frame, then treat the problem as one of a single particle of mass μ undergoing a force $\boldsymbol{F}_{12}$. Scattering angles, see our discussion of scattering problems below, can also be expressed in this frame. Before we proceed to our definition of the CoM frame we need to set up the expression for the energy in terms of the relative and CoM coordinates.

Kinetic and total Energy. The kinetic energy and momenta also have analogues in center-of-mass coordinates. We have defined the total linear momentum as

$$\boldsymbol{P} = \sum_{i=1}^{N} m_i \frac{d\boldsymbol{r}_i}{dt} = M \dot{\boldsymbol{R}}.$$

For the relative momentum $\boldsymbol{q}$, we have that the time derivative of $\boldsymbol{r}$ is

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{r}}_2,$$

We know also that the momenta $\boldsymbol{p}_1 = m_1 \dot{\boldsymbol{r}}_1$ and $\boldsymbol{p}_2 = m_2 \dot{\boldsymbol{r}}_2$. Using these expressions we can rewrite

$$\dot{\boldsymbol{r}} = \frac{\boldsymbol{p}_1}{m_1} - \frac{\boldsymbol{p}_2}{m_2},$$

which gives

$$\dot{\boldsymbol{r}} = \frac{m_2\boldsymbol{p}_1 - m_1\boldsymbol{p}_2}{m_1 m_2},$$

and dividing both sides with M we have

$$\frac{m_1 m_2}{M}\dot{\boldsymbol{r}} = \frac{m_2\boldsymbol{p}_1 - m_1\boldsymbol{p}_2}{M}.$$

Introducing the reduced mass $\mu = m_1 m_2/M$ we have finally

$$\mu\dot{\boldsymbol{r}} = \frac{m_2\boldsymbol{p}_1 - m_1\boldsymbol{p}_2}{M}.$$

And $\mu\dot{\boldsymbol{r}}$ defines the relative momentum $\boldsymbol{q} = \mu\dot{\boldsymbol{r}}$.

With these definitions we can then calculate the kinetic energy in terms of the relative and CoM coordinates.

We have that

$$K = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2},$$

and with $\boldsymbol{p}_1 = m_1\dot{\boldsymbol{r}}_1$ and $\boldsymbol{p}_2 = m_2\dot{\boldsymbol{r}}_2$ and using

$$\dot{\boldsymbol{r}}_1 = \dot{\boldsymbol{R}} + \frac{m_2}{M}\dot{\boldsymbol{r}},$$

and

$$\dot{\boldsymbol{r}}_2 = \dot{\boldsymbol{R}} - \frac{m_1}{M}\dot{\boldsymbol{r}},$$

we obtain after squaring the expressions for $\dot{\boldsymbol{r}}_1$ and $\dot{\boldsymbol{r}}_2$

$$K = \frac{(m_1+m_2)\dot{\boldsymbol{R}}^2}{2} + \frac{(m_1+m_2)m_1 m_2\dot{\boldsymbol{r}}^2}{2M^2},$$

which we simplify to

$$K = \frac{\dot{\boldsymbol{P}}^2}{2M} + \frac{\mu\dot{\boldsymbol{q}}^2}{2}.$$

Below we will define a reference frame, the so-called CoM-frame, where $\boldsymbol{R} = 0$. This is going to simplify our equations further.

Conservation of Angular Momentum. The angular momentum (the total one) is the sum of the individual angular momenta. In our case we have two bodies only, meaning that our angular momentum is defined as

$$\boldsymbol{L} = \boldsymbol{r}_1 \times \boldsymbol{p}_1 + \boldsymbol{r}_2 \times \boldsymbol{p}_2,$$

and using that $m_1\dot{\boldsymbol{r}}_1 = \boldsymbol{p}_1$ and $m_2\dot{\boldsymbol{r}}_2 = \boldsymbol{p}_2$ we have

$$\boldsymbol{L} = m_1\boldsymbol{r}_1 \times \dot{\boldsymbol{r}}_1 + m_2\boldsymbol{r}_2 \times \dot{\boldsymbol{r}}_2.$$

We define now the CoM-Frame where we set $\boldsymbol{R}=0$. This means that the equations for $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in terms of the relative motion simplify and we have

$$\boldsymbol{r}_1=\frac{m_2}{M}\boldsymbol{r},$$

and

$$\boldsymbol{r}_2=-\frac{m_1}{M}\boldsymbol{r}.$$

resulting in

$$\boldsymbol{L}=m_1\frac{m_2}{M}\boldsymbol{r}\times\frac{m_2}{M}\boldsymbol{r}+m_2\frac{m_1}{M}\boldsymbol{r}\times\frac{m_1}{M}\dot{\boldsymbol{r}}.$$

We see that can rewrite this equation as

$$\boldsymbol{L}=\boldsymbol{r}\times\mu\dot{\boldsymbol{r}}=\mu\boldsymbol{r}\times\dot{\boldsymbol{r}}.$$

If we now use a central force, we have that

$$\mu\dot{\boldsymbol{r}}=\boldsymbol{F}(\boldsymbol{r})=f(r)\hat{\boldsymbol{r}},$$

and inserting this in the equation for the angular momentum we have

$$\boldsymbol{L}=\boldsymbol{r}\times f(r)\hat{\boldsymbol{r}},$$

which equals zero since we are taking the cross product of the vector $\boldsymbol{r}$ with itself. Angular momentum is thus conserved and in addition to the total linear momentum being conserved, we know that energy is also conserved with forces that depend only on position and the relative coordinate only.

Since angular momentum is conserved, we can idealize the motion of our two objects as two bodies moving in a plane spanned by the relative coordinate and the relative momentum. The angular momentum is perpendicular to the plane spanned by these two vectors.

It means also, since $\boldsymbol{L}$ is conserved, that we can reduce our problem to a motion in say the xy-plane. What we have done then is to reduce a two-body problem in three-dimensions with six degrees of freedom (the six coordinates of the two objects) to a problem defined entirely by the relative coordinate in two dimensions. We have thus moved from a problem with six degrees of freedom to one with two degrees of freedom only.

Since we deal with central forces that depend only on the relative coordinate, we will show below that transforming to polar coordinates, we cna find analytical solution to the equation of motion

$$\mu\dot{\boldsymbol{r}}=\boldsymbol{F}(\boldsymbol{r})=f(r)\hat{\boldsymbol{r}}.$$

Note the boldfaced symbols for the relative position $\boldsymbol{r}$. Our vector $\boldsymbol{r}$ is defined as

$$\boldsymbol{r}=x\boldsymbol{e}_1+y\boldsymbol{e}_2$$

and introducing polar coordinates $r \in [0, \infty)$ and $\phi \in [0, 2\pi]$ and the transformation

$$r = \sqrt{x^2 + y^2},$$

and $x = r\cos\phi$ and $y = r\sin\phi$, we will rewrite our equation of motion by transforming from Cartesian coordinates to Polar coordinates. By so doing, we end up with two differential equations which can be solved analytically (it depends on the form of the potential).

What follows now is a rewrite of these equations and the introduction of Kepler's laws as well.