MATH 161, Autumn 2024
SCRIPT 3: Introducing a Continuum

This sheet introduces a continuum C through a series of axioms.

Axiom 1. *A continuum is a nonempty set C.*

We often refer to elements of C as *points*.

Definition 3.1. Let X be a set. An *ordering* on the set X is a subset $<$ of $X \times X$, with elements $(x, y) \in <$ written as $x < y$, satisfying the following properties:

(a) *(Trichotomy)*

For all $x, y \in X$ exactly one of the following holds: $x < y,\ y < x$ or $x = y$.

(b) *(Transitivity)* For all $x, y, z \in X$, if $x < y$ and $y < z$ then $x < z$.

Remarks 3.2. a) In mathematics "or" is understood to be inclusive unless stated otherwise. So in a) above, the word "exactly" is needed.

b) $x < y$ may also be written as $y > x$.

c) By $x \leq y$, we mean $x < y$ or $x = y$; similarly for $x \geq y$.

Axiom 2. *A continuum C has an ordering $<$.*

Definition 3.3. If $A \subset C$ is a subset of C, then a point $a \in A$ is a *first* point of A if, for every element $x \in A$, either $a < x$ or $a = x$. Similarly, a point $b \in A$ is called a *last* point of A if, for every $x \in A$, either $x < b$ or $x = b$.

Lemma 3.4. *If A is a nonempty, finite subset of a continuum C, then A has a first and last point.*

Proof. We prove this by induction on $|A| = n \in \mathbb{N}$.

For the base case, $n = 1$, so let the singleton set $A = \{x\}$. Thus, x is both the first and last point of A.

For the inductive step, we want to show that if the set A with cardinality n, denoted A_n, has a first and last point, then A_{n+1} has a first and last point.

Consider $A' = A_{n+1} \setminus \{x\}$ for some $x \in A_{n+1}$. Then, we know that $|A'| = n$, so A' has a first point (let this be F) and a last point (let this be L).

Then, consider $x_0 \in A_{n+1}$. By the ordering $<$ on C, we know that either $x_0 < F$ or $x_0 = F$ or $x_0 > F$.

If $x < F$, then x is the first point of A_{n+1}. If $x = F$, then $x \in A_{n+1} \setminus \{x\}$ and this is a contradiction. If $x > F$, then F is the first point of A_{n+1}.

Therefore, if a set with cardinality n has a first and last point, then a set with cardinality $n+1$ has a first and last point. This completes the proof.

QED

Theorem 3.5. *Suppose that A is a set of n distinct points in a continuum C, or, in other words, $A \subset C$ has cardinality n. Then symbols $a_1, \ldots, a_n$ may be assigned to each point of A so that $a_1 < a_2 < \cdots < a_n$, i.e. $a_i < a_{i+1}$ for $1 \leq i \leq n-1$.*

Proof. We prove this by induction on n.

For the base case, $n = 1$, so let the singleton set $A = \{a\}$. Then, $a = a_1$ and this completes the base case.

For the inductive step, consider the set A_{n+1} with cardinality $n+1$. By Lemma 3.4, we know that A_{n+1} has a last point (let this be denoted by a_{max}). Define $A' = A \setminus \{a_{max}\}$. Then, we know that $|A'| = n$.

By the inductive hypothesis, A' is a set of cardinality n, hence A' can be written as $a_1 < a_2 < \cdots < a_n$. Then, we know that A can be written as $a_1 < a_2 < \cdots < a_n < a_{max}$. This completes the proof.

QED

Definition 3.6. If $x, y, z \in C$ and either (i) both $x < y$ and $y < z$ or (ii) both $z < y$ and $y < x$, then we say that y is *between* x and z.

Corollary 3.7. *Of three distinct points in a continuum, one must be between the other two.*

Proof. Let $x, y, z \in C$ such that $x \neq y, x \neq z, y \neq z$.

Then, let $A \subset C$ such that $A = \{x, y, z\}$. We know that A can be expressed as $\{a_1, a_2, a_3\}$, and moreover these can be written as $a_1 < a_2 < a_3$. Then, $a_1 < a_2$ and $a_2 < a_3$, so a_2 is between a_1 and a_3.

QED

Axiom 3. *A continuum C has no first or last point.*

In the next exercise we show that the integers and the rationals both have orderings that satisfy Axioms 1-3.

Exercise 3.8. a) We define a relation $<$ on $\mathbb{Z}$ by $m < n$ if $n = m + c$ for some $c \in \mathbb{N}$. Show that, $\mathbb{Z}$, with the ordering $<$, satisfies Axioms 1-3.

Proof. We know that $-1 \in \mathbb{Z}$, hence $\mathbb{Z} \neq \emptyset$, so it satisfies Axiom 1.

For Axiom 2, we want to show that transitivity holds for the ordering $<$. We want to show that if $x_1 < x_2$ and $x_2 < x_3$, then $x_1 < x_3$.

We know that $x_2 = x_1 + c_1$ and $x_3 = x_2 + c_2$. Hence, $x_3 = x_1 + c_1 + c_2$. Let $c_3 = c_1 + c_2$, then $x_3 = x_1 + c_3$, hence $x_3 > x_1$.

We know that $\mathbb{Z}$ goes from negative infinity to positive infinity, so it has no first or last point, so it satisfies Axiom 3. This completes the proof.

QED

b) Show that, for any $p = \left[\frac{a}{b}\right] \in \mathbb{Q}$, there is some $(a_1, b_1) \in p$ with $0 < b_1$.

Proof. We know that $b \neq 0$, so we proceed by cases.

Case 1: $b > 0$ and we are done.

Case 2: $b < 0$. Then, let $a' = -a$ and $b' = -b$. We want to show that $[\frac{a}{b}] = [\frac{a'}{b'}]$.

$a \cdot -b = -a \cdot b = -ab$, hence $ab' \sim ba'$, hence $(a, b) \sim (a', b')$, hence $[\frac{a}{b}] = [\frac{a'}{b'}]$. Then, $p = [\frac{a'}{b'}]$ and this completes the proof.

QED

c) We define a relation $<_{\mathbb{Q}}$ on $\mathbb{Q}$ as follows. For $p, q \in \mathbb{Q}$, let $(a_1, b_1) \in p$ be such that $0 < b_1$, and let $(a_2, b_2) \in q$ be such that $0 < b_2$. Then we define $p <_{\mathbb{Q}} q$ if $a_1 b_2 < a_2 b_1$. Show that $<_{\mathbb{Q}}$ is a well-defined relation on $\mathbb{Q}$.

Proof. We want to show that if $[\frac{a}{b}] = [\frac{a'}{b'}]$ and $[\frac{c}{d}] = [\frac{c'}{d'}]$ with $b, b', d, d' > 0$ and $ad < bc$, then $a'd' < b'c'$.

We know that $ab' = a'b$ and $cd' = c'd$. Hence $ab'dd' = a'bdd'$, where we are multiplying by dd' since $d, d' > 0$.

Then, $ab'dd' = adb'd' < bcb'd'$. Therefore $ab'dd' < bcb'd'$, hence $ab'dd' < cd'bb'$, hence $ab'dd' < c'dbb'$.

Hence, $a'd'bd < b'c'bd$, hence $a'd' < b'c'$. This completes the proof.

QED

d) Show that $\mathbb{Q}$, with the ordering $<_{\mathbb{Q}}$, satisfies Axioms 1-3.

Proof. We know that $[1/2] \in \mathbb{Q}$, hence $\mathbb{Q} \neq \emptyset$, so it satisfies Axiom 1.

For Axiom 2, we want to show that transitivity holds for the ordering $<$. We want to show that if $[\frac{a}{b}] < [\frac{c}{d}]$ and $[\frac{c}{d}] < [\frac{e}{f}]$, then $[\frac{a}{b}] < [\frac{e}{f}]$.

Since $ad < bc$ and $cf < de$, we know that $adf < bcf$ and $bcf < bde$, hence $adf < bde$ and $af < be$.

We know that $\mathbb{Q}$ goes from negative infinity to positive infinity, so it has no first or last point, so it satisfies Axiom 3. This completes the proof.

QED

Definition 3.9. If $a, b \in C$ and $a < b$, then the set of points between a and b is called a *region*, denoted by $\underline{ab}$.

Warning 3.10. One often sees the notation (a, b) for regions. We will reserve the notation (a, b) for ordered pairs in a product $A \times B$. These are very different things.

Theorem 3.11. *If x is a point of a continuum C, then there exists a region $\underline{ab}$ such that $x \in \underline{ab}$.*

Proof. By the axioms of a continuum, we know that C does not have a first point, hence there exists $a \in C$ such that $a < x$. Similarly, we know that C does not have a last point, hence there exists $b \in C$ such that $x < b$.

Since $a < x < b$, we know that there exists a region $\underline{ab}$ such that $x \in \underline{ab}$. This completes the proof.

QED

We now come to one of the most important definitions of this course:

Definition 3.12. Let A be a subset of a continuum C. A point p of C is called a *limit point* of A if every region R containing p has nonempty intersection with $A \setminus \{p\}$. Explicitly, this means:

$$\text{for every region } R \text{ with } p \in R, \text{ we have } R \cap (A \setminus \{p\}) \neq \emptyset.$$

Notice that we do not require that a limit point p of A be an element of A. We will use the notation $LP(A)$ to denote the set of limit points of A.

Theorem 3.13. *If $A \subset B$, then $LP(A) \subset LP(B)$.*

Proof. Suppose $x \in LP(A)$. Then, for every region R with $x \in R$, there exists $y \in R \cap (A \setminus \{x\})$ such that $y \neq x, y \in R, y \in A, y \in B$, where the last statement follows since $A \subset B$.

Then, we know that since $y \in R, y \neq x, y \in B$, therefore $y \in R \cap (B \setminus \{x\})$. Hence, $R \cap (B \setminus \{x\}) \neq \emptyset$, therefore $x \in LP(B)$.

Hence, $LP(A) \subset LP(B)$. This completes the proof.

QED

Definition 3.14. If $\underline{ab}$ is a region in a continuum C, then $C \setminus (\{a\} \cup \underline{ab} \cup \{b\})$ is called the *exterior* of $\underline{ab}$ and is denoted by $\operatorname{ext} \underline{ab}$.

Lemma 3.15. *If $\underline{ab}$ is a region in a continuum C, then*

$$\operatorname{ext} \underline{ab} = \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}.$$

Proof. Suppose $x \in \text{ext } \underline{ab}$. Then, $x \notin \{a\}, x \notin \{b\}, x \in \underline{ab}$. Hence, either $x < a$ or $b < x$. Therefore, $x \in \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$. Hence, $\operatorname{ext} \underline{ab} \subset \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$.

Suppose $x \in \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$. Then, $x \notin \{a\} \cup \underline{ab} \cup \{b\}$. Hence, $x \in \operatorname{ext} \underline{ab}$. Hence, $\{x \in C \mid x < a\} \cup \{x \in C \mid b < x\} \subset \operatorname{ext} \underline{ab}$.

This completes the proof.

QED

Lemma 3.16. *No point in the exterior of a region is a limit point of that region. No point of a region is a limit point of the exterior of that region.*

Proof. Suppose $x \in \operatorname{ext} \underline{ab}$.

Case 1: $x < a$. Let $c \in C$ such that $c < x < a$. Then, $\underline{ca} \cap \underline{ab} = \emptyset = \underline{ca} \cap \underline{ab} \setminus \{x\}$. Hence, $x \notin LP(\underline{ab})$.

Case 2: $x > b$. Let $c \in C$ such that $b < x < c$. Then, $\underline{bc} \cap \underline{ab} = \emptyset = \underline{bc} \cap \underline{ab} \setminus \{x\}$. Hence, $x \notin LP(\underline{ab})$.

This proves that no point in the exterior of a region is a limit point of that region.

Next, suppose $x \in \underline{ab}$. Then, $x \notin \operatorname{ext} \underline{ab}$. Hence, $\operatorname{ext} \underline{ab} \setminus \{x\} = \operatorname{ext} \underline{ab}$. Therefore, $\underline{ab} \cap (\operatorname{ext} \underline{ab} \setminus \{x\}) = \underline{ab} \cap \operatorname{ext} \underline{ab} = \emptyset$. Hence, $x \notin LP(\operatorname{ext} \underline{ab})$.

This proves that no point of a region is a limit point of the exterior of that region.

QED

Theorem 3.17. *If two regions have a point x in common, their intersection is a region containing x.*

Proof. Let $x \in \underline{ab}, x \in \underline{cd}$, and let $e = \max\{a, c\}, f = \min\{b, d\}$. Then, we know that $a < x < b$ and $c < x < d$, hence $\max\{a, c\} < \min\{b, d\}$, hence $e < f$. We want to show that $\underline{ab} \cap \underline{cd} = \underline{ef}$.

Suppose $y \in \underline{ab} \cap \underline{cd}$. Then, $a < y < b$ and $c < y < d$, hence $e < y < f$, hence $y \in \underline{ef}$, hence $(\underline{ab} \cap \underline{cd}) \subset \underline{ef}$.

Suppose $y \in \underline{ef}$. Then, $e < y < f$, hence $\max\{a, c\} < y < \min\{b, d\}$. Hence, $y \in \underline{ab}$ and $y \in \underline{cd}$, hence $y \in \underline{ab} \cap \underline{cd}$. Therefore, $\underline{ef} \subset (\underline{ab} \cap \underline{cd})$.

QED

Corollary 3.18. *If n regions $R_1, \ldots, R_n$ have a point x in common, then their intersection $R_1 \cap \cdots \cap R_n$ is a region containing x.*

Proof. We prove this by induction on n.

For the base case, we want to show that if two regions R_1, R_2 have a point x in common, then their intersection $R_1 \cap R_2$ is a region containing x. We know this from Theorem 3.17.

For the inductive step, we want to show that if n regions $R_1, \ldots, R_n$ have a point x in common and if R_{n+1} is a region containing x, then the intersection $R_1 \cap \cdots \cap R_n \cap R_{n+1}$ is a region containing x.

By the inductive hypothesis, let $M = R_1 \cap \cdots \cap R_n$ be a region containing x. Then, we want to show that $M \cap R_{n+1}$ is a region containing x. We know that M and R_{n+1} are regions that contain x, so this follows from Theorem 3.17, and this completes the proof.

QED

Theorem 3.19. *Let A, B be subsets of a continuum C. Then $LP(A \cup B) = LP(A) \cup LP(B)$.*

Proof. First, we show that $LP(A) \cup LP(B) \subset LP(A \cup B)$. By Theorem 3.13, we know that $A \subset A \cup B$, so $LP(A) \subset LP(A \cup B)$. Similarly, $B \subset A \cup B$, so $LP(B) \subset LP(A \cup B)$. Hence, $LP(A) \cup LP(B) \subset LP(A \cup B)$.

Next, we show that $LP(A \cup B) \subset LP(A) \cup LP(B)$. Suppose $x \in LP(A \cup B)$, then we want to show that $x \in LP(A) \cup LP(B)$. Suppose $x \notin LP(A) \cup LP(B)$. This is equivalent to $x \notin LP(A), x \notin LP(B)$.

Hence, there exists a region R_A such that $x \in R_A$ and $R_A \cap A \setminus \{x\} = \emptyset$, and there exists a region R_B such that $x \in R_B$ and $R_B \cap B \setminus \{x\} = \emptyset$.

Let a region $R = R_A \cap R_B$. Then, $R \subset R_A$. Hence, $(R \cap A \setminus \{x\}) \subset (R_A \cap A \setminus \{x\})$. Since the RHS is empty, the LHS must also be empty. Similarly, $R \subset R_B$. Hence, $(R \cap B \setminus \{x\}) \subset (R_B \cap B \setminus \{x\})$. Since the RHS is empty, the LHS must also be empty.

$R \cap (A \cup B) \setminus \{x\} = (R \cap A \setminus \{x\}) \cup (R \cap B \setminus \{x\})$. Since both sets on the RHS are empty, their union must be empty, hence the LHS must be empty. However, this contradicts our original assumption that $R \cap (A \cup B) \setminus \{x\} \neq \emptyset$, since $x \in LP(A \cup B)$.

QED

Corollary 3.20. *Let $A_1, \ldots, A_n$ be n subsets of a continuum C. Then p is a limit point of $(A_1 \cup \cdots \cup A_n)$ if, and only if, p is a limit point of at least one of the sets A_k.*

Proof. We prove this by strong induction on n.

For the base case, let $n = 2$. Then, by Theorem 3.19, we know that $LP(A_1 \cup A_2) = LP(A_1) \cup LP(A_2)$.

For the inductive step, suppose $LP(A_1 \cup A_2 \cup \cdots \cup A_n) = LP(A_1) \cup LP(A_2) \cup \cdots \cup LP(A_n)$. Then, $LP(A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}) = LP(A_1 \cup A_2 \cup \cdots \cup A_n) \cup LP(A_{n+1}) = LP(A_1) \cup LP(A_2) \cup \cdots \cup LP(A_n) \cup LP(A_{n+1})$.

This completes the proof.

QED

Theorem 3.21. *If p and q are distinct points of a continuum C, then there exist disjoint regions R and S containing p and q, respectively.*

Proof. We know that $p \neq q$. Since a continuum has an ordering $<$, suppose without loss of generality that $p < q$. Since a continuum does not have a first or last point, let $a, b \in C$ such that $a < p, b > q$.

Case 1: $\exists x \in C$ such that $p < x < q$. Then, let region $R = \underline{ax}$ and let region $S = \underline{xb}$.

Since $a < p < x$, we know that $p \in R$. Since $x < q < b$, we know that $q \in S$.

Suppose R and S are not disjoint, i.e., there exists some $y \in C$ such that $y \in R, y \in S$. Then, $a < y < x$ and $x < y < b$, i.e., $y < x$ and $x < y$, and this contradicts the trichotomy of the ordering $<$. Hence, R and S are disjoint.

Case 2: $\nexists x \in C$ such that $p < x < q$. Then, let region $R = \underline{aq}$ and let region $S = \underline{pb}$.

Since $a < p < q$, we know that $p \in R$. Since $p < q < b$, we know that $q \in S$.

Suppose R and S are not disjoint, i.e., there exists some $y \in C$ such that $y \in R, y \in S$. Then, $a < y < q$ and $p < y < b$, i.e., $p < y < q$, and this contradicts our original assumption that $\nexists y \in C$ such that $p < y < q$. Hence, R and S are disjoint.

This completes the proof.

QED

Corollary 3.22. *A subset of a continuum C consisting of one point has no limit points.*

Proof. Let $A \subset C$ such that $A = \{a\}$. We want to show that $LP(A) = \emptyset$.

Suppose $LP(A) \neq \emptyset$, then let $p \in LP(A)$. It suffices to show that there exists a region R such that $R \cap (A \setminus \{p\}) = \emptyset$.

Case 1: $p = a$. Then, $A \setminus \{p\} = A \setminus \{a\} = \{a\} \setminus \{a\} = \emptyset$. Hence, $R \cap \emptyset = \emptyset$.

Case 2: $p \neq a$. Then, $A \setminus \{p\} = A$. Hence, we want to show that there exists a region R that contains p such that $R \cap A = \emptyset$. It suffices to show that there exists a region R such that $p \in R, a \notin R$.

Subcase 2a: $p > a$. Then, let $b \in C$ such that $b > p$ and let $R = \underline{ab}$. Since $a < p < b$, $p \in R$ and $a \notin R$.

Subcase 2b: $p < a$. Then, let $b \in C$ such that $b < p$ and let $R = \underline{ba}$. Since $b < p < a$, $p \in R$ and $a \notin R$.

This completes the proof.

QED

Theorem 3.23. *A finite subset A of a continuum C has no limit points.*

Proof. Let $A_n \subset C$ such that $|A_n| = n$. Then, we prove this theorem by induction on n.

For the base case, $n = 1$, i.e., A consists of one point. Corollary 3.22 completes the base case.

For the inductive step, we want to show that if $LP(A_n) = \emptyset$, then $LP(A_{n+1}) = \emptyset$.

We know that any finite subset of a continuum has a last point. Hence, let a_{max} denote the last point of A_{n+1}. Then, we know that $|A_{n+1} \setminus \{a_{max}\}| = n$. Hence, by the inductive hypothesis, $LP(A_{n+1} \setminus \{a_{max}\}) = \emptyset$. We also know from Corollary 3.22 that $LP(\{a_{max}\}) = \emptyset$.

By Theorem 3.19, $LP(A_{n+1}) = LP(A_{n+1} \setminus \{a_{max}\}) \cup LP(a_{max}) = \emptyset \cup \emptyset = \emptyset$. This completes the proof.

QED

Corollary 3.24. *If A is a finite subset of a continuum C and $x \in A$, then there exists a region R, containing x, such that $A \cap R = \{x\}$.*

Proof. Since A is finite, $LP(A) = \emptyset$. Hence, there exists some region R such that $x \in R$ and $R \cap A \setminus \{x\} = \emptyset$.

We can write $A \cap R = A \cap (R \setminus \{x\} \cup \{x\}) = (A \cap (R \setminus \{x\})) \cup (A \cap \{x\}) = \emptyset \cup \{x\} = \{x\}$.

This completes the proof.

QED

Theorem 3.25. *If p is a limit point of A and R is a region containing p, then the set $R \cap A$ is infinite.*

Proof. Assume $R \cap A$ is finite. Then, $LP(R \cap A) = \emptyset$, hence $p \notin LP(R \cap A)$.

Hence, there exists a region R' such that $p \in R'$ and $R' \cap R \cap A \setminus \{p\} = \emptyset$.

Then, $p \in R', p \in R$, hence $R' \cap R$ is a region containing p, and let this be S, i.e., $p \in S$.

Then, $S \cap A \setminus \{p\} = \emptyset$, hence $p \notin LP(A)$.

QED

Exercise 3.26. Find realizations of a continuum $(C, <)$. That is, find concrete sets C endowed with a relation $<$ satisfying all of the axioms (so far). Are they the same? What does "the same" mean here?

Proof. $\mathbb{Q}, \mathbb{Z}$ are realizations of a continuum.

QED

Additional Exercises

In all exercises you are expected to prove your answer, unless explicitly stated otherwise.

1. Show that for all $p, q \in \mathbb{Q}$ such that $p <_{\mathbb{Q}} q$, there is some $r \in \mathbb{Q}$ with $p <_{\mathbb{Q}} r <_{\mathbb{Q}} q$.

   Proof. Let $r = \frac{p+q}{2}$. Then, $p < q$, hence $2p < p + q$, hence $\frac{2p}{2} < \frac{p+q}{2}$, hence $p < r$.

   Next, $p < q$, hence $p + q < 2q$, hence $\frac{p+q}{2} < \frac{2q}{2}$, hence $r < q$.

   Therefore, $p < r < q$. This completes the proof.

   QED

2. By identifying $n \in \mathbb{Z}$ with $[\frac{n}{1}] \in \mathbb{Q}$ we can think of $\mathbb{Z}$ as a subset of $\mathbb{Q}$. We shall write n to mean $[\frac{n}{1}]$. Show that for all $p \in \mathbb{Q}$, there is some $n \in \mathbb{Z}$ such that $p <_{\mathbb{Q}} n$.

Proof. Let $p = [\frac{a}{b}]$ for $a \in \mathbb{Z}, b \in \mathbb{N}$.

Case 1: If $a = 0$, let $n = 1$. Then, $p = [\frac{0}{1}] < 1 = n$.

Case 2: If $a > 0$, let $n = a$. Then, $p = [\frac{a}{b}] < a = n$.

Case 3: If $a < 0$, let $n = 0$. Then, $p = [\frac{a}{b}] < 0 = n$.

This completes the proof.

QED

3.

Definition 3.27. Given two sets A and B, we say that $A \subsetneq B$ if $A \subset B$ and $A \neq B$.

For each set X and subset $<_X \subset X \times X$, determine if $<_X$ is an ordering:

(a) Let $X = \wp(\mathbb{N})$ and $<_X = \{(A, B) \in \wp(\mathbb{N}) \times \wp(\mathbb{N}) | A \subset B\}$.

(b) Let $X = \big\{\{x \in \mathbb{N} | x \leq n\} \in \wp(\mathbb{N}) | n \in \mathbb{N}\big\}$ and $<_X = \{(A, B) \in X \times X | A \subsetneq B\}$.

(c) Let $X = \{f \subset \mathbb{N} \times \mathbb{N} | f \text{ is a function}\}$ and $<_X = \{(f, g) \in X \times X | f(n) < g(n) \text{ for all } n \in \mathbb{N}\}$.

(d) Let $X = \{f \subset \mathbb{N} \times \mathbb{N} | f \text{ is a function}\}$ and $<_X = \{(f, g) \in X \times X | f(n) \leq g(n) \text{ for all } n \in \mathbb{N} \text{ and there exists } n \in \mathbb{N} \text{ such that } f(n) < g(n)\}$.

(For the purposes of this exercise, "$<$" means the usual ordering on $\mathbb{N}$, i.e. if $m, n \in \mathbb{N}$ then $m < n$ if and only if $n = m + k$ for some $k \in \mathbb{N}$.)

4. Which of the following pairs of a set and an ordering satisfy Axioms 1,2, and 3 (and are thus examples of a "continuum")?

   i) $C_1 = \{[n] \in \mathcal{P}(\mathbb{N}) \mid n \in \mathbb{N} \cup \{0\}\}$, where, for $[n], [m] \in C_1$, we say $[n] <_{C_1} [m]$ if $[n] \subsetneq [m]$;

   ii) $C_2 = \mathbb{Z}$, where $<_{\mathbb{Z}}$ is the usual ordering on $\mathbb{Z}$;

   iii) $C_3 = \mathbb{Z} \times \mathbb{N}$, where, for $(a, b), (x, y) \in \mathbb{Z} \times \mathbb{N}$ we say that $(a, b) <_3 (x, y)$ if $a < x$ or if $a = x$ and $b < y$.

5. Find, without proof, the exterior of each region.

   (a) Let C_2 and $<_{\mathbb{Z}}$ be as in Exercise 5. Let $R = \underline{38}$.

   (b) Let C_3 and $<_3$ be as in Exercise 5. Let $R = \underline{(-2, 5)(-2, 10)}$.

   (c) Let C_3 and $<_3$ be as in Exercise 5. Let $R = \underline{(-2, 5)(1, 10)}$.

6. Find the limit points of each region.

   (a) Let C_2 and $<_{\mathbb{Z}}$ be as in Exercise 5. Let $R = \underline{38}$.

   (b) Let C_3 and $<_3$ be as in Exercise 5. Let $R = \underline{(-2, 5)(-2, 10)}$.

   (c) Let C_3 and $<_3$ be as in Exercise 5. Let $R = \underline{(-2, 5)(1, 10)}$.

7. Let A and B be realizations of the continuum, with orderings $<_A, <_B$, respectively. We say that A and B are *isomorphic* if there is a bijection $f : A \longrightarrow B$ such that

$$a_1 <_A a_2 \Longrightarrow f(a_1) <_B f(a_2).$$

Show that $\mathbb{Z}$ and $\mathbb{Q}$ (with the orderings given in Scripts 1 and 2) are realizations of the continuum that are *not* isomorphic.

Proof. Suppose $f : \mathbb{Q} \to \mathbb{Z}$ such that $a_1 <_{\mathbb{Q}} a_2 \Longrightarrow f(a_1) <_{\mathbb{Z}} f(a_2)$ and f is bijective.

Let $a_1 = [\frac{0}{1}]$ and $a_2 = [\frac{1}{1}]$. Then, we know by the ordering $<_{\mathbb{Q}}$ that $a_1 <_{\mathbb{Q}} a_2$.

Let $f(a_1) = z_1$ and $f(a_2) = z_2$, then we know by our original assumption that $z_1 <_{\mathbb{Z}} z_2$.

Then, for all q such that $q \in \underline{01}$, we have that $f(q) \in \underline{z_1 z_2}$. However, there are finitely many points $z \in \mathbb{Z}$ in the region $\underline{z_1 z_2}$, but infinitely many points $q \in \mathbb{Q}$ in the region $\underline{01}$, hence by the pigeonhole principle f cannot be injective, hence it cannot be bijective. This is a contradiction.

QED

8. Suppose that $R_1, R_2, \ldots$ are regions in a continuum C and that $x \in LP(\bigcup_{i=1}^{\infty} R_i)$. Is it true that there exists $i \in \mathbb{N}$ such that $x \in LP(R_i)$? Does this answer change if we add the additional hypothesis that there exists $k \in \mathbb{N}$ such that $x \in R_k$?

9. Show that any continuum C satisfying Axioms 1-3 is infinite and write C as a union of regions in C.

Proof. Assume continuum C is finite. By Axiom 1, $C \neq \emptyset$. By Axiom 2 for the ordering and Theorem 3.5, C can be written as $a_1 < a_2 < \cdots < a_n$, where $n = |C|$. Then, a_1 is the first point of C and a_n is the last point of C, and this violates Axiom 3, which says that C has no first or last point. Hence, C is infinite.

By Theorem 3.11, if $p \in C$, then there exists a region R_p such that $p \in R_p$. Then, we know by Axiom of Choice that we can select points p in a continuum C, hence $C = \bigcup_{p \in C} R_p$.

QED

10. Prove that if $A \subset \mathbb{Z}$, A has no limit points.

Proof. We know that if $A \subset \mathbb{Z}$, then $LP(A) \subset LP(\mathbb{Z})$, hence it suffices to show that $LP(\mathbb{Z}) = \emptyset$.

Assume $LP(\mathbb{Z}) \neq \emptyset$, then let $p \in LP(\mathbb{Z})$. We want to show that there exists some region R such that $p \in R, R \cap \mathbb{Z} \setminus \{p\} = \emptyset$.

Let $R = \underline{(p-1)(p+1)}$. Then, we know that $p - 1 < p < p + 1$, hence $p \in R$.

If p is a limit point of $\mathbb{Z}$, then by Theorem 3.25, $R \cap \mathbb{Z}$ is infinite, and this is a contradiction.

QED

11. Let $i^2 = -1$ and define $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$. Define an ordering on $\mathbb{Z}[i]$ by $a+bi < c+di$ if, and only if, either $a < c$, or $a = c$ and $b < d$.

    (a) Prove that $<$ is indeed an ordering. (So you must verify the properties in Definition 3.1. Note that you may assume the usual properties of $\mathbb{Z}$ - see Script 0 for full details.)

    (b) Prove that, with this ordering, $\mathbb{Z}[i]$ satisfies Axioms 1,2 and 3, so is a realization of the continuum as defined so far.

    *****Food For Thought*****

    1 - Suppose $C_1, C_2, \ldots$ are continua, where $<_i$ is the order of C_i. Prove that the following are continua:

    a - $C_1 \times C_2$ equipped with lexicographic order $<_L$, that is,

    $(x, y) <_L (u, v)$ if $x <_1 u$ or ($x = u$ and $y <_2 v$).

    Proof. Since $C_1 \neq \emptyset, C_2 \neq \emptyset$, we know that $C_1 \times C_2 \neq \emptyset$. This satisfies Axiom 1.

    transitive trichotomy

    Let $(p, q) \in C_1 \times C_2$. Since C_1 has no first point, there exists $p' \in C_1$ such that $p' < p$. Then, $(p', q) <_L (p, q)$. Since $(p, q) \in C_1$ is arbitrary, $C_1 \times C_2$ has no first point.

    Let $(p, q) \in C_1 \times C_2$. Since C_2 has no last point, there exists $q' \in C_2$ such that $q < q'$. Then, $(p, q) <_L (p, q')$. Since $(p, q) \in C_1$ is arbitrary, $C_1 \times C_2$ has no last point. This satisfies Axiom 3.

    QED

    b - $\prod_{i=1}^{n} C_i$ (i.e., $C_1 \times \cdots \times C_n$) equipped with lexicographic order $<_L$, that is,

    $(x_1, \ldots, x_n) <_L (y_1, \ldots, y_n)$ if there exists $i \in [n]$ such that $x_j = y_j$ for every $j \in [i-1]$ and $x_i <_i y_i$.

    Proof. strong induction on n

    QED

    c - $\prod_{i \in \mathbb{N}} C_i$ (i.e., the set of infinite sequences $(x_i)_{i \in \mathbb{N}}$ such that $x_i \in C_i$ for every $i \in \mathbb{N}$) equipped with lexicographic order $<_L$, that is,

    $(x_i)_{i \in \mathbb{N}} <_L (y_i)_{i \in \mathbb{N}}$ if there exists $i \in \mathbb{N}$ such that $x_j = y_j$ for every $j \in [i-1]$ and $x_i <_i y_i$.

    2 - Using the constructions above of continua (along with the fact that $\mathbb{Z}$ and $\mathbb{Q}$ are continua), decide how many "different" continua you can get considering $C_1 \times \cdots \times C_n$ where each C_i is either $\mathbb{Z}$ or $\mathbb{Q}$. What about if you also consider $\prod_{i \in \mathbb{N}} C_i$ where each C_i is either $\mathbb{Z}$ or $\mathbb{Q}$?