MATH 161, Autumn 2024
SCRIPT 5: Connectedness and Boundedness

We introduce a new axiom for a continuum C and derive many interesting properties from it. From now on, we will always assume that C satisfies axiom 4 (as well as axioms 1,2 and 3).

Axiom 4. *A continuum is connected.*

Theorem 5.1. *The only subsets of a continuum C that are both open and closed are $\emptyset$ and C.*

Proof. Assume $M \subset C$ such that $M \neq C, M \neq \emptyset$, and M is both open and closed. Then, $C \setminus M$ must be both open and closed.

Since we know that $M \subset C$, there must be some $x \in C$ such that $x \notin M$, i.e., $x \in C \setminus M$.

Then, $C = M \cup (C \setminus M)$, where M and $C \setminus M$ are nonempty, disjoint, and open. Hence, C is disconnected, and this contradicts Axiom 4.

QED

Corollary 5.2. *Every region is infinite.*

Proof. Let $R = \underline{ab}$ be a finite region. We know that all regions are open, hence R is open. Finite sets are closed, hence R is closed.

$a \in C, a \notin R$, hence $R \neq C$. We know that all regions are nonempty, hence $R \neq \emptyset$.

Then, R is a set that is nonempty, not equal to C, and is both open and closed. This is a contradiction.

QED

Corollary 5.3. *Every point of C is a limit point of C.*

Proof. Let $x \in C$, then we want to show that $x \in LP(C)$. We know that every region is infinite, hence any region that contains x must contain other points.

Hence, for all regions R such that $x \in R$, we have that $R \cap (C \setminus \{x\}) \neq \emptyset$. Hence, $x \in LP(C)$. This completes the proof.

QED

Corollary 5.4. *Every point of the region $\underline{ab}$ is a limit point of $\underline{ab}$.*

Proof. Suppose there exists some $x \in \underline{ab}$ such that $x \notin LP(\underline{ab})$. Then, there exists a region $R \in C$ such that $x \in R$ and $R \cap (\underline{ab} \setminus \{x\}) = \emptyset$.

Then, $S = R \cap \underline{ab}$ is a region such that $S = \{x\}$.

Hence, S is finite, but all regions are infinite. This is a contradiction.

QED

We will now introduce boundedness. The first definition should be intuitively clear. The second is subtle and powerful.

Definition 5.5. Let X be a subset of C. A point u is called an *upper bound* of X if for all $x \in X$, $x \leq u$. A point l is called a *lower bound* of X if for all $x \in X$, $l \leq x$. If there exists an upper bound of X, then we say that X is *bounded above.* If there exists a lower bound of X, then we say that X is *bounded below.* If X is bounded above and below, then we simply say that X is *bounded.*

Definition 5.6. Let X be a subset of C. We say that u is a *least upper bound* of X and write $u = \sup X$ if:

1. u is an upper bound of X, and
2. if u' is an upper bound of X, then $u \leq u'$.

We say that l is a *greatest lower bound* and write $l = \inf X$ if:

1. l is a lower bound of X, and
2. if l' is a lower bound of X, then $l' \leq l$.

The notation sup comes from the word *supremum*, which is another name for least upper bound. The notation inf comes from the word *infimum*, which is another name for greatest lower bound.

Exercise 5.7. If $\sup X$ exists, then it is unique, and similarly for $\inf X$.

Proof. We want to show that if $u = \sup X, u' = \sup X$, then $u = u'$.
Since u is an upper bound and u' is a supremum, we have that $u \leq u'$.
Since u' is an upper bound and u is a supremum, we have that $u' \leq u$.
Hence, $u = u'$. Analogous reasoning for infimum completes the proof.

QED

Exercise 5.8. If X has a first point L, then $\inf X$ exists and equals L. Similarly, if X has a last point U, then $\sup X$ exists and equals U.

Proof. Since L is a first point, for all $x \in X, L \leq x$. Thus, L is a lower bound. Hence, if L' is a lower bound, $L' \leq L$. Hence, $\inf X = L$.
Analogous reasoning for infimum completes the proof.

QED

Exercise 5.9. For this exercise, we assume that $C = \mathbb{R}$. Find $\sup X$ and $\inf X$ for each of the following subsets of $\mathbb{R}$, or state that they do not exist. You need not give proofs.

1. $X = \mathbb{N}$

   Proof. sup does not exist. $\inf = 1$.

   QED

2. $X = \mathbb{Q}$

   Proof. sup and inf do not exist.

   QED

3. $X = \{\frac{1}{n} \mid n \in \mathbb{N}\}$

   Proof. $\sup = 1$ and $\inf = 0$.

   QED

4. $X = \{x \in \mathbb{R} \mid 0 < x < 1\}$

   Proof. $\sup = 1$ and $\inf = 0$.

   QED

5. $X = \{3\} \cup \{x \in \mathbb{R} \mid -7 \leq x \leq -5\}$

   Proof. $\sup = 3$ and $\inf = -7$.

   QED

The following lemma is extremely useful when dealing with suprema; an analogous statement can be made for infima.

Lemma 5.10. *Suppose that X is a subset of C and $s = \sup X$ exists. If $p < s$, then there exists an $x \in X$ such that $p < x \leq s$.*

Proof. Suppose there is no $x \in X$ such that $p < x$, then for all $x \in X$, $x \leq p$. Then, p is an upper bound of X.

Since $p < s$, we have an upper bound less than $\sup X$, and this is a contradiction.

Hence, there must exist an $x \in X$ such that $p < x \leq s$.

QED

Theorem 5.11. *Let $a < b$. The least upper bound and greatest lower bound of the region $\underline{ab}$ are:*

$$\sup \underline{ab} = b \quad \textit{and} \quad \inf \underline{ab} = a.$$

Proof. For all $x \in \underline{ab}, x \leq b$. Hence, b is an upper bound of $\underline{ab}$. Suppose $u \in \underline{ab}$ is an upper bound such that $u < b$. Since regions are nonempty, there exists $m \in \underline{ub}$, i.e., $u < m < b$, hence u is not an upper bound. Thus, $\sup \underline{ab} = b$.

For all $x \in \underline{ab}, x \geq a$. Hence, a is a lower bound of $\underline{ab}$. Suppose $l \in \underline{ab}$ is a lower bound such that $l > a$. Since regions are nonempty, there exists $m \in \underline{al}$, i.e., $a < m < l$, hence l is not a lower bound. Thus, $\inf \underline{ab} = a$.

QED

Lemma 5.12. *Let C satisfy Axioms 1-3, but not necessarily Axiom 4. Suppose also that all regions of C are nonempty. Let X be a subset of C and suppose $\sup X$ exists. Then $\sup X \in \overline{X}$. Similarly, $\inf X \in \overline{X}$.*

Proof. We know that $\overline{X} = \{x \in C \mid \text{ for all regions } R \text{ containing } x, R \cap X \neq \emptyset\}$. Hence, it suffices to show that for all regions R containing $\sup X$, $R \cap X \neq \emptyset$, and similarly for $\inf X$.

Let R be a region containing $\sup X$. Then R can be written as $\underline{ab}$, such that $a < \sup X < b$. By Lemma 5.10, there exists $x \in X$ such that $a < x \leq \sup X$, hence $x \in R \cap X$, hence $R \cap X \neq \emptyset$.

QED

Corollary 5.13. *Let X be a subset of a connected continuum. Suppose that* $\sup X$ *exists. Then* $\sup X \in \overline{X}$. *Similarly,* $\inf X \in \overline{X}$.

Proof. By Corollary 5.2, all regions of a connected continuum are infinite, hence they are nonempty. Then, by Lemma 5.12, $\sup X \in \overline{X}$ and $\inf X \in \overline{X}$.

QED

Corollary 5.14. *Both a and b are limit points of the region $\underline{ab}$.*

Proof. We know that $a = \inf \underline{ab}$ and $b = \sup \underline{ab}$, hence $a, b \in \overline{\underline{ab}}$.

Thus, $a, b \in \underline{ab} \cup LP(\underline{ab})$, but $a, b \notin \underline{ab}$. Hence, $a, b \in LP(\underline{ab})$.

QED

Let $[a, b]$ denote the closure $\overline{\underline{ab}}$ of the region $\underline{ab}$.

Corollary 5.15. $[a, b] = \{x \in C \mid a \leq x \leq b\}$.

Proof. We know that $a, b \in LP(\underline{ab})$, hence $(\{a\} \cup \{b\} \cup \underline{ab}) \subset [a, b]$.

Also, $C \setminus (\{a\} \cup \{b\} \cup \underline{ab}) = \text{ext}(\underline{ab})$. We know that $LP(\underline{ab}) \cap \text{ext}(\underline{ab}) = \emptyset$.

Hence, $[a, b] = \{a\} \cup \{b\} \cup \underline{ab} = \{x \in C \mid a \leq x \leq b\}$.

QED

Lemma 5.16. *Let $X \subset C$ and define:*

$$\Psi(X) = \{x \in C \mid x \textit{ is not an upper bound of } X\}.$$

Then $\Psi(X)$ is open. Define:

$$\Omega(X) = \{x \in C \mid x \textit{ is not a lower bound of } X\}.$$

Then $\Omega(X)$ is open.

Proof. Let $x \in \Psi(X)$. Then, there exists some $y \in X$ such that $y > x$.

Construct $\underline{ay} = \{k \in C \mid a < k < y\}$ such that $a < x$ (since C has no first point). Then, $x \in \underline{ay}$.

It suffices to show that $\underline{ay} \subset \Psi(X)$ (by Theorem 4.9). Let $k \in \underline{ay}$, then $a < k < y$, hence there exists $y \in X$ such that $y > k$, hence k is not an upper bound of X. Thus, $k \in \Psi(X)$.

Analogous reasoning for $\Omega(X)$ completes the proof.

QED

Theorem 5.17. *Suppose that X is nonempty and bounded above. Then* $\sup X$ *exists. Similarly, if X is nonempty and bounded below, then* $\inf X$ *exists.*

Proof. Let $A = \{a \in C \mid a \text{ is an upper bound of } X\}$. Since X is bounded above, $A \neq \emptyset$ and we also know that $X \neq \emptyset$.

Then, by the definition of upper bound, there exists $x \in X$ such that for all $a \in A$, $x \leq a$. Hence, x is a lower bound of A.

Let $B = \{b \in C \mid b \text{ is a lower bound of } A\}$. Since $x \in B, B \neq \emptyset$. We know from Lemma 5.16 that B is closed.

Let $k \in C$ and suppose $k \notin A$, i.e., k is not an upper bound of X. Then, there exists $x_0 \in X$ such that $x_0 > k$.

For all $a \in A$, $x_0 \leq a$, hence $k \leq a$, hence $k \in B$.

Let $s \in A \cap B$. We want to show that $s = \sup X$.

$s \in A$ implies s is an upper bound of X. $s \in B$ implies s is a lower bound of A. Hence, for all $a \in A$, $a \geq s$, where a is an upper bound of X.

Hence, s is the least upper bound of X, hence $s = \sup X$.

Analogous reasoning for $\inf X$ completes the proof.

QED

Corollary 5.18. *Every nonempty closed and bounded set has a first point and a last point.*

Proof. Let X be nonempty, closed and bounded. Then, $X = \overline{X}$.

Since X is bounded above and bounded below, $\sup X$ exists and $\inf X$ exists.

Since $\sup X, \inf X \in \overline{X}$, we have that $\sup X, \inf X \in X$. Hence, $\sup X$ is the last point of X and $\inf X$ is the first point of X.

QED

Exercise 5.19. Is this true for $\mathbb{Q}$?

Proof. QED

Additional Exercises

1. For this exercise, we assume that $C = \mathbb{R}$. Find $\sup X$ and $\inf X$ for each of the following subsets of $\mathbb{R}$, or state that they do not exist. You should give proofs.

   (a) $X = \{-2^n \mid n \in \mathbb{N}\}$
   (b) $X = \{x \mid a \leq x \leq b\}$
   (c) $X = \{-1\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$
   (d) $X = \bigcup_{k \geq 0} \{\frac{k}{n} \mid n \in \mathbb{N}\}$.

2. Prove that if A, B are subsets of a continuum C and $a \leq b$, for all $a \in A, b \in B$, then

$$\sup A \leq \inf B.$$

   Proof. Since for all $a \in A, b \in B, b \geq a$, we have that b is an upper bound of A. By definition of sup, we know that $\sup A \leq b$ for all $b \in B$.

   Since $b \geq \sup A$ for all $b \in B$, we know that $\sup A$ is a lower bound of B. Then, by definition of inf, we have that $\inf B \geq \sup A$. This completes the proof.

   QED

3. Let $a, b \in C$ with $a < b$. Show that $\underline{ab}$ is a continuum.

Proof. By Corollary 5.2, every region is infinite, hence $\underline{ab}$ is nonempty. This satisfies Axiom 1.

Since $\underline{ab} \subset C$, the ordering $<_C$ on C restricts $\underline{ab}$, i.e., $\underline{ab}$ is equipped with the ordering $<_C$. This satisfies Axiom 2.

Suppose $\underline{ab}$ has a first point, and let this be denoted by m. Then, we know that $a < m < b$. Consider the region $\underline{am}$. Since every region is infinite, there exists some $x \in \underline{am}$, i.e., $a < x < m$. This contradicts our original assumption that $\underline{ab}$ has a first point. Analogous reasoning applies for the last point. This satisfies Axiom 3.

Suppose $\underline{ab}$ is disconnected. Then we can write $\underline{ab} = A \cup B$, where A and B are disjoint, nonempty and open.

For the first case, suppose there exist $u, v \in \underline{ab}$ such that $\underline{au} \subseteq A$ and $\underline{vb} \subseteq B$. Then we want to show that $A \cup \{x \in C \mid x \leq a\}$ and $B \cup \{x \in C \mid x \geq b\}$ are non-empty disjoint open subsets of C whose union is C.

$a \in A \cup \{x \in C \mid x \leq a\}$ and $b \in B \cup \{x \in C \mid x \geq b\}$, hence they are nonempty.

Since $\underline{au} \subseteq A$, we have that $A \cup \{x \in C \mid x \leq a\} = A \cup \{x \in C \mid x < u\}$. This final expression is the union of two open sets, hence it is open. Analogously, $B \cup \{x \in C \mid x \geq b\} = B \cup \{x \in C \mid x > v\}$ is open.

Consider $y \in A$. By our original assumption, $A \cap B = \emptyset$, hence $y \notin B$. Also, since $\underline{vb} \subseteq B$, we have that $\underline{vb} \cap A = \emptyset$, hence $y \notin \{x \in C \mid x > v\}$. Next consider $y \in \{x \in C \mid x < u\}$. Since A and B are disjoint, we have that $u < v$, hence we know that $y \notin \{x \in C \mid x > v\}$. Since $\underline{au} \subseteq A$, hence $y \notin B$. Hence, we have that $A \cup \{x \in C \mid x < u\}$ and $B \cup \{x \in C \mid x > v\}$ are disjoint.

Take some $k \in C$. If $k \in \underline{ab}$, then $k \in A \cup B$. If $k \in C \setminus \underline{ab}$, then either $k \leq a$ in which case $k \in \{x \in C \mid x < u\}$, or $k \geq b$ in which case $k \in \{x \in C \mid x > v\}$. Hence, $A \cup \{x \in C \mid x < u\} \cup B \cup \{x \in C \mid x > v\} = C$.

To consider the general case, let $u \in A$ and $v \in B$ and without loss of generality suppose that $u < v$. We define the sets $A' = \{x \in \underline{ab} \mid x \in A, x < v\} \cup \underline{au}$ and $B' = \{x \in \underline{ab} \mid x \in B, x > u\} \cup \underline{vb}$. We want to show that A' and B' are non-empty disjoint open subsets of $\underline{ab}$ whose union is $\underline{ab}$. This reduces the general case to the previous case.

We have that $A' = \{x \in \underline{ab} \mid x \in A, x < v\} \cup \underline{au} = \{A \cap \underline{av}\} \cup \underline{au}$. Analogously, $B' = \{x \in \underline{ab} \mid x \in B, x > u\} \cup \underline{vb} = \{B \cap \underline{ub}\} \cup \underline{vb}$. We know that $A, \underline{av}$ and $\underline{au}$ are open, hence A' is open. Similarly, $B, \underline{ub}$ and $\underline{vb}$ are open, hence B' is open.

$u \in A'$ and $v \in B'$, hence A' and B' are nonempty.

Consider $y \in A \cap \underline{av}$. A and B are disjoint, hence $y \notin B \cap \underline{ub}$, and $\underline{av} \cap \underline{vb} = \emptyset$, hence $y \notin B'$. Next consider $y \in \underline{au}$. We know that $\underline{au} \cap \underline{ub} = \emptyset$ and $\underline{au} \cap \underline{vb} = \emptyset$, hence $y \notin B'$. Thus, A' and B' are disjoint.

Hence, if $\underline{ab}$ is disconnected, then the continuum C must be disconnected and this contradicts Axiom 4. Hence, $\underline{ab}$ is connected. This satisfies Axiom 4.

This completes the proof.

QED

4. Assume that C satisfies Axioms 1, 2 and 3. Do not assume that C satisfies Axiom 4. Suppose also that

   (a) every nonempty subset X of C that is bounded above has a supremum

   (b) all regions of C are nonempty.

   Show that C is connected.

   Hint: Argue by contradiction.

   Proof. Suppose C is disconnected. Then, we can write $C = A \cup B$, where A, B are disjoint, nonempty and open.

   Case 1: A is bounded above. Then, by condition (a), we know that $\sup A$ exists. Let $s = \sup A$.

   Subcase 1a: Suppose $s \in A$. Since A is open, there exists a region $R \subset A$ such that $s \in R$. Let $R = \underline{pq}$, then we have that $p < s < q$. Then consider the region $\underline{sq} \subset A$. Since all regions are nonempty by condition (b), there exists $y \in \underline{sq}$. Then, $y \in A, y > s$, and this contradicts the assumption that s is the supremum of A.

   Subcase 1b: Suppose $s \in B$. Since B is open, there exists a region $R \subset B$ such that $s \in R$. Let $R = \underline{pq}$, then we have that $p < s < q$. Then consider the region $\underline{ps} \subset B$. Since all regions are nonempty by condition (b), there exists $y \in \underline{ps}$, i.e. $p < y < s$. Then, we claim that y is an upper bound of A, i.e., for all $a \in A, a \leq y$. Suppose there exists some a_0 such that $a_0 > y$. Since $s = \sup A, s \geq a_0$. But $a_0 \in A, s \in B$ hence $a_0 \neq s$, hence $s > a_0$. Then we have that $y < a_0 < s$, i.e., $a \in \underline{ys}$. However, $\underline{ys} \subset B$ and this is a contradiction.

   This completes case 1. Next, we will show that the general case can be reduced to case 1.

   Case 2: A is not bounded above. Let $a \in A, b \in B$, and without loss of generality suppose $a < b$. Then, define new sets $A' = A \cap \{x \in C \mid x < b\}$ and $B' = B \cup \{x \in C \mid x > b\}$. We want to show that A' is bounded above, and that A' and B' are nonempty, disjoint and open sets such that $A' \cup B' = C$.

   B is open and $\{x \in C \mid x > b\}$ is open, hence B' is the union of two open sets and is therefore open. A is open and $\{x \in C \mid x < b\}$ is open, hence A' is the intersection of two open sets and is therefore open.

   $b \in B$, hence $b \in B'$, hence B' is nonempty. $a \in A$ and $a < b$ hence $a \in A'$, hence A' is nonempty.

   Suppose $m \in A'$. Then, $m \in A$ implies $m \notin B$ and $m \in \{x \in C \mid x < b\}$ implies $m \notin \{x \in C \mid x > b\}$. Hence, $m \notin B'$, thus $A' \cap B' = \emptyset$.

   We can rewrite $A' = \{x \in A \mid x < b\}$. Hence, b is an upper bound of A', hence A' is bounded above.

   We can write $\{x \in C \mid x > b\} = \{x \in B \mid x > b\} \cup \{x \in A \mid x > b\}$. Then, $A' \cup B' = B \cup \{x \in A \mid x < b\} \cup \{x \in A \mid x > b\} = B \cup A = C$ (since $a \neq b$).

   This completes the proof.

   QED