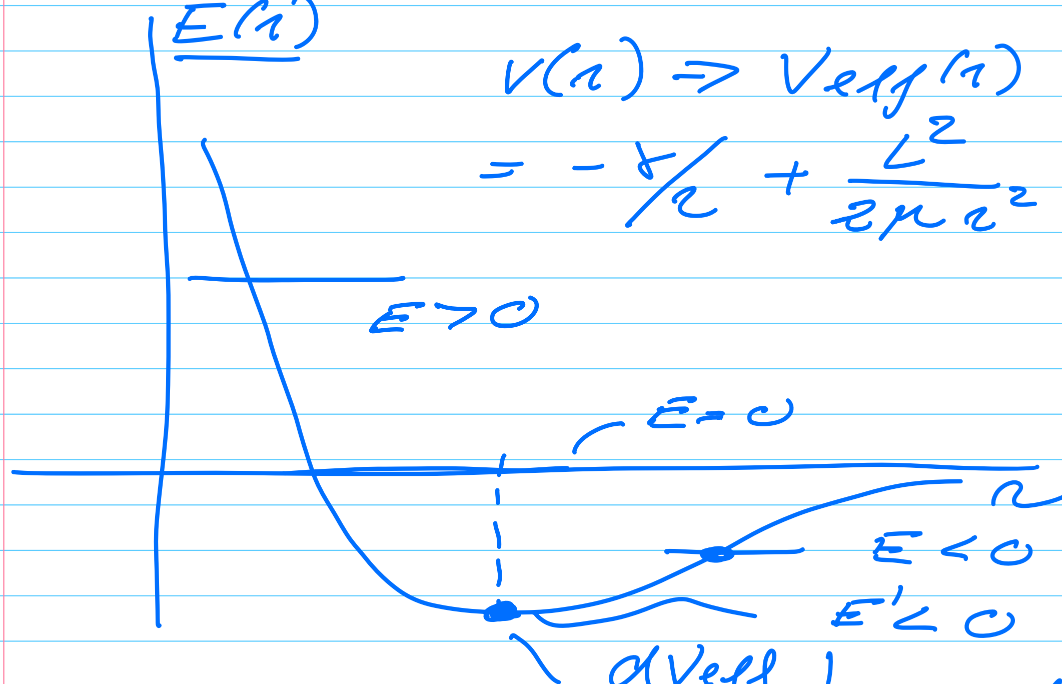
PHY 321, MARCH 29, 2023

$$V(r) \Rightarrow V_{eff}(r) = -\gamma/r + \frac{L^2}{2\mu r^2}$$



$$\left.\frac{dV_{eff}}{dr}\right|_{r_{min}} = 0$$

$$\Rightarrow F_{eff}(r_{min}) = \mu \ddot{r} = 0$$

$$K = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right)$$

$$\frac{d\phi}{dt} = \dot{\phi} = \frac{L}{\mu r^2} = \text{const}$$

$r = r_{min}$

$$r^2 = r_{min}^2$$

Kinetic with circular motion is conserved as function of t.

$$V = \frac{1}{2} k (x^2 + y^2) = \frac{1}{2} k r^2$$

$$V(r_{min}) = \frac{1}{2} k r_{min}^2$$

$$V = -\gamma / r = -\gamma / \sqrt{x^2+y^2}$$

$$V(r_{min}) = -\gamma / r_{min} = \text{const.}$$

$$\mu \ddot{r} = -\gamma / r^2 + \frac{L^2}{\mu r^3}$$

$$V_{eff}(r) = -\gamma / r + \frac{L^2}{2 \mu r^2}$$

$$\frac{d\phi}{dt} = \dot{\phi} = \frac{L}{\mu r^2}$$

– First trick Define $r = \frac{1}{u}$

we want r as function of ϕ $r(t) \Rightarrow r(\phi)$

$$\frac{dr}{d\phi} = -\frac{1}{u^2} \frac{du}{d\phi}$$

– Second trick

rewrite $\frac{d}{dt}$ in term $\frac{d}{d\phi}$

chain rule

$$\frac{d}{dt} = \frac{d\phi}{dt}\cdot\frac{d}{d\phi} = \underbrace{\dot{\phi}}_{\frac{L}{\mu r^2}}\frac{d}{d\phi}$$

$$= \frac{L}{\mu r^2}\frac{d}{d\phi} = \frac{L}{\mu}u^2\frac{d}{d\phi}$$

radial velocity

$$\frac{dr}{dt} = \dot{r} = \frac{L u^2}{\mu}\frac{d}{d\phi}\left[\frac{1}{u}\right]$$

$$= -\frac{L}{\mu}\frac{du}{d\phi}$$

$$\frac{d}{dt}\left[\frac{dr}{dt}\right] = \ddot{r}$$

$$= \frac{d}{dt}\left[-\frac{L}{\mu}\frac{du}{d\phi}\right]$$

$$\ddot{r} = -\frac{L^2 u^2}{\mu^2}\frac{d^2 u}{d\phi^2}$$

$$= \overbrace{\frac{F(r)}{\mu}}^{-\frac{k u^2}{\mu}} + \overbrace{\frac{L^2}{\mu r^3}}^{\frac{L^2 u^3}{\mu}}$$

$$F(r) = -\gamma/r^2 = -\gamma u^2$$

$$\boxed{\frac{d^2u}{d\phi^2} = +\frac{\gamma\mu}{L^2} - u}$$

assume $\gamma = 0$

$$\frac{d^2u}{d\phi^2} = -u \quad \Bigg| \quad \frac{d^2x}{dt^2} = -w_0^2 x(t)$$

$$\downarrow$$

$$u(\phi) = A\cos\phi + B\sin\phi$$

$$= C\cos(\phi - \delta)$$

$$r(\phi) = \frac{1}{u(\phi)} = \frac{1}{C\cos(\phi-\delta)}$$

$$w(\phi) = u(\phi) - \gamma\mu/L^2$$

$$\frac{d^2w}{d\phi^2} = -w$$

$$w(\phi) = D\cos(\phi - \delta)$$

$$u(\phi) = \frac{\mu\gamma}{L^2} + D\cos(\phi - \delta)$$

scale $\delta = 0$

$$u(\phi) = \mu\gamma/L^2 + D\cos\phi$$

$$= \frac{\mu \gamma}{L^2}\left(1 + \varepsilon \cos\phi\right)$$

↑ eccentricity

$$\varepsilon = D L^2 / \gamma \mu$$

Define $C = L^2 / \mu \gamma$

$$u(\phi) = \frac{1}{r(\phi)} \Rightarrow$$

$$\boxed{r(\phi) = \frac{C}{1 + \varepsilon \cos\phi}}$$

$\varepsilon = 0 \Rightarrow r(\phi) = C$

circular motion ☺

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{L}{\mu r^2} = \frac{L}{\mu C^2}$$

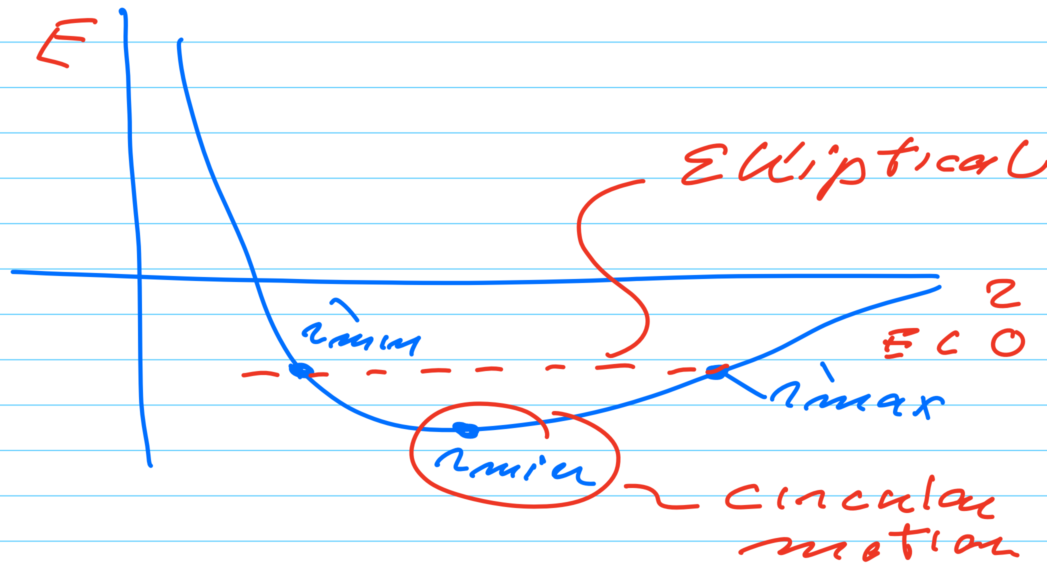
$$r(\phi) = \frac{C}{1 + \varepsilon \cos\phi}$$

$\varepsilon > 0 \quad (0 < \varepsilon < 1)$

$\frac{dr}{d\phi} = 0$, which ϕ give r_{max} and r_{min}?

(i) $\phi = 0 \Rightarrow r(\phi) = \dfrac{c}{1+\varepsilon} = r_{min}$

(ii) $\phi = \pi \Rightarrow r(\phi) = \dfrac{c}{1-\varepsilon} = r_{max}$



$x = r\cos\phi \quad , \quad y = r\sin\phi$

$$r = \frac{c}{1+\varepsilon\cos\phi}$$

$$r(1+\varepsilon\cos\phi) = c$$

$$r + \varepsilon x = c$$

$$r = c - \varepsilon x \qquad \text{square}$$

$$r^2 = c^2 + \varepsilon^2 x^2 + 2\varepsilon x$$

$$x^2 + y^2 = \text{——} \quad , \quad \text{——}$$

$$x^2(1-\varepsilon^2) + 2c\,\varepsilon x + y^2 = c^2$$