

# MATH 16110/50 – Honors Calculus I (IBL)

## Midterm Exam

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**Student:** \_\_\_\_\_

**Instructions:** This exam is worth a total of 30 points. However, you will find that the problems add up to a total of 51 points. This means that to get the maximum score you do not need to solve all problems. You should still solve as many questions as possible so that even if some of your solutions are wrong, you can still get maximum score. The more symbols \* an item has, the harder it is. The symbol <sup>H</sup> indicates that a hint is available for the item on the final page of the exam.

You are allowed to use any result from class or your homework except when the problem is explicitly asking you to solve that specific result from class or your homework. You are also allowed to use standard facts about natural numbers and integers. You can (and should) use other items from this exam to prove your current item (even if you did not solve the item you are using) as long as you do not form a dependency loop (e.g., if you used 2c in your proof of 2d, then you are not allowed to use 2d in your proof of 2c).

# 1 Injectivity and surjectivity [17 points]

Let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be functions.

- a) [1 point] State the definitions of “injective”, “surjective” and “bijective”.
- b) [3 points] Prove that if  $f$  and  $g$  are injective, then so is  $g \circ f$ .
- c) [3 points] Prove that if  $f$  and  $g$  are surjective, then so is  $g \circ f$ .
- d) [2 points] Prove that if  $f$  and  $g$  are bijective, then so is  $g \circ f$ .
- e) [3 points] Prove that if  $g \circ f$  is injective, then so is  $f$ .
- f) [3 points] Prove that if  $g \circ f$  is surjective, then so is  $g$ .
- g) [2 points<sup>H</sup>] Find an example of  $f$  and  $g$  such that  $g \circ f$  is bijective but neither  $f$  nor  $g$  are bijective.

## 2 Countable sets [17 points]

a) [1 point] State the definition of “countable”.

b) [4 points<sup>H</sup>] Prove that  $\mathbb{N} \times \mathbb{N}$  is countable.

c) [5 points<sup>H</sup>] Define  $\mathbb{N}^n$  by induction in  $n \in \mathbb{N}$  as follows: we let  $\mathbb{N}^1 \stackrel{\text{def}}{=} \mathbb{N}$  and for  $n \in \mathbb{N}$ , we let  $\mathbb{N}^{n+1} \stackrel{\text{def}}{=} \mathbb{N}^n \times \mathbb{N}$ .

Prove that for every  $n \in \mathbb{N}$ , the set  $\mathbb{N}^n$  is countable.

d) [7 points<sup>\*\*H</sup>] A function  $f: \mathbb{N} \rightarrow \mathbb{N} \cup \{0\}$  is *eventually zero* if there exists  $n_f \in \mathbb{N}$  such that for every  $m \geq n_f$ , we have  $f(m) = 0$ . Let  $\ell_{00}$  be the set of all functions  $f: \mathbb{N} \rightarrow \mathbb{N} \cup \{0\}$  that are eventually zero.

Prove that  $\ell_{00}$  is countable.

### 3 Partial orders and well-foundedness [17 points]

Recall that a relation on a set  $X$  is a set  $R$  such that  $R \subseteq X \times X$ . For  $x, y \in X$ , let us use the notation “ $x \triangleleft_R y$ ” to mean “ $(x, y) \in R$ ”.

We say that  $R \subseteq X \times X$  is a *partial order* on  $X$  if all of the following hold:

**Reflexivity:** For every  $x \in X$ , we have  $x \triangleleft_R x$ .

**Antisymmetry:** For every  $x, y \in X$ , if  $x \triangleleft_R y$  and  $y \triangleleft_R x$ , then  $x = y$ .

**Transitivity:** For every  $x, y, z \in X$ , if  $x \triangleleft_R y$  and  $y \triangleleft_R z$ , then  $x \triangleleft_R z$ .

- a) [1 point] State the definition of “ $A \subseteq B$ ”.
- b) [4 points] Prove that if  $Z$  is a set and  $\mathcal{P}(Z) \stackrel{\text{def}}{=} \{A \subseteq Z\}$  is the set of all subsets of  $Z$ , then the relation  $R_Z \stackrel{\text{def}}{=} \{(A, B) \mid A \subseteq B \subseteq Z\}$  is a partial order on  $\mathcal{P}(Z)$ .
- c) [6 points<sup>\*H</sup>] For  $A \subseteq X$  and  $x \in X$ , we say that  $x$  is *R-minimal* in  $A$  if  $x \in A$  and for every  $y \in A$ , if  $y \triangleleft_R x$ , then  $x = y$ .

A partial order  $R \subseteq X \times X$  is *well-founded* if for every non-empty subset  $A$  of  $X$ , there exists an *R-minimal* element in  $A$ .

Prove that the relation  $R_Z$  of item (b) is well-founded if and only if  $Z$  is finite.

- d) [6 points<sup>\*\*H</sup>] An *infinite decreasing sequence* for  $R$  is an infinite sequence of elements  $x_1, x_2, \dots$  in  $X$  such that for every  $n \in \mathbb{N}$ , we have  $x_n \neq x_{n+1}$  and  $x_{n+1} \triangleleft_R x_n$ .

Prove that a partial order  $R$  on a non-empty set  $X$  is well-founded if and only if there does *not* exist any infinite decreasing sequence for  $R$ .

## Hints

**Problem 1(g):** Try simple examples in which  $A$ ,  $B$  and  $C$  have at most two elements each.

**Problem 2(b):** You may find useful a result from class that says that if there exists an injection  $g: A \rightarrow B$  and  $B$  is countable, then  $A$  is countable.

**Problem 2(c):** Use induction and establish a bijection between  $A^{n+1}$  and  $\mathbb{N} \times \mathbb{N}$ .

**Problem 2(d):** Consider an enumeration  $p_1, p_2, \dots$  of the prime numbers and the function  $g: \ell_{00} \rightarrow \mathbb{N}$  given by

$$g(f) \stackrel{\text{def}}{=} \prod_{i=1}^{n_f} p_i^{f(i)}$$

(where an empty product is interpreted as 1). You are allowed to use the Fundamental Theorem of Arithmetic (every natural number has a unique decomposition into a product of prime numbers up to permutation of the factors).

**Problem 3(c):** How does  $\subseteq$  behave with cardinality?

**Problem 3(d):** Prove both directions by their contra-positive, i.e., show that  $R$  is *not* well-founded if and only if there exists an infinite decreasing sequence for  $R$ .