conservative forces

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$$E = \underbrace{K}_{\text{kinetic energy}} + \underbrace{V}_{\text{potential energy}}$$

conservative forces $\Rightarrow$ conservation of energy

$$\frac{dE}{dt} = 0$$

(i) $\vec{F} = F(\vec{r})$

(ii) $\vec{\nabla} \times \vec{F}$: 3-dim

$$\vec{i}\left(\frac{\partial F_y}{\partial z} - \frac{\partial}{\partial y}F_z\right) + \vec{j}\left(\frac{\partial}{\partial x}F_z - \frac{\partial}{\partial z}F_x\right)$$

$$+ \vec{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial}{\partial y}F_x\right) = 0$$

(iii) $\vec{F} = -\vec{\nabla} V(\vec{r})$

(ii) $\vec{F} = F(x)\vec{i} = -kx\vec{i}$

$$\vec{\nabla} \times \vec{F} = 0$$

Conservation of linear Momentum

$$\vec{F}_{total} = \sum_{i=1}^{N} m_i \vec{a}_i = \sum_{i=1}^{N} \vec{F}_i$$

$$\vec{F}_i = \vec{F}_i^{ext} + F_i^{int}$$

$$F_i^{int} = \sum_{j \neq i}^{N} \vec{F}_{ij}$$

$$\vec{F}_{ij} = - \frac{G M_{\odot} M_E}{(|\vec{r}_{\underset{i}{\odot}} - \vec{r}_{\underset{j}{E}}|)^3} \cdot (\vec{r}_{\underset{i}{\odot}} - \vec{r}_{\underset{j}{E}})$$

$N = 2$

$$\sum_{i=1}^{N=2} F_i^{int} = \sum_{i=1}^{N=2} \sum_{j \neq i}^{N=2} \vec{F}_{ij}$$

$$\vec{F}_{12} + \vec{F}_{21} = 0$$

Newton's 3rd law

$$\vec{F}_{12} = -\vec{F}_{21}$$

$N = 3$

$$\sum_{i=1}^{3} \sum_{j \neq i}^{3} \vec{F}_{ij} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{23} + \vec{F}_{21} + \vec{F}_{31} + \vec{F}_{32}$$

$$\sum_{i=1}^{N} \vec{F}_i^{\,int} = \sum_{i=1}^{N} \sum_{j \neq i}^{N} \vec{F}_{ij}$$

$$= \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}}^{N} \left[\vec{F}_{ij} + \vec{F}_{ji} \right]$$

$$= \sum_{i=1}^{N} \sum_{j>i}^{N} \left(\vec{F}_{ij} + \vec{F}_{ji} \right)$$

$$\vec{F}_{ij} = - \vec{F}_{ji}$$

$$\sum_{i=1}^{N} \vec{F}_i^{\,int} = 0 \quad !$$

$$\vec{P}^{\,total} = \sum_{i=1}^{N} m_i \vec{v}_i = \sum_{i=1}^{N} \vec{P}_i$$

$$\frac{d\vec{P}^{\,total}}{dt} = \sum_{i=1}^{N} m_i \frac{d\vec{v}_i}{dt}$$

$$= \sum_{i=1}^{N} m_i \vec{a}_i$$

$$= \sum_{i=1}^{N} \vec{F}_i^{\,net}$$

$$\vec{F}_i^{\,net} = \vec{F}_i^{\,ext} + \vec{F}_i^{\,int}$$

$$\vec{F}_i^{net} = F_i^{int}$$

$$\sum_{i=1}^{N} F_i^{int} = 0$$

$$\frac{d\vec{P}^{total}}{dt} = 0 \Rightarrow$$

Momentum is conserved (constant of motion)

if $\vec{F}_i^{net} = \underbrace{\vec{F}_i^{ext}}_{\neq 0} + \vec{F}_i^{int}$

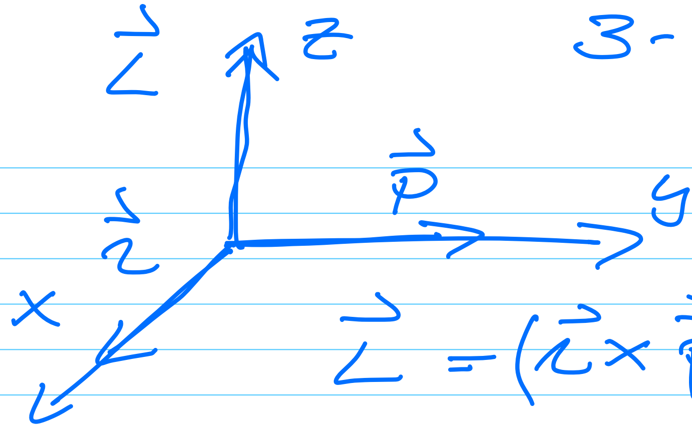
$$\vec{F}^{total} = \sum_{i=1}^{N} \left(\vec{F}_i^{ext} + \vec{F}_i^{int} \right)$$

$$= \sum_{i=1}^{N} \vec{F}_i^{ext} \neq 0$$

$$\Rightarrow \frac{d\vec{P}^{total}}{dt} \neq 0$$

Angular Momentum

3-Dim



$$\vec{L} = (\vec{r} \times \vec{p}) = m(\vec{r} \times \vec{v})$$

$$\frac{d\vec{L}}{dt} = 0 \ ?$$

$$\frac{d}{dt}(a \times b)$$

Exercise 4 from hw 1

$$\frac{dL}{dt} = \underline{m} \left\{ \left(\overbrace{\frac{d\vec{r}}{dt}}^{\vec{v}} \times \vec{v} \right) + \left(\vec{r} \times \underbrace{\frac{d\vec{v}}{dt}}_{\vec{a}} \right) \right\}$$

$$m\vec{a} = \vec{F}$$

$$= m \cdot \underbrace{\vec{v} \times \vec{v}}_{=0} + \vec{r} \times \vec{F}$$

$$\vec{r} \parallel \vec{F} \rightarrow = 0$$

if not, then

$$\frac{d\vec{L}}{dt} = \underbrace{\vec{\tau}}_{\text{Torque}} = \vec{r} \times \vec{F} \neq 0$$

$$\vec{F}_i = \vec{F}_i^{\,total} = \underbrace{\vec{F}_i^{\,ext}}_{=0} + \vec{F}_i^{\,int}$$

$$\vec{L} = \sum_{i=1}^{N} \vec{l}_i \qquad \frac{d\vec{l}_i}{dt} = \vec{r}_i \times \vec{F}_i$$

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{N} \frac{d\vec{l}_i}{dt} = \sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i$$

(internal forces only)

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{N} \vec{r}_i \times \Big(\sum_{j \neq i}^{N} \vec{F}_{ij} \Big)$$

$$= \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \Big\{ \vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \underbrace{\vec{F}_{ji}}_{-\vec{F}_{ij}} \Big\}$$

$$= \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \left(\vec{r}_i - \vec{r}_j \right) \times \vec{F}_{ij}$$

Example: Gravitational

$$\vec{F}_{ij} \propto \left(\vec{r}_i - \vec{r}_j \right)$$

$$(\vec{r}_i-\vec{r}_j)\times(\vec{r}_i-\vec{r}_j)=0 \;!$$

$$\Rightarrow \quad \frac{d\vec{L}}{dt}=0 \;!$$

angular momentum is conserved.