FYS 321, FEB 20, 2023

Euler's method

$$\frac{dx}{dt} = f(x,t)$$

$t \Rightarrow t_i \qquad x \Rightarrow x_i$

Taylor expand

$$x(t+\Delta t) \Rightarrow x(t_i+\Delta t)$$

$$= x_{i+1} = \underbrace{x(t_i)}_{=x_i} + \Delta t\, x'|_{t_i}$$

$$+ \quad O(\Delta t^2)$$

Euler's method

$$x_{i+1} = x_i + \Delta t\, x'(t_i)$$

$$x' = \frac{dx}{dt}\Big|_{t=t_i} = v_i$$

$$\left.\begin{aligned} x_{i+1} &= x_i + \Delta t\, v_i \\ v_{i+1} &= v_i + \Delta t\, a_i \end{aligned}\right| \; O((\Delta t)^2)$$

Euler-Cromer:

$$v_{i+1} = v_i + \Delta t\, a_i \quad (O(\Delta t^2))$$

$$x_{i+1} = x_i + \Delta t\,(v_i + \Delta t\, a_i)$$

$$\underbrace{\qquad\qquad}_{v_{i+1}}$$

$(O(\Delta t^3))$

Euler's method

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 1 & a_i \Delta t \end{bmatrix} \begin{bmatrix} x_i \\ v_i \end{bmatrix}$$

Example harmonic oscillator

$$\frac{d^2 x}{dt^2} = - \omega_0^2 x(t)$$

$$\omega_0^2 = k/m$$

$$\frac{dx}{dt} = v_x \quad \wedge \quad \frac{dv_x}{dt} = -\omega_0^2 x$$

Back to $\hat{A}$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \Delta t \\ -\omega_0^2 \Delta t & 1 \end{bmatrix}}_{\hat{A}} \begin{bmatrix} x_i \\ v_i \end{bmatrix}$$

in order to conserve energy, det(A)

$$\det(A) = 1 + \omega^2 \Delta t^2$$

Explicit Δt dependence

Euler-Cromer

$$x_{i+1} = x_i + [v_i - \omega_0^2 x_i \Delta t]\Delta t$$

$$v_{i+1} = v_i + \Delta t(-\omega_0^2 x_i)$$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 - \omega_0^2 \Delta t^2 & \Delta t \\ -\omega_0^2 \Delta t & 1 \end{bmatrix}}_{\hat{A}} \begin{bmatrix} x_i \\ v_i \end{bmatrix}$$

$$\det(\hat{A}) = 1 - \omega_0^2 \Delta t^2 + \omega_0^2 (\Delta t)^2 = \underline{1}$$

Velocity-Verlet

$$x(t+\Delta t) \longrightarrow x_{i+1}$$

$$= x_i + \Delta t \underbrace{x'(t_i)}_{v_i}$$

$$+ \frac{\Delta t^2}{2!} \underbrace{x''(t_i)}_{a_i} + O(\Delta t^3)$$

$$\simeq x_i + \Delta t\, v_i + \frac{\Delta t^2}{2} a_i$$

$$v_{i+1} = v_i + \Delta t \underbrace{v_i'}_{a_i}$$

?

$$+ \frac{\Delta t^2}{2!} v_i'' + O(\Delta t^3)$$

$$v_i'' \simeq \frac{v_{i+1}' - v_i'}{\Delta t}$$

$$= \frac{a_{i+1} - a_i}{\Delta t}$$

$$v_{i+1} = v_i + \Delta t\, a_i$$

$$+ \frac{\Delta t}{2} (a_{i+1} - a_i)$$

$$v_{i+1} = v_i + \frac{\Delta t}{2}\left[a_{i+1} + a_i\right]$$

$$x_{i+1} = x_i + \Delta t\, v_i + \frac{\Delta t^2}{2} a_i$$

error (truncation)

$O(\Delta t^3)$

need to find a_i to get x_{i+1}

applies only if a_i depends only on x_i!

$x_{i+1} \Rightarrow a_{i+1}$ and then v_{i+1}