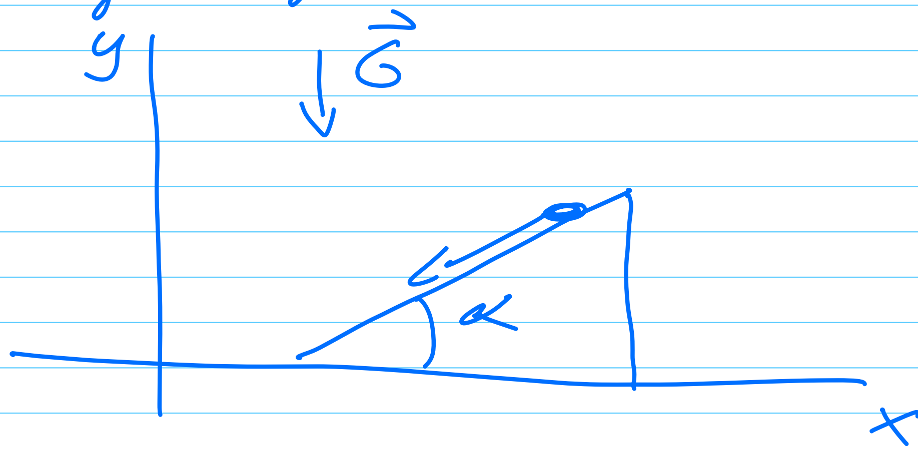
PHY 321, APRIL 19, 2023

Lagrangian with constraints



$$g(x,y) = y - x\tan\alpha = 0$$

$$y/x = \tan\alpha$$

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy + \lambda g(x,y)$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

$$-\lambda\tan\alpha - m\ddot{x} = 0$$

$$m\ddot{x} = -\lambda\tan\alpha \qquad (*)$$

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = 0$$

$$-m\ddot{y} + mg + \lambda = 0 \quad (**)$$

$y = x \cdot \tan\alpha \Rightarrow$

$\ddot{y} = \ddot{x}\tan\alpha$

multiply $(*)$ with $\tan\alpha$

$$m\ddot{x}\tan\alpha = -\lambda\tan^2\alpha$$

subtract $(**)$ $\Rightarrow$

$m\ddot{y} = m\ddot{x}\tan\alpha$

$\lambda\tan^2\alpha + \lambda - mg = 0$

$\Rightarrow \quad \lambda = mg\cos^2\alpha$ force

$$\ddot{x} = -g\sin\alpha\cos\alpha$$

$$\ddot{y} = -g\sin^2\alpha$$

integrate wrt time

$\dot{x}$ and $\dot{y}$

integrate again

$$x(t) = x_0 + \underbrace{\dot{x}_0 \cdot t}_{v_{0x}} - \frac{1}{2}g t^2 \sin\alpha \times \cos\alpha$$

$$y(t) = y_0 + \underbrace{\dot{y}_0 t}_{v_{0y}} - \frac{1}{2}g t^2 \sin^2\alpha$$

How to derive $\lambda g(x, y)$?

Example

$$f(x_1, x_2) = -3x_1^2 - 6x_1x_2 - 5x_2^2 + 7x_1 + 5x_2$$

$$g(x_1 x_2) = 0 = x_1 + x_2 - 5$$

$$\mathcal{L} = f(x_1 x_2) + \lambda g(x_1 x_2)$$

?

minimize $f(x_1 x_2)$ subject to $g(x_1 x_2) = 0$, satisfied by

$$\tilde{x} = \{\tilde{x}_1, \tilde{x}_2\}$$

necessary condition

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0$$

Taylor expansion

$$g(\tilde{x}_1 + dx_1, \tilde{x}_2 + dx_2) = 0$$

$$g(\tilde{x}_1 + dx_1, \tilde{x}_2 + dx_2) \simeq$$

$$g(\tilde{x}_1, \tilde{x}_2) + \left.\frac{\partial g}{\partial x_1}\right|_{\tilde{x}_1 \tilde{x}_2} dx_1$$

$$+ \left.\frac{\partial g}{\partial x_2}\right|_{\tilde{x}_1 \tilde{x}_2} dx_2$$

we require that

$$\boxed{dg = \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 = 0}$$

at $(\tilde{x}_1, \tilde{x}_2)$

$$\frac{\partial g}{\partial x_2} \neq 0$$

$$dx_2 = -\frac{\partial g/\partial x_1}{\partial g/\partial x_2} dx_1$$

$$df = \left[\frac{\partial f}{\partial x_1} - \frac{\partial g/\partial x_1}{\partial g/\partial x_2} \frac{\partial f}{\partial x_2}\right] \times dx_1$$

must be satisfied by all dx_1

Define $\lambda = -\frac{\partial f}{\partial x_2} \Big/ \frac{\partial g}{\partial x_2}$

$$df = 0 = \left[\frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1}\right] dx_1$$

$$\Rightarrow \quad d f' = d[f + \lambda g]$$

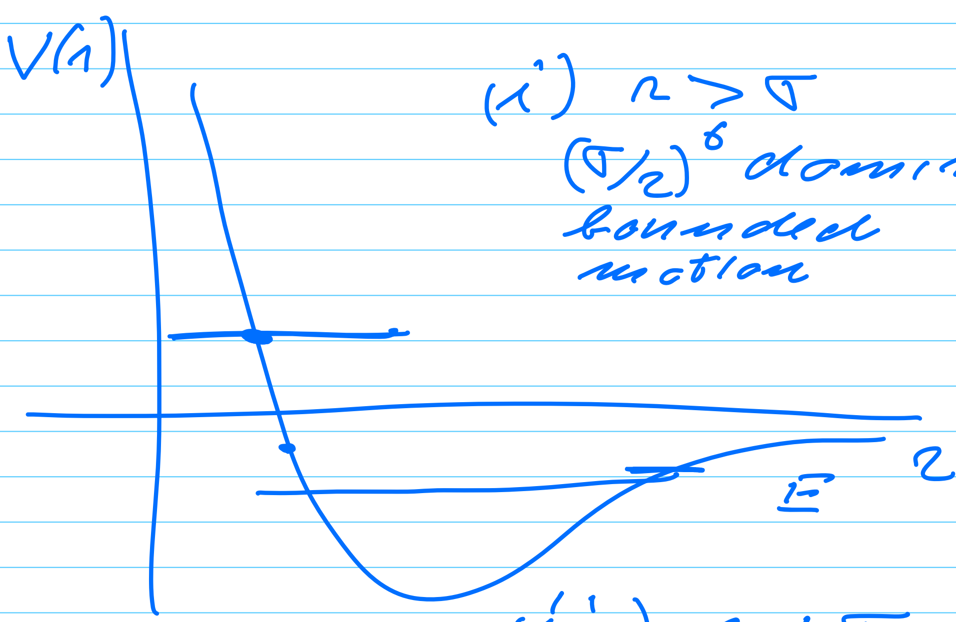
$$f'(x_1 x_2) = f(x_1, x_2) + \lambda g(x_1 x_2)$$

Example 5

Ex 5 in hw9

$$V(r) = \frac{4\varepsilon}{V_0}\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right]$$

$$r = |\vec{r}_1 - \vec{r}_2| \qquad \sigma > 0$$



(i) $r > \sigma$

$(\sigma/r)^6$ dominates

bounded motion

(ii) $r < \sigma$

$\left(\frac{\sigma}{r}\right)^{12}$ dominates

$$\vec{\nabla} \times \vec{F} = \vec{\nabla} \times (-\vec{\nabla} V(r))$$

$$=$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$\propto \quad \vec{r} \times \vec{F}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\vec{F} = -\vec{\nabla} V(r) = f(r)\,\vec{r}$$

$$\mathcal{L} = \frac{1}{2}\mu(v_x^2 + v_y^2)$$

$$- V(r)$$

$$x, y \Rightarrow r, \phi$$

$$x = r \cdot \cos\phi \qquad y = r \cdot \sin\phi$$

$$x, y \in (-\infty, +\infty)$$

$$r \in [0, \infty) \qquad \phi \in [0, 2\pi]$$

$$\frac{1}{2}\mu(v_x^2 + v_y^2) = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2)$$

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0$$

$$\frac{24\varepsilon\sigma^6}{r^7}\left[-\frac{2\sigma^6}{r^6}+1\right]$$

$$+\mu r\dot{\phi}^2 - \mu\ddot{r}$$

$$\ddot{r} = r\dot{\phi}^2 + \frac{24\varepsilon}{\mu r}\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right]$$

$$\dot{\phi} = \frac{L}{\mu r^2}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0$$

$$= -\mu r^2\ddot{\phi}$$

$$\ddot{\phi} = 0$$