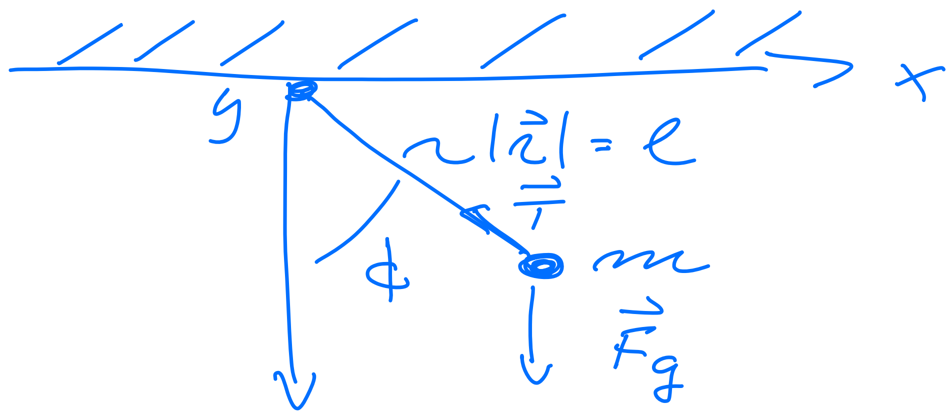
PHY 321, APRIL 27, 2022

Calculus of variations and Lagrangian formalism.

Monday: pendulum



$$y = r\cos\phi \quad \wedge \quad x = r\sin\phi$$

$$\mathcal{L} = \mathcal{L}(r, \dot{r}, \phi, \dot{\phi}, t)$$

$$= \frac{1}{2} m \left[\dot{r}^2 + r^2\dot{\phi}^2\right]$$

$$- mgr\cos\phi$$

$$+ \lambda (r - l)$$

$$\ddot{\phi} = -g/l \sin\phi$$

$$\lambda = ml\dot{\phi}^2 + mg\cos\phi$$

quick reminder:

$$K = \frac{1}{2} m \left(v_x^2 + v_y^2 \right)$$

$$x = r\cos\phi \quad \wedge \quad y = r\sin\phi$$

$$\frac{dx}{dt} = v_x = \frac{d(r\cos\phi)}{dt}$$

$$v_y = \frac{d(r\sin\phi)}{dt}$$

$$v_x^2 = \left(\frac{dx}{dt}\right)^2 =$$

$$\left(\frac{d(r\cos\phi)}{dt} = \underbrace{\frac{dr}{dt}}_{\dot r} \cos\phi - \sin\phi \times \underbrace{\frac{d\phi}{dt}}_{\dot\phi} \cdot r \right)$$

$$= \dot r^2 \cos^2\phi + r^2 \dot\phi^2 \sin^2\phi - 2 r \dot r \dot\phi \cos\phi \cdot \sin\phi$$

$$\left(\frac{dy}{dt}\right)^2 = \dot r^2 \sin^2\phi + r^2 \dot\phi^2 \cos^2\phi + 2 r \dot r \dot\phi \cos\phi \sin\phi$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = v_x^2 + v_y^2$$

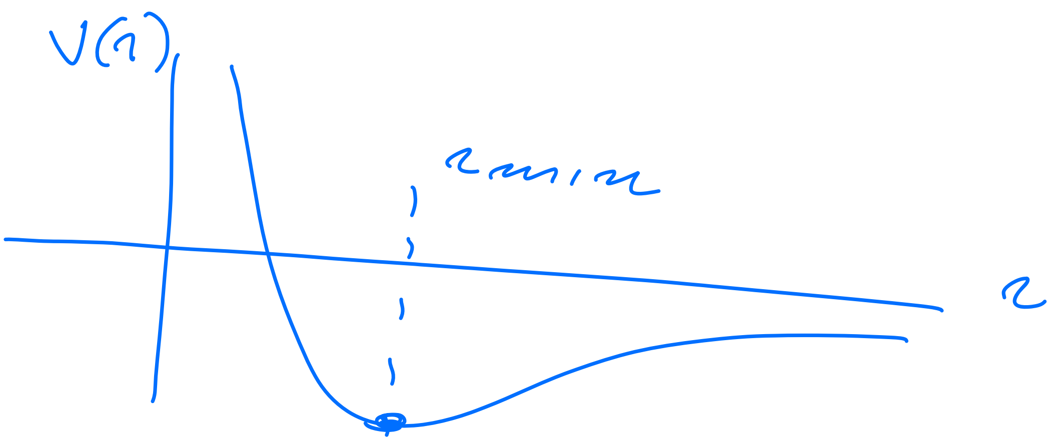
$$= \dot{r}^2 + r^2\dot{\phi}^2$$

<u>Example (hw 6)</u>

Lennard-Jones $V(r)$

$$V(r) = V_0\left(\left(\frac{a}{r}\right)^{12} - \left(\frac{b}{r}\right)^6\right)$$

$$r = |\vec{r}_1 - \vec{r}_2|$$



$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)$$

no dependence on ϕ

Euler-Lagrange eqs:

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$\frac{d}{dt}\left(\mu r^2 \dot{\phi}\right)$$

$$\dot{\phi} = \frac{L}{\mu r^2}$$

$$\frac{d}{dt} L = 0$$

$$\frac{\partial \mathcal{L}}{\partial r} = r\mu\dot{\phi}^2 + 6V_0\left[-\frac{2a^{12}}{r^{13}} - \frac{b^6}{r^7}\right]$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \mu \cdot \dot{r}$$

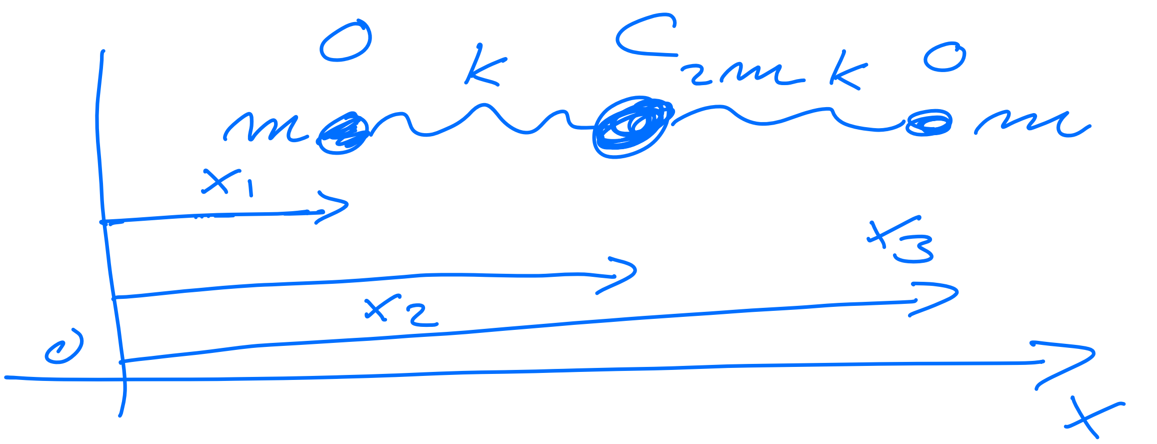
$$\frac{d}{dt}(\mu \cdot \dot{r}) = \mu \ddot{r} \Rightarrow$$

$$\mu \ddot{r} = \mu r \phi + 6V_0\left[\frac{[\text{illegible}]}{r^{13}} - \frac{b^6}{r^7}\right]$$

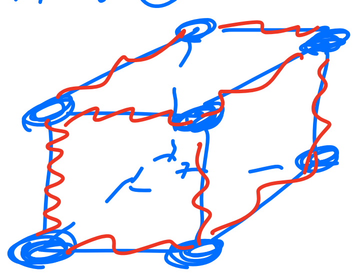
Example 2

Coupled Harmonic oscillator

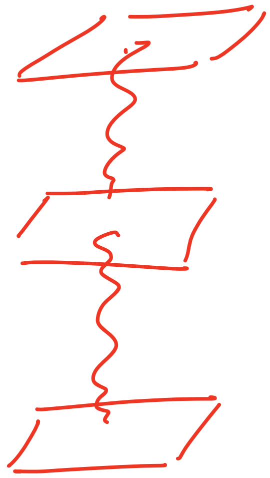
1-Dim Model for CO_2



Material



Earthquake protection



$$\mathcal{L} = \frac{1}{2} m \dot{x}_1^2 + m \dot{x}_2^2 + \frac{1}{2} m \dot{x}_3^2 - \frac{k}{2}(x_2 - x_1)^2 - \frac{k}{2}(x_3 - x_2)^2$$

$$\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = 0$$

$$i = 1, 2, 3$$

$$m \ddot{x}_1 = -k(x_1 - x_2)$$

$$2m \ddot{x}_2 = -k(x_2 - x_1) - k(x_2 - x_3)$$

$$= -k(2x_2 - x_1 - x_3)$$

$$m \ddot{x}_3 = -k(x_3 - x_1)$$

$$q_1 = x_1 - x_2$$

$$q_3 = x_3 - x_2$$

$$\vec{R} = \frac{\sum_{i=1}^{3} m_i \vec{r}_i}{\sum_{i=1}^{3} m_i}$$

$$\vec{R} \Rightarrow X = \frac{x_1 m + 2m x_2 + m x_3}{4m}$$

$$= \frac{x_1 + 2x_2 + x_3}{4}$$

$$x_2 = \frac{4X - x_1 - x_3}{2}$$

$$x_1 = \frac{3q_1 - q_3 + 4X}{2}$$

$$x_3 = \frac{3q_3 - q_1 + 4X}{2}$$

$$\mathcal{L} = K - V$$

$$= \frac{3m}{8}\left(\dot{q}_1^2 + \dot{q}_3^2\right) - \frac{m \dot{q}_1 \dot{q}_3}{4} + 2m\dot{X}^2 - \frac{k}{2}\left(q_1^2 + q_3^2\right)$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

$$\Rightarrow \frac{d}{dt}(m\dot{x}) = 0$$

$$\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = 0$$

q_1 : $\quad -\frac{3}{4} m \ddot{q}_1 + \frac{m}{4} \ddot{q}_3 = -k q_1$

q_3 : $\quad -\frac{3}{4} m \ddot{q}_3 + \frac{1}{4} m \ddot{q}_1 = -k q_3$

$$q_1 = A e^{i\omega t}$$

$$q_3 = B e^{i\omega t}$$

$$\omega^2 = k/m$$

q_1