The objective of this advanced topic is to prove the Schröder–Bernstein Theorem (without using the Axiom of Choice). This theorem makes precise the intuition that when there exists an injection from A to B, then the cardinality (size) of A is less or equal than that of B (even when they are potentially infinite sets) by showing that if there exist injections from A to B and from B to A, then there must exist a bijection between the sets.

Theorem 0.1 (Schröder–Bernstein)**.** *Let A and B be sets and suppose there exist injections $f\colon A \to B$ and $g\colon B \to A$. Then there exists a bijection $h\colon A \to B$.*

Here is the step-by-step to prove Theorem 0.1: throughout this section, we fix A, B, f and g as in the statement of the theorem. Furthermore, we will make a slight abuse of notation and denote by $f^{-1}\colon f(A) \to A$ and $g^{-1}\colon g(B) \to B$ the inverse functions of f and g when we restrict their codomains to their image (after this restriction, the resulting functions are bijective).

Define a sequence of sets $(C_n)_{n\in\mathbb{N}}$ inductively as follows. Let $C_1 \overset{\text{def}}{=} A \setminus g(B)$ and for every $n \in \mathbb{N}$, let $C_{n+1} \overset{\text{def}}{=} g(f(C_n))$. We also let

$$C \overset{\text{def}}{=} \bigcup_{n\in\mathbb{N}} C_n, \qquad\qquad D \overset{\text{def}}{=} f(C).$$

Define $h\colon A \to B$ by

$$h(a) \overset{\text{def}}{=} \begin{cases} f(a), & \text{if } a \in C,\\ g^{-1}(a), & \text{if } a \notin C.\end{cases}$$

Lemma 0.2. *h is well-defined, that is, if $a \in A \setminus C$, then $a \in g(B)$ (i.e., $A \setminus C \subseteq g(B)$).*

Proof. If $a \in A \setminus C$, we have that $a \in A, a \notin C$. Hence, we know that $a \in A, a \notin \bigcup_{n\in\mathbb{N}} C_n$, thus $a \notin C_1$. Hence, $a \notin A \setminus g(B)$, but $a \in A$, hence $a \in g(B)$.

QED

Lemma 0.3. *We have $g^{-1}(A \setminus C) = B \setminus D$. (Hint: it might be easier to prove the equivalent statement $B \setminus g^{-1}(A \setminus C) = D$.)*

Proof. We prove this by double containment.

Below is the first containment.

Let $x \in f(C)$, then since $f : A \to B$, we know that $x \in B$. Then, we want to show that $x \notin g^{-1}(A \setminus C)$. We know that $x \in f(C)$, hence $x \in f(C_i)$ for some $i \in \mathbb{N}$. Then, $C_{i+1} = g(f(C_i))$, hence $g(x) \in C_{i+1}$, hence $g(x) \in C$. Thus, $g(x) \notin A \setminus C$, hence $x \notin g^{-1}(A \setminus C)$.

Below is the second containment.

Let $x \in B$ such that $x \notin g^{-1}(A \setminus C)$, then we want to show that $x \in f(C)$. We have that $g(x) \notin A \setminus C$, hence $g(x) \in C$. Then, there exists C_i such that $g(x) \in C_i$ for some $i \in \mathbb{N}$.

If $i = 1$, then $g(x) \in C_1$, i.e., $g(x) \in A \setminus g(B)$ and this is a contradiction. Hence we know that $i > 1$.

Then, $g(x) \in C_i$ and $C_i = g(f(C_{i-1}))$, hence $g(x) \in g(f(C_{i-1}))$. Since g is injective, we have that $x \in f(C_{i-1})$, hence $x \in f(C)$.

This completes the proof.

QED

Lemma 0.4. *If $a_1, a_2 \in A$ are such that $h(a_1) = h(a_2)$ then either both a_1 and a_2 are in C or both a_1 and a_2 are in $A \setminus C$.*

Proof. Suppose without loss of generality that $a_1 \in C, a_2 \in A \setminus C$ and $h(a_1) = h(a_2)$. Then, $h(a_1) = f(a_1)$ and $h(a_2) = g^{-1}(a_2)$ by the definition of h. Hence, $f(a_1) = g^{-1}(a_2)$.

We know by Lemma 1.3 that $f(C) \cap g^{-1}(A \setminus C) = \emptyset$. Hence, $f(a_1) \notin g^{-1}(A \setminus C)$, i.e., $g^{-1}(a_2) \notin g^{-1}(A \setminus C)$ where $a_2 \in A \setminus C$. This is a contradiction.

QED

Put the lemmas above together to prove that h is bijective (i.e., Theorem 0.1 holds).

Proof. Define $h \colon A \to B$ by

$$h(a) \overset{\text{def}}{=} \begin{cases} f(a), & \text{if } a \in C, \\ g^{-1}(a), & \text{if } a \notin C. \end{cases}$$

First we want to show that h is well defined. We know this by Lemma 1.2.

Next we want to show that h is surjective.

Case 1: if $b \in D$, then $b \in f(C)$. Since f is injective, $\exists! a_0 \in C$ such that $b = f(a_0)$, then $b = h(a_0)$.

Case 2: if $b \in B \setminus D$, then by Lemma 1.3, $b \in g^{-1}(A \setminus C)$. Since g^{-1} is injective, $\exists! a' \in A \setminus C$ such that $b = g^{-1}(a')$, then $b = h(a')$.

If $a_1, a_2 \in A$ such that $h(a_1) = h(a_2)$, then, by Lemma 1.4,

Case 1: both $a_1, a_2 \in C$. Then, $h(a_1) = f(a_1)$, and $h(a_2) = f(a_2)$ by definition, but f is injective so this implies $a_1 = a_2$ hence h is injective.

Case 2: both $a_1, a_2 \in A \setminus C$. Then, $h(a_1) = g^{-1}(a_1)$, and $h(a_2) = g^{-1}(a_2)$ by definition, but g^{-1} is injective so this implies $a_1 = a_2$ hence h is injective.

Thus, we know that h is both injective and surjective, hence it is bijective.

QED