PHY 321, APRIL 20, 2022

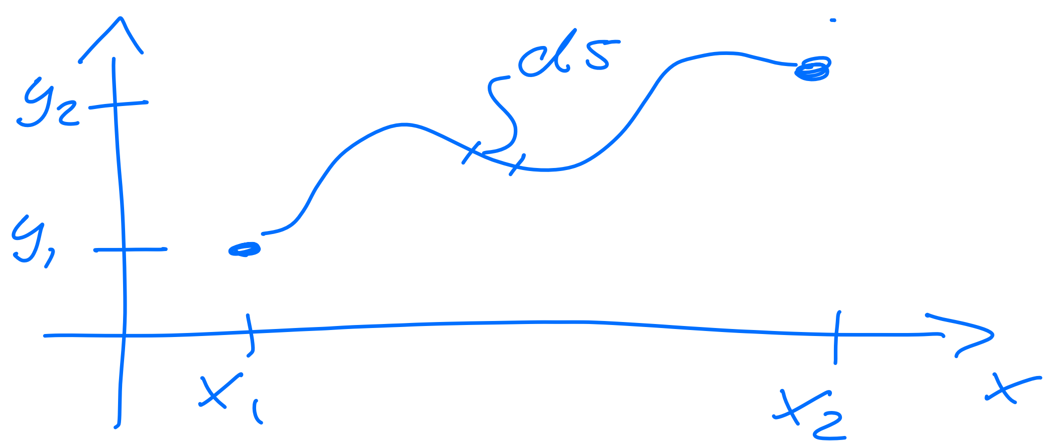
action

$$S = \int_{t_1}^{t_2} \mathcal{L}\left(x, \underbrace{\frac{dx}{dt}}_{v}, t\right) dt$$

$$\mathcal{L} = K - V$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v} = 0$$

Example: shortest between two points (x_1, y_1) and (x_2, y_2)



$$l = \int_{x_1}^{x_2} ds \qquad ds = \sqrt{dx^2 + dy^2}$$

$$dy = \frac{dy}{dx}\,dx = y'\,dx$$

$$ds = \sqrt{1+(y')^2}\,dx$$

$$L(s) = \int_{x_1}^{x_2} \sqrt{1+(y')^2}\,dx$$

Euler-Lagrange:

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'} = 0$$

$$f = (1+(y')^2)^{1/2}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0 \qquad \mathcal{L} = f$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$\frac{\partial f}{\partial y'} = c \quad \text{(constant)}$$

$$f = \sqrt{(1+(y')^2} \quad \Rightarrow$$

$$\frac{y'}{(1+(y')^2)^{1/2}} = C$$

$$(y')^2 = c^2(1+(y')^2) \Rightarrow$$

$$(y')^2(1-c^2) = c^2 \Rightarrow$$

$$(y')^2 = D = \frac{c^2}{1-c^2}$$

$y' = \pm\sqrt{D} = A$ (constant)

$\Rightarrow$

$$\frac{dy}{dx} = A \Rightarrow$$

$$\boxed{\begin{array}{l} y(x) = Ax + B \\ \text{straight line} \end{array}}$$

Example 2 : Energy conservation

$$\mathcal{L}(x, v, t) = \frac{1}{2}mv^2 - V(x)$$

$$\frac{dx}{dt} = v$$

note on t dependence

$$S = \int_{t_1}^{t_2} \mathcal{L}(x, v, t)\, dt$$

stationary quantity

$$\frac{dS}{dt} = 0 \implies \frac{d\mathcal{L}}{dt} = 0$$

need :

$$\frac{\partial \mathcal{L}}{\partial v} \frac{dv}{dt} + \frac{\partial \mathcal{L}}{\partial x} \frac{dx}{dt}$$

Euler-Lagrange eqs :

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial v} = 0$$

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial v} \frac{d^2 x}{dt^2} + \left[\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial v} \right] \frac{dx}{dt}$$

$$= \frac{d}{dt} \left[\frac{dx}{dt} \frac{\partial \mathcal{L}}{\partial v} \right]$$

$$= \frac{d}{dt}\left[\dot{v}\,\frac{\partial \mathcal{L}}{\partial v}\right]$$

$$= \frac{d\mathcal{L}}{dt} \implies$$

$$\frac{d}{dt}\left[v\frac{\partial \mathcal{L}}{\partial v} - \mathcal{L}\right] = 0$$

$$\mathcal{L} = \frac{1}{2}mv^2 - V(x)$$

$$\frac{d}{dt}\left[mv^2 - \frac{1}{2}mv^2 + V(x)\right]$$

$$= \frac{d}{dt}\left[\ E\ \right] = 0$$

Energy is conserved

<u>Example 3</u>

$$(x, y) \Rightarrow (r, \phi)$$

$$\mathcal{L}(x, \dot{x}, y, \dot{y}, t) \Rightarrow$$

$$\mathcal{L}(r, \dot{r}, \phi, \dot{\phi}, t) =$$

$$\frac{1}{2} m \left(\dot r^2 + r^2 \dot\phi^2 \right) - V(r,\phi)$$

<u>r-equation</u>

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot r} = 0$$

$$m r \dot\phi^2 \underbrace{- \frac{dV}{dr}}_{F_r} = m \ddot r$$

$$\boxed{\ddot r = r\dot\phi^2 + F_r/m}$$

ϕ-equation

$$\frac{\partial L}{\partial \phi} = \frac{d}{dt}\frac{\partial L}{\partial \dot\phi}$$

$$\vec\nabla V = \frac{\partial V}{\partial r}\hat r + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat\phi$$

$$F_\phi = -\frac{dV}{d\phi}$$

$$r F_\phi = \frac{d}{dt}\left(m r^2 \dot\phi \right)$$

dt

$$V(r,\phi) = V(r)$$

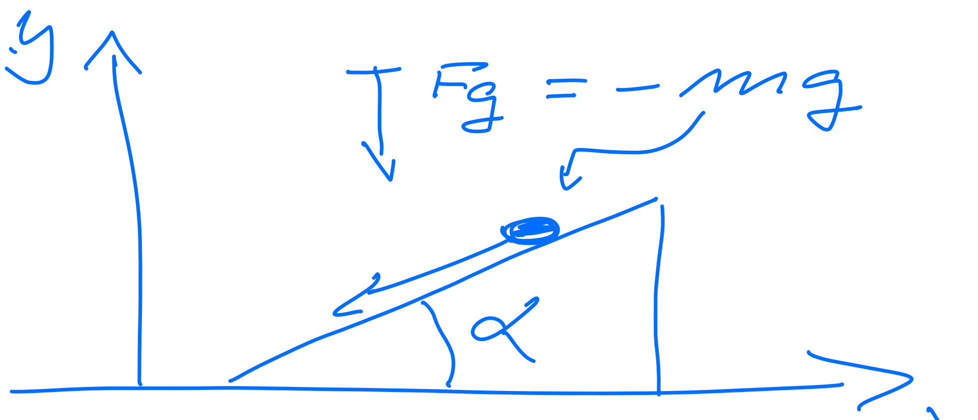
$$F_\phi = ?$$

$$F_\phi = 0$$

$$0 = \frac{d}{dt}\left(m r^2 \dot{\phi}\right)$$

$$\Rightarrow \frac{d}{dt} L = 0$$

Variational calculus & constrained motion



$$\boxed{y = x \cdot \tan\alpha}$$