PHY 321, FEB 27, 2023

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

$$\omega_0 = \sqrt{k/m}$$ natural frequency

$$\gamma = \frac{b}{2m\omega_0}$$

$$\tau = \omega_0 \cdot t$$ Dimensionless time

$$\frac{d^2x}{d\tau^2} + 2\gamma\frac{dx}{d\tau} + x = 0$$

$$x(\tau) = A\cos(\underbrace{\omega_0 \cdot t}_{\tau}) + B\sin(\omega_0 \cdot t)$$

$$= A\cos(\tau) + B\sin(\tau)$$

HW 1

$$\cos(\omega_0 t) = \frac{1}{2}\left[e^{i\omega_0 t} + e^{-i\omega_0 t}\right]$$

$$\sin(\omega_0 t) = \frac{1}{2i}\left[e^{i\omega_0 t} - e^{-i\omega_0 t}\right]$$

$$x(\tau) = Ce^{i\tau} + De^{-i\tau}$$

$$x(\tau) = (C+D)\cos(\tau) + i(C-D)\sin(\tau)$$

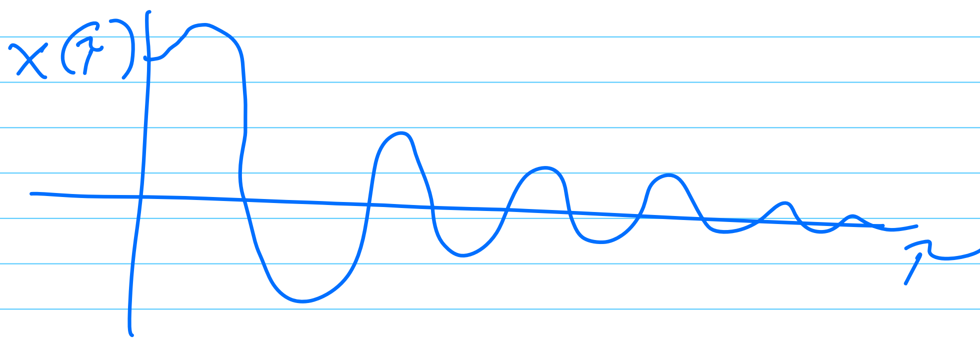
$$C+D = A \quad \Big| \quad i(C-D) = B$$

$$\frac{d^2 x(\tau)}{d\tau^2} = \frac{d^2 e^{i\tau}}{d\tau^2} = -e^{i\tau} = -x(\tau)$$

$$\frac{d^2 x}{d\tau^2} = -x(\tau)$$

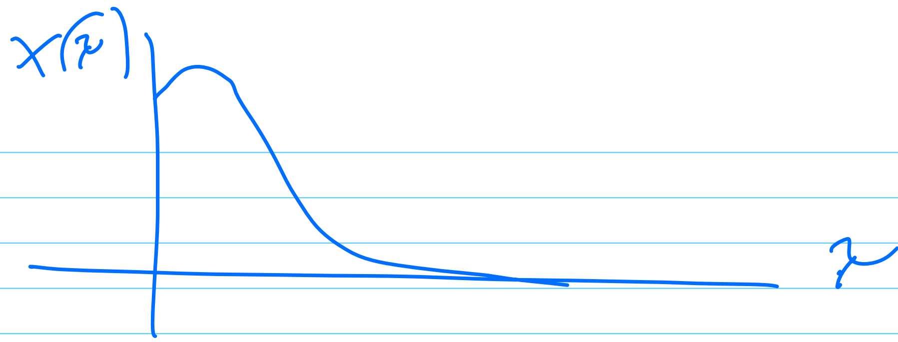
(i) underdamping

$\gamma < 1$



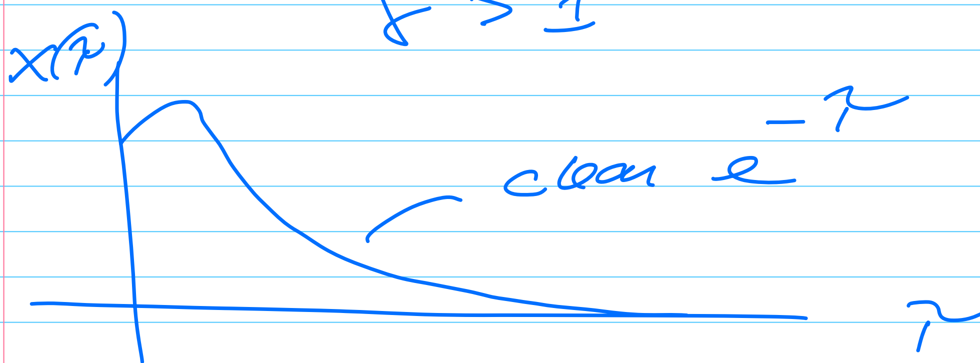
(ii) critical damping

$$\gamma = 1 \quad \left(= \frac{b}{2m\omega_0} \right)$$



(iii) overdamping

$\gamma > 1$



no external force

$$\frac{d^2x}{d\tau^2} + 2\gamma \frac{dx}{d\tau} + x = 0$$

general solution

$$x(\tau) = A e^{r\tau}$$

r is not yet specified.

$$A e^{r\tau} r^2 + 2\gamma r A e^{r\tau} +$$

$$A e^{r\tau} = 0$$

$$A e^{r\tau}(r^2 + 2\gamma r + 1) = 0$$

$$r^2 + 2\gamma r + 1 = 0 \implies$$

$$r = -\gamma \pm \sqrt{\gamma^2 - 1}$$

$$r_1 = -\gamma + \sqrt{\gamma^2 - 1}$$

$$r_2 = -\gamma - \sqrt{\gamma^2 - 1}$$

$$X(\tau) = A_1 e^{r_1 \tau} + A_2 e^{r_2 \tau}$$

$$= A_1 e^{-\gamma\tau + \tau\sqrt{\gamma^2-1}}$$

$$+ A_2 e^{-\gamma\tau - \tau\sqrt{\gamma^2-1}}$$

$\gamma > 0$, real

give exponential decay,

(ii) underdamping

$$\gamma < 1$$

$$\gamma' = \sqrt{\gamma^2 - 1}$$

$$r_1 = -\gamma + i\gamma'$$

$$r_2 = -\gamma - i\gamma'$$

$$x(\tau) = A_1 e^{-\gamma\tau} e^{i\gamma'\tau} + A_2 e^{-\gamma\tau} e^{-i\gamma'\tau}$$

$$= 2(A_1+A_2) e^{-\gamma\tau} \cos(\gamma'\tau) + 2i(A_1-A_2) e^{-\gamma\tau} \sin(\gamma'\tau)$$

$$B_1 = (A_1+A_2)/2 \qquad B_2 = \frac{i}{2}(A_1-A_2)$$

$$\boxed{x(\tau) = B_1 e^{-\gamma\tau} \cos\gamma'\tau + B_2 e^{-\gamma\tau} \sin(\gamma'\tau)}$$

exp decay + oscillations due to cos & sin

(ii) $\gamma = 1$, critical damping.

$$\gamma' = \sqrt{\gamma^2-1} = 0$$

$$x(\tau) = B_1 e^{-\gamma\tau} + B_2 e^{-\gamma\tau}$$

equal

add new solution

$$x(\tau) = \tau e^{-\gamma\tau} = x(\tau)$$

$$\frac{dx}{d\tau} = e^{-\gamma\tau} - \gamma\tau e^{-\gamma\tau}$$

$$\frac{d^2x}{d\tau^2} = -\gamma e^{-\gamma\tau} - \gamma e^{-\gamma\tau} + \gamma^2\tau e^{-\gamma\tau}$$

$$\frac{d^2x}{d\tau^2} + 2\gamma\frac{dx}{d\tau} + x = 0$$

$$= (\gamma^2 - 1) \times (-1)\tau e^{-\gamma\tau} = 0$$

$$\Longrightarrow \quad \gamma^2 = 1 \quad \Longrightarrow$$

$$\gamma = \pm 1$$

$$\gamma = \frac{b}{2m\omega_0} > 0$$

$$x(\tau) = Ae^{-\gamma\tau} + B\tau e^{-\gamma\tau}$$

dominates when

γ is small

(iii) $\gamma > 1 \Rightarrow$

$\gamma' > 0$

$\gamma' = \sqrt{\gamma^2 - 1}$

$$x(\tau) = A_1 e^{-(\gamma - \sqrt{\gamma^2-1})\tau} + A_2 e^{-(\gamma + \sqrt{\gamma^2-1})\tau}$$

exponential decay,

$$\underbrace{-\gamma + \sqrt{\gamma^2-1}}_{\text{goes to } 0} \qquad \gamma >> 1$$

$\sqrt{\gamma^2-1} \simeq \gamma$

$-\gamma - \sqrt{\gamma^2-1} \simeq -2\gamma$

$$x(\tau) \simeq A_1 + A_2 e^{-2\gamma\tau}$$