Rodeo algorithm and quantum computing

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What is this talk about?

Rodeo Algorithm

In PRL **127** (2021), Choi *et al.,* presented a stochastic quantum computing algorithm that can prepare any eigenvector of a quantum Hamiltonian within a selected energy interval $[E - \epsilon, E + \epsilon]$.
In order to reduce the spectral weight of all other eigenvectors by a suppression factor δ, the required computational effort scales as $O[|\log \delta|/(p\epsilon)]$, where p is the squared overlap of the initial state with the target eigenvector. The method uses auxiliary qubits to control the time evolution of the Hamiltonian minus some tunable parameter E. With each auxiliary qubit measurement, the amplitudes of the eigenvectors are multiplied by a stochastic factor that depends on the proximity of their energy to E. In this manner, one can converge to the target eigenvector with exponential accuracy in the number of measurements.

What is this talk about?

Rodeo Algorithm, more

In addition to preparing eigenvectors, the method can also compute the full spectrum of the Hamiltonian. For energy eigenvalue determination with error ϵ, the computational scaling is $O[(\log \epsilon)^2/(p\epsilon)]$. For eigenstate preparation, the computational scaling is $O(\log \Delta/p)$, where Δ is the magnitude of the orthogonal component of the residual vector. The speed for eigenstate preparation is exponentially faster than that for phase estimation or adiabatic evolution.

The three steps of quantum control

1. **Quantum correlations:** Understanding and preparing an initial state
2. **Quantum dynamics:** Controlled evolution towards a desired final state
3. **Quantum measurements:** Measuring and characterizing the final state

The Rodeo algorithm, single qubit setup

Consider first a Hadamard transformation on a single qubit

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

and a phase-rotation matrix

$$R = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\imath t(E_{\mathrm{obj}}-E)} \end{bmatrix}.$$

We have then

$$H^{\dagger}RH = \begin{bmatrix} \frac{1}{2}+\frac{1}{2}e^{-\imath t(E_{\mathrm{obj}}-E)} & \frac{1}{2}-\frac{1}{2}e^{-\imath t(E_{\mathrm{obj}}-E)} \\ \frac{1}{2}-\frac{1}{2}e^{-\imath t(E_{\mathrm{obj}}-E)} & \frac{1}{2}+\frac{1}{2}e^{-\imath t(E_{\mathrm{obj}}-E)} \end{bmatrix}.$$

The Rodeo algorithm, single qubit start state

Let us start in the state

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and perform these unitary operations

$$H^{\dagger} R H \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} e^{-\imath t(E_{\mathrm{obj}} - E)} \\ \frac{1}{2} + \frac{1}{2} e^{-\imath t(E_{\mathrm{obj}} - E)} \end{bmatrix}.$$

We project then back to the $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ state

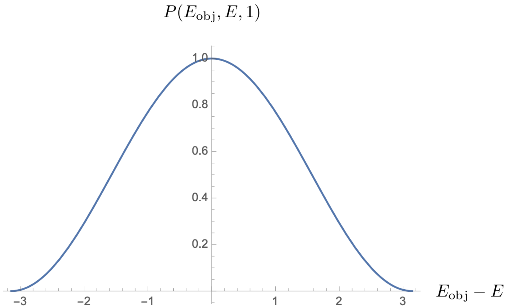
$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} H^{\dagger} R H \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} + \frac{1}{2} e^{-\imath t(E_{\mathrm{obj}} - E)} \end{bmatrix}.$$

Projection of single qubit start state

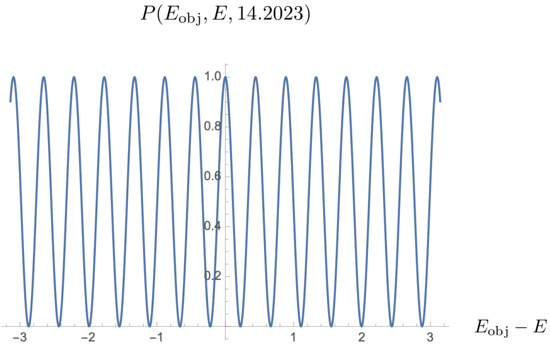
The above projection is done via quantum measurement and the success probability is

$$P(E_{\rm obj}, E, t) = \left| \frac{1}{2} + \frac{1}{2} e^{-\imath t(E_{\rm obj} - E)} \right|^2 = \cos^2 \left[\frac{t(E_{\rm obj} - E)}{2} \right]$$

Convergence pattern

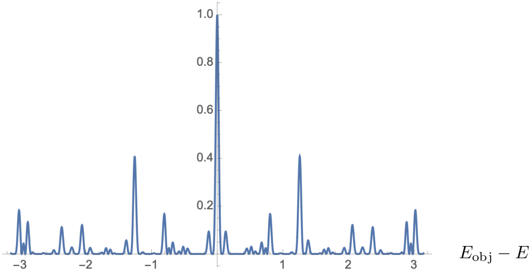


Convergence pattern



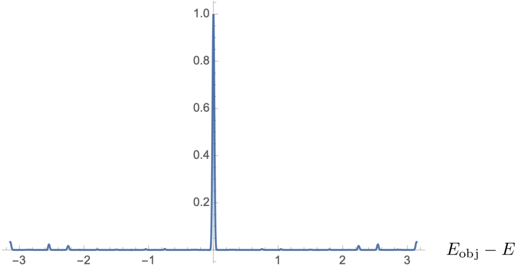
Convergence pattern

$$\prod_{k=1}^{5} P(E_{\text{obj}}, E, t_k) \qquad |t_k| < 50$$



Convergence pattern

$$\prod_{k=1}^{10} P(E_{\mathrm{obj}}, E, t_k) \qquad |t_k| < 50$$



Expanding to more qubits

Let us couple this qubit, which we call the **ancilla** qubit, to another system that we call the **object.** We also promote the 2×2 matrices to become 2×2 matrices of operators acting on the object, that is

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \to \frac{1}{\sqrt{2}} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} \\ \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix},$$

and the phase-rotation matrix

$$R = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\imath t(E_{\mathrm{obj}} - E)} \end{bmatrix} \to \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & e^{-\imath t(E_{\mathrm{obj}} - E)} \end{bmatrix}.$$

Expanding to more qubits, same transformations

$$H^\dagger R H = \frac{1}{\sqrt{2}} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} \\ \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & e^{-\imath t(E_{\mathrm{obj}} - E)} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} \\ \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix},$$

The Rodeo algorithm, another start state

Let us start in the state

$$\begin{bmatrix} 0 \\ |\psi_I\rangle \end{bmatrix},$$

and we perform the operations and then measure if the **arena** qubit is in the $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ state, that is we project then back to the $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ state

$$\begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{I} \end{bmatrix} H^{\dagger} R H \begin{bmatrix} 0 \\ |\psi_I\rangle \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} + \frac{1}{2} e^{-\imath t(E_{\mathrm{obj}} - E)} |\psi_I\rangle \end{bmatrix}.$$

By repeated successful measurements with random values of t, we reduce the spectral weight of eigenvectors with energies that do not match E.

Probability of success

The success probability of measuring all N ancilla qubits in the $|1\rangle$ state is given by product,

$$P_N = \prod_{n=1}^{N} \cos^2\left[(E_{\text{obj}} - E)\tfrac{t_n}{2}\right].$$

Averaging over the Gaussian random times with *STD* value σ we get

$$P_N = \left[\tfrac{1}{2} + \tfrac{1}{2} e^{-\imath t(E_{\text{obj}} - E)}\right]^N.$$

The convergence is exponential. For N cycles of the rodeo algorithm, the suppression factor for undesired energy states is $1/4N$

The Rodeo algorithm, as a figure

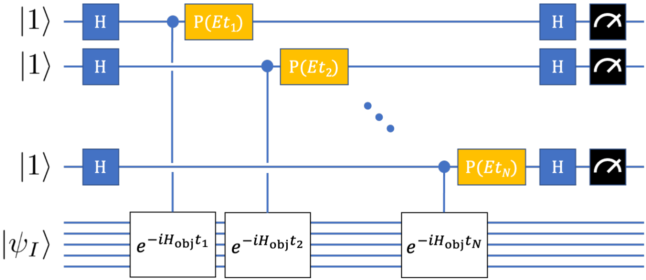


Figure: (color online) **Circuit diagram for the rodeo algorithm.** The object system starts in an arbitrary state $|\psi_I\rangle$. Each of the ancilla qubits are initialized in the state $|1\rangle$ and operated on by a Hadamard gate H. We use each ancilla qubit $n = 1, \cdots, N$ for the controlled time evolution of the object Hamiltonian, H_{obj}, for time t_n. This is followed by a phase rotation $\mathrm{P}(Et_n)$ on ancilla qubit n, another Hadamard gate H, and then measurement.

More general expression

We will refer to the Hamiltonian of interest as the object Hamiltonian, $H_{\rm obj}$, and the linear space which it acts upon the object system. By assumption, the object system starts in some initial state $|\psi_I\rangle$, which in general will update after each measurement.

We will use auxiliary or ancilla qubits coupled to the object system. In the following we use the standard terminology, ancilla qubits. But Choi *et al.,* also mention that this collection of ancilla qubits is also informally called the **rodeo arena.**

More general expression

In order to illustrate the effect of these gate operations, let us explicitly write out the operation for one cycle of the rodeo algorithm with one ancilla qubit. Starting from the initial state $|1\rangle \otimes |\psi_I\rangle$ and performing one rodeo cycle, we obtain

$$
\begin{bmatrix} \left[\frac{\boldsymbol{I}}{2} - \frac{1}{2}e^{-i(H_{\mathrm{obj}}-E)t_1}\right]|\psi_I\rangle \\ \left[\frac{\boldsymbol{I}}{2} + \frac{1}{2}e^{-i(H_{\mathrm{obj}}-E)t_1}\right]|\psi_I\rangle \end{bmatrix} =
$$

$$
\begin{bmatrix} \frac{\boldsymbol{I}}{\sqrt{2}} & \frac{\boldsymbol{I}}{\sqrt{2}} \\ \frac{\boldsymbol{I}}{\sqrt{2}} & \frac{-\boldsymbol{I}}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & Ie^{iEt_1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & e^{-iH_{\mathrm{obj}}t_1} \end{bmatrix} \begin{bmatrix} \frac{\boldsymbol{I}}{\sqrt{2}} & \frac{\boldsymbol{I}}{\sqrt{2}} \\ \frac{\boldsymbol{I}}{\sqrt{2}} & \frac{-\boldsymbol{I}}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ |\psi_I\rangle \end{bmatrix},
$$

where ***I*** is the identity operator on the object system.

Success probabilities

We note that H_{obj} commutes with all of our gates, and so we can describe the action of the rodeo algorithm for each individual eigenvector of H_{obj} with energy E_{obj}. In that case, the probability of measuring the ancilla qubit n in the $|1\rangle$ state is

$$\cos^2\left[(E_{\text{obj}} - E)\tfrac{t_n}{2}\right] = \left|\tfrac{1}{2} + \tfrac{1}{2}e^{-i(E_{\text{obj}}-E)t_n}\right|^2 .$$

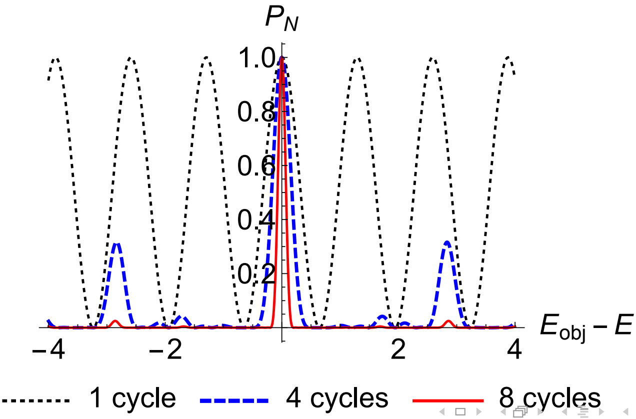
The success probability of measuring all N ancilla qubits in the $|1\rangle$ state is given by product,

$$P_N = \prod_{n=1}^{N} \cos^2\left[(E_{\text{obj}} - E)\tfrac{t_n}{2}\right] .$$

If we now take random values of t_n, we have an energy filter for $E_{\text{obj}} = E$. The geometric mean of $\cos^2\theta$ when sampled uniformly over all θ is equal to $\frac{1}{4}$. Therefore the spectral weight for any $E_{\text{obj}} \neq E$ is suppressed by a factor of $\frac{1}{4^N}$ for large N.

Measurement probability

Here we plot the probability P_N of measuring the $|1\rangle$ state for all ancilla qubits versus $E_{\mathrm{obj}} - E$ for 1 (dotted black line), 4 (dashed blue line), and 8 (solid red line) cycles with Gaussian random values of t_n with root-mean-square value $t_{\mathrm{RMS}} = 10$. If we use a Gaussian approximation for P_N near its maximum value of 1 at $E_{\mathrm{obj}} = E$, we find that the width of the peak scales as $O[1/(\sqrt{N}t_{\mathrm{RMS}})]$.



Measurement probability

Let ϵ be the desired energy resolution of our rodeo algorithm such that all energy eigenvectors outside of the interval $[E-\epsilon, E+\epsilon]$ are exponentially suppressed. If we choose t_{RMS} to scale proportionally with $1/\epsilon$, then we achieve the desired energy filtering with energy resolution ϵ. The actual peak width will be a factor of $1/\sqrt{N}$ narrower than ϵ, but that is needed to get exponential suppression as a function of N for all energies E_{obj} further than ϵ from E.

Measurement probability

In the next figure we plot $\ln P_N$ versus N for Gaussian random values of t_n using several values for $\theta_{\rm RMS} \equiv (E_{\rm obj} - E)\frac{t_{\rm RMS}}{2}$. We show $\theta_{\rm RMS} = 0.5$ (open circles), $\theta_{\rm RMS} = 1.0$ (open triangles), $\theta_{\rm RMS} = 2.0$ (open diamonds), and $\theta_{\rm RMS} = 3.0$ (open squares). We present the predicted asymptotic scaling, $\ln P_N = -N \ln 4$, with filled circles. We see that for $\theta_{\rm RMS}$ greater than 1, the expected asymptotic scaling is achieved. Therefore, if $t_{\rm RMS}$ is larger than twice the inverse spacing between energy levels, then P_N scales as $\frac{1}{4^N}$ for large N.

Measurement probability

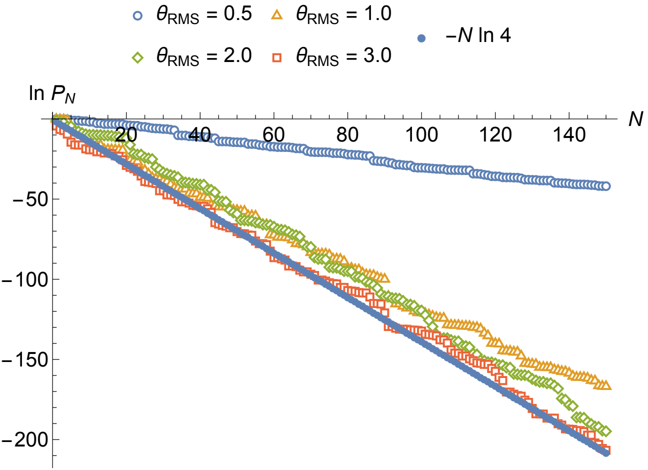


Figure: (color online) **Asymptotic scaling.** We plot ln P_N versus N for Gaussian random values of t_n with several selected values for $\theta_{\rm RMS} \equiv (E_{\rm obj} - E)\frac{t_{\rm RMS}}{2}$. We show $\theta_{\rm RMS} = 0.5$ (open circles), $\theta_{\rm RMS} = 1.0$ (open triangles), $\theta_{\rm RMS} = 2.0$ (open diamonds), and $\theta_{\rm RMS} = 3.0$ (open squares). We