Parametric matrix models

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Outline

Quantum control

The three steps of quantum control

1. **Quantum correlations:** Understanding and preparing an initial state
2. **Quantum dynamics:** Controlled evolution towards a desired final state
3. **Quantum measurements:** Measuring and characterizing the final state

Quantum control and this talk

Last week we discussed the so-called Rodeo algorithm as a way to prepare an initial state and/or find the eigenpairs of a system. This week we will look at how to control the time-evolution of a system. In so doing, we will study

1. **Quantum dynamics:** Controlled evolution towards a desired final state
   - The Baker–Campbell–Hausdorff (BCH) formula
   - Combining Exponentials of Non-commuting Operators and the Lie-Trotter formula (Trotterization)
   - Parametric matrix models as a way to compute the Lie-Trotter formula, see `https://www.nature.com/articles/s41467-025-61362-4`, Cook, Jammooa, MHJ, Lee and Lee

Motivation: Non-commuting Exponentials

- In quantum mechanics and Lie theory, we often encounter operators X and Y that do not commute ($[X, Y] \neq 0$).
- We want to find an effective operator Z such that: $e^X e^Y = e^Z$, for X, Y in a Lie algebra . If X and Y commute, then simply $Z = X + Y$. If not, Z includes additional correction terms.
- **BCH Formula:** $Z = \log(e^X e^Y)$ is given by an infinite series in X, Y and their commutators. It provides a systematic expansion to combine exponentials of non-commuting operators .
- **Use Cases:** Combines two small transformations into one. Fundamental in connecting Lie group multiplication with Lie algebra addition, time-evolution with split Hamiltonians, etc.

Commutators and Lie Algebra

- The **commutator** of two operators is $[X, Y] = XY - YX$.
- For a Lie algebra (common for operators in quantum mechanics), commutators of algebra elements remain in the algebra.
- The BCH formula asserts Z can be expressed entirely in terms of X, Y, and nested commutators like $[X, [X, Y]]$, $[Y, [X, Y]]$, etc. – no other independent products appear .
- Notation: It's useful to denote $\mathrm{ad}_X(Y) := [X, Y]$. Then nested commutators are iterated adjoint actions (e.g. $\mathrm{ad}_X^2(Y) = [X, [X, Y]]$, etc.).
- We assume familiarity with basic Lie algebra identities (Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$) which will simplify nested commutators.

BCH Expansion: First Terms

For $Z = \log\left(e^X e^Y\right)$, the expansion begins:

$$\begin{aligned} Z = X &+ Y \\ &+ \frac{1}{2}[X, Y] \\ &+ \frac{1}{12}\left([X,[X,Y]] + [Y,[Y,X]]\right) \\ &- \frac{1}{24}[Y,[X,[X,Y]]] + \cdots \end{aligned}$$

- The series alternates between symmetric and antisymmetric nested commutators at higher orders .
- All higher-order terms involve nested commutators of X and Y only. No ordinary products without commutators appear (ensuring Z lies in the same Lie algebra) .
- The coefficients $1/2, 1/12, 1/24, \ldots$ are fixed numerical values (involving Bernoulli numbers for higher terms). These were first worked out explicitly by Dynkin (1947) in general .

Series Characteristics

- The BCH series is generally infinite. In most cases, there is **no closed-form finite expression** for Z in terms of a finite number of terms .
- Each increasing order introduces more deeply nested commutators. For example:
  - 1st order: $X + Y$
  - 2nd order: $[X, Y]$
  - 3rd order: $[X, [X, Y]]$, $[Y, [X, Y]]$
  - 4th order: $[Y, [X, [X, Y]]]$, $[X, [Y, [Y, X]]]$, etc.
- The number of independent commutator terms grows rapidly with order. (All such terms up to 6th order are listed in the literature , but it becomes cumbersome beyond a few orders.)
- Fortunately, many practical scenarios require only the first few terms for approximation.
- If X and Y are “small” (e.g. small matrices or small time-step in evolution), the series converges and truncating after a few terms can give a good approximation .

Derivation: Outline (up to Third Order)

- **Method:** Compare power series of $e^X e^Y$ and e^Z and solve for Z order-by-order .
- Expand both sides:

$$e^X e^Y = I + X + Y + \frac{1}{2}(X^2 + XY + YX + Y^2) + \frac{1}{6}(X^3 + \cdots) + \cdots$$

$$e^Z = I + Z + \frac{1}{2}Z^2 + \frac{1}{6}Z^3 + \cdots$$

  where $Z = X + Y + A_2 + A_3 + \cdots$ (with A_n = terms of order n in X, Y).
- **First order:** Match linear terms: $Z^{(1)} = X + Y$. So far $Z = X + Y$.
- next slide

Derivation: Outline (up to Third Order)

- **Second order:** The $e^X e^Y$ expansion has $\frac{1}{2}(XY + YX)$ at order 2. Meanwhile e^Z gives $\frac{1}{2}(X+Y)^2 = \frac{1}{2}(X^2 + XY + YX + Y^2)$. The extra X^2 and Y^2 terms match on both sides, but $XY + YX$ vs $XY + YX$ is already present. However, note that $XY + YX$ cannot simplify to $2XY$ unless $XY = YX$. The discrepancy appears at this order .
- Thus, we postulate Z has a second-order correction $A_2 = \frac{1}{2}[X, Y]$ to account for the difference:

  $$XY + YX = (X+Y)^2 - X^2 - Y^2 = XY + YX,$$

  but including A_2 in Z yields new cross terms when squaring Z:

  $$\frac{1}{2}(X + Y + A_2)^2 = \frac{1}{2}(X^2 + XY + YX + Y^2 + [X, Y]).$$

  which adds the $[X, Y]$ term we need. So $A_2 = \frac{1}{2}[X, Y]$.
- **Third order:** Now include A_2 and match cubic terms. There will be terms involving X^2Y, XY^2, etc. The mismatch yields terms $[X, [X, Y]]$ and $[Y, [X, Y]]$. By similar (though more involved)