Susceptibility of the Ising model

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Consider the Ising model on a finite lattice of size $V = L^d$. There are 2^V configurations $\{\sigma_x\}_x$, $\sigma_x = \pm 1$. The Hamiltonian is

$$H = -\sum_{\langle x,y \rangle} \sigma_x \sigma_y, \quad Z = \sum_{\sigma} e^{-\beta H}. \tag{1}$$

The susceptibility is

$$\chi := \langle \mathcal{X} \rangle = \frac{1}{Z} \sum_{\sigma} \mathcal{X} e^{-\beta H}, \quad \mathcal{X} := \frac{1}{V^2} \sum_{x,y} \sigma_x \sigma_y. \tag{2}$$

Note that

$$\mathcal{X} = \left(\frac{1}{V} \sum_{x} \sigma_x \right)^2 \geq 0 \tag{3}$$

meaning that $\mathcal{X} \geq 0$ for *every* configuration $\{\sigma_x\}_x$. Moreover clearly $\mathcal{X} \leq 1$. We can measure χ directly for small V, by summing over all 2^V configurations. The results are shown in Fig. 1.

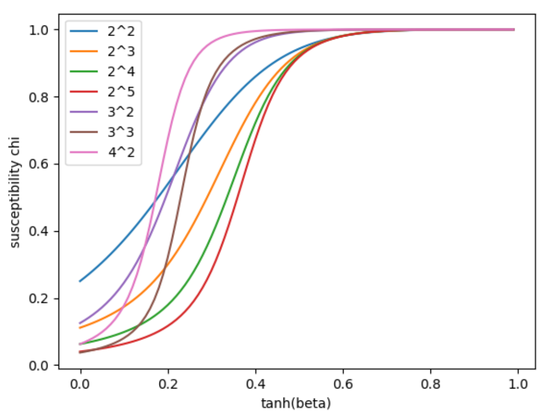

Figure 1: Susceptibility for a range of different lattices, scanning over $\tanh(\beta)$.

Moreover notice that for any observable $\mathcal{O}$

$$\frac{\partial \langle \mathcal{O} \rangle}{\partial \beta} = \langle \mathcal{O} \rangle E - \langle \mathcal{O} H \rangle \quad \Rightarrow \quad \frac{\partial \chi}{\partial \beta} = \chi \cdot E - \langle \mathcal{X} H \rangle. \tag{4}$$

Using this, we can compute $\partial_\beta \chi$ directly too; see Fig. 2. Finally we can plot the energy, see Fig. 3.

As for analytics, we can show that

$$\chi \underset{\beta \to 0}{\sim} \frac{1}{V} + \frac{2d}{V} \tanh(\beta) + \ldots \quad \text{and} \quad \chi \underset{\beta \to \infty}{\sim} 1. \tag{5}$$

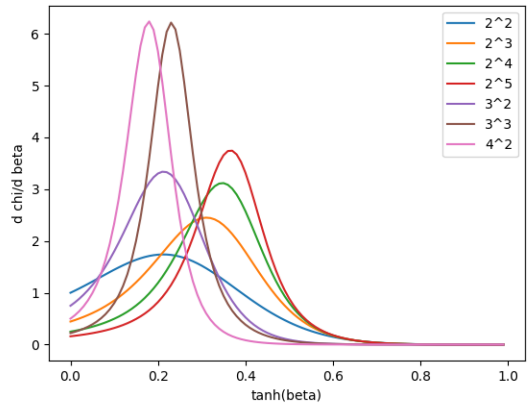

Figure 2: Derivative of the susceptibility (right) for a range of different lattices, scanning over $\tanh(\beta)$.

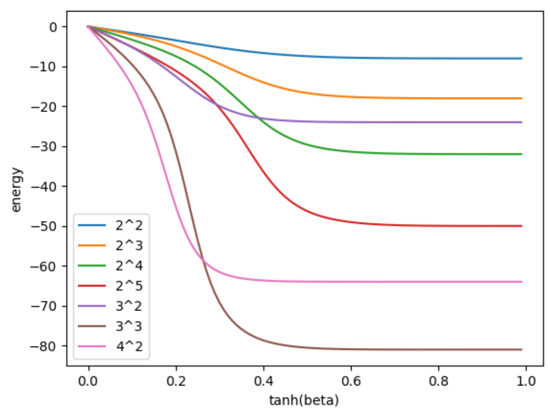

Figure 3: Energy $\langle H \rangle$ for a range of different lattices, scanning over $\tanh(\beta)$.

For the energy we have

$$E \underset{\beta \to 0}{\sim} 0 \quad \text{and} \quad E \underset{\beta \to \infty}{\sim} -dV. \tag{6}$$

This seems to agree with the data.