$SO(3)$ singlets

MJH

June 30, 2022

In Hamiltonian truncation (or in other physics settings), we often work with tensor products of $SO(3)$ representations:

eq:bas

$$|\ell_1, m_1; \ldots; \ell_N, m_N\rangle = |\ell_1, m_1\rangle \otimes \cdots \otimes |\ell_N, m_N\rangle. \tag{1}$$

We have in mind that such states correspond to N *different* particles, so e.g. the state $|0, 0; 1, -1\rangle$ is different from $|1, -1; 0, 0\rangle$. It is often useful or even crucial to organize such states into irreps of $SO(3)$. That is to say that we want to construct tensors $T^{(\lambda)}(m_1, \ldots, m_N)$ such that

$$|L, j\rangle\!\rangle := \sum_{m_i} T_L^{(j)}(m_1, \ldots, m_N)|\ell_1, m_1; \ldots, \ell_N, m_N\rangle \tag{2}$$

transforms as the lowest-weight state of a spin-L multiplet.[1] Recall that acting on a single-particle state $|\ell, m\rangle$ we have

$$J_-|\ell, m\rangle = \gamma_{\ell,m}|\ell, m-1\rangle \quad \text{and} \quad J_z|\ell, m\rangle = m|\ell, m\rangle \tag{3}$$

with $\gamma_{\ell,m} = \sqrt{(\ell+m)(\ell-m+1)}$. Hence the tensor $T_L^{(j)}$ should obey

eq:ia

$$(m_1 + \ldots + m_N + L)\, T_L^{(j)}(m_i) = 0, \tag{4a}$$

eq:ib

$$\gamma_{\ell_1, m_1+1}\, T^{(j)}(m_1+1, m_2, \ldots, m_N) + \ldots + \gamma_{\ell_N, m_N+1}\, T_L^{(j)}(m_1, \ldots, m_{N-1}, m_N+1) = 0. \tag{4b}$$

The j index labels different tensors, which can be normalized such that

$$\sum_{m_i} T^{(j)}(m_i) T^{(j')}(m_i) = \delta_{j,j'}. \tag{5}$$

As a matter of principle these equations can be solved (they form a linear system), but this can be a rather long procedure. Instead we can form solutions constructively.

For $N = 2$, it is well-understood how to organize basis states of the form (1) into irreps, using the Clebsch-Gordan coefficients. If ℓ_1, ℓ_2 are given, then the respective two-particle states fall into irreps of the following spins:

$$[\ell_1] \otimes [\ell_2] = \sum_{j=|\ell_1-\ell_2|}^{\ell_1+\ell_2} [j] \tag{6}$$

If $[L]$ is included in this tensor product, then

$$|L\rangle\!\rangle := \sum_{m_1, m_2} \langle \ell_1, m_1; \ell_2, m_2 | L, -L\rangle\, |\ell_1, m_1; \ell_2, m_2\rangle, \tag{7}$$

[1] Notice that rotations only act within tensor products $[\ell_1] \otimes \cdots \otimes [\ell_N]$. Hence we can consider the spins $(\ell_1, \ldots, \ell_N)$ to be fixed.

is the desired lowest-weight state. Recall that the CG coefficients by construction obey

$$(m_1 + m_2 - M)\langle \ell_1, m_1; \ell_2, m_2 | L, M\rangle = 0, \qquad \text{eq:cga} \tag{8a}$$

$$\begin{aligned}\gamma_{L,M}\langle \ell_1, m_1; \ell_2, m_2 | L, M-1\rangle &= \gamma_{\ell_1,m_1+1}\langle \ell_1, m_1+1; \ell_2, m_2 | L, M\rangle \\ &\quad + \gamma_{\ell_2,m_2+1}\langle \ell_1, m_1; \ell_2, m_2+1 | L, M\rangle. \qquad \text{eq:cgb}\end{aligned} \tag{8b}$$

When $M = -L$ the LHS of the last equation vanishes, as it should.

For $N \geq 3$ we can concatenate CG coefficients. For example, fix external spins (ℓ_1, ℓ_2, ℓ_3) and let

$$T_L^{(j)}(m_i) := \sum_{\mu=-j}^{j} \langle \ell_1, m_1; \ell_2, m_2 | j, \mu\rangle \langle j, \mu; \ell_3, m_3 | L, -L\rangle. \tag{9}$$

This tensor is non-zero provided that $\{\ell_1, \ell_2, j\}$ and $\{j, \ell_3, L\}$ both obey the triangle inequality. We should check that the equations (4) are satisfied. Indeed

$$(m_1 + m_2)T_L^{(j)} = \sum_{\mu} \mu \langle \ell_1, m_1; \ell_2, m_2 | j, \mu\rangle \langle j, \mu; \ell_3, m_3 | L, -L\rangle = -(m_3 + L)T_L^{(j)} \tag{10}$$

after using Eq. (8a) twice, hence (4a) is satisfied. Likewise

$$\begin{aligned}\gamma_{\ell_1,m_1+1}\, T_L^{(j)}(m_1+1, m_2, m_3) + (1 \leftrightarrow 2) &= \sum_{\mu} \gamma_{j,\mu+1}\langle \ell_1, m_1; \ell_2, m_2 | j, \mu\rangle \langle j, \mu+1; \ell_3, m_3 | L, -L\rangle \\ &= -\gamma_{\ell_3,m_3+1} \sum_{\mu} \langle \ell_1, m_1; \ell_2, m_2 | j, \mu\rangle \langle j, \mu; \ell_3, m_3+1 | L, -L\rangle \\ &= -\gamma_{\ell_3,m_3+1} T_L^{(j)}(m_1, m_2, m_3+1).\end{aligned}$$

We applied (8b) twice, and in the first line we shifted the summation index $\mu \mapsto \mu + 1$. Hence (4b) is satisfied as well. Finally, we can check that the tensors with different values of j are different:

$$\begin{aligned}\sum_{m_i} T_L^{(j)}(m_i) T_L^{(j')}(m_i) &= \sum_{\mu,\mu'} \sum_{m_1,m_2} \langle \ell_1, m_1; \ell_2, m_2 | j, \mu\rangle \langle \ell_1, m_1; \ell_2, m_2 | j', \mu'\rangle \\ &\quad \times \sum_{m_3} \langle j, \mu; \ell_3, m_3 | L, -L\rangle \langle j', \mu'; \ell_3, m_3 | L, -L\rangle.\end{aligned} \tag{11}$$

By orthogonality of the CG coefficients, the sum over m_1, m_2 evalues to $\delta_{j,j'}\delta_{\mu,\mu'}$ whence

$$\sum_{m_i} T_L^{(j)}(m_i) T_L^{(j')}(m_i) = \delta_{j,j'} \sum_{\mu,m_3} \langle j, \mu; \ell_3, m_3 | L, -L\rangle \langle j, \mu; \ell_3, m_3 | L, -L\rangle = \delta_{j,j'}. \tag{12}$$

For $N \geq 4$ the same pattern applies. In complete generality we have

$$\begin{aligned}T_L^{(j_1,\ldots,j_{N-2})}(m_1, \ldots, m_N) &:= \sum_{\mu_i} \langle \ell_1, m_1; \ell_2, m_2 | j_1, \mu_1\rangle \prod_{k=1}^{N-3} \langle j_k, \mu_k; \ell_{k+2}, m_{k+2} | j_{k+1}, \mu_{k+1}\rangle \\ &\quad \times \langle j_{N-2}, \mu_{N-2}; \ell_N, m_N | L, -L\rangle.\end{aligned} \tag{13}$$

which is non-vanishing if $\{\ell_1, \ell_2, j_1\}$, $\{j_{N-2}, \ell_N, L\}$ and all $\{j_k, \ell_{k+2}, j_{k+1}\}$ all obey the triangle inequality.