The Welfare Benefits of Early Termination in Relationship Banking Contracts

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Abstract

This paper explores the welfare benefits from relationship banking that arise from the information gleaned by banks through monitoring. If monitoring reveals not only current output, but also new information about future payoffs, lenders can shield themselves from future losses through early termination of lending agreements. In a competitive lending environment, banks shift the benefits of early termination to borrowers through the lending terms, improving not only the overall expected payoff of projects, but also the welfare of borrowers. Numerical results reveal that the benefits of long-term relationships based on the information revealed in monitoring could be substantial.

Keywords: Relationship banking, costly state verification, contracts, financial frictions.

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1 Introduction

Why do banks find it worthwhile to use multiple repayments in a debt contract? A common argument is that missing a repayment is a strong signal that the borrower is unable to meet future obligations. Thus, having multiple repayments can serve as an early warning signal that a firm is not profitable and banks can terminate underperforming loans early to avoid future losses. Thus, repayments and monitoring by the bank are tools used to not only verify the accuracy of the *current* cash flows of a borrower, but are also used to update the expectation of meeting *future* obligations. Despite the intuitive appeal of this argument, most models of debt contracts featuring costly monitoring lack the ability to investigate the role of repayments and monitoring in updating expected future cash flows.

A related question is why banks find it worthwhile to engage in long-term relationships with lenders. One of the primary roles of banks is to be "relationship lenders," that rely on cultivating and maintaining long-lasting relationships with customers. **?** defines relationship banking as "the provision of financial service by a financial intermediary that

i. invests in obtaining customer-specific information, often proprietary in nature; and

ii. evaluates the profitability of these investments through multiple interactions with the same customer over time and/or across products."

An unanswered question in the research on relationship banking is what kind of customer-specific information banks are looking for. While it is undoubtedly true that banks want to be kept abreast of the current financial situation of their borrowers, banks should want to also know how their loans will perform in the future. In fact, to avoid a sunk-cost fallacy, information about future payoffs should be the *only* information that is important in maintaining a long-term relationship. Monitoring the current cash flows of a firm is useful in inducing truth-telling, but cannot be used to sustain a contract long-term.

Following this motivation, I investigate a multi-period costly state verification (CSV) environment where cash flows from a project consist of both short-term and long-term components. The short-term component of the cash flows is independent across periods, while the long-term component is the same in every period. Therefore, cash flow in any period reveals critical information about cash flows in future periods; repayments in this setup serve a dual purpose. First, they enforce truth-telling from the borrower as in an ordinary CSV model. Second, they inform the lender about the future viability of the project. Receiving a payment from the borrower tells the lender that the borrower is likely to make good on future payments. Conversely, not receiving a payment tells the lender that there may be long-term financial distress that the lender should investigate through monitoring. Finally, the lender has the power to act on the long-term financial viability of the project revealed through monitoring by terminating the lending relationship early.

This paper is related to several strands of literature. First, the importance of bank-borrower relationships have been well documented in the literature. For example, banks tend to provide more credit to lenders as the length of a relationship increases (**???**), indicating that banks find longer relationships to be valuable. In addition, announcing a major bank loan or line of credit is associated with an increase in the price of firm equity (**????????**), indicating that equity holders believe that having a relationship with a bank signals that a firm is financially sound.

The reasons banks find relationships with borrowers to be worthwhile are discussed in **?** and are summarized here. Repeated interactions generate proprietary information about borrowers that banks can leverage by cross-selling products, offering lower loan rates, and giving other financial benefits to their current customers. Customers may also find it worthwhile to engage in long-term relationships with banks because of switching costs, where finding a new financial intermediary takes time and financial expenses. Relationship banking can add to the relative switching costs since banks that have interacted with customers have information that is not available to other lenders, allowing banks to earn monopoly rents on current customers (**???**). This paper adds to this literature by highlighting the importance of long-term information to relationship banking.

My model also relates to the screening function of banks as in **?**, where banks use information to screen potential loans and avoid adverse selection of borrowers. While there is no adverse selection before loan contracts are written in my model, after the first repayment adverse selection is introduced because borrowers have limited liability and therefore have no incentive to cancel projects that have a negative expected future payoff. The lender uses monitoring after the first repayment as a costly screening device to avoid this adverse selection problem. Thus, my paper highlights that screening is an ongoing process in banks, rather than a single operation that happens at the beginning of the contracting process.

In addition to contributing to the literature on relationship banking, this paper is also related to the investigation of multi-period CSV contracts. These papers focus on discovering and cataloging under what conditions features that are seen in real-world debt contracts will arise (**??????**). For simplicity, they largely concentrate on the case when output is independent across periods. Therefore, they do not allow for early termination under the usual assumptions of this literature. Thus, this paper adds to this literature by discussing how multiperiod CSV contracts can be justified when borrowers are able to make use of long-term information about firms' finances.

Finally, this paper fits in to the wider literature on financial frictions in long-term lending relationships. Unlike this study, the literature on financial frictions in long-term relationships focuses primarily on difficulties arising from the process by which borrowers and lenders find each other (**????**). These papers use search and matching to model lender-borrower interactions. Long-term relationships are optimal in this setting because of the cost of finding a new financier after separation in much the same way that long-term relationships can be justified by switching costs in the relationship banking literature. While frictions in finding new lenders are important, this study highlights frictions that arise within the banking relationships themselves, abstracting away from the matching process.

I first construct a benchmark "relationship banking" contract that has the features under investigation — a long-term component to stochastic cash flows, the generation of proprietary information by lenders, and early termination by the lender. I derive the expected payoffs from this contract for very general distributional assumptions and demonstrate the relationship between the contractual payments and the probability of terminating the contract. I also demonstrate that a key feature of such a contract is its ability to avoid future losses from low-performing projects. In particular, when monitoring costs are relatively low, the expected cost of monitoring in the intermediate period is far outweighed by the benefit of identifying and terminating low-performing projects. To demonstrate this, I construct an alternative transactional "one-shot" contract that does not have an intermediate payment, deriving the expected payoffs from it as well. I prove that the one-shot contract leads to worse welfare for borrowers, and that this is primarily driven by the decrease in the expected payoff of the project when the borrower is unable to terminate low-performing projects early. Thus, I demonstrate that the relationship banking contract which produces proprietary information about borrowers in interim periods is superior to a transactional one-shot contract that does not.

Next, I perform a series of experiments by numerically solving for the optimal contract under different assumptions. First, I determine the welfare gains from a relationship banking contract compared to the one-shot contract. I find them to be substantial, with borrower payoff being 31% higher under relationship banking. Next, I solve the optimal contract under different assumptions about the relative importance of the long-term component of output. I find that the more important the long-term component of output is, the greater the benefit to relationship banking. Finally, I perform a robustness test on the assumption that repayments must be positive. I find that, while the lender prefers to make the first repayment negative when the long-term component of output is relatively unimportant, as the long-term component of output becomes more important, positive payoffs are sustainable in both periods. In addition, the welfare gain from having a negative payment in the first period is generally small.

The rest of this chapter proceeds as follows: Section 2 sets up the modeling environment and the contracting problem. Section 3 contains propositions that show that a debt contract with an intermediate payment and cancellation dominates one without either of those features. Section 4 demonstrates some properties of the debt contract using numerical analysis.

2 The Model

Before I explain the contracting environment in detail, I first provide an overview. A borrower has access to a unique, risky project that will pay off over time. Some of the risk in the project is period-specific, or short-term. Some of the risk in the project is persistent, or long-term. Both the borrower and the lender are risk-neutral in my setup. Risk Imposing risk neutrality allows me to investigate the implications of this contracting environment independent of risk-sharing considerations, though the results I derive should carry over to the case where either party is risk-averse. Additionally, risk neutrality is not an unusual assumption in the multi-period CSV literature, and can be justified by assuming the lender is able to hedge any risk from this project and the borrower is less averse to risk than an ordinary household because of their entrepreneurial nature.

The borrower seeks outside funding for her project which is offered by a number of lenders in perfect competition with one another. Both the borrower and the lender know the distributions of the output shocks beforehand, but the actual values of the shocks are not observed by the borrower until they arrive. Since part of the risk is persistent, when the output of the project is observed for the first time borrowers will know the value of the long-term shock prior to its arrival in subsequent periods. Lenders, however, can only see the shocks after incurring a fixed monitoring cost, giving borrowers an informational advantage. Monitoring reveals both the period-specific and the persistent shock and lenders use this information to determine if projects should continue or be canceled. The problem the lender faces is to choose a set of repayments that will satisfy the lender's participation constraint, induce truth-telling from the borrower, and maximize borrower surplus. Now I turn to the formal modeling environment. For reference, a simplified game tree of the contract is presented in Figure 1

2.1 Agents

The contracting environment consists of a borrower who has special access to a unique project and a large number of lenders. The project has an initial investment period ($t = 0$) and pays off over periods $t = 1, 2$. The borrower is endowed with $n < 1$ resources at the beginning of $t = 0$, but the project requires an investment of one unit of resources. In addition, assume that the project has stochastic output Y_t in $t = 1, 2$. The borrower has access to an alternative investment with a net interest rate of r^b, while the lenders have access to a different alternative investment with a net interest rate of r^l. Assume that the expected payoff of the project is greater than the expected payoff to the alternative investments, so there is a surplus to be gained by investing in the project.

2.2 The Project

The stochastic payoff of the project takes the form of the sum of two random variables such that $Y_t = X_t + S$ for $t = 1, 2$. X_t is the "short-term" component to output and is independently and identically distributed (IID) for $t = 1, 2$ and S is the "long-term" component whose realization is the same for both periods, $s_1 = s_2 = s$. In addition, assume that S is independent of X_t. For the numerical solution to the contracting problem, I will assume specific distributions for X_t and S, but for now, I make no further assumptions about the distributions of the project payoff.[1] Denote the probability density function (PDF) and cumulative distribution function (CDF) of a random variable W by $f_W(\cdot)$ and $F_W(\cdot)$. Therefore, $f_Y(y_t)$ is the convolution of X_t and S:

$$f_Y(y_t) = \int_{-\infty}^{\infty} f_S(s) f_X(y_t - s) ds$$

with CDF

$$F_Y(y_t) = \int_{-\infty}^{y_t} \int_{-\infty}^{\infty} f_S(s) f_X(y_t - s) ds dy_t$$

One other assumption that I make is that while the borrower can store the proceeds of the project between periods, they cannot consume the proceeds until after the project has finished. As a result, the borrower does not get a true payoff that they can spend on consumption until the second period. This assumption was made in order to remove an enforcement motivation for long-term contracting. **?** considers the possibility that the borrower could abscond with the proceeds before the contract ends and bases a justification for long-term contracting on balancing the need for lenders to be repaid and incentivizing

[1] A common assumption imposed on payoffs is that they must be weakly positive. This is in line with the notion of limited liability on the part of the borrower and lender. Ordinarily, limited liability is an essential assumption in CSV models, as the ability to use arbitrarily high punishments or rewards allows the lender to completely avoid the cost of monitoring (**?**). While I still impose limited liability on the part of the borrower, limited liability on the part of the lender is not necessary when there is an additional benefit to monitoring through the continuation mechanism.

continuation of the project. The borrower is incentivized to keep the project going by promising future rewards. The assumption that lenders must be fully repaid before the borrower can consume eliminates this justification for long-term contracting so that early termination based on expected future payoffs can be explored in isolation. In the case of **?**, long-term contracting is based on the threat by borrowers that the project will be terminated early. In contrast, my model uses early termination as an incentive to the lender to engage long-term contracting, as it can increase the expected payoff of projects. In my setup, borrower is indifferent to early termination as only contracts with negative future expected payoffs are terminated. Because the borrower has limited liability, contracts with a negative expected future payoff are viewed by her as identical to contracts with a zero expected future payoff. Thus early termination serves to relax the lender's participation constraint.

2.3 The Debt Contract

The borrower must borrow $b \equiv 1 - n$ in $t = 0$ in order to invest in the project. The lender operates in a perfectly competitive environment and both parties are risk-neutral, so she must offer a contract that maximizes the borrower's expected payoff, conditional on breaking even in expectation, and the borrower truthfully reporting the proceeds of the project in each period. The lender cannot directly observe the project's payoff without incurring a fixed monitoring cost, γ, each time she monitors the project. However, if the project is monitored in period t, both x_t and s are separately revealed to the lender at that time. Monitoring, therefore, reveals information about future payoffs as well as current payoffs if monitoring occurs before $t = 2$. After monitoring, it may be revealed that the long-term component is so low that the contract's expected future payoff is less than the expected future monitoring costs. In particular, if the lender knows that thee long-term component s is less than some cutoff s^*, then future payoffs are expected to be negative. In this case, the lender has the option of terminating the project. Once the project is terminated, all future payoffs to the borrower and the lender are zero.

In addition to the termination condition, the contract includes a schedule of payments, $d_1(\cdot)$, and $d_2(\cdot)$. $d_t(\cdot)$ cannot be made dependent on y_t unless monitoring occurs, but I assume that $d_t(\cdot)$ can be made dependent on the report of current net worth the borrower makes to the lender, $l_1(y_1)$ and $l_2(y_1 + y_2 - a_1)$. The lender chooses cutoffs, a_1 and a_2 such that if the borrower reports that their current net worth is less than a_1 or a_2 the lender will monitor the project and set $d_1 = y_1$ or $d_2 = y_1 + y_2 - d_1$, but if the borrower reports that their current net worth is greater than a_1 or a_2, the lender sets the repayments $d_1 = a_1$ and $d_2 = a_2$. Such a contract specifies $\{a_1, a_2, s^*\}$ for a given borrowing amount, b.

This is a variation on the standard debt contract that has two payment periods instead of one. It is useful to examine a contract with these features because it resembles those used in real-world debt markets. I do not prove that this is the optimal debt contract. However, so long as renegotiation is not allowed in the first period, this contract design can be shown to dominate a contract that uses a single payment in the second period. Allowing renegotiation in the first period may lead to better outcomes, but doing so means the contract more closely resembles equity, rather than debt, since the lender's payoff becomes more dependent on the performance of the project. As this paper is focused on debt contracts, I do not consider the case where renegotiation can take place.

We can characterize the optimization problem as follows: Let V be the borrower's share of the proceeds of the project and let Π_1 and Π_2 be the lender's share of the proceeds of the project in periods 1 and 2. The lender's contracting problem is to choose a_1 and a_2 to maximize the present value of the discounted expected payoffs of the project:

$$\frac{E(V)}{(1+r^b)^2} - n$$

subject to their break-even constraint

$$\frac{E(\Pi_1) + E(\Pi_2)}{(1+r^l)^2} - b \geq 0.$$

2.4 Expected Payoffs

Next, I turn to the expected payoffs to the borrower and the lender, starting with the second period and working backward to the first period. Determining the expected payoffs relies on the convolution and truncation of the random variables that make up the project payoff.

2.4.1 Period 2 after monitoring in period 1

In the second period, if monitoring has occurred in the first period the lender sees s. Therefore, their expected payoff is conditional on s. If the lender observes s below a cutoff point, s^*, the lender will choose to cancel the project and the borrower and lender will receive a payoff of zero in the second period. If the expected social surplus is greater than zero, they will choose to continue the project.

Determination of s^*. Suppose that after monitoring, the lender observes $S = s$. Then the CDF of $Y_2|s$ is

$$F_{Y|S=s}(y_2) = F_X(y_2 - s)$$

and the PDF of Y_2 is

$$f_{Y|S=s}(y_2) = f_X(y_2 - s)$$

Positive expected social surplus in the second period requires that

$$\mu_{Y|S=s} - \gamma F_{Y|S=s}(a_2) \geq 0$$

This means that there is a particular s, call it s^*, that defines the cutoff point between the lender terminating the project or allowing it to continue:

$$\mu_X + s^* = \gamma F_X(a_2 - s^*) \tag{2.1}$$

s^* is an implicit function of a_2, $s^*(a_2)$. Depending on the distribution of X, I may or may not be able to solve for an explicit $s^*(a_2)$, but there are some properties I can determine from this setup as-is, which I explore in Proposition 1.

Proposition 1. *a) $s^* \in [-\mu_X, \gamma - \mu_X]$. b) Assuming that $F_X(x)$ is differentiable in x, $s^*(a_2)$ is nondecreasing in a_2.*

Proof. a) Follows immediately from $F_X(x) \in [0, 1]$ and 2.1. b) Taking the total derivative of (2.1) with respect to a_2 yields

$$\frac{\partial s^*(a_2)}{\partial a_2} = \frac{\gamma f_X(a_2 - s^*(a_2))}{1 + \gamma f_X(a_2 - s^*(a_2))},$$

which is weakly positive for any a_2.

□

The immediate consequence of part (b) of Proposition 1 is that increasing the second payment increases the probability of terminating the project. Thus, a_2 is tied not only to the expected payoffs of the borrower and the lender, but also to the expected value of the project itself. Therefore, a_2 plays a dual role in this contract. Not only does a_2 determine when monitoring will occur in the second period, but it also influences the expected overall payoff of the project itself. A change in a_2 has two opposing effects on the probability of terminating the project: 1. Increasing a_2 increases the expected payoff of the project through by increasing s^*. This increased the probability of terminating the project after monitoring in the first period, increasing the overall expected value of the project. 2. Increasing a_2 allows the lender to decrease a_1, which decreases the probability that the project will be terminated and decreases the overall expected value of the project. The lender must balance these two opposing forces in choosing a_1.

Distribution of Second Period Returns Conditional on S. Using $s^*(a_2)$, I can say that

$$E_2(\Pi_2|S=s) = \begin{cases} \int_{-\infty}^{a_2}(y_2-\gamma)\, f_{Y|S=s}(y_2)dy_2 + a_2\int_{a_2}^{\infty} f_{Y|S=s}(y_2)dy_2 & \text{if } s \geq s^*, \\ 0 & \text{if } s < s^*. \end{cases} \tag{2.2}$$

is the expected payoff to the lender for a known value of s. Similarly, the expected payoff of the borrower for a known value of s is

$$E_2(V|S=s) = \begin{cases} \int_{a_2}^{\infty}(y_2-a_2)\, f_{Y|S=s}(y_2)dy_2 & \text{if } s \geq s^*, \\ 0 & \text{if } s < s^* \end{cases}.. \tag{2.3}$$

Distribution of Y_2 Conditional on Monitoring To calculate the distribution of Y_2 given there will be monitoring in the first period, the lender uses the distribution of s given $y_1 < a_1$. That is,

$$f_{Y_2|Y_1<a_1}(y_1) = \int_{-\infty}^{\infty} f_{S|Y_1<a_1}(s) f_X(y_2-s)ds.$$

Furthermore, the lender knows that they will cancel the project if they observe $s < s^*$. Therefore, the distribution of Y_2 conditional on monitoring in the first period is a mixed random variable, where with probability $F_{S|Y_1<a_1}(s^*)$, $Y_2 = 0$ and with probability $1 - F_{S|Y_1<a_1}(s^*)$ Y_2 is a draw from the distribution of $X_2 + S|\,(Y_1 < a_1, S \geq s^*)$. Define $\hat{S} \equiv S|\,(Y_1 < a_1, S \geq s^*)$, then,

$$f_{Y_2|Y_1<a_1}(y_2) = \begin{cases} F_{S|Y_1<a_1}(s^*) & \text{for } Y_2 = 0 \\ \left[1 - F_{S|Y_1<a_1}(s^*)\right]\int_{-\infty}^{\infty} f_{\hat{S}}(s) f_X(y_2-s)ds & \text{for } Y_2 \neq 0 \end{cases}$$

To find $f_{\hat{S}}(t)$, define $A = \{S : S < t\}$, $B = \{S : s \geq s^*\}$, $C = \{Y_1 : Y_1 < a_1\}$. Now,

$$\begin{aligned} F_{\hat{S}}(t) &= Pr(A|B\cap C) = Pr(A\cap B\cap C)/Pr(B\cap C) \\ &= \begin{cases} \frac{\int_{-\infty}^{a_1}\int_{s^*}^{t} f_S(s)f_X(y_1-s)dsdy_1}{\int_{-\infty}^{a_1}\int_{s^*}^{\infty} f_S(s)f_X(y_1-s)dsdy_1} & \text{for } t \geq s^* \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Then, taking the derivative with respect to t, I find

$$f_{\hat{S}}(t) = \begin{cases} \frac{\int_{-\infty}^{a_1} f_S(t)f_X(y_1-t)dy_1}{\int_{-\infty}^{a_1}\int_{s^*}^{\infty} f_S(s)f_X(y_1-s)dsdy_1} & \text{for } t \geq s^* \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Abusing the notation somewhat, conditional on monitoring and on continuing the project, the PDF of Y_2 is

$$f_{Y_2|\hat{S}}(y_2) = \int_{s^*}^{\infty} f_{\hat{S}}(s) f_X(y_2 - s) ds. \tag{2.5}$$

I also need to find $F_{S|Y_1<a_1}(s^*)$, which is

$$Pr(B^c|C) = \frac{Pr(B^C \cap C)}{Pr(C)} = \frac{\int_{-\infty}^{a_1} \int_{-\infty}^{s^*} f_S(s) f_X(y_1 - s) ds dy_1}{F_Y(a_1)}$$

and the (generalized) PDF of Y_2 is

$$f_{Y_2|Y_1<a_1}(y_2) = \begin{cases} \left[1 - F_{S|Y_1<a_1}(s^*)\right] \int_{s^*}^{\infty} f_{\hat{S}}(s) f_X(y_2 - s) ds & \text{for } Y_2 \neq 0 \\ F_{S|Y_1<a_1}(s^*) & \text{for } Y_2 = 0. \end{cases}$$

The expected payoff of the lender in the second period, condtional on monitoring in the first period is

$$E_0(\Pi_2|y_1 < a_1) = \left[1 - F_{S|Y_1<a_1}(s^*)\right] \left[\int_{-\infty}^{a_2} (y_2 - \gamma) f_{Y_2|\hat{S}}(y_2) dy_2 + a_2 \int_{a_2}^{\infty} f_{Y_2|\hat{S}}(y_2) dy_2\right]. \tag{2.6}$$

Similarly, the expected total payoff for the borrower, conditional on monitoring occurring in the first period is

$$E_0 (V|y_1 < a_1) = \left[1 - F_{S|Y_1<a_1}(s^*)\right] \left[\int_{a_2}^{\infty} (y_2 - a_2) f_{Y_2|\hat{S}}(y_2) dy_2\right]. \tag{2.7}$$

Note that if monitoring occurs in the first period, the lender confiscates all wealth from the first period, so borrower wealth in the second period does not include any leftover proceeds from the first period.

2.4.2 Period 2 with no monitoring in period 1

If monitoring does not occur in period 1, then the value of S remains private information to the borrower. However, just like in the case with monitoring, $y_1 \geq a_1$ allows the lender to update their beliefs about the lender's wealth in the second period. Specifically, the distributions of X_1 and S need to be updated to account for this information. $f_{X_1|Y_1>a_1}(x_1)$ can be derived as follows:

First, define $A = \{X_1 : X_1 < x_1\}$, $B = \{X_1 + S : Y_1 \geq a_1\}$. Then the CDF of $X_1|Y_1 > a_1$ is

$$\begin{aligned} F_{X|Y_1 \geq a_1}(x_1) &= Pr(X_1 < x_1 | X_1 + S \geq a_1) = Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} \\ &= \frac{\int_{a_1}^{\infty} \int_{-\infty}^{x_1} f_X(t) f_S(y_1 - t) dt dy_1}{1 - F_Y(a_1)}. \end{aligned}$$

Taking the derivative with respect to x_1, I have

$$f_{X|Y_1 \geq a_1}(x_1) = \frac{\int_{a_1}^{\infty} f_X(x_1) f_S(y_1 - x_1) dy_1}{1 - F_Y(a_1)}.$$

Finding $f_{S|Y_1 \geq a_1}(s)$ is similar: Define $A = \{S : S < s\}$ and $B = \{Y_1 : Y_1 \geq a_1\}$. Then the CDF of $S|Y_1 \geq a_1$ is

$$\begin{aligned} F_{S|Y_1 \geq a_1}(s) &= Pr(S < s | Y_1 \geq a_1) = Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} \\ &= \frac{\int_{a_1}^{\infty} \int_{-\infty}^{s} f_S(t) f_X(y_1 - t) dt dy_1}{1 - F_Y(a_1)} \end{aligned}$$

and

$$f_{S|Y_1 \geq a_1}(s) = \frac{\int_{a_1}^{\infty} f_S(s) f_X(y_1 - s) dy_1}{1 - F_Y(a_1)}.$$

If monitoring does not occur in the first period, then the borrower will have retained earnings that the lender must include in their optimization. Borrower wealth in the second period will be

$$\begin{aligned} w &= (y_1 - a_1)(1 + r^b) + y_2 \\ &= (x_1 - a_1)(1 + r^b) + s(1 + r^b) + s + x_2 \\ &= x_1(1 + r^b) + s(2 + r^b) + x_2 - a_1(1 + r^b). \end{aligned}$$

Setting aside a_1 for a moment, let $U = Y_1(1 + r^b) + Y_2$. I can calculate the PDF of U as the convolution of three random variables, $X_1(1 + r^b)$, $S(2 + r^b)$, and X_2, keeping in mind that X_1 and S are conditional on $Y_1 \geq a_1$. Let $S^r = S(2 + r^b)$ and $X_1^r = X_1(1 + r^b)$. Then

$$f_{S^r|Y_1 \geq a_1}(s^r) = f_{S|Y_1 \geq a_1}\left(s^r \left[2 + r^b\right]^{-1}\right) \left[2 + r^b\right]^{-1}$$

and

$$f_{X^r|Y_1 \geq a_1}(x_1^r) = f_{X|Y_1 \geq a_1}\left(x_1^r \left[1 + r^b\right]^{-1}\right) \left[1 + r^b\right]^{-1}.$$

Next,

$$f_{U|Y_1\geq a_1}(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{S^r|Y_1\geq a_1}(s^r) f_{X^r|Y_1\geq a_1}\left(z-s^r\right) f_{X_1|Y_1\geq a_1}(u-z)ds^r dz. \tag{2.8}$$

Let $a_{1r} = a_1(1+r^b)$, then wealth in the second period will be $w = u - a_{1r}$. To find $f_W(w)$, I can use the PDF of $u = w + a_{1r}$:

$$\begin{aligned} f_{W|Y_1\geq a_1}(w) =& f_V(w+a_{1r}) \\ =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{S^r|Y_1\geq a_1}(s^r) f_{X^r|Y_1\geq a_1}\left(z-s^r\right) f_{X_1|Y_1\geq a_1}(w+a_{1r}-z)ds^r dz. \end{aligned} \tag{2.9}$$

The CDF of w is

$$\begin{aligned} F_{W|Y_1\geq a_1}(w) =& F_V(w+a_{1r}) \\ =& \int_{-\infty}^{w+a_{1r}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{S^r|Y_1\geq a_1}(s^r) f_{X^r|Y_1\geq a_1}\left(z-s^r\right) f_{X_1|Y_1\geq a_1}(u-z)ds^r dz dv. \end{aligned} \tag{2.10}$$

In the second period, conditional on no monitoring occurring in the first period, the lender's expected payoff is then

$$E_0(\Pi_2|y_1\geq a_1) = a_2\left[1-F_{W|Y_1\geq a_1}(a_2)\right] + \int_{-\infty}^{a_2} w f_{W|Y_1\geq a_1}(w)dw - \gamma F_{W|Y_1\geq a_1}(a_2).$$

Conditional on no monitoring in the first period, the borrower's total payoff will be

$$E_0(V|y_1\geq a_1) = \int_{a_2}^{\infty} w f_{W|Y_1\geq a_1}(w)dw - a_2\int_{a_2}^{\infty} f_{W|Y_1\geq a_1}(w)dw.$$

2.4.3 Period 2 Returns

Now that I have the payoff conditional on either monitoring or not monitoring, I can combine them to say

$$\begin{aligned} E_0(\Pi_2) = & F_Y(a_1)\left[1-F_{S|Y_1<a_1}(s^*)\right]\times \\ & \left[\int_{-\infty}^{a_2}\left(y_2-\gamma\right) f_{Y_2|\hat{S}}(y_2)dy_2 + a_2\int_{a_2}^{\infty} f_{Y_2|\hat{S}}(y_2)dy_2\right] \\ & + \left[1-F_Y(a_1)\right]\left[a_2\left[1-F_{W|Y_1\geq a_1}(a_2)\right] + \int_{-\infty}^{a_2} w f_{W|Y_1\geq a_1}(w)dw - \gamma F_{W|Y_1\geq a_1}(a_2)\right] \end{aligned} \tag{2.11}$$

and the borrower's total payoff will be

$$E_0(V) = F_Y(a_1)\left[1 - F_{S|Y_1<a_1}(s^*)\right]\left[\int_{a_2}^{\infty}(y_2 - a_2)\, f_{Y_2|\hat{S}}(y_2)dy_2\right] + \left[1 - F_Y(a_1)\right]\left[\int_{a_2}^{\infty} w f_{W|Y_1\geq a_1}(w)dw - a_2\int_{a_2}^{\infty} f_{W|Y_1\geq a_1}(w)dw\right]. \tag{2.12}$$

2.4.4 Period 1

In the first period, the expected payoff to the lender is relatively straightforward. If $y_1 \geq a_1$, the lender will get a_1. If $y_1 < a_1$, the lender gets $y_1 - \gamma$. Then the lender's expected payoff in the first period is

$$E_0(\Pi_1) = \int_{-\infty}^{a_1}(y_1 - \gamma)\, f_Y(y_1)dy_1 + a_1\int_{a_1}^{\infty} f_Y(y_1)dy_1. \tag{2.13}$$

2.5 Contracting Problem

The lender's problem is to choose a contract using the following maximization:

$$\max_{\{a_1,a_2\}} \frac{E_0(V)}{(R^b)^2} - n \tag{2.14}$$

subject to

$$\frac{E_0(\Pi_1)}{R^l} + \frac{E_0(\Pi_2)}{(R^l)^2} - b \geq 0 \tag{2.15}$$

and

$$\mu_X + s^* = \gamma F_X(a_2 - s) \tag{2.16}$$

where $E_0(\Pi_1)$, $E_0(\Pi_2)$, and $E_0(V)$are given by (2.11), (2.12), (2.13), and R^b and R^l are the gross interest rates on savings available to the borrower and lender. Note that the first constraint will bind in equilibrium.

3 Welfare Gains from Intermediate Payments

How does this contract compare to one where there is only one payment at $t = 2$? Intuitively, having no intermediate payment means that the lender cannot take advantage of information about the two different shocks in the project, but must instead rely on the overall distribution of the project returns for contracting. To formalize this, I construct a one-shot contract with only one payment in the second period. Throughout this section, I assume that $r^l = r^b = 0$.

3.1 One-shot contract

Suppose the lender cannot make a contract with an intermediate payment, but must rely on a contract with only a single payment at $t = 2$, which I label a_2 for comparability with the two-payment contract and call this setup a one-shot contract. The borrower's wealth under a one-shot contract will be $\hat{w} = y_1 + y_2$. Having no intermediate period means that the contracting problem is simply

$$\max_{a_2} E_0(V) - n \tag{3.1}$$

subject to

$$E_0(\Pi) - b \geq 0 \tag{3.2}$$

where

$$E_0(V) = \int_{a_2}^{\infty} (\hat{w} - a_2) f_{\hat{W}}(\hat{w}) d\hat{w} \tag{3.3}$$

and

$$E_0(\Pi) = \int_{-\infty}^{a_2} (\hat{w} - \gamma) f_{\hat{W}}(\hat{w}) d\hat{w} + a_2 \int_{a_2}^{\infty} f_{\hat{W}}(\hat{w}) d\hat{w}. \tag{3.4}$$

This setup is nearly identical to that of **?**. Proposition 2 shows that such a contract is dominated by a contract with an intermediate payment and cancellation under certain circumstances.

Proposition 2. *Assume that S has support $\mathbb{R}$. Then there exists a contract with an intermediate payment under which the borrower is strictly better off than the optimal one-shot contract.*

Proof. First, suppose $\hat{a}_2$ is the optimal repayment under the one-shot contract. Next, compare the total expected payoff net of monitoring costs for the two contracts assuming they both use $\hat{a}_2$ as the second payment. Under the relationship banking contract, the total expected payoff net of monitoring costs is

$$\begin{aligned} E(Y) \equiv & E(Y_1) + E(Y_2|Y_1 \geq a_1)[1 - F_Y(a_1)] \\ & + E(Y_2|Y_1 < a_1, s > s^*)[1 - F_{S|Y_1<a_1}(s^*)]F_Y(a_1) \\ & - \gamma F_Y(a_1) - \gamma F_{Y_2|\hat{S}}(\hat{a}_2) F_Y(a_1)[1 - F_{S|Y_1<a_1}(s^*)] \\ & - \gamma F_{W|Y_1 \geq a_1}(\hat{a}_2)[1 - F_Y(a_1)] \end{aligned} \tag{3.5}$$

Under the one-shot contract, the total expected payoff net of monitoring costs is

$$E(\hat{Y}) \equiv E(Y_1) + E(Y_2) - \gamma F_W(\hat{a}_2) \tag{3.6}$$

In order to make an "apples-to-apples" comparison of the two, I expand $E(\hat{Y})$ using the law of total probability:

$$\begin{aligned}
E(\hat{Y}) =& E(Y_1) + E(Y_2|Y_1 \geq a_1)[1 - F_Y(a_1)] + E(Y_2|Y_1 < a_1)F_Y(a_1) \\
& - \gamma F_{W|Y_1 \geq a_1}(\hat{a}_2)[1 - F_Y(a_1)] - \gamma F_{W|Y_1 < a_1}(\hat{a}_2)F_Y(a_1) \\
=& E(Y_1) + E(Y_2|Y_1 \geq a_1)[1 - F_Y(a_1)] + E(Y_2|Y_1 < a_1, s \geq s^*)[1 - F_{S|Y_1<a_1}(s^*)]F_Y(a_1) \\
& + E(Y_2|Y_1 < a_1, s < s^*)F_{S|Y_1<a_1}(s^*)F_Y(a_1) \\
& - \gamma F_{W|Y_1 \geq a_1}(a_2)[1 - F_Y(a_1)] - \gamma F_{W|Y_1 < a_1}(\hat{a}_2)F_Y(a_1)
\end{aligned} \tag{3.7}$$

Now,

$$\begin{aligned}
E(Y) - E(\hat{Y}) =& - E(Y_2|Y_1 < a_1, s < s^*)F_{S|Y_1<a_1}(s^*)F_Y(a_1) \\
& - \gamma F_Y(a_1) - \gamma F_{Y_2|\hat{S}}(a_2)F_Y(a_1)[1 - F_{S|Y_1<a_1}(s^*)] \\
& + \gamma F_{W|Y_1<a_1}(a_1)F_Y(a_1)
\end{aligned} \tag{3.8}$$

The first term in 3.8 is how much the expected payoff is increased by introducing termination. Recall that, using the definition of s^*, the lender terminates any contract where $E(Y_2|s) < 0$. Therefore, $E(Y_2|Y_1 < a_1, s < s^*)$ is *negative*, and subtracting it from the expected payoff results in an increase.

The remaining terms of 3.8 are the difference in monitoring costs. Notice that $F_Y(a_1)$ is a common factor in all three terms. If S has support $\mathbb{R}$, then Y_1 also has support $\mathbb{R}$, and the lender can choose an a_1 such that the monitoring costs are outweighed by the increase in the expected payoff. Since $\hat{a}_2$ is the optimal repayment for the one-shot contract, it must be the case that there exists a two-period contract that dominates the one-shot contract.

□

Proposition 2 says that a two-period contract with cancellation dominates a one-shot contract. While I have only proved the proposition for the case when the sample space of S is the whole real line, the same should hold true for more limited sample spaces, as well. In addition, the proposition makes no assumptions about monitoring costs or borrowing amounts. *Any* optimal one-shot contract is dominated by a two-shot contract, regardless of monitoring costs or debt levels. This comes with the important caveat that the first payment may not be positive. The lender may find it worthwhile to subsidize the borrower in the first period, so long as the lender expects to be repaid in the second period. However, there is an easy corollary to Proposition 2:

Corollary 1. *Suppose that the supports of S and X are the nonnegative real line. Then there exists a contract with non-negative intermediate repayment under which the borrower is weakly better off than the optimal one-shot contract.*

Proof. All the results from Proposition 2 carry through with the restricted supports. No further proof is necessary. □

Corollary 1 is promising, but there is a significant caveat. using (2.1), it can be shown that a necessary condition for s^* to be positive is

$$F_X(a_2 - s^*) \geq \frac{\mu_X}{\gamma}. \tag{3.9}$$

Thus, the relative value of μ_X and γ is important in determining if s^* will be positive. If s^* is negative and the support of X and S is the positive real line, then the borrower will be no better off with early termination.

However, an important question that the proposition does not answer is how significant are the welfare gains from a two-period contract? Without specifying distributions for X_t and S, it is impossible to say for sure. In the next section, I make some reasonable assumptions about the distributions of X_t and S. This allows me to numerically solve for the optimal one-shot and two-period contracts and calculate the difference in welfare between the two.

4 Numerical Calculation of Welfare Gains

To solve the contract numerically, I impose the following: Let $r^l = r^b = 0$. Let $X_t \sim N(\mu_x, \sigma_x)$ and $S \sim N(\mu_s, \sigma_s)$. Then

$$X_t + S_t \sim N\left(\mu_y = \mu_x + \mu_s, \sigma_y = \sqrt{\sigma_x^2 + \sigma_s^2}\right)$$

and

$$X_1 + X_2 + 2S \sim N\left(2\mu_y, \sqrt{2\sigma_y^2\left(1 + \rho_{YY}\right)}\right),$$

where ρ_{YY} is the correlation between Y_1 and Y_2.

The Normal Distribution. The main result of this paper does not rely on particular distributions for X_t and S. For numerical calculation, many different choices of distribution could have been made. Why the normal distribution? Note from the equations in 2.4 that a number of distributions must be derived conditional on various truncations of Y_t, X_t and S. While the normal CDF does not have a closed form solution, all mathematical software programs have fast approximation algorithms available for it. Moreover, the multivariate normal distribution has useful properties that allow many of the conditional distributions used in the contract to be expressed as transformations of the underlying distributions. This reduces (though does not eliminate) the need for numerical approximation in calculating optimal contracts for particular parameters. Throughout this section I refer to results in Appendix A and B, where derivations of the conditional distributions can be found.

Distributions of the Payoffs. It is useful to define standardized distributions first. Let $Z_1 = (Y_1 - \mu_y)/\sigma_y$, $Z_2 = (Y_2 - \mu_y)/\sigma_y$, $Z_s = (S - \mu_s)/\sigma_s$, $Z_U = (Y_1 + Y_2 - 2\mu_y)/\sqrt{2\sigma_y^2(1+\rho_{YY})}$. Additionally, standardize the following values: $k_1 = (a_1 - \mu_y)/\sigma_y$, $k_2 = (a_2 - \mu_y)/\sigma_y$, $k_U = (a_1 + a_2 - \mu_u)/\sigma v$, and $k_s = (s^* - \mu_s)/\sigma_s$. I will find the necessary conditional distributions in terms of these standardizations.

Beginning with the second period after the borrower has made the required repayment a_1, I find 2.8 in terms of normal distributions. The goal is to find the PDF of z_U conditional on $Y_1 \geq a_1$. Finding the covariance between Y_1 and V (or equivalently Z_1 and z_U) will allow me to use some properties in Appendix A. Define $\rho_{YU} = Corr(Y_1, U)$. Then, $\rho_{YU} = \left(\sigma_y^2 + \sigma_s^2\right)/\sqrt{2}\sigma_y^2$. To see this, I will first find the covariance between Y_1 and U.

$$\begin{aligned}
cov(Y_1, U) &= E\left[(V - \mu_U)(Y_1 - \mu_y)\right] \\
&= E\left[(Y_1 + Y_2 - 2\mu_y)(Y_1 - \mu_y)\right] \\
&= E\left[Y_1^2 - \mu_y Y_1 + Y_2 Y_1 - Y_2\mu_Y - 2\mu_y Y_1 + 2\mu_y^2\right] \\
&= E\left[\left(Y_1^2 - 2\mu_y Y_1 + \mu_y^2\right) + \left(Y_1 Y_2 - \mu_y Y_1 - \mu_y Y_2 + \mu_y^2\right)\right] \\
&= E\left[(Y_1 - \mu_y)^2\right] + E\left[(Y_1 - \mu_y)(Y_2 - \mu_y)\right] \\
&= \sigma_y^2 + cov\left(Y_1, Y_2\right).
\end{aligned}$$

The covariance between Y_1 and Y_2 is

$$\begin{aligned}
cov(Y_1, Y_2) &= E\left[(Y_1 - \mu_y)(Y_2 - \mu_y)\right] \\
&= E\left[Y_1 Y_2 - \mu_y Y_2 - \mu_y Y_1 - \mu_y^2\right] \\
&= E\left[Y_1 Y_2\right] - \mu_y^2 = E\left[(x_1 + s)(x_2 + s)\right] - (\mu_x + \mu_s)^2 \\
&= E\left[x_1 x_2\right] + E\left[x_2 S\right] + E\left[x_1 S\right] + E\left[S^2\right] - \mu_x^2 - 2\mu_x\mu_s - \mu_s^2 \\
&\quad \text{because } X_1, X_2 \text{ are IID and } S \text{ is independent of both } x_1 \text{ and } x_2, \\
&= E[S^2] - \mu_s^2 = \sigma_s^2.
\end{aligned}$$

Thus,

$$cov(Y_1, V) = \sigma_y^2 + \sigma_s^2.$$

Now,

$$\rho_{YU} = \frac{\sigma_y^2 + \sigma_s^2}{\sigma_y \sigma_U},$$

where $\sigma_U = \sqrt{2\sigma_y^2 + 2\rho_{YY}\sigma_y}$.

Therefore, Z_1 and z_U form a bivariate standard normal pair with correlation coefficient ρ_{YU}. Let $\Phi_{\rho_{YU}}(\cdot,\cdot)$ $\phi_{\rho_{YU}}(\cdot,\cdot)$ be the CDF and PDF of $\{Z_1, z_U\}$. Let $\Phi(\cdot)$ and $\phi(\cdot)$ be the CDF and PDF from a univariate standard normal. Using the results in A.4, I can say that

$$f_{z_U|Z_1 \geq k_1}(z_U) = \frac{\phi(z_U)\left[1 - \Phi\left(\frac{k_1 - \rho_{YU} z_U}{\sqrt{1-\rho_{YU}}}\right)\right]}{1 - \Phi(k_1)} \tag{4.1}$$

and

$$F_{z_U|Z_1 \geq k_1}(z_U) = \frac{\Phi_{\rho_{YU}}(z_U, \infty) - \Phi_{\rho_{YU}}(z_U, k_1)}{1 - \Phi(k_1)}. \tag{4.2}$$

When calculating the payoffs in the second period, conditional on monitoring not occurring, I can dispense with the convolution of three random variables and instead use (4.1) and (4.2). The PDF does not enter the payoffs directly, but instead enters through the integrals $\int_{k_2}^{\infty} z_U f_{z_U|Z_1 \geq k_1}(z_U)$ and $\int_{-\infty}^{k_2} z_U f_{z_U|Z_1 \geq k_1}(z_U)$. That is,

$$\begin{aligned}
\int_{k_2}^{\infty} z_U f_{z_U|Z_1 \geq k_1}(z_U) &= \frac{\int_{k_2}^{\infty} z_U \phi(z_U)\left[1 - \Phi\left(\frac{k_1 - \rho_{YU} z_U}{\sqrt{1-\rho_{YU}}}\right)\right] dz_U}{1 - \Phi(k_1)} \\
&= \frac{\int_{k_2}^{\infty} z_U \phi(z_U) dz_U - \int_{k_2}^{\infty} z_U \phi(z_U)\Phi\left(\frac{k_1 - \rho_{YU} z_U}{\sqrt{1-\rho_{YU}}}\right) dz_U}{1 - \Phi(k_1)}.
\end{aligned}$$

The first integral is

$$\int_{k_2}^{\infty} z_U \phi(z_U) dz_U = \phi(k_2),$$

but the second integral is more difficult. Let $A = k_1/\sqrt{1-\rho_{YU}}$, $B = -\rho_{YU}/\sqrt{1-\rho_{YU}}$, and $D = \sqrt{1+B^2}$. Then,

$$\int_{k_2}^{\infty} z_U \phi(z_U)\Phi\left(\frac{k_1 - \rho_{YU} z_U}{\sqrt{1-\rho_{YU}}}\right) dz_U = \int_{k_2}^{\infty} z_U \phi(z_U)\Phi(A + B z_1)\, dz_U.$$

? lists the definite integral

$$\int z_U \phi\left(z\right)\left(A + B z_U\right) dz_U = \frac{B}{D}\phi\left(\frac{A}{D}\right)\Phi\left(z_U D + \frac{AB}{D}\right) - \phi\left(z_U\right)\Phi\left(A + B z_U\right) + C.$$

Then,

$$\begin{aligned}\int_{k_2}^{\infty} z_U \phi(z_U)\Phi\left(A + B z_1\right) dz_U &= \frac{B}{D}\phi\left(\frac{A}{D}\right)\Phi\left(z_U D + \frac{AB}{D}\right) - \phi\left(z_U\right)\Phi\left(A + B z_U\right)|_{k_2}^{\infty} \\ &= \frac{B}{D}\phi\left(\frac{A}{D}\right) - \frac{B}{D}\phi\left(\frac{A}{D}\right)\Phi\left(k_2 D + \frac{AB}{D}\right) + \phi\left(k_2\right)\Phi\left(A + B k_2\right)\end{aligned}$$

and

$$\int_{k_2}^{\infty} z_U f_{z_U|Z_1 \geq k_1}(z_U) = \frac{\phi\left(k_2\right)\left[1 - \Phi\left(A + B k_2\right)\right] + \frac{B}{D}\phi\left(\frac{A}{D}\right)\Phi\left(k_2 D + \frac{AB}{D}\right) - \frac{B}{D}\phi\left(\frac{A}{D}\right)}{1 - \Phi(k_1)}. \quad (4.3)$$

Next,

$$\int_{-\infty}^{k_2} z_U f_{z_U|Z_1 \geq k_1}(z_U) dz_U = \int_{-\infty}^{\infty} z_U f_{z_U|Z_1 \geq k_1}(z_U) dz_U - \int_{k_2}^{\infty} z_U f_{z_U|Z_1 \geq k_1}(z_U) dz_U.$$

The second integral is solved. For the first, using results from A.5.3, I have

$$\begin{aligned}\int_{-\infty}^{\infty} z_U f_{z_U|Z_1 \geq k_1}(z_U) dz_U &= E(z_U | Z_1 \geq k_1) \\ &= \frac{\rho_{YU}\phi(k_1)}{\left[1 - \Phi(k_1)\right]}.\end{aligned}$$

Then,

$$\begin{aligned}\int_{-\infty}^{k_2} z_U f_{z_U|Z_1 \geq k_1}(z_U) dz_U = &\frac{\rho_{YU}\phi(k_1)}{\left[1 - \Phi(k_1)\right]} \\ &- \frac{\phi\left(k_2\right)\left[1 - \Phi\left(A + B k_2\right)\right] + \frac{B}{D}\phi\left(\frac{A}{D}\right)\Phi\left(k_2 D + \frac{AB}{D}\right) - \frac{B}{D}\phi\left(\frac{A}{D}\right)}{1 - \Phi(k_1)}.\end{aligned} \quad (4.4)$$

When monitoring occurs, the primary formula I need to work with is (2.5). To construct this equation for my chosen distributions, I recognize that $\boldsymbol{Z} = \left[Z_1, Z_2, Z_S\right]^T$ is a multinormal random vector with mean $\mu = [0,0,0]^T$ and correlation matrix

$$\Sigma = \begin{bmatrix} 1 & \rho_{YY} & \rho_{YS} \\ \rho_{YY} & 1 & \rho_{YS} \\ \rho_{YS} & \rho_{YS} & 1 \end{bmatrix}.$$

ρ_{YY} is the correlation between Y_1 and Y_2 and ρ_{YS} is the correlation between Y_t and S. $\rho_{YS} = \sigma_s/\sigma_y$, which I can show by first calculating the covariance

$$\begin{aligned} Cov(Y_2, S) &= E\left[(X_2 + S - \mu_s - \mu_x)(S - \mu_s)\right] \\ &= E\left[\left(S^2 - 2\mu_s S + \mu_s^2\right) + \left(X_2^2 - \mu_x S - \mu_s X_2 + \mu_x \mu_x\right)\right] \\ &= \sigma_s^2 + Cov(X_2, S) = \sigma_S^2. \end{aligned}$$

Then the correlation is

$$\rho_{YS} = \frac{\sigma_s^2}{\sigma_s \sigma_y} = \frac{\sigma_s}{\sigma_y}.$$

I also show that $\rho_{YY} = \sigma_s^2/\sigma_y^2$.

$$\begin{aligned} Cov(Y_1, Y_2) &= E\left[(X_2 + S - \mu_s - \mu_x)(X_1 + S - \mu_s - \mu_x)\right] \\ &= E\left[\left(X_1 X_2 - \mu_x X_2 - \mu_x X_1 + \mu_x^2\right) + \left(S^2 - 2\mu_s S + \mu_s^2\right)\right] \\ &+ E\left(X_2 S - \mu_s X_2 - \mu_x S + \mu_x \mu_s\right) \\ &= Cov(X_1, X_2) + \sigma_s^2 + Cov(X_2, S) \\ &= \sigma_s^2 \end{aligned}$$

and

$$\rho_{YY} = \frac{\sigma_s^2}{\sigma_y^2}.$$

Now I can apply the results in appendix B to say

$$F_{Z_2|Z_1<k_1, Z_s>k_s}(z_2) = \frac{\Phi_{\rho_{YY}}(k_1, z_2) - \Phi_\Sigma([k_1, z_2, k_s])}{\Phi(k_1) - \Phi_{\rho_{YS}}(k_1, k_s)} \tag{4.5}$$

and

$$f_{Z_2|Z_1<k_1, Z_s>k_s}(z_2) = \frac{\int_{-\infty}^{k_1} \phi_{\rho_{YY}}(z_1, z_2)\, dz_1 - \int_{-\infty}^{k_1}\int_{-\infty}^{k_s} \phi_\Sigma(z_1, z_2, z_s)\, dz_s dz_1}{\Phi(k_1) - \Phi_{\rho_{YS}}(k_1, k_s)}.$$

The first integral is in Appendix A.4, and the second integral can be simplified in a similar manner, where I separate the distribution into a marginal distribution times a conditional distribution:

$$\phi_\Sigma(z_1, z_2, z_s) = \phi(z_2) f_{z_1,z_s|z_2}(z_1, z_s)$$

The conditional distribution $f_{z_1,z_s|z_2}(z_1, z_s)$ can be determined by first reordering the random vector $\mathbb{Z} = [Z_1, Z_S, Z_2]$, which rearranges the covariance matrix to

$$\Sigma = \begin{bmatrix} 1 & \rho_{YS} & \rho_{YY} \\ \rho_{YS} & 1 & \rho_{YS} \\ \rho_{YY} & \rho_{YS} & 1 \end{bmatrix}.$$

Next, partitioning the covariance matrix of $\boldsymbol{Z}$ into $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$, where $\Sigma_{11} = \begin{bmatrix} 1 & \rho_{YS} \\ \rho_{YS} & 1 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} \rho_{YY} \\ \rho_{YS} \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} \rho_{YY} & \rho_{YS} \end{bmatrix}$, and $\Sigma_{22} = [1]$. Then $\begin{bmatrix} Z_1 & Z_s \end{bmatrix}^T |z_2$ is a normal random vector with mean

$$\Sigma_{12}\Sigma_{22}^{-1} z_2 = \begin{bmatrix} \rho_{YY} z_2 \\ \rho_{YS} z_2 \end{bmatrix}$$

and covariance matrix

$$\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = \begin{bmatrix} 1 - \rho_{YY}^2 & \rho_{YS}(1 - \rho_{YY}) \\ \rho_{YS}(1 - \rho_{YY}) & 1 - \rho_{YS}^2 \end{bmatrix}$$

(**?**). The correlation between Z_1 and Z_s conditional on z_2 is

$$\rho_{Y_1 S|Y_2} = \frac{\rho_{YS}(1 - \rho_{YY})}{\sqrt{(1 - \rho_{YS}^2)(1 - \rho_{YY}^2)}},$$

and

$$f_{Z_2|Z_1<k_1, Z_s>k_s}(z_2) = \frac{\phi(z_2)\left[\Phi\left(\frac{k_1 - \rho_{YY} z_2}{\sqrt{1-\rho_{YY}^2}}\right) - \Phi_{\rho_{Y_1 S|Y_2}}\left(\frac{k_1 - \rho_{YY} z_2}{\sqrt{1-\rho_{YY}^2}}, \frac{k_s - \rho_{YS} z_2}{\sqrt{1-\rho_{YS}^2}}\right)\right]}{\Phi(k_1) - \Phi_{\rho_{YS}}(k_1, k_s)}. \tag{4.6}$$

Once again, rather than working directly with $f_{Z_2|Z_1<k_1, Z_s>k_s}(z_2)$, this PDF enters the payoffs through the integrals $\int_{-\infty}^{k_2} z_2 f_{Z_2|Z_1<k_1, Z_s>k_s}(z_2)\, dz_2$ and $\int_{k_2}^{\infty} z_2 f_{Z_2|Z_1<k_1, Z_s>k_s}(z_2)$. It is likely that these integrals can be expressed in terms of PDFs and CDFs of univariate normal random variables, but fast approximations of the bivariate normal CDF are available so it is not necessary to find them for this paper.

The final standardized formula I need is the probability that $S < s^*$, conditional on $Y_1 < a_1$, or $Pr(Z_s < k_s | Z_1 < k_1)$. Z_1 and Z_s are jointly normal with correlation ρ_{YS}.

Therefore, using results from A.4.4,

$$F_{Z_s|Z_1<k_1}(k_s) = \frac{\Phi_{\rho_{YS}}(k_s, k_1)}{\Phi(k_1)}. \tag{4.7}$$

Now, I can write the conditional distributions in terms of the standardized conditional distributions in (4.2), (4.3),(4.4),(4.5), (4.6) and (4.7):

$$\begin{aligned}
\int_{-\infty}^{a_2} w f_{W|Y_1\geq a_1}(w)dw &= \int_{-\infty}^{a_2+a_1} (u+a_1) f_{U|Y_1\geq a_1}(u)du \\
&= \int_{-\infty}^{a_2+a_1} u f_{U|Y_1\geq a_1}(u)du + a_1 \int_{-\infty}^{a_2+a_1} f_{U|Y_1\geq a_1}(u)du \\
&= \int_{-\infty}^{a_2+a_1} (\sigma_U z_U + \mu_U) f_{U|Y_1\geq a_1}(u)du + a_1 \int_{-\infty}^{a_2+a_1} f_{U|Y_1\geq a_1}(u)du \\
&= \sigma_U \int_{-\infty}^{k_U} z_U f_{z_U|Z_1\geq k_1}(z_U)dz_U + (\mu_U + a_1) \int_{-\infty}^{k_U} f_{z_U|Z_1\geq k_1}(z_U)du \\
&= \sigma_U \int_{-\infty}^{k_U} z_U f_{z_U|Z_1\geq k_1}(z_U)dz_U + (\mu_U + a_1) F_{z_U|Z_1\geq k_1}(k_U),
\end{aligned} \tag{4.8}$$

where $k_U = (a_1 + a_2)/\sigma_U$. Similarly, I can find

$$\int_{a_2}^{\infty} w f_{W|Y_1\geq a_1}(w)dw = \sigma_U \int_{k_U}^{\infty} z_U f_{z_U|Z_1\geq k_1}(z_U)dz_U + (\mu_U + a_1)\left[1 - F_{z_U|Y_1\geq a_1}(k_U)\right], \tag{4.9}$$

$$\int_{-\infty}^{a_2} y_2 f_{Y_2|Y_1<a_1}(y_2)dy_2 = \sigma_y \int_{-\infty}^{k_2} z_2 f_{Z_2|Z_1<k_1}(z_2)dz_2 + \mu_y F_{Z_2|Z_1<k_1}(k_2), \tag{4.10}$$

$$\int_{a_2}^{\infty} y_2 f_{Y_2|Y_1<a_1}(y_2)dy_2 = \sigma_y \int_{k_2}^{\infty} z_2 f_{Z_2|Z_1<k_1}(z_2)dz_2 + \mu_y \left[1 - F_{Z_2|Z_1<k_1}(k_2)\right] \tag{4.11}$$

and in the first period

$$\int_{-\infty}^{a_1} y_1 f_y(y_1)dy_1 = \sigma_y \int_{-\infty}^{k_1} z_1 \phi(z_1)dz_1 + \mu_y \Phi(y_1) \tag{4.12}$$

and

$$\int_{a_1}^{\infty} y_1 f_y(y_1)dy_1 = \sigma_y \int_{k_1}^{\infty} z_1 \phi(z_1)dz_1 + \mu_y \left[1 - \Phi(y_1)\right]. \tag{4.13}$$

Equations (4.8) through (4.13), along with (4.2), (4.5), (4.7), and $F_y(y_1)$ are used to program the numerical maximization problem in (2.14) through (2.16) in Julia 1.6.0 (**?**). Several experiments and their results are presented, along with discussion.

4.1 Results

To demonstrate the welfare gains from a contract with an intermediate payment, I use the same parameterization and solve for the optimal contract with and without an intermediate payment and early termination. Table 1 has the initial parameterization of the contract model. The parameters are set in such a way that the project will have a positive expected value, so the borrower is incentivized to come to an agreement. Additionally, I fix $\sigma_y = 0.75$ and $\sigma_s = 0.7$. This ensures that the project is risky enough that the lender will want to monitor and that a sufficient proportion of the risk in the project happens in the long-term component of the shock, making monitoring in the first period desirable. Fixing these two standard deviations determines the standard deviation of X. I set $b = 0.5$ to allow the borrowing amount to have sufficient influence on the contract terms. Finally, I set the net interest rates to zero.

Table 2 shows some of the key results from this experiment. Comparing the one-shot contract to the two-payment contract, note that a_2 is significantly lower in the two-payment contract (0.899 versus 1.055). Recall from Proposition 2 that the probability of termination decreases as a_2 increases. This incentivizes the lender to lower a_2. Note that monitoring costs are not significantly different between the one-shot and two-payment contracts (0.046 versus 0.046)[2]. However, the gross return of the two-payment project is much higher (1.2 versus 1.271). Thus, termination increases the expected payoff of the project significantly. Consequently, the net return (gross return minus monitoring costs) is greater in the two-payment contract. Finally, as the lender breaks even in expectation, her payoff is zero for both contracts and all the profits go to the borrower. Borrower payoff is 31% higher in the two payment contract (0.224 versus 0.154)but the gross and net returns from the project are much larger with an intermediate payment.

To ensure that the results do not depend on the borrowing amount, I solve for the optimal one-shot and two-payment contracts for a grid over the range of b. Figure 2 shows the results of this experiment. The two-payment contract dominates the one-shot contract for any b, and both contracts show a decline in borrower welfare as the amount borrowed increases due to increasing monitoring costs. Intuitively, as the borrowing amount increases, the lender needs a greater gross expected return in order to meet their participation constraint (2.15). This is to be expected and is in line with previous research (**??**). Interestingly, the lender meets their tighter participation constraint by increasing a_2, whereas a_1 increases slightly, then decreases, and finally levels off.

[2]Monitoring costs are slightly larger with two payments, but the difference is far below the precision of the table.

4.2 Relative importance of long-term risk

In order to assess the relative importance of long-term risk versus short-term risk, I fix $\sigma_Y = 0.75$. Then I test $\sigma_S = 0.5, 0.6, 0.7$. Since $\sigma_Y = \sqrt{\sigma_X + \sigma_S}$, fixing $\sigma_Y = 0.75$ and changing σ_S means that σ_X must decrease for each increase in σ_S. Thus, the overall project riskiness remains the same, but the relative importance of the long-term shock increases as σ_S increases. Table 3 shows how the standard deviation of X and the correlations change for different choices of σ_S. Notice that increasing σ_S also increases the correlation between first and second period payoffs.

Table 4 shows some results from changing the relative importance of the long-term shock. As σ_S increases, borrower payoff increases, rising from 0.168 when $\sigma_S = 0.5$ to 0.225 when $\sigma_S = 0.7$. While monitoring costs decrease somewhat as σ_S increases, the primary channel through which the borrower is better off is the increase in gross return. As σ_S increases, the probability that the project will be scrapped more than doubles, from 0.094 when $\sigma_S = 0.5$ to 0.186 when $\sigma_S = 0.7$.

Next, as in the comparison with the one-shot contract, I solve the optimal contract for each chosen σ_S for a grid over b in order to see how the welfare from the contract changes with the amount that needs to be borrowed. Figure 3 presents the results of these three contracts. As the borrowing amount increases, the borrower's welfare decreases uniformly, nearing zero for the lowest value of σ_S. This is primarily achieved through the increase in the probability of scrapping.

4.2.1 Assumption of nonnegative repayments

Notice that the intermediate repayment a_1 is zero for $\sigma_S = 0.5$ and $\sigma_S = 0.6$. Since the underlying distributions are normal, a repayment of zero does not correspond to a zero probability of monitoring, and it may be the case that the lender would prefer to subsidize the borrower in this period in order to lower the expected monitoring cost. If that is the case, then there may be substantial welfare gains for allowing the repayments to be negative. Table 5 and Figure 4 give the optimal contracts for $\sigma_s = 0.5$. Table 5 gives results for the case when $b = 0.5$ and 4 shows how the optimal contract changes with b. Relaxing this assumption only leads to a small improvement; borrower welfare increases by about 1%. This indicates that the nonnegativity constraint on payments does not significantly affect the results. In addition, the difference between the two is driven primarily by a decrease in monitoring costs, as the gross return is actually decreased by allowing the payment to go negative.

5 Conclusion

In this paper, I have demonstrated that discovering information about future payoffs is a crucial component of a lender's monitoring decisions, as it allows lenders to terminate low-performing loans before incurring losses. Relationship banking delivers superior results when the long-term components of output is relatively important compared to the short-term component. When there are both long- and short-term shocks to output, the information that is revealed in monitoring is important to the contracting decision and to welfare gains. So long as monitoring costs are sufficiently small, welfare is improved in this setup when lenders require an intermediate payment. Numerical calculations show that the welfare improvement depends critically on the relative importance of the long-term and short-term risk of the project. With greater long-term risk, welfare is significantly improved for all debt levels, while greater short-term risk reduces the gains to welfare.

6 Tables and Figures

6.1 Tables

Table 1: Initial parameterization

Parameter	Value	Justification
μ_y	0.6	Ensure positive expected value of the poject
σ_y	0.75	Ensure there is enough risk
μ_s	0.0	Isomorphic with μ_y
σ_s	0.7	Ensure a significant proportion of the risk is in the long-term shock.
μ_x	0.6	Fixed by μ_y and μ_s
σ_x	0.27	Fixed by σ_s and σ_y
μ_v	1.2	Fixed by μ_y
σ_v	1.45	Fixed by σ_s and σ_y
ρ_{YV}	0.97	Determined by σ_s and σ_x
ρ_{YS}	0.93	Determined by σ_s and σ_x
ρ_{YY}	0.87	Determined by σ_s and σ_x
b	0.5	Ensure that the borrowing amount is important.
r^l, r^b	0.0	Ensure that results do not depend on positive interest rates.

Table 2: Results from one- and two-payment contracts

	Contract type	
Result	One-shot	Two-payment
a_1	N/A	0.295
a_2	1.055	0.899
s^*	N/A	-0.500
Gross return	1.2	1.271
Monitoring costs	0.046	0.046
Net return	1.154	1.225
Lender payoff	0.000	0.000
Borrower payoff	0.154	0.224

Table 3: Standard deviations and correlations for different values of σ_s

σ_y	σ_s	σ_x	σ_v	ρ_{YY}	ρ_{YS}	ρ_{YV}
0.750	0.500	0.559	1.339	0.444	0.667	0.809
0.750	0.600	0.450	1.444	0.640	0.800	0.852
0.750	0.700	0.269	1.559	0.871	0.933	0.900

Table 4: Welfare differences for different levels of long-term risk

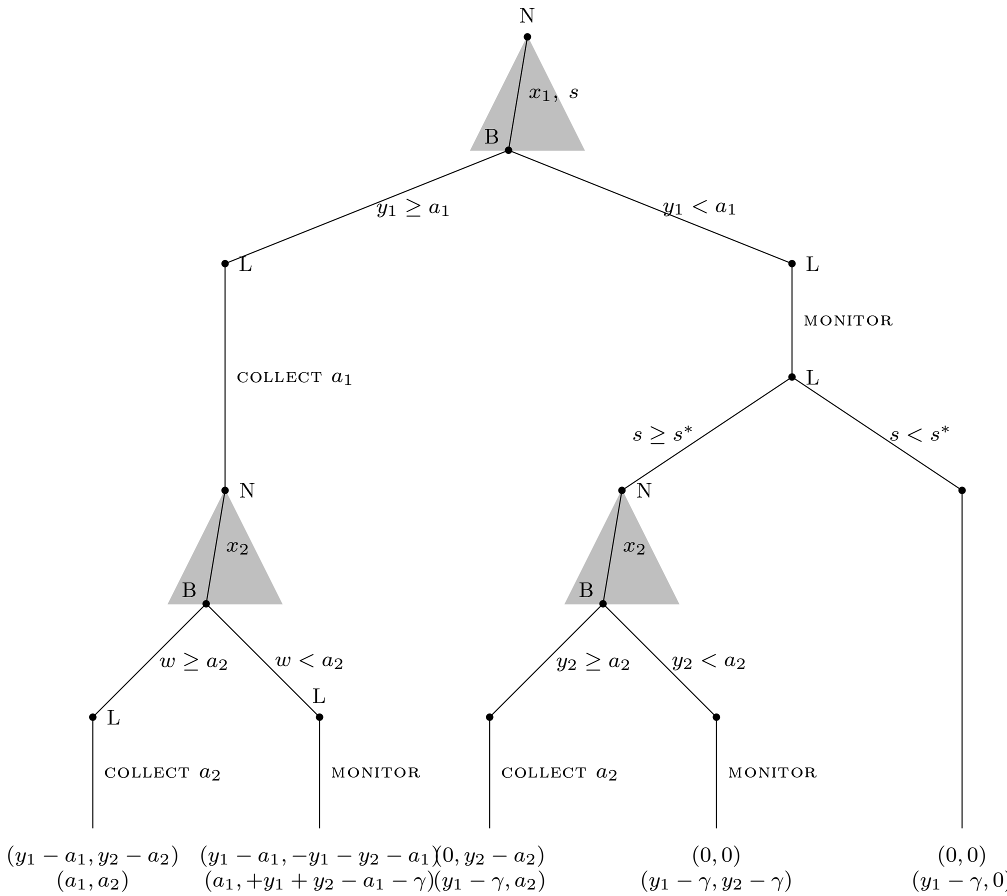
Result	$\sigma_S = 0.5$	$\sigma_S = 0.6$	$\sigma_S = 0.7$
a_1	0.000	0.000	0.030
a_2	0.914	0.925	0.899
S cutoff	-0.507	-0.503	-0.500
Probability of Scrapping	0.094	0.134	0.186
Gross return	1.220	1.241	1.271
Monitoring Costs	0.052	0.049	0.046
Net return	1.168	1.192	1.225
Lender payoff	-0.000	-0.000	-0.000
Borrower payoff	0.168	0.192	0.225

Table 5: Results from relaxing the nonnegativity constraint on repayments

Result	Benchmark	No lower limits on payments
a_1	-0.000	-0.249
a_2	0.914	1.155
S cutoff	-0.507	-0.503
Gross return	1.220	1.216
Monitoring costs	0.052	0.047
Net return	1.168	1.170
Lender payoff	-0.000	-0.000
Borrower payoff	0.168	0.170

6.2 Figures

Figure 1: Simplified game tree for the contract with two payments and cancellation



Note: Node labels: N - Nature, L - Lender B - Borrower. y_1 and y_2 are the total payoffs of the project in periods 1 and 2. $w = y_1 + y_2 - a_1$ is total borrower wealth in period 2. This figure assumes for simplicity that interest rates are zero. Payoffs for the borrower are on top. Some decisions have been truncated or omitted to only show the optimal option. Gray triangles represent continuous outcomes.

Figure 2: Features and results from a one-shot and two-payment contract with $\sigma_Y = 0.75$ and $\sigma_S = 0.7$

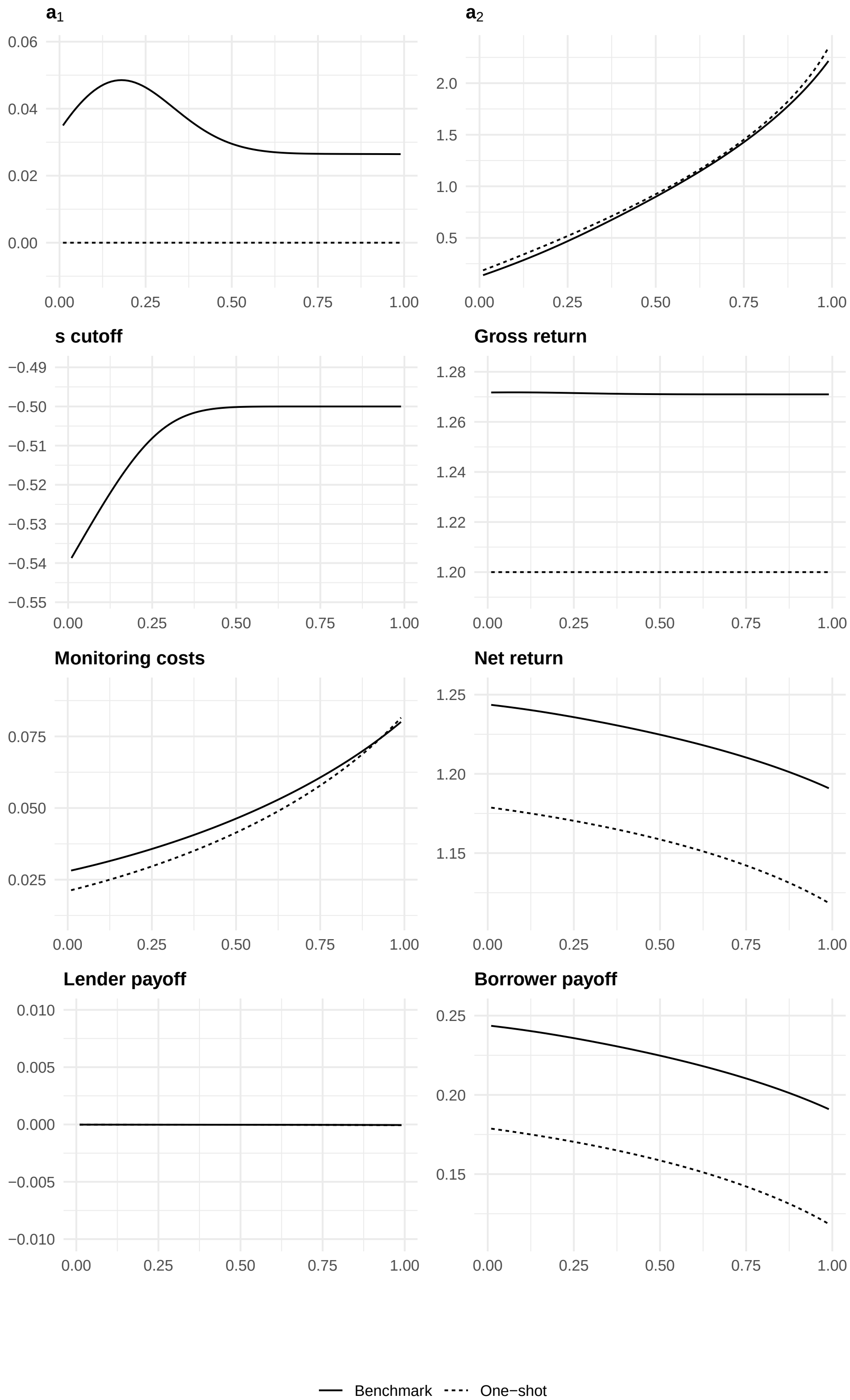


Figure 3: Features and results from changing the relative importance of σ_S and σ_X and from relaxing the nonnegativity assumption on a_1

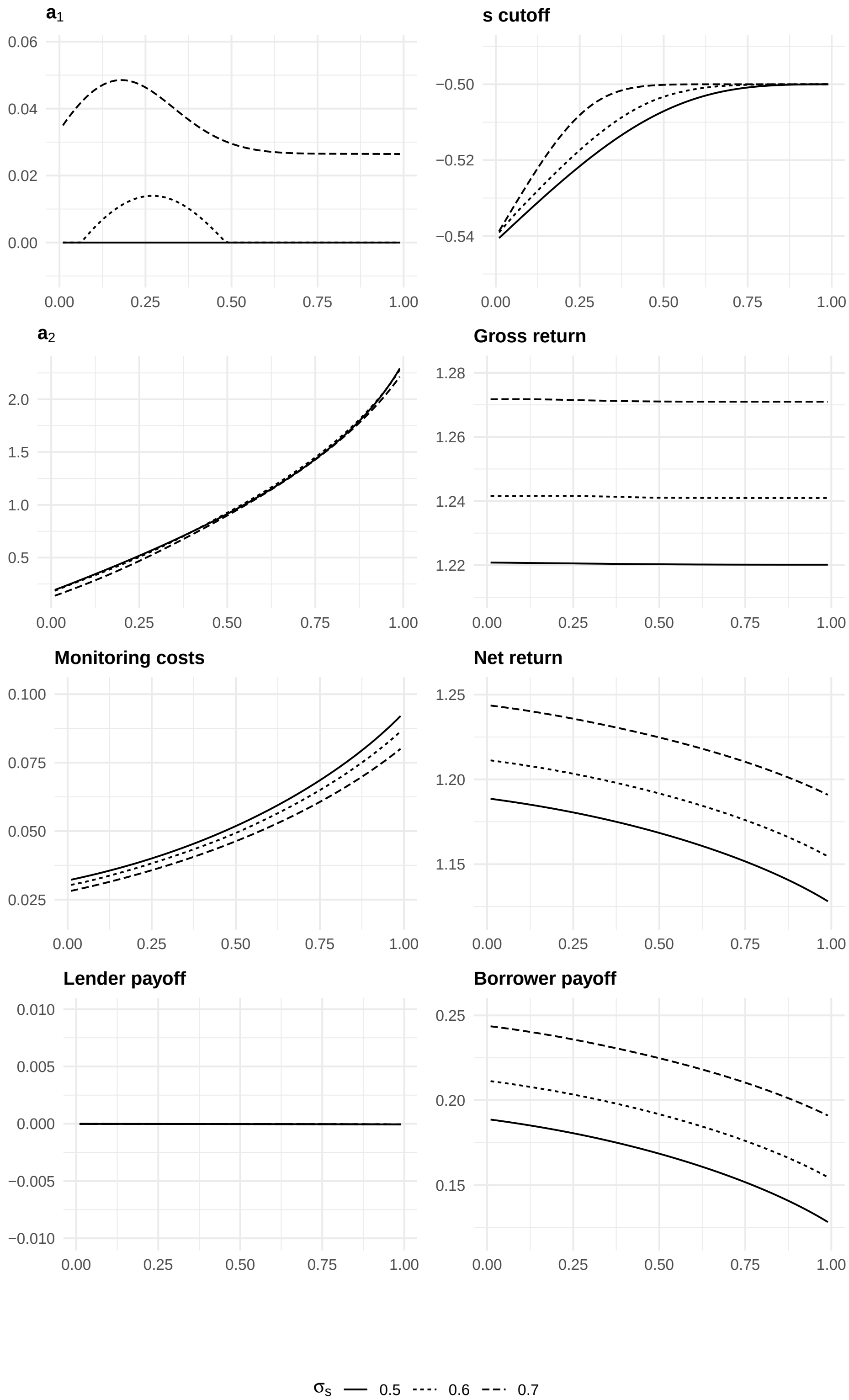
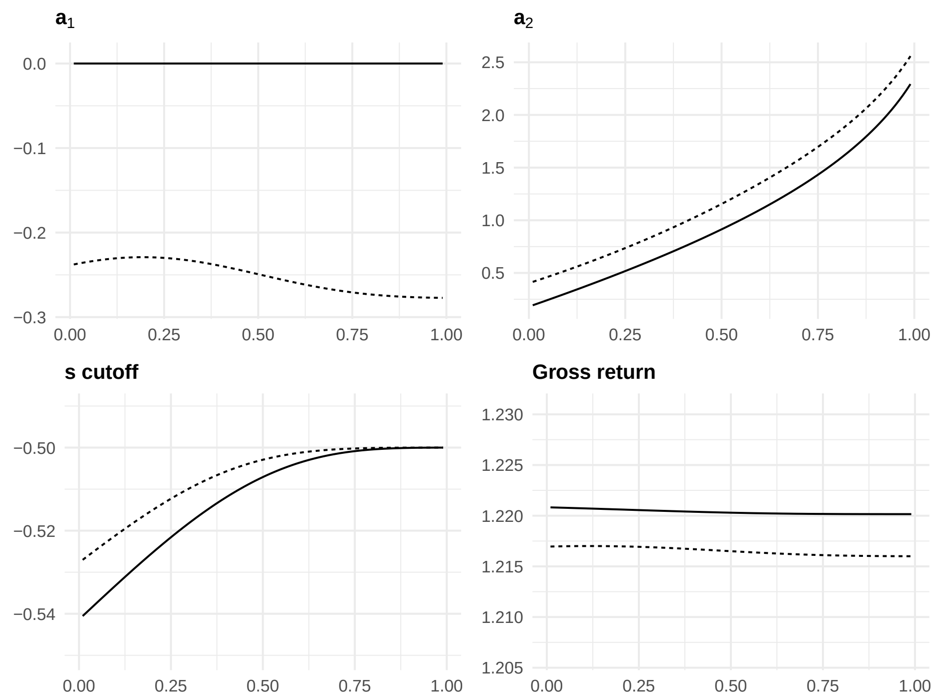
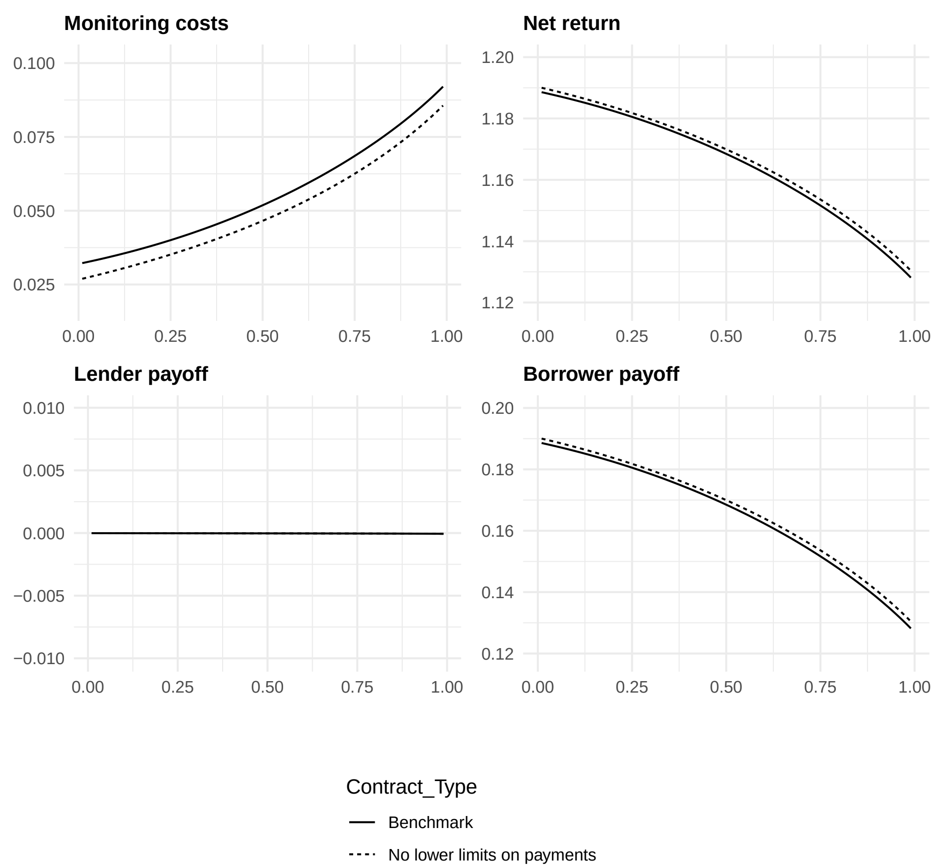


Figure 4: Features and results from relaxing the nonnegativity constraint on payments

(a) Features



(b) Results



Appendices

A The Bivariate Normal Distribution

A number of the results in the main body rely on the properties of the bivariate correlated normal distribution. In this appendix, I derive the marginal and conditional distributions and expectations used in the main body. For more information on the bivariate normal distribution, see **?**.

A.1 Definitions

Let $Z_2 \sim N(0,1)$, $Z_1 \sim N(0,1)$ with correlation ρ Additionally, let $Z_2 + Z_1 \sim N(0, 2\rho)$. Then Z_2 and Z_1 form a bivariate normal random vector $\begin{bmatrix} Z_2 \\ Z_1 \end{bmatrix}$ with mean vector $\mu_{\mathbf{z}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and covariance matrix $\Sigma_{z_2 z_1} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$. Define the bivariate CDF of (z_2, z_1) as the joint probability of $\{Z_2 | Z_2 \geq z_2\}$ and $\{Z_1 | Z_1 \geq z_1\}$ and notate it as $\Phi_\rho(z_2, z_1)$. We can find this probability using the joint pdf of (z_2, z_1), which we define as

$$\phi_\rho(z_2, z_1) = \frac{1}{2\pi\sqrt{(1-\rho^2)}} exp\left\{ -\frac{z_2^2 - 2z_2 z_1 + z_1^2}{2(1-\rho^2)} \right\}.$$

The joint probability can be found as

$$\Phi_\rho(z_2, z_1) = \int_{-\infty}^{z_2} \int_{-\infty}^{z_1} \phi_\rho(z_2, z_1) dz_1 dz_2.$$

The contract requires us to find conditional distributions and expectations from the bivariate normal. We discuss each of these with increasing complexity.

A.2 Marginal Density

The marginal density of Z_2 is

$$f(z_2) = \int_{-\infty}^{\infty} \phi_\rho(z_2, z_1) dz_1 = \frac{1}{\sqrt{2\pi}} exp\left\{ -\frac{z_2^2}{2} \right\}$$

which is simply a standard normal density. Let $f_Z(z_2) = \phi(z_2)$ and the same for z_1.

A.3 Conditional Densities

Define the density of $Z_2|Z_1 = z_1$ as $f_{Z_2|Z_1}(z_2)$. Using the fact that the conditional density is the the joint density divided by the marginal density, we have

$$f_{Z_2|Z_1}(z_2) = \frac{\phi_\rho(z_2, z_1)}{\phi(z_1)} = \frac{\frac{1}{2\pi\sqrt{(1-\rho_y^2)}} exp\left\{-\frac{z_2^2-2z_2z_1+z_1^2}{2(1-\rho_y^2)}\right\}}{\frac{1}{\sqrt{2\pi}} exp\left\{-\frac{z_1^2}{2}\right\}} = \frac{1}{\sqrt{2\pi(1-\rho^2)}} exp\left\{-\frac{1}{2}\frac{(z_2-\rho z_1)^2}{1-\rho^2}\right\}$$

Which is a normal distribution with mean $z_1\rho$ and variance $(1-\rho^2)$. Therefore,

$$(Z_2|Z_1 = z_1) \sim N\left(z_1\rho, (1-\rho^2)\right) \tag{A.1}$$

and similarly for Z_1.

A.4 Conditioning on an interval

For the contract, we need to derive the distribution of a normal random variable conditional on various truncation conditions. We start with the distribution of a normal random variable conditional on the lower truncation of itself.

A.4.1 Lower Truncation

Define the PDF of Z_1 conditional on its lower truncation as $f_{Z_1|Z_1\geq k}(z_1)$. Let $A = \{Z_1|Z_1 \leq z_1\}$, $B = \{Z_1|Z_1 \geq k\}$.Then we have

$$F_{Z_1|Z_1\geq k}(z_1) = P(Z_1 \leq z_1|Z_1 > k) = P(A|B) = P(AB)/P(B)$$

$AB = \{Z_1|k < Z_1 \leq z_1\}$for $z_1 \geq k$ and $AB = \emptyset$ for $z_1 < k$, therefore,

$$F_{Z_1|Z_1\geq k}(z_1) = \begin{cases} \frac{\Phi(z_1)-\Phi(k)}{1-\Phi(k)} & \text{for } z_1 \geq k \\ 0 & \text{for } z_1 < k \end{cases}.$$

Taking the derivative with respect to z_1, we can find the conditional pdf of z_1:

$$\begin{aligned} f_{Z_1|Z_1\geq k}(z_1) &= \frac{\partial F_{Z_1|Z_1\geq k}(z_1)}{\partial z_1} \\ &= \begin{cases} \frac{\phi(z_1)}{1-\Phi(k)} & \text{for } z_1 \geq k \\ 0 & \text{for } z_1 < k \end{cases} \end{aligned} \tag{A.2}$$

Generalizing this result to any normal random variable, define

$$X = \mu_x Z_1 + \sigma_x$$

Then

$$f_{X|X\geq a} = \frac{1}{\sigma}\frac{\phi(z_1)}{1-\Phi(k)},$$

where $k = \frac{a-\mu_x}{\sigma_x}$ is the standardized value of the lower truncation condition, a.

A.4.2 Upper Truncation

Next, define the PDF of an upper truncated standard normal as $f_{Z_1|Z_1<z_1}(z_1)$. This distribution can be derived similar to the lower truncation. Let $A = \{Z_1|Z_1 \leq z_1\}$ and $B = \{Z_1|Z_1 < k\}$. Then we have

$$F_{Z_1|Z_1<k}(z_1) = P(Z_1 \leq z_1|Z_1 < k) = P(A|B) = P(AB)/P(B)$$

$AB = \{Z_1|Z_1 \leq z_1\}$for $z_1 < k$ and $AB = \emptyset$ for $z_1 < k$, therefore,

$$F_{Z_1|Z_1<k}(z_1) = \begin{cases} \frac{\Phi(z_1)}{\Phi(k)} & \text{for } z_1 < k \\ 0 & \text{otherwise} \end{cases} \tag{A.3}$$

Taking the derivative with respect to z_1, we get

$$f_{Z_1|Z_1<k}(z_1) = \begin{cases} \frac{\phi(z_1)}{\Phi(k)} & \text{for } z_1 < k \\ 0 & \text{otherwise} \end{cases}$$

We can generalize this result using X:

$$f_{X|X<b} = \frac{1}{\sigma}\frac{\phi(z_1)}{\Phi(k)}$$

where $k = \frac{b-\mu_x}{\sigma_x}$ is the standardized value of the upper truncation point b.

A.4.3 Density of Z_2 conditional on the lower truncation of Z_1

The conditional density of $Z_2|Z_1 \geq k$ is found similarly. Let $A = \{Z_2|Z_2 \leq z_2\}$ and $B = \{Z_1|Z_1 \geq k\}$. Then we have

$$F_{Z_2|Z_1\geq k}(z_2) = P(Z_2 \leq z_2|Z_1 > k) = P(A|B) = P(AB)/P(B)$$

Using the fact that $P(A) = P(AB) + P(AB^c)$:

$$P(AB)/P(B) = \frac{P(A) - P(AB^c)}{P(B)}$$

Since Z_2, Z_1 are standard normal, we have

$$F_{Z_2|Z_1 \geq k}(z_2) = \frac{\Phi_\rho(z_2, \infty) - \Phi_\rho(z_2, k_1)}{1 - \Phi(k_1)}$$

Note that I have kept the second argument to $\Phi_{\rho_y}(\cdot, \cdot)$, even though integrating out z_1 results in the marginal density of z_2 — a standard normal random variable. This is to emphasize that it is possible to differentiate with respect to z_2 and find the conditional density function:

$$\begin{aligned} f_{Z_2|Z_1>k_1}(z_2) &= \frac{\partial F_{z_2|Z_1 \geq k}(z_2)}{\partial z_2} \\ &= \frac{\int_{-\infty}^{\infty} \phi_\rho(z_2, z_1) dz_1 - \int_{-\infty}^{k_1} \phi_\rho(z_2, z_1) dz_1}{1 - \Phi(k_1)} \\ &= \frac{\int_{k_1}^{\infty} \phi_\rho(z_2, z_1) dz_1}{1 - \Phi(k_1)} \\ &= \frac{\int_{k_1}^{\infty} \phi(z_2) f_{z_1|z_2}(z_1)\, dz_1}{1 - \Phi(k_1)} \\ & \frac{\phi(z_2) \left[1 - \Phi\left(\frac{k_1 - \rho z_2}{\sqrt{1-\rho^2}}\right)\right]}{1 - \Phi(k_1)} \end{aligned} \tag{A.4}$$

where I have made use of the conditional density in **??**.

A.4.4 Density of Z_2 conditional on the upper truncation of Z_1

Let $A = \{Z_2 | Z_2 \leq z_2\}$ and $B = \{Z_1 | Z_1 < k\}$. Then we have

$$F_{Z_2|Z_1<k}(z_2) = P(Z_2 \leq z_2 | Z_1 < k) = P(A|B) = P(AB)/P(B)$$

Now,

$$P(AB) = Pr(Z_2 \leq z_2, Z_1 < k) = \Phi_\rho(z_{2,} k)$$

then,

$$F_{Z_2|Z_1<k}(z_2) = \frac{\Phi_\rho(z_{2,} k)}{\Phi(k)} \tag{A.5}$$

Taking the derivative with respect to z_2,

$$f_{Z_2|Z_1<k}(z_2) = \frac{\int_{-\infty}^{k} \phi_\rho(z_2, z_1)dz_1}{\Phi(k)}$$
$$= \frac{\phi\left(z_2\right)\Phi\left(\frac{k-\rho z_2}{\sqrt{1-\rho^2}}\right)}{\Phi(k)} \tag{A.6}$$

A.5 Conditional expected values

Now that I know the conditional densities, I can calculate conditional expected values:

A.5.1 Expected value of a lower truncated normal

$E(Z_1|Z_1 \geq k)$: Using (A.2) we can find the expected value of a lower truncated normal:

$$E(Z_1|Z_1 \geq k) = \int_k^\infty z_1 f_{Z_1|Z_1\geq k}(z_1)dz_1 = \frac{\int_k^\infty z_1\phi(z_1)dz_1}{1-\Phi(k)}$$

Taking the numerator by itself, we have

$$\int_k^\infty z_1\phi(z_1)dz_1 = \int_k^\infty z_1 \frac{1}{\sqrt{2\pi}} exp\left\{-\frac{1}{2}z_1^2\right\} dz_1$$
$$= \frac{1}{\sqrt{2\pi}}\left(-exp\left\{-\frac{1}{2}z_1^2\right\}|_k^\infty\right)$$
$$= \frac{1}{\sqrt{2\pi}} exp\left\{-\frac{1}{2}k^2\right\} = \phi(k)$$

Therefore,

$$E(Z_1|Z_1 \geq k) = \frac{\phi(k)}{1-\Phi(k)} \tag{A.7}$$

Similarly for the case of any normal random variable X:

$$E(X|X \geq a) = \mu_x + \sigma_x \frac{\phi(k)}{1-\Phi(k)}$$

where $k = \frac{a-\mu_x}{\sigma_x}$

A.5.2 Expected value of an upper truncated normal

$E(Z_1|Z_1 < k)$: Using (A.3), we can find the expected value of an upper truncated normal:

$$E(Z_1|Z_1 < k) = \int_{-\infty}^{k} z_1 f_{Z_1|Z_1<k}(z_1)dz_1 = \frac{\int_{-\infty}^{k} z_1\phi(z_1)dz_1}{\Phi(k)}$$

Following the same steps as above, we find

$$E(Z_1|Z_1 < k) = -\frac{\phi(k)}{\Phi(k)} \tag{A.8}$$

and similarly for any normal random variable X:

$$E(X|X < b) = \mu_x - \sigma_x \frac{\phi(k)}{\Phi(k)}$$

where $k = \frac{b-\mu_x}{\sigma_x}$.

A.5.3 Expected value over a normal, conditional on the upper truncation of another normal.

We also need $E(Z_2|Z_1 \geq k)$.Using A.4:

$$\begin{aligned}
E(Z_2|Z_1 \geq k) &= \frac{\int_{-\infty}^{\infty} z_2 \left[\int_k^{\infty} \phi_\rho(z_2, z_1)dz_1\right] dz_2}{1 - \Phi(k)} \\
&= \frac{\int_{-\infty}^{\infty} z_2 \left[\int_k^{\infty} \phi_\rho(z_2, z_1)dz_1\right] dz_2}{1 - \Phi(k)} \\
&= \frac{\int_{-\infty}^{\infty} z_2 \left[\int_k^{\infty} \phi(z_1)f(z_2|z_1)dz_1\right] dz_2}{1 - \Phi(k)} \\
&\quad \text{Changing the order of integration} \\
&= \frac{\int_k^{\infty} \phi(z_1) \left[\int_{-\infty}^{\infty} z_2 f(z_2|z_1)dz_2\right] dz_1}{1 - \Phi(k)}
\end{aligned}$$

Here we can recognize from section A.3 that

$$(Z_2|Z_1 = z_1) \sim N\left(z_1\rho, 1 - \rho^2\right)$$

and the inner integral is just the expected value of $Z_2|Z_1 = z_1$:

$$
\begin{aligned}
&= \frac{\int_k^{\infty} \phi(z_1)\rho z_1 dz_2}{1-\Phi(k_1)} \\
&= \rho\frac{\int_k^{\infty} z_1\phi(z_1)dz_1}{1-\Phi(k)} \\
&= \rho E(Z_1|Z_1 \geq k) \\
E(Z_2|Z_1 \geq k) &= \rho\frac{\phi(k)}{1-\Phi(k)} \qquad \text{(A.9)}
\end{aligned}
$$

This can be generalized to any two correlated normal random variables. Let $X_1 \sim N(\mu_1, \sigma_1)$ and $X_2 \sim N(\mu_2, \sigma_2)$ be two normal random variables with correlation coefficient ρ_x. Then

$$
E(X_2|X_1 \geq x_1) = \mu_2 + \rho_x\sigma_2\frac{\phi(k)}{1-\Phi(k)}, \qquad \text{(A.10)}
$$

where $k = \frac{x_1-\mu_1}{\sigma_1}$

A.5.4 Expected value over a normal, conditional on the upper truncation of another normal.

Using, we have

$$
\begin{aligned}
E(Z_2|Z_1 \leq k_1) &= \frac{\int_{-\infty}^{\infty} z_2 \int_{-\infty}^{k} \phi_\rho(z_2, z_1)dz_1dz_2}{\Phi(k)} \\
&= \frac{\int_{-\infty}^{k} \phi(z_1)\int_{-\infty}^{\infty} z_2\phi\left(z_2|z_1\right)dz_2dz_1}{\Phi(k)}
\end{aligned}
$$

The inner integral is just the expected value of z_2 conditional on $Z_1 = z_1$, which, from (A.1) is ρz_1, therefore,

$$
\begin{aligned}
E(Z_2|Z_1 \leq k_1) &= \frac{\rho\int_{-\infty}^{k} z_1\phi(z_1)dz_1}{\Phi(k)} \\
&= \rho E\left(Z_1|Z_1 \leq k\right) \\
&= -\rho\frac{\phi(k)}{\Phi(k)}
\end{aligned}
$$

For any two correlated normal random variables X_1 and X_2, we have

$$
E(X_2|X_1 \geq x_1) = \mu_2 - \rho\sigma_2\frac{\phi(k)}{\Phi(k)},
$$

where $k = \frac{x_1 - \mu_1}{\sigma_1}$

B The Trivariate Normal

A random vector $\boldsymbol{Z}$ of size k is considered multinormal if it has a joint density function

$$f_Z(\boldsymbol{z}) = \frac{1}{(2\pi)^{d/2} \det(\boldsymbol{\Sigma})^{1/2}} exp\left\{-\frac{1}{2}(\boldsymbol{z}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{z}-\boldsymbol{\mu})\right\}$$

where $\boldsymbol{\Sigma}$ is a $k \times k$ positive definite covariance matrix and $\boldsymbol{\mu}$ is a vector of means μ_i for $i = 1...d$. When the diagonal elements of $\boldsymbol{\Sigma}$ are all 1 and $\mu_i = 0$ for all i, then $\boldsymbol{Z}$ is a *standard correlated multivariate normal*. Denote the CDF of the standard correlated normal with covariance matrix $\boldsymbol{\Sigma}$ as $\Phi_\Sigma(\boldsymbol{z})$ and the PDF as $\phi_\Sigma(\boldsymbol{z})$. We restrict our focus to the case when $d = 3$, also known as the *trivariate standard normal.*

B.1 Conditional Distributions from the Truncated Trivariate Normal

Suppose we know that a trivariate standard normal with covariance matrix Σ is truncated with lower limits $\boldsymbol{a} = [a_1, a_2, a_3]$ and upper limits $\boldsymbol{b} = [b_1, b_2, b_3]$, where any of the elements of $\boldsymbol{a}$ or $\boldsymbol{b}$ could be infinite. The marginal CDF of Z_i is $Pr(Z_i < z_i | \boldsymbol{a} \leq \boldsymbol{Z} \leq \boldsymbol{b})$. Let $A = \{Z_i : Z_i < z_i\}$, $B = \{\boldsymbol{Z} : \boldsymbol{a} \leq \boldsymbol{Z} \leq \boldsymbol{b}\}$. Then,

$$\begin{aligned} F_{Z_i|\boldsymbol{a}\leq\boldsymbol{Z}\leq\boldsymbol{b}}(z_i) &= \frac{P(A|B)}{P(B)} \\ &= \begin{cases} 0 & \text{if } z_i < a_i \\ \frac{\int_{\boldsymbol{a}}^{\boldsymbol{b}^*} \phi_\Sigma(\boldsymbol{z}) d\boldsymbol{z}}{\int_{\boldsymbol{a}}^{\boldsymbol{b}} \phi_\Sigma(\boldsymbol{z}) d\boldsymbol{z}} & \text{if } a_i \leq z_i \leq b_i \\ 1 & \text{if } z_i > b_i \end{cases} \end{aligned}$$

where $\boldsymbol{b}^*$ is $\boldsymbol{b}$, but with element b_i replaced with z_i.

Taking the derivative with respect to z_i, we find

$$f_{Z_i|\boldsymbol{a}\leq\boldsymbol{Z}\leq\boldsymbol{b}}(z_i) = \begin{cases} 0 & \text{if } z_i < a_1 \text{ or } z_i > b_1 \\ \frac{\int_{\boldsymbol{a}}^{\boldsymbol{b}^{-i}} \phi_\Sigma(\boldsymbol{z}) d\boldsymbol{z}^{-i}}{\int_{\boldsymbol{a}}^{\boldsymbol{b}} \phi_\Sigma(\boldsymbol{z}) d\boldsymbol{z}} & \end{cases}$$

where $\boldsymbol{b}^{-i}$ and $\boldsymbol{z}^{-i}$ are $\boldsymbol{b}$ and $\boldsymbol{z}$ but with element i removed.

The general form above can be made more explicit when we know which of the elements of $\boldsymbol{a}$ and $\boldsymbol{b}$ are infinite. For example, in the main body we use the CDF and PDF of Z_2 conditional on the upper truncation of Z_1 and the lower truncation of Z_3. That is, $\boldsymbol{a} = [-\infty, -\infty, a_3]$, $\boldsymbol{b} = [b_1, \infty, \infty]$, and $i = 1$, i. e. $F_{Z_2|Z_1>b_1, Z_3<b_3}(z_1)$. Denote $\rho_{ij} = corr(Z_i, Z_j)$. Then,

$$\begin{aligned}
F_{Z_2|Z_1>b_1,Z_3<b_3}(z_1) &= Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} \\
&= \frac{\int_{-\infty}^{b_1} \int_{-\infty}^{z_2} \int_{a_3}^{\infty} \phi_\Sigma\left(\boldsymbol{z}\right) d\boldsymbol{z}}{\int_{-\infty}^{b_1} \int_{-\infty}^{\infty} \int_{a_3}^{\infty} \phi_\Sigma\left(\boldsymbol{z}\right) d\boldsymbol{z}} \\
&= \frac{\int_{-\infty}^{b_1} \int_{-\infty}^{z_2} \int_{-\infty}^{\infty} \phi_\Sigma\left(\boldsymbol{z}\right) d\boldsymbol{z} - \int_{-\infty}^{b_1} \int_{-\infty}^{z_2} \int_{-\infty}^{a_3} \phi_\Sigma\left(\boldsymbol{z}\right) d\boldsymbol{z}}{\int_{-\infty}^{b_1} \int_{-\infty}^{\infty} \phi_{\rho_{23}}\left(b_2, b_3\right) dz_3 dz_1 - \int_{-\infty}^{b_1} \int_{-\infty}^{a_3} \phi_{\rho_{13}}\left(b_2, b_3\right) dz_3 dz_1} \\
&= \frac{\Phi_{\rho_{12}}\left(b_1, z_2\right) - \Phi_\Sigma\left(\left[b_1, z_2, a_3\right]\right)}{\Phi\left(b_1\right) - \Phi_{\rho_{13}}\left(b_1, a_3\right)} \qquad \text{(B.1)}
\end{aligned}$$

and

$$f_{Z_2|Z_1>b_1,Z_3<b_3}(z_1) = \frac{\int_{-\infty}^{b_1} \phi_{\rho_{12}}\left(z_1, z_2\right) dz_1 - \int_{-\infty}^{b_1} \int_{-\infty}^{a_3} \phi_\Sigma\left(\left[z_1, z_2, z_3\right]\right) dz_1 dz_3}{\Phi\left(b_1\right) - \Phi_{\rho_{13}}\left(b_1, a_3\right)} \qquad \text{(B.2)}$$