Solving ordinary Differential equations (ODEs): Meet the Runge-Kutta family!

$$\frac{dy}{dt} = f(t,y)$$

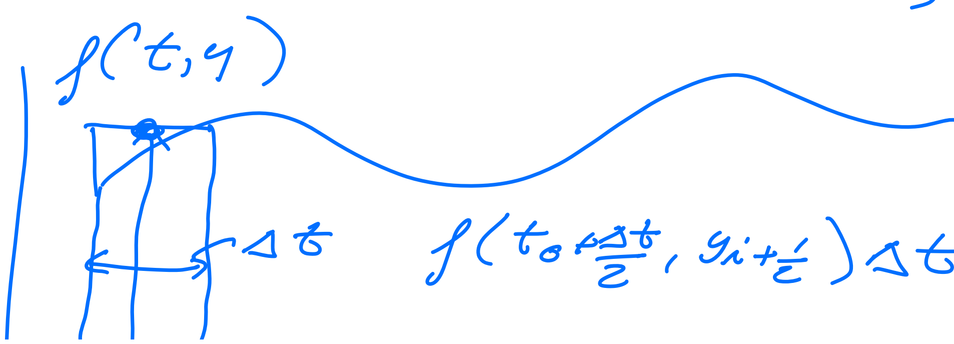
$$y(t) = \int f(t,y)\,dt$$

$$y(t) \Rightarrow y(t_i) = y_i$$

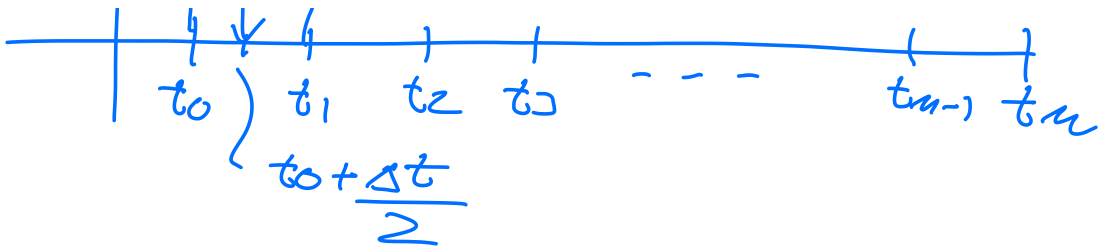
$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} f(t,y)\,dt$$

- Rectangular rule

$$\int_{t_i}^{t_{i+1}} f(t,y)\,dt = \Delta t\, f(t_{i+1/2}, y_{i+1/2}) + O(\Delta t^3)$$



$$f\left(t_0 + \frac{\Delta t}{2}, y_{i+\frac{1}{2}}\right)\Delta t$$

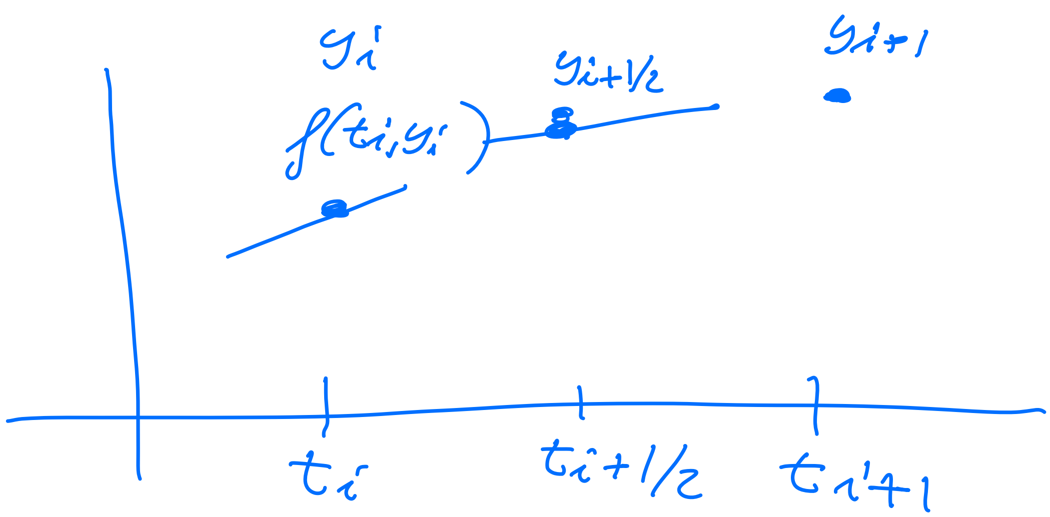


$$y_{i+1} = y_i + \Delta t\, f(t_{i+1/2}, y_{i+1/2})$$

unknown

$$y_{i+1/2} = y_i + \frac{\Delta t}{2} f(t_i, y_i)$$

Euler's method



$$k_1 = \Delta t\, f(t_i, y_i)$$

$$k_2 = \Delta t\, f\left(t_{i+1/2}, y_i + \frac{k_1}{2}\right)$$

$$\boxed{y_{i+1} = y_i + k_2}$$

RK2

Runge-Kutta 4 = RK4

Simpson's Rule

$$\int_{t_i}^{t_{i+1}} f(t,y)\,dt = \frac{\Delta t}{6}\Big[f(t_i, y_i) + 4 f(t_{i+1/2}, y_{i+1/2}) + f(t_{i+1}, y_{i+1}) \Big] + O(\Delta t^5)$$

$$y_{i+1} = y_i + \frac{\Delta t}{6}\Big[f(t_i, y_i) + 2 f(t_{i+1/2}, y_{i+1/2}) + 2 f(t_{i+1/2}, y_{i+1/2}) + f(t_{i+1}, y_{i+1}) \Big]$$

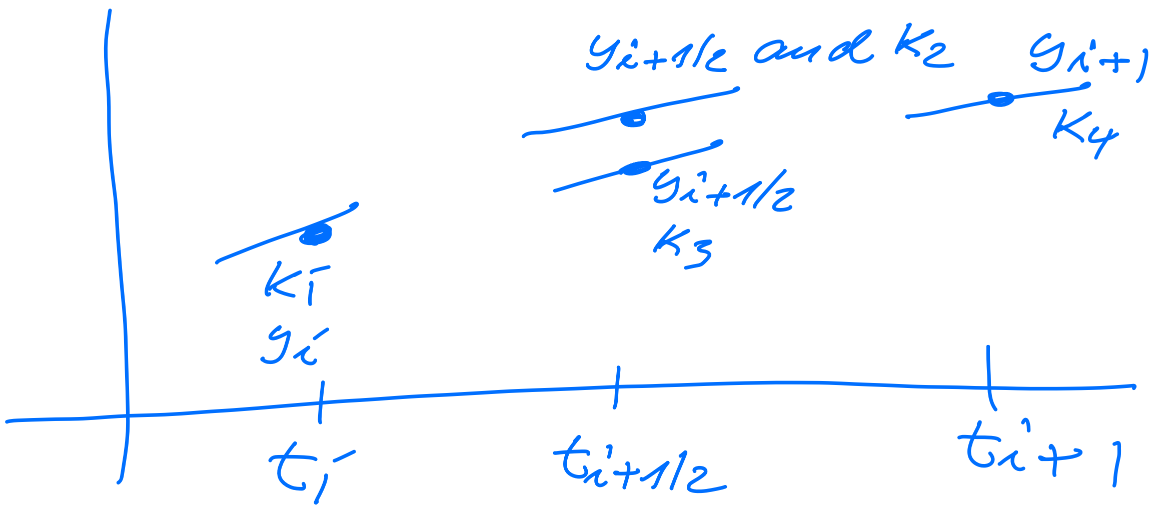
$$y_{i+1/2} = y_i + \frac{\Delta t}{2} f(t_i, y_i)$$

$$k_1 = \Delta t\, f(t_i, y_i)$$

$$k_2 = \Delta t\, f\left(t_{i+1/2},\ y_i + \frac{k_1}{2}\right)$$

$$k_3 = \Delta t\, f\left(t_{i+1/2},\ y_i + \frac{k_2}{2}\right)$$

$$k_4 = \Delta t\, f\left(t_{i+1},\ y_i + k_3\right)$$



$$y_{i+1} = y_i + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

RK 4 $\quad (O(\Delta t^5))$