PHY 321, APRIL 10, 2023

Lagrangian mechanics

4 ingredients

- calculus of variations
- Lagrangian $\mathcal{L} = K - V$
- Euler-Lagrange equations
- constrained motion

Top-down examples

Example $\underline{1}$

1-Dim Harmonic oscillator

$$K = \frac{1}{2} m v^2$$

$$V = \frac{1}{2} k x^2$$

$$\mathcal{L} = \frac{1}{2} m v^2 - \frac{1}{2} k x^2 = \mathcal{L}(x, v, t)$$

Euler-Lagrange eqs:

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial v} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x} = -kx$$

$$\frac{\partial \mathcal{L}}{\partial v} = m \cdot v$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v} = m \cdot \frac{dv}{dt} = ma$$

$$-kx - ma = 0 \quad \Rightarrow$$

$$\boxed{ma = -kx}$$

Example 2 : Gravitational problem in polar coordinates

$$K = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2} r^2 \mu \dot{\phi}^2$$

$$V = -\gamma/r \qquad r = \sqrt{x^2 + y^2}$$

$$\mathcal{L} = \mathcal{L}(r, \dot{r}, \phi, \dot{\phi}, t)$$

$$= \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2} r^2 \mu \dot{\phi}^2 + \gamma/r$$

$$\frac{\partial \mathcal{L}}{\partial r} = r \mu \dot{\phi}^2 - \gamma/r^2$$

$$\dot{\phi} = \frac{L}{\mu r^2}$$

$$\frac{\partial L}{\partial r} = \frac{l^2}{\mu r^3} - \gamma/r^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{d}{dt}(\mu \dot{r})$$

$$= \mu \frac{d^2 r}{dt^2} = \mu a_r$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} =$$

$$\frac{l^2}{\mu r^3} - \gamma/r^2 - \mu a_r \Rightarrow$$

$$\mu \ddot{r} = \frac{l^2}{\mu r^3} - \gamma/r^2$$

$$= F_{eff}(r)$$

$$= -\frac{dV_{eff}}{dr}$$

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0$$

$$\overset{\shortparallel}{0} - \frac{d}{dt}(r^2 \mu \dot{\phi})$$

$$\dot{\phi} = \frac{L}{r^2 \mu}$$

$$0 - \frac{d}{dt} L = 0$$

$$0 - 0 = 0$$

<u>Example 3</u>

$$\mathcal{L} = \frac{1}{2}\mu\left(v_x^2 + v_y^2\right) + \frac{\gamma}{\sqrt{x^2+y^2}}$$

$$= \mathcal{L}(x, \dot{x}, y, \dot{y}, t)$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v_x} = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v_y} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x} = -\gamma \frac{x}{\left(\sqrt{x^2+y^2}\right)^3}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v_x} = \frac{d}{dt}\mu v_x$$

$$= \mu a_x$$

$$\mu a_x = -\frac{\gamma x}{\left(\sqrt{x^2+y^2}\right)^3} = F_x$$

$$\mu a_y = F_y = -\gamma \frac{y}{\left(\sqrt{x^2+y^2}\right)^3}$$

Example 4

assume $\frac{d\mathcal{L}}{dt} = 0$

$$d\mathcal{L} = \frac{\partial \mathcal{L}}{\partial v} dv + \frac{\partial \mathcal{L}}{\partial x} dx$$

$$\frac{d\mathcal{L}}{dt} = 0 = \frac{\partial \mathcal{L}}{\partial v}\frac{dv}{dt} + \frac{\partial \mathcal{L}}{\partial x}\frac{dx}{dt}$$

Euler-Lagrange eq!

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v}$$

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial v}\frac{dv}{dt} + \left[\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v}\right] \times \frac{dx}{dt}$$

$$\frac{d^2x}{dt^2}$$

$$= \frac{d}{dt}\left[\frac{dx}{dt} \cdot \frac{\partial \mathcal{L}}{\partial v}\right]$$

$$= \frac{d}{dt}\left[v \frac{\partial \mathcal{L}}{\partial v}\right]$$

$$\frac{d}{dt}\left[\underbrace{v \frac{\partial \mathcal{L}}{\partial v} - \mathcal{L}}_{= E}\right] = 0$$

$$\frac{\partial \mathcal{L}}{\partial v} = \frac{1}{2} m v \cdot 2 = m \cdot v$$

$$\frac{d}{dt}\left[m \cdot v^2 - \frac{1}{2} m v^2 + V\right]$$

$$= \frac{d}{dt}\left[\frac{1}{2}mv^2+V\right]$$

$$= \frac{d}{dt}E = 0$$

- Principle of least action + calculus of variations:

variational calculus deals with finding min or max of a quantity that can be expressed as an integral.

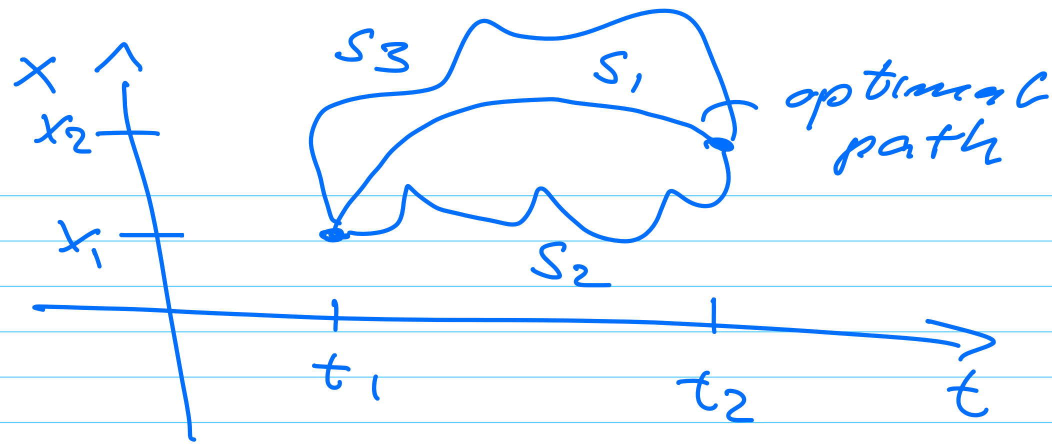
Define the so-called action

$$S = \int_{x_1}^{x_2} f\left(y, \frac{dy}{dx}, x\right) dx$$

optimizing S can be used to show that the shortest distance between two points is a line,

in our case

$$S = \int_{t_1}^{t_2} \mathcal{L}(x, v, t)\,dt$$



we want "the path" where -S- is at its smallest,