MATH 16110/50 – Honors Calculus I (IBL)
Midterm Exam

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Student: ______________________________

Instructions: This exam is worth a total of 30 points. However, you will find that the problems add up to a total of 51 points. This means that to get the maximum score you do not need to solve all problems. You should still solve as many questions as possible so that even if some of your solutions are wrong, you can still get maximum score. The more symbols * an item has, the harder it is. The symbol ^{H} indicates that a hint is available for the item on the final page of the exam.

You are allowed to use any result from class or your homework except when the problem is explicitly asking you to solve that specific result from class or your homework. You are also allowed to use standard facts about natural numbers and integers. You can (and should) use other items from this exam to prove your current item (even if you did not solve the item you are using) as long as you do not form a dependency loop (e.g., if you used 2c in your proof of 2d, then you are not allowed to use 2d in your proof of 2c).

1 Injectivity and surjectivity [17 points]

Let $f\colon A \to B$ be a function.

a) [1 point] For $X \subseteq A$, state the definition of $f(X)$ (the image of X under f).

b) [1 point] For $Y \subseteq B$, state the definition of $f^{-1}(Y)$ (the preimage of Y under f).

c) [1 point] State the definitions of "injective" and "surjective".

d) [2 points] Prove that for every $Y \subseteq B$, we have $f(f^{-1}(Y)) \subseteq Y$.

e) [4 points] Prove that f is surjective if and only if for every $Y \subseteq B$, we have $f(f^{-1}(Y)) = Y$.

f) [2 points] Prove that for every $X \subseteq A$, we have $f^{-1}(f(X)) \supseteq X$.

g) [6 points^H] Prove that f is injective if and only if for every $X \subseteq A$, we have $f^{-1}(f(X)) = X$.

2 Uncountable sets [17 points]

a) [1 point] State the definition of "uncountable".

b) [5 points^H] Recall that the *power set of a set* A is the set

$$\mathcal{P}(A) \stackrel{\text{def}}{=} \{B \subseteq A\}$$

of all subsets of A. Prove that for every set A, there does *not* exist a surjection $f\colon A \to \mathcal{P}(A)$

c) [4 points^H] Let $\{0,1\}^{\mathbb{N}}$ be the set of all functions of the form $f\colon \mathbb{N} \to \{0,1\}$. Prove that $\{0,1\}^{\mathbb{N}}$ is uncountable.

d) [7 points^{*H}] For $n \in \mathbb{N}$, let $[n]^{\mathbb{N}}$ be the set of all functions of the form $f\colon \mathbb{N} \to [n]$. Prove that if $n \geq 2$, then $[n]^{\mathbb{N}}$ is uncountable.

3 Ultrafilters [17 points]

Recall that for a set X, the power set $\mathcal{P}(X) \stackrel{\text{def}}{=} \{A \subseteq X\}$ of X is the set of subsets of X.

An *ultrafilter* on X is a set $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying the following:

Non-triviality: $\mathcal{F}$ is non-empty and $\varnothing \notin \mathcal{F}$.

Closure under intersections: If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Upward closure: If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.

Maximality: For every $A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

For $x \in X$, we define

$$\mathcal{F}_x \stackrel{\text{def}}{=} \{A \subseteq X \mid x \in A\}.$$

a) [1 point] State the definition of "$A \subseteq B$".

b) [4 points] Prove that if X is a non-empty set and $x \in X$, then $\mathcal{F}_x$ is an ultrafilter on X.

c) [6 points^{*H}] Suppose $\mathcal{F}$ is an ultrafilter on $\mathbb{N}$ such that for every $n \in \mathbb{N}$, the set $\{m \in \mathbb{N} \mid m > n\}$ is an element of $\mathcal{F}$. Prove that for every $x \in \mathbb{N}$, we have $\mathcal{F} \neq \mathcal{F}_x$.

d) [6 points^{**H}] Let $\mathcal{F}$ be an ultrafilter on X. Prove that there exists a finite set A in $\mathcal{F}$ if and only if there exists $x \in X$ such that $\mathcal{F} = \mathcal{F}_x$.

Hints

Problem 1(g): Prove both directions by their contra-positive, that is, prove that f is *not* injective if and only if there exists $X \subseteq A$ such that $f^{-1}(f(X)) \neq X$.

Problem 2(b): Suppose toward a contradiction that $f\colon A \to \mathcal{P}(A)$ is a surjection and consider the set $B \stackrel{\text{def}}{=} \{a \in A \mid a \notin f(a)\}$.

Problem 2(c): Construct a bijection between $\{0,1\}^{\mathbb{N}}$ and $\mathcal{P}(\mathbb{N})$ and use item (b).

Problem 2(d): Recall that we saw in class that if $f\colon A \to B$ is an injection and B is countable, then A is countable. When $n \geq 2$, construct an injection $\{0,1\}^{\mathbb{N}} \to [n]^{\mathbb{N}}$.

Problem 3(c): What is the smallest set in $\mathcal{F}_x$? What happens when you intersect it with other sets?

Problem 3(d): For the hard direction, try an induction on the cardinality of A.