PHY321: Harmonic Oscillations, Damping, Resonances and time-dependent Forces

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Aims and Overarching Motivation

Monday. Damped oscillations. Analytical and numerical solutions **Reading suggestion**: Taylor sections 5.4-5.5.

Wednesday. No lecture, study day

Friday. Driven oscillations and resonances with examples. **Reading suggestion**: Taylor sections 5.5-5.6.

Damped Oscillators

We consider only the case where the damping force is proportional to the velocity. This is counter to dragging friction, where the force is proportional in strength to the normal force and independent of velocity, and is also inconsistent with wind resistance, where the magnitude of the drag force is proportional the square of the velocity. Rolling resistance does seem to be mainly proportional to the velocity. However, the main motivation for considering damping forces proportional to the velocity is that the math is more friendly. This is because the differential equation is linear, i.e. each term is of order x, $\dot{x}$, $\ddot{x} \cdots$, or even terms with no mention of x, and there are no terms such as x^2 or $x\ddot{x}$. The equations of motion for a spring with damping force $-b\dot{x}$ are

$$m\ddot{x} + b\dot{x} + kx = 0. \tag{1}$$

Harmonic Oscillator, Damping

Just to make the solution a bit less messy, we rewrite this equation as

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \quad \beta \equiv b/2m, \ \omega_0 \equiv \sqrt{k/m}. \tag{2}$$

Both β and ω have dimensions of inverse time. To find solutions (see appendix C in the text) you must make an educated guess at the form of the solution. To do this, first realize that the solution will need an arbitrary normalization A because the equation is linear. Secondly, realize that if the form is

$$x = Ae^{rt} \tag{3}$$

that each derivative simply brings out an extra power of r. This means that the Ae^{rt} factors out and one can simply solve for an equation for r. Plugging this form into Eq. (2),

$$r^2 + 2\beta r + \omega_0^2 = 0. \tag{4}$$

Harmonic Oscillator, Solutions of Damped Motion

Because this is a quadratic equation there will be two solutions,

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}. \tag{5}$$

We refer to the two solutions as r_1 and r_2 corresponding to the $+$ and $-$ roots. As expected, there should be two arbitrary constants involved in the solution,

$$x = A_1 e^{r_1 t} + A_2 e^{r_2 t}, \tag{6}$$

where the coefficients A_1 and A_2 are determined by initial conditions.

The roots listed above, $\sqrt{\omega_0^2 - \beta_0^2}$, will be imaginary if the damping is small and $\beta < \omega_0$. In that case, r is complex and the factor ert will have some oscillatory behavior. If the roots are real, there will only be exponentially decaying solutions. There are three cases:

Underdamped: $\beta < \omega_0$

$$\begin{aligned} x &= A_1 e^{-\beta t} e^{i\omega' t} + A_2 e^{-\beta t} e^{-i\omega' t}, \ \ \omega' \equiv \sqrt{\omega_0^2 - \beta^2} \\ &= (A_1 + A_2)e^{-\beta t}\cos\omega' t + i(A_1 - A_2)e^{-\beta t}\sin\omega' t. \end{aligned} \tag{7}$$

Here we have made use of the identity $e^{i\omega' t} = \cos\omega' t + i\sin\omega' t$. Because the constants are arbitrary, and because the real and imaginary parts are both solutions individually, we can simply consider the real part of the solution alone:

$$x = B_1 e^{-\beta t} \cos \omega' t + B_2 e^{-\beta t} \sin \omega' t, \tag{8}$$
$$\omega' \equiv \sqrt{\omega_0^2 - \beta^2}.$$

Critical dampling: $\beta = \omega_0$

In this case the two terms involving r_1 and r_2 are identical because $\omega' = 0$. Because we need to arbitrary constants, there needs to be another solution. This is found by simply guessing, or by taking the limit of $\omega' \to 0$ from the underdamped solution. The solution is then

$$x = A e^{-\beta t} + B t e^{-\beta t}. \tag{9}$$

The critically damped solution is interesting because the solution approaches zero quickly, but does not oscillate. For a problem with zero initial velocity, the solution never crosses zero. This is a good choice for designing shock absorbers or swinging doors.

Overdamped: $\beta > \omega_0$

$$x = A_1 \exp -(\beta + \sqrt{\beta^2 - \omega_0^2})t + A_2 \exp -(\beta - \sqrt{\beta^2 - \omega_0^2})t \tag{10}$$

This solution will also never pass the origin more than once, and then only if the initial velocity is strong and initially toward zero.

Given b, m and ω_0, find $x(t)$ for a particle whose initial position is $x = 0$ and has initial velocity v_0 (assuming an underdamped solution).

The solution is of the form,

$$x = e^{-\beta t} \left[A_1 \cos(\omega' t) + A_2 \sin \omega' t \right],$$
$$\dot{x} = -\beta x + \omega' e^{-\beta t} \left[-A_1 \sin \omega' t + A_2 \cos \omega' t \right].$$
$$\omega' \equiv \sqrt{\omega_0^2 - \beta^2}, \quad \beta \equiv b/2m.$$

From the initial conditions, $A_1 = 0$ because $x(0) = 0$ and $\omega' A_2 = v_0$. So

$$x = \frac{v_0}{\omega'} e^{-\beta t} \sin \omega' t.$$

Harmonic Oscillator, Solutions

Consider a single solution with no arbitrary constants, which we will call a *particular solution*, $x_p(t)$. It should be emphasized that this is **A** particular solution, because there exists an infinite number of such solutions because the general solution should have two arbitrary constants. Now consider solutions

to the same equation without the driving term, which include two arbitrary constants. These are called either *homogenous solutions* or *complementary solutions*, and were given in the previous section, e.g. Eq. (8) for the underdamped case. The homogenous solution already incorporates the two arbitrary constants, so any sum of a homogenous solution and a particular solution will represent the *general solution* of the equation. The general solution incorporates the two arbitrary constants A and B to accommodate the two initial conditions. One could have picked a different particular solution, i.e. the original particular solution plus any homogenous solution with the arbitrary constants A_p and B_p chosen at will. When one adds in the homogenous solution, which has adjustable constants with arbitrary constants A' and B', to the new particular solution, one can get the same general solution by simply adjusting the new constants such that $A' + A_p = A$ and $B' + B_p = B$. Thus, the choice of A_p and B_p are irrelevant, and when choosing the particular solution it is best to make the simplest choice possible.

Harmonic Oscillator, Particular Solution

To find a particular solution, one first guesses at the form,

$$x_p(t) = D \cos(\omega t - \delta), \tag{11}$$

and rewrite the differential equation as

$$D \left\{ -\omega^2 \cos(\omega t - \delta) - 2\beta\omega \sin(\omega t - \delta) + \omega_0^2 \cos(\omega t - \delta) \right\} = \frac{F_0}{m} \cos(\omega t). \tag{12}$$

One can now use angle addition formulas to get

$$D \left\{ (-\omega^2 \cos\delta + 2\beta\omega \sin\delta + \omega_0^2 \cos\delta) \cos(\omega t) \right. \tag{13}$$
$$\left. +(-\omega^2 \sin\delta - 2\beta\omega \cos\delta + \omega_0^2 \sin\delta) \sin(\omega t) \right\} = \frac{F_0}{m} \cos(\omega t).$$

Both the cos and sin terms need to equate if the expression is to hold at all times. Thus, this becomes two equations

$$D \left\{ -\omega^2 \cos\delta + 2\beta\omega \sin\delta + \omega_0^2 \cos\delta \right\} = \frac{F_0}{m} \tag{14}$$
$$-\omega^2 \sin\delta - 2\beta\omega \cos\delta + \omega_0^2 \sin\delta = 0.$$

After dividing by $\cos\delta$, the lower expression leads to

$$\tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}. \tag{15}$$

Solving with Driven Oscillations

Using the identities $\tan^2 + 1 = \csc^2$ and $\sin^2 + \cos^2 = 1$, one can also express $\sin\delta$ and $\cos\delta$,

$$
\begin{aligned}
\sin\delta &= \frac{2\beta\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}}, \\
\cos\delta &= \frac{(\omega_0^2 - \omega^2)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}}
\end{aligned}
\tag{16}
$$

Inserting the expressions for $\cos\delta$ and $\sin\delta$ into the expression for D,

$$
D = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}}.
\tag{17}
$$

For a given initial condition, e.g. initial displacement and velocity, one must add the homogenous solution then solve for the two arbitrary constants. However, because the homogenous solutions decay with time as $e^{-\beta t}$, the particular solution is all that remains at large times, and is therefore the steady state solution. Because the arbitrary constants are all in the homogenous solution, all memory of the initial conditions are lost at large times, $t >> 1/\beta$.

The amplitude of the motion, D, is linearly proportional to the driving force (F_0/m), but also depends on the driving frequency ω. For small β the maximum will occur at $\omega = \omega_0$. This is referred to as a resonance. In the limit $\beta \to 0$ the amplitude at resonance approaches infinity.

Alternative Derivation for Driven Oscillators

Here, we derive the same expressions as in Equations (11) and (17) but express the driving forces as

$$
F(t) = F_0 e^{i\omega t},
\tag{18}
$$

rather than as $F_0 \cos\omega t$. The real part of F is the same as before. For the differential equation,

$$
\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t},
\tag{19}
$$

one can treat $x(t)$ as an imaginary function. Because the operations d^2/dt^2 and d/dt are real and thus do not mix the real and imaginary parts of $x(t)$, Eq. (19) is effectively 2 equations. Because $e^{\omega t} = \cos\omega t + i\sin\omega t$, the real part of the solution for $x(t)$ gives the solution for a driving force $F_0 \cos\omega t$, and the imaginary part of x corresponds to the case where the driving force is $F_0 \sin\omega t$. It is rather easy to solve for the complex x in this case, and by taking the real part of the solution, one finds the answer for the $\cos\omega t$ driving force.

We assume a simple form for the particular solution

$$x_p = De^{i\omega t}, \tag{20}$$

where D is a complex constant.
From Eq. (19) one inserts the form for x_p above to get

$$D\left\{-\omega^2 + 2i\beta\omega + \omega_0^2\right\}e^{i\omega t} = (F_0/m)e^{i\omega t}, \tag{21}$$

$$D = \frac{F_0/m}{(\omega_0^2 - \omega^2) + 2i\beta\omega}.$$

The norm and phase for $D = |D|e^{-i\delta}$ can be read by inspection,

$$|D| = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}, \qquad \tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}. \tag{22}$$

This is the same expression for δ as before. One then finds $x_p(t)$,

$$
\begin{aligned}
x_p(t) &= \Re\frac{(F_0/m)e^{i\omega t - i\delta}}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \\
&= \frac{(F_0/m)\cos(\omega t - \delta)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}.
\end{aligned} \tag{23}
$$

This is the same answer as before. If one wished to solve for the case where $F(t) = F_0\sin\omega t$, the imaginary part of the solution would work

$$
\begin{aligned}
x_p(t) &= \Im\frac{(F_0/m)e^{i\omega t - i\delta}}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \\
&= \frac{(F_0/m)\sin(\omega t - \delta)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}.
\end{aligned} \tag{24}
$$

Damped and Driven Oscillator

Consider the damped and driven harmonic oscillator worked out above. Given F_0, m, β and ω_0, solve for the complete solution $x(t)$ for the case where $F = F_0\sin\omega t$ with initial conditions $x(t = 0) = 0$ and $v(t = 0) = 0$. Assume the underdamped case.

The general solution including the arbitrary constants includes both the homogenous and particular solutions,

$$x(t) = \frac{F_0}{m}\frac{\sin(\omega t - \delta)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} + A\cos\omega' te^{-\beta t} + B\sin\omega' te^{-\beta t}.$$

The quantities δ and ω' are given earlier in the section, $\omega' = \sqrt{\omega_0^2 - \beta^2}$, $\delta = \tan^{-1}(2\beta\omega/(\omega_0^2 - \omega^2))$. Here, solving the problem means finding the arbitrary constants A and B. Satisfying the initial conditions for the initial position and velocity:

$$
\begin{aligned}
x(t = 0) = 0 &= -\eta \sin \delta + A, \\
v(t = 0) = 0 &= \omega \eta \cos \delta - \beta A + \omega' B, \\
\eta &\equiv \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2}}.
\end{aligned}
$$

The problem is now reduced to 2 equations and 2 unknowns, A and B. The solution is

$$
A = \eta \sin \delta, \quad B = \frac{-\omega \eta \cos \delta + \beta \eta \sin \delta}{\omega'}. \tag{25}
$$

Resonance Widths; the Q factor

From the previous two sections, the particular solution for a driving force, $F = F_0 \cos \omega t$, is

$$
\begin{aligned}
x_p(t) &= \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2 \beta^2}} \cos(\omega_t - \delta), \tag{26} \\
\delta &= \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right).
\end{aligned}
$$

If one fixes the driving frequency ω and adjusts the fundamental frequency $\omega_0 = \sqrt{k/m}$, the maximum amplitude occurs when $\omega_0 = \omega$ because that is when the term from the denominator $(\omega_0^2 - \omega^2)^2 + 4\omega^2 \beta^2$ is at a minimum. This is akin to dialing into a radio station. However, if one fixes ω_0 and adjusts the driving frequency one minimize with respect to ω, e.g. set

$$
\frac{d}{d\omega}\left[(\omega_0^2 - \omega^2)^2 + 4\omega^2 \beta^2\right] = 0, \tag{27}
$$

and one finds that the maximum amplitude occurs when $\omega = \sqrt{\omega_0^2 - 2\beta^2}$. If β is small relative to ω_0, one can simply state that the maximum amplitude is

$$
x_{\max} \approx \frac{F_0}{2m\beta\omega_0}. \tag{28}
$$

$$
\frac{4\omega^2 \beta^2}{(\omega_0^2 - \omega^2)^2 + 4\omega^2 \beta^2} = \frac{1}{2}. \tag{29}
$$

For small damping this occurs when $\omega = \omega_0 \pm \beta$, so the $FWHM \approx 2\beta$. For the purposes of tuning to a specific frequency, one wants the width to be as small as possible. The ratio of ω_0 to $FWHM$ is known as the *quality* factor, or Q factor,

$$Q \equiv \frac{\omega_0}{2\beta}. \tag{30}$$