Rodeo algorithm and quantum computing

Morten Hjorth-Jensen

Department of Physics and Center for Computing in Science Education, University of Oslo, Norway

qGap seminar September 30, 2025

What is this talk about?

Rodeo Algorithm

In PRL **127** (2021), Choi *et al.,* presented a stochastic quantum computing algorithm that can prepare any eigenvector of a quantum Hamiltonian within a selected energy interval $[E - \epsilon, E + \epsilon]$.
In order to reduce the spectral weight of all other eigenvectors by a suppression factor δ, the required computational effort scales as $O[|\log \delta|/(p\epsilon)]$, where p is the squared overlap of the initial state with the target eigenvector. The method uses auxiliary qubits to control the time evolution of the Hamiltonian minus some tunable parameter E. With each auxiliary qubit measurement, the amplitudes of the eigenvectors are multiplied by a stochastic factor that depends on the proximity of their energy to E. In this manner, one can converge to the target eigenvector with exponential accuracy in the number of measurements.

What is this talk about?

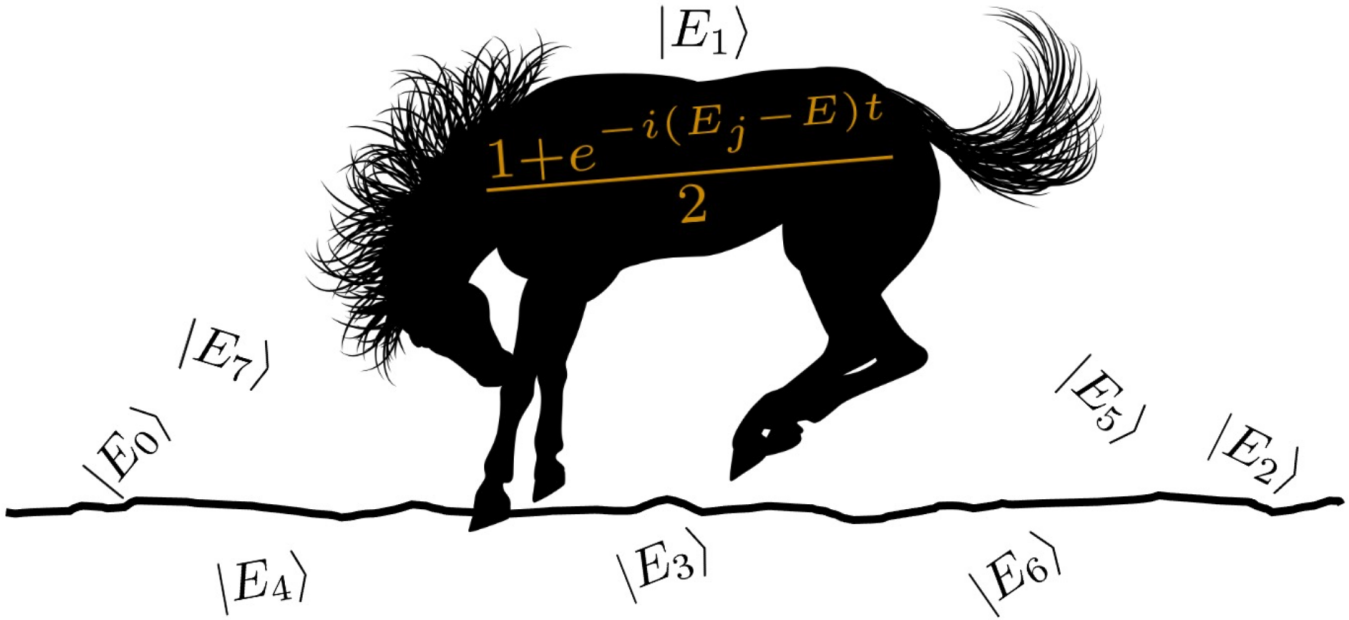
Rodeo Algorithm, more

In addition to preparing eigenvectors, the method can also compute the full spectrum of the Hamiltonian. For energy eigenvalue determination with error ϵ, the computational scaling is $O[(\log \epsilon)^2/(p\epsilon)]$. For eigenstate preparation, the computational scaling is $O(\log \Delta/p)$, where Δ is the magnitude of the orthogonal component of the residual vector. The speed for eigenstate preparation is exponentially faster than that for phase estimation or adiabatic evolution.

The three steps of quantum control

1. **Quantum correlations:** Understanding and preparing an initial state
2. **Quantum dynamics:** Controlled evolution towards a desired final state
3. **Quantum measurements:** Measuring and characterizing the final state

Rodeo algorithm



Kenneth Choi, D.L., Joey Bonitati, Zhengrong Qian, Jacob Watkins, PRL 127, 040505 (2021)

Why is the Rodeo algorithm interesting?

The rodeo algorithm operates by shaking off all other states until only the target eigenvector remains.

1. The rodeo algorithm has the advantage that it can be applied to any quantum Hamiltonian
2. It is a recursive algorithm that achieves exponential convergence in the number of cycles
3. It can be used to compute the full energy spectrum as well as prepare any energy eigenstate.

The Rodeo algorithm, as a figure

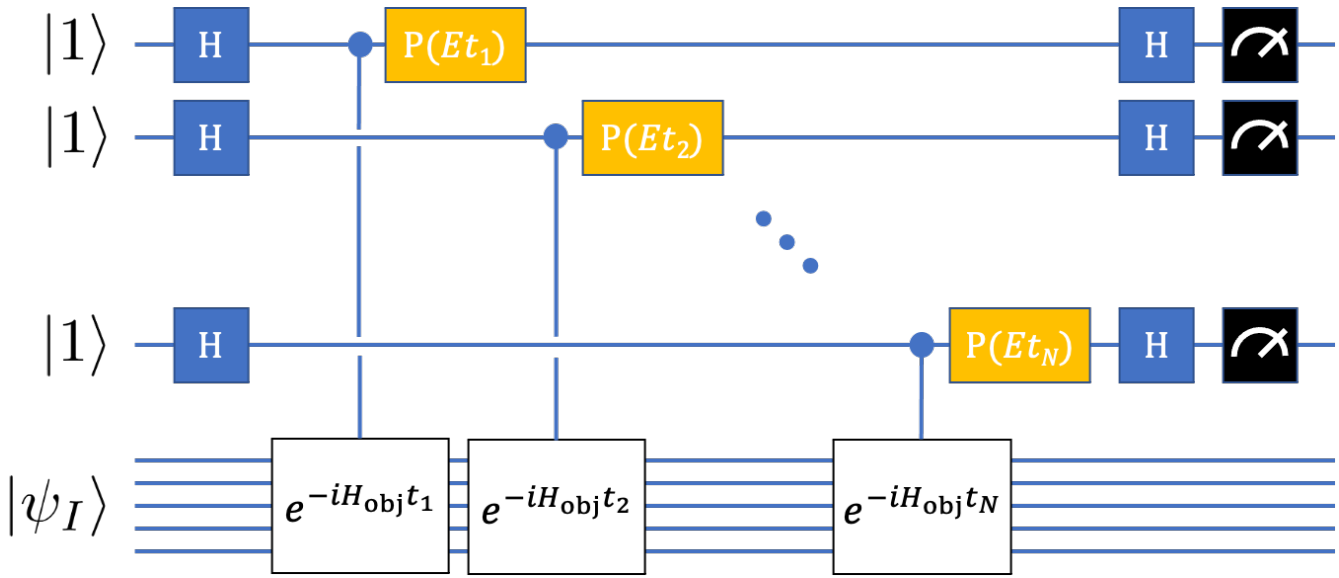


Figure: (color online) **Circuit diagram for the rodeo algorithm.** The object system starts in an arbitrary state $|\psi_I\rangle$. Each of the ancilla qubits are initialized in the state $|1\rangle$ and operated on by a Hadamard gate H. We use each ancilla qubit $n = 1, \cdots, N$ for the controlled time evolution of the object Hamiltonian, $H_{\rm obj}$, for time t_n. This is followed by a phase rotation P(Et_n) on ancilla qubit n, another Hadamard gate H, and then measurement.

The Rodeo algorithm, single qubit setup

Consider first a Hadamard transformation on a single qubit

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

and a phase-rotation matrix

$$P = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\imath t(E_{\rm obj}-E)} \end{bmatrix}.$$

We have then

$$H^\dagger P H = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{-\imath t(E_{\rm obj}-E)} & \frac{1}{2} - \frac{1}{2}e^{-\imath t(E_{\rm obj}-E)} \\ \frac{1}{2} - \frac{1}{2}e^{-\imath t(E_{\rm obj}-E)} & \frac{1}{2} + \frac{1}{2}e^{-\imath t(E_{\rm obj}-E)} \end{bmatrix}.$$

The Rodeo algorithm, single qubit start state

Let us start in the state

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and perform these unitary operations

$$H^\dagger P H \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} e^{-\imath t(E_{\text{obj}} - E)} \\ \frac{1}{2} + \frac{1}{2} e^{-\imath t(E_{\text{obj}} - E)} \end{bmatrix}.$$

We project then back to the $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ state

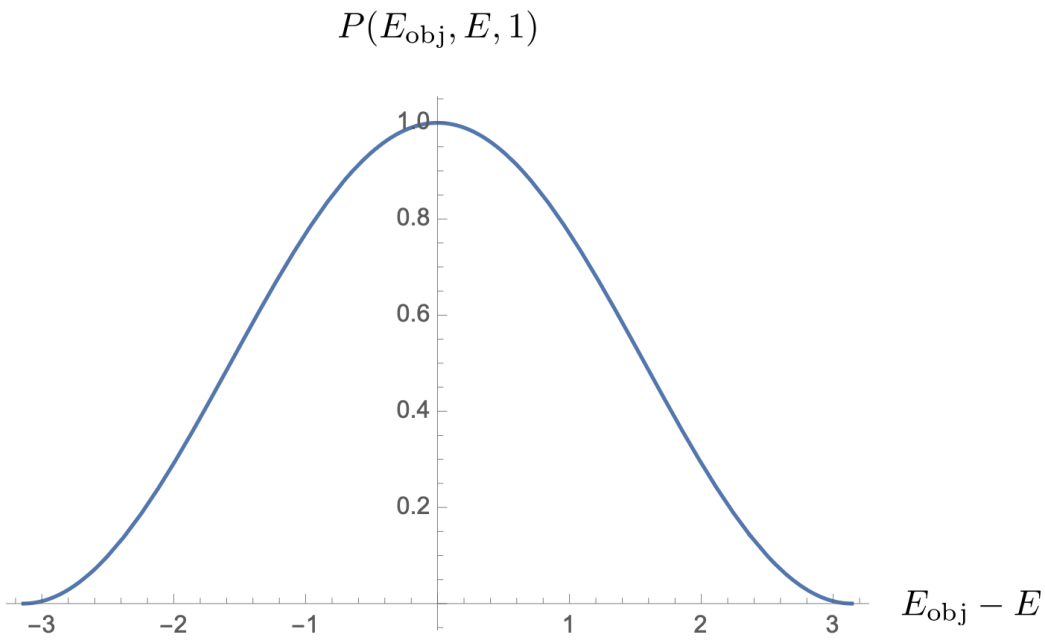
$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} H^\dagger P H \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} + \frac{1}{2} e^{-\imath t(E_{\text{obj}} - E)} \end{bmatrix}.$$

Projection of single qubit start state

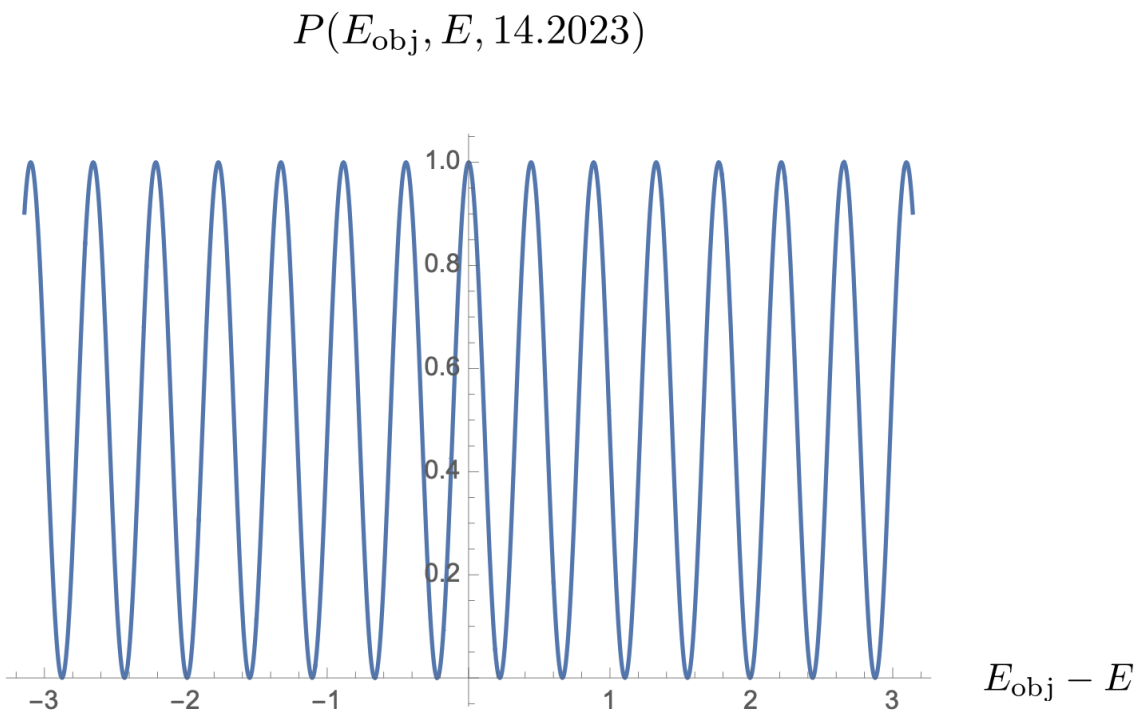
The above projection is done via quantum measurement and the success probability is

$$\text{Prob}(E_{\text{obj}}, E, t) = \left| \frac{1}{2} + \frac{1}{2} e^{-\imath t(E_{\text{obj}} - E)} \right|^2 = \cos^2 \left[\frac{t(E_{\text{obj}} - E)}{2} \right]$$

Convergence pattern

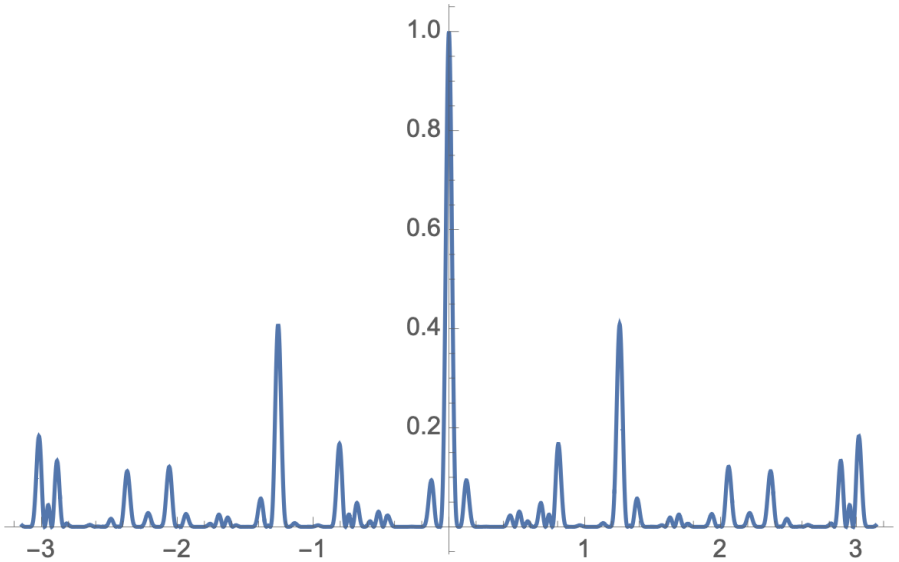


Convergence pattern



Convergence pattern

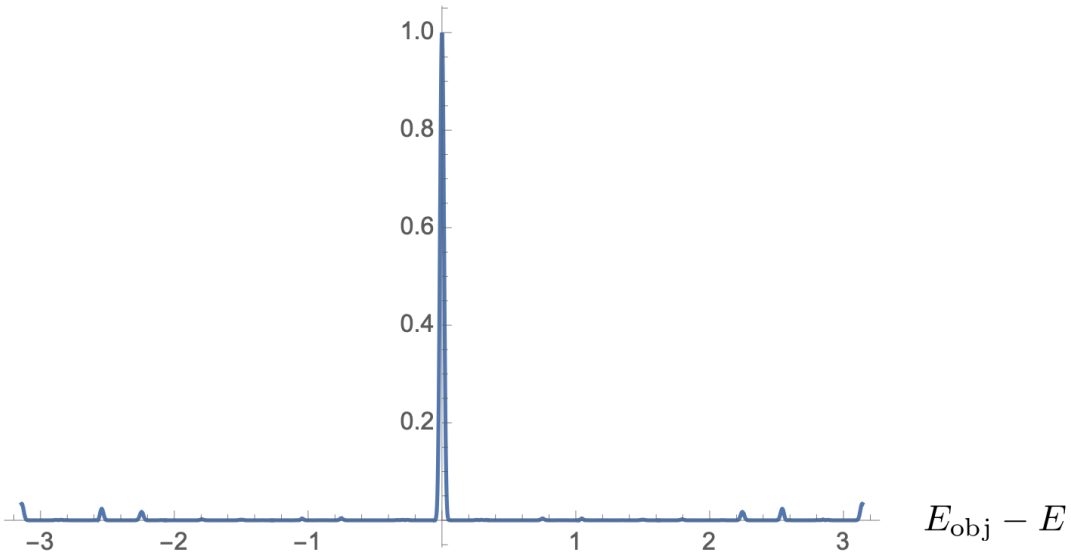
$$\prod_{k=1}^{5} P(E_{\mathrm{obj}}, E, t_k) \qquad |t_k| < 50$$



$E_{\mathrm{obj}} - E$

Convergence pattern

$$\prod_{k=1}^{10} P(E_{\text{obj}}, E, t_k) \qquad |t_k| < 50$$



Expanding to more qubits

Let us couple this qubit, which we call the **ancilla** qubit, to another system that we call the **object**. We also promote the 2×2 matrices to become 2×2 matrices of operators acting on the object, that is

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} \\ \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix},$$

and the phase-rotation matrix

$$P = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\imath t(E_{\rm obj}-E)} \end{bmatrix} \rightarrow \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & e^{-\imath t(E_{\rm obj}-E)} \end{bmatrix}.$$

Expanding to more qubits, same transformations

$$H^\dagger P H = \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} \\ \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix}\begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & e^{-\imath t(E_{\rm obj}-E)} \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} \\ \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix},$$

The Rodeo algorithm, another start state

Let us start in the state

$$\begin{bmatrix} 0 \\ |\psi_I\rangle \end{bmatrix},$$

and we perform the operations and then measure if the **ancilla** qubit is in the $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ state, that is we project then back to the $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ state

$$\begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{I} \end{bmatrix} H^\dagger P H \begin{bmatrix} 0 \\ |\psi_I\rangle \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} + \frac{1}{2} e^{-\imath t(E_{\text{obj}} - E)} |\psi_I\rangle \end{bmatrix}.$$

By repeated successful measurements with random values of t, we reduce the spectral weight of eigenvectors with energies that do not match E.

Probability of success

The success probability of measuring all N ancilla qubits in the $|1\rangle$ state is given by product,

$$\mathrm{Prob}_N = \prod_{n=1}^{N} \cos^2\left[(E_{\mathrm{obj}} - E)\tfrac{t_n}{2}\right].$$

Averaging over the Gaussian random times with STD value σ we get

$$\mathrm{Prob}_N = \left[\tfrac{1}{2} + \tfrac{1}{2}e^{-\imath t(E_{\mathrm{obj}}-E)}\right]^N.$$

The convergence is exponential. For N cycles of the rodeo algorithm, the suppression factor for undesired energy states is $1/4N$

The Rodeo algorithm, as a figure

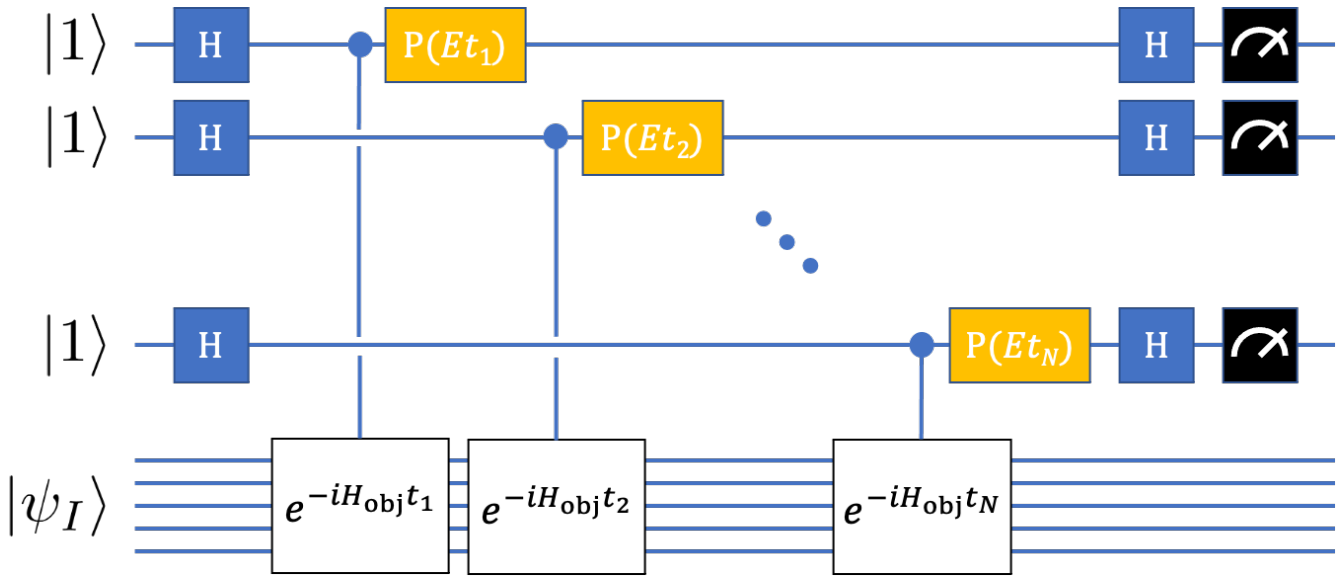


Figure: (color online) **Circuit diagram for the rodeo algorithm.** The object system starts in an arbitrary state $|\psi_I\rangle$. Each of the ancilla qubits are initialized in the state $|1\rangle$ and operated on by a Hadamard gate H. We use each ancilla qubit $n = 1, \cdots, N$ for the controlled time evolution of the object Hamiltonian, $H_{\rm obj}$, for time t_n. This is followed by a phase rotation P(Et_n) on ancilla qubit n, another Hadamard gate H, and then measurement.

More general expression

- The Hamiltonian of interest is labelled as the object Hamiltonian, H_{obj}, and the linear space which it acts upon the object system.
- By assumption, the object system starts in some initial state $|\psi_I\rangle$, which in general will update after each measurement.
- Use the auxiliary or ancilla qubits coupled to the object system.

More general expression

In order to illustrate the effect of these gate operations, let us explicitly write out the operation for one cycle of the rodeo algorithm with one ancilla qubit. Starting from the initial state $|1\rangle \otimes |\psi_I\rangle$ and performing one rodeo cycle, we obtain

$$\begin{bmatrix} \left[\frac{\boldsymbol{I}}{2} - \frac{1}{2}e^{-i(H_{\rm obj}-E)t_1}\right]|\psi_I\rangle \\ \left[\frac{\boldsymbol{I}}{2} + \frac{1}{2}e^{-i(H_{\rm obj}-E)t_1}\right]|\psi_I\rangle \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\boldsymbol{I}}{\sqrt{2}} & \frac{\boldsymbol{I}}{\sqrt{2}} \\ \frac{\boldsymbol{I}}{\sqrt{2}} & \frac{-I}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & Ie^{iEt_1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & e^{-iH_{\rm obj}t_1} \end{bmatrix} \begin{bmatrix} \frac{\boldsymbol{I}}{\sqrt{2}} & \frac{\boldsymbol{I}}{\sqrt{2}} \\ \frac{\boldsymbol{I}}{\sqrt{2}} & \frac{-I}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ |\psi_I\rangle \end{bmatrix},$$

where $\boldsymbol{I}$ is the identity operator on the object system.

Success probabilities

We note that H_{obj} commutes with all of our gates, and so we can describe the action of the rodeo algorithm for each individual eigenvector of H_{obj} with energy E_{obj}. In that case, the probability of measuring the ancilla qubit n in the $|1\rangle$ state is

$$\cos^2\left[(E_{\text{obj}} - E)\frac{t_n}{2}\right] = \left|\frac{1}{2} + \frac{1}{2}e^{-i(E_{\text{obj}}-E)t_n}\right|^2 .$$

The success probability of measuring all N ancilla qubits in the $|1\rangle$ state is given by the product

$$\text{Prob}_N = \prod_{n=1}^{N} \cos^2\left[(E_{\text{obj}} - E)\frac{t_n}{2}\right] .$$

If we now take random values of t_n, we have an energy filter for $E_{\text{obj}} = E$. The geometric mean of $\cos^2\theta$ when sampled uniformly over all θ is equal to $\frac{1}{4}$. Therefore the spectral weight for any $E_{\text{obj}} \neq E$ is suppressed by a factor of $\frac{1}{4^N}$ for large N.

Measurement probability

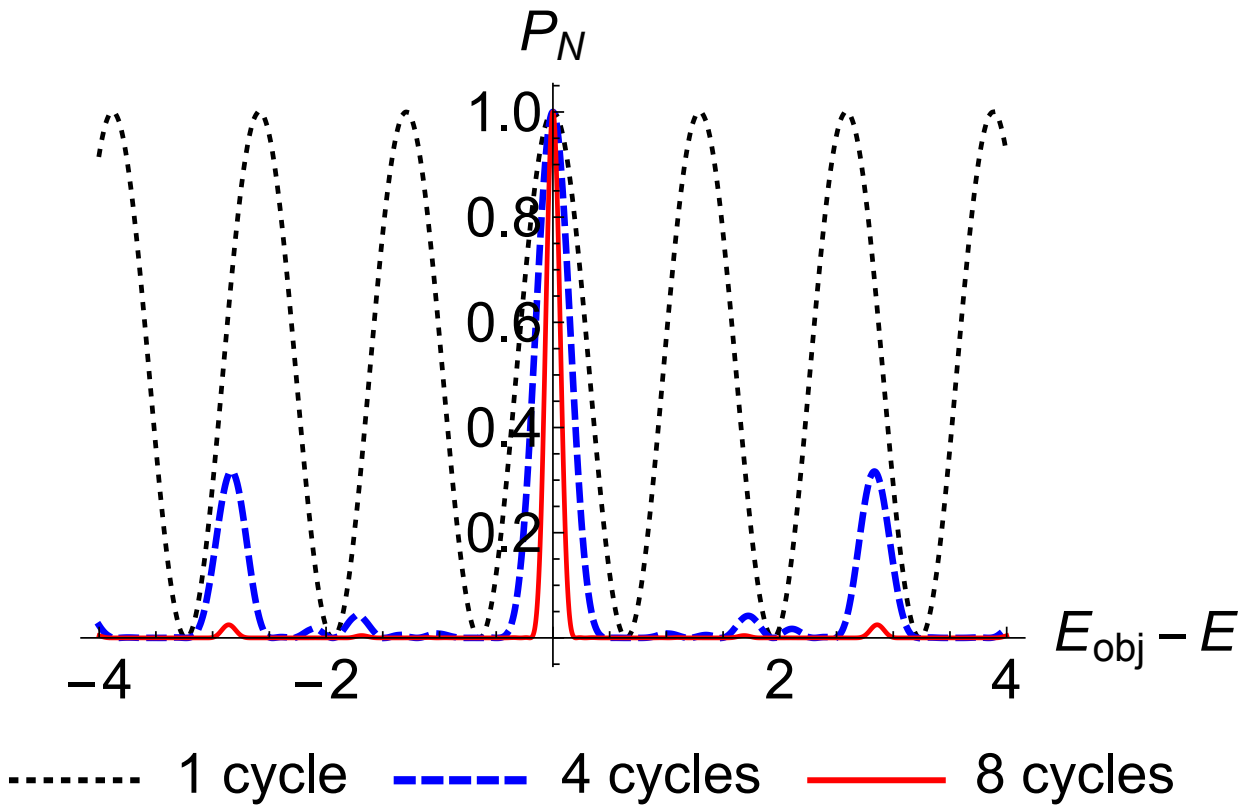


Figure: **Measurement probability.** Probability Prob_N of measuring the $|1\rangle$ state for all ancilla qubits versus $E_{\text{obj}} - E$ for 1 (dotted black line), 4 (dashed blue line), and 8 (solid red line) cycles with Gaussian random values of t_n with $t_{\text{RMS}} = 10$. If we use a Gaussian approximation for Prob_N near its maximum value of 1 at $E_{\text{obj}} = E$, we find that the width of the peak scales as $O[1/(\sqrt{N} t_{\text{RMS}})]$.

Measurement probability

Let ϵ be the desired energy resolution of our rodeo algorithm such that all energy eigenvectors outside of the interval $[E - \epsilon, E + \epsilon]$ are exponentially suppressed.

- If we choose t_{RMS} to scale proportionally with $1/\epsilon$, then we achieve the desired energy filtering with energy resolution ϵ.
- The actual peak width will be a factor of $1/\sqrt{N}$ narrower than ϵ, but that is needed to get exponential suppression as a function of N for all energies E_{obj} further than ϵ from E.

Measurement probability

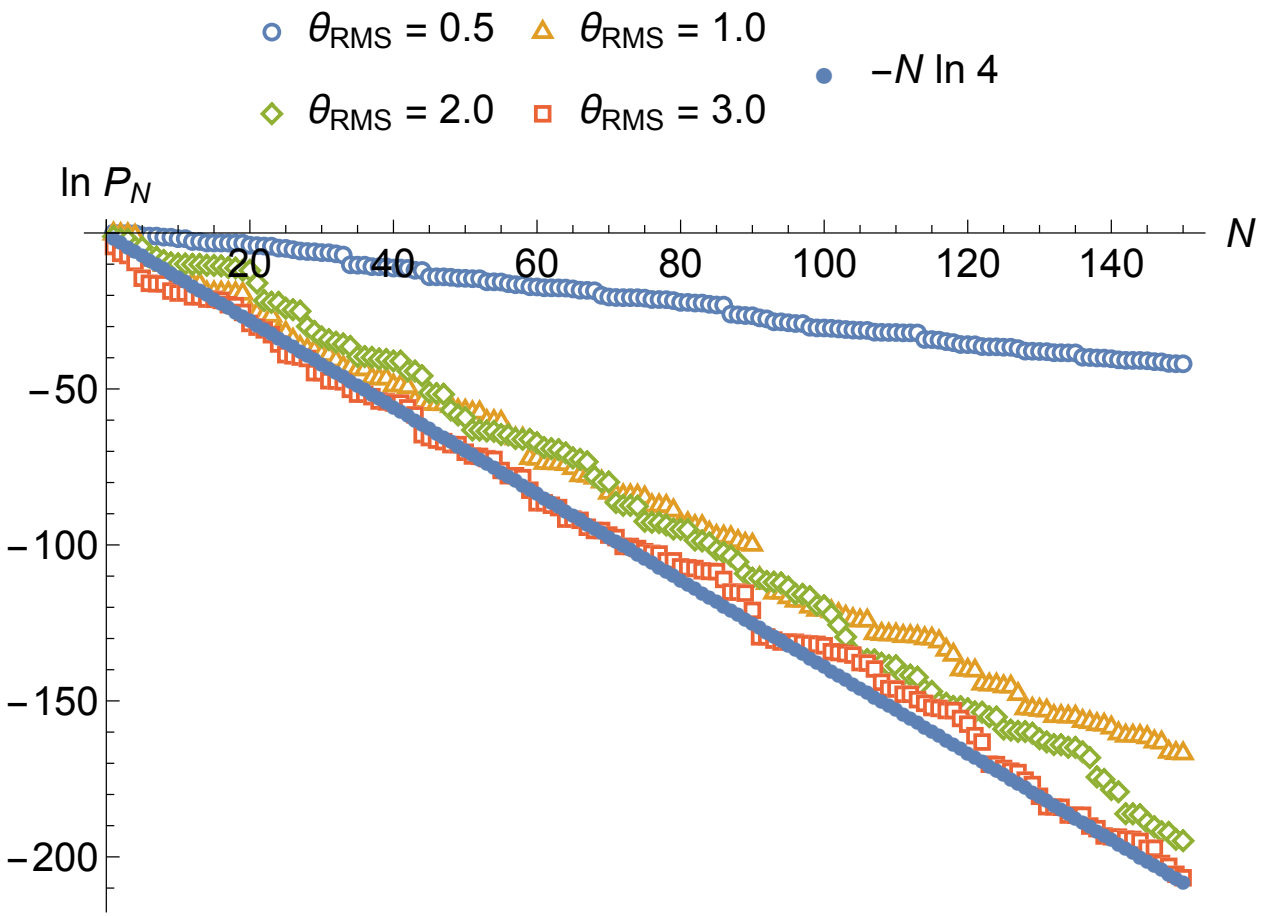


Figure: **Asymptotic scaling.** Plot of $\ln \mathrm{Prob}_N$ versus N for Gaussian random values of t_n with several selected values for $\theta_{\mathrm{RMS}} \equiv (E_{\mathrm{obj}} - E)\frac{t_{\mathrm{RMS}}}{2}$. $\theta_{\mathrm{RMS}} = 0.5$ (open circles), $\theta_{\mathrm{RMS}} = 1.0$ (open triangles), $\theta_{\mathrm{RMS}} = 2.0$ (open diamonds), and $\theta_{\mathrm{RMS}} = 3.0$ (open squares). Predicted asymptotic scaling, $\ln \mathrm{Prob}_N = -N \ln 4$, with filled circles.

Measurement probability

From the previous figure we see that for θ_{RMS} greater than 1, the expected asymptotic scaling is achieved. Therefore, if t_{RMS} is larger than twice the inverse spacing between energy levels, then Prob_N scales as $\frac{1}{4^N}$ for large N.

Arbitrary initial state $|\psi_I\rangle$

- Let E_j be the energy eigenvalue nearest to E, and let $|E_j\rangle$ be the corresponding eigenvector. In the limit of large N, the probability that we measure the $|1\rangle$ state N times in a row is $p\mathrm{Prob}_N$, where p is the overlap probability of the initial state with $|E_j\rangle$, and Prob_N is the success probability for $E_{\mathrm{obj}} = E_j$.
- By tuning E equal to E_j, this probability becomes p. If one requires that the spectral weights of all other energy eigenvectors outside the interval $[E - \epsilon, E + \epsilon]$ are suppressed by a factor δ, then the computational effort scaling for the rodeo algorithm is $Nt_{\mathrm{RMS}}/p = O[|\log\delta|/(p\epsilon)]$.

Find any given energy eigenvalue E_j with error ϵ

- The search process involves $O(\log \epsilon)$ sequential scans of the energy, each scan sweeping over an energy range that is some constant factor K smaller than the previous scan.
- Each scan is performed for several evenly spaced values of E, with a fixed number of rodeo cycles, and t_{RMS} a factor of K larger than that used for the previous scan.
- The total time evolution required will scale as $O(1/\epsilon)$, and the factor of $(\log \epsilon)^2/p$ comes from the required statistics needed to perform the energy scans successfully with high probability. The resulting performance as a function of ϵ is close to the $O(1/\epsilon)$ bound set by the Heisenberg uncertainty principle.

Prepare any given eigenstate $|E_j\rangle$ with a residual orthogonal component that has magnitude Δ

- Keep $t_{\rm RMS}$ fixed but large enough that we are filtering out only the desired eigenstate.
- Perform $N = O(\log \Delta)$ cycles of the rodeo algorithm, must be multiplied by $1/p$ for the number of measurements required.
- Keep E centered on the peak maximum associated with E_j. Re-centering E with each cycle requires only a constant overall factor in the computational cost that is independent of p and Δ.

Applications to the Heisenberg model

As a first application of the rodeo algorithm, Choi et al., considered the spin-$\frac{1}{2}$ Heisenberg model in a uniform magnetic field with 10 sites forming a closed one-dimensional chain. The Hamiltonian has the form

$$H_{\text{obj}} = J \sum_{\langle j,k \rangle} \vec{\sigma}_j \cdot \vec{\sigma}_k + h \sum_j \sigma_j^z,$$

where J is the exchange coupling, $\vec{\sigma}_j$ are the Pauli matrices on site j, $\langle j, k \rangle$ indicates nearest neighbors, and h is the coupling to a uniform magnet ic field in the z direction. We consider the antiferromagnetic case with values $J = 1$ and $h = 3$. For our initial state we use an alternating tensor product state,

$$|\psi_I\rangle = |0101010101\rangle .$$

Since the initial state has a high degree of symmetry, one can expect that the initial state to have nonzero overlap with a relatively small number of energy eigenstates.

Heisenberg model

Initial-state spectral function and state preparation. The example shown below is for a 1D Heisenberg chain with ten sites, antiferromagnetic interactions, and uniform magnetic field.

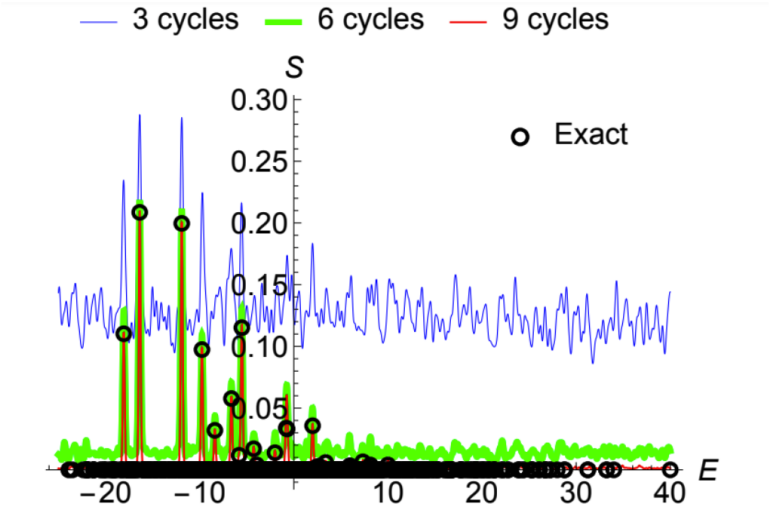


FIG. 4. (color online) **Initial-state spectral function for the Heisenberg model.** We plot the initial-state spectral function using the rodeo algorithm for the Heisenberg spin chain with 3 (thin blue line), 6 (thick green line), and 9 (medium red line) cycles. We have averaged over 20 sets of Gaussian random values for t_n with $t_{\mathrm{RMS}} = 5$. For comparison, we also show the exact initial-state spectral function with black open circles.

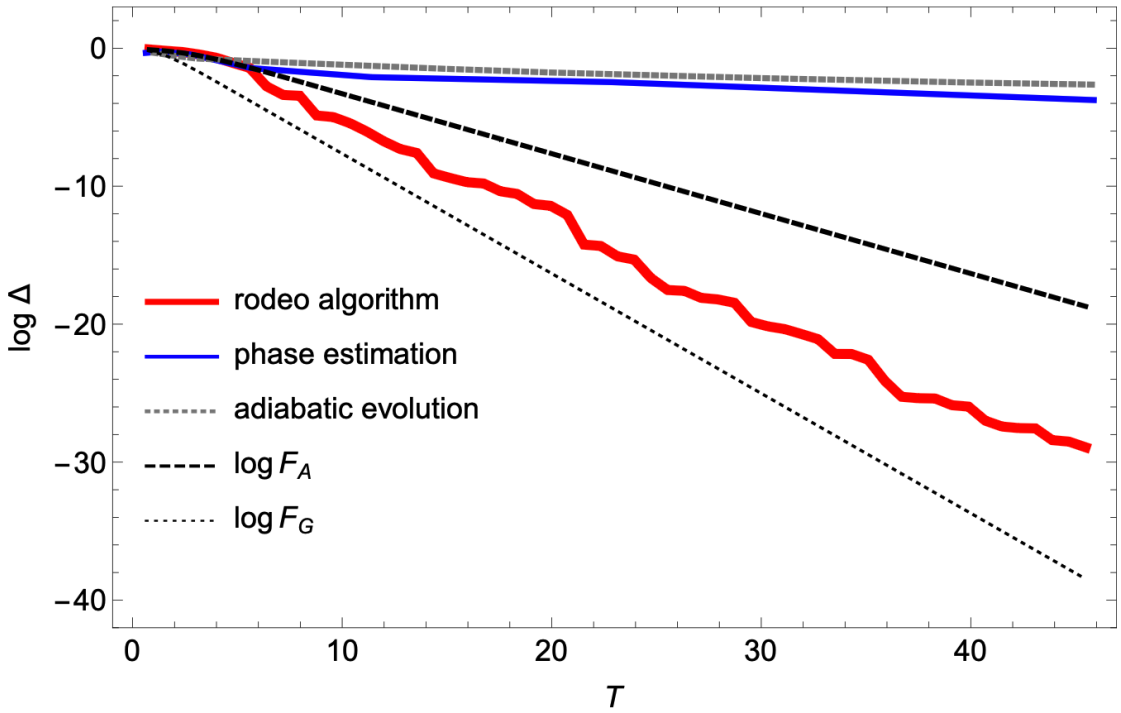
$$|\psi_{\mathrm{init}}\rangle = |0101010101\rangle$$

TABLE I. Overlap probability with energy eigenvector $|E_j\rangle$ after N cycles of the rodeo algorithm using Gaussian random values for t_n with $t_{\mathrm{RMS}} = 5$ and $E = E_j$.

E_j	$N=0$	$N=3$	$N=6$	$N=9$
−18.1	0.110	0.746	0.939	0.997
−16.4	0.209	0.841	0.993	1.000
−11.9	0.200	0.629	0.889	0.999
−9.76	0.0974	0.488	0.903	0.999
−8.38	0.0320	0.467	0.832	0.993
−6.63	0.0577	0.309	0.818	0.996
−5.81	0.0118	0.179	0.637	0.817
−5.52	0.115	0.456	0.766	0.997
−4.26	0.0171	0.144	0.696	0.995
−3.95	0.00401	0.0430	0.343	0.952
−2.00	0.0139	0.158	0.593	0.942
−0.802	0.0338	0.216	0.545	0.594
−0.704	0.0331	0.286	0.540	0.585
2.00	0.0357	0.371	0.925	0.994
2.42	0.00235	0.0122	0.0874	0.521
2.68	0.00291	0.0845	0.639	0.929
3.39	0.00592	0.0360	0.754	0.943
5.96	0.00336	0.0951	0.559	0.981
7.33	0.00650	0.184	0.792	0.978
8.13	0.00393	0.0832	0.665	0.841
8.24	0.00105	0.0275	0.142	0.289
10.0	0.00397	0.0128	0.295	0.902

Log of wave function error

Comparison with other well-known algorithms. Let Δ be the norm of the error in the wave function.



$$F_A \equiv \sqrt{2^{-N}(1-p)/[p+2^{-N}(1-p)]}$$

$$F_G \equiv \sqrt{4^{-N}(1-p)/[p+4^{-N}(1-p)]}$$

Implementation on real quantum computer

Eur. Phys. J. A (2024) 60:151
https://doi.org/10.1140/epja/s10050-024-01373-9

THE EUROPEAN PHYSICAL JOURNAL A

Regular Article - Theoretical Physics

Demonstration of the rodeo algorithm on a quantum computer

Zhengrong Qian[1,a], Jacob Watkins[1,b], Gabriel Given[1,c], Joey Bonitati[1,d], Kenneth Choi[2,e], Dean Lee[1,f]

Using IBM Q devices, we implement the rodeo algorithm for a one qubit Hamiltonian. We consider a random Hamiltonian of the form

$$H_{\rm obj} = H^{(0)} = -0.08496I - 0.89134X + 0.26536Y + 0.57205Z$$

We use mid-circuit measurements without resets for the ancilla qubit