MATH 161, Autumn 2024
SCRIPT 2: The Rationals

Note that all of the following proofs should be straightforward consequences of properties of the integers; you should be sure to make it clear what facts about the integers you are using. See Script 0 for a list of the defining properties of $\mathbb{Z}$.

Definition 2.1. Let X be a nonempty set. A *relation* R on X is a subset of $X \times X$. The statement $(x, y) \in R$ is read as 'x is related to y by the relation R,' and is often denoted $x \sim y$.

A relation is *reflexive* if $x \sim x$ for all $x \in X$.
A relation is *symmetric* if $y \sim x$ whenever $x \sim y$.
A relation is *transitive* if $x \sim z$ whenever $x \sim y$ and $y \sim z$.
A relation is an *equivalence relation* if it is reflexive, symmetric and transitive.

Exercise 2.2. Determine which of the following are equivalence relations.

a) Any set X with the relation $=$. So $x \sim y$ if and only if $x = y$.

   True, since $x = x$ so it is reflexive, $x = y$ implies $y = x$ so it is symmetric, and if $x = y$ and $y = z$ then $x = z$ so it is transitive. Hence it is an equivalence relation.

b) $\mathbb{Z}$ with the relation $<$.

   False, since $1 \not< 1$ so it is not reflexive (it is symmetric and not transitive).

c) Any subset X of $\mathbb{Z}$ with the relation $\leq$. So $x \sim y$ if and only if $x \leq y$.

   False, since $1 \leq 2$ but $2 \not\leq 1$ so it is not symmetric (it is reflexive and transitive; it is symmetric in the special cases where $X = \emptyset$ or $|X| = 1$).

d) $X = \mathbb{Z}$ with $x \sim y$ if and only if $y - x$ is divisible by 5.

   True, since $x - x = 0$ is divisible by 5 so it is reflexive. $y - x = -(x - y)$ so if $y - x$ is divisible by 5, then $x - y$ is divisible by 5, so it is symmetric. If $\frac{y-x}{5} \mod 1 \equiv 0$ and $\frac{z-y}{5} \mod 1 \equiv 0$, then $\frac{z-x}{5} \mod 1 = (\frac{z-y}{5} + \frac{y-x}{5}) \mod 1 \equiv 0$ so it is transitive.

e) $X = \{(a, b) \mid a, b \in \mathbb{Z}, b \neq 0\}$ with the relation $\sim$ defined by

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad ad = bc.$$

   True, since $ab = ba$ so it is reflexive, $ad = bc$ implies $cb = da$ so it is symmetric, and if $ad = bc$ and $cf = de$ then $af = be$ since $adcf = bcde$, so it is transitive. Hence it is an equivalence relation.

Remark 2.3. A *partition* of a set is a collection of non-empty disjoint subsets whose union is the original set. Any equivalence relation on a set creates a partition of that set by collecting into subsets all of the elements that are equivalent (related) to each other. When the partition of a set arises from an equivalence relation in this manner, the subsets are referred to as *equivalence classes.* (See Exercises 2 and 3 in the Additional Exercises section below.)

Remark 2.4. If we think of the set X in 2.2e) as representing the collection of all fractions whose denominators are not zero, then the relation $\sim$ may be thought of as representing the equivalence of two fractions.

Definition 2.5. As a set, the *rational numbers*, denoted $\mathbb{Q}$, are the equivalence classes in the set $X = \{(a,b) \mid a, b \in \mathbb{Z}, b \neq 0\}$ under the equivalence relation $\sim$ as defined in 2.2e). If $(a,b) \in X$, we denote the equivalence class of this element as $\left[\frac{a}{b}\right]$. So

$$\left[\frac{a}{b}\right] = \{(x_1, x_2) \in X \mid (x_1, x_2) \sim (a,b)\} = \{(x_1, x_2) \in X \mid x_1 b = x_2 a\}.$$

Then,

$$\mathbb{Q} = \left\{\left[\frac{a}{b}\right] \mid (a,b) \in X\right\}.$$

Exercise 2.6. $\left[\frac{a}{b}\right] = \left[\frac{a'}{b'}\right] \iff (a,b) \sim (a',b')$.

Proof. Let $x, x' \in [\frac{a}{b}]$. Then, $(x,x') \sim (a,b)$. By transitivity, we know that $(x,x') \sim (a',b')$. Therefore, $(x,x') \in [\frac{a'}{b'}]$. Hence, $[\frac{a}{b}] \subseteq [\frac{a'}{b'}]$. Analogously, $[\frac{a'}{b'}] \subseteq [\frac{a}{b}]$. Therefore, $[\frac{a'}{b'}] = [\frac{a}{b}]$, hence $(a,b) \sim (a',b')$.

QED

Definition 2.7. We define the binary operations addition and multiplication on $\mathbb{Q}$ as follows. If $\left[\frac{a}{b}\right]$ and $\left[\frac{c}{d}\right]$ are in $\mathbb{Q}$, then:

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{ad+bc}{bd}\right]$$

$$\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{ac}{bd}\right].$$

We use the notation $+_{\mathbb{Q}}$ and $\cdot_{\mathbb{Q}}$ to represent addition and multiplication in $\mathbb{Q}$ so as to distinguish these operations from the usual addition $(+)$ and multiplication $(\cdot)$ in $\mathbb{Z}$.

We want to know that addition and multiplication are "well-defined", by which we mean that if we change the representatives of the classes $\left[\frac{a}{b}\right]$ and $\left[\frac{c}{d}\right]$, then does this not change the resulting classes on the right-hand-side of the equalities in the definition. To prove the next theorem first you will need to formulate a precise statement about what needs to be checked.

Theorem 2.8. *Addition and multiplication in $\mathbb{Q}$ are well-defined.*

Proof for Addition. We want to show that if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$ then $[\frac{ad+bc}{bd}] = [\frac{a'd'+b'c'}{b'd'}]$. Hence, we want to show that $(ad+bc)b'd' = (a'd'+b'c')bd$.

The LHS above is equivalent to $adb'd' + bcb'd' = ab'dd' + cd'bb' = a'bdd' + c'dbb' = (a'd' + b'c')bd$ = RHS. This completes the proof.

QED

Proof for Multiplication. We want to show that if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$ then $[\frac{ac}{bd}] = [\frac{a'c'}{b'd'}]$. Hence, we want to show that $acb'd' = a'c'bd$.

The LHS above is equivalent to $acb'd' = a'bcd' = a'bc'd = a'c'bd$ = RHS. This completes the proof.

QED

Theorem 2.9. *a)* **Commutativity of addition**

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{c}{d}\right] +_{\mathbb{Q}} \left[\frac{a}{b}\right] \textit{ for all } \left[\frac{a}{b}\right], \left[\frac{c}{d}\right] \in \mathbb{Q}.$$

Proof. LHS $= [\frac{a}{b}] + [\frac{c}{d}] = [\frac{ad+bc}{bd}]$.

RHS $= [\frac{c}{d}] + [\frac{a}{b}] = [\frac{cb+da}{db}]$. Hence, LHS = RHS.

QED

b) **Associativity of addditon**

$$\left(\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right]\right) +_{\mathbb{Q}} \left[\frac{e}{f}\right] = \left[\frac{a}{b}\right] +_{\mathbb{Q}} \left(\left[\frac{c}{d}\right] +_{\mathbb{Q}} \left[\frac{e}{f}\right]\right) \textit{ for all } \left[\frac{a}{b}\right], \left[\frac{c}{d}\right], \left[\frac{e}{f}\right] \in \mathbb{Q}.$$

Proof. LHS $= [\frac{ad+bc}{bd}] + [\frac{e}{f}] = [\frac{adf+bcf+bde}{bdf}] = [\frac{a}{b}] + [\frac{cf+de}{df}] =$ RHS.

QED

c) **Existence of an additive identity**

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{0}{1}\right] = \left[\frac{a}{b}\right], \textit{ for all } \left[\frac{a}{b}\right] \in \mathbb{Q}.$$

Proof. LHS $= [\frac{a \cdot 1 + b \cdot 0}{b \cdot 1}] = [\frac{a}{b}] =$ RHS.

QED

d) **Existence of additive inverses**

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{-a}{b}\right] = \left[\frac{0}{1}\right], \textit{ for all } \left[\frac{a}{b}\right] \in \mathbb{Q}.$$

Proof. LHS $= [\frac{a \cdot b + (-a) \cdot b}{b \cdot b}] = [\frac{ab-ab}{b^2}] = [\frac{0}{b^2}] = [\frac{0}{1}] =$ RHS.

QED

e) **Commutativity of multiplication**

$$\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{c}{d}\right] \cdot_{\mathbb{Q}} \left[\frac{a}{b}\right] \textit{ for all } \left[\frac{a}{b}\right], \left[\frac{c}{d}\right] \in \mathbb{Q}.$$

Proof. LHS $= [\frac{a \cdot c}{b \cdot d}] = [\frac{c \cdot a}{d \cdot b}] =$ RHS by commutativity of multiplication in $\mathbb{Z}$.

QED

f) **Associativity of multiplication**

$$\left(\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{c}{d}\right]\right) \cdot_{\mathbb{Q}} \left[\frac{e}{f}\right] = \left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left(\left[\frac{c}{d}\right] \cdot_{\mathbb{Q}} \left[\frac{e}{f}\right]\right) \textit{ for all } \left[\frac{a}{b}\right], \left[\frac{c}{d}\right], \left[\frac{e}{f}\right] \in \mathbb{Q}.$$

Proof. LHS $= [\frac{a \cdot c}{b \cdot d}] \cdot [\frac{e}{f}] = [\frac{a \cdot c \cdot e}{b \cdot d \cdot f}] = [\frac{c \cdot e}{d \cdot f}] \cdot [\frac{a}{b}] =$ RHS.

QED

g) **Existence of a multiplicative identity**

$$\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{1}{1}\right] = \left[\frac{a}{b}\right], \textit{ for all } \left[\frac{a}{b}\right] \in \mathbb{Q}.$$

Proof. LHS $= [\frac{a \cdot 1}{b \cdot 1}] = [\frac{a}{b}] =$ RHS.

QED

h) **Existence of multiplicative inverses for nonzero elements**

$$\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{b}{a}\right] = \left[\frac{1}{1}\right], \textit{ for all } \left[\frac{a}{b}\right] \in \mathbb{Q} \textit{ such that } \left[\frac{a}{b}\right] \neq \left[\frac{0}{1}\right].$$

Proof. LHS $= [\frac{a \cdot b}{b \cdot a}] = [\frac{1}{1}] =$ RHS, since $(ab)(1) = (ba)(1)$, hence $(ab, ba) \sim (1, 1)$.

QED

i) **Distributivity**

$$\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left(\left[\frac{c}{d}\right] +_{\mathbb{Q}} \left[\frac{e}{f}\right]\right) = \left(\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{c}{d}\right]\right) +_{\mathbb{Q}} \left(\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{e}{f}\right]\right), \textit{ for all } \left[\frac{a}{b}\right], \left[\frac{c}{d}\right], \left[\frac{e}{f}\right] \in \mathbb{Q}.$$

Proof. LHS $= [\frac{a}{b}] \cdot [\frac{cf+de}{df}] = [\frac{acf+ade}{bdf}] = [\frac{acbf+aebd}{bdbf}] = [\frac{ac}{bd}] + [\frac{ae}{bf}] =$ RHS.

QED

Theorem 2.10. $\mathbb{Q}$ *is countable.*

Hint: look back at Script 1

Proof. Let $f \colon \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ be defined as $f([\frac{a}{b}]) = (a, b)$, for $[\frac{a}{b}] \in \mathbb{Q}$, where $a \in \mathbb{Z}$, $b \in \mathbb{N}$.
We know that $\mathbb{Z} \times \mathbb{N}$ is countable, hence it suffices to show that f is injective.
Suppose $(a, b) \not\sim (a', b')$, i.e., $ab' \neq a'b$.
Case 1: $a = a', b \neq b'$. Then, $f([\frac{a}{b}]) = (a, b) \neq (a, b') = f([\frac{a'}{b'}])$.
Case 2: $a \neq a', b = b'$. Then, $f([\frac{a}{b}]) = (a, b) \neq (a', b) = f([\frac{a'}{b'}])$.
Case 3: $a \neq a', b \neq b'$. Then, $f([\frac{a}{b}]) = (a, b) \neq (a', b') = f([\frac{a'}{b'}])$, since we know that $ab' \neq a'b$.

QED

We will now lose the equivalence class notation and simply refer to elements of $\mathbb{Q}$ as usual. So for example, if we refer to 0, we really mean $[\frac{0}{1}]$, but this distinction should no longer be necessary or relevant.

Additional Exercises

In all exercises you are expected to prove your answer, unless explicitly stated otherwise.

1. Prove that for every $q = \left[\frac{a}{b}\right] \in \mathbb{Q}$, there is exactly one element $(a_0, b_0) \in q$ such that a_0 and b_0 have no common factors and $b_0 > 0$.

Proof. First, we want to show that there is at least one such element.

If $b < 0$, then we know that $q = [\frac{a'}{b'}]$ for some $a' \in \mathbb{Z}, b' \in \mathbb{N}$.

Next, if $gcd(a', b') = 1$, then we are done. If $gcd(a', b') = m > 1$, then let $a_0 = \frac{a'}{m}$ and let $b_0 = \frac{b'}{m}$.

We know that $a_0 \cdot b' = \frac{a'b'}{m} = b_0 \cdot a'$, hence $(a_0, b_0) \sim (a', b')$. Therefore, we can write $q = [\frac{a_0}{b_0}]$.

Second, we want to show that there is at most one such element.

Suppose there are two such elements, (a_0, b_0) and (a_1, b_1) such that $a_0 \neq a_1, b_0 \neq b_1$. Since $gcd(a_0, b_0) = gcd(a_1, b_1) = 1$, this implies $a_0 \cdot b_1 \neq a_1 \cdot b_0$. Then, by our initial assumption, $q = [\frac{a_0}{b_0}]$ and $q = [\frac{a_1}{b_1}]$, hence $(a_0, b_0) \sim (a_1, b_1)$, which implies $a_0 \cdot b_1 = a_1 \cdot b_0$ and this is a contradiction.

QED

2. Let $\sim$ be an equivalence relation on X. Let $[x]$ be the equivalence class of x. Show that

   (a) $[x] = [y]$ if and only if $x \sim y$

   (b) for all $x, y \in X$,
   $$[x] \cap [y] = \begin{cases} [x] = [y] & \text{if } x \sim y \\ \emptyset & \text{if } x \not\sim y. \end{cases}$$

   (c) $X = \bigcup_{x \in X} [x]$

3. For the equivalence relations found in Exercise 2.2, what are the equivalence classes?