MATH 161, Autumn 2024
SCRIPT 4: The Topology of a Continuum

In this sheet we give a continuum C a topology. Roughly speaking, this is a way to describe how the points of C are 'glued together'.

Definition 4.1. A subset of a continuum is *closed* if it contains all of its limit points.

Theorem 4.2. *The sets $\emptyset$ and C are closed.*

Proof. $LP(\emptyset) = \emptyset \subset \emptyset$, hence $\emptyset$ contains all of its limit points, hence it is closed.

$LP(C) = \{p \in C \mid \forall R : R \cap C \setminus \{p\} \neq \emptyset\} \subset C$, hence C contains all of its limit points, hence it is closed.

QED

Theorem 4.3. *A subset of C containing a finite number of points is closed.*

Proof. Let A be a subset of C containing a finite number of points. Then, we know that $LP(A) = \emptyset$, hence $\emptyset \subset A$, hence A is closed.

QED

Definition 4.4. Let X be a subset of C. The *closure* of X is the subset $\overline{X}$ of C defined by:

$$\overline{X} = X \cup LP(X).$$

Theorem 4.5. *$X \subset C$ is closed if and only if $X = \overline{X}$.*

Proof. First, we want to show that if $X = \overline{X}$, then X is closed. We have that $X = X \cup LP(X)$, hence for all $p \in LP(X), p \in X$, hence $LP(X) \subset X$, hence X is closed.

Next, we want to show that if X is closed, then $X = \overline{X}$. We have that $LP(X) \subset X$, hence $\overline{X} = X \cup LP(X) = X$.

QED

Theorem 4.6. *Let $X \subset C$. Then $\overline{X}$ is closed. (Equivalently, $\overline{X} = \overline{\overline{X}}$.)*

Proof. It suffices to show that $X = \overline{X}$.

First, we want to show that $\overline{X} \subset \overline{\overline{X}}$. We know that $\overline{X} \subset \overline{X} \cup LP(\overline{X})$. This completes the first containment.

Next, we want to show that $\overline{\overline{X}} \subset \overline{X}$. Hence, we want to show that $\overline{X} \cup LP(\overline{X}) \subset \overline{X}$, i.e., $\overline{X} \cup LP(X \cup LP(X)) \subset \overline{X}$, i.e., $\overline{X} \cup LP(X) \cup LP(LP(X)) \subset \overline{X}$. Hence, it suffices to show that $LP(LP(X) \subset LP(X)$.

Let $x \in LP(LP(X))$, then we want to show that $x \in LP(X)$. Let R be a region such that $x \in R$, then we want to show that $R \cap X \setminus \{x\} = \emptyset$.

Since $x \in LP(LP(X))$, we know that $R \cap LP(X) \setminus \{x\} \neq \emptyset$. Then let $y \in R \cap LP(X) \setminus \{x\}$, i.e., $y \neq x, y \in R, y \in LP(X)$.

Then, there exist infinitely many $z_i \neq y$ such that $z_i \in X \cap R$, hence there is some $\tilde{z} \neq x$ such that $\tilde{z} \in R \cap X$. Hence, $R \cap X \setminus \{x\} \neq \emptyset$. Hence, $x \in LP(X)$.

This completes the proof.

QED

Definition 4.7. A subset G of a continuum C is *open* if its complement $C \setminus G$ is closed.

Theorem 4.8. *The sets $\emptyset$ and C are open.*

Proof. $C \setminus \emptyset = C$, and we know from Theorem 4.2 that C is closed, hence $\emptyset$ is open.

$C \setminus C = \emptyset$, and we know from Theorem 4.2 that $\emptyset$ is closed, hence C is open.

QED

The following is a very useful criterion to determine whether a set of points is open.

Theorem 4.9. *Let $G \subset C$. Then G is open if and only if for all $x \in G$, there exists a region R such that $x \in R \subset G$.*

Proof. First, we want to show that if G is open, then for all $x \in G$, there exists a region R such that $x \in R \subset G$. Let $x \in G$, then we want to construct a region R such that $x \in R \subset G$.

We know that G is open, hence $C \setminus G$ is closed. Hence $x \notin LP(C \setminus G)$. Hence, there exists a region R such that $x \in R$ and $R \cap (C \setminus G) \setminus \{x\} = \emptyset$.

We know that $x \in G$, hence $x \notin C \setminus G$, hence $(C \setminus G) \setminus x = C \setminus G$. Therefore, we know that there exists a region R such that $x \in R$ and $R \cap C \setminus G = \emptyset$.

Hence, for all $r \in R, r \notin C \setminus G, r \in G$. Therefore, $R \subset G$.

Next, we want to show that if there exists a region R such that $x \in R \subset G$ for all $x \in G$, then G is open. We want to show that $LP(C \setminus G) \subset (C \setminus G)$. Hence, it suffices to show that for all $x \in G, x \notin LP(C \setminus G)$.

Suppose there is some $g \in G$ such that $g \in LP(C \setminus G)$. This implies that for all regions R such that $g \in R$, $R \cap (C \setminus G) \setminus \{g\} \neq \emptyset$.

We know that there exists a region R_g such that $g \in R_g \subset G$. Hence, $R_g \cap (C \setminus G) = \emptyset$. Then, we have constructed a region, namely R_g, such that $g \in R_g$ and $R_g \cap (C \setminus G) \setminus \{g\} = \emptyset$. This is a contradiction.

This completes the proof.

QED

Corollary 4.10. *Every region R is open. Every complement of a region, $C \setminus R$, is closed.*

Proof by Contradiction. We want to show that $LP(C \setminus R) \subset C \setminus R$, i.e., we want to show that $\forall p \in LP(C \setminus R) : p \in C \setminus R$.

Suppose, for sake of contradiction, that there is some $x \in LP(C \setminus R)$ such that $x \notin C \setminus R$. Since $x \in C, x \notin C \setminus R$, hence $x \in R$. Also, since $x \notin C \setminus R$, hence $C \setminus R \setminus \{x\} = C \setminus R$.

Then, R is a region such that $x \in R$ and $R \cap (C \setminus R) \setminus \{x\} = R \cap (C \setminus R) = \emptyset$. This means $x \notin LP(C \setminus R)$, and this is a contradiction.

QED

Proof by Theorem 4.9. We want to show that for all $x \in R$, there exists a region R' such that $x \in R' \subseteq R$. We let $R' = R$ and this completes the proof.

QED

Corollary 4.11. *Let $G \subset C$. Then G is open if and only if for all $x \in G$, there exists a subset $V \subset G$ such that $x \in V$ and V is open.*

Proof. In the forward direction, we want to show that if G is open, then for all $x \in G$ there exists $V \subset G$ such that $x \in V$ and V is open. We know that there exists a region $R \subset G$ such that $x \in R$. Every region is open, hence let $V = R$ and this completes the forward direction.

In the backward direction, let $x \in G$, then by assumption we know that there exists $V \subset G$ such that V is open and $x \in V$. Then, by Theorem 4.9, $\exists$ region R such that $x \in R \subset V \subset G$, hence G is open. This completes the backward direction.

QED

Corollary 4.12. *Let $a \in C$. Then the sets $\{x \in C \mid x < a\}$ and $\{x \in C \mid a < x\}$ are open.*

Proof. Let $A = \{x \in C \mid x < a\}$ and let $x \in A$. Since C has no first point, let $y \in A$ such that $y < x < a$. Then, $x \in \underline{ya} \subset A$. Hence, A is open by Theorem 4.9.

Let $B = \{x \in C \mid a < x\}$ and let $x \in B$. Since C has no last point, let $y \in A$ such that $a < x < y$. Then, $x \in \underline{ay} \subset B$. Hence, B is open by Theorem 4.9.

QED

Theorem 4.13. *Let G be a nonempty open set. Then G is the union of a collection of regions.*

Proof. By Theorem 4.9, if G is open, then for all $x \in G$, there exists a region R_x such that $x \in R_x \subset G$.

Then, $G = \bigcup_{x \in G} x \subset \bigcup_{x \in G} R_x = G$.

QED

Exercise 4.14. Do there exist subsets $X \subset C$ that are neither open nor closed?

Proof. Yes. Let $M = \{x \in \mathbb{Q} \mid x > 0 \text{ and } x \leq 1\}$.

Then, $0 \in LP(M), 0 \notin M$, hence M is not closed.

$1 \in LP(C \setminus M), 1 \notin C \setminus M$, hence $C \setminus M$ is not closed, hence M is not open.

QED

Theorem 4.15. *Let $\{X_\lambda\}$ be an arbitrary nonempty collection of closed subsets of a continuum C. Then the intersection $\bigcap_\lambda X_\lambda$ is closed.*

Proof. We prove this using Corollary 4.16 (for which a standalone proof is provided).

Let $G_\lambda = C \setminus X_\lambda$, where $\forall \lambda : G_\lambda$ is open. Then, $\bigcup_\lambda G_\lambda = \bigcup_\lambda (C \setminus X_\lambda) = C \setminus (\bigcap_\lambda X_\lambda)$ is open.

Hence, $\bigcap_\lambda X_\lambda$ is closed.

QED

Corollary 4.16. *Let $\{G_\lambda\}$ be an arbitrary nonempty collection of open subsets of a continuum C. Then the union $\bigcup_\lambda G_\lambda$ is open.*

Proof. Let $x \in \bigcup_\lambda G_\lambda$, then $\exists \lambda_0$ such that $x \in G_{\lambda_0}$. Since by definition G_{λ_0} is open, there exists a region R such that $x \in R, R \subset G_{\lambda_0}$, hence $R \in \bigcup_\lambda G_\lambda$.

Therefore, $\bigcup_\lambda G_\lambda$ is open.

QED

Theorem 4.17. *Let $\{G_1, \ldots, G_n\}$ be a finite nonempty collection of open subsets of a continuum C. Then the intersection $G_1 \cap \cdots \cap G_n$ is open.*

Proof. We prove this using Corollary 4.18 (for which a standalone proof is provided).

We know that for all $i \in [n], G_i$ is open, hence $C \setminus G_i$ is closed.

Then, $C \setminus (G_1 \cap \cdots \cap G_n) = (C \setminus G_1) \cup \cdots (C \setminus G_n)$, which we know is a finite union of closed sets, so it is closed.

Since $C \setminus (G_1 \cap \cdots \cap G_n)$ is closed, $G_1 \cap \cdots \cap G_n$ is open.

QED

Corollary 4.18. *Let* $\{X_1, \ldots, X_n\}$ *be a finite nonempty collection of closed subsets of a continuum* C*. Then the union* $X_1 \cup \cdots \cup X_n$ *is closed.*

Proof. We know that limit points are commutative over finite unions. Hence,

$$LP(\bigcup_{i=1}^{n} X_i) = \bigcup_{i=1}^{n} LP(X_i) \subset \bigcup_{i=1}^{n} X_i.$$

Therefore, $LP(\bigcup_{i=1}^{n} X_i) \subset \bigcup_{i=1}^{n} X_i$, hence $\bigcup_{i=1}^{n} X_i$ is closed.

QED

Exercise 4.19. Is it necessarily the case that the intersection of an infinite number of open sets is open? Is it possible to construct an infinite collection of open sets whose intersection is not open? Equivalently, is it possible to construct an infinite collection of closed sets whose union is not closed?

Proof for Intersections. $C \setminus \left[\bigcap_{n\in\mathbb{N}} \underline{(-\frac{1}{n})(\frac{1}{n})}\right]$ is not closed, since 0 is a limit point that is not contained in the set.

Therefore, $\left[\bigcap_{n\in\mathbb{N}} \underline{(-\frac{1}{n})(\frac{1}{n})}\right]$ is not open, even though $\underline{(-\frac{1}{n})(\frac{1}{n})}$ is a region (i.e., it is open).

QED

Proof for Unions. $\bigcup_{n\in\mathbb{N}} \left[C \setminus \underline{(-\frac{1}{n})(\frac{1}{n})}\right]$ is not closed, even though $C \setminus \underline{(-\frac{1}{n})(\frac{1}{n})}$ is closed for all $n \in \mathbb{N}$.

QED

Theorem 4.13 says that every nonempty open set is the union of a collection of regions. This necessary condition for open sets is also sufficient:

Corollary 4.20. *Let* $G \subset C$ *be nonempty. Then* G *is open if and only if* G *is the union of a collection of regions.*

Proof. For the forward direction, we know by Theorem 4.13 that if G is a nonempty open subset of C, then G is the union of a collection of regions.

For the backward direction, we want to show that if G is the union of a collection of regions, then G is open.

We know that every region is open, and the union of open sets is open, hence G is open.

QED

Corollary 4.21. *If* $\underline{ab}$ *is a region in* C*, then* $\mathrm{ext}\,\underline{ab}$ *is open.*

Proof. $\operatorname{ext}\underline{ab} = \{x \in C \mid x < a\} \cup \{x \in C \mid a < x\}$.

We know by Corollary 4.12 that $\{x \in C \mid x < a\}$ and $\{x \in C \mid a < x\}$ are open, and a union of open sets is open.

Hence, $\operatorname{ext}\underline{ab}$ is open.

QED

Definition 4.22. Let C be a continuum. We say that C is *disconnected* if it may be written as $C = A \cup B$, where A and B are disjoint, non-empty open sets. C is *connected* if it is not disconnected.

Theorem 4.23. *Let C be a connected continuum. Let $x, y \in C$, with $x < y$. Then there exists $z \in C$ such that $x < z < y$. In particular, all regions of a connected continuum are nonempty.*

Proof. Assume there does not exist $z \in C$ such that $x < z < y$. Let $A = \{m \in C \mid m < y\}$ and let $B = \{m \in C \mid m > x\}$.

Then, we want to show that $C = A \cup B$, where A and B are disjoint, nonempty and open.

We know that $A \subset C, B \subset C$, hence $A \cup B \subset C$. Suppose $C \not\subset A \cup B$, i.e., $\exists p \in C$ such that $p \notin A \cup B$. Since $p \notin A, p \geq y$. Since $p \notin B, p \leq x$. Thus, $y \leq p \leq x$, and this contradicts the assumption that $x < y$.

By Corollary 4.12, A and B are open.

Suppose $A \cap B \neq \emptyset$. Then, $\exists q \in C$ such that $q \in A, q \in B$, i.e., $q < y, q > x$, however this contradicts our original assumption. Hence A and B are disjoint.

$x \in A, y \in B$, hence A and B are nonempty.

Hence, if there does not exist $z \in C$ such that $x < z < y$, then C must be disconnected.

This completes the proof.

QED

Exercise 4.24. Let C be a connected continuum and $a \in C$. Prove that $C \setminus \{a\}$ is a disconnected continuum.

Proof. $C \setminus \{a\}$ is a continuum, since it is nonempty, has the same ordering as C, and has no first or last point.

Let $A = \{x \in C \mid x < a\}$ and $B = \{x \in C \mid x > a\}$.

By trichotomy of the ordering, $C \setminus \{a\} = A \cup B$, and A, B are disjoint.

We know that A is open in C. Let $x \in A$, then there exists a region $R = \underline{pq}$ such that $x \in R \subset C$. If $q \neq a$, then we are done.

If $q = a$, then we know that there exists a point s such that $q < s < a$. Hence, $\underline{ps}$ is a region in $C \setminus \{a\}$ that contains x. Hence, A is open in $C \setminus A$.

By analogous reasoning for B, the proof is completed.

QED

Exercise 4.25. Must every realization of a continuum be disconnected? Think about the realizations of the continuum from Exercise 3.26. Are they connected/disconnected?

Proof. $\mathbb{Z}$ and $\mathbb{Q}$ are disconnected. $\mathbb{R}$ is connected.

QED

1. Prove that if $S \subset C$ then

$$\overline{S} = \{x \in C \mid \text{ for all } R \text{ containing } x, R \cap S \neq \emptyset\}.$$

This is the best way to think of closure - in general, splitting into S and its limit points is a very inefficient approach.

Proof. By definition, $\overline{S} = S \cup LP(S)$. Hence, we want to show that $S \cup LP(S) = \{x \in C \mid \text{ for all } R \text{ containing } x, R \cap S \neq \emptyset\}$.

For the first containment, suppose $x \in S \cup LP(S)$.

Case 1: $x \in S$. Then, we know that for every region R containing x, $x \in R \cap S$, hence $R \cap S \neq \emptyset$. Hence, $[S \cup LP(S)] \subset \{x \in C \mid \text{ for all } R \text{ containing } x, R \cap S \neq \emptyset\}$.

Case 2: $x \notin S$. Then, since $x \in S \cup LP(S)$, we know that $x \in LP(S)$. Hence, for every region R containing x, $R \cap S \setminus \{x\} \neq \emptyset$. Since $x \notin S$, we know that $S \setminus \{x\} = S$, hence $R \cap S \neq \emptyset$. Hence, $[S \cup LP(S)] \subset \{x \in C \mid \text{ for all } R \text{ containing } x, R \cap S \neq \emptyset\}$.

For the second containment, suppose $x \in C$ such that for all regions R containing $x, R \cap S \neq \emptyset$.

Case 1: $x \in S$. Then, $x \in S \cup LP(S)$, hence $\{x \in C \mid \text{ for all } R \text{ containing } x, R \cap S \neq \emptyset\} \subset [S \cup LP(S)]$.

Case 2: $x \notin S$, i.e., $x \in C \setminus S$. Then, $S = S \setminus \{x\}$, and since for every region R that contains $x, R \cap S \neq \emptyset$, hence for every region R that contains $x, R \cap S \setminus \{x\} \neq \emptyset$. Hence, $x \in LP(S)$, hence $\{x \in C \mid \text{ for all } R \text{ containing } x, R \cap S \neq \emptyset\} \subset [S \cup LP(S)]$.

This completes the proof.

QED

2. Find an example of a continuum C and a subset that is both open and closed, other than $\emptyset$ and C.

3. Prove that if $X \subset Y$ then $\overline{X} \subset \overline{Y}$. If $\overline{X} = \overline{Y}$, is it necessarily true that $X = Y$?

Proof. We want to show that if $A \subset B$, then $\overline{A} \subset \overline{B}$.

Let $a \in \overline{A}$, then for every region R that contains a, we have that $R \cap A \neq \emptyset$. Since $A \subset B$, then for any particular R, we have that $(R \cap A) \subset (R \cap B)$.

Hence, we know that for every region R that contains a, $R \cap B \neq \emptyset$. Hence, $a \in \overline{B}$. Thus, $\overline{A} \subset \overline{B}$.

It is not necessarily true that if $\overline{X} = \overline{Y}$, then $X = Y$. We present a counterexample.

Let $X = \underline{ab}$ and $Y = C \setminus \{\text{ext}(\underline{ab})\}$. Then, $\overline{X} = \overline{Y} = \underline{ab} \cup \{a\} \cup \{b\}$, but $X \neq Y$.

QED

4. Suppose $A, B \subset C$. Prove that
$$\overline{A \cap B} \subset \overline{A} \cap \overline{B}.$$
Can one expect equality in general? Why or why not?

5. Prove that the set of limit points of a set $A \subset C$ is a closed set.

6. Let $A \subset C$. We say that $x \in A$ is an *interior* point of A if there is a region R such that $x \in R \subset A$. We let $int(A) = \{a \in A \mid a \text{ is an interior point of } A\}$.

   (a) Let $A = \underline{ab}$, for $a, b \in C, a < b$. Find $int(A)$.

   Proof. $C \setminus \{\text{ext}(\underline{ab}) \cup a \cup b\}$. QED

   (b) Show that A is open if, and only if, $A = int(A)$.

   Proof. First, we want to show that if $A = int(A)$, then A is open. A standalone proof that $int(A)$ is open is given in part (c) below, hence if $A = int(A)$, then A is open.

   Second, we want to show that if A is open, then $A = int(A)$. If A is open, we know that for all $a \in A$, there exists a region $S_a \subset A$ such that $a \in S_a$. Since $S_a \subset A$, we know that for any $s \in S_a, s \notin \text{ext}(\underline{ab}), s \neq a, s \neq b$. Hence, $A = int(A)$.

   QED

   (c) Show that $int(A)$ is open.

   Proof. Since $int(A) = C \setminus \{\text{ext}(\underline{ab}) \cup \{a\} \cup \{b\}\}$, it suffices to show that $\{\text{ext}(\underline{ab}) \cup \{a\} \cup \{b\}\}$ is closed.

   Let $p \in LP(\text{ext}(\underline{ab}) \cup a \cup b)$, then we want to show that $p \in \{\text{ext}(\underline{ab}) \cup a \cup b\}$.

   By commutativity of limit points over finite unions, we know that $p \in LP(\text{ext}(\underline{ab})) \cup LP(a) \cup LP(b)$. However, $LP(a) = LP(b) = \emptyset$, hence $p \in LP(\text{ext}(\underline{ab}))$.

   If $LP(\text{ext}(\underline{ab})) = \emptyset$, then $\{\text{ext}(\underline{ab}) \cup a \cup b\}$ is closed, and we are done. Else, we know that $\text{ext}(\underline{ab})$ is open, hence $p \notin \text{ext}(\underline{ab})$.

   Assume $p \in \underline{ab}$. Then, there exists a region R, namely $R = \underline{ab}$ such that $R \cap (\text{ext}(\underline{ab}) \cup a \cup b) = \emptyset$. This is a contradiction. Hence, $p \in \{a, b\}$. Therefore, $LP(\text{ext}(\underline{ab}) \cup a \cup b) \subset \{a, b\} \subset \{\text{ext}(\underline{ab}) \cup a \cup b\}$, hence $\{\text{ext}(\underline{ab}) \cup a \cup b\}$ is closed.

   QED

7. Let A, B be subsets of C. Either prove or give a counterexample to each of the following:
$$\begin{aligned} int(A \cap B) &= int(A) \cap int(B) \\ int(A \cup B) &= int(A) \cup int(B). \end{aligned}$$

   Proof for Intersections. Let $x \in int(A \cap B)$. Then, by definition of interior, there exists a region $R \subset A \cap B$ such that $x \in R$.

   Since $A \cap B \subset A$, we know that $R \subset A$, hence there exists a region, namely R, such that $x \in R \subset A$. Hence, $x \in int(A)$.

Since $A \cap B \subset B$, we know that $R \subset B$, hence there exists a region, namely R, such that $x \in R \subset B$. Hence, $x \in int(B)$.

Thus, $x \in int(A) \cap int(B)$. This completes the first containment, i.e., $int(A \cap B) \subset [int(A) \cap int(B)]$.

Let $y \in int(A) \cap int(B)$. There exist regions $R \subset A$ and $R_0 \subset B$ such that $y \in R, y \in R_0$. Then, $S = R \cap R_0$ is a region such that $y \in S$ and $S \subset A \cap B$. Hence, $y \in int(A \cap B)$.

This completes the second containment, i.e., $[int(A) \cap int(B)] \subset int(A \cap B)$.

This completes the proof.

QED

Counter-Example for Unions. Let $A, B \subset \mathbb{Q}$ such that $A = \underline{01} \cup \{1\}$ and $B = \underline{12}$.

Then, $A \cup B = \underline{02}$, hence $1 \in int(A \cup B), 1 \notin int(A), 1 \notin int(B)$.

QED

8. Let C be a continuum.

   (a) Show that if A, B are subsets of C such that $A \subset B$ then, if B is closed, $\overline{A} \subset B$.

   Proof. B is closed, hence $B = \overline{B}$, hence it suffices to show that if $A \subset B$, then $\overline{A} \subset \overline{B}$.
   Let $a \in \overline{A}$, then for every region R that contains a, we have that $R \cap A \neq \emptyset$. Since $A \subset B$, then for any particular R, we have that $(R \cap A) \subset (R \cap B)$.
   Hence, we know that for every region R that contains a, $R \cap B \neq \emptyset$. Hence, $a \in \overline{B}$. Thus, $\overline{A} \subset \overline{B}$.

   QED

   (b) Show that if A, B are subsets of C such that $A \subset B$ then, if A is open, $A \subset int(B)$.

   Proof. If A is open, then for every $a \in A$, there exists a region $R \subset A$ such that $a \in R$. Since $R \subset A$ and $A \subset B$, hence $R \subset B$.
   Then, for every $a \in A$, there exists a region $R \subset B$ such that $a \in R$.
   Hence, $A \subset int(B)$.

   QED

   (c) Let $A \subset C$ and $\mathcal{F}_{\mathcal{A}} = \{B \subset C \mid B \text{ is a closed set containing } A\}$. Show that $\overline{A} = \bigcap_{B \in \mathcal{F}_{\mathcal{A}}} B$.

   Proof. For the first containment, let $a \in \overline{A}$ and let $B \subset C$ be such that $A \subset B$ and B is closed.
   Case 1: $a \in A$, then $a \in B$, hence $a \in \bigcap_{B \in \mathcal{F}_{\mathcal{A}}} B$.
   Case 2: $a \notin A$, i.e., $a \in LP(A)$. Since $A \subset B$, $LP(A) \subset LP(B)$, hence $a \in LP(B)$. Since B is closed, $LP(B) \subset B$, hence $a \in \bigcap_{B \in \mathcal{F}_{\mathcal{A}}} B$.
   This completes the first containment, i.e., $\overline{A} \subset \bigcap_{B \in \mathcal{F}_{\mathcal{A}}} B$.

For the second containment, let $b \in \bigcap_{B \in \mathcal{F}_A} B$. Then, we know that $\overline{A}$ is a closed set containing A, hence $A \in \mathcal{F}_A$, hence $b \in \overline{A}$.

This completes the second containment, i.e., $\bigcap_{B \in \mathcal{F}_A} B \subset \overline{A}$.

This completes the proof.

QED

(d) Formulate and prove a result analogous to c) for $int(A)$.

Proof. We want to show that $int(A) = \bigcup_{B \in \mathcal{F}_A} B$, where $\mathcal{F}_A = \{B \subset C \mid B \subset A \text{ and } B \text{ is open}\}$.

$int(A)$ is an open set contained in A, hence if $x \in int(A)$, then $x \in \bigcup_{B \in \mathcal{F}_A} B$. This completes the first containment, i.e., $int(A) \subset \bigcup_{B \in \mathcal{F}_A} B$.

If $x \in \bigcup_{B \in \mathcal{F}_A} B$, then there exists at least one $B \in \mathcal{F}_A$ such that $x \in B$. Let this be B_0. Since B_0 is open, it is a union of regions, hence there exists some region R such that $x \in R \subset B_0 \subset A$, hence $x \in int(A)$. This completes the second containment, i.e., $\bigcup_{B \in \mathcal{F}_A} B \subset int(A)$.

QED

9. Prove that

   i) $\{0\}$ is closed but not open in $\mathbb{Q}$.

   ii) Prove that <u>01</u> is open but not closed in $\mathbb{Q}$.

10. Consider the set $X = \{x \in \mathbb{Q} \mid x \leq 0\} \cup \{x \in \mathbb{Q} \mid 0 < x \text{ and } x^2 \leq 2\}$. Determine whether this set is open/closed/both/neither in $\mathbb{Q}$.

Proof. We want to show that there does not exist $x \in \mathbb{Q}$ such that $x^2 = 2$. Suppose $x \in \mathbb{Q}$ such that $x^2 = 2$.

Then, we can write $x = \frac{p}{q}$, where $p \in \mathbb{Z}, q \in \mathbb{N}$ such that q is as small as possible, i.e., $\gcd(p, q) = 1$. Since $x > 1$, we know that $p > q$.

Then, $x^2 = \frac{p^2}{q^2} = 2$. Hence, $p^2 = 2 \cdot q^2$. Since the RHS is even, the LHS must be even. Since p^2 must be even, p must be even, hence we can write $p = 2 \cdot k$.

Then, $(2 \cdot k)^2 = 4 \cdot k^2 = 2 \cdot q^2$. Hence, q^2 must also be even, hence q is also even, but both p and q are even, contradicting the assumption that they were coprime.

First, we want to show that X is open. It suffices to show that for every point $x \in X$, there exists a region contained in X that contains x. This is equivalent to finding a point $y \in X$ such that $y > x$. If $x \leq 0$, then $1 \in X, 1 > x$.

If $x > 0$, we can write $x = \frac{a}{b}$, with $a, b > 0$ and $\frac{a^2}{b^2} < 2$. This implies that $a^2 \leq 2b^2 - 1$. Then, $a \geq 1$. Suppose $y = \frac{a + \frac{1}{4a}}{b}$. Then, we want to show that $\frac{(a + \frac{1}{4a})^2}{b^2} < 2$. This is equivalent to showing that $(a + \frac{1}{4a})^2 < 2b^2$. This is equivalent to showing that $a^2 < 2b^2 - \frac{1}{2} - \frac{1}{16a^2}$. It suffices to show that $\frac{1}{2} + \frac{1}{16a^2} < 1$. We know that $a \geq 1$, so $\frac{1}{16a^2} \leq \frac{1}{16}$. This proves that X is open, i.e., $\mathbb{Q} \setminus X$ is closed.

Similarly, for $\mathbb{Q} \setminus X$, we want to show that for every $x \in \mathbb{Q} \setminus X$, there exists some $y \in \mathbb{Q} \setminus X$ such that $y < x$. Consider $y = \frac{a - \frac{1}{4a}}{b}$. We want to show that $(a - \frac{1}{4a})^2 > 2b^2$. Expanding, we want to show that $a^2 + \frac{1}{16a^2} - \frac{1}{2} > 2b^2$. Analogously to the above, we have that $a^2 + 1 \geq 2b^2$. Hence, we want to show that $a^2 - 2b^2 > \frac{1}{2} - \frac{1}{16a^2}$. We know that $LHS \geq 1$, and it is therefore trivial that $\frac{1}{2} - \frac{1}{16a^2} < 1$. This proves that $\mathbb{Q} \setminus X$ is open, i.e., X is closed.

Since X is clopen and $X \neq \emptyset$, $X \neq \mathbb{Q}$, this is how we know that $\mathbb{Q}$ is disconnected.

QED

11. Let $R = \underline{ab}$ be a region in C. Must it hold that $\overline{R} = \{x \in C \mid a \leq x \leq b\}$? Either prove or find a counterexample.

12. For each part, give an *explanation*, but you do not need to give a full proof.

    (a) Give an example of a continuum C and a non-empty, closed set $A \subset C$ with no limit points.

    (b) Give an example of a continuum C and a non-empty, closed set $A \subset C$ with exactly one limit point.

    (c) Give an example of a continuum C and a non-empty, closed set $A \subset C$ with a countably infinite set of limit points.

    (d) Give an example of a continuum C and a non-empty, open set $A \subset C$ with no limit points.

    (e) Give an example of a continuum C and a non-empty, open set $A \subset C$ with exactly one limit point.[1]

    (f) Give an example of a continuum C and a non-empty, open set $A \subset C$ with a countably infinite set of limit points.

13. Let $X \subset C$. Then if $A \subset X$, we use $\overline{A}^X$ to denote the closure of A in X, i.e.

$$\overline{A}^X = A \cup \{x \in X \mid x \text{ is a limit point of A}\} = (\overline{A} \cap X).$$

We say that A is *closed in* X if $\{x \in X \mid x \text{ is a limit point of A}\} \subset A$. (Thus A is closed in X if it contains all its limit points in X.) We say that A is *open in* X if its complement is closed in X.

We define X to be disconnected if there are non-empty disjoint sets $A, B \subset X$ that are *open in* X such that $X = A \cup B$.

    (a) Prove that X is disconnected if, and only if, there are non-empty subsets A and B such that $X = A \cup B$ and $\overline{A}^X \cap B = \emptyset, \overline{B}^X \cap A = \emptyset$.

    (b) Prove that X is disconnected if, and only if, there are non-empty disjoint sets $A, B \subset X$ that are *closed in* X such that $X = A \cup B$.

[1] Hint: There are no such examples if $C = \mathbb{Q}$ or $\mathbb{Z}$ (think about why). However, one may find an example by looking at a well-chosen subset of $\mathbb{Q}$.