PHY 321, APRIL 12, 2023

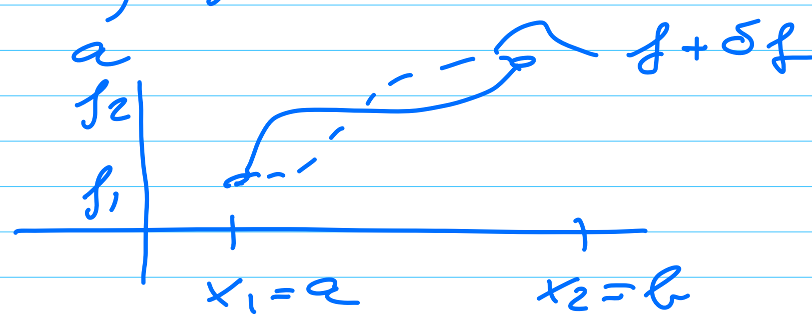
- calculus of variations
- Lagrangian $\mathcal{L} = K - V$
- Derive the Euler-Lagrange eq
- constrained motion

$$\mathcal{L} = \mathcal{L}(x, v, t) = \mathcal{L}(x, \dot{x}, t)$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v}$$

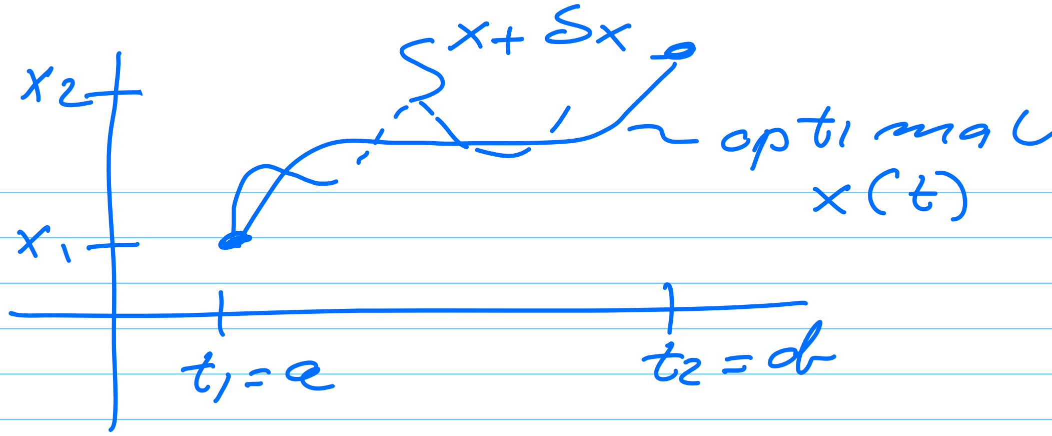
Calculus of variation, a variation of function f, δf, $f \in [a, b]$

$$\int_a^b f(x) h(x) dx = 0$$



then $f(x) = 0$

$$S = \int_a^b \mathcal{L}(x, v, t) dt$$



$x(t_1 = a) = x_1$ fixed
$x(t_2 = b) = x_2$ — " —

$x(t) + \delta x(t)$

$\delta x(t_1) = \delta x(t_2) = 0$

$v(t) = \dfrac{dx}{dt}$

$v(t) + \delta v(t)$

$\delta v(t_1) = \delta v(t_2) = 0$

$$S_{\delta x} = \int_a^b \mathcal{L}(x + \delta x, v + \delta v, t)\, dt$$

$$\mathcal{L}(x + \delta x, v + \delta v, t) = \mathcal{L}(x, v, t)$$
$$+ \frac{\partial \mathcal{L}}{\partial x} \delta x + \frac{\partial \mathcal{L}}{\partial v} \delta v$$
$$+ O(\delta x^2) + O(\delta v^2) + O(\delta x \delta v)$$

assume δx & δv are small and neglect higher order terms

$$S_{\delta x} - S = \delta S = 0 =$$

$$\int \mathcal{L}(x+\delta x, v+\delta v, t)\,dt$$

$$- \int \mathcal{L}(x, v, t)\,dt$$

$$= \int_a^b \left[\frac{\partial \mathcal{L}}{\partial x} \delta x + \frac{\partial \mathcal{L}}{\partial v} \delta v \right] dt$$

$$= 0$$

$$\delta v = \frac{d}{dt} \delta x$$

$$= \int_a^b \left[\frac{\partial \mathcal{L}}{\partial x} \delta x + \frac{\partial \mathcal{L}}{\partial v} \frac{d}{dt} (\delta x) \right] \times dt$$

integrate by parts

$$= \frac{\partial L}{\partial v} \delta x \Big|_a^b \quad (\to 0)$$

$$+ \int_a^b \left[\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial v} \right] \delta x \, dt$$

$$\delta x(a) = \delta x(b) = 0$$

$$\int_a^b \left[\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial v} \right] \delta x \, dt = 0$$

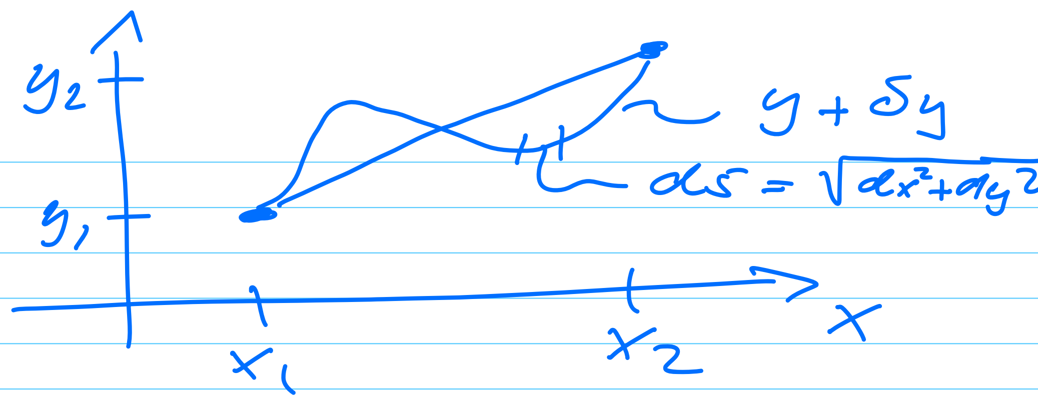
δx is an arbitrary continuous function, using variational calculus lemma

$$\Rightarrow \quad \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial v} = 0$$

Euler-Lagrange eq.

Example 1, shortest distance between 2 points

(x_1, y_1) & (x_2, y_2)



$$dy = \frac{dy}{dx}\, dx = y'\, dx$$

$$\sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\, dx$$

$$\mathcal{L} = \mathcal{L}(y, y', x)$$

$$= \sqrt{1 + y'^2}$$

$$S = \int_{x_1}^{x_2} \mathcal{L}(y, y', x)\, dx$$

$$\underbrace{\frac{\partial \mathcal{L}}{\partial y}}_{=\,0} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial y'} = 0$$

$$\Rightarrow \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial y'} = 0$$

$$\Rightarrow \quad \frac{\partial \ell}{\partial y'} = 0$$

$$\frac{\partial \ell}{\partial y'} = \frac{y'}{(1+y'^2)^{1/2}} = C$$

$$y'^2 = (1+y'^2)\,C^2$$

$$y'^2(1-C^2) = C^2 \quad \Rightarrow$$

$$y'^2 = \frac{C^2}{1-C^2} = D^2$$

$$y' = \pm D \quad \Rightarrow$$

$$y = \pm Dx + B$$

straight line !

Example 2

$$\mathcal{L} = K - V$$

$$K = \frac{1}{2} m v^2 \qquad x = x(t)$$

$$v = v(t)$$

$$V = V(x)$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v}$$

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{dV}{dx} = F(x)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v} = \frac{d}{dt}(m \cdot v)$$

$$= m \cdot \frac{dv}{dt} = m\frac{d^2x}{dt^2} =$$

$$m \cdot a$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v}$$

$$F(x) - ma = 0 \Rightarrow$$

$$\boxed{ma = F}$$