Homework 4 – Decision Analytics for Business & Policy
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a)

1. Objective Function: The objective is to minimize the total distance to cover all n cities by visiting each city only once and returning to the original city. For every origin city i and destination city j, the objective function includes the distance i.e. D_{ij} in the total distance summation only if the origin and destination cities have a path connecting them i.e. when $x_{ij} = 1$.

   D_{ij}: the distance between city i and city j for i, j = 1, …, n
   $x_{ij} = 1$, if city i and j are connected, 0 otherwise

$$\min_{x_{ij}} \sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij}x_{ij}$$

Constraints:

i. For each pair of cities (i, j), if there is a path connecting origin city i to destination city j, then the path should be two way to allow for travel from origin city j to destination city i.

$$x_{ij} = x_{ji} \quad \text{for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, n$$

ii. For every origin city i, if destination city is the same, then there is no path required to connect a city to itself.

$$x_{ii} = 0 \quad \text{for all } i = 1, \ldots, n$$

iii. For every origin city i, the total number of destination cities it is connected to should be exactly equal to 2. This means each city will have a path coming into it and a path going out of it.

$$\sum_{j=1}^{n} x_{ij} = 2 \quad \text{for all } i = 1, \ldots, n$$

iv. Without the fourth constraint, there is a possibility of having an optimal solution with sub tours instead of a single loop connecting all the cities. Therefore, for any subset S for n cities, there should only be one loop connecting all the cities, which is enforced by the logic that the number of (i, j) pairs in that subset S is less than equal to $2(|S_{11}| - 1)$. The 2 on the right-hand side allows x_{ij} and x_{ji} to both be equal to 1, whereas the 1 is subtracted to restrict this sub-tour containing cities within the subset from closing, and instead connect it to one large loop.

$$\boxed{\sum_{i \in S} \sum_{j \in S} x_{ij} \leq 2(|S| - 1) \quad \text{for every non-empty subset of cities } S \subset \{1, 2, \ldots, n\}}$$

v. For every pair of origin city i and destination city j, if there is a path connecting them, then x_{ij} should be 1, otherwise 0 i.e. only take binary values.

$$\boxed{x_{ij} \in \{0, 1\} \quad \text{for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, n}$$

2. It will be a difficult problem because there will be n! possible tours for gurobi to scan through and select the shortest one.

<u>Number of constraints:</u>

i) n^2 constraints

$$\boxed{x_{ij} = x_{ji} \quad \text{for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, n}$$

ii) n constraints

$$\boxed{x_{ii} = 0 \quad \text{for all } i = 1, \ldots, n}$$

iii) n constraints

$$\boxed{\sum_{j=1}^{n} x_{ij} = 2 \quad \text{for all } i = 1, \ldots, n}$$

iv) $(2^n - 2)$ constraints

$$\boxed{\sum_{i \in S} \sum_{j \in S} x_{ij} \leq 2(|S| - 1) \quad \text{for every non-empty subset of cities } S \subset \{1, 2, \ldots, n\}}$$

v) n^2 constraints

$$\boxed{x_{ij} \in \{0, 1\} \quad \text{for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, n}$$

Therefore, total constraints = $2(n^2 + n) + (2^n - 2)$ in this problem!

b) According to Gurobi's solution, the shortest distance to visit each city exactly once and return to the origin city is 38 miles. However, since this counts distances twice owing to the first constraint above, where $x_{ij} = x_{ji}$, the total distance for this tour is the sum of below i.e. 19 miles.

- Visit from city 1 to city 3 at a distance of 4
- Visit from city 3 to city 2 at a distance of 4
- Visit from city 2 to city 5 at a distance of 3
- Visit from city 5 to city 4 at a distance of 6
- Visit from city 4 to city 5 at a distance of 6
- Visit from city 4 to city 1 at a distance of 2