PHY 321, APRIL 3, 2023

COM-frame $\vec{R} = 0$

$x = r\cos\phi \wedge y = r\sin\phi$

$r \in [0, \infty) \quad \phi \in [0, 2\pi]$

$$\mu \ddot{r} = \mu a_r = F(r) + \frac{l^2}{\mu r^3}$$

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{l^2}{\mu r^2}$$

$\left.\begin{array}{l} l \text{ - conserved} \\ P \text{ - conserved} \end{array}\right\}$ internal forces

$\vec{F}(r) = f(r)\vec{r}$

$$K = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right)$$

$$V(r) = -\gamma/r^\beta$$

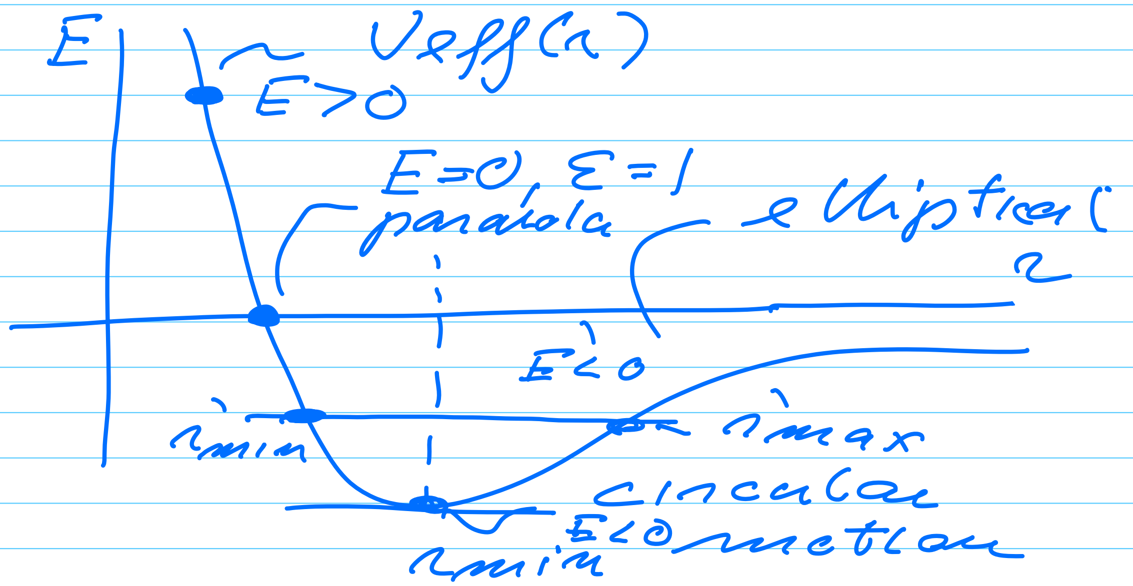
$$V_{eff}(r) = V(r) + \frac{l^2}{2\mu r^2}$$

$$r(\phi) = \frac{c}{1 + \varepsilon\cos\phi}$$

$$c = l^2/\mu\gamma$$

$$\varepsilon = A \frac{L^2}{\gamma \mu}$$ constant

(c) $\varepsilon = 0 \Rightarrow$ circular motion



how to find r_{min}?

$$\left.\frac{dV_{eff}}{dr}\right|_{r=r_{min}} = 0$$

$$V_{eff} = -\gamma/_{r^\beta} + \frac{L^2}{2\mu r^2}$$

$$\boxed{r_{min}^{2-\beta} = \frac{L^2}{\gamma \beta \mu}}$$

gravitational $\beta = 1$

$$r_{min} = \frac{L^2}{8\mu}$$

$\beta = 2, 3$, part 1 asks about stable orbits

(ii) $0 < \varepsilon < 1$

$$r_{min} = \frac{c}{1+\varepsilon}$$

$$r_{max} = \frac{c}{1-\varepsilon}$$

t-period $= 2\pi$

$$d = \frac{c\varepsilon}{1-\varepsilon^2} \qquad a = \frac{c^2}{(1-\varepsilon^2)^2}$$

$$b = a\sqrt{(1-\varepsilon^2)}$$

$$\left(\frac{x+d}{a}\right)^2 + \frac{y^2}{b^2} = 1$$

$$r_{max} = a + d$$

Energy consideration

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{eff}(r)$$

$$= \frac{1}{2}\mu\dot{r}^2 + V(r) + \frac{L^2}{2\mu r^2}$$

at r_{min} $\quad \frac{dr}{dt} = \dot{r} = 0$

$$E(r = r_{min}) = -\frac{\gamma}{r_{min}} + \frac{L^2}{2\mu r^2_{min}}$$

$$= \frac{1}{2r_{min}}\left[\frac{L^2}{\mu r_{min}} - 2\gamma\right]$$

$$\left(r_{min} = \frac{c}{1+\varepsilon} = \frac{L^2}{\gamma\mu(1+\varepsilon)}\right)$$

(insert r_{min})

$$= \frac{\gamma\mu}{2L^2}\left[\gamma(1+\varepsilon) - 2\gamma\right]$$

$$= \underbrace{\frac{\gamma^2\mu}{2L^2}}(1+\varepsilon)(\varepsilon-1)$$

$$= B\,(\varepsilon^2-1)$$

$\mu > 0$, $L^2 > 0$, $\gamma > 0$

$\Rightarrow B > 0$

if $E < 0 \Rightarrow$

$(\varepsilon^2-1) < 0 \Rightarrow$

$\varepsilon < 1$

(i) $\varepsilon = 0$; circular

(ii) $0 < \varepsilon < 1$; elliptical

(iii) $E = 0 \Rightarrow (\varepsilon^2-1) = 0$

$\Rightarrow \varepsilon = +1$

which motion?

$$r(\phi) = \frac{c}{1+\varepsilon\cos\phi} = \frac{c}{1+\cos\phi}$$

Diverges when $\phi = \pm\pi$

$r(1+\cos\phi) = c \qquad r\cos\phi = x$

$r = c - x$

square $\quad r^2 = x^2+y^2$

$x^2+y^2 = c^2 - 2cx + x^2$

$y^2 = c^2 - 2cx \quad \Rightarrow$

$$x(y) = \frac{c}{2} - y^2/2c$$

$f(y) = x(y) = a_0 + a_1 y + a_2 y^2$

$a_1 = 0 \qquad a_0 = c/2 \qquad a_2 = -\frac{1}{2c}$

parabolic motion,

(iv) $E > 0 \qquad \dot{r}\big|_{r=r_{min}} = 0$

$E = B(\varepsilon^2 - 1)$

$\Rightarrow \varepsilon = ?$

$$\boxed{\varepsilon > 1}$$

$$r(\phi) = \frac{c}{1+\varepsilon\cos\phi}$$

Denominator vanishes when

$$\cos\phi = -1/\varepsilon$$

$$x = r\cos\phi$$

$$r = c - \varepsilon x$$

square

$$x^2+y^2 = c^2 + \varepsilon^2 x^2 - 2c\varepsilon x$$

$$x^2(1-\varepsilon^2) + y^2 + 2c\varepsilon x = c^2$$

$$x^2(\varepsilon^2-1) - y^2 - 2c\varepsilon x = -c^2$$

complete squares for x

$$\delta = \frac{c\varepsilon}{\varepsilon^2-1} \qquad \alpha = \frac{c}{\varepsilon^2-1}$$

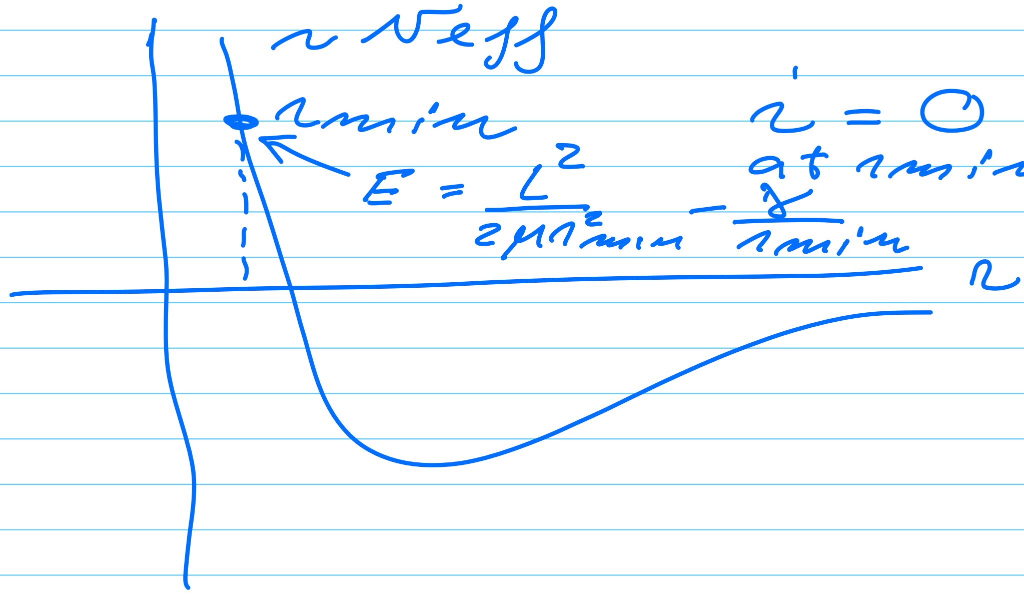
$$\delta = \varepsilon\alpha \qquad \beta = \frac{c}{\sqrt{\varepsilon^2-1}}$$

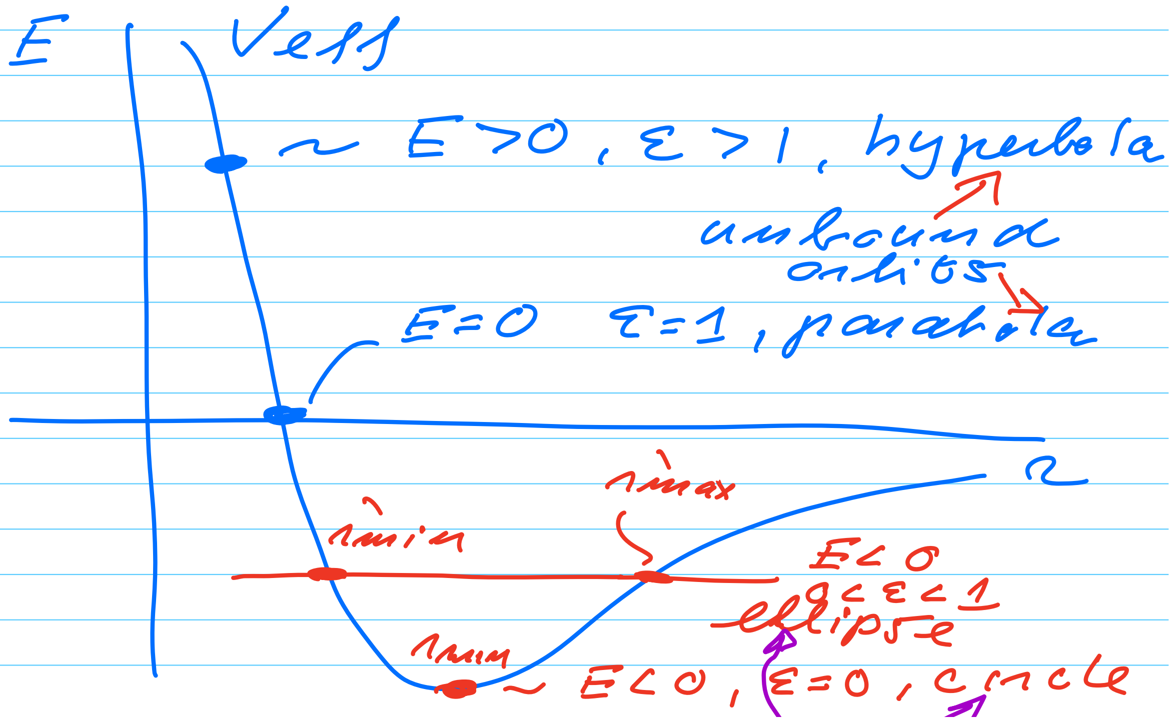
$$(x-\delta)^2(\varepsilon^2-1) - y^2 - c^2 + \frac{\varepsilon^2 c^2}{\varepsilon^2-1}$$

$$\frac{(x-\delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$$

$E > 0 \qquad \varepsilon > 1$

$\Rightarrow$ hyperbolic motion.





bound
orbits