# Introduction

The objective of this assignment is to perform the numerical calculation of the heat equation in one spatial dimension, defined in Equation 1. This equation describes the diffusion of a passive scalar in a field for any number of spatial dimensions.

Equation . 1D Heat Equation.

For the domain of this assignment is a simulated wall of a given thickness originally at a given temperature. The wall has a known diffusion of the heat transfer or temperature difference. In the transient simulation, the sides of the wall are suddenly increased to a given temperature, bringing the temperature of the wall up to some value. These values are summarized in Table 1.

Table . Conditions of Transient Analysis.

|  |  |
| --- | --- |
| Parameter | Value |
| (L) Thickness of the simulated wall | 0.3 [m] |
| (T0) Initial uniform temperature of the wall | 100 [K] |
| (α) Heat diffusivity | 3e-6 [m2/s] |
| (Tw) Sudden wall temperature | 300 [K] |
| (dx) Spatial mesh spacing | 0.015 [m] |

# Forward in Time, Central in Space Solution (FTCS)

## FTCS Method

This method is a simple explicit formulation of a discretized heat equation. The step is a central second order derivative in space for a forward step in time, hence the name, described in Equation 2.

Equation . FTCS Discretization of 1D Heat Equation. n is the time step, and j is the spatial point.

This scheme’s stability is determined by a parameter, , although some references refer to it as . The FTCS scheme is stable for .

Equation . S Parameter for FTCS.

The accuracy of the scheme can come from its Truncation Error (TE). This TE would suggest that the method is first (1st) order accurate in time and second (2nd) order accurate in space

Equation . Truncation Error for FTCS.

Now, in Equation 4, there is a term in curved brackets. As the authors of [1] point out, there is a numerical phenomenon where if , then the 1st error term becomes zero, and thus the FTCS method becomes 2nd order in time and 4th order in space accurate.

## Calculation Method

If one changes the FTCS formula with the stability parameter, , the discretization of the heat equation becomes:

Equation . FTCS Discretization of 1D Heat Equation with Substituted Stability Factor.

Taking this a step further, the known values can all be moved to the right-hand side (RHS) of the equation, and the unknowns on the left-hand side (LHS) of the equation.

Equation . FTCS Discretization of 1D Heat Equation as Used in Linear Equation.

Thus, if this equation becomes a linear equation, it becomes:

Equation . FTCS Discretization Arranged in a Linear Equation.

Which can be re-arranged as:

Equation . FTCS Discretization Decomposed for Time and Spatial Gradient Components.

# Crank-Nicolson (CN) Solution

## CN Method

The Crank-Nicolson method is effectively central in time while being central in space. The discretization of the heat equation is outlined in Equation 9.

Equation . CN Discretization of 1D Heat Equation. n is the time step, and j is the spatial point.

The stability parameter, defined in Equation 10, is the same, although the discretization is unconditionally stable.

Equation . S Parameter for CN.

The truncation error of this method is a little more complicated than for FTCS. CN truncation error will be second (2nd) order in space and second (2nd) order in time, but the second order in time comes from a higher derivative in the Taylor series. Also unlike FTCS, there is no value of where the truncation error discretely drops.

Equation . Truncation Error for CN.

## Calculation Method

If one changes the CN formula with the stability parameter, , the discretization of the heat equation becomes:

Equation . CN Discretization of 1D Heat Equation with Substituted Stability Factor.

Taking this a step further, the known values can all be moved to the right-hand side (RHS) of the equation, and the unknowns on the left-hand side (LHS) of the equation.

Equation . CN Discretization of 1D Heat Equation as Used in Linear Equation.

Becoming a linear equation, it becomes:

Equation . CN Discretization Arranged in a Linear Equation.

Which can be re-arranged as:

Equation . CN Discretization Decomposed for Time and Spatial Gradient Components.

# Works Cited

1. Anderson, D. A., Tannehill, J. C, Pletcher, R. H., Munipalli, R., and Shankar, V. (2021). Computational Fluid Mechanics and Heat Transfer. *Series in Computation and Physical Processes in Mechanics and Thermal Sciences.* 4th Edition, CRC Press.