PHY 321, APRIL 25, 2022

Variational calculus, constraints and Lagrangians

$$\mathcal{L}(x, v, t) \rightarrow \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)$$

$$\vec{q} = [q_1, q_2, q_3, \ldots q_N]$$

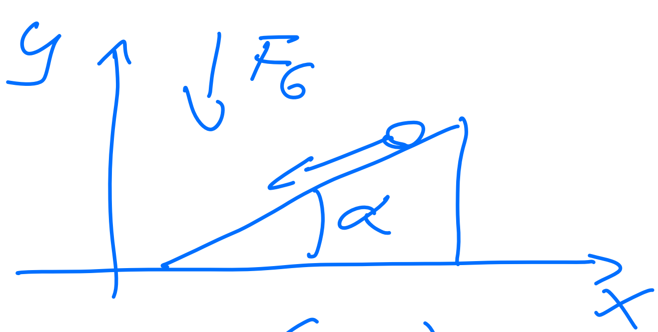
$$\mathcal{L}(x, v_x, y, v_y, t) \rightarrow \mathcal{L}(r, \dot{r}, \phi, \dot{\phi}, t)$$

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$$

Euler-Lagrange eq.

constraints:

holonomic $g(q_1, q_2, q_3, \ldots q_N) = 0$



$$g(x, y) = y - x\tan\alpha = 0$$

$$+\lambda \tan^2\alpha + \lambda - mg = 0 \Rightarrow$$

$$\lambda = mg\cos^2\alpha$$

$$\ddot{x} = -g\sin\alpha\cos\alpha$$

$$\ddot{y} = -g\sin^2\alpha$$

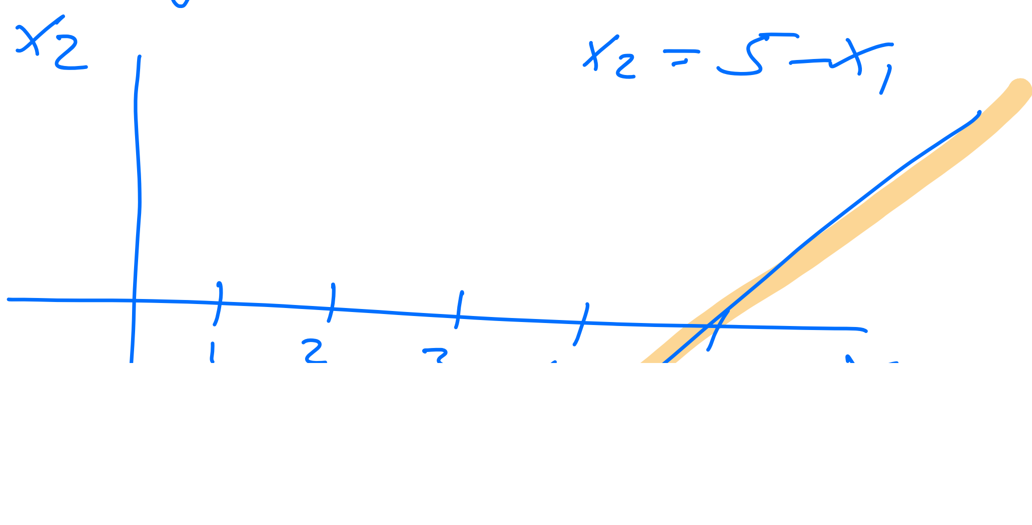
$$\mathcal{L}(x, v_x, y, v_y, t) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy + \lambda(y - x\tan\alpha)$$

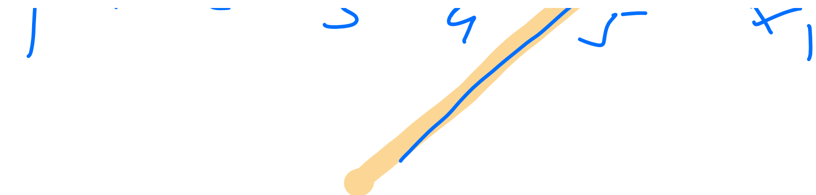
Example:

want to minimize

$$f(x_1, x_2) = -3x_1^2 - 6x_1x_2 - 5x_2^2 + 7x_1 + 5x_2$$

subject to $x_1 + x_2 = 5$





$$f(x_1, x_2) \longrightarrow h(x_1) =$$

$$-3x_1^2 - 6x_1(5-x_1) - 5(5-x_1)^2$$

$$+7x_1 + 5(5-x_1)$$

$$= -2x_1^2 + 22x_1 - 100$$

optimal value

$$\frac{dh}{dx_1} = 0 = -4x_1 + 22 \Rightarrow$$

$$x_1 = 11/2$$

$$\Rightarrow x_2 = -1/2$$

Lagrangian with constraint

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2)$$

$$+ \lambda g(x_1, x_2)$$

$$g(x_1, x_2) = x_1 + x_2 - 5 = 0$$

$$\left(\frac{\partial}{\partial x_i} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\right)\mathcal{L} = 0$$

$$= -6x_1 - 6x_2 + 7 + \lambda = 0$$

$$x_1 + x_2 = \frac{1}{6}(7+\lambda) = 5$$

$$\Rightarrow \quad \lambda = 23$$

$$\left(\frac{\partial}{\partial x_2} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_2} \right) \mathcal{L} = 0$$

$$= -6x_1 - 10x_2 + 5 + \lambda = 0$$

$$x_1 = 11/2 \,, \quad x_2 = -1/2$$

Lagrangian multipliers

Example: minimize

$f(x_1, x_2)$ subject to

$g(x_1, x_2) = 0$

satisfied by $\tilde{x} = [\tilde{x}_1, \tilde{x}_2]$

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0$$

$- \qquad \partial x_1 \qquad\qquad \partial x_2$

dx_1 and dx_2 are variations that are on the constraint

$g(\tilde{x}_1 + dx_1, \tilde{x}_2 + dx_2)$

assume small.

Taylor expand g:

$$g(\tilde{x}_1 + dx_1, \tilde{x}_2 + dx_2) \simeq g(\tilde{x}_1, \tilde{x}_2) + \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2$$

we require that

$$dg = \frac{\partial g}{\partial x_1} dx + \frac{\partial g}{\partial x_2} dx_2 = 0$$

at $(\tilde{x}_1, \tilde{x}_2)$

assume $\frac{\partial g}{\partial x_2} \neq 0$

$$dx_2 = -\left(\frac{\partial g}{\partial x_?} \Big/ \right) dx_1$$

$$\left(\partial x_1 \Big/ \frac{\partial g}{\partial x_2} \right)$$

$$df = \left[\frac{\partial f}{\partial x_1} - \frac{\partial g/\partial x_1}{\partial g/\partial x_2} \frac{\partial f}{\partial x_2} \right] \times dx_1$$

must be satisfied by all dx_1-values $\Rightarrow$

$$\left[\frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial x_1} \right]_{(x_1^*, x_2^*)} = 0$$

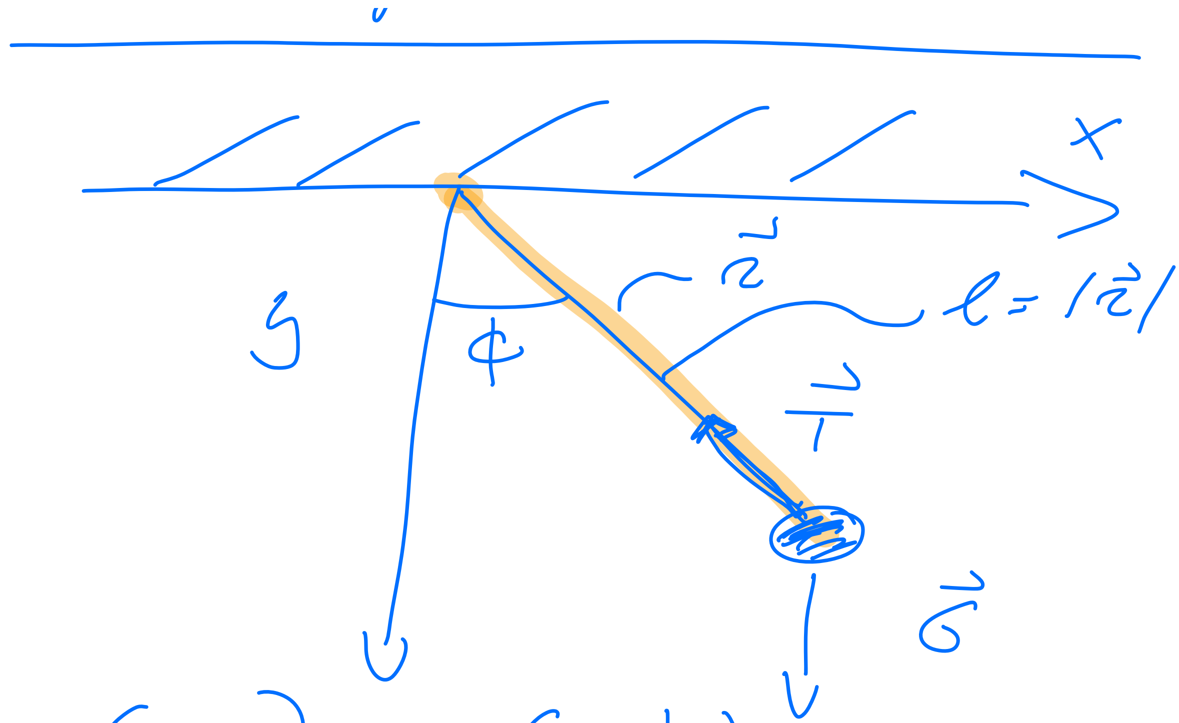
Define a parameter

$$\lambda = -\frac{\partial f}{\partial x_2} \Big/ \frac{\partial g}{\partial x_2} \Rightarrow$$

$$df = 0 \quad \left[\frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1} \right] = 0$$

Lagrangian multiplier.

Example (hw 9)



$$(x, y) \Rightarrow (r, \phi)$$

$$y = r\cos\phi \;\wedge\; x = r\sin\phi$$

constraint $r = \ell$

$$\mathcal{L}(r, \dot{r}, \phi, \dot{\phi}, t) =$$

$$\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + mgr\cos\phi$$

$$+ \lambda (r - \ell)$$

$$\left[\frac{\partial}{\partial r} - \frac{d}{dt} \frac{\partial}{\partial \dot{r}} \right] \mathcal{L} = 0$$

$$\frac{\partial}{\partial r} \mathcal{L} = m r \dot{\phi}^2 + mg\cos\phi + \lambda$$

$$\frac{\partial L}{\partial \dot r} = m\dot r \Rightarrow \frac{d}{dt} m\dot r = m\ddot r$$

$$m\ddot r = m r \dot\phi^2 + m g \cos\phi + \lambda$$

$$\frac{\partial L}{\partial \phi} = - m g r \sin\phi$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = m r^2 \ddot\phi \Rightarrow$$

$$\cancel{m} r^{\cancel{2}} \ddot\phi = - \cancel{m} g \cancel{r} \sin\phi$$

$$\boxed{\ddot\phi = - g/r \sin\phi}$$

$$= - g/l \sin\phi$$

$$= - \omega_0^2 \sin\phi$$

$$\omega_0^2 = g/l$$

$$0 = m r \dot\phi^2 + m g \cos\phi + \lambda \Rightarrow$$

$$\lambda = - m r \dot\phi^2 - m g \cos\phi$$

when solving

$\ddot{\phi} = -\omega_0^2 \sin\phi \rightarrow$

we find ϕ and $\dot{\phi}$