PHY 321, MARCH 30, 2022

L = constant

CoM - frame $\vec{R} = 0$

x, y, z for $\vec{r} = \vec{r}_1 - \vec{r}_2 \longrightarrow$

$x, y \longrightarrow$ $x \in (-\infty, +\infty)$

$y \in (-\infty, +\infty)$

$r \in [0, \infty)$

$\phi \in [0, 2\pi]$

$$x = r\cos\phi \quad \wedge \quad y = r\sin\phi$$

$$\frac{d\phi}{dt} = \dot{\phi} = \frac{L}{\mu r^2} \qquad \mu = \frac{m_1 m_2}{M}$$

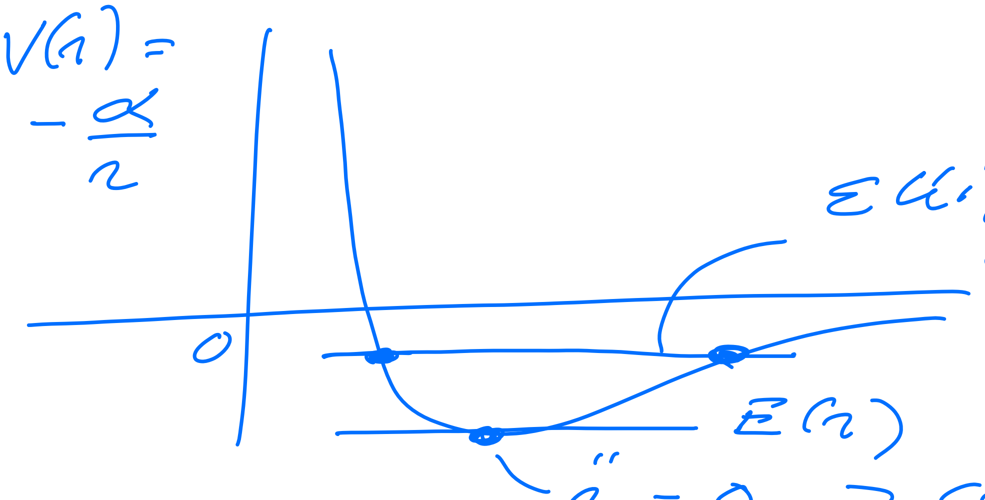
$$\mu \frac{d^2 r}{dt^2} = F(r) + \frac{L^2}{\mu r^3}$$

$$E(r,\phi) = V(r) + \underbrace{\frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2)}_{\text{kinetic}}$$

$$= \frac{1}{2}\mu\dot{r}^2 + \underbrace{V(r) + \frac{1}{2}\frac{L^2}{2\mu r^2}}$$

$$V_{eff}(r) = E(r)$$

$V(r) = -\frac{\alpha}{r}$



$\ddot{r} = 0 \Rightarrow$ constant velocity circular motion

$$\mu \ddot{r} = F(r) + \frac{L^2}{\mu r^3}$$

$$V(r) = -\frac{\alpha}{r} \Rightarrow F(r) = -\frac{\alpha}{r^2}$$

we want r as function of ϕ

– First trick:

Define $u = \frac{1}{r}$

$$\frac{dr}{d\phi} = -\frac{1}{u^2}\frac{du}{d\phi}$$

- Second trick:

rewrite $\frac{d}{dt}$ in terms of $\frac{d}{d\phi}$

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi}$$

$$\left(\dot{\phi} = \frac{L}{\mu r^2} \right)$$

$$= \frac{L}{\mu r^2}\frac{d}{d\phi}$$

$$u = \frac{1}{r}$$

$$\frac{d}{dt} = \frac{L}{\mu} u^2 \frac{d}{d\phi}$$

radial velocity $\frac{dr}{dt} = \dot{r}$

$$= \frac{L u^2}{\mu}\frac{d}{d\phi}\left(\frac{1}{u}\right)$$

$$= -\frac{L}{\mu}\frac{du}{d\phi}$$

$$\frac{d^2 r}{dt^2} = \ddot{r} = \frac{d}{dt}(\dot{r})$$

$$= \frac{L u^2}{\mu} \frac{d}{d\phi} \left[\dot{r} \right]$$

$$= - \frac{L^2 u^2}{\mu^2} \frac{d^2 u}{d\phi^2}$$

$$= \frac{F(r = \frac{1}{u})}{\mu} + \frac{L^2}{\mu^2 r^3}$$

$$\frac{L^2 u^3}{\mu^2}$$

Divide with L^2 and u^2

$$\frac{d^2 u}{d\phi^2} = - \frac{F(u)\mu}{L^2 u^2} - u$$

$$V(r) = - \frac{\alpha}{r} \Rightarrow F(r) = - \frac{\alpha}{r^2}$$

$$= - \alpha u^2$$

$$\frac{d^2 u}{} \quad + \mu\alpha \quad \mu$$

$$\frac{\quad}{d\phi^2} = + \frac{\quad}{L^2} - u$$

$$\frac{\mu\alpha}{L^2} = 0$$

$$\frac{d^2u}{d\phi^2} = -u \quad \Big/ \quad \frac{d^2x}{dt^2} = -\omega_0^2 x$$

$$u = C\cos\phi + D\sin\phi$$

$$= A\cos(\phi - \delta)$$

$$r(\phi) = \frac{1}{u(\phi)} = \frac{1}{A\cos(\phi-\delta)}$$

$$w(\phi) = u(\phi) - \frac{\mu\alpha}{L^2}$$

$$\frac{d^2w}{d\phi^2} = -w$$

$$w = B\cos(\phi - \delta)$$

$$u(\phi) = B\cos(\phi-\delta) + \frac{\mu\alpha}{\quad}$$

$L \; L$

Scale $\delta = 0$

$$u(\phi) = \frac{\mu\alpha}{L^2} + B\cos(\phi)$$

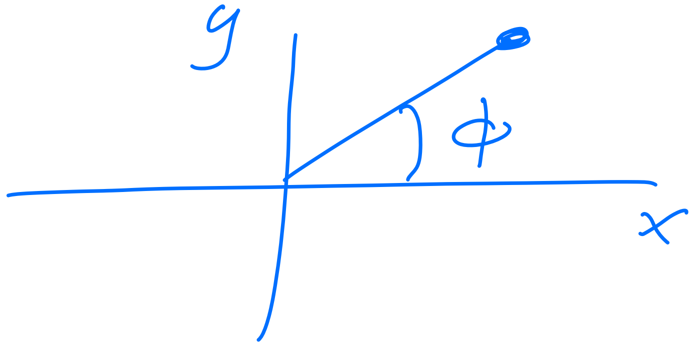
$$= \frac{\mu\alpha}{L^2}\left(1+\varepsilon\cos\phi\right)$$

$$\varepsilon = \frac{BL^2}{\alpha\mu}$$

$$\alpha = G m_1 m_2$$

$$c = \frac{L^2}{\mu\alpha}$$

$$\boxed{r(\phi) = \frac{c}{1+\varepsilon\cos\phi}}$$



$\varepsilon = 0 \Rightarrow$ circular motion.

Min and Max values of $r(\phi)$?

$$r(\phi) = \frac{c}{1+\varepsilon\cos\phi}$$

$$\frac{dr}{d\phi} = 0 = \frac{c\,\varepsilon\sin\phi}{(1+\varepsilon\cos\phi)^2}$$

$\varepsilon \neq 1$ ($\varepsilon = 1$, parabolic motion)

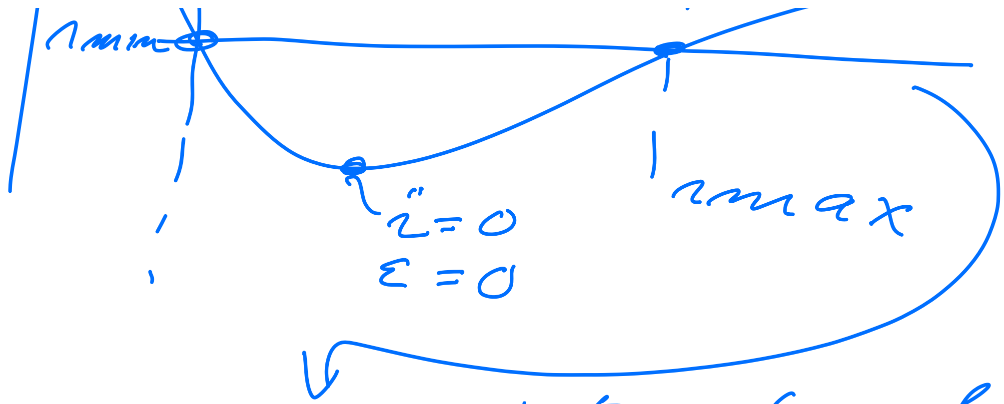
$\phi = 0 \vee \phi = \pi$

$$\phi = 0 \Rightarrow r(\phi) = \frac{c}{1+\varepsilon} = r_{min}$$

$$\phi = \pi \Rightarrow r(\phi) = \frac{c}{1-\varepsilon} = r_{max}$$

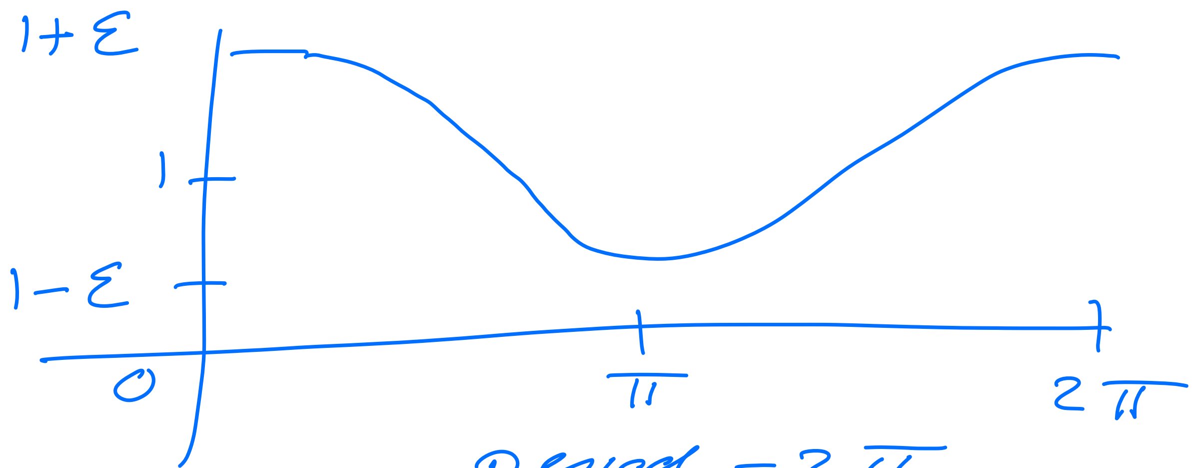
$\varepsilon \neq 1$, $\varepsilon > 0$

$\boxed{0 < \varepsilon < 1}$



elliptical orbits

$$r(\phi) = \frac{c}{1+\varepsilon\cos\phi}$$



period $= 2\pi$