A scalar field on the strip (and AdS_2) using Hamiltonian truncation

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Consider a scalar field with mass μ^2 on $[0, L]$, having boundary conditions $\phi(0) = \phi(L) = 0$. The field admits a mode decomposition (exercise!)

$$\phi(t=0,x) = \sum_{n=1}^{\infty} f_n(x) \left[a_n + a_n^\dagger \right] \tag{1}$$

with appropriately normalized wavefunctions

$$f_n(x) = \frac{1}{\sqrt{\omega_n L}} \sin\!\Big(\frac{n\pi x}{L}\Big), \quad \omega_n = \sqrt{\Big(\frac{n\pi}{L}\Big)^2 + \mu^2}. \tag{2}$$

The ω_n play the role of energies: the free Hamiltonian of the system is

$$H_0 = \sum_{n=1}^{\infty} \omega_n \, a_n^\dagger a_n. \tag{3}$$

We will turn on a ϕ^2 interaction:

$$H = H_0 + gV, \quad V = \int_0^L dx \, {:}\phi^2(t=0,x){:}. \tag{4}$$

This corresponds to shifting the mass as $\mu^2 \mapsto \mu^2 + 2g$. We expect that the gap between the vacuum and the first excited state becomes

$$\sqrt{\Big(\frac{\pi}{L}\Big)^2 + \mu^2 + 2g} = \omega_1 + \frac{g}{\omega_1} - \frac{g^2}{2\omega_1^3} + \mathrm{O}(g^3). \tag{5}$$

Likewise, we can compute the Casimir energy to order g^2. We will use that

$$\Delta E_{\text{casimir}} = -g^2 \int_0^\infty d\tau \, \langle \Omega | V(\tau) V(0) | \Omega \rangle \tag{6}$$

where $V(\tau)$ is the perturbing operator evolving in Euclidean time:

$$V(\tau) = e^{H_0 \tau} V(0) e^{-H_0 \tau}. \tag{7}$$

Using Wick's theorem, we have

$$\langle \Omega | V(\tau) V(0) | \Omega \rangle = 2 \int_0^L dx \, dx' \, G_E(\tau, x|0, x')^2, \quad G_E(\tau, x|0, x') = \sum_n f_n(x) f_n(x') e^{-\omega_n \tau}. \tag{8}$$

Using this, the integral (**??**) can be computed (numerically).

From the point of view of Hamiltonian truncation, we can express V in terms of creation and annihilation operators:

$$V = \sum_{m,n} A_{mn} \left[a_m^\dagger a_n^\dagger + 2a_m^\dagger a_n + a_m a_n\right] \quad \text{with} \quad A_{mn} = \int_0^L dx\, f_m(x) f_n(x). \tag{9}$$

For the case at hand $A_{mn} = \delta_{mn}/(2\omega_m)$, but in other geometries, the matrix A_{mn} may also have off-diagonal terms.

Using HT, we want to reproduce $\Delta E_{\text{Casimir}}$ and the correction to the first energy gap, to order g^2. Recall from quantum mechanics that the correction to the energy of a state $|i\rangle$ is given by

$$\Delta E_i^{(2)} = -\sum_{n \neq i} \frac{|\langle n|V|i\rangle|^2}{E_n - E_i} \tag{10}$$

which can be recast as

$$\Delta E_i^{(2)} = -\int_0^\Lambda dE\, \frac{\rho_i(E)}{E - E_i}, \quad \rho_i(E) = \sum_{n \neq i} \delta(E - E_n) |\langle n|V|i\rangle|^2. \tag{11}$$

In the accompanying script `run.py` you can compute these spectral densities for both the vacuum state $|\Omega\rangle$ and the first excited state $|1\rangle = a_1^\dagger |\Omega\rangle$, and check that the correct observables are computed when the cutoff Λ is sufficiently large.

Formally, the quantization of a scalar field on AdS_2 of radius R looks very much like the case above, with energies

$$n = 0, 1, 2, \ldots : \quad \omega_n^{\text{AdS}} = \frac{\Delta + n}{R}, \quad \Delta = \frac{1}{2} + \sqrt{\frac{1}{4} + \mu^2 R^2}. \tag{12}$$

The accompanying script also supports the same calculation in AdS. If you compare the spectral densities in AdS_2 and on the strip, what do you notice?