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Nonlinear analysis of hydrodynamic instability in laminar flames—I. Derivation of basic equations

G. I. SIVASHINSKY

Department of Mathematical Sciences, Division of Applied Mathematics, Tel-Aviv University, Ramat-Aviv, Israel

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Abstract—An asymptotic nonlinear integrodifferential equation is derived for spontaneous instability of the plane front of a laminar flame.

If the combustible mixture is deficient in the light component, spontaneous instability will lead to self-turbulization of the flame, and the flame front assumes a strongly nonstationary cellular structure.

If there is an excess of the light component, spontaneous instability produces stationary, irregular wrinkles on the flame front, and the flame continues to propagate in a laminar regime.

It is shown that in all cases spontaneous instability of the flame implies an increase in its propagation velocity.

Introduction

IT IS WELL known that the difficulties involved in setting up a theory of flame propagation in turbulent flow of a gaseous combustible mixture are due largely to the problem of spontaneous instability of laminar flames. This question was discussed in detail by Markstein (1964), and also in a recent survey of turbulent combustion.†

Owing to the intrinsic instability of a laminar flame front, the turbulent field of the incoming flow causes (in the linear approximation) unlimited amplification of long-wave disturbances of the front. This effect is a serious obstacle to developing a general theory of flame propagation by direct asymptotic expansion in powers of the amplitude of the external turbulent pulsations.

In the context of a flame progagating freely in *open space*, nonlinearity is evidently the only factor inhibiting unlimited amplification of disturbances. Therefore, any consistent theory of turbulent flame propagation must take it into consideration.

At the present time, effective nonlinear analysis of hydrodynamic instability is feasible only in cases that can be formulated in terms of bifurcation theory. In such cases one can often accomplish asymptotic analysis in the neighborhood of a bifurcation point (a certain value of the parameter, representing the transition from absolute stability to instability), and this makes it possible to describe certain essentially nonlinear effects.

†See Libby and Williams (1976); also Andrews, Bradley and Lwakabamba (1975); Williams (1970).

As we shall see later, spontaneous instability of a laminar flame does indeed fall into this category of problems.

It is now known that spontaneous instability of a flame is due primarily to two destabilizing effects: (i) thermal expansion of the gas passing through the flame front, and (ii) interaction of the diffusion and heat-conduction processes within the flame front structure.

The destabilizing effect of thermal expansion was noticed by Landau (1944), who treated the flame as a density discontinuity in an ideal fluid, traveling at constant velocity relative to the gas. It is clear that this effect is always present in flame of exothermal reactions.

The second destabilizing effect was described in a qualitative framework by Zel'dovich (1944), and later subjected to a quantitative analysis by Barenblatt, Zel'dovich, Istratov (1962) and Sivashinsky (1977). It is found that the destabilizing effect occurs if the diffusivity of the component limiting the reaction exceeds the thermal conductivity of the mixture. In other words, the second effect appears only in mixtures in which the more mobile (i.e. more easily diffusing) component is present in small concentrations.

These two destabilizing effects have been corroborated by a great number of experiments studying laminar flame stability.†

If the mixture is deficient in the heavy component (e.g. lean mixtures of heavy hydrocarbons with air), the flame will experience only hydrodynamic instability. In that case the flame front is criss-crossed by wrinkles. The distribution of these wrinkles over the surface of the flame front is extremely nonuniform, depending on the initial disturbance imparted to the smooth flame front. If there are no persistent external disturbances of sufficiently high amplitude (e.g. turbulence in the incoming flow), the resulting corrugated flame front is highly stable to small disturbances and the initial pattern of the wrinkles is well preserved. In a freely propagating spherical flame in a stationary gas, the wrinkles grow and deepen as the flame front expands, though their number remains the same. Such corrugated flames have been produced artificially by passing flames through wire gauze (Markstein, 1964). When the flame passes through the gauze, the latter leaves a characteristic periodic pattern of wrinkles on the front, and these are clearly preserved in further propagation of the flame.

The appearance of inhomogeneity on the flame front is accompanied by a marked increase in the mean propagation velocity of the flame, compared with the case of a smooth plane, flame front. Note that if there is no external turbulence a corrugated flame propagates in a clear-cut laminar regime.

If there is an excess of the heavy component (e.g. rich mixtures of heavy hydrocarbons and air) the effect of hydrodynamic instability is superimposed on that of diffusional-thermal instability. Considering a freely propagating, expanding spherical flame in an initially stationary gaseous mixture, one discerns three main stages of spontaneous instability (Istratov and Librovich, 1969). At first, the spherical flame front is covered with widely spaced, irregular wrinkles. Only

†See, e.g. Markstein (1964), Schelkin and Troshin (1964), Istratov and Librovich (1969), Palm-Leis and Strehlow (1969).

hydrodynamic instability of the type described above can arise at this stage. Then, when the front exceeds a certain critical radius, its entire surface becomes covered with uniformly distributed weak protuberances, somewhat reminiscent in appearance of regular hexagons of equal size. With further increase in flame radius, the amplitude of the protuberances increases, their regular hexagonal shape disappears, but their typical characteristic dimension remains unchanged. This stage is that of cellular flame structure. The size of the cells depends only on the chemical composition of the combustible mixture and may vary over a wide interval: from fractions of a millimeter to a centimeter.

As in the case of noncellular corrugated flames, the wrinkles in a cellular flame front are sharper toward the combustion products. Since the characteristic dimension of the cells remains the same as the flame propagates, the number of cells increases continually as the radius of the sphere increases. For some time following the appearance of cells, the flame continues to propagate in a quasistationary laminar regime. With further increase in radius, the flame becomes unstable and gradual self-turbulence sets in. The turbulent flame front shows the same dimension of inhomogeneity as the quasistationary cellular front; in fact, it may be regarded as an essentially nonstationary cellular flame in a state of constant, irregular pulsation, with the cells breaking up and merging at random. The turbulent cellular structure gradually destroys the initial wrinkles, which were due to hydrodynamic instability.

We emphasize that under conditions of diffusional-thermal instability the flame spontaneously becomes turbulent even when the incoming flow shows no signs of turbulence far from the flame.

The transition of the flame from the corrugated noncellular to the turbulent regime of propagation is characterized by a constant increase in its mean propagation velocity. With increase in the radius of the sphere, this velocity gradually overshoots a certain maximum corresponding to saturation in the development of turbulence. This maximum may apparently be treated as the normal velocity of a "plane" turbulent flame front propagating in a self-turbulent regime.

The purpose of study in this paper is nonlinear analysis of hydrodynamic flame instability due to thermal expansion of the gas and to transport effects.

Adopting the simplest possible assumptions concerning diffusion and chemical kinetics, the following asymptotic equation describing the evolution of a disturbed plane flame front was derived:

$$F_t + 4(1+\epsilon)^2\nabla^4 F + \epsilon\nabla^2 F + \frac{1}{2}(\nabla F)^2 = \frac{(1-\sigma)}{8\pi^2}\int_{-\infty}^{\infty} |\mathbf{k}|\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{z})} F(\mathbf{z}, t)\, d\mathbf{k}\, d\mathbf{z} \qquad (1)$$

Here $F = F(\mathbf{x}, t)$ is the dimensionless perturbation of the flame front surface in units of the width l_T of thermal structure of flame, $\mathbf{x}$ = dimensionless space coordinates in units of l_T, t = dimensionless time in units of l_T/U_b, U_b = normal velocity of the flame (velocity of a plane flame front relative to the combustion products), $\epsilon = (L_0 - L)/(1 - L_0)$, $1/2N(1-\sigma)(1-L_0) = 1$, N = dimensionless activation energy in units of R^0T_b, R^0 = universal gas constant, T_b = adiabatic temperature of combustion products, σ = coefficient of thermal expansion of the gas ($\sigma < 1$), L = Lewis number for the component of the mixture limiting the reaction.

Equation (1) covers the entire range of diffusional, thermal and hydrodynamic effects in the flame. At $L > L_0$ one can also consider an asymptotic situation that leads to the equation

$$F_t + \epsilon\nabla^2 F + \frac{1}{2}(\nabla F)^2 = \frac{(1-\sigma)}{8\pi^2}\int_{-\infty}^{\infty} |\mathbf{k}|\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{z})}F(\mathbf{z}, t)\, d\mathbf{k}\, d\mathbf{z}, \qquad \epsilon < 0. \tag{2}$$

Asymptotic analysis is based on the conditions that (i) the Lewis number is near a certain critical value $L_0 (L_0 < 1)$, (ii) the thermal expansion of the gas is weak and (iii) the nondimensional activation energy is a large number.

Flame model and fundamental equations

Despite the fact that real combustion mixtures contain more than two components and the reactions involve many steps—these effects play a secondary role in the formation of cellular (or noncellular) flame structure and the onset of self-turbulence; they may therefore be disregarded in a qualitative explanation of the relevant physical mechanism. Below we shall adopt the simple hydrodynamic flame model proposed by Istratov and Librovich (1966) in their work on the effect of transport processes on the hydrodynamic stability of a plane flame front.

In this model the flame is described exclusively by the diffusion equation for the concentration of the component M limiting the combustion reaction, by the heat conduction equation and the Navier–Stokes equations. The transport coefficients are assumed to be constant.

The reaction zone is assumed to be narrow, concentrated on a certain surface (the flame front) where the heat and mass flows are discontinuous and the concentration of the limiting component vanishes.

A model of this type may be derived rigorously from the general equations of combustion theory, for the case, say, of a bimolecular reaction with high activation energy and high heat release, when one of the reactants is present in minute concentrations. One must also assume in that case, that the initial fresh mixture consists entirely of reacting components, or that the reactants are weak traces in some homogeneous, "inert" gas (Fristrom and Westenberg, 1965). The flame is assumed to propagate slowly in open space (quasi-isobaricity). The condition that the transport coefficients are constant is not essential for the subsequent analysis, and is easily eliminated if necessary.

Such an elementary model is of course incapable of describing the possible effects of multicomponent diffusion or the transition from combustion to detonation. Nevertheless, for all its simplicity, the model of Istratov and Librovich (1966) proves adequate to describe cellular structure, wrinkles, self-turbulence and the resultant increase in the mean propagation velocity of the flame. In a full gas-dynamical model the latter effect should ultimately produce a detonation wave.

Below is the framework of the above model, we shall present a non-linear analysis of the stability of a propagating plane flame front. Subject to suitable choice of dimensionless variables (see Introduction), the system of equations for the flame may be written as follows:

Diffusion:

$$\rho\frac{\partial C}{\partial t}+\rho U_i\frac{\partial C}{\partial x_i}=\frac{1}{L}\frac{\partial}{\partial x_i}\frac{\partial C}{\partial x_i}-Q\delta_F. \tag{3}$$

Heat conduction:

$$\rho\frac{\partial T}{\partial t}+\rho U_i\frac{\partial T}{\partial x_i}=\frac{\partial}{\partial x_i}\frac{\partial T}{\partial x_i}+(1-\sigma)Q\delta_F. \tag{4}$$

Continuity:

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho U_i}{\partial x_i}=0. \tag{5}$$

Momentum:

$$\frac{\partial \rho U_i}{\partial t}+\frac{\partial \rho U_i U_k}{\partial x_k}=\frac{\partial P_{ik}}{\partial x_k}, \tag{6}$$

where

$$P_{ik}=\nu\left(\frac{\partial U_i}{\partial x_k}+\frac{\partial U_k}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial U_m}{\partial x_m}\right)-P\delta_{ik}.$$

Dynamic incompressibility condition:

$$\rho=\frac{1}{T} \tag{7}$$

$$Q=\exp\frac{1}{2}N(T-1), \tag{8}$$

where δ_F is the surface delta-function

$$\delta_F=\sqrt{[1+(\nabla F)^2]}\,\delta(x_1-F), \tag{9}$$

$x_1=F(x_2, x_3, t)$ is the perturbation of the plane flame front ($x_1=0$).

T is the dimensionless temperature, referred to adiabatic temperature T_b of combustion products; C = dimensionless concentration of the limiting component, referred to its initial concentration C_0; ρ = dimensionless density, referred to density ρ_b of combustion products; P-dimensionless pressure, referred to $\rho_b U_b^2$; U_i = gas velocity referred to U_b; ν = Prandtl number. For a justification of the eqn (8) defining the reaction rate intensity, we refer the reader to Sivashinsky (1975).

Far ahead of the flame front ($x_1 \to -\infty$) the temperature of the gas is equal to the temperature T_0 of the fresh mixture, while the concentration of the combustible component is constant and equal to C_0. Beyond the flame front ($x_1 \to +\infty$) the concentration is zero, and the temperature of the gas is equal to the adiabatic temperature T_b of the combustion products. In the dimensionless variables introduced above, these conditions may be written thus

$$\begin{gathered} T(-\infty, x_2, x_3, t) = \sigma, \qquad C(-\infty, x_2, x_3, t) = 1, \\ T(+\infty, x_2, x_3, t) = 1, \qquad C(x_1, x_2, x_3, t) \equiv 0 \qquad \text{for} \qquad x_1 \geq F(x_2, x_3, t). \end{gathered} \tag{10}$$

The following relations are satisfied on the flame front ($x_1 = F$) (see (3), (4), (8)–(10))

$$\begin{gathered} [\nabla T \cdot \mathbf{n}] + (1-\sigma)Q = 0, \qquad [\nabla C \cdot \mathbf{n}] - LQ = 0, \\ [T] = 0, \qquad C = 0 \end{gathered} \tag{11}$$

where $\mathbf{n}$ is the normal to the surface $x_1 = F(x_2, x_3, t)$. For the undisturbed plane flame front ($F = 0$) we have

$$\begin{gathered} Q^0 = 1; \qquad T^0 = \sigma + (1-\sigma)\exp x_1, 1; \qquad C^0 = 1 - \exp Lx_1, 0; \\ P^0 = 2 - \sigma + (1-\sigma)\left(\frac{4}{3}\nu - 1\right)\exp x_1, 1, \end{gathered} \tag{12}$$

for $x < 0$ and $x > 0$, respectively

$$U_1{}^0 = T^0, \qquad \rho^0 = (T^0)^{-1}, \qquad U_2{}^0 = U_3{}^0 = 0,$$

(The superscript 0 indicates the solution of the one-dimensional stationary problem.)

For the further analysis, it is convenient to transform to a coordinate system attached to the surface of the disturbed flame front

$$x_1 = x + F(y, z, t), \qquad x_2 = y, \qquad x_3 = z. \tag{13}$$

Auxiliary model problem. Linear stability analysis

Before discussing nonlinear instability of a plane flame front, we shall find it extremely useful to carry out the classical, linear analysis of the stability of one-dimensional propagation of a plane flame front to infinitesimal periodic disturbances. Solution of this relatively simpler problem will provide us with valuable information for the subsequent asymptotic nonlinear analysis.

However, it is well known that linear stability analysis in the general case involves a mathematical problem of great difficulty: to determine conditions for the existence of a nontrivial solution to a system comprising a large number of ordinary linear differential equations with variable coefficients. The variability of the coefficients is due to the temperature-dependence of the density, and this constitutes the main difficulty, precluding reduction of the problem to the investigation of a certain algebraic equation.†

However, since the aim of our linear analysis is only to obtain some idea of the orders of magnitude involved, it will be quite sufficient to consider a certain approximate flame model, which is amenable to analytical solution. This model will be a "brute-force" combination of the hydrodynamic model of Landau (1944) and the diffusional-thermal model of Zel'dovich *et al.* (1944) and Barenblatt *et al.* (1962). In other words, we shall treat the flame as a discontinuity jump in a non-viscous, incompressible fluid, but without neglecting the effects of diffusion, heat conduction and temperature dependence of the reaction rate. The model will clearly show both hydrodynamic and diffusional-thermal destabilizing effects.

As the density is now constant both ahead of and behind the front, it is easy to derive an algebraic relation between the stability index ω, the disturbance wavenumber k, and the parameters σ, L and N. Indeed, set

$$\rho_- \equiv 1/\sigma \quad \text{for} \quad x < 0, \qquad \rho_+ \equiv 1 \quad \text{for} \quad x > 0, \qquad \text{and also} \quad \nu = 0. \tag{14}$$

In the linear approximation, the solution to the perturbed system of equations (transformed) (3)–(8) is sought in the form‡

$$\begin{aligned} F = D\exp(iky + \omega t), \qquad C &= C^0 + F\tilde{s}(x), \\ T = T^0 + F\tilde{\theta}(x), \qquad P &= P^0 + F\tilde{p}(x), \\ U_1 = T^0 + F\tilde{u}(x), \qquad U_2 &= F\tilde{v}(x). \end{aligned} \tag{15}$$

where D is the amplitude of small periodic perturbations. From (20), (22)–(25), (30), (31) we obtain the system of ordinary differential equations for the functions $\tilde{s}(x)$, $\tilde{\theta}(x)$, $\tilde{p}(x)$, $\tilde{u}(x)$, $\tilde{v}(x)$:

$$A_i(\tilde{s}(x), \tilde{\theta}(x), \tilde{p}(x), \tilde{u}(x), \tilde{v}(x)) = 0 \qquad i = 1, 2, 3, 4, 5 \tag{16}$$

For brevity's sake, we shall not give the explicit form of (16).

The mathematical problem of stability may be stated thus: determine the spectrum $\omega = \omega(k)$ in such a way that the problem (16) has a nontrivial solution which goes to zero as $x \to \pm\infty$.

We shall seek conditions under which $Re\,\omega$ is nonnegative, i.e. the flame is unstable. The solution of system (16) presents no mathematical difficulties, and we shall therefore skip the intermediate steps and present the final result:

The rate of stability parameter $\omega = \omega(k)$ is determined from the algebraic

†A survey of the research on stability of flames under small perturbations, may be found in Istratov and Librovich (1965) (see also Andrews *et al.* (1975)).

‡To simplify matters, we shall confine ourselves to the case of two-dimensional disturbances, i.e. the derivatives with respect to z are assumed to vanish everywhere.

equation

$$\frac{1}{2}N(1-\sigma)=\frac{kB[R(1,\sigma)+R(1,1)]}{(B-\omega+k\sigma)[L-1+R(1,\sigma)-R(L,\sigma)]}+\frac{[L-R(L,\sigma)][R(1,\sigma)+R(1,1)]}{2[L-1+R(1,\sigma)-R(L,\sigma)]}, \tag{17}$$

where

$$R(L,\sigma)=\sqrt{(L^2+4k^2+4k\omega/\sigma)}.$$

For $x<0$

$$\tilde{u}(x)=B\exp kx, \tag{18}$$

where

$$B=\frac{(1-\sigma)(\omega^2+2k\omega+\sigma k^2)}{2(k+\omega)}.$$

We shall not present the cumbersome expressions for the remaining quantities $\tilde{p}, \tilde{v}, \tilde{u}(x>0)$. In the next section we shall give asymptotic expressions for them—the only information necessary for the subsequent analysis.

The first term on the right of (17) clearly represents the perturbation of the flame due to thermal expansion, while the second is due to the diffusional-thermal effect.

We first consider the case in which the concentration and temperature fields are similar, $L=1$. This classical case is of particular interest, since when $L=1$ the temperature at the exit from the reaction zone is not affected by the disturbance of the front but remains equal to T_b. The coefficient of the activation energy vanishes (see (17)) and no longer figures in the solution. On the other hand (Zel'dovich and Barenblatt, 1959), when $L=1$ the diffusion and conduction processes exercise only a stabilizing effect on the flame front. Thus, in the case of similarity the hydrodynamic instability effect appears in its simplest and purest form (i.e. uncontaminated by diffusional-thermal instability or by temperature-dependence of the reaction rate).

When $L=1$ and $\omega=0$, we obtain from (17)

$$k_0=\frac{(1-\sigma)(3-\sigma)}{4(2-\sigma)}. \tag{19}$$

Even without the effect of high activation energy and viscosity, this value of the critical wavenumber testifies to the powerful stabilizing effect of heat conduction. Indeed, for a typical thermal expansion coefficient, say $\sigma=0.2$, we have

$$k_0\simeq 0.31, \qquad \Lambda_0=2\pi l_T/k_0\simeq 20.2 l_T.$$

Thus, near the stability limit the characteristic disturbance wavelength considerably exceeds the width l_T of the thermal structure of the flame. Here thermal expansion of the gas is the sole destabilizing factor. As $\sigma\to 1$, therefore, the instability index $Re\,\omega$ will tend to zero. Thus $\sigma = 1$ is a bifurcation point of the problem.

On the other hand, we see from (19) that as $\sigma \to 1$

$$k_0 \simeq \frac{1}{2}(1-\sigma). \tag{20}$$

Thus, in the neighborhood of the bifurcation point the structure of the disturbed flame is not only quasistationary ($\omega \ll 1$) but also quasi-one-dimensional ($k \ll 1$).

By comparing (19) and (20), one can derive a rough estimate of the error in the principal term of the asymptotic expansion with respect to $(1-\sigma)$. For example, for $\sigma = 0.2$ the relative error is $\sim 30\%$. To improve the approximation, we must of course proceed to higher approximations relative to $(1-\sigma)$.

Consider the second extreme case

$$L = 1, \qquad k \sim \omega \ll 1.$$

From (17) we obtain the asymptotic case of Landau (1944)

$$\omega \simeq \frac{k\{\sqrt{[\sigma + \sigma^2(1-\sigma)]} - \sigma\}}{1+\sigma}, \tag{21}$$

or as $\sigma \to 1$

$$\omega \simeq \frac{1}{2}(1-\sigma)k. \tag{22}$$

Equations (20) and (22) give an indication of the correct scaling factor for ω and k in the case $\sigma \to 1$:

$$k = (1-\sigma)K, \qquad \omega = (1-\sigma)^2\Omega. \tag{23}$$

From this and (17) we obtain the following asymptotic equation:

$$\Omega = \frac{1}{2}K - K^2 \qquad \text{or} \qquad \omega = \frac{1}{2}(1-\sigma)k - k^2. \tag{24}$$

Equations (24) provides a corroboration of the well-known Markstein (1964) condition, which relates the propagation rate of the flame to the curvature of the front.

In the general case, $L \neq 1$, we can also scale the variables in accordance with (23) and isolate the principal term of the asymptotic expansion as $\sigma \to 1$, $N \to \infty$, $(1-\sigma)N < \infty$. By (17),

$$\frac{K^2 - 1/2KL + \Omega L}{K^2} = \frac{1}{2}N(1-\sigma)(1-L),$$

or

$$\omega = \frac{1}{2}(1-\sigma)k - \frac{1}{L}\left[1 - \frac{1}{2}N(1-\sigma)(1-L)\right]k^2. \tag{25}$$

Like (24), eqn (25) may be written as a linearization of Markstein's condition (1964):

$$\frac{\partial F}{\partial t}=\frac{1}{L}\left[1-\frac{1}{2}N(1-\sigma)(1-L)\right]\frac{\partial^2 F}{\partial y^2}+u'(0,y,t), \tag{26}$$

where $u'(x, y, t)$ is the perturbation of the x-component of the velocity of the gas.

For condition (26) to be well-posed, the coefficient of the second derivative must be positive. In other words, the Lewis number L should satisfy the condition

$$L>L_0\equiv 1-\frac{2}{N(1-\sigma)}. \tag{27}$$

Thus, for finite K, Ω, $N(1-\sigma)$, the asymptotic treatment at $\sigma\to 1$, $N\to\infty$ is admissible only for sufficiently large Lewis numbers ($L>L_0$).

To describe the case $L<L_0$, we begin with the asymptotic case $\sigma\to 1$ when ω, k and $N(1-\sigma)$ are finite.

In the first approximation, the term representing hydrodynamic instability drops out of the expression on the right of eqn (17). We obtain an equation describing only the effect of the diffusion and conduction effects on flame stability:

$$\frac{1}{2}N(1-\sigma)=\frac{R(1,1)[L-R(L,1)]}{L-1+R(1,1)-R(L,1)}. \tag{28}$$

At $\omega=0$, we obtain from (28) an expression for the critical wavenumber:

$$k_0=\frac{1}{2}\sqrt{\left[\frac{1}{2}N(1-\sigma)(1-L)-1\right]}. \tag{29}$$

Thus, in direct opposition to the previous asymptotic case, neutral stability is possible here only at sufficiently small Lewis numbers ($L<L_0$).

At $\omega\sim k^2\ll 1$, we obtain from (28)

$$\omega\simeq\left[\frac{1}{2}N(1-\sigma)(1-L)-1\right]k^2. \tag{30}$$

Thus, at $L<L_0$ a plane flame front is unstable to long-wave disturbances. $L=L_0$ is clearly a bifurcation point for the diffusional-thermal problem.

We consider the situation when L is close to L_0, or

$$\frac{1}{2}N(1-\sigma)(1-L)-1=\epsilon,\quad (\epsilon\ll 1). \tag{31}$$

Let us see what happens to eqn (28) in this asymptotic case. Equations (29) and

(30) indicate the correct scaling factor for the parameters ω and k:

$$k = \sqrt{\epsilon}\tilde{K}, \qquad \omega = \epsilon^2\tilde{\Omega}. \tag{32}$$

From this and (28) we obtain the following asymptotic relation between $\tilde{\Omega}$ and $\tilde{K}$:

$$\tilde{\Omega} = \tilde{K}^2 - 4\tilde{K}^4,$$

or

$$\omega \simeq \left[\frac{1}{2}N(1-\sigma)(1-L) - 1\right]k^2 - 4k^4. \tag{33}$$

At $L < L_0$ the flame is stable only to short-wave disturbances ($k > k_0$), and unstable to long-wave disturbances. Here there is a number k_c corresponding to the most rapid amplification of disturbances (maximum ω):

$$k_c = \frac{\sqrt{2}}{4}k_0. \tag{34}$$

In terms of the disturbed flame front, eqn (33) may be written as

$$\frac{\partial F}{\partial t} = -4\frac{\partial^4 F}{\partial y^4} + \left[1 - \frac{1}{2}N(1-\sigma)(1-L)\right]\frac{\partial^2 F}{\partial y^2}. \tag{35}$$

Condition (35) is well-posed for any values of L, since the coefficient of the fourth derivative is always negative.

The essential limitation of the result (35) is that the destabilizing effect of thermal expansion does not appear in the asymptotic relations (33), (35). In order to bring this effect into play, while retaining the advantages of eqn (35), we must consider the asymptotic situation as $\sigma \to 1$, $\epsilon \to 0$ simultaneously. To find a suitable uniform asymptotic relationship, we use the scaling factors of (32) and eqn (17). Letting $\epsilon \to 0$, $\sigma \to 1$ in (17), we obtain

$$-\frac{(1-\sigma)}{2\epsilon\sqrt{\epsilon}}\frac{1}{\tilde{K}} + \frac{\tilde{\Omega} + 4\tilde{K}^4}{\tilde{K}^2} = 0. \tag{36}$$

Hence it is evident that if the hydrodynamic term is to be regained as $\sigma \to 1$, the relevant orders of magnitude must satisfy the condition

$$(1-\sigma) \sim \epsilon\sqrt{\epsilon} \qquad \text{or} \qquad \frac{1}{2}N(1-\sigma)(1-L) - 1 = \alpha(1-\sigma)^{2/3}. \tag{37}$$

α is a certain constant of the order of unity. In terms of the original variables ω and k, eqn (36) may be written as

$$\omega + 4k^4 + \left[\frac{1}{2}N(1-\sigma)(1-L) - 1\right]k^2 - \frac{1}{2}(1-\sigma)k = 0, \tag{38}$$

or, in terms of the disturbed flame front,

$$\frac{\partial F}{\partial t}+4\frac{\partial^4 F}{\partial y^4}+\left[\frac{1}{2}N(1-\sigma)(1-L)-1\right]\frac{\partial^2 F}{\partial y^2}=u'(0,y,t). \tag{39}$$

Equation (39) is a generalization of Markstein's condition (26); this uniform asymptotic treatment includes both hydrodynamic and dissipative effects.

Choice of characteristic dimensions for nonlinear asymptotic analysis of flame stability

The results of the preceding section amount to two different asymptotic situations, which reduce to eqns (26) and (39) on the flame front. The first case (26) holds when $N\to\infty$, $\sigma\to 1$ in such a way that the product $N(1-\sigma)$ remains finite. Here the characteristic scaling factors of the time t and space variable y (along the flame front) are seen from (23) to be $(1-\sigma)^{-2}$ and $(1-\sigma)^{-1}$, respectively.

Using (16), we have the following asymptotic expression for the perturbations of the hydrodynamic quantities, the temperature and the concentration as $\sigma\to 1$:

$$\tilde{u}_-=-\tilde{p}_-=-i\tilde{v}_-=\frac{1}{2}(1-\sigma)k\exp kx. \tag{40}$$

$$\tilde{u}_+=-\tilde{p}_+=i\tilde{v}_+=\frac{1}{2}(1-\sigma)k\exp -kx. \tag{41}$$

$$\begin{aligned}\tilde{s}_-&=k^2x\exp x, &\tilde{\theta}_-&=-(1-\sigma)k^2x\exp x-(1-\sigma)(1-L)k^2\exp x\\ \tilde{s}_+&=0, &\tilde{\theta}_+&=-(1-\sigma)(1-L)k^2\exp[-(\omega+k^2)x].\end{aligned} \tag{42}$$

Note that if the incoming hydrodynamic flow to the flame front is irrotational (see (40)), the flow will remain irrotational behind the flame front too, in the first approximation relative to $(1-\sigma)$ (see (41)). Generally speaking, this property does not hold for arbitrary $(1-\sigma)$ (Landau, 1944).

By (40)–(42), along the normal to the flame front there are three characteristic zones (layers), whose widths are of orders 1, $(1-\sigma)^{-1}$ and $(1-\sigma)^{-2}$. The first zone corresponds to the thermal structure of the flame, of width l_T. The second corresponds to the depth of hydrodynamic disturbances due to distortion of the flame front. In the third zone, the disturbances are dissipated by the effect of transport processes. We use the results obtained for the model problem and introduce suitable characteristic time and space variables:

$$\begin{aligned}&\tau=(1-\sigma)^2t,\ \eta=(1-\sigma)y,\ \zeta=(1-\sigma)z,\\ &x=x \text{ (thermal structure of flame)};\\ &\xi=(1-\sigma)x \text{ (hydrodynamic structure of flame)};\\ &\chi=(1-\sigma)^2x \text{ (dissipation zone of thermal and hydrodynamic disturbances)}.\end{aligned} \tag{43}$$

We shall use the variables (43) for an asymptotic nonlinear analysis of the stability problem, based on the exact eqns (3)–(8).

By contrast, the information gained from the linear analysis is no longer sufficient to enable us to choose the characteristic scaling factor for the amplitude of the disturbed flame front. In this case Landau's model (1944) turns out to be useful, provided thermal expansion of the gas is excluded. In a model of this kind, the flame front is simply a surface moving at a constant unit velocity relative to the stationary gas. Thus the disturbed surface $F(y, z, t)$ of the plane flame front satisfies the equation

$$1 - F_t = \sqrt{(1 + F_y^2 + F_z^2)}. \tag{44}$$

Using (43), we can transform eqn (44) to the form

$$1 - (1-\sigma)^2 F_\tau = \sqrt{[1 + (1-\sigma)^2 F_\eta^2 + (1-\sigma)^2 F_\zeta^2]}$$

or

$$(1-\sigma)^2 F_\tau + \frac{1}{2}(1-\sigma)^2 F_\eta^2 + \frac{1}{2}(1-\sigma)^2 F_\zeta^2 = o((1-\sigma)^2). \tag{45}$$

Hence it is clear that in the nonlinear asymptotic analysis we must treat F as a quantity of the order of unity as $\sigma \to 1$. We shall assume henceforth that this result, based on a Landau-type model, remains valid for our fundamental problem, which is based on eqns (3)–(8):

$$F(\tau, \eta, \zeta) \sim 1. \tag{46}$$

Now, on the basis of (46) and the results (40)–(42) of the linear analysis, we can estimate the orders of magnitude of the perturbations of velocity, pressure temperature and concentration:

$$u' \sim v' \sim w' \sim p' \sim (1-\sigma)^2, \qquad s' \sim (1-\sigma)^2; \qquad \theta' \sim (1-\sigma)^3. \tag{47}$$

We now turn to the second asymptotic situation (39), again considering the limit as $N \to \infty$, $\sigma \to 1$ with $N(1-\sigma)$ remaining finite. At the same time, we assume that the Lewis number L is close to a certain L_0 such that the order relation (37) is satisfied. The time t and space variable y (along the front) have characteristic scales

$$t \sim \epsilon^{-2} \sim (1-\sigma)^{-4/3} \qquad \text{and} \qquad y \sim z \sim \epsilon^{-1/2} \sim (1-\sigma)^{-1/3}$$

respectively (see 32).

As in the first asymptotic situation, there are again three characteristic zones along the normal to the flame front; their widths are of orders 1, $(1-\sigma)^{-1/3}$, $(1-\sigma)^{-4/3}$, respectively. Thus the characteristic time and space variables are:

$$\begin{aligned}&\tau = (1-\sigma)^{4/3}t,\ \eta = (1-\sigma)^{1/3}y,\ \zeta = (1-\sigma)^{1/3}z;\\ &x = x \text{ (thermal structure)};\\ &\xi = (1-\sigma)^{1/3}x \text{ (hydrodynamic structure)};\\ &\chi = (1-\sigma)^{2/3}x \text{ (dissipation zone)}.\end{aligned} \tag{48}$$

To determine the characteristic scaling factors for the amplitude of the disturbed front, we again use eqn (44). Using (48), we can write eqn (44) as

$$(1-\sigma)^{4/3}F_\tau + \frac{1}{2}(1-\sigma)^{2/3}F_\eta{}^2 + \frac{1}{2}(1-\sigma)^{2/3}F_\zeta^2 = o((1-\sigma)^{2/3}). \tag{49}$$

Thus, in order to retain the nonlinear terms as $\sigma \to 1$ we must stipulate that $F(\tau, \eta, \zeta)$ is of the order of $(1-\sigma)^{2/3}$:

$$F \sim (1-\sigma)^{2/3}. \tag{50}$$

Using this and eqns (40)–(42), we have the following relations for the perturbations of hydrodynamic variables, temperature and concentration:

$$u' \sim v' \sim w' \sim p' \sim (1-\sigma)^2; \qquad s' \sim (1-\sigma)^{4/3}; \qquad \theta' \sim (1-\sigma)^{7/3}. \tag{51}$$

Derivation of nonlinear equation for disturbed flame front. First asymptotic case

In the first asymptotic case, in view of (47) we set

$$\begin{aligned}T = T^0 + (1-\sigma)^3\theta; \qquad C = C^0 + (1-\sigma)^2 S; \qquad U_1 = T^0 + (1-\sigma)^2 u;\\ U_2 = (1-\sigma)^2 v; \qquad U_3 = (1-\sigma)^2 w; \qquad P = P^0 + (1-\sigma)^2 p\end{aligned} \tag{52}$$

$$\frac{1}{2}N(1-\sigma) = \beta \tag{53}$$

where

$$T^0 = 1 + (1-\sigma)I(x), \qquad P^0 = 1 + (1-\sigma)\Pi(x),$$

$$I(x) = \exp x - 1, \qquad \Pi(x) = 1 + \left(\frac{4}{3}\nu - 1\right)\exp x \qquad \text{for } x < 0, \tag{54}$$

$$I(x) = \Pi(x) = 0 \qquad \text{for } x > 0.$$

We shall look for asymptotic expansions of θ, S, u, v, w, p, F in the form

$$\theta = \theta_0 + (1-\sigma)\theta_1 + \cdots, \qquad S = S_0 + (1-\sigma)S_1 + \cdots, \qquad u = u_0 + (1-\sigma)u_1 + \cdots$$

$$v = v_0 + (1-\sigma)v_1 + \cdots, \qquad w = w_0 + (1-\sigma)w_1 + \cdots, \qquad p = p_0 + (1-\sigma)p_1 + \cdots,$$

$$F = F_0 + (1-\sigma)F_1 + \cdots. \tag{55}$$

In the thermal structure zone of the flame (x), the first approximation that follows from eqns (3)–(8) and from (43), (52)–(55) is

$$[u_0 - (F_0)_\tau]I_x + (\theta_0)_x = (\theta_0)_{xx} - [(F_0)_{\eta\eta} + (F_0)_{\zeta\zeta}]I_x + [(F_0)_\eta{}^2 + (F_0)_\zeta{}^2]I_{xx} + \left[\beta\theta_0 + \frac{1}{2}(F_0)_\eta{}^2 + \frac{1}{2}(F_0)_\zeta{}^2\right]\delta(x). \tag{56}$$

$$L[u_0 - (F_0)_\tau]C_x^0 + L(S_0)_x = (S_0)_{xx} - [(F_0)_{\eta\eta} + (F_0)_{\zeta\zeta}]C_x^0 + [(F_0)_\eta{}^2 + (F_0)_\zeta{}^2]C_{xx}^0 - L\left[\beta\theta_0 + \frac{1}{2}(F_0)_\eta{}^2 + \frac{1}{2}(F_0)_\zeta{}^2\right]\delta(x) \tag{57}$$

$$[u_0 - (F_0)_\tau]_x = 0 \tag{58}$$

$$[2u_0 - (F_0)_\tau]_x = \left[-p_0 + \frac{4}{3}\nu(u_0)_x\right]_x \tag{59}$$

$$(v_0)_{xx} = \nu(v_0)_x - (F_0)_\eta\left(\frac{1}{3}\nu I_x - \Pi\right)_x \tag{60}$$

$$(w_0)_{xx} = \nu(w_0)_x - (F_0)_\zeta\left(\frac{1}{3}\nu I_x - \Pi\right)_x \tag{61}$$

By (58)–(61),

$$u_0(x, \eta, \zeta, \tau) = f(\eta, \zeta, \tau) \qquad \text{for } -\infty < x < \infty \tag{62}$$

$$v_0(x, \eta, \zeta, \tau) = g(\eta, \zeta, \tau) \qquad \text{for } x > 0$$

and (63)

$$v_0(x, \eta, \zeta, \tau) = g(\eta, \zeta, \tau) + (F_0)_\eta - (F_0)_\eta \exp x \qquad \text{for } x < 0$$

$$w_0(x, \eta, \zeta, \tau) = h(\eta, \zeta, \tau) \qquad \text{for } x > 0$$

and (64)

$$w_0(x, \eta, \zeta, \tau) = h(\eta, \zeta, \tau) + (F_0)_\zeta - (F_0)_\zeta \exp x \qquad \text{for } x < 0$$

$$p_0(x, \eta, \zeta, \tau) = e(\eta, \zeta, \tau) \qquad \text{for } -\infty < x < \infty. \tag{65}$$

The actual form of the functions f, g, h, e will depend on the matching conditions which link (62)–(65) to the external solution describing the hydrodynamic structure of the disturbed flame (variable ξ):

$$u_0(\xi, \eta, \zeta, \tau) \to f(\eta, \zeta, \tau) \qquad \text{for } \xi \to \pm 0 \tag{66}$$

$$\begin{aligned} v_0(\xi, \eta, \zeta, \tau) &\to g(\eta, \zeta, \tau) && \text{for } \xi \to +0 \\ v_0(\xi, \eta, \zeta, \tau) &\to g(\eta, \zeta, \tau) + (F_0)_\eta && \text{for } \xi \to -0 \end{aligned} \tag{67}$$

$$\begin{aligned} w_0(\xi, \eta, \zeta, \tau) &\to h(\eta, \zeta, \tau) && \text{for } \xi \to +0 \\ w_0(\xi, \eta, \zeta, \tau) &\to h(\eta, \zeta, \tau) + (F_0)_\zeta && \text{for } \xi \to -0 \end{aligned} \tag{68}$$

$$p_0(\xi, \eta, \zeta, \tau) \to e(\eta, \zeta, \tau) \qquad \text{for } \xi \to \pm 0 \tag{69}$$

we now obtain from (56), (57), (62)

$$S_0(x,\eta,\zeta,\tau)=[L(F_0)_\eta^2+L(F_0)_\zeta^2-(F_0)_{\eta\eta}-(F_0)_{\zeta\zeta}+L(F_0)_\tau-Lf]x\exp Lx$$
$$\text{for } x<0 \quad \text{and} \quad S_0(x,\eta,\zeta,\tau)\equiv 0 \quad \text{for } x>0 \tag{70}$$

$$\theta_0(x,\eta,\zeta,\tau)=\theta_0(0,\eta,\zeta,\tau)\exp x - [(F_0)_\eta^2+(F_0)_\zeta^2-(F_0)_{\eta\eta}-(F_0)_{\zeta\zeta}+(F_0)_\tau-f]x\exp x$$
$$\text{for } x<0 \quad \text{and} \quad \theta_0(x,\eta,\zeta,\tau)\equiv\theta_0(0,\eta,\zeta,\tau) \quad \text{for } x>0. \tag{71}$$

Integrating (56), (57) in the neighborhood of $x=0$, we obtain the following expressions for the discontinuities of the derivatives:

$$\begin{aligned}[(S_0)_x]&=\beta L\theta_0-\frac{1}{2}L(F_0)_\eta^2-\frac{1}{2}L(F_0)_\zeta^2,\\ [(\theta_0)_x]&=-\beta\theta_0+\frac{1}{2}(F_0)_\eta^2+\frac{1}{2}(F_0)_\zeta^2.\end{aligned} \tag{72}$$

In view of (66), (70), (71), eqns (72) form a system of two equations for $\theta_0(0,\eta,\zeta,\tau)$ and $F_0(\eta,\zeta,\tau)$. Eliminating $\theta_0(0,\eta,\zeta,\tau)$, we obtain

$$(F_0)_\tau=\frac{1}{L}[1+\beta(L-1)][(F_0)_{\eta\eta}+(F_0)_{\zeta\zeta}]-\frac{1}{2}(F_0)_\eta^2-\frac{1}{2}(F_0)_\zeta^2+f \tag{73}$$

where $f(\eta,\zeta,\tau)=u_0(0,\eta,\zeta,\tau)$ (see 66).

To determine $u_0(0,\eta,\tau)$, we consider the zone of hydrodynamic structure (variable ξ). By (4)–(7), the principal term of the asymptotic expansion (55) gives

$$\begin{aligned}&(u_0)_\xi+(p_0)_\xi=0, \qquad (v_0)_\xi+(p_0)_\eta=0\\ &(w_0)_\xi+(p_0)_\zeta=0, \qquad (u_0)_\xi+(v_0)_\eta+(w_0)_\zeta=0.\end{aligned} \tag{74}$$

From (66)–(69) we obtain the conditions that must hold at $\xi=0$.

$$\begin{aligned}u_0(+0,\eta,\zeta,\tau)&=u_0(-0,\eta,\zeta,\tau),\\ p_0(+0,\eta,\zeta,\tau)&=p_0(-0,\eta,\zeta,\tau),\\ v_0(+0,\eta,\zeta,\tau)&=v_0(-0,\eta,\zeta,\tau)-(F_0)_\eta,\\ w_0(+0,\eta,\zeta,\tau)&=w_0(-0,\eta,\zeta,\tau)-(F_0)_\zeta.\end{aligned} \tag{75}$$

We see that the hydrodynamic variables may be expressed in terms of the flame front disturbance $F_0(\eta,\zeta,\tau)$ by solving the simple linear problem (74), (75). We require a solution of (74), (75) that is bounded for $|\xi|<\infty$.

As we are interested in this paper in *intrinsic* instability of the flame, caused by initial disturbances of the front, we must exclude all *external* disturbances; to this end, we assume that the incoming flow to the flame is irrotational, i.e. for $\xi<0$:

$$(u_0)_\eta - (v_0)_\xi = 0, \qquad (u_0)_\zeta - (w_0)_\xi = 0, \qquad (v_0)_\zeta - (w_0)_\eta = 0. \tag{76}$$

Solving problem (74)–(75) by Fourier transforms, we obtain

$$(u_0)_\mp = \frac{1}{8\pi^2} \iiiint_{-\infty}^{\infty} \sqrt{(k^2+l^2)}\, e^{ik(\eta-\eta')+il(\zeta-\zeta')\pm\sqrt{(k^2+l^2)}\xi} F_0(\eta', \zeta', \tau)\, d\eta'\, d\zeta'\, dk\, dl. \tag{77}$$

The signs (+) and (−) refer to the zones $\xi > 0$ and $\xi < 0$, respectively. Note that the flow at $\xi > 0$ is irrotational (for the principal term of the asymptotic expansion).

Using (77), we can write (73) as an integrodifferential equation for F_0:†

$$F_\tau = \frac{1}{L}[1+\beta(L-1)]\nabla^2 F - \frac{1}{2}(\nabla F)^2 + \frac{1}{8\pi^2}\int_{-\infty}^{\infty} |\mathbf{k}|\, e^{i\mathbf{k}(\boldsymbol{\eta}-\boldsymbol{\eta}')} F(\boldsymbol{\eta}', t)\, d\mathbf{k}\, d\boldsymbol{\eta}' \quad \boldsymbol{\eta} = (\eta, \zeta). \tag{78}$$

This equation is clearly meaningful only when $L > L_0$. Otherwise, the sign of the coefficient of the second derivative is negative and the corresponding initial-value problem is not well-posed.

Nonlinear equation for flame front. Second asymptotic case

In the first asymptotic case, the first-approximation equations were sufficient to yield an equation of type (2) for the flame front. In the second asymptotic case the situation is less trivial. Here the first approximation enables us to find only a branch point of the general bifurcation problem. The equation of the flame front is then derived as the condition for solvability of the second-approximation system of equations.

In view of (51), in the second approximation we must set

$$\begin{gathered} T = T^0 + (1-\sigma)^{7/3}\theta, \qquad C = C^0 + (1-\sigma)^{4/3} S, \\ U_1 = T^0 + (1-\sigma)^2 u, \qquad U_2 = (1-\sigma)^2 v, \qquad U_3 = (1-\sigma)^2 w, \\ P = P^0 + (1-\sigma)^2 p, \qquad F = (1-\sigma)^{2/3}\Phi \end{gathered} \tag{79}$$

where C^0, T^0, and P^0 are defined in (54). In view of (37), we express the Lewis number L as

$$L = L_0 + \alpha(L_0 - 1)(1-\sigma)^{2/3}. \tag{80}$$

In the thermal structure zone (variable x), the system of equations for heat conduction and diffusion (3), (4) may be written, up to terms of order $o((1-$

†We have omitted the index "0", which is not necessary here.

$\sigma)^{2/3}$), as follows:

$$\theta_x - \theta_{xx} + I_x(\Phi_{\eta\eta} + \Phi_{\zeta\zeta}) = (1-\sigma)^{2/3}(\theta_{\eta\eta} + \theta_{\zeta\zeta}) + (1-\sigma)^{2/3} I_{xx}(\Phi_\eta{}^2 + \Phi_\zeta{}^2) + (1-\sigma)^{2/3}(\Phi_\tau - u) I_x + \left[\beta\theta + \frac{1}{2}(1-\sigma)^{2/3}(\Phi_\eta{}^2 + \Phi_\zeta{}^2)\right]\delta(x) \quad (81)$$

$$LS_x - S_{xx} + C_x{}^0(\Phi_{\eta\eta} + \Phi_{\zeta\zeta}) = (1-\sigma)^{2/3}(S_{\eta\eta} + S_{\zeta\zeta}) + (1-\sigma)^{2/3} C_{xx}^0(\Phi_\eta{}^2 + \Phi_\zeta{}^2) + L(1-\sigma)^{2/3}(\Phi_\tau - u) C_x{}^0 - L\left[\beta\theta + \frac{1}{2}(1-\sigma)^{2/3}(\Phi_\eta{}^2 + \Phi_\zeta{}^2)\right]\delta(x). \quad (82)$$

A solution of system (81), (82) will be sought as an asymptotic expansion in powers of $(1-\sigma)^{2/3}$, i.e.

$$\begin{aligned} \Phi &= \Phi_0 + (1-\sigma)^{2/3}\Phi_1 + \cdots, & \theta &= \theta_0 + (1-\sigma)^{2/3}\theta_1 + \cdots, \\ S &= S_0 + (1-\sigma)^{2/3} S_1 + \cdots, & u &= u_0 + (1-\sigma)^{2/3} u_i + \cdots. \end{aligned} \quad (83)$$

By virtue of (80), the undisturbed concentration distribution C^0 may also be expanded in powers of $(1-\sigma)^{2/3}$

$$C^0 = C_0{}^0 + (1-\sigma)^{2/3} C_1{}^0 + \cdots. \quad (84)$$

It is readily seen that solution of the hydrodynamic problem for the second asymptotic case reduced in the first approximation to solution of eqns (58)–(61), (74), as done in the first asymptotic case.

Clearly, the orders of magnitude of Φ_0, ξ, η, τ are determined by eqns (48), (50) and (79). Thus, as in the preceding section,

$$u_0(x, y, z, \tau) = f(\eta, \zeta, \tau) \quad (85)$$

where $f(\eta, \zeta, \tau)$ is given in terms of $\Phi_0(\eta, \zeta, \tau)$ by

$$f(\eta, \zeta, \tau) = \frac{1}{8\pi^2}\int_{-\infty}^{\infty} |\mathbf{k}|\, e^{i\mathbf{k}(\boldsymbol{\eta}-\boldsymbol{\eta}')}\Phi_0(\boldsymbol{\eta}', \tau)\, d\boldsymbol{\eta}'\, d\mathbf{k}. \quad (86)$$

In the first approximation, system (81), (82) may be written as

$$\begin{aligned} (\theta_0)_x - (\theta_0)_{xx} + I_x[(\Phi_0)_{\eta\eta} + (\Phi_0)_{\zeta\zeta}] &= \beta\theta_0\delta(x) \\ L_0(S_0)_x - (S_0)_{xx} + (C_0{}^0)_x[(\Phi_0)_{\eta\eta} + (\Phi_0)_{\zeta\zeta}] &= -\beta L_0\theta_0\delta(x). \end{aligned} \quad (87)$$

The bounded solution of system (87) satisfying the continuity condition at $x = 0$ has the form

$$S_0 = -[(\Phi_0)_{\eta\eta} + (\Phi_0)_{\zeta\zeta}]x \exp L_0 x \qquad \text{for } x < 0$$

$$\theta_0 = [(\Phi_0)_{\eta\eta} + (\Phi_0)_{\zeta\zeta}]\left(x + \frac{1 - L_0}{L_0}\right)\exp x \qquad \text{for } x < 0$$

$$\theta_0 = \frac{1-L_0}{L_0}[(\Phi_0)_{\eta\eta} + (\Phi_0)_{\zeta\zeta}], \qquad \text{for } x > 0$$

$$S_0 \equiv 0 \qquad \text{for } x > 0. \tag{88}$$

The conditions for discontinuity of the derivatives at $x = 0$ imply a necessary condition for the existence of a nontrivial solution:

$$\beta(1 - L_0) = 1. \tag{89}$$

Thus, we have obtained the critical Lewis number L_0, corresponding to a bifurcation point of the problem. Note that the fact that the disturbance θ_0 is not damped out as $x \to +\infty$ is due to the existence of an external region $x \sim (1-\sigma)^{-2/3}$ (with characteristic space variable $\chi = (1-\sigma)^{2/3}x$) in which θ_0 is exponentially damped (see (42)).

In the second approximation, system (81), (82) becomes

$$\begin{aligned}(\theta_1)_x - (\theta_1)_{xx} + I_x[(\Phi_1)_{\eta\eta} + (\Phi_1)_{\zeta\zeta}] &= (\theta_0)_{\eta\eta} + (\theta_0)_{\zeta\zeta} + I_{xx}[(\Phi_0)_\eta^2 + (\Phi_0)_\zeta^2] \\ &+ I_x[(\Phi_0)_\tau - u_0] + \left[\beta u_1 + \frac{1}{2}(\Phi_0)_\eta^2 + \frac{1}{2}(\Phi_0)_\zeta^2\right]\delta(x)\end{aligned} \tag{90}$$

$$\begin{aligned}L_0(S_1)_x - (S_1)_{xx} + (C_0{}^0)_x[(\Phi_1)_{\eta\eta} + (\Phi_1)_{\zeta\zeta}] &= \alpha(1-L_0)(S_0)_x - (C_1{}^0)_x[(\Phi_0)_{\eta\eta} + (\Phi_0)_{\zeta\zeta}] \\ &+ (S_0)_{\eta\eta} + (S_0)_{\zeta\zeta} + (C_0{}^0)_{xx}[(\Phi_0)_\eta^2 + (\Phi_0)_\zeta^2] + L_0(C_0{}^0)_x[(\Phi_0)_\tau - u_0] \\ &- \left[L_0\beta\theta_1 + \frac{1}{2}L_0(\Phi_0)_\eta^2 + \frac{1}{2}L_0(\Phi_0)_\zeta^2 + \alpha\beta(L_0 - 1)\theta_0\right]\delta(x).\end{aligned} \tag{91}$$

The second-approximation solution of (90), (91) satisfying the continuity condition at $x = 0$ is

$$\begin{aligned}\theta_1 &= \frac{1}{2}[(\Phi_0)_{\eta\eta\eta\eta} + 2(\Phi_0)_{\eta\eta\zeta\zeta} + (\Phi_0)_{\zeta\zeta\zeta\zeta}]x^2 \exp x + \left\{\left(2 - \frac{1}{L_0}\right)[(\Phi_0)_{\eta\eta\eta\eta}\right. \\ &\left. + 2(\Phi_0)_{\eta\eta\zeta\zeta} + (\Phi_0)_{\zeta\zeta\zeta\zeta}] - (\Phi_0)_\eta^2 - (\Phi_0)_\zeta^2 - (\Phi_0)_\tau + u_0 + (\Phi_1)_{\eta\eta} + (\Phi_1)_{\zeta\zeta}\right\}x \exp x \\ &+ \theta_1(0)\exp x, \qquad \text{for } x < 0\end{aligned} \tag{92}$$

$$\begin{aligned}S_1 &= \frac{1}{2L_0}[(\Phi_0)_{\eta\eta\eta\eta} + 2(\Phi_0)_{\eta\eta\zeta\zeta} + (\Phi_0)_{\zeta\zeta\zeta\zeta}]x^2 \exp L_0 X \\ &+ \frac{\alpha(1-L_0)}{2L_0}[(\Phi_0)_{\eta\eta} + (\Phi_0)_{\zeta\zeta}]x^2 \exp L_0 x \\ &+ [L_0(\Phi_0)_\eta^2 + L_0(\Phi_0)_\zeta^2 + L_0(\Phi_0)_\tau - L_0 u_0 - (\Phi_1)_{\eta\eta} - (\Phi_1)_{\zeta\zeta}]x \exp L_0 x \\ &- \frac{1}{L_0^2}[(\Phi_0)_{\eta\eta\eta\eta} + 2(\Phi_0)_{\eta\eta\zeta\zeta} + (\Phi_0)_{\zeta\zeta\zeta\zeta}]x \exp L_0 x, \qquad \text{for } x < 0\end{aligned} \tag{93}$$

$$\theta_1 = \theta_1(0) + \left(\frac{1}{L_0} - 1\right)x[(\Phi_0)_{\eta\eta\eta\eta} + 2(\Phi_0)_{\eta\eta\zeta\zeta} + (\Phi_0)_{\zeta\zeta\zeta\zeta}]. \tag{94}$$

$$S \equiv 0, \qquad x > 0.$$

To define $\theta_1(0)$, Φ_0 and Φ_1, we appeal to the conditions for discontinuity of the derivatives at $x = 0$. These conditions may be expressed as a system of two linear equations in $(\Phi_1)_{\eta\eta} + (\Phi_1)_{\zeta\zeta}$ and $\theta_1(0)$

$$\begin{aligned}(\beta - 1)\theta_1(0) - (\Phi_1)_{\eta\eta} - (\Phi_1)_{\zeta\zeta} &= \frac{3L_0 - 2}{L_0}[(\Phi_0)_{\eta\eta\eta\eta} + 2(\Phi_0)_{\eta\eta\zeta\zeta} + (\Phi_0)_{\zeta\zeta\zeta\zeta}] \\ &\quad - \frac{1}{2}(\Phi_0)_\eta^2 - \frac{1}{2}(\Phi_0)_\zeta^2 - (\Phi_0)_\tau + u_0, \\ \beta L_0\theta_1(0) - (\Phi_1)_{\eta\eta} - (\Phi_1)_{\zeta\zeta} &= \frac{1}{L_0}[(\Phi_0)_{\eta\eta\eta\eta} + 2(\Phi_0)_{\eta\eta\zeta\zeta} + (\Phi_0)_{\zeta\zeta\zeta\zeta}] \\ &\quad - L_0(\Phi_0)_\tau + L_0u_0 + (\Phi_0)_{\eta\eta} + (\Phi_0)_{\zeta\zeta}.\end{aligned} \tag{95}$$

As the determinant of this system vanishes (see, 89), it will be solvable at $\Phi_0 \not\equiv const.$ only if the following condition holds:

$$\begin{aligned}(\Phi_0)_\tau + \frac{1 + 3L_0}{L_0^2}[(\Phi_0)_{\eta\eta\eta\eta} + 2(\Phi_0)_{\eta\eta\zeta\zeta} + (\Phi_0)_{\zeta\zeta\zeta\zeta}] \\ + \frac{\alpha}{L_0}[(\Phi_0)_{\eta\eta} + (\Phi_0)_{\zeta\zeta}] + \frac{1}{2}[(\Phi_0)_\eta^2 + (\Phi_0)_\zeta^2] = u_0.\end{aligned} \tag{96}$$

Together with (85), (86), this yields a nonlinear integrodifferential equation for $\Phi_0(\eta, \zeta, \tau)$. If $\beta \gg 1$ (or $L_0 \simeq 1$), this equation has the form:†

$$F_t + 4\nabla^4F + \epsilon\nabla^2F + \frac{1}{2}(\nabla F)^2 = \frac{(1-\sigma)}{8\pi^2}\int_{-\infty}^{\infty} |\mathbf{k}|\, e^{i\mathbf{k}(\boldsymbol{\eta}-\boldsymbol{\eta}')}\Phi(\boldsymbol{\eta}', \tau)\,\mathrm{d}\mathbf{k}\,\mathrm{d}\boldsymbol{\eta}'. \tag{97}$$

Now in terms of the natural dimensionless variables (in units of l_T and the time interval l_T/U_b) eqn (97) may be transformed as follows:

$$F_t + 4\nabla^4F + \epsilon\nabla^2F + \frac{1}{2}(\nabla F)^2 = \frac{(1-\sigma)}{8\pi^2}\int_{-\infty}^{\infty} |\mathbf{k}|\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{z})}F(\mathbf{z}, t)\,\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{z}. \tag{98}$$

Improvement of asymptotic expansions

As we have already mentioned, when $L = 1$ the thermal expansion coefficient σ of the gas is the only parameter in the stability problem for a plane flame front. In that case, in the vicinity of $\sigma = 1$ there is a single correct nonlinear asymptotic form

†We have omitted the index "0".

for the disturbance of the front, leading to the equation

$$F_t - \nabla^2 F + \frac{1}{2}(\nabla F)^2 = \frac{(1-\sigma)}{8\pi^2}\int_{-\infty}^{\infty} |\mathbf{k}|\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{z})} F(\mathbf{z}, t)\, d\mathbf{k}\, d\mathbf{z}. \tag{99}$$

We now examine the appearance of eqn (98) when $L = 1$. As in eqn (99), the coefficient of the second derivative is -1. However, as the term involving the fourth derivative does not vanish, eqn (98) does not degenerate into eqn (99). This discrepancy is not surprising. Equation (98) was derived on the assumption that L lies near L_0. Thus, extrapolation to the value $L = 1$ ($L_0 < 1$) is generally speaking not legitimate, and may lead to significant errors.

Nevertheless, it turns out that in regard to the second asymptotic case one can modify the scaling factors (37), (48), (50) in such a way that the resulting asymptotic equation for F is a good approximation in a larger neighborhood of L_0 and, in particular, at $L = 1$. To this end, the parameter α, the normalized variables τ, η, ζ and the disturbance Φ of the flame front must be defined as follows:

$$\begin{gathered} \alpha = \epsilon(1+\epsilon)^{-2/3}(1-\sigma)^{-2/3}, \qquad \tau = (1-\sigma)^{4/3}(1+\epsilon)^{-2/3}t \\ \eta = (1-\sigma)^{1/3}(1+\epsilon)^{-2/3}y, \qquad \zeta = (1-\sigma)^{1/3}(1+\epsilon)^{-2/3}z, \\ \Phi = (1+\epsilon)^{-2/3}(1-\sigma)^{-2/3}F. \end{gathered} \tag{100}$$

These transformations again yield an equation of type (97) for the principal asymptotic term of the function Φ. Now, however, in terms of the natural dimensionless variables (in units of l_T and the time interval l_T/U_b), eqn (97) becomes an equation of type (1).

Equation (1) is correct for any Lewis number, and in particular for $L = L_0$ and $L = 1$. At $L = 1$, the coefficient of the fourth derivative vanishes and eqn (99) results.

Note that the Lewis number of a gaseous mixture never exceeds $L = \sqrt{2}$.

The relation linking the stability index ω to the wave-number k for eqn (1), in the case of small harmonic disturbances, is

$$\omega = \frac{(1-\sigma)}{2}k - \left(\frac{L-L_0}{1-L_0}\right)k^2 - 4\left(\frac{1-L}{1-L_0}\right)^2 k^4. \tag{101}$$

Figure 1 illustrates the shape of the curve $\omega = \omega(k)$ for $L \gtrless L_0$ and fixed parameters N and σ.

On solution of the nonlinear equations. Some physical conclusions

To investigate the evolution in time of the disturbed flame front, we must consider an initial-value problem for eqns (1), (2), (78), (98). Although there are no effective analytical tools for solution of this nonlinear problem, it seems possible to conduct a qualitative investigation, obtaining certain estimates of physical interest. For example, it would be interesting to estimate the pro-

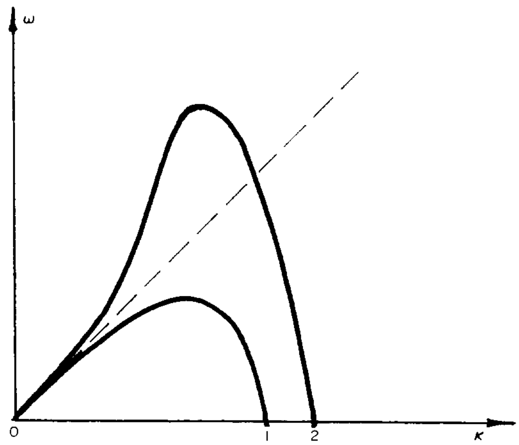

Fig. 1. Stability index ω plotted against wavenumber k. Curve 1—$L \geq L_0$, curve 2—$L < L_0$, dashed line—approximation of Landau (1944).

pagation velocity and amplitude of the disturbed flame front at times far removed from the origin, when the solution has "forgotten" the details of the initial disturbances and its behavior has become regular. This is clearly a profound mathematical problem, far beyond the scope of the present article.

As a preliminary step, numerical solution of eqns (1), (2), (78), (98) is quite admissible and at the same time provides a great deal of information. Analysis of the corresponding finite-difference scheme and computer solution of the problem will be the subject of the second part of this paper, written in collaboration with Michelson (1978). Here we shall give a brief description of the main conclusions following from the numerical work. We shall limit ourselves to the case of a two-dimensional flame, as it is the simplest case and the result is obtained most rapidly.

For the sake of brevity we shall discuss only eqn (97), since eqn (78) may be obtained from it as a long-wave asymptotic form as $\alpha \to -\infty$. Equation (97) contains a single dimensionless parameter α (see (100)) whose value governs the entire (qualitative) variety of possible situations. Linear stability analysis of a plane flame front (Fig. 1) has shown that, whatever the value of α, there is always a critical wavelength l_c, corresponding to the maximum amplification rate of the initial harmonic disturbance (i.e. maximum ω). Extrapolating this result to the general nonlinear case, one would expect that for any α the flame front will have the shape of some cellular (wrinkled) surface with cells of characteristic dimension l_c.

Solution of eqn (97) shows, however, that this is not always so. The type of instability displayed by the flame front changes entirely as the parameter α goes through a certain critical value α_c,

$$\alpha_c \simeq -0.3. \tag{102}$$

At $\alpha > \alpha_c$, following an arbitrary initial disturbance, the solution of eqn (97) ultimately settles down to a steady, stationary regime:

$$\Phi(\tau, \eta) \rightarrow \tilde{\Phi}(\eta) - V\tau, \qquad \tau > \infty. \tag{103}$$

The limiting shape $\tilde{\Phi}(\eta)$ of the flame front is a warped surface, composed as it were of pieces of parabola-like arcs convex toward the fresh mixture (Fig. 2). The velocity V is practically independent of $\alpha(\alpha < \alpha_c)$, being approximately equal to 0.18.

The positive sign of V indicates that the velocity of the corrugated front is greater than the normal velocity of the plane flame front. The shape of the steady-state front, i.e. the dimensions of the arc-shaped wrinkles and their distribution over the front, depends essentially on the initial random disturbance. The very existence of different limiting configurations of the front points to their stability under small disturbances. Numerical solution shows (Michelson and Sivashinsky, (1978) that the dimensions of the individual wrinkles may consider-

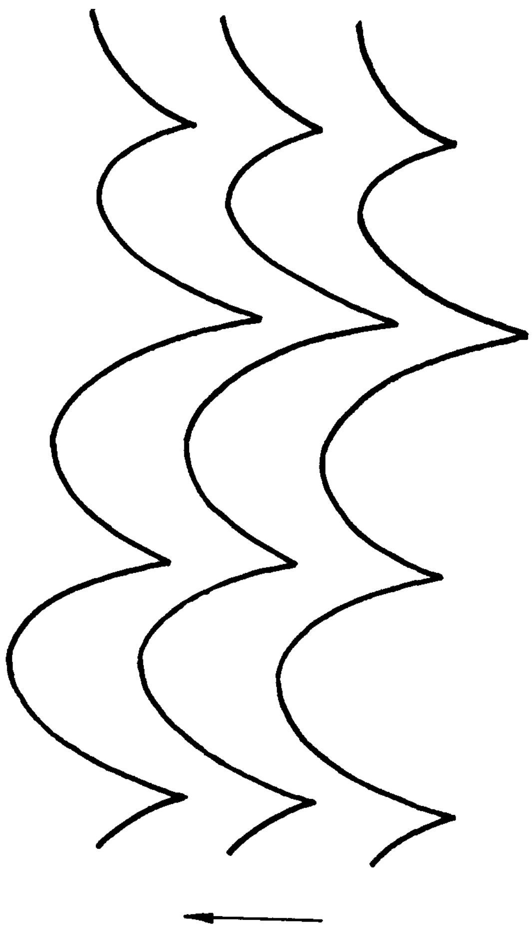

Fig. 2. General view of a wrinkled laminar flame front ($\alpha < \alpha_c$).

ably exceed the value l_c predicted by linear analysis. Here the only role of l_c is formation of the structure (radius of curvature) of the angles that arise in the front at points where contiguous wrinkles join up.

Away from these angular points, the shape of the steady stationary front may be fully described within the unstructured model of Landau (1944). As $(1-\sigma)\to 0$, the Landau model clearly leads to the equation

$$\Phi_\tau + \frac{1}{2}(\Phi_\eta)^2 = \frac{1}{4\pi}\iint\limits_{-\infty}^{\infty} |k|\, e^{ik(\eta-\zeta)}\Phi(\zeta,\tau)\, d\zeta\, dk. \tag{104}$$

In the nonstationary case ($\Phi_\tau \neq -V$), the initial-value problem for eqn (104) is obviously not well-formed, because of the rapid amplification of high-frequency disturbances. Nevertheless, it is easy to regularize the equation, for example by expressing the flame velocity as a phenomenological function of the curvature of the front (Markstein, 1964).

This result implies that the well-known Landau paradox (1944) concerning absolute instability of a flame front may be partly explained in the framework of the original Landau model. As we now see, Landau's model allows the flame to assume a stationary form with distorted wrinkles. In the framework of the unstructured Landau model, such configurations are obviously unstable only to very short-wave disturbances; such instability is eliminated by merely including dissipative effects. Overly large wrinkles will of course be unstable even when transport effects are taken into consideration. However, thanks to the strong nonlinear effect of the disturbed-flow geometry (as shown by numerical experiments), the dimensions of the individual wrinkles may considerably exceed the stability limit ($2\pi/k_0$) predicted by linear analysis for a plane flame front. Thus, the source of the Landau paradox is the complete linearization of the equations for the disturbed flame, which eliminates a major stabilizing factor.

The inclusion of transport effects regularizes the nonstationary Landau model. However, the characteristic dimension l_c introduced by transport effects has almost no significance for the shape of the steady flame front or for the dimensions of the individual wrinkles.

In this sense, the Markstein model (1964) provides a complete solution to the problem of nonstationary formation of a corrugated noncellular flame front. However, in contrast to the predictions of the linear theory, which is based on the existence of l_c, the Markstein model does not describe the formation of cellular flame structure. This is also in full accordance with well-known experimental data. As we saw in the section on Auxiliary Model Problem, Markstein's phenomenological model is valid only for sufficiently large Lewis numbers, i.e. when flame structure is not cellular. To summarize: when $\alpha < \alpha_c$, the plane flame front is transformed as a result of *spontaneous* instability into a stationary distorted, corrugated front, but the flame continues to propagate in a laminar regime. Figure 3 is a plot of the function $\alpha = \alpha(\epsilon)$, which shows that flames with Lewis number greater than unity (i.e. $\epsilon > -1$) are always of the noncellular wrinkled type.

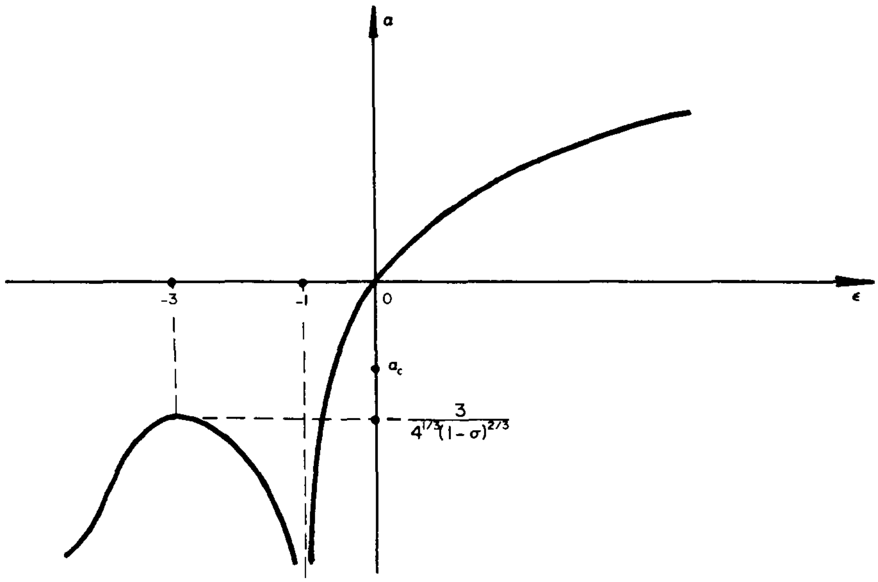

Fig. 3. Parameter α plotted against ϵ.

The situation is quite different when $\alpha > \alpha_c$. Here only the second-asymptotic eqns (1), (98) may be used. Following an initial disturbance, the solution of eqns (1), (98) ultimately settles down to a steady but essentially *nonstationary* regime. The front assumes the form of a random corrugated surface, comprising cells of more or less equal dimensions, constantly breaking up and recombining in a chaotic fashion (Fig. 4). As in the case $\alpha < \alpha_c$, there are clearly defined protuberances pointing toward the burnt-out gas. The whole of this turbulent front propagates at a certain constant mean velocity. As when $L > L_0$, the propagation velocity is greater than normal. We emphasize here that the turbulent front we have described owes its formation exclusively to spontaneous instability of the flame; it has nothing to do with external factors. According to the formulation of the problem, the incoming hydrodynamic flow far ahead of the flame front is uniform and homogeneous, with a constant flow rate.

In contradistinction to the situation for laminar noncellular flames ($\alpha < \alpha_c$), when $\alpha > \alpha_c$ the mean propagation velocity $V = V(\alpha)$ depends essentially on α:

$$V(-0.15) \simeq 0.18, \qquad V(0) \simeq 0.6, \qquad V(0.75) \simeq 1.6 \tag{105}$$

for

$$\alpha \gg 1, \qquad V(\alpha) \simeq 0.3\alpha^3. \tag{106}$$

The asymptotic relation (106) corresponds to a purely diffusional-thermal flame model, in which the effect of thermal expansion of the gas is ignored.

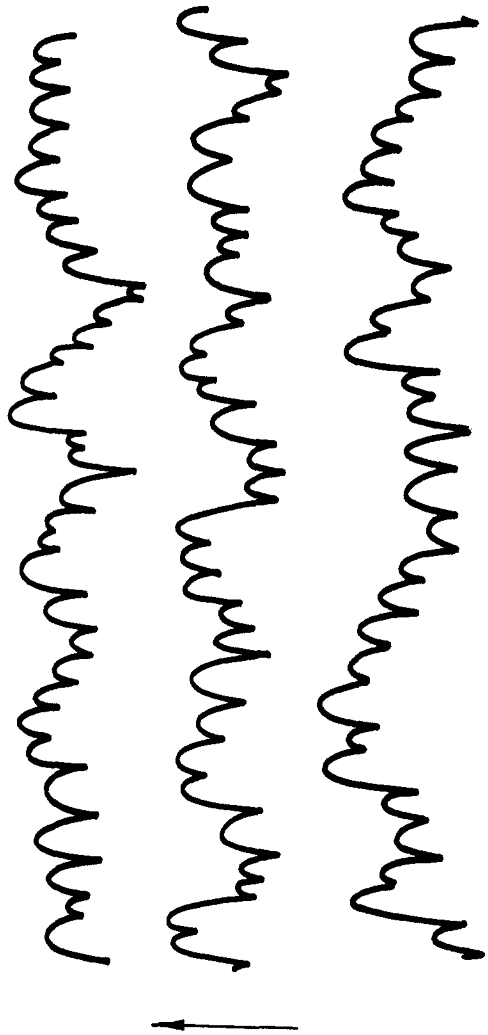

Fig. 4. General view of a turbulent cellular flame front ($\alpha > \alpha_c$).

Recall that in terms of the natural dimensionless coordinates

$$\langle F_t \rangle = -(1-\sigma)^2 V(\alpha). \tag{107}$$

An interesting outcome of the hydrodynamic model is that, if thermal expansion is taken into account ($\sigma \neq 1$), the flame may assume turbulent cellular structure even for nonpositive ϵ (i.e. $L \geq L_0$). Since the diffusional-thermal instability is possible only when $\epsilon > 0$, it follows that if $\alpha_c < \alpha \leq 0$ the onset of cellular structure depends exclusively on hydrodynamic instability. This effect is observed, for example, when $\epsilon = 0$. Equation (1) is then of the form

$$F_t + 4\nabla^4 F + \frac{1}{2}(\nabla F)^2 = \frac{(1-\sigma)}{8\pi^2}\int_{-\infty}^{\infty} |\mathbf{k}|\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{z})} F(\mathbf{z}, t)\, d\mathbf{k}\, d\mathbf{z}. \tag{108}$$

In mathematical terms, the new formation of cellular structure is apparently due to the fact that the stabilizing effect of the dissipative operator $(-4\nabla^4)$ is *weaker* than the analogous effect of the operator $(+\nabla^2)$.

For sufficiently large negative α, the stabilizing effect of the operator $(+\nabla^2)$ completely suppresses the tendency for cells of dimension l_c to form, as predicted by the linear stability analysis. In this situation, only wrinkled noncellular flames are formed. At small negative α, the major factor in stabilization of the flame is the weaker operator $(-4\nabla^4)$, which preserves the tendency toward formation of cellular structure.

Conclusion

The sole aim of our analysis of *linear* stability was to determine the order of magnitude of the disturbances. We therefore confined attention to a simplified mathematical model, neglecting, among other things, the viscosity (14). This was no longer the case in the asymptotic nonlinear analysis, where viscosity was included along with thermal conduction and diffusion. For it is well known that in gases these three effects are of the same order of magnitude, and it would be physically unjustifiable to neglect viscosity compared with the other transport effects.

However, despite the appearance of viscosity in the initial equations (6) and in the disturbances of the hydrodynamic variables (59), (60), the *principal* term of the asymptotic form of the flame-front equation (eqns (1), (2)) turns out to be independent of the Prandtl number ν. Thus, in phenomena of hydrodynamic instability the viscosity effect is as it were secondary. Of course, the Prandtl number ν must appear in approximations of higher order in $(1-\sigma)$.

It is interesting to compare this conclusion with the results of the Istratov–Librovich asymptotic linear analysis (Istratov and Librovich, 1966). In that work, where the same initial model (3)–(8) was used, the stabilizing role of viscosity was evident even in the first approximation. This does not contradict our result, since the asymptotic approach of Istratov and Librovich is based on a different choice of expansion parameter. Their point of departure is the assumption that the width l_T is small compared with the wavelength of the front disturbance, the parameter $(1-\sigma)$ being assumed finite. In our analysis, however, it is assumed not only that the wavenumber k is small but also that

$$k \sim (1-\sigma) \sim N^{-1} \ll 1.$$

We also note that, although our linear analysis was based on a flame model that makes no allowance for viscosity and for the continuous temperature-dependence of density, the asymptotic relations (25), (26), (38), (39) are nevertheless correct from the standpoint of the exact flame model (3)–(8).

The integrodifferential equations derived above for the disturbed flame front comprise only the principal terms of the expansions in question. In this sense, the equations provide a good description of weak turbulence only, or of a weakly distorted flame front. To extend the range within which the results are applicable and to confirm the various extrapolations, it would be highly interesting to compute also higher approximations relative to the parameter $(1-\sigma)$.

In addition to eqns (1), (2) for F_0, such an approach would require successive introduction of linear equations for $F_1, F_2, \ldots$, with coefficients which may be treated as given functions defined by previous approximations. It is clear that the flame front may be described at each step of the approximation procedure by a certain explicit function of the space coordinates, which certainly excludes disruption of the turbulent front. Disruption of the flame front and formation of islets of combustible mixture all over the flame are familiar phenomena in turbulent propagation of flames. The regime of spontaneous self-turbulence in a quiescent gas (or laminar flow) is characterized, on the other hand, by a continuous turbulent front (Libby and Williams, 1976; Palm-Leis and Strehlow, 1969).

The nonlinear equations of spontaneous instability (1), (2) may now serve as a basis for derivation of equations for a flame front propagating in an *external* turbulent flow. Of the many situations possible here, the simplest is probably interaction of the flame front with an incoming vortex wave of low intensity (Markstein, 1964).

Note that eqns (1), (2) were not derived by considering an initial-value problem for the equation system (3)–(9), but by isolating the "low-frequency" time-asymptotic case $(t \sim (1-\sigma)^{-4/3})$. In principle, this is the only asymptotic situation of physical interest, representing as it does a limit regime. A more thorough mathematical treatment must be based on the general initial-value problem. This means that, apart from the interval $(t \sim (1-\sigma)^{-4/3})$ one should also consider the initial time interval $t \sim 1$, in which the disturbances of temperature, concentration and flame front adjust to one another. At $t = 0$ these disturbances must satisfy certain arbitrary, incompatible initial values.

In regard to the ideas discussed here, the reader is referred to the papers of Matkowsky (1970a and 1970b).

To end this article we note that spontaneous flame instability is also of interest from the general physicomathematical point of view. It turns out that a flame, though a fully deterministic physical system, is under certain conditions liable spontaneously to become turbulent, as happens in flow of viscous fluids at large Reynolds numbers. In contrast to the situation obtaining in the flow of a classical viscous fluid, in the case of eqn (1) turbulence may arise even in the one-dimensional case.

At $\sigma = 1$ the corresponding random solutions will obviously have the properties of statistical homogeneity and isotropy. Equation (1) can thus describe simpler types of turbulence than the Navier–Stokes equation, and in this sense it may provide a useful mathematical model for studies of the onset and essence of turbulence-type phenomena.

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References

Andrews G. E., Bradley, D. and Lwakabamba, S. B. (1975) Turbulence and turbulent flame propagation—a critical appraisal. *Combustion and Flame* **24**, 285–403.

Barenblatt, G. I., Zel'dovich, Ya. B. and Istratov, A. G. (1962) On heat and diffusion effects in stability of laminar flames. *J. Appl. Mech. Tech. Phys.* (*PMTF*) **4**, 21–26 (in Russian).

Fristrom, R. M. and Westenberg, A. A. (1965) *Flame Structure*. McGraw-Hill, New York.

Istratov, A. G. and Librovich V. B. (1966a) The effect of transport processes on the stability of a plane flame front. *J. Appl. Math. Mech.* (*PMM*) **30**(3), 541–547.

Istratov, A. G. and Librovich, V. B. (1966b) Stability of Flames. *Hydromechanics*—1965. Acad. Sc. USSR. Institute of Scientific Information. (English translations, FSTC-HT-23-952-68).

Istratov, A. G. and Librovich, V. B. (1969) On the stability of gasodynamic discontinuities associated with chemical reactions. The case of spherical flame. *Astronautica Acta* **14**, 453–467.

Landau, L. D. (1944) On the theory of slow combustion. *J. Exp. Theor. Phys.* (*ZhETF*) **14**(6), 240–245 (in Russian).

Libby, P. A. and Williams, F. A. (1976) Turbulent flows involving chemical reactions, *Annual Review of fluid Mechanics* **8**, 351–376.

Markstein, G. H. (Editor) (1964) *Non-steady Flame Propagation*. Pergamon Press, Oxford.

Matkowsky, B. J. (1970a) A simple nonlinear dynamic stability problem. *Bull. Amer. Math. Soc.* **76**(3), 620–625.

Matkowsky, B. J. (1970b) Nonlinear dynamic stability: a formal theory. *SIAM J. Appl. Math.* **18**(4), 872–883.

Michelson, D. M. and Sivashinsky, G. I. (1978) Nonlinear analysis of hydrodynamic instability in laminar flames. Part 2. Numerical Experiments. *Acta Astronautica* (in press).

Palm-Leis, A. and Strehlow, R. A. (1969) On the propagation of turbulent flames. *Combustion and Flame* **13**, 111–129.

Schelkin, R. I. and Troshin, Y. K. (1964) Gas Dynamics of Combustion, Nauka, Moscow (English translations, NASA TT F231).

Sivashinsky, G. I. (1975) The structure of Bunsen flames. *J. Chem. Phys.* **62**(2), 638–643.

Sivashinsky, G. I. (1977) Diffusional-thermal theory of cellular flames. *Combustion Sc. Tech.* **15**(3/4), 137–145.

Williams, F. A. (1970) An approach to turbulent flame theory. *J. Fluid Mech.* **40**(2), 401–421.

Zel'dovich, Ya. B. (1944) Theory of Combustion and Detonation of Gases. *Acad. Sc. USSR*, Moscow.

Zel'dovich, Ya. B. and Barenblatt, G. I. (1959) Theory of flame propagation. *Combustion and Flame* 3(1), 61–74.

Appendix A

Nomenclature

- B see eqn (18)
- C_0 concentration of the limiting component
- C dimensionless concentration (referred to C_0)
- E activation energy
- e see eqn (65)
- F flame front surface (referred to l_T)
- f see eqn (62)
- g see eqn (63)
- h see eqn (64)
- I see eqn (54)
- K see eqn (23)
- $\tilde{K}$ see eqn (32)
- k dimensionless wavenumber (referred to $1/l_T$)
- L Lewis number

l_T	thermal thickness of flame
N	dimensionless activation energy $(=E/R^0T_b)$
P	dimensionless pressure (referred to $\rho_b U_b^2$)
p	see eqns (52), (79)
$\tilde{p}$	see eqn (15)
p'	perturbation of pressure
Q	strength of surface source
R^0	universal gas constant
R	see eqn (17)
S	see eqn (52), (79)
$\tilde{S}$	see eqn (15)
S'	perturbation of concentration
T	dimensionless temperature (referred to T_b)
T_b	adiabatic temperature of combustion products
T_0	temperature of fresh mixture
t	dimensionless time (referred to l_T/U_b)
U_i	dimensionless gas velocity (referred to U_b)
U_b	velocity of a plane flame front relative to the burned gas
u, v, w	see eqns (52), (79)

$\tilde{u}, \tilde{v}$	see eqn (15)
u', v', w'	perturbation of velocity
V	velocity of corrugated flame front
x, y, z	dimensionless space coordinates (referred to l_T)
$\mathbf{x}$	(y, z)
α	see eqns (37), (100)
β	see eqn (53)
ϵ	see eqn (31)
ζ, η	see eqns (43), (48), (100)
θ	see eqns (52), (79)
ν	Prandtl number
ξ	see eqns (43), (48)
ρ	dimensionless density (referred to ρ_b)
ρ_b	density of combustion products
σ	temperature (density) ratio $(=T_0/T_b)$
τ	see eqns (43), (48), (100)
Φ	see eqns (79), (100)
Ω	see eqns (23)
$\tilde{\Omega}$	see eqn (32)
ω	rate of stability parameter
Π	see eqn (54)
χ	see eqn (43), (48)