Modelli computazionali per la scuola superiore

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Unique opportunities

Computing competence represents a central element in scientific problem solving, from basic education and research to essentially almost all advanced problems in modern societies.

Computing competence **enlarges the body of tools available to students** and scientists beyond classical tools and **allows for a more generic handling of problems**. Focusing on algorithmic aspects **results in deeper insights** about scientific problems.

Why is computing competence important?

The impact of the computer on mathematics and science is tremendous: **science and industry now rely on solving mathematical problems through computing**.

- Computing can increase the relevance in education by solving more realistic problems earlier.
- Computing through programming can be excellent training of creativity.
- Computing can enhance the understanding of abstractions and generalization.
- Computing can decrease the need for special tricks and tedious algebra, and shifts the focus to problem definition, visualization, and "what if" discussions.

Come introdurre metodi numerici nelle scuole superiori: Un modello per prede e predatori

The population dynamics of a simple predator-prey system is a classical example shown in many biology textbooks when ecological systems are discussed. The system contains all elements of the scientific method:

- The set up of a specific hypothesis combined with
- the experimental methods needed (one can study existing data or perform experiments)
- analyzing and interpreting the data and performing further experiments if needed
- trying to extract general behaviors and extract eventual laws or patterns
- develop mathematical relations for the uncovered regularities/laws and test these by per forming new experiments

Lepre e lince nella baia di Hudson

Lots of data about populations of hares and lynx collected from furs in Hudson Bay, Canada, are available. It is known that the populations oscillate. Why? We shall demonstrate the scientific method by

1. plotting the data
2. derive a simple model for the population dynamics
3. (fitting parameters in the model to the data)
4. using the model predict the evolution other predator-pray systems

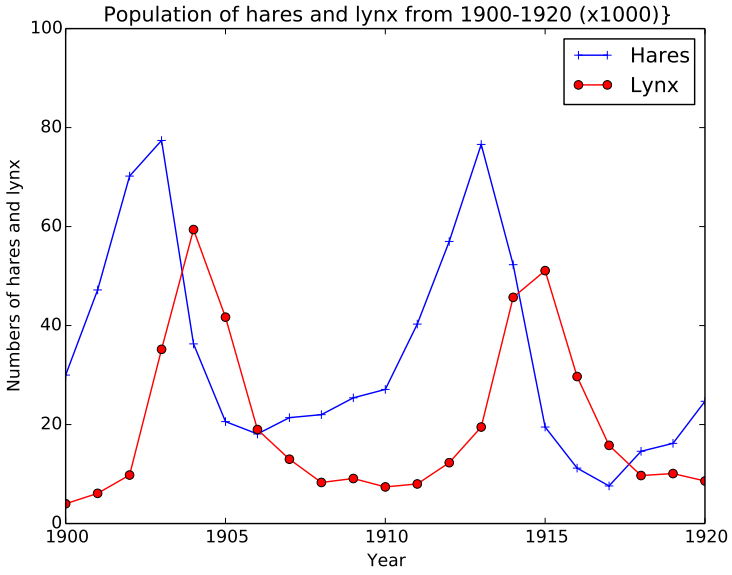
Hudson bay data

Anno	Lepri (x1000)	Linci (x1000)
1900	30.0	4.0
1901	47.2	6.1
1902	70.2	9.8
1903	77.4	35.2
1904	36.3	59.4
1905	20.6	41.7
1906	18.1	19.0
1907	21.4	13.0
1908	22.0	8.3
1909	25.4	9.1
1910	27.1	7.4
1911	40.3	8.0
1912	57	12.3
1913	76.6	19.5
1914	52.3	45.7
1915	19.5	51.1
1916	11.2	29.7
1917	7.6	15.8
1918	14.6	9.7
1919	16.2	10.1
1920	24.7	8.6

Plotting the data

```
import numpy as np
from  matplotlib import pyplot as plt

# Load in data file
data = np.loadtxt('Hudson_Bay.dat', delimiter=',', skiprows=1)
# Make arrays containing x-axis and hares and lynx populations
year = data[:,0]
hares = data[:,1]
lynx = data[:,2]

plt.plot(year, hares ,'b-+', year, lynx, 'r-o')
plt.axis([1900,1920,0, 100.0])
plt.xlabel(r'Year')
plt.ylabel(r'Numbers of hares and lynx ')
plt.legend(('Hares','Lynx'), loc='upper right')
plt.title(r'Population of hares and lynx from 1900-1920 (x1000)}')
plt.savefig('Hudson_Bay_data.pdf')
plt.savefig('Hudson_Bay_data.png')
plt.show()
```

Hares and lynx in Hudson bay from 1900 to 1920



Why now create a computer model for the hare and lynx populations?

- We see oscillations in the data
- What causes cycles to slow or speed up?
- What affects the amplitude of the oscillation or do you expect to see the oscillations damp to a stable equilibrium?
- With a model we can better *understand the data*
- More important: we can understand the ecology dynamics of predator-pray populations

The traditional (top-down) approach

The classical way (in all books) is to present the Lotka-Volterra equations:

$$\frac{dH}{dt} = H(a - bL)$$
$$\frac{dL}{dt} = -L(d - cH)$$

Here,

- H is the number of preys
- L the number of predators
- a, b, d, c are parameters

Most books quickly establish the model and then use considerable space on discussing the qualitative properties of this *nonlinear system of ordinary differential equations* (which cannot be solved analytically)

The "new" discrete bottom-up approach

The bottom-up approach.

- Start with experimental data and discuss the methods which have been used to collect the data, the assumptions, the electronic devices, the aims etc. That is, expose the students to the theory and assumptions behind the data that have been collected and motivate for the scientific method.
- Where appropriate the students should do the experiment(s) needed to collect the data.
- The first programming tasks are to read and visualize the data to see if there are patterns or regularities. This strengthens a research-driven intuition.
- Now we want to increase the understanding through modeling.
- Most of the biology lies in the *derivation* of the model. We shall focus on an intuitive discrete approach that leads to difference equations that can be programmed *and solved* directly.

Basic (computer-friendly) mathematics notation

- Time points: $t_0, t_1, \ldots, t_m$
- Uniform distribution of time points: $t_n = n\Delta t$
- H^n: population of hares at time t_n

- L^n: population of lynx at time t_n
- We want to model the changes in populations, $\Delta H = H^{n+1} - H^n$ and $\Delta L = L^{n+1} - L^n$ during a general time interval $[t_{n+1}, t_n]$ of length $\Delta t = t_{n+1} - t_n$

Basic dynamics of the population of hares, vita e morte

The population of hares evolves due to births and deaths

$$\Delta H = a\Delta t H^n$$

However, hares have an additional loss in the population because they are eaten by lynx. All the hares and lynx can form $H \cdot L$ pairs in total. When such pairs meet during a time interval Δt, there is some small probablity that the lynx will eat the hare. So in fraction $b\Delta t HL$, the lynx eat hares. This loss of hares and must be accounted for: subtracted in the equation for hares:

$$\Delta H = a\Delta t H^n - b\Delta t H^n L^n$$

Basic dynamics of the population of lynx

We assume that the primary growth for the lynx population depends on sufficient food for raising lynx kittens, which implies an adequate source of nutrients from predation on hares. Thus, the growth of the lynx population does not only depend of how many lynx there are, but on how many hares they can eat. In a time interval $\Delta t HL$ hares and lynx can meet, and in a fraction $b\Delta t HL$ the lynx eats the hare. All of this does not contribute to the growth of lynx, again just a fraction of $b\Delta t HL$ that we write as $d\Delta t HL$. In addition, lynx die just as in the population dynamics with one isolated animal population, leading to a loss $-c\Delta t L$.

The accounting of lynx then looks like

$$\Delta L = d\Delta t H^n L^n - c\Delta t L^n$$

Evolution equations

By writing up the definition of ΔH and ΔL, and putting all assumed known terms H^n and L^n on the right-hand side, we have

$$H^{n+1} = H^n + a\Delta t H^n - b\Delta t H^n L^n$$

$$L^{n+1} = L^n + d\Delta t H^n L^n - c\Delta t L^n$$

Note:

- These equations are ready to be implemented!
- But to start, we need H^0 and L^0
  (which we can get from the data)
- We also need values for a, b, d, c

Adapt the model to the Hudson Bay case

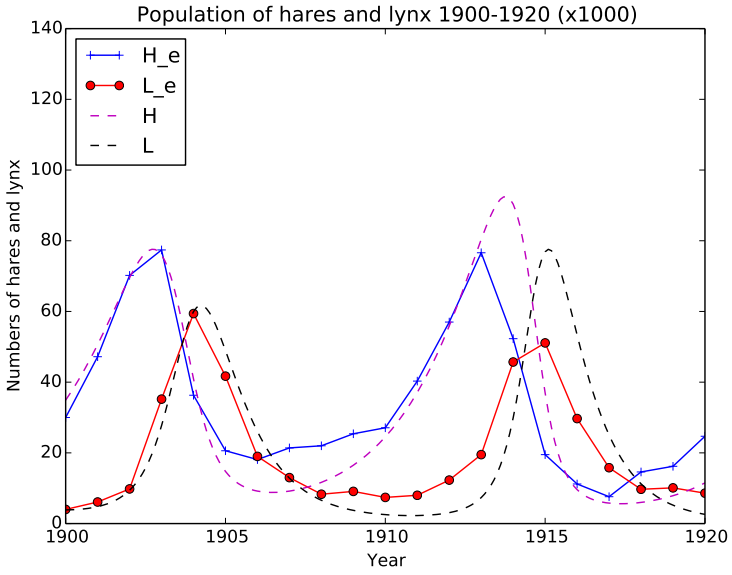
- As always, models tend to be general - as here, applicable to "all" predator-pray systems
- The critical issue is whether the *interaction* between hares and lynx is sufficiently well modeled by constHL
- The parameters a, b, d, and c must be estimated from data
- Measure time in years
- $t_0 = 1900$, $t_m = 1920$

The program

```
import numpy as np
import matplotlib.pyplot as plt

def solver(m, H0, L0, dt, a, b, c, d, t0):
    """Solve the difference equations for H and L over m years
    with time step dt (measured in years."""

    num_intervals = int(m/float(dt))
    t = np.linspace(t0, t0 + m, num_intervals+1)
    H = np.zeros(t.size)
    L = np.zeros(t.size)

    print 'Init:', H0, L0, dt
    H[0] = H0
    L[0] = L0

    for n in range(0, len(t)-1):
        H[n+1] = H[n] + a*dt*H[n] - b*dt*H[n]*L[n]
        L[n+1] = L[n] + d*dt*H[n]*L[n] - c*dt*L[n]
    return H, L, t

# Load in data file
data = np.loadtxt('Hudson_Bay.csv', delimiter=',', skiprows=1)
# Make arrays containing x-axis and hares and lynx populations
t_e = data[:,0]
H_e = data[:,1]
L_e = data[:,2]
```

```
# Simulate using the model
H, L, t = solver(m=20, H0=34.91, L0=3.857, dt=0.1,
                 a=0.4807, b=0.02482, c=0.9272, d=0.02756,
                 t0=1900)

# Visualize simulations and data
plt.plot(t_e, H_e, 'b-+', t_e, L_e, 'r-o', t, H, 'm--', t, L, 'k--')
plt.xlabel('Year')
plt.ylabel('Numbers of hares and lynx')
plt.axis([1900, 1920, 0, 140])
plt.title(r'Population of hares and lynx 1900-1920 (x1000)')
plt.legend(('H_e', 'L_e', 'H', 'L'), loc='upper left')
plt.savefig('Hudson_Bay_sim.pdf')
plt.savefig('Hudson_Bay_sim.png')
plt.show()
```

The plot



Le equazioni di Newton con forze gravitazionali

```
import numpy as np
from math import *
import matplotlib.pyplot as plt

def solver(m, dt, t0):
    """Solve the difference equations for H and L over m years
```

```
    with time step dt (measured in years."""

    num_intervals = int(m/float(dt))
    t = np.linspace(t0, t0 + m, num_intervals+1)
    x = np.zeros(t.size)
    y = np.zeros(t.size)
    vx = np.zeros(t.size)
    vy = np.zeros(t.size)
    r = np.zeros(t.size)
    v = np.zeros(t.size)


    x[0] = 1.0
    y[0] = 0.0
    vx[0] = 2.0*pi
    vy[0] = 0.0
    pi4 = 4.0*pi*pi
    for n in range(0, len(t)-1):
        x[n+1] = x[n] + dt*vx[n]
        y[n+1] = y[n] + dt*vy[n]
        r3 = (x[n]*x[n]+y[n]*y[n])**3
        vx[n+1] = vx[n] -dt*pi4*x[n]/r3
        vy[n+1] = vy[n] -dt*pi4*y[n]/r3
        v[n+1] = sqrt(vx[n+1]*vx[n+1]+vy[n+1]*vy[n+1])
        r[n+1] = sqrt(x[n+1]*x[n+1]+y[n+1]*y[n+1])

    return r, v, t
# Simulate using the model
m =20 # 20 years
dt =0.01 # stepsize
t0 =0.0
r, v, t = solver(m, dt,t0)
# Visualize simulations and data
plt.plot(r, v, 'b-+')
plt.xlabel('r')
plt.ylabel('velocity')
plt.axis([0, 2.0, 0, 4*pi])
plt.title(r'Velocity versus position')
plt.savefig('SunEarth.pdf')
plt.show()
```