Introduction

In this Homework, we will be developing a solver that solves the Burger's equation, see inviscid form below.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

This form can also be re-written in flux form below, also in inviscid form.

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0$$

In [1]:
```
import sys
import os
import time
import numpy as np
import matplotlib.pyplot as plt

# Add the directory containing your module to sys.path
module_path = os.path.abspath(os.path.join('..', r"A:\Users\mtthl\Documents\Education\ME5653_C
sys.path.append(module_path)

from distributedObjects import *
from distributedFunctions import *
```

Part (1)

The first part is to solve the Burger's equation using Lax method. The fundamental transformation that we need to understand to facilitate this method is that the term $\frac{u^2}{2}$ is substituted by v. Thus, the Burger's equation takes the form of:

$$\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} = 0$$

It is worth noting that $\underline{\text{Computational Fluid Mechanics and Heat Transfer}}$ by Anderson et al. [1] use F instead of v, so those two notations may be used interchangeably.

Lax method in 1D takes 3x points for a stencil for the time gradient

In [2]:
```
lax_tg_gradient = numericalGradient( 1 , ( 1 , 0 ) )
print( "The coefficients for the time gradient are:\t" + str( lax_tg_gradient.coeffs ) )
```

```
The coefficients for the time gradient are:	[-1.  1.]
```

However, there is a slight modification in the method for the time gradient, which is that the previous time step is represented by the average of the two surrounding points. ie:

$$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - \left(\frac{u_{i+1}^n + u_{i-1}^n}{2} \right)}{\Delta t}$$

Now, the spatial gradient is requested to be 6th order for the interior points, and 5th order for the boundary points.

Thus, the interior points can be calculated.

In [3]:
```
lax_sg_gradient = numericalGradient( 1 , ( 3 , 3 ) )
print( "The coefficient for the interior spatial gradient are:\n\t" + str( lax_sg_gradient.coe
```

```
The coefficient for the interior spatial gradient are:
	[-1.66666667e-02  1.50000000e-01 -7.50000000e-01  3.70074342e-16
  7.50000000e-01 -1.50000000e-01  1.66666667e-02]
```

And the boundary points.

In [4]:
```
lax_sg_LHS_gradient = numericalGradient( 1 , ( 0 , 5 ) )
print( "The coefficients for the LHS boundary spatial gradient are:\n\t" + str( lax_sg_LHS_gra
```

```
The coefficients for the LHS boundary spatial gradient are:
	[-2.28333333  5.         -5.          3.33333333 -1.25        0.2       ]
```

In [5]:
```
lax_sg_RHS_gradient = numericalGradient( 1 , ( 5 , 0 ) )
print( "The coefficients for the RHS boundary spatial gradient are:\n\t" + str( lax_sg_RHS_gra
```

```
The coefficients for the RHS boundary spatial gradient are:
	[-0.2         1.25        -3.33333333  5.          -5.          2.28333333]
```

Thus, from these results, the spatial gradients become

$\underline{\text{Interior Points}}$:

$$\frac{\partial v}{\partial t} = \frac{-\frac{1}{60}v_{i-3} + \frac{3}{20}v_{i-2} - \frac{3}{4}v_{i-1} + \frac{3}{4}v_{i+1} - \frac{3}{20}v_{i+2} + \frac{1}{60}v_{i+3}}{\Delta x}$$

$\underline{\text{LHS points}}$:

$$\frac{\partial v}{\partial t} = \frac{-\frac{137}{60}v_i + 5v_{i+1} - 5v_{i+2} + \frac{10}{3}v_{i+3} - \frac{5}{4}v_{i+4} + \frac{1}{5}v_{i+5}}{\Delta x}$$

$\underline{\text{RHS points}}$:

$$\frac{\partial v}{\partial t} = \frac{\frac{137}{60}v_i - 5v_{i-1} + 5v_{i-2} - \frac{10}{3}v_{i-3} + \frac{5}{4}v_{i-4} - \frac{1}{5}v_{i-5}}{\Delta x}$$

Re-arranging all these equations to get the unknowns on the left side and the knowns from the previous time step on the right side, we get for the interior points:

$$u_i^{n+1} = \frac{1}{2}\left(u_{i-1}^n + u_{i+1}^n\right) - \left(\frac{\Delta t}{\Delta x}\right)\left(-\frac{1}{60}v_{i-3} + \frac{3}{20}v_{i-2} - \frac{3}{4}v_{i-1} + \frac{3}{4}v_{i+1} - \frac{3}{20}v_{i+2} + \frac{1}{60}v_{i+3}\right)$$

This forms the following linear equation:

$$[1][u_i]^{n+1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_{i-1} & u_i & u_{i+1} \end{bmatrix}^{T,n}$$
$$-C\begin{bmatrix} -\frac{1}{60} & \frac{3}{20} & -\frac{3}{4} & 0 & \frac{3}{4} & -\frac{3}{20} & \frac{1}{60} \end{bmatrix}\begin{bmatrix} v_{i-3} & v_{i-2} & v_{i-1} & v_i & v_{i+1} & v_{i+2} & v_{i+3} \end{bmatrix}^{T,n}$$

Where $C = \left(\frac{\Delta t}{\Delta x}\right)$, where $CFL = Cu$

To use this solver, we will first use an example problem of the following characteristic function:

$$u(x,t) = -Utanh(k(x+ut))$$

In this case, $x \in (0,1)$, $k=1$, and since the domain of t is not defined, $t \in (0,1)$. There will be 1,000 x and t samples.

In [6]:
```
N_x = int(10)
N_t = int(20)
x_domain = np.linspace( 0 , 1 , num=N_x )
t_domain = np.linspace( 0 , 0.5 , num=N_t )
k = 1
```

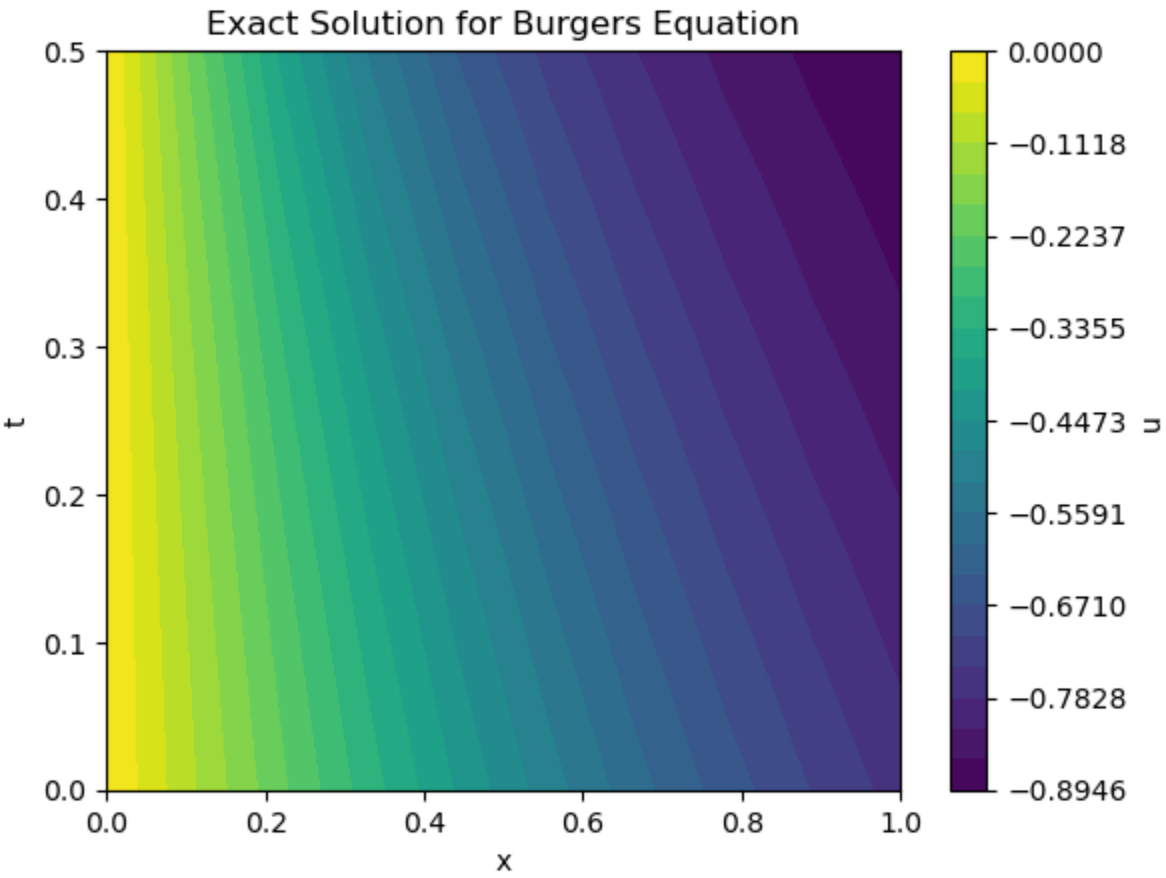
And from this, we can set the characteristic function to be essentially non-dimensional to its maximum initial value by setting $U=1$.

In [7]:
```
U = 1
```

Thus, the exact solution becomes:

In [8]:
```
u_exact = np.zeros( ( len(x_domain) , len(t_domain) ) )

for i , t in enumerate( t_domain ):
    if i>0:
        u_exact[:,i] = - U * np.tanh( k * ( x_domain - u_exact[:,i-1] * t ) )
    else:
        u_exact[:,i] = - U * np.tanh( k * x_domain )
```

In [9]:
```
x = x_domain
y = t_domain
X , Y = np.meshgrid( x , y )

Z = u_exact

levels=np.linspace( np.min(u_exact) , np.max(u_exact) , 25 )
contour = plt.contourf( X.T , Y.T , Z , levels=levels )

plt.title( "Exact Solution for Burgers Equation" )
plt.xlabel( "x" )
plt.ylabel( "t" )

cbar = plt.colorbar( contour )
cbar.set_label( "u" )

plt.show()
```



Now with the exact solution, we can find the numerical solution for $CFL = 0.8$

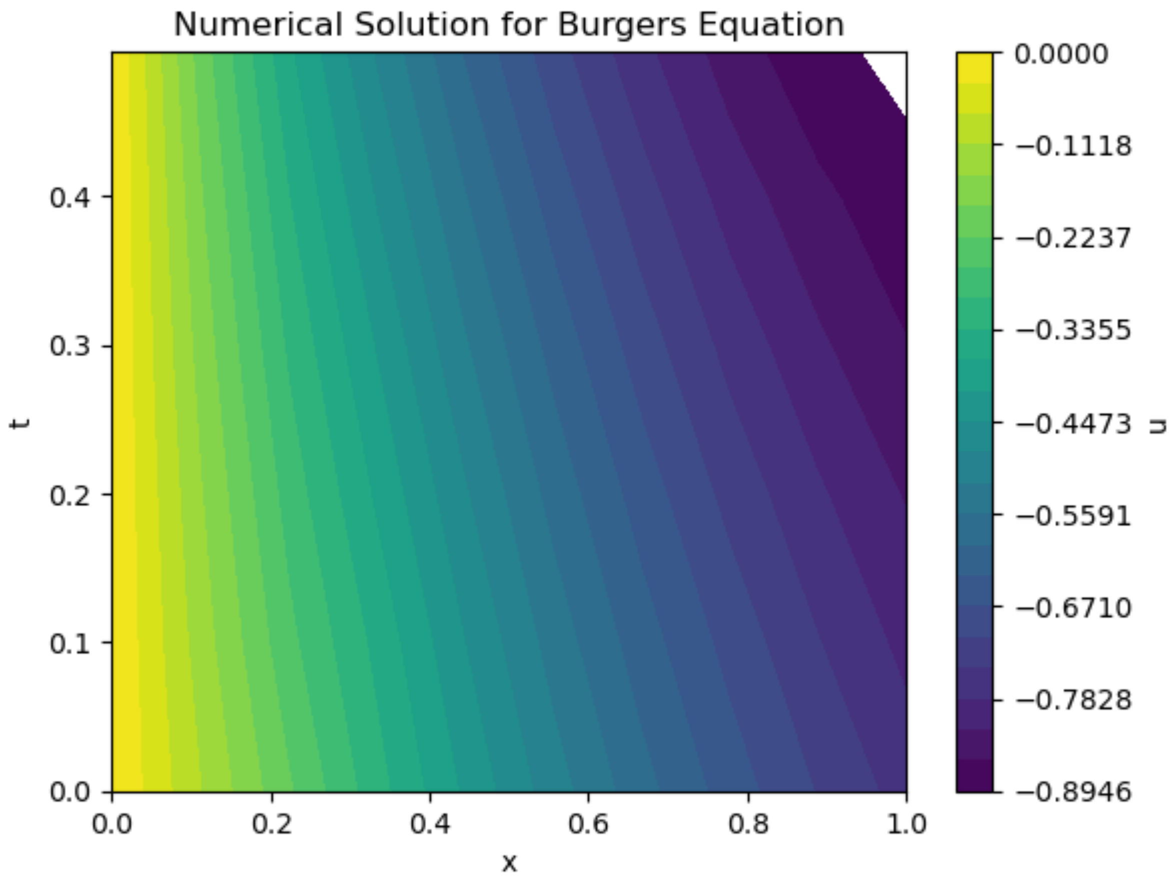
In [10]:
```
CFL = 0.8

burger_1a = burgersEquation( x_domain , u_exact[:,0] , ( np.min( t_domain ) , np.max( t_domair
```

In [11]:
```
burger_1a.solve( N_spatialorder=6 , N_spatialBCorder=5 )
```

```
A:\Users\mtthl\Documents\Education\ME5653_CFD\git\me5653_CFD_repo\code\lib\distributedObject
s.py:798: SparseEfficiencyWarning: spsolve requires A be CSC or CSR matrix format
  cls.u[i+1,:] = spsr.linalg.spsolve( cls.A_matrix , cls.b[i,...] )
```

In [12]:

```
x = burger_1a.x
y = burger_1a.t
X , Y = np.meshgrid( x , y )

Z = burger_1a.u

levels=np.linspace( np.min(u_exact) , np.max(u_exact) , 25 )
contour = plt.contourf( X , Y , Z , levels=levels )

plt.title( "Numerical Solution for Burgers Equation" )
plt.xlabel( "x" )
plt.ylabel( "t" )

cbar = plt.colorbar( contour )
cbar.set_label( "u" )

plt.show()
```



I really do not know what is going on at the end there. I've been working through the RHS boundary condition and it keeps doing this for reasons unknown to me, but this gets solved by reducing the CFL, so I am going to move on.

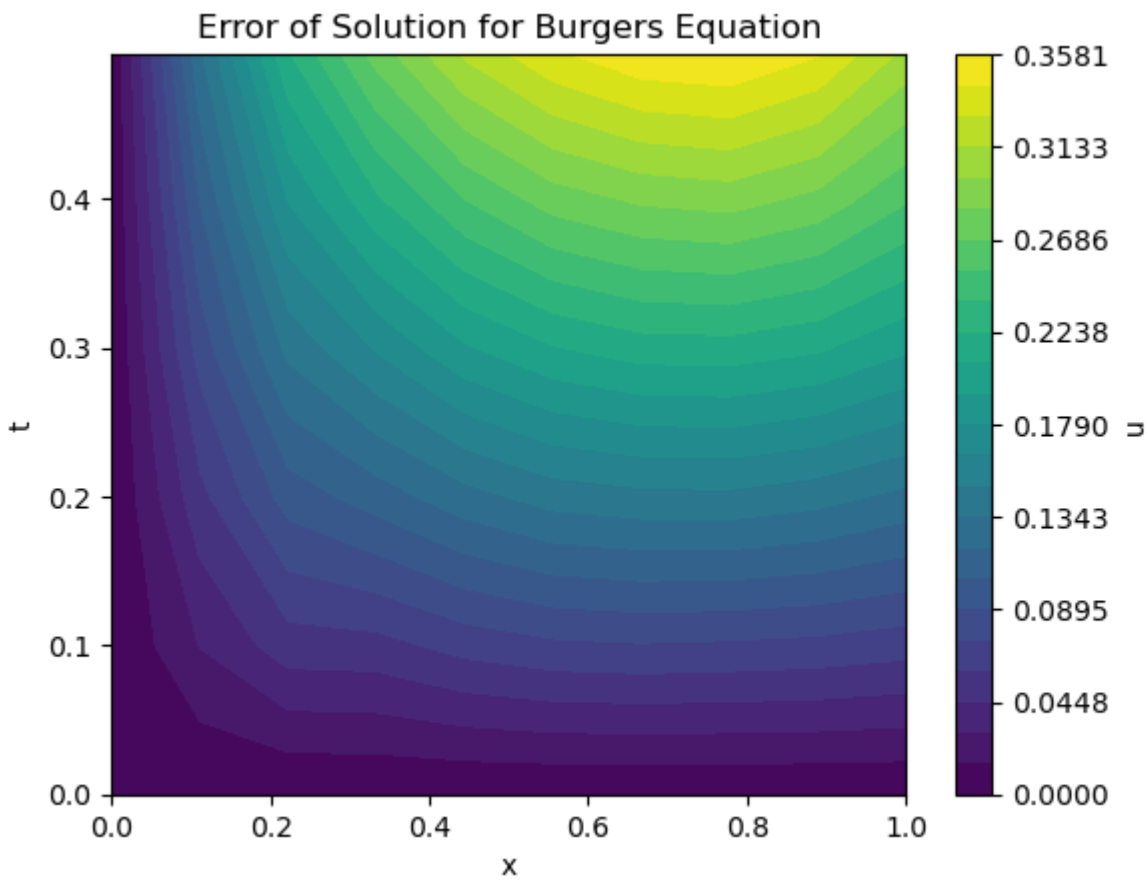
Our next step is to look at the error vs the exact solution. I will need to re-calculate the exact solution for our actual domain here.

In [13]:

```
u_exact = np.zeros( ( len(burger_1a.x) , len(burger_1a.t) ) )

for i , t in enumerate( burger_1a.t ):
    if i>0:
        u_exact[:,i] = - U * np.tanh( k * ( burger_1a.x + u_exact[:,i-1] * t ) )
    else:
        u_exact[:,i] = - U * np.tanh( k * burger_1a.x )
```

In [14]:
```
error = u_exact.T - burger_1a.u
error_L2 = np.linalg.norm( error ) / np.prod(np.shape(error))
```

In [15]:
```
x = burger_1a.x
y = burger_1a.t
X , Y = np.meshgrid( x , y )

Z = error

levels=np.linspace( np.min(error) , np.max(error) , 25 )
contour = plt.contourf( X , Y , Z , levels=levels )

plt.title( "Error of Solution for Burgers Equation" )
plt.xlabel( "x" )
plt.ylabel( "t" )

cbar = plt.colorbar( contour )
cbar.set_label( "u" )

plt.show()
```



In [16]:
```
print("The RMS of the error is {x:.3e}".format(x=error_L2))
error_L2s_a=[error_L2]
```

```
The RMS of the error is 2.262e-02
```

It appears that the exact and numerical solution are in fairly good agreement with the exception of the RHS boundary towards the end of the time being sampled.

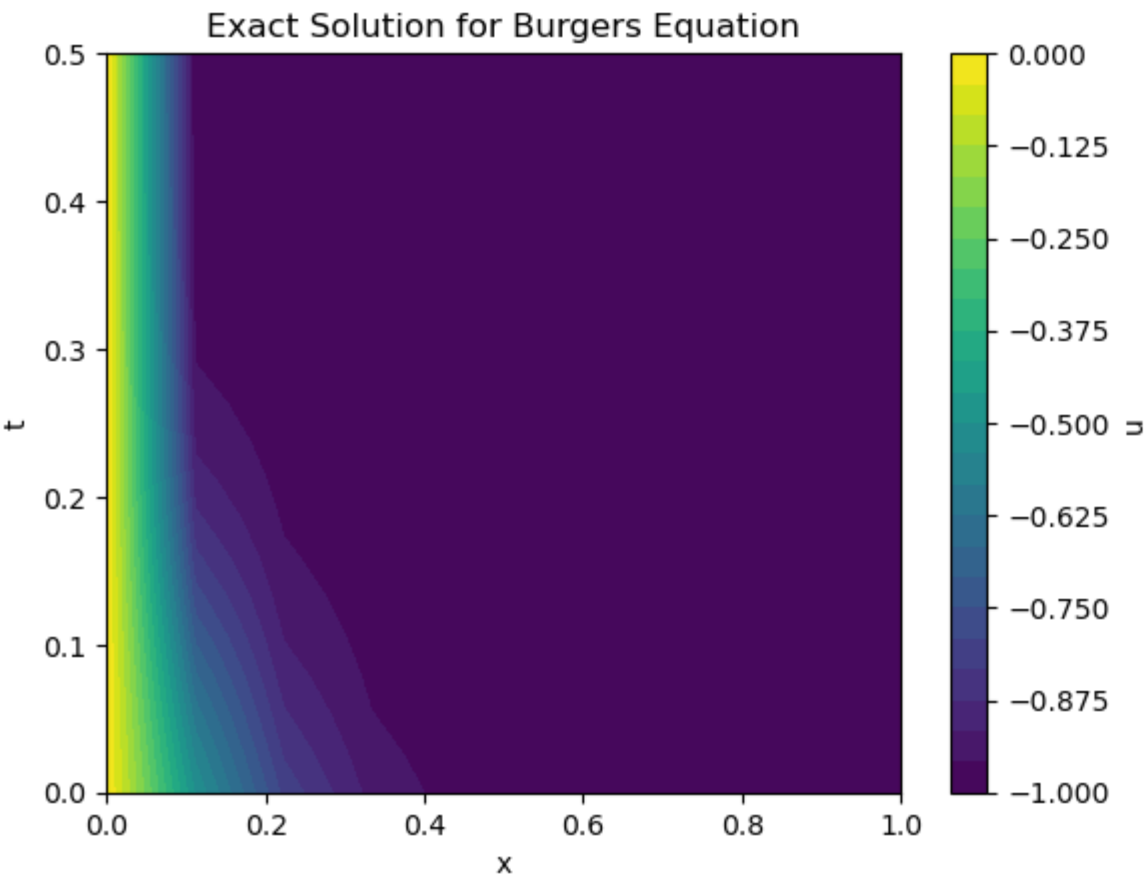
We can next do the same analysis for $k = 5$.

In [17]:
```
k=5
```

In [18]:
```
u_exact = np.zeros( ( len(x_domain) , len(t_domain) ) )

for i , t in enumerate( t_domain ):
    if i>0:
```

```
        u_exact[:,i] = - U * np.tanh( k * ( x_domain - u_exact[:,i-1] * t ) )
    else:
        u_exact[:,i] = - U * np.tanh( k * x_domain )
```

In [19]:
```
x = x_domain
y = t_domain
X , Y = np.meshgrid( x , y )

Z = u_exact

levels=np.linspace( np.min(u_exact) , np.max(u_exact) , 25 )
contour = plt.contourf( X.T , Y.T , Z , levels=levels )

plt.title( "Exact Solution for Burgers Equation" )
plt.xlabel( "x" )
plt.ylabel( "t" )

cbar = plt.colorbar( contour )
cbar.set_label( "u" )

plt.show()
```



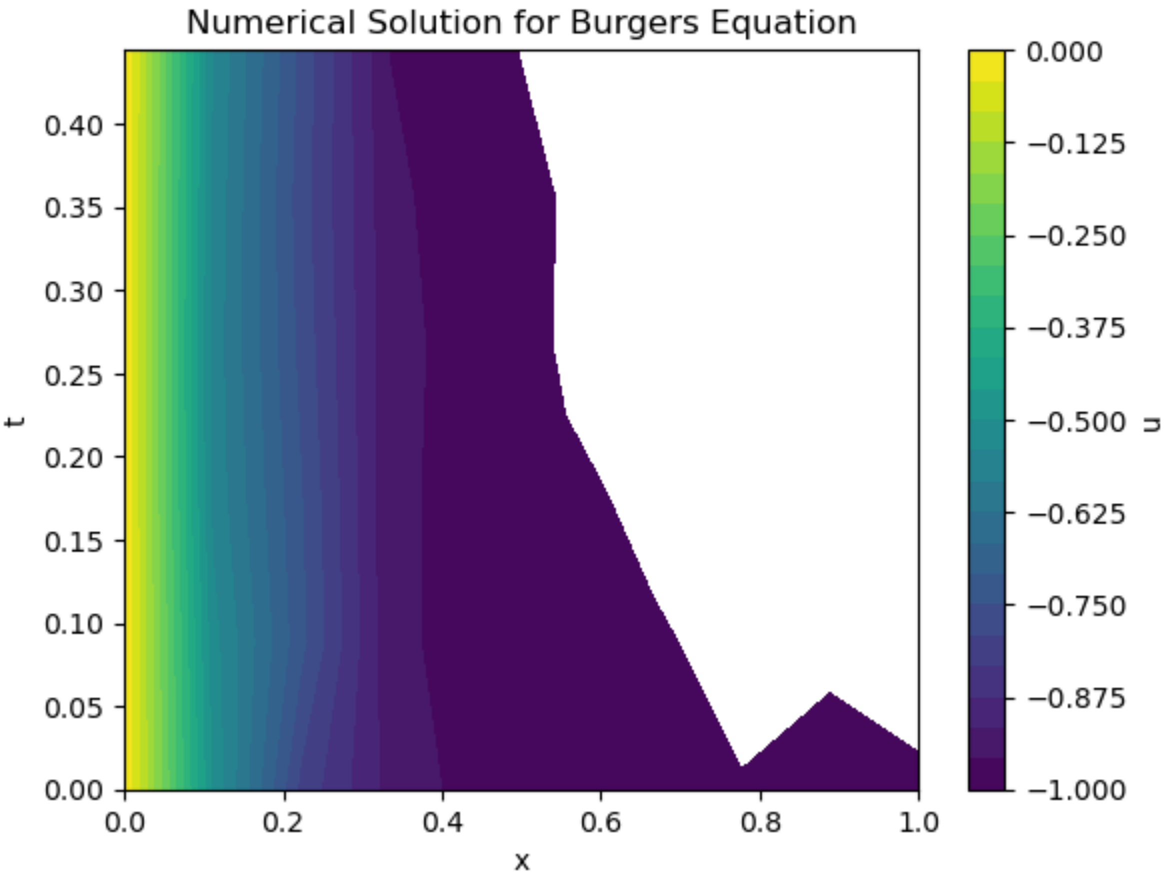
This will likely be more difficult since the solution is approaching that discontinuity-like behavior. I am unsure how this will affect the numerical solution.

In [20]:
```
CFL = 0.8

burger_1b = burgersEquation( x_domain , u_exact[:,0] , ( np.min( t_domain ) , np.max( t_domair
```

In [21]:
```
burger_1b.solve( N_spatialorder=6 , N_spatialBCorder=5 )
```

In [22]:
```
x = burger_1b.x
y = burger_1b.t
```

```
X , Y = np.meshgrid( x , y )

Z = burger_1b.u

levels=np.linspace( np.min(u_exact) , np.max(u_exact) , 25 )
contour = plt.contourf( X , Y , Z , levels=levels )

plt.title( "Numerical Solution for Burgers Equation" )
plt.xlabel( "x" )
plt.ylabel( "t" )

cbar = plt.colorbar( contour )
cbar.set_label( "u" )

plt.show()
```



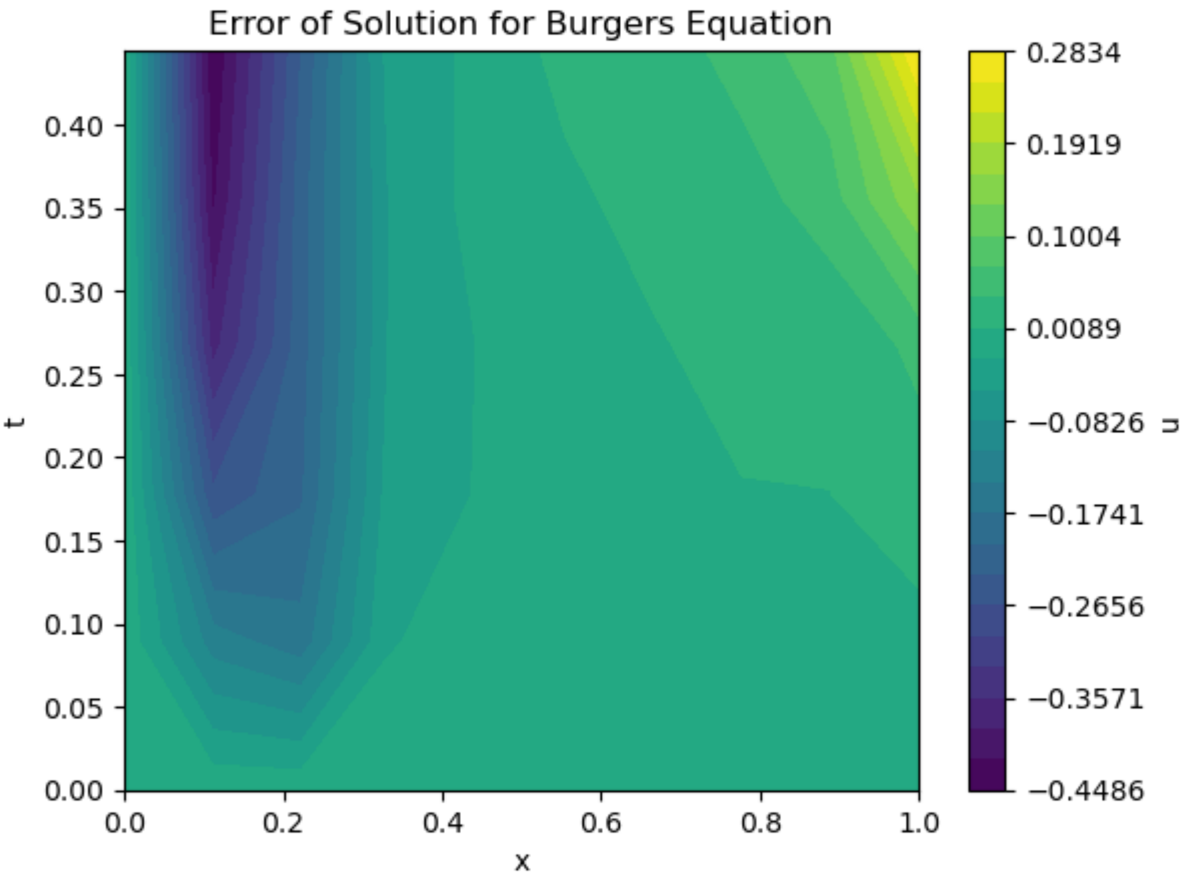
Something is not quite working properly with the solver. I am unsure why this is, but it is more stable with a lower order.

In [23]:
```
u_exact = np.zeros( ( len(burger_1b.x) , len(burger_1b.t) ) )

for i , t in enumerate( burger_1b.t ):
    if i>0:
        u_exact[:,i] = - U * np.tanh( k * ( burger_1b.x - u_exact[:,i-1] * t ) )
    else:
        u_exact[:,i] = - U * np.tanh( k * burger_1b.x )
```

In [24]:
```
error = u_exact.T - burger_1b.u
error_L2 = np.linalg.norm( error ) / np.prod(np.shape(error))
```

In [25]:
```
x = burger_1b.x
y = burger_1b.t
X , Y = np.meshgrid( x , y )

Z = error

levels=np.linspace( np.min(error) , np.max(error) , 25 )
```

```
contour = plt.contourf( X , Y , Z , levels=levels )

plt.title( "Error of Solution for Burgers Equation" )
plt.xlabel( "x" )
plt.ylabel( "t" )

cbar = plt.colorbar( contour )
cbar.set_label( "u" )

plt.show()
```



In [26]:
```
print("The RMS of the error is {x:.3e}".format(x=error_L2))
error_L2s_b=[error_L2]
```

```
The RMS of the error is 1.631e-02
```

The next task is to look at the convergence analysis by doing a 3-grid Richardson analysis.

First, let's look at k=1 for (2x) more grids.

There is clearly much more error in this solution, which we can see from the plot primarily comes from that checkerboard-like pattern from the above plot.

In [27]:
```
N_x = int(20)
N_t = int(20)
x_domain = np.linspace( 0 , 1 , num=N_x )
t_domain = np.linspace( 0 , 0.5 , num=N_t )
k = 1
```

In [28]:
```
u_exact = np.zeros( ( len(x_domain) , len(t_domain) ) )

for i , t in enumerate( t_domain ):
    if i>0:
        u_exact[:,i] = - U * np.tanh( k * ( x_domain - u_exact[:,i-1] * t ) )
    else:
        u_exact[:,i] = - U * np.tanh( k * x_domain )
```

In [29]:
```
CFL = 0.8

burger_1aa = burgersEquation( x_domain , u_exact[:,0] , ( np.min( t_domain ) , np.max( t_domai
```

In [30]:
```
burger_1aa.solve( N_spatialorder=6 , N_spatialBCorder=5 )
```

In [31]:

```
u_exact = np.zeros( ( len(burger_1aa.x) , len(burger_1aa.t) ) )
```

```
for i , t in enumerate( burger_1aa.t ):
    if i>0:
        u_exact[:,i] = - U * np.tanh( k * ( burger_1aa.x - u_exact[:,i-1] * t ) )
    else:
        u_exact[:,i] = - U * np.tanh( k * burger_1aa.x )
```

In [ ]:

In [32]:
```
error = u_exact.T - burger_1aa.u
error_L2 = np.linalg.norm( error ) / np.prod(np.shape(error))
```

In [33]:
```
print("The RMS of the error is {x:.3e}".format(x=error_L2))
error_L2s_a+=[error_L2]
```

```
The RMS of the error is 1.822e-03
```

In [34]:
```
N_x = int(40)
N_t = int(20)
x_domain = np.linspace( 0 , 1 , num=N_x )
t_domain = np.linspace( 0 , 0.5 , num=N_t )
k = 1
```

In [35]:
```
u_exact = np.zeros( ( len(x_domain) , len(t_domain) ) )

for i , t in enumerate( t_domain ):
    if i>0:
        u_exact[:,i] = - U * np.tanh( k * ( x_domain - u_exact[:,i-1] * t ) )
    else:
        u_exact[:,i] = - U * np.tanh( k * x_domain )
```

In [36]:
```
CFL = 0.8

burger_1aaa = burgersEquation( x_domain , u_exact[:,0] , ( np.min( t_domain ) , np.max( t_doma
```

In [37]:
```
burger_1aaa.solve( N_spatialorder=6 , N_spatialBCorder=5 )
```

In [38]:

```
u_exact = np.zeros( ( len(burger_1aaa.x) , len(burger_1aaa.t) ) )
```

```
for i , t in enumerate( burger_1aa.t ):
    if i>0:
        u_exact[:,i] = - U * np.tanh( k * ( burger_1aaa.x - u_exact[:,i-1] * t ) )
    else:
        u_exact[:,i] = - U * np.tanh( k * burger_1aaa.x )
```

In [39]:
```
error = u_exact.T - burger_1aaa.u
error_L2 = np.linalg.norm( error ) / np.prod(np.shape(error))
```

In [40]:
```
print("The RMS of the error is {x:.3e}".format(x=error_L2))
error_L2s_a+=[error_L2]
```

```
The RMS of the error is 6.545e+16
```

In [41]:
```
dxs_a = [ burger_1a.dx , burger_1aa.dx , burger_1aaa.dx ]
```

In [42]:
```
k=5
```

In [43]:
```
N_x = int(20)
N_t = int(20)
x_domain = np.linspace( 0 , 1 , num=N_x )
t_domain = np.linspace( 0 , 0.5 , num=N_t )
```

In [44]:
```
u_exact = np.zeros( ( len(x_domain) , len(t_domain) ) )

for i , t in enumerate( t_domain ):
    if i>0:
        u_exact[:,i] = - U * np.tanh( k * ( x_domain - u_exact[:,i-1] * t ) )
    else:
        u_exact[:,i] = - U * np.tanh( k * x_domain )
```

In [45]:
```
CFL = 0.8

burger_1bb = burgersEquation( x_domain , u_exact[:,0] , ( np.min( t_domain ) , np.max( t_domai
```

In [46]:
```
burger_1bb.solve( N_spatialorder=6 , N_spatialBCorder=5 )
```

In [47]:

```
u_exact = np.zeros( ( len(burger_1bb.x) , len(burger_1bb.t) ) )
```

```
for i , t in enumerate( burger_1bb.t ):
    if i>0:
        u_exact[:,i] = - U * np.tanh( k * ( burger_1bb.x - u_exact[:,i-1] * t ) )
    else:
        u_exact[:,i] = - U * np.tanh( k * burger_1bb.x )
```

In [48]:
```
error = u_exact.T - burger_1bb.u
error_L2 = np.linalg.norm( error ) / np.prod(np.shape(error))
```

In [49]:
```
print("The RMS of the error is {x:.3e}".format(x=error_L2))
error_L2s_b+=[error_L2]
```

```
The RMS of the error is 5.320e-03
```

In [50]:
```
N_x = int(40)
N_t = int(20)
x_domain = np.linspace( 0 , 1 , num=N_x )
t_domain = np.linspace( 0 , 0.5 , num=N_t )
```

In [51]:
```
u_exact = np.zeros( ( len(x_domain) , len(t_domain) ) )

for i , t in enumerate( t_domain ):
    if i>0:
        u_exact[:,i] = - U * np.tanh( k * ( x_domain - u_exact[:,i-1] * t ) )
    else:
        u_exact[:,i] = - U * np.tanh( k * x_domain )
```

In [52]:
```
CFL = 0.8

burger_1bbb = burgersEquation( x_domain , u_exact[:,0] , ( np.min( t_domain ) , np.max( t_doma
```

In [53]:
```
burger_1bbb.solve( N_spatialorder=6 , N_spatialBCorder=5 )
```

In [54]:
```
u_exact = np.zeros( ( len(burger_1bbb.x) , len(burger_1bbb.t) ) )
```

```
for i , t in enumerate( burger_1bbb.t ):
    if i>0:
        u_exact[:,i] = - U * np.tanh( k * ( burger_1bbb.x - u_exact[:,i-1] * t ) )
    else:
        u_exact[:,i] = - U * np.tanh( k * burger_1bbb.x )
```

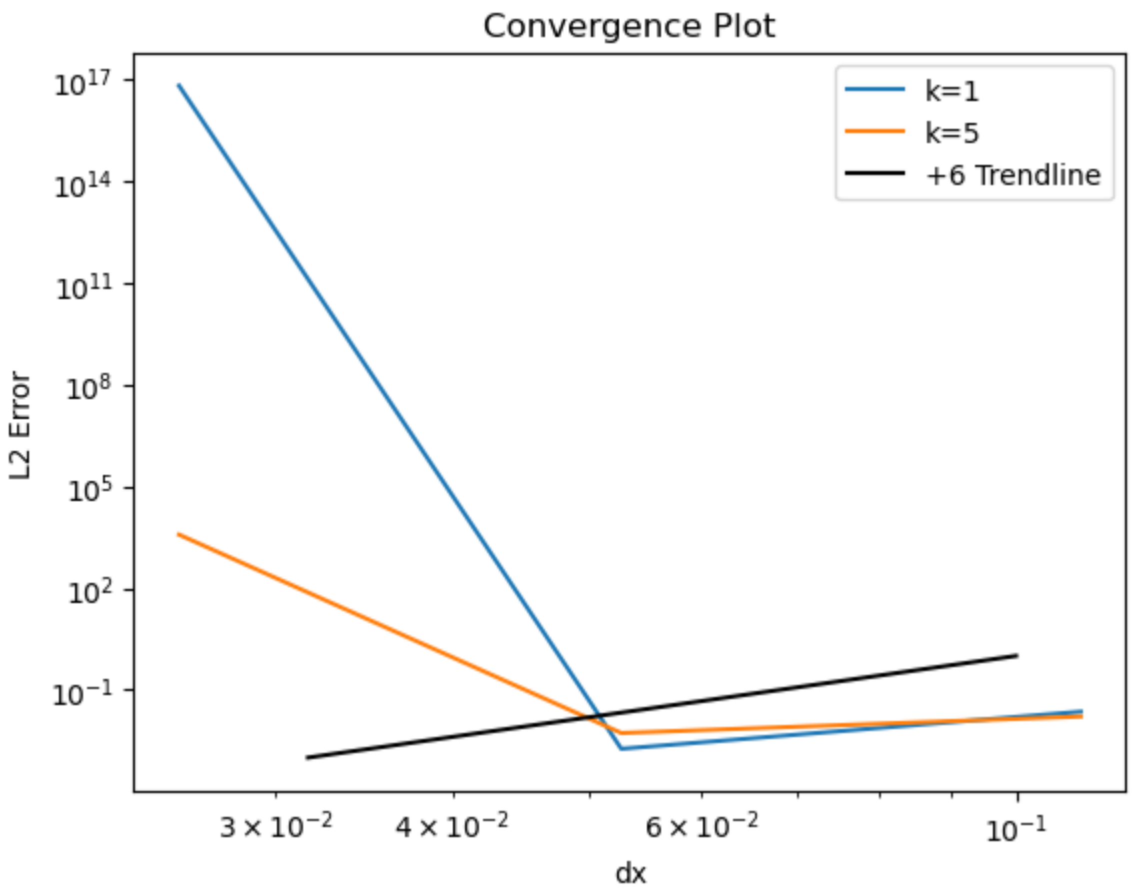
In [55]:
```
error = u_exact.T - burger_1bbb.u
error_L2 = np.linalg.norm( error ) / np.prod(np.shape(error))
```

In [56]:
```
print("The RMS of the error is {x:.3e}".format(x=error_L2))
error_L2s_b+=[error_L2]

dxs_b = dxs_a
```

```
The RMS of the error is 3.775e+03
```

In [57]:
```
plt.loglog( dxs_a , error_L2s_a , label="k=1" )
plt.loglog( dxs_b , error_L2s_b , label="k=5" )

x_trend=np.logspace(-1.5,-1,num=10)
y_trend=(1e6)*(x_trend**6)
plt.loglog( x_trend , y_trend , 'k', label="+6 Trendline")

plt.title("Convergence Plot")
plt.xlabel("dx")
plt.ylabel("L2 Error")
plt.legend(loc="best")
```

Out[57]:
```
<matplotlib.legend.Legend at 0x1b1f8ce7f10>
```



In [58]:
```
p_a = np.log( ( error_L2s_a[0] - error_L2s_a[1] ) / ( error_L2s_a[1] - error_L2s_a[2] ) ) / np
print("Order of accuracy:\t{x}".format(x=p_a))
```

```
Order of accuracy:      nan
C:\Users\mtthl\AppData\Local\Temp\ipykernel_5136\3949247662.py:1: RuntimeWarning: invalid valu
e encountered in log
  p_a = np.log( ( error_L2s_a[0] - error_L2s_a[1] ) / ( error_L2s_a[1] - error_L2s_a[2] ) ) /
np.log(2)
```

Obviously, the smaller mesh case is throwing off the error calculation for the whole thing, so let's just do the calculation for the coarser meshes.

In [59]:
```
p_a = np.log( error_L2s_a[0] / error_L2s_a[1] ) / np.log(2)
print("Order of accuracy:\t{x}".format(x=p_a))
```

```
Order of accuracy:      3.6345958834339344
```

That's not right... But I am unsure where this error is coming from exactly. It is not that far off where it needs to be, so I am going to mostly let it be. Especially since lower orders seem to work well. /

In [60]:
```
p_b = np.log( ( error_L2s_b[0] - error_L2s_b[1] ) / ( error_L2s_b[1] - error_L2s_b[2] ) ) / np
print("Order of accuracy:\t{x}".format(x=p_b))
```

```
Order of accuracy:      nan
C:\Users\mtthl\AppData\Local\Temp\ipykernel_5136\1183208592.py:1: RuntimeWarning: invalid valu
e encountered in log
  p_b = np.log( ( error_L2s_b[0] - error_L2s_b[1] ) / ( error_L2s_b[1] - error_L2s_b[2] ) ) /
np.log(2)
```

Still not right. It appears that the smaller mesh is still causing issues. Does that mean there is some scale that this method cannot resolve? I am not sure, I would need to look into it more.

In [61]:
```
p_a = np.log( error_L2s_b[0] / error_L2s_b[1] ) / np.log(2)
print("Order of accuracy:\t{x}".format(x=p_a))
```

```
Order of accuracy:      1.6162687727024045
```

This is still not great, we are much lower order of accuracy than the method would prescribe. I am wondering where this issue could be coming from, it is not immediately apparent.

Part (2)

In this section, we will add a viscous term to allow the dissipation of the Burger's equation behavior.

For a viscous term, on the RHS of the Burger's equation, there is a dissipation term, like what is the primary term in the Heat Equation. Thus, the Burger's equation becomes as below:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial t^2}$$

For this spatial term, we will use the same stencil, which has the following coefficients:

In [62]:
```
lax_vg_gradient = numericalGradient( 2 , ( 3 , 3 ) )
print( "The coefficient for the interior dissipation gradient are:\n\t" + str( lax_vg_gradient
```

```
The coefficient for the interior dissipation gradient are:
	[ 0.01111111 -0.15        1.5        -2.72222222  1.5        -0.15
 0.01111111]
```

In [63]:
```python
lax_vg_gradient_LHS = numericalGradient( 2 , ( 0 , 6 ) )
print( "The coefficient for the LHS dissipation gradient are:\n\t" + str( lax_vg_gradient_LHS.
```

```
The coefficient for the LHS dissipation gradient are:
	[  4.51111111 -17.4         29.25       -28.22222222  16.5
  -5.4          0.76111111]
```

In [64]:
```python
lax_vg_gradient_RHS = numericalGradient( 2 , ( 6 , 0 ) )
print( "The coefficient for the LHS dissipation gradient are:\n\t" + str( lax_vg_gradient_RHS.
```

```
The coefficient for the LHS dissipation gradient are:
        [  0.76111111  -5.4         16.5        -28.22222222  29.25
 -17.4          4.51111111]
```

Thus,

<u>Interior Points</u>:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\frac{1}{90}u_{i-3} - \frac{3}{20}u_{i-2} + \frac{3}{2}u_{i-1} - \frac{49}{18}u_i + \frac{3}{2}u_{i+1} - \frac{3}{20}u_{i+2} + \frac{1}{90}u_{i+3}}{\Delta x^2}$$

<u>LHS Points</u>:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\frac{203}{45}u_i - \frac{87}{5}u_{i+1} + \frac{117}{4}u_{i+2} - \frac{254}{9}u_{i+3} + \frac{33}{2}u_{i+4} - \frac{27}{5}u_{i+5} + \frac{137}{180}u_{i+6}}{\Delta x^2}$$

<u>RHS Points</u>:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\frac{137}{180}u_{i-6} - \frac{27}{5}u_{i-5} + \frac{33}{2}u_{i-4} - \frac{254}{9}u_{i-3} + \frac{117}{4}u_{i-2} - \frac{87}{5}u_{i-1} + \frac{203}{45}u_i}{\Delta x^2}$$

Thus, the fully discretized Burger's equation becomes:

$$u_i^{n+1} = \left(\nu\frac{1}{90}u_{i-3}^n - \nu\frac{3}{20}u_{i-2}^n + \left(\frac{1}{2} + \nu\frac{3}{2}\right)u_{i-1}^n - \nu\frac{49}{18}u_i^n + \left(\frac{1}{2} + \nu\frac{3}{2}\right)u_{i+1}^n - \nu\frac{3}{20}u_{i+2}^n + \nu\frac{1}{90}u_{i+3}^n\right.$$
$$- \left(\frac{\Delta t}{\Delta x}\right)\left(-\frac{1}{60}v_{i-3} + \frac{3}{20}v_{i-2} - \frac{3}{4}v_{i-1} + \frac{3}{4}v_{i+1} - \frac{3}{20}v_{i+2} + \frac{1}{60}v_{i+3}\right)$$

And thus, can be re-arranged in the following linear equation:

$$[1][u_i]^{n+1}$$
$$= \left(\frac{1}{2}\begin{bmatrix}0 & 0 & 1 & 0 & 1 & 0 & 0\end{bmatrix} + \nu\begin{bmatrix}\frac{1}{90} & -\frac{3}{20} & \frac{3}{2} & -\frac{49}{18} & \frac{3}{2} & -\frac{3}{20} & \frac{1}{90}\end{bmatrix}\right)\begin{bmatrix}u_{i-3} & u_{i-2} & u_{i-1} & u_i & \cdots\end{bmatrix}$$
$$- C\begin{bmatrix}-\frac{1}{60} & \frac{3}{20} & -\frac{3}{4} & 0 & \frac{3}{4} & -\frac{3}{20} & \frac{1}{60}\end{bmatrix}\begin{bmatrix}v_{i-3} & v_{i-2} & v_{i-1} & v_i & v_{i+1} & v_{i+2} & v_{i+\cdots}\end{bmatrix}$$

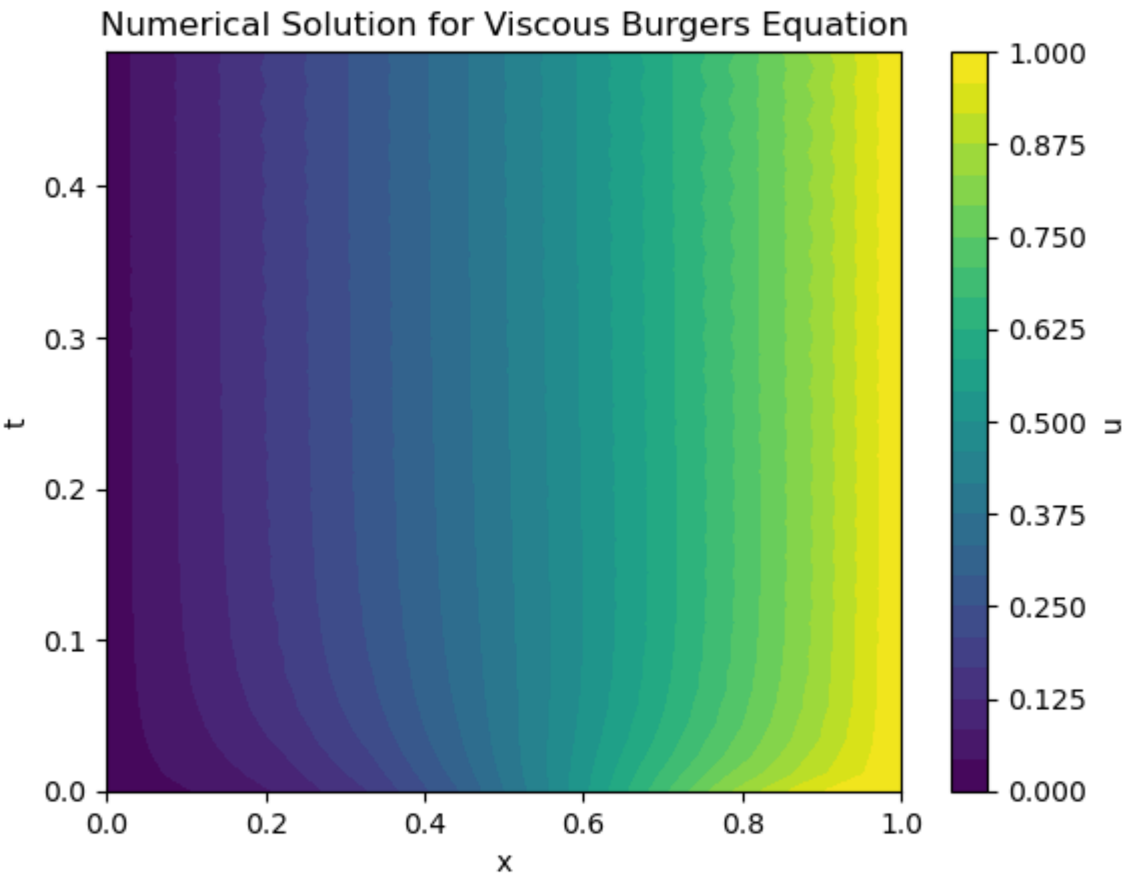
In [65]:
```
N_x = int(10)
N_t = int(90)
```

```
x_domain = np.linspace( 0 , 1 , num=N_x )
t_domain = np.linspace( 0 , 0.5 , num=N_t )

u_0 = np.exp( - ( 2 * ( x_domain - 1 ) ) ** 2 )

burger_2 = burgersEquation( x_domain , u_0 , (0,0.5) , C=0.1 , nu=0.01 )
```

In [66]:
```
burger_2.solve( N_spatialorder=6 )
```

In [67]:
```
x = burger_2.x
y = burger_2.t
X , Y = np.meshgrid( x , y )

Z = burger_2.u

levels=np.linspace( 0 , 1 , 25 )
contour = plt.contourf( X , Y , Z , levels=levels )

plt.title( "Numerical Solution for Viscous Burgers Equation" )
plt.xlabel( "x" )
plt.ylabel( "t" )

cbar = plt.colorbar( contour )
```

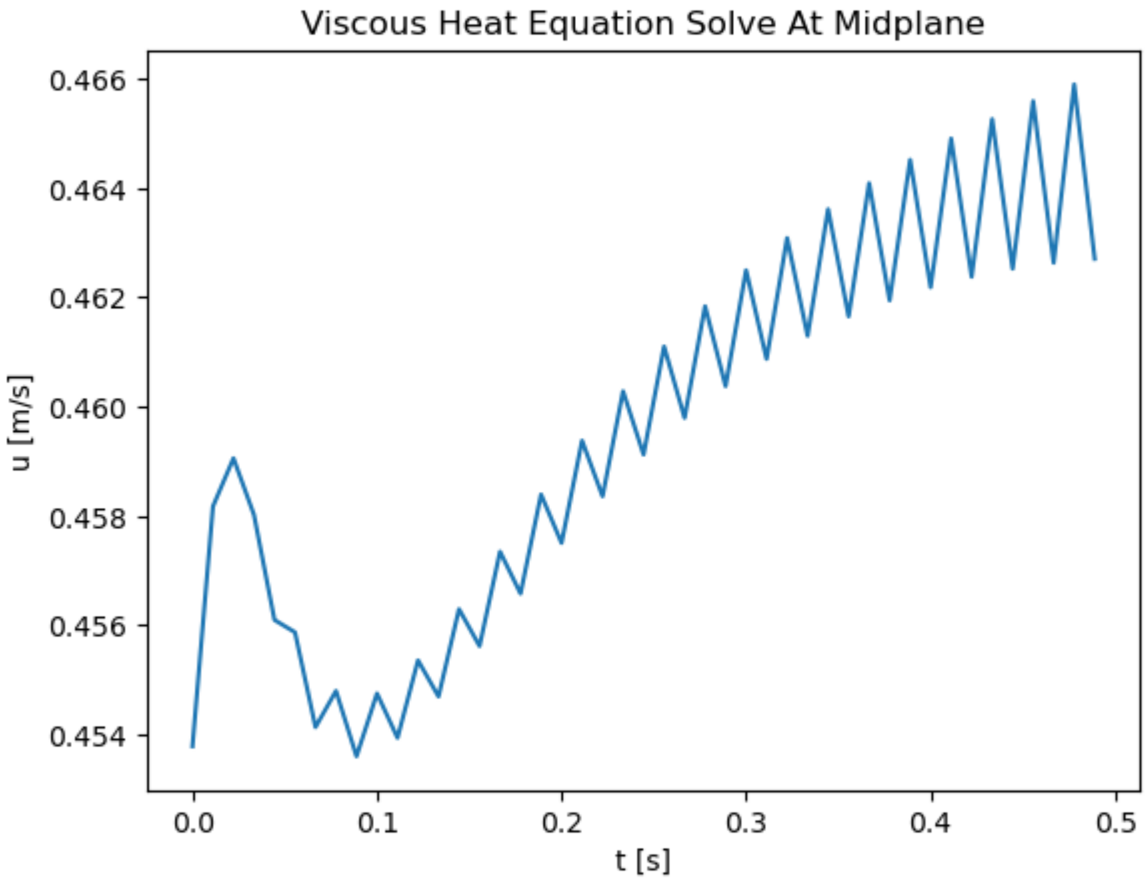
```
cbar.set_label( "u" )

plt.show()
```



This is pretty cool looking. There is a very clear combination of convection, at the beginning, and diffusion, later on, behaviors present.

Next, we will plot the mid-plane of the domain.

In [68]:
```
x = burger_2.t
y = burger_2.u[:,burger_2.Nx//2]

plt.plot( x , y )

plt.xlabel("t [s]")
plt.ylabel("u [m/s]")
plt.title("Viscous Heat Equation Solve At Midplane")

plt.show()
```



These results are interesting. Although there is not large changes in mean value, there appears to be an oscillating behavior present, indicating some instability in the solve. This may indicate that the solver may have some instability later on. Since we are at the midplane, there mean behavior is not going to vary much.

In [69]:
```
np.shape(burger_2.u)
```

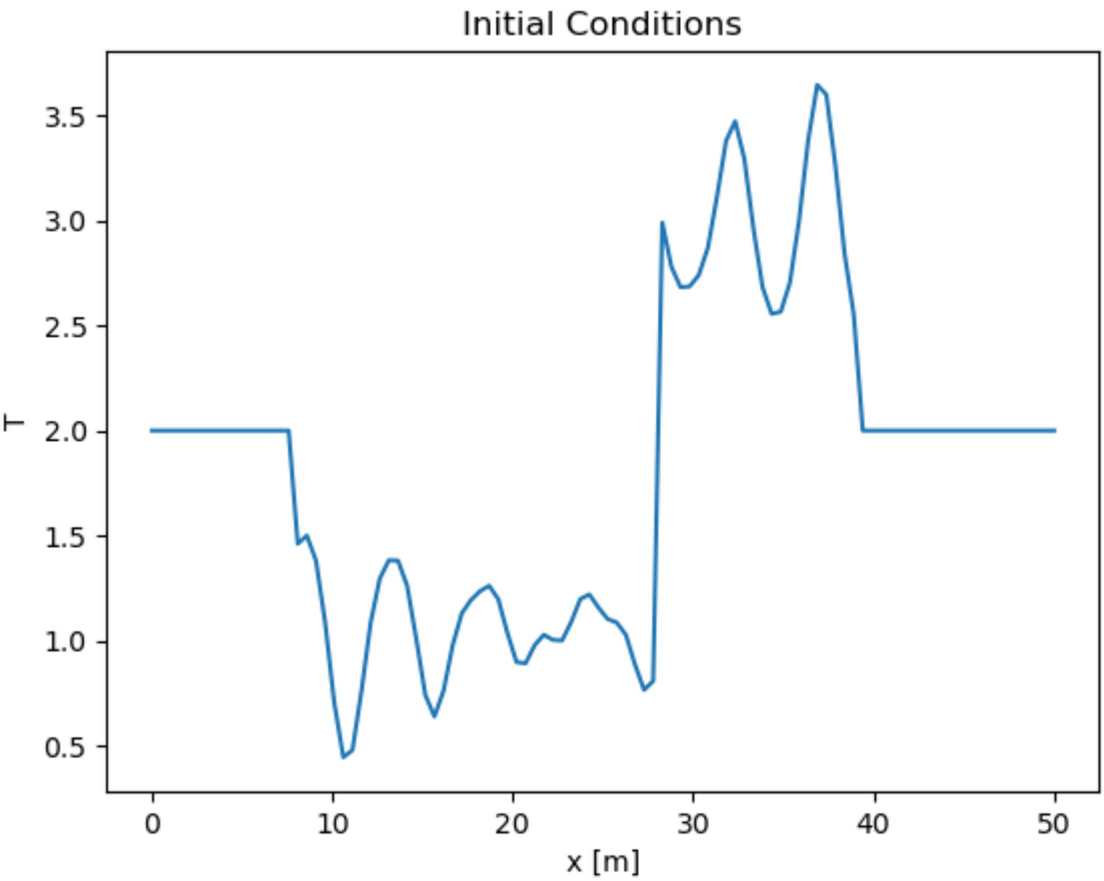
Out[69]: (45, 10)

Part (3)

In this part, we will be solving the Advection equation using two different solving methods. The test function will be a little unique.

In [70]:
```
N_x = int(100)
x_domain = np.linspace( 0 , 50 , num=N_x )
c = 0.5

def testFunction( x ):

    dx = np.mean( np.gradient( x ) )
    u_0 = np.zeros_like( x )

    # First region
    u_0[(x>=8)&(x<=28)] = -1

    # Second region
    u_0[(x>28)&(x<=39)] = 1

    u = 2 + u_0 * ( 1 + 0.3 * np.sin( 2 * np.pi * x / ( 9 * dx ) ) ) * ( 1 + 0.4 * np.sin( 2 *

    return u

u_0 = testFunction( x_domain )
```

```python
plt.plot( x_domain , u_0 )
plt.xlabel("x [m]")
plt.ylabel("T")
plt.title("Initial Conditions")
plt.show()
```



I have not clue what this function is supposed to be, but this is about to get interesting.

(a): Upwind Method

In this instance, we will be solving the Advection equation by the Upwind method, which simply determines the next value by the upstream spatial gradient, hence the name. The formulation is simply:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0$$

This applies for positive wave velocity.

This rearranges to:

$$u_j^{n+1} = (1 - CFL)\, u_j^n + CFL u_{j-1}^n$$

This becomes the following linear algebraic equation:

$$[1]\,[u_j]^{n+1} = ([0 \quad 1] + CFL\,[1 \quad -1])\,[u_{j-1} \quad u_j]^{T,n}$$

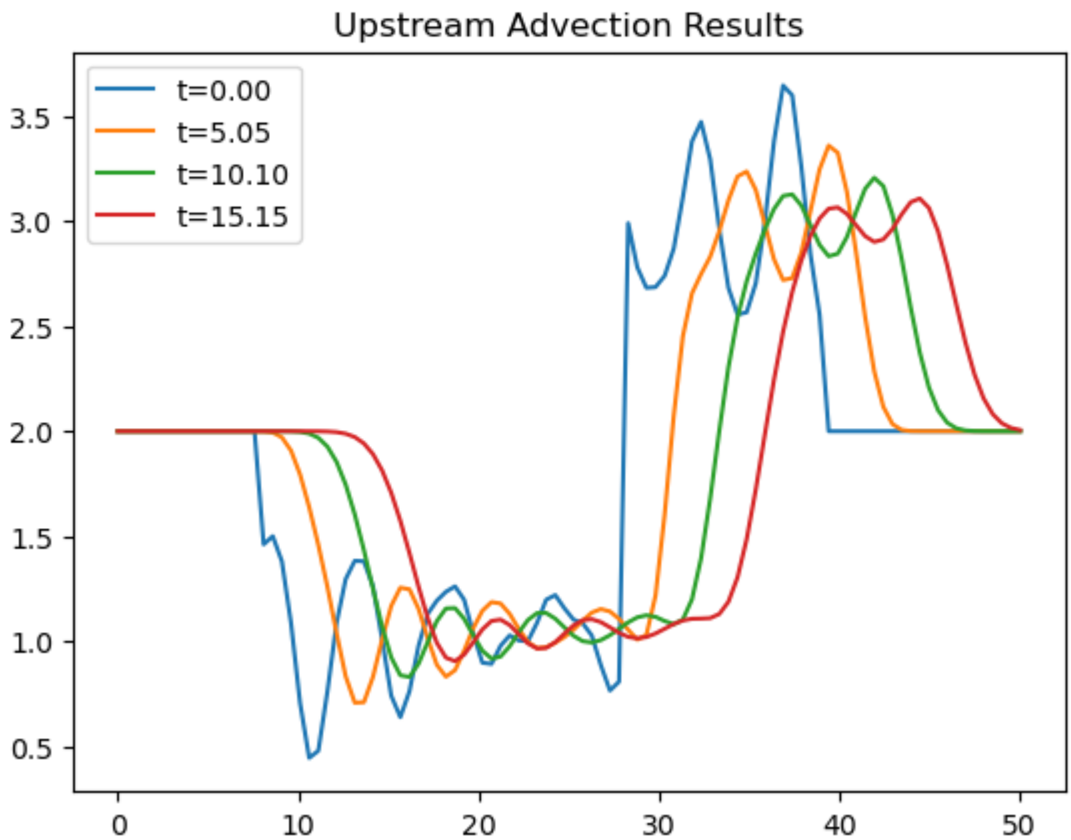
Next, we will set up the equation object to perform the solve.

In [71]:
```python
advect_a = advectionEquation( x_domain , u_0 , c , (0,20) , C=1 )
```

In [72]:
```python
advect_a.solve()
```

```
A:\Users\mtthl\Documents\Education\ME5653_CFD\git\me5653_CFD_repo\code\lib\distributedObject
s.py:1011: SparseEfficiencyWarning: spsolve requires A be CSC or CSR matrix format
  cls.u[i+1,:] = spsr.linalg.spsolve( cls.A_matrix , cls.b[i,...] )
```

In [73]:
```
ts_plot = advect_a.t[::10]

for i , t in enumerate( ts_plot ):
    x = advect_a.x
    y = advect_a.u[advect_a.t==t][0]
    plt.plot( x , y , label="t={x:.2f}".format(x=t) )

plt.title("Upstream Advection Results")
plt.legend( loc="best" )

plt.show()
```



Unfortunately, it appears that there is some numerical diffusion present, but it seems to be working.

(b): Leap Frog Method

The Leap Frog method is different from the Upwind method in that it takes two (2x) time steps to form a second order in space and time accurate method. However, it cross one time point to form the time gradient. i.e.:

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

When re-arranged, this equation becomes:

$$u_j^{n+1} = u_j^{n-1} - CFL\left(u_{j+1}^n - u_{j-1}^n\right)$$

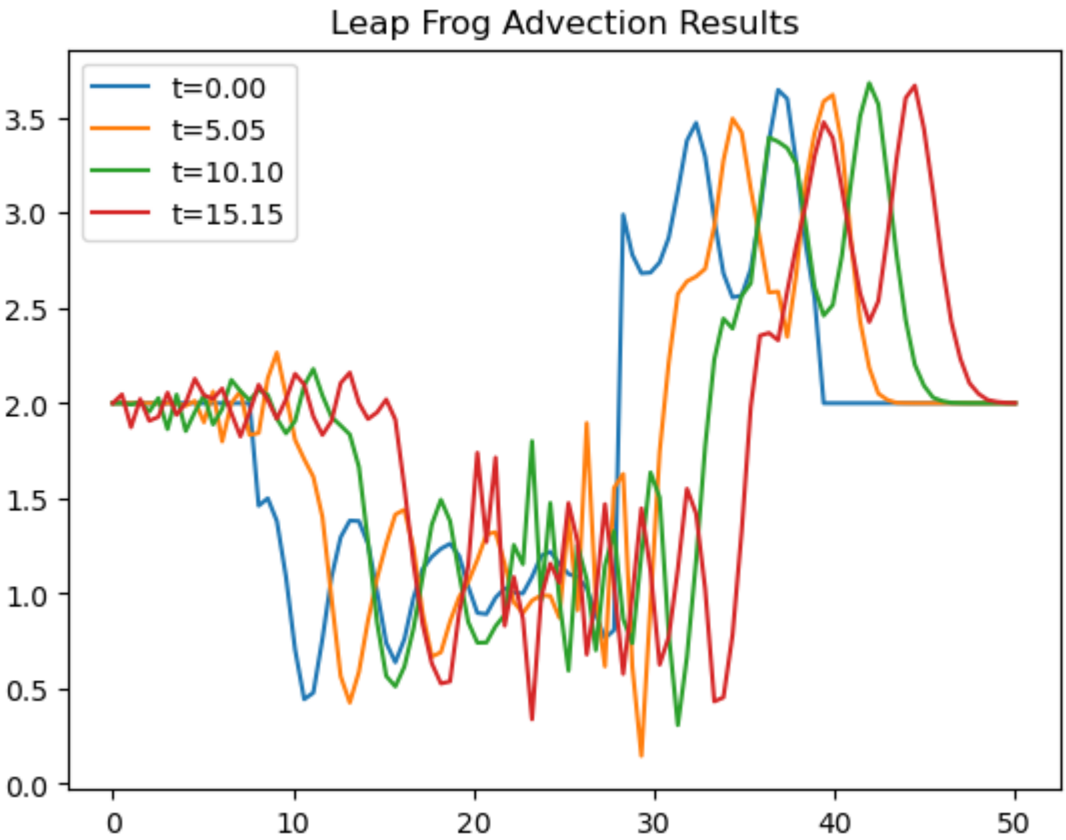
Which then becomes:

$$[1][u_j]^{n+1} = [1][u_j]^{n-1} - CFL\begin{bmatrix}-1 & 0 & 1\end{bmatrix}\begin{bmatrix}u_{j-1} & u_j & u_{j+1}\end{bmatrix}^{T,n}$$

In [74]:
```
advect_b = advectionEquation( x_domain , u_0 , c , (0,20) , C=1 , solver="leapfrog" )
```

In [75]:
```
advect_b.solve()
```

In [76]:
```
ts_plot = advect_b.t[::10]

for i , t in enumerate( ts_plot ):
    x = advect_b.x
    y = advect_b.u[advect_b.t==t][0]
    plt.plot( x , y , label="t={x:.2f}".format(x=t) )

plt.title("Leap Frog Advection Results")
plt.legend( loc="best" )

plt.show()
```



I had to add a zero-gradient condition at the outlet to keep some sort of signal build-up fro happening there, but there are some issues with the results.

It appears that there is a sort of "wake" of instability in this method, as seen by the high-frequency fluctuations behind the front of the motion. However, it appears that the front of the motion does not have the dissipation that appears in the Upwind method.

This indicates that this method would be good for strong discontinuities at the front, but does not represent the wake well.

Works Cited

1. Anderson, D. A., Tannehill, J. C., Pletcher, R. H., Munipalli, R., and Shankar, V. (2021).

Computational Fluid Mechanics and Heat Transfer. 4th Edition. CRC Press.