The Euler-Bernoulli Beam Model

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Contents

1 Derivation of the Euler-Bernoulli Beam Equation

The Euler-Bernoulli beam is a simplified model of beam bending. It accounts for small, time-varying deformations of a beam which is bending in one direction. It does not account for the deformation of the beam cross-section, like Timoshenko beams, but it is also simpler to deal with.

The Euler-Bernoulli beam equation is applicable for non-rotating beams like the one shown in Figure 1, where the xyz coordinate system is located at the geometrical center (centroid) of the left-most face of the beam, and where the z-axis is defined to be the long axis of the beam. Here we will derive the equations which describe the bending of the beam in the x-direction as a result of some force per unit length $f(z,t)$ as shown in Figure 2. Later, we will consider the case where the beam is subject to some constraints, such as those shown in Figure 3, which depicts a beam that is cantilevered at the origin, and free at the end. Intuitively, we expect the beam to deform like in Figure 4, where the magnitude of the deflection is obviously dependent on the magnitude and distribution of the force $f(z,t)$, as well as the constraints.

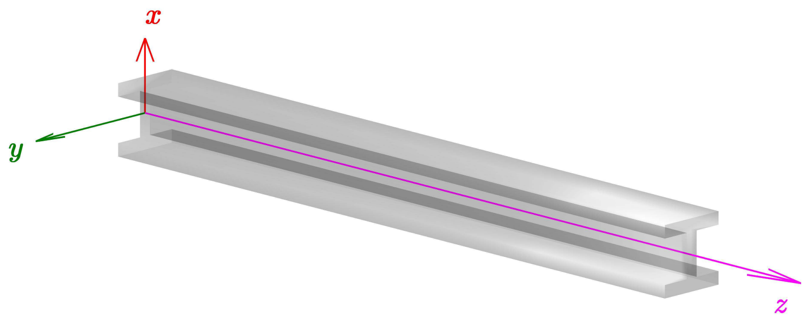


Figure 1: Coordinate-system definition for a non-rotating I-beam.

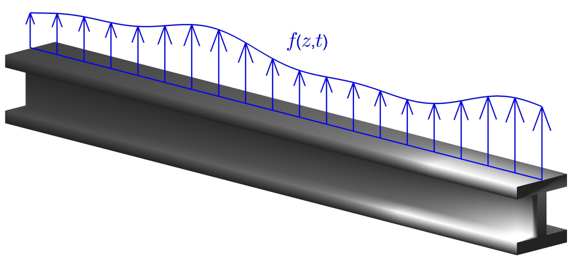


Figure 2: A standard I-Beam with an external force per unit length $f(z,t)$.

1.1 Translational Equation of Motion

To derive the deflection equation, we begin by examining the translational motion of an infinitesimally-thin cross-sectional slice of the beam, as shown in orange in Figure 5, where we only allow for forces and bending to occur in the body-fixed x-direction (see Figures 1-4). Specifically, letting $u(z,t)$ denote the displacement of the geometrical center of the cross-sectional slice at the point z, as shown in Figure 6, and assuming that the xyz coordinate system is an inertial coordinate system (a non-rotating coordinate system), we have that the translational equation of motion for the centroid of the beam slice is given by

$$\rho A \frac{\partial^2 u}{\partial t^2}(z,t)dz = -\Big[V(z+0.5dz,t) - V(z-0.5dz,t)\Big] + f(z,t)dz \tag{1}$$

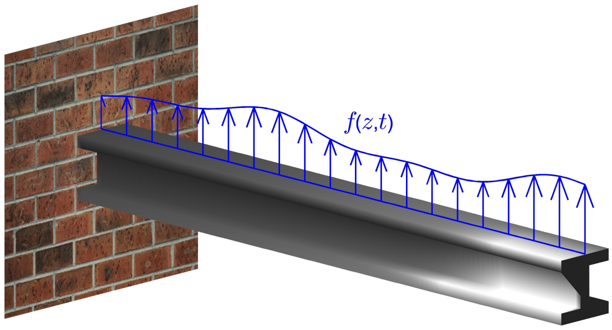


Figure 3: A cantilevered-free I-Beam with an external force per unit length $f(z,t)$.

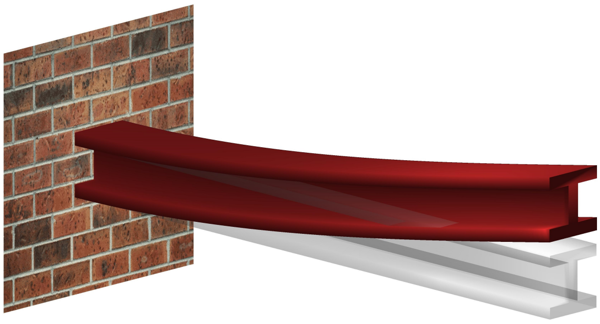


Figure 4: A cantilevered-free I-Beam deflecting in response to an external force.

where ρ denotes the density of the beam slice, A denotes the area of the beam slice in the xy-plane, dz denotes the thickness of the beam slice (in the z-direction), $V(z,t)$ denotes the shear stress in the beam, and $f(z,t)$ denotes the external force per unit length on the beam, as shown in Figure 7. Note that (1) is simply the result of Newton's laws, where the mass of the beam slice is given by $\rho A dz$, the total external force is the force per unit length $f(z,t)$ multiplied by the length dz, and the shear stresses $V(z,t)$ act like external forces from the point of view of the free-body diagram. Hence dividing (1) by dz, and letting $dz \to 0$, we have that

$$\rho A \frac{\partial^2 u}{\partial t^2}(z,t) = -\frac{\partial V}{\partial z}(z,t) + f(z,t) \tag{2}$$

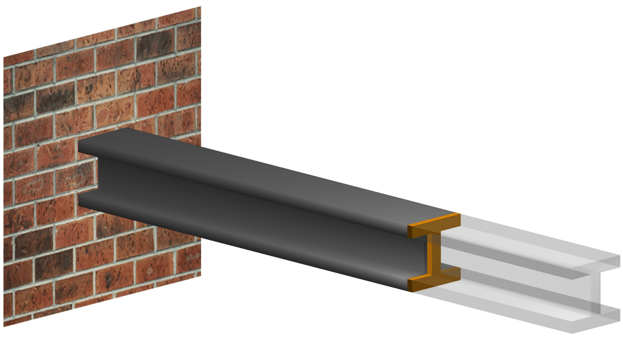


Figure 5: Infinitesimally-thin cross-sectional slice of a cantilevered-free I-Beam shown in orange.

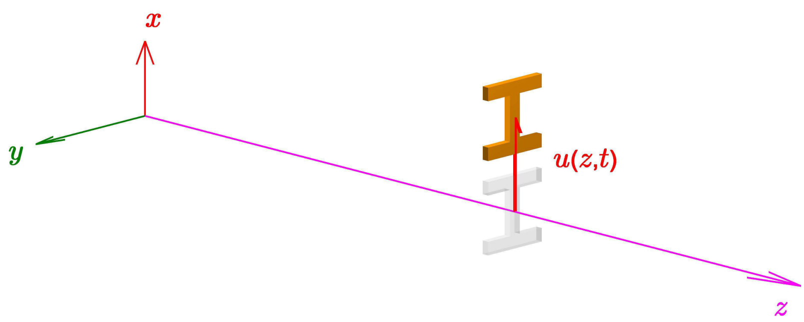


Figure 6: Deflection of an infinitesimally-thin cross-sectional slice of a cantilevered-free I-Beam.

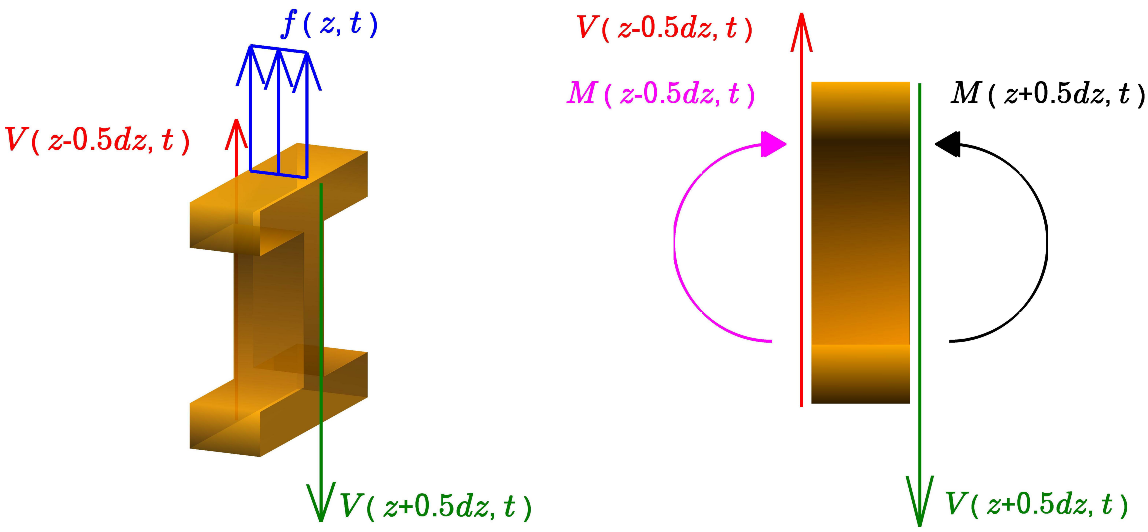


Figure 7: Free-body-diagram of an infinitesimally-thin cross-sectional slice of a cantilevered-free I-Beam, where the right figure shows the side-view.

1.2 Rotational Equation of Motion

Next, we examine the rotational equations of motion of the beam slice. Specifically, assuming that the slice is not rotating, we have that all of the moments acting on the slice must sum to zero, that is, from the rightmost figure of Figure 7, which shows a side-view of the beam, we have that the net moment about the positive y-axis is given by

$$M(z+0.5dz,t) - M(z-0.5dz,t) - 0.5dz\Big[V(z+0.5dz,t) + V(z-0.5dz,t)\Big] = 0 \tag{3}$$

where M denotes the internal moment in the beam. Hence dividing by dz and letting $dz \to 0$, we have that

$$\frac{\partial M}{\partial z}(z,t) = V(z,t) \tag{4}$$

1.3 Beam Strain

In the next two sections, we are going to develop an expression for the internal moment M in terms of the displacement u. To accomplish this, we will use Hooke's law to relate the stress σ and strain ε in the beam, that is,

$$\sigma = E\varepsilon \tag{5}$$

where E denotes the Young's modulus of the beam material. Specifically, from Figure 8, which shows an exaggerated side-view of the beam slice under bending, we have that

$$\delta\theta(z,t) = \theta(z + 0.5dz, t) - \theta(z - 0.5dz, t) \tag{6}$$

where the strain ε at a distance w from the neutral axis is approximately given by

$$\varepsilon = \frac{L(w,z,t) - L(0,z,t)}{L(0,z,t)} = \frac{(\rho - w)\,\delta\theta(z,t) - \rho\delta\theta(z,t)}{\rho\delta\theta(z,t)} = -\frac{w}{\rho} = -w\frac{\delta\theta(z,t)}{L(0,z,t)} \tag{7}$$

Furthermore, since dz is an infinitesimal element, from Figure 8 we have that

$$L(0,z,t) = \sec\left(\theta(z,t)\right) dz \tag{8}$$

$$\frac{\partial u}{\partial z}(z,t) = \tan\left(\theta(z,t)\right) \tag{9}$$

Therefore, from (6), (7), and (8), we find that

$$\varepsilon = -w\cos\left(\theta(z,t)\right)\frac{\partial\theta}{\partial z}(z,t) \tag{10}$$

where computing the derivative of (9), we have that

$$\frac{\partial^2 u}{\partial^2 z}(z,t) = \sec^2\left(\theta(z,t)\right)\frac{\partial\theta}{\partial z}(z,t) \tag{11}$$

and hence from (10) and (11),

$$\varepsilon = -w\cos^3\left(\theta(z,t)\right)\frac{\partial^2 u}{\partial^2 z}(z,t) \tag{12}$$

Additionally, assuming that θ is small, it follows that

$$\varepsilon = -x\frac{\partial^2 u}{\partial^2 z}(z,t) \tag{13}$$

1.4 Beam Stress

Finally, let $\delta M(x,z,t)$ denote the moment about the y-axis of the beam due to an axial stress $\sigma(x,z,t)$ on an area $\delta A(x,z,t)$ as shown in Figure 9. Then

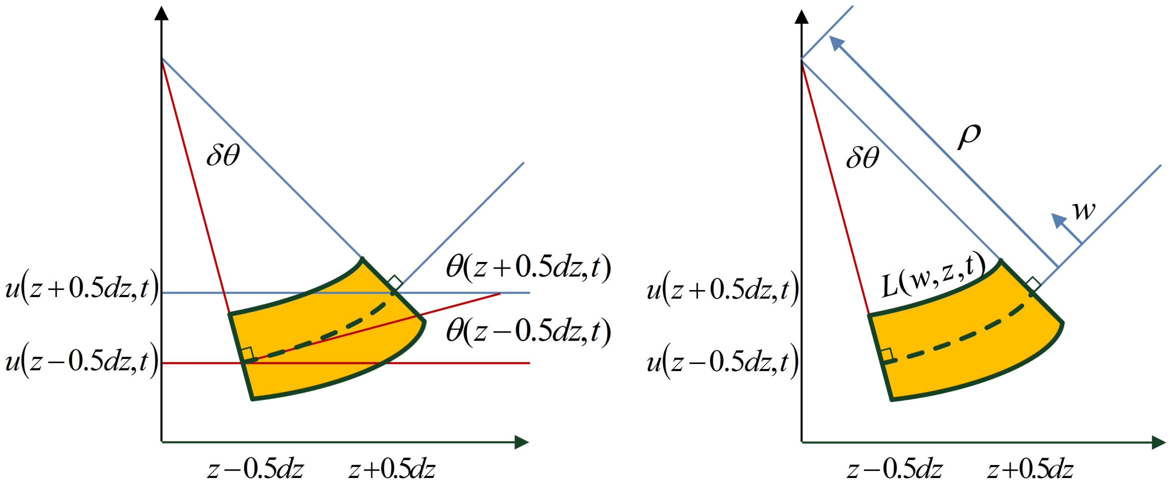
$$\delta M(x,z,t) = -x\sigma(x,z,t)\delta A(x,z,t) \tag{14}$$

where the negative sign in (14) denotes the fact that a positive stress σ at a positive height x above the neutral axis creates a negative moment about the y-axis. Hence from (5), (13), and (14), we have that

$$\delta M(x,z,t) = -xE\varepsilon(x,z,t)\delta A(x,z,t) = E\Big[x^2\delta A(x,z,t)\Big]\frac{\partial^2 u}{\partial^2 z}(z,t) \tag{15}$$

Furthermore, upon integrating (15) over the x-axis, we find that the internal moment M as a function of z and t is given by

$$M(z,t) = \int \delta M(x,z,t)dx = E\left[\int x^2\delta A(x,z,t)dx\right]\frac{\partial^2 u}{\partial^2 z}(z,t) \tag{16}$$



(a) Angles used in stress and strain relations.

(b) Angles and lengths used in stress and strain relations.

Figure 8: Strain diagrams and angles used in the derivation of the strains. The dotted line represents the neutral axis, that is, the axis running through the beam which undergoes no strain.

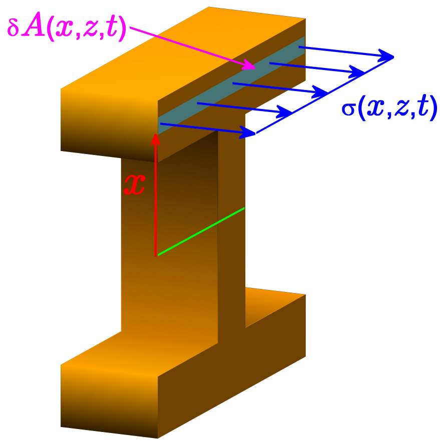


Figure 9: Stress on an infinitesimal area of the cross-sectional slice.

where the term in parentheses is called the *area moment of inertia*, and is given by (see Figure 9)

$$I(z) \triangleq \int x^2 \delta A(x, z, t) dx \tag{17}$$

Hence for a beam with a uniform cross-section, the area moment of inertia is constant, and we find that

$$M(z, t) = EI \frac{\partial^2 u}{\partial^2 z}(z, t) \tag{18}$$

1.5 The Euler-Bernoulli Beam Equation

Combining the translational (2), rotational (4), and constitutive (18) equations, we find that the Euler-Bernoulli beam equation is given by

$$\rho A \frac{\partial^2 u}{\partial t^2}(z, t) = -EI \frac{\partial^4 u}{\partial^4 z}(z, t) + f(z, t) \tag{19}$$

which assumes that the beam is homogenous, non-rotating, and with a constant cross-section. Furthermore, (19) is only valid for small beam deformations.

2 The Euler-Bernoulli Beam

The dynamic equation of an Euler-Bernoulli beam, which we derived in the previous section, is given by

$$\rho A \frac{\partial^2 u}{\partial t^2}(z,t) + EI \frac{\partial^4 u}{\partial z^4}(z,t) = f(z,t) \tag{20}$$

where ρ represents the beam density, A represents the cross-sectional area, $u(z,t)$ represents the position of the neutral axis in the x-direction (the axis which undergoes no strain), E denotes Young's modulus, I denotes the area moment of inertia, and $f(z,t)$ represents the force per unit length acting on the beam. Although not explicitly displayed in (20), we will always assume that the beam has a length of ℓ.

2.1 Beam Constraints

There are four types of constraints that are typically considered for beams, namely: *cantilevered*, *pinned*, *sliding*, and *free* constraints. Note that a *cantilevered* constraint is sometimes referred to as a *fixed* or *clamped* constraint. The mathematical representations of these constraints are shown in Table 1.

Constraint Type at $z = p$	Mathematical Constraint Representation	
Cantilevered	$u(p) = 0$	$\frac{\partial u}{\partial z}(p) = 0$
Pinned	$u(p) = 0$	$\frac{\partial^2 u}{\partial^2 z}(p) = 0$
Sliding	$\frac{\partial u}{\partial z}(p) = 0$	$\frac{\partial^3 u}{\partial^3 z}(p) = 0$
Free	$\frac{\partial^2 u}{\partial^2 z}(p) = 0$	$\frac{\partial^3 u}{\partial^3 z}(p) = 0$

Table 1: Four common beam constraints and their mathematical representation.

To determine a well-defined solution of (20), we must usually define constraints at two points along a beam, which we usually do at the origin and end of the beam. When a beam has a constraint at the origin and at the end, it is commonly referred to by the types of its constraints. For instance, if a beam is cantilevered at the origin and free at the end, then the beam is called a *cantilevered-free beam.* If the beam is cantilevered at both the origin and at the tip, then the beam is called a *cantilevered-cantilevered beam*, and so forth.

3 Cantilevered-Free Beam Deflection with a Static Point Force

Here we consider a cantilevered-free beam with a static point force F at $z = \gamma$. Hence the Euler-Bernoulli beam equation (20) becomes

$$EI \frac{\partial^4 u}{\partial z^4}(z) = F\delta(z - \gamma) \tag{21}$$

Note that since the beam is cantilevered at the origin, we have that

$$u(0) = 0, \qquad \frac{\partial u}{\partial z}(0) = 0 \tag{22}$$

and hence the beam deflection between $z = 0$ and $z = \gamma$ is of the form

$$u(z) = u_3 z^3 + u_2 z^2, \qquad 0 \leq z < \gamma \tag{23}$$

Furthermore, since the end of the beam is free, it follows that

$$\frac{\partial^2 u}{\partial z^2}(\ell) = 0, \qquad \frac{\partial^3 u}{\partial^3 z}(\ell) = 0 \tag{24}$$

Thus the beam deflection between $z = \gamma$ and $z = \ell$ is of the form

$$u(z) = u_1 z + u_0, \qquad \gamma < z \leq \ell \tag{25}$$

Next, integrating (21), it follows that

$$\frac{\partial^3 u}{\partial z^3}(\gamma^+) - \frac{\partial^3 u}{\partial z^3}(\gamma^-) = \frac{F}{EI} \tag{26}$$

where $\gamma^- < \gamma$ and $\gamma^+ > \gamma$ are points that are infinitesimally before and after γ. Hence from (23) and (25) it follows that

$$-6u_3 = \frac{F}{EI} \tag{27}$$

Thus the beam deflection is of the form

$$u(z) = \begin{cases} -\dfrac{F}{6EI} z^3 + u_2 z^2, & 0 \leq z < \gamma \\ u_1 z + u_0, & \gamma < z \leq \ell \end{cases} \tag{28}$$

Finally, from (21), note that the delta function represents an instantaneous change in the third derivative of $u(z)$ at $z = \gamma$. However, all of the lower derivatives should be continuous across the point $z = \gamma$, that is, we should find that

$$u(\gamma^-) = u(\gamma^+), \qquad \frac{\partial u}{\partial z}(\gamma^-) = \frac{\partial u}{\partial z}(\gamma^+), \qquad \frac{\partial^2 u}{\partial^2 z}(\gamma^-) = \frac{\partial^2 u}{\partial z^2}(\gamma^+) \tag{29}$$

Hence from (28)

$$-\frac{F}{6EI}\gamma^3 + u_2 \gamma^2 = u_1 \gamma + u_0 \tag{30}$$

$$-3\frac{F}{6EI}\gamma^2 + 2u_2 \gamma = u_1 \tag{31}$$

$$-6\frac{F}{6EI}\gamma + 2u_2 = 0 \tag{32}$$

and therefore, we have that

$$u_0 = -\frac{F\gamma^3}{6EI}, \qquad u_1 = \frac{F\gamma^2}{2EI}, \qquad u_2 = \frac{F\gamma}{2EI} \tag{33}$$

or simply

$$u(z) = \begin{cases} \dfrac{F}{6EI}\left[3\gamma - z\right] z^2, & 0 \leq z \leq \gamma \\ \dfrac{F\gamma^2}{6EI}\left[3z - \gamma\right], & \gamma \leq z \leq \ell \end{cases} \tag{34}$$

4 Beam Deflection with a Static Point Force

A summary of beam deflections in response to a static point force F at $z = \gamma$ are shown in Table 2 with various constraint combinations at the origin $z = 0$ and the end $z = \ell$. The fact that the sliding-sliding, sliding-free, and free-free beams do not have a solution should not be surprising since there is no position constraint, and hence the beam would keep moving towards infinity in response to a constant point force on the beam. Perhaps surprising is the fact that the pinned-sliding beam has a solution, but the pinned-free beam does not.

Figure 10 shows the beam deformations when $\ell = 1$, $\frac{F}{EI} = 1$, and the force is applied at the midway point of the beam, that is, $\gamma = \ell/2$. Figure 10 shows the deformations of cantilevered-cantilevered (CC), cantilevered-pinned (CP), and pinned-pinned (PP) beams on the left, and cantilevered-sliding (CS), cantilevered-free (CF), and pinned-sliding (PS) beams on the right. Note that the parameters used in both the left and right plots are the same, however, the scale of the deformations are approximately 10 times larger for the beam types on the right hand plot. Note that the cantilevered-cantilevered and pinned-pinned beams are symmetric since the constraints on both ends of those beams are equal, and the force occurs at the midpoint. Figure 11 shows the beam deformations when $\ell = 1$, $\frac{F}{EI} = 1$, and the force is applied at the beam endpoint, that is, $\gamma = \ell$.

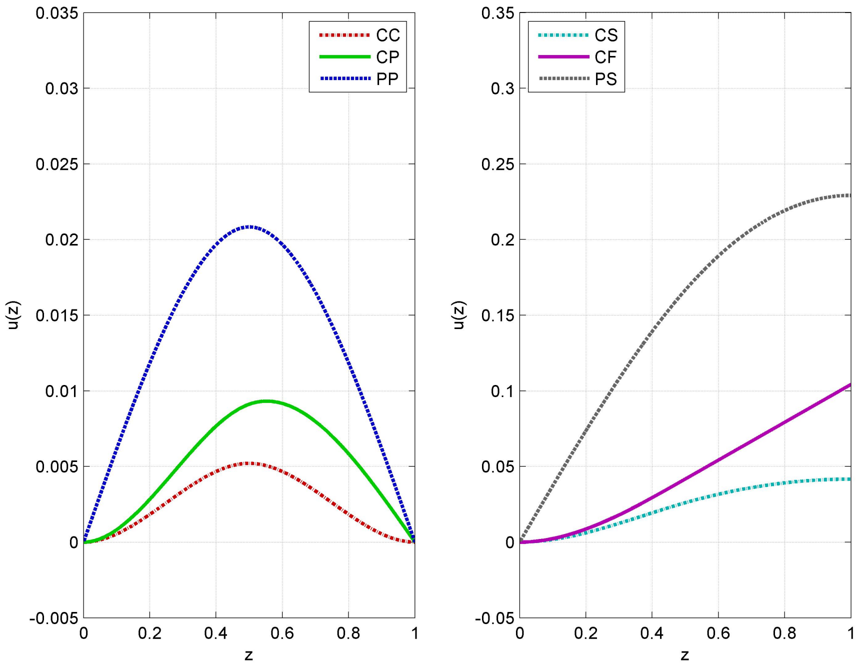


Figure 10: Beam deformations when $\ell = 1$, $\frac{F}{EI} = 1$, and the force is applied at the midway point of the beam, that is, $\gamma = \ell/2$. CC denotes the cantilevered-cantilevered beam, CP denotes the cantilevered-pinned beam, PP denotes the pinned-pinned beam, CS denotes the cantilevered-sliding beam, CF denotes the cantilevered-free beam, and PS denotes the pinned-sliding beam.

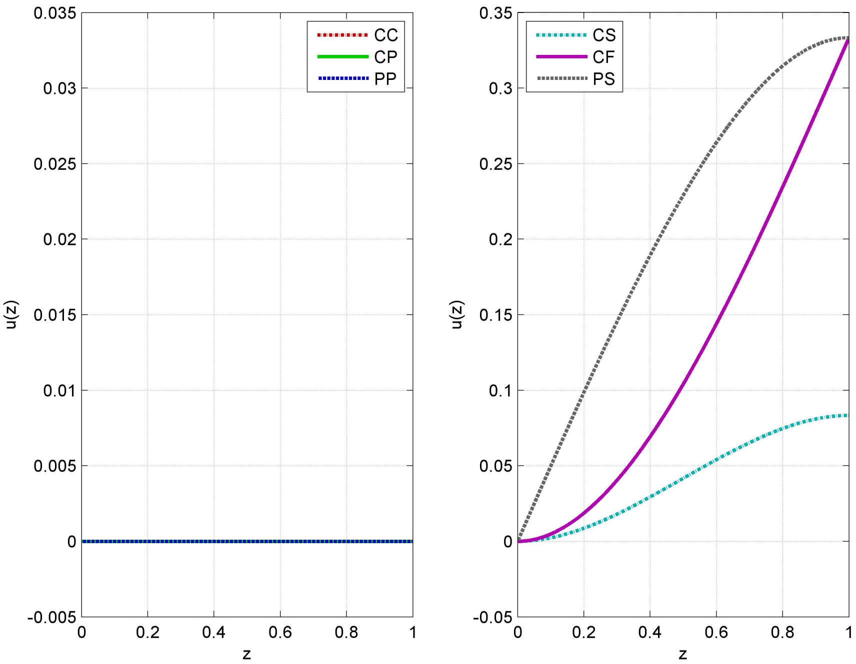


Figure 11: Beam deformations when $\ell = 1$, $\frac{F}{EI} = 1$, and the force is applied at the beam endpoint, that is, $\gamma = \ell$. CC denotes the cantilevered-cantilevered beam, CP denotes the cantilevered-pinned beam, PP denotes the pinned-pinned beam, CS denotes the cantilevered-sliding beam, CF denotes the cantilevered-free beam, and PS denotes the pinned-sliding beam.

Constraint at $z=0$	Constraint at $z=\ell$	Beam Deflection in the Region: $0 \le z \le \gamma$	Beam Deflection in the Region: $\gamma \le z \le \ell$
Cantilevered	Cantilevered	$-\frac{F}{6\ell^3 EI} z^2 (\ell-\gamma)^2 (\ell z - 3\ell\gamma + 2\gamma z)$	$-\frac{F}{6\ell^3 EI} \gamma^2 (\ell-z)^2 (\ell\gamma - 3\ell z + 2\gamma z)$
	Pinned	$-\frac{F}{12\ell^3 EI} z^2 (\ell-\gamma)(3\ell\gamma^2 - 6\ell^2\gamma + 2\ell^2 z - \gamma^2 z + 2\ell\gamma z)$	$-\frac{F}{12\ell^3 EI} \gamma^2 (\ell-z)(2\ell^2\gamma + 3\ell z^2 - 6\ell^2 z - \gamma z^2 + 2\ell\gamma z)$
	Sliding	$-\frac{F}{12\ell EI} z^2 (2\ell z - 6\ell\gamma + 3\gamma^2)$	$-\frac{F}{12\ell EI} \gamma^2 (2\ell\gamma - 6\ell z + 3z^2)$
	Free	$\frac{F}{6EI} z^2 (3\gamma - z)$	$-\frac{F}{6EI} \gamma^2 (\gamma - 3z)$
Pinned	Cantilevered	$-\frac{F}{12\ell^3 EI} z (\ell-\gamma)^2 (2\ell z^2 - 3\ell^2\gamma + \gamma z^2)$	$-\frac{F}{12\ell^3 EI} \gamma (\ell-z)^2 (2\ell\gamma^2 - 3\ell^2 z + \gamma^2 z)$
	Pinned	$-\frac{F}{6\ell EI} z (\ell-\gamma)(\gamma^2 - 2\ell\gamma + z^2)$	$-\frac{F}{6\ell EI} \gamma (\ell-z)(\gamma^2 - 2\ell z + z^2)$
	Sliding	$-\frac{F}{6EI} z (3\gamma^2 - 6\ell\gamma + z^2)$	$-\frac{F}{6EI} \gamma (\gamma^2 - 6\ell z + 3z^2)$
	Free	No Solution	No Solution
Sliding	Cantilevered	$\frac{F}{12\ell EI} (\ell-\gamma)^2 (2\ell\gamma + \ell^2 - 3z^2)$	$\frac{F}{12\ell EI} (\ell-z)^2 (2\ell z + \ell^2 - 3\gamma^2)$
	Pinned	$\frac{F}{6EI} (\ell-\gamma)(2\ell\gamma + 2\ell^2 - \gamma^2 - 3z^2)$	$\frac{F}{6EI} (\ell-z)(2\ell z + 2\ell^2 - 3\gamma^2 - z^2)$
	Sliding	No Solution	No Solution
	Free	No Solution	No Solution
Free	Cantilevered	$\frac{F}{6EI} (\ell-\gamma)^2 (2\ell + \gamma - 3z)$	$\frac{F}{6EI} (\ell-z)^2 (2\ell - 3\gamma + z)$
	Pinned	No Solution	No Solution
	Sliding	No Solution	No Solution
	Free	No Solution	No Solution

Table 2: Deflection of a beam of length ℓ in response to a static point force F at $z=\gamma$.

5 Eigenmodes and Eigenfrequencies

The eigenmodes and eigenfrequencies of a Euler-Bernoulli beam together make a basis for all of the solutions of the unforced Euler-Bernoulli beam equation

$$\rho A \frac{\partial^2 u}{\partial t^2}(z,t) + EI \frac{\partial^4 u}{\partial z^4}(z,t) = 0 \tag{35}$$

subject to the spatial boundary constraints of the beam being considered (see Table 1 for some examples). Specifically, suppose that $\phi_1(z), \phi_2(z), \ldots$ form a mutually orthogonal set of basis functions over $z \in [0, \ell]$, where $\phi_1(z), \phi_2(z), \ldots$ satisfy the normalization constraints

$$\int_0^\ell \phi_1^2(z)dz = 1, \quad \int_0^\ell \phi_2^2(z)dz = 1, \quad \ldots \tag{36}$$

and, for each positive integer j, there exists $\psi_j(t)$ such that

$$u_j(z,t) = \psi_j(t)\phi_j(z) \tag{37}$$

is a solution of (35). Then every solution of (35) which satisfies the spatial boundary constraints can be written in the form

$$u(z,t) = \sum_{j=1}^{\infty} \psi_j(t)\phi_j(z) \tag{38}$$

where $\phi_j(z)$ is called the j^{th} *eigenmode of the beam*, and together $\phi_1(z), \phi_2(z), \ldots$ are called the *eigenmodes of the beam*. Note that since $\phi_1(z), \phi_2(z), \ldots$ satisfy the spatial boundary conditions of the beam being considered, such as (22) and (24) for a cantilevered-free beam, the eigenmodes are unique to the type of beam. Furthermore, the normalization constraints (36) are only enforced so that there is no scale ambiguity between $\psi_j(t)$ and $\phi_j(z)$ in the product $\psi_j(t)\phi_j(z)$.

5.1 Solutions of the Unforced Beam Equation

Here we will develop conditions for the solutions of (35) of the form (37), before proceeding to develop the specific form of the eigenmodes in the next section.

First, let $u_j(z,t)$ be a solution of (35) of the form (37). Then

$$\rho A \frac{\partial^2 \psi_j}{\partial t^2}(t)\phi_j(z) + EI\psi_j(t)\frac{\partial^4 \phi_j}{\partial z^4}(z) = 0 \tag{39}$$

Specifically, from (39) it follows that

$$\frac{\rho A}{EI}\frac{1}{\psi_j(t)}\frac{\partial^2 \psi_j}{\partial t^2}(t) = -\frac{1}{\phi_j(z)}\frac{\partial^4 \phi_j}{\partial z^4}(z) \tag{40}$$

where, since the left-hand side of (40) is only a function of t, and the right-hand side of (40) is only a function of z, it follows that both sides must be equal to some constant value, which we arbitrarily choose to be $\left(2\pi \bar{f}_j\right)^2/\ell^4$, that is,

$$\frac{\rho A}{EI}\frac{1}{\psi_j(t)}\frac{\partial^2 \psi_j}{\partial t^2}(t) = -\frac{1}{\phi_j(z)}\frac{\partial^4 \phi_j}{\partial z^4}(z) = \frac{\left(2\pi \bar{f}_j\right)^2}{\ell^4} \tag{41}$$

Thus, if $u_j(z,t) = \psi_j(t)\phi_j(z)$ is a solution of (35), then the components $\psi_j(t)$ and $\phi_j(z)$ satisfy

$$\frac{\partial^2 \psi_j}{\partial t^2}(t) - \frac{\left(2\pi \bar{f}_j\right)^2 EI}{\ell^4 \rho A}\psi_j(t) = 0 \tag{42}$$

$$\frac{\partial^4 \phi_j}{\partial z^4}(z) + \frac{\left(2\pi \bar{f}_j\right)^2}{\ell^4}\phi_j(z) = 0 \tag{43}$$

and therefore $\psi_j(t)$ and $\phi_j(z)$ are of the form

$$\psi_j(t) = \alpha_j \cos\left(2\pi t \bar{f}_j \sqrt{\frac{EI}{\rho A \ell^4}}\right) + \beta_j \sin\left(2\pi t \bar{f}_j \sqrt{\frac{EI}{\rho A \ell^4}}\right) \tag{44}$$

$$\phi_j(z) = a_j \cos\left(z\sqrt{\frac{2\pi \bar{f}_j}{\ell^2}}\right) + b_j \sin\left(z\sqrt{\frac{2\pi \bar{f}_j}{\ell^2}}\right) + c_j \exp\left(+z\sqrt{\frac{2\pi \bar{f}_j}{\ell^2}}\right) + d_j \exp\left(-z\sqrt{\frac{2\pi \bar{f}_j}{\ell^2}}\right) \tag{45}$$

Note that although the choice of constant $\left(2\pi \bar{f}_j\right)^2/\ell^4$ in (41) might seem a little strange, in the next section we will show that $\bar{f}_j$ has a special interpretation. Specifically, when we calculate the beam eigenmodes in the next section, we will show that $\bar{f}_j$ is the j^{th} *specific eigenfrequency of the beam*, which is a scaled version of the j^{th} *eigenfrequency of the beam*.

Note that if $\phi_1(z), \phi_2(z), \ldots$ satisfy the normalization constraint (36) and the orthogonality constraint

$$\int_0^\ell \phi_i(z)\phi_j(z)dz = 0, \qquad i,j \in \mathbb{Z}^+, \quad i \neq j \tag{46}$$

then one complete set of basis functions which satisfy (35) are given by the Fourier series, that is,

$$\phi_{2j-1}(z) = \sqrt{\frac{2}{\ell}}\cos\left(\frac{2\pi j z}{\ell}\right) \tag{47}$$

$$\phi_{2j}(z) = \sqrt{\frac{2}{\ell}}\sin\left(\frac{2\pi j z}{\ell}\right) \tag{48}$$

which are obtained by using the values

$$a_{2j-1} = b_{2j} = \sqrt{\frac{2}{\ell}} \tag{49}$$

$$a_{2j} = b_{2j-1} = c_j = d_j = 0 \tag{50}$$

$$\bar{f}_{2j-1} = \bar{f}_{2j} = 2\pi j^2 \tag{51}$$

in (45). Furthermore, since the Fourier series forms a complete basis, it follows that there exists at least one complete set of basis functions $\phi_1(z), \phi_2(z), \ldots$ which satisfy (35). Thus every general solution of (35) can be written in the form (38), where $\phi_j(z)$ are the Fourier series basis functions (47) and (48).

5.2 Cantilevered-Free Beam Eigenmodes

In the previous section, we showed that there was at least one complete set of basis functions $\phi_1(z), \phi_2(z), \ldots$ such that every solution of (35) can be written in the form (38). Here we take a different approach. Namely, here we show that there exists a unique set of basis functions $\phi_1(z), \phi_2(z), \ldots$ which satisfy the unforced beam equation (35), the normalization constraints (36), the orthogonality constraints (46), and the cantilevered-free beam boundary conditions

$$u_j(0,t) = 0, \qquad \frac{\partial u_j}{\partial z}(0,t) = 0, \qquad \frac{\partial^2 u_j}{\partial^2 z}(\ell,t) = 0, \qquad \frac{\partial^3 u_j}{\partial^3 z}(\ell,t) = 0 \tag{52}$$

These basis functions $\phi_1(z), \phi_2(z), \ldots$ are called the *eigenmodes of the beam.* Specifically, they are called the *eigenmodes of the cantilevered-free beam* since they satisfy the cantilevered-free boundary conditions.

To compute the eigenmodes, first note that since $u_j(z,t) = \psi_j(t)\phi_j(z)$, the eigenmodes satisfy

$$\phi_j(0) = 0, \qquad \frac{\partial \phi_j}{\partial z}(0) = 0, \qquad \frac{\partial^2 \phi_j}{\partial^2 z}(\ell) = 0, \qquad \frac{\partial^3 \phi_j}{\partial^3 z}(\ell) = 0 \tag{53}$$

Specifically, computing the derivatives of $\phi_j(z)$ with respect to z, where $\phi_j(z)$ is given by (45), we find that

$$\phi_j(z) = +a_j\left[\frac{\sigma_j}{\ell}\right]^0 \cos\left(\frac{z\sigma_j}{\ell}\right) + b_j\left[\frac{\sigma_j}{\ell}\right]^0 \sin\left(\frac{z\sigma_j}{\ell}\right) + c_j\left[\frac{\sigma_j}{\ell}\right]^0 \exp\left(\frac{z\sigma_j}{\ell}\right) + d_j\left[\frac{\sigma_j}{\ell}\right]^0 \exp\left(-\frac{z\sigma_j}{\ell}\right) \tag{54}$$

$$\frac{\partial \phi_j}{\partial z}(z) = -a_j\left[\frac{\sigma_j}{\ell}\right]^1 \sin\left(\frac{z\sigma_j}{\ell}\right) + b_j\left[\frac{\sigma_j}{\ell}\right]^1 \cos\left(\frac{z\sigma_j}{\ell}\right) + c_j\left[\frac{\sigma_j}{\ell}\right]^1 \exp\left(\frac{z\sigma_j}{\ell}\right) - d_j\left[\frac{\sigma_j}{\ell}\right]^1 \exp\left(-\frac{z\sigma_j}{\ell}\right) \tag{55}$$

$$\frac{\partial^2 \phi_j}{\partial^2 z}(z) = -a_j\left[\frac{\sigma_j}{\ell}\right]^2 \cos\left(\frac{z\sigma_j}{\ell}\right) - b_j\left[\frac{\sigma_j}{\ell}\right]^2 \sin\left(\frac{z\sigma_j}{\ell}\right) + c_j\left[\frac{\sigma_j}{\ell}\right]^2 \exp\left(\frac{z\sigma_j}{\ell}\right) + d_j\left[\frac{\sigma_j}{\ell}\right]^2 \exp\left(-\frac{z\sigma_j}{\ell}\right) \tag{56}$$

$$\frac{\partial^3 \phi_j}{\partial^3 z}(z) = +a_j\left[\frac{\sigma_j}{\ell}\right]^3 \sin\left(\frac{z\sigma_j}{\ell}\right) - b_j\left[\frac{\sigma_j}{\ell}\right]^3 \cos\left(\frac{z\sigma_j}{\ell}\right) + c_j\left[\frac{\sigma_j}{\ell}\right]^3 \exp\left(\frac{z\sigma_j}{\ell}\right) - d_j\left[\frac{\sigma_j}{\ell}\right]^3 \exp\left(-\frac{z\sigma_j}{\ell}\right) \tag{57}$$

where

$$\sigma_j \triangleq \sqrt{2\pi \bar{f}_j} \tag{58}$$

Hence from (53) and (54)-(57), we find that the coefficients a_j, b_j, c_j and d_j must satisfy

$$0 = +a_j + c_j + d_j \tag{59}$$

$$0 = +b_j + c_j - d_j \tag{60}$$

$$0 = -a_j \cos\left(\sqrt{2\pi\bar{f}_j}\right) - b_j \sin\left(\sqrt{2\pi\bar{f}_j}\right) + c_j \exp\left(+\sqrt{2\pi\bar{f}_j}\right) + d_j \exp\left(-\sqrt{2\pi\bar{f}_j}\right) \tag{61}$$

$$0 = +a_j \sin\left(\sqrt{2\pi\bar{f}_j}\right) - b_j \cos\left(\sqrt{2\pi\bar{f}_j}\right) + c_j \exp\left(+\sqrt{2\pi\bar{f}_j}\right) - d_j \exp\left(-\sqrt{2\pi\bar{f}_j}\right) \tag{62}$$

However, the equations (59)-(62) only yield a solution when $\bar{f}_j$ satisfies

$$\cos\left(\sqrt{2\pi\bar{f}_j}\right) \cosh\left(\sqrt{2\pi\bar{f}_j}\right) = -1 \tag{63}$$

It turns out that, for every type of beam, $\bar{f}_j$ must satisfy an equation similar to (63). We call this equation the *characteristic equation of the beam.* Specifically, since we are considering a cantilevered-free beam, (63) is called the *characteristic equation of a cantilevered-free beam.*

In this case, we find that one set of coefficients a_j, b_j, c_j, and d_j which satisfy the constraints (59)-(62) are given by

$$a_j = +\sin\left(\sqrt{2\pi\bar{f}_j}\right) + \sinh\left(\sqrt{2\pi\bar{f}_j}\right) \tag{64}$$

$$b_j = -\cos\left(\sqrt{2\pi\bar{f}_j}\right) - \cosh\left(\sqrt{2\pi\bar{f}_j}\right) \tag{65}$$

$$c_j = -\frac{1}{2}\sin\left(\sqrt{2\pi\bar{f}_j}\right) + \frac{1}{2}\cos\left(\sqrt{2\pi\bar{f}_j}\right) + \frac{1}{2}\exp\left(-\sqrt{2\pi\bar{f}_j}\right) \tag{66}$$

$$d_j = -\frac{1}{2}\sin\left(\sqrt{2\pi\bar{f}_j}\right) - \frac{1}{2}\cos\left(\sqrt{2\pi\bar{f}_j}\right) - \frac{1}{2}\exp\left(+\sqrt{2\pi\bar{f}_j}\right) \tag{67}$$

Hence the unnormalized eigenmodes for a cantilevered-free beam, that is, before applying the normalization constraints (36), are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\left[\sin\left(\sqrt{2\pi\bar{f}_j}\right) + \sinh\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\cos\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) - \cosh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right] \\ &- \left[\cos\left(\sqrt{2\pi\bar{f}_j}\right) + \cosh\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\sin\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) - \sinh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right]\end{aligned} \tag{68}$$

After applying the normalization constraints (36), we find that the eigenmodes of a cantilevered-free beam are given by

$$\phi_j(z) = \frac{\bar{\phi}_j(z)}{\sqrt{\ell}\left|\sin\left(\sqrt{2\pi\bar{f}_j}\right) + \sinh\left(\sqrt{2\pi\bar{f}_j}\right)\right|} \tag{69}$$

Note that the first five positive solutions of (63) are

$$\bar{f}_1 = 0.55959, \qquad \bar{f}_2 = 3.5069, \qquad \bar{f}_3 = 9.8194, \qquad \bar{f}_4 = 19.242, \qquad \bar{f}_5 = 31.809 \tag{70}$$

Furthermore, since

$$\cosh\left(\sqrt{2\pi\bar{f}_j}\right) = \frac{1}{2}\left[\exp\left(+\sqrt{2\pi\bar{f}_j}\right) + \exp\left(-\sqrt{2\pi\bar{f}_j}\right)\right] \tag{71}$$

it follows that $\cosh\left(\sqrt{2\pi\bar{f}_j}\right)$ approaches zero very rapidly. Hence, aside from the first two or three solutions of (63), the rest of the solutions are approximately the same as the roots of $\cos\left(\sqrt{2\pi\bar{f}_j}\right)$. Thus as a general rule, one could say that the solutions of (63) are given by

$$\bar{f}_1 = 0.55959, \qquad \bar{f}_2 = 3.5069, \qquad \bar{f}_3 = 9.8194, \qquad \bar{f}_4 = \frac{49\pi}{8}, \qquad \bar{f}_5 = \frac{81\pi}{8}, \qquad \ldots \tag{72}$$

where even the first three solutions are close to the roots of $\cos\left(\sqrt{2\pi\bar{f}_j}\right)$, that is, $\pi/8 = 0.39270$, $9\pi/8 = 3.5343$, and $25\pi/8 = 9.8175$.

Finally, letting $\bar{f}_j$ denote the j^{th} positive solution of (63), and letting $\phi_j(z)$ be given by (69), where $\bar{\phi}_j(z)$ is given by (68), we have that every solution $u(z,t)$ of (35) which satisfies the cantilevered-free beam boundary conditions (52) is of the form

$$u(z,t) = \sum_{j=1}^{\infty}\left[\alpha_j \cos\left(2\pi t\bar{f}_j\sqrt{\frac{EI}{\rho A\ell^4}}\right) + \beta_j \sin\left(2\pi t\bar{f}_j\sqrt{\frac{EI}{\rho A\ell^4}}\right)\right]\phi_j(z) \tag{73}$$

where the coefficients α_j and β_j are determined from some other constraints of the problem at hand.

Note that the *eigenfrequencies* f_j are the frequencies of the time-varying components, that is, the j^{th} *eigenfrequency* f_j is given by

$$f_j = \bar{f}_j\sqrt{\frac{EI}{\rho A\ell^4}} \tag{74}$$

and corresponds to the frequency of the vibration of the j^{th} eigenmode $\phi_j(z)$. From (74), we can finally see why our choice of the constant $\left(2\pi\bar{f}_j\right)^2/\ell^4$ in (41) was justified. Specifically, $\bar{f}_j$ is called the j^{th} *specific eigenfrequency of the beam* since it is an invariant property of the type of beam being considered. To obtain the eigenfrequencies of the beam, we simply multiply the specific eigenfrequency $\bar{f}_j$ by the constants of the problem, see (74).

6 Beam Eigenmodes, Eigenfrequencies, and Characteristic Equations

In the previous section, we computed the eigenmodes and eigenfrequencies of a cantilevered-free beam. Here we compute the eigenmodes and eigenfrequencies of all of the 16 beam types.

First, recall that every solution of the unforced beam equation (35) is of the form

$$u(z,t) = \sum_{j=1}^{\infty} \left[\alpha_j \cos\left(2\pi t \bar{f}_j \sqrt{\frac{EI}{\rho A \ell^4}}\right) + \beta_j \sin\left(2\pi t \bar{f}_j \sqrt{\frac{EI}{\rho A \ell^4}}\right)\right] \phi_j(z) \tag{75}$$

where the eigenmodes $\phi_j(z)$ and eigenfrequencies f_j are of the form

$$\phi_j(z) = a_j \cos\left(z\sqrt{\frac{2\pi \bar{f}_j}{\ell^2}}\right) + b_j \sin\left(z\sqrt{\frac{2\pi \bar{f}_j}{\ell^2}}\right) + c_j \exp\left(+z\sqrt{\frac{2\pi \bar{f}_j}{\ell^2}}\right) + d_j \exp\left(-z\sqrt{\frac{2\pi \bar{f}_j}{\ell^2}}\right) \tag{76}$$

$$f_j = \bar{f}_j \sqrt{\frac{EI}{\rho A \ell^4}} \tag{77}$$

Furthermore, the coefficients a_j, b_j, c_j, and d_j as well as $\bar{f}_j$ are determined by applying the spatial boundary constraints to the beam, and by applying the normalization constraint (36) to the eigenmodes. For instance, after applying the boundary constraints of a cantilevered-free beam, we found that $\bar{f}_j$ must satisfy the *characteristic equation* (63). Furthermore, after applying the boundary conditions (53) and normalization constraints (36), we found that the eigenmodes are of the form (69).

In general, every beam type has its own *characteristic equation*, *eigenmodes*, and *eigenfrequencies*, which will be different from those for a cantilevered-free beam, that is, different from (63) and (69). The characteristic equation of each type of beam is displayed in Table 3. The first five specific eigenfrequencies of each type of beam are shown in Table 4. The form of the unnormalized eigenmodes $\bar{\phi}_j(z)$ of each type of beam, that is, before applying the normalization constraints (36), are displayed in Table 5. The normalizations which need to be applied to $\bar{\phi}_j(z)$ to obtain the eigenmodes are shown in Table 6. Specifically, to obtain the eigenmodes $\phi_j(z)$ of a beam, one would obtain the unnormalized eigenmode $\bar{\phi}_j(z)$ from Table 5, and the normalization n_j from Table 6, after which we find that the eigenmodes which satisfy the normalization constraints (36) are given by

$$\phi_j(z) = \frac{1}{\sqrt{n_j}} \bar{\phi}_j(z) \tag{78}$$

Constraint at $z=0$	Constraint at $z=\ell$	Characteristic Equation	Approximate $\bar{f}_j$ for large j
Cantilevered	Cantilevered	$\cos\left(\sqrt{2\pi\bar{f}_j}\right)\cosh\left(\sqrt{2\pi\bar{f}_j}\right)=+1$	$\frac{\pi}{2}\left(j+\frac{1}{2}\right)^2$
	Pinned	$\tan\left(\sqrt{2\pi\bar{f}_j}\right)=+\tanh\left(\sqrt{2\pi\bar{f}_j}\right)$	$\frac{\pi}{2}\left(j+\frac{1}{4}\right)^2$
	Sliding	$\tan\left(\sqrt{2\pi\bar{f}_j}\right)=-\tanh\left(\sqrt{2\pi\bar{f}_j}\right)$	$\frac{\pi}{2}\left(j-\frac{1}{4}\right)^2$
	Free	$\cos\left(\sqrt{2\pi\bar{f}_j}\right)\cosh\left(\sqrt{2\pi\bar{f}_j}\right)=-1$	$\frac{\pi}{2}\left(j+\frac{1}{2}\right)^2$
Pinned	Cantilevered	$\tan\left(\sqrt{2\pi\bar{f}_j}\right)=+\tanh\left(\sqrt{2\pi\bar{f}_j}\right)$	$\frac{\pi}{2}\left(j+\frac{1}{4}\right)^2$
	Pinned	$\sin\left(\sqrt{2\pi\bar{f}_j}\right)=0$	$\frac{\pi}{2}\left(j\right)^2$
	Sliding	$\cos\left(\sqrt{2\pi\bar{f}_j}\right)=0$	$\frac{\pi}{2}\left(j+\frac{1}{2}\right)^2$
	Free	$\tan\left(\sqrt{2\pi\bar{f}_j}\right)=+\tanh\left(\sqrt{2\pi\bar{f}_j}\right)$	$\frac{\pi}{2}\left(j+\frac{1}{4}\right)^2$
Sliding	Cantilevered	$\tan\left(\sqrt{2\pi\bar{f}_j}\right)=-\tanh\left(\sqrt{2\pi\bar{f}_j}\right)$	$\frac{\pi}{2}\left(j-\frac{1}{4}\right)^2$
	Pinned	$\cos\left(\sqrt{2\pi\bar{f}_j}\right)=0$	$\frac{\pi}{2}\left(j+\frac{1}{2}\right)^2$
	Sliding	$\sin\left(\sqrt{2\pi\bar{f}_j}\right)=0$	$\frac{\pi}{2}\left(j\right)^2$
	Free	$\tan\left(\sqrt{2\pi\bar{f}_j}\right)=-\tanh\left(\sqrt{2\pi\bar{f}_j}\right)$	$\frac{\pi}{2}\left(j-\frac{1}{4}\right)^2$
Free	Cantilevered	$\cos\left(\sqrt{2\pi\bar{f}_j}\right)\cosh\left(\sqrt{2\pi\bar{f}_j}\right)=-1$	$\frac{\pi}{2}\left(j+\frac{1}{2}\right)^2$
	Pinned	$\tan\left(\sqrt{2\pi\bar{f}_j}\right)=+\tanh\left(\sqrt{2\pi\bar{f}_j}\right)$	$\frac{\pi}{2}\left(j+\frac{1}{4}\right)^2$
	Sliding	$\tan\left(\sqrt{2\pi\bar{f}_j}\right)=-\tanh\left(\sqrt{2\pi\bar{f}_j}\right)$	$\frac{\pi}{2}\left(j-\frac{1}{4}\right)^2$
	Free	$\cos\left(\sqrt{2\pi\bar{f}_j}\right)\cosh\left(\sqrt{2\pi\bar{f}_j}\right)=+1$	$\frac{\pi}{2}\left(j+\frac{1}{2}\right)^2$

Table 3: The characteristic equation of various beam types.

Constraint at $z = 0$	Constraint at $z = \ell$	$\bar{f}_1$	$\bar{f}_2$	$\bar{f}_3$	$\bar{f}_4$	$\bar{f}_5$
Cantilevered	Cantilevered	0	3.5608	9.8155	19.242	31.809
	Pinned	0	2.4539	7.9522	16.592	28.373
	Sliding	0	0.89020	4.8106	11.879	22.089
	Free	0.55959	3.5069	9.8194	19.242	31.809
Pinned	Cantilevered	0	2.4539	7.9522	16.592	28.373
	Pinned	0	1.5708	6.2832	14.137	25.133
	Sliding	0.39270	3.5343	9.8175	19.242	31.809
	Free	0	2.4539	7.9522	16.592	28.373
Sliding	Cantilevered	0	0.89020	4.8106	11.879	22.089
	Pinned	0.39270	3.5343	9.8175	19.242	31.809
	Sliding	0	1.5708	6.2832	14.137	25.133
	Free	0	0.89020	4.8106	11.879	22.089
Free	Cantilevered	0.55959	3.5069	9.8194	19.242	31.809
	Pinned	0	2.4539	7.9522	16.592	28.373
	Sliding	0	0.89020	4.8106	11.879	22.089
	Free	0	3.5608	9.8155	19.242	31.809

Table 4: The first five specific eigenfrequencies of various beam types. Note the true eigenfrequencies can be obtained from the equation (77).

Constraint at $z = 0$	Constraint at $z = \ell$	Form of the Unnormalized Eigenmodes $\bar{\phi}_j(z)$
Cantilevered	Cantilevered	$\left[\cos\left(\sqrt{2\pi\bar{f}_j}\right) - \cosh\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\sinh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) - \sin\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right] + \left[\sin\left(\sqrt{2\pi\bar{f}_j}\right) - \sinh\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\cos\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) - \cosh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right]$
	Pinned	$\left[\cos\left(\sqrt{2\pi\bar{f}_j}\right) - \cosh\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\sinh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) - \sin\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right] + \left[\sin\left(\sqrt{2\pi\bar{f}_j}\right) - \sinh\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\cos\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) - \cosh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right]$
	Sliding	$\left[\sin\left(\sqrt{2\pi\bar{f}_j}\right) + \sinh\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\sinh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) - \sin\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right] - \left[\cos\left(\sqrt{2\pi\bar{f}_j}\right) - \cosh\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\cos\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) - \cosh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right]$
	Free	$\left[\cos\left(\sqrt{2\pi\bar{f}_j}\right) + \cosh\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\sinh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) - \sin\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right] + \left[\sin\left(\sqrt{2\pi\bar{f}_j}\right) + \sinh\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\cos\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) - \cosh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right]$
Pinned	Cantilevered	$\sin\left(\sqrt{2\pi\bar{f}_j}\right)\sinh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) - \sinh\left(\sqrt{2\pi\bar{f}_j}\right)\sin\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)$
	Pinned	$\sin\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)$
	Sliding	$\sin\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)$
	Free	$\sin\left(\sqrt{2\pi\bar{f}_j}\right)\sinh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) + \sinh\left(\sqrt{2\pi\bar{f}_j}\right)\sin\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)$
Sliding	Cantilevered	$\cos\left(\sqrt{2\pi\bar{f}_j}\right)\cosh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) - \cosh\left(\sqrt{2\pi\bar{f}_j}\right)\cos\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)$
	Pinned	$\cos\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)$
	Sliding	$\cos\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)$
	Free	$\cos\left(\sqrt{2\pi\bar{f}_j}\right)\cosh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) + \cosh\left(\sqrt{2\pi\bar{f}_j}\right)\cos\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)$
Free	Cantilevered	$\left[\cos\left(\sqrt{2\pi\bar{f}_j}\right) + \cosh\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\sin\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) + \sinh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right] - \left[\sin\left(\sqrt{2\pi\bar{f}_j}\right) + \sinh\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\cos\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) + \cosh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right]$
	Pinned	$\left[\cos\left(\sqrt{2\pi\bar{f}_j}\right) + \cosh\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\sin\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) + \sinh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right] - \left[\sin\left(\sqrt{2\pi\bar{f}_j}\right) + \sinh\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\cos\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) + \cosh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right]$
	Sliding	$\left[\sin\left(\sqrt{2\pi\bar{f}_j}\right) - \sinh\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\sin\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) + \sinh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right] + \left[\cos\left(\sqrt{2\pi\bar{f}_j}\right) + \cosh\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\cos\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) + \cosh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right]$
	Free	$\left[\cos\left(\sqrt{2\pi\bar{f}_j}\right) - \cosh\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\sin\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) + \sinh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right] + \left[\sinh\left(\sqrt{2\pi\bar{f}_j}\right) - \sin\left(\sqrt{2\pi\bar{f}_j}\right)\right]\left[\cos\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right) + \cosh\left(z\sqrt{\frac{2\pi\bar{f}_j}{\ell^2}}\right)\right]$

Table 5: The unnormalized eigenmodes of various beam types.

7 Time-Varying Beam Deflection when a Point Force is Released

In Section 4, we computed the beam deflections in response to a static point force F at $z = \gamma$. Specifically, Table 2 summarized the deflections with various constraint combinations at the origin $z = 0$ and the end $z = \ell$. Here we will determine the time-varying response when the force F is released, that is, we compute the solution $u(z,t)$ of

$$\rho A \frac{\partial^2 u}{\partial t^2}(z,t) + EI \frac{\partial^4 u}{\partial z^4}(z,t) = F\delta(z-\gamma), \qquad t \le 0 \tag{79}$$

$$\rho A \frac{\partial^2 u}{\partial t^2}(z,t) + EI \frac{\partial^4 u}{\partial z^4}(z,t) = 0, \qquad t > 0 \tag{80}$$

7.1 A Cantilevered-Free Beam

Here we assume that the beam is a cantilevered-free beam. Hence from Table 2, we have that the beam deformation at time $t = 0$ will be given by

$$u(z,0) = \begin{cases} \dfrac{Fz^2}{6EI}\left[3\gamma - z\right], & 0 \le z \le \gamma \\[2ex] \dfrac{F\gamma^2}{6EI}\left[3z - \gamma\right], & \gamma \le z \le \ell \end{cases} \tag{81}$$

Furthermore, since the beam is initially at rest, the time derivative of the beam deflection will initially be zero, that is,

$$\frac{\partial u}{\partial t}(z,0) = 0 \tag{82}$$

However, after the force is released, the beam deflection will satisfy the unforced beam equation (80). Hence the time-varying beam deflection after the force is released can be decomposed into the infinite sum of its eigenmodes, that is, the solution $u(z,t)$ of (79)-(80) will be of the form

$$u(z,t) = \sum_{j=1}^{\infty} \left[\alpha_j \cos\left(2\pi t \bar{f}_j \sqrt{\frac{EI}{\rho A \ell^4}}\right) + \beta_j \sin\left(2\pi t \bar{f}_j \sqrt{\frac{EI}{\rho A \ell^4}}\right)\right] \phi_j(z), \qquad t > 0 \tag{83}$$

where $\phi_j(z)$ are the cantilevered-free eigenmodes (69), and where α_j and β_j must be determined so that $u(z,t)$ satisfies the initial conditions (81) and (82). Specifically, α_j and β_j must satisfy

$$u(z,0) = \sum_{j=1}^{\infty} \alpha_j \phi_j(z) = \begin{cases} \dfrac{Fz^2}{6EI}\left[3\gamma - z\right], & 0 \le z \le \gamma \\[2ex] \dfrac{F\gamma^2}{6EI}\left[3z - \gamma\right], & \gamma \le z \le \ell \end{cases} \tag{84}$$

$$\frac{\partial u}{\partial t}(z,0) = \sum_{j=1}^{\infty} \beta_j \left[2\pi \bar{f}_j \sqrt{\frac{EI}{\rho A \ell^4}}\right] \phi_j(z) = 0 \tag{85}$$

To determine β_j, we multiply (85) by $\phi_j(z)$, and integrate from $z = 0$ to $z = \ell$, that is,

$$\int_0^{\ell} \phi_j(z) \left(\sum_{i=1}^{\infty} \beta_i \left[2\pi \bar{f}_i \sqrt{\frac{EI}{\rho A \ell^4}}\right] \phi_i(z) \right) dz = 0 \tag{86}$$

Constraint at $z = 0$	Constraint at $z = \ell$	Eigenmode Normalization Constant $n_j = \int\limits_0^\ell \bar{\phi}_j^2(z)dz$ such that $\phi_j(z) = \frac{1}{\sqrt{n_j}}\bar{\phi}_j(z)$
Cantilevered	Cantilevered	$\ell\left[\sin\left(\sqrt{2\pi\bar{f}_j}\right) - \sinh\left(\sqrt{2\pi\bar{f}_j}\right)\right]^2$
	Pinned	$\ell\left[\sin\left(\sqrt{2\pi\bar{f}_j}\right) - \sinh\left(\sqrt{2\pi\bar{f}_j}\right)\right]^2$
	Sliding	$\ell\left[\cos\left(\sqrt{2\pi\bar{f}_j}\right) - \cosh\left(\sqrt{2\pi\bar{f}_j}\right)\right]^2$
	Free	$\ell\left[\sin\left(\sqrt{2\pi\bar{f}_j}\right) + \sinh\left(\sqrt{2\pi\bar{f}_j}\right)\right]^2$
Pinned	Cantilevered	$-\frac{\ell}{2}\left[\sin^2\left(\sqrt{2\pi\bar{f}_j}\right) - \sinh^2\left(\sqrt{2\pi\bar{f}_j}\right)\right]$
	Pinned	$\frac{\ell}{2}$
	Sliding	$\frac{\ell}{2}$
	Free	$-\frac{\ell}{2}\left[\sin^2\left(\sqrt{2\pi\bar{f}_j}\right) - \sinh^2\left(\sqrt{2\pi\bar{f}_j}\right)\right]$
Sliding	Cantilevered	$+\frac{\ell}{2}\left[\cos^2\left(\sqrt{2\pi\bar{f}_j}\right) + \cosh^2\left(\sqrt{2\pi\bar{f}_j}\right)\right]$
	Pinned	$\frac{\ell}{2}$
	Sliding	$\frac{\ell}{2}$
	Free	$+\frac{\ell}{2}\left[\cos^2\left(\sqrt{2\pi\bar{f}_j}\right) + \cosh^2\left(\sqrt{2\pi\bar{f}_j}\right)\right]$
Free	Cantilevered	$\ell\left[\sin\left(\sqrt{2\pi\bar{f}_j}\right) + \sinh\left(\sqrt{2\pi\bar{f}_j}\right)\right]^2$
	Pinned	$\ell\left[\sin\left(\sqrt{2\pi\bar{f}_j}\right) + \sinh\left(\sqrt{2\pi\bar{f}_j}\right)\right]^2$
	Sliding	$\ell\left[\cos\left(\sqrt{2\pi\bar{f}_j}\right) + \cosh\left(\sqrt{2\pi\bar{f}_j}\right)\right]^2$
	Free	$\ell\left[\sin\left(\sqrt{2\pi\bar{f}_j}\right) - \sinh\left(\sqrt{2\pi\bar{f}_j}\right)\right]^2$

Table 6: Normalization constant n_j such that the eigenmode $\phi_j(z) = \frac{1}{\sqrt{n_j}}\bar{\phi}_j(z)$ satisfies the normalization constraint (36), where the unnormalized eigenmode $\bar{\phi}_j(z)$ is given in Table 5.

Hence recalling that the eigenmodes are mutually orthogonal (46), it follows that

$$\beta_j \left[2\pi \bar{f}_j \sqrt{\frac{EI}{\rho A \ell^4}} \right] \int_0^{\ell} \phi_j^2(z) dz = 0 \tag{87}$$

and therefore the coefficients β_j are all zero, that is,

$$\beta_j = 0 \tag{88}$$

Finally, to determine α_j, we follow that same procedure. Specifically, we multiply (84) by $\phi_j(z)$, and integrate from $z = 0$ to $z = \ell$, that is,

$$\int_0^{\ell} \phi_j(z) \left(\sum_{i=1}^{\infty} \alpha_i \phi_i(z) \right) dz = \int_0^{\gamma} \phi_j(z) \left(\frac{Fz^2}{6EI} \left[3\gamma - z \right] \right) dz + \int_0^{\gamma} \phi_j(z) \left(\frac{F\gamma^2}{6EI} \left[3z - \gamma \right] \right) dz \tag{89}$$

Hence recalling that the eigenmodes are normalized and mutually orthogonal, we have that

$$\alpha_j = \int_0^{\gamma} \phi_j(z) \left(\frac{Fz^2}{6EI} \left[3\gamma - z \right] \right) dz + \int_0^{\gamma} \phi_j(z) \left(\frac{F\gamma^2}{6EI} \left[3z - \gamma \right] \right) dz \tag{90}$$

Furthermore, since the cantilevered-free eigenmodes are given by Tables (5) and (6) as

$$\begin{aligned} \phi_j(z) = {} & \frac{1}{\sqrt{n_j}} \left[\sin\left(\sqrt{2\pi \bar{f}_j} \right) + \sinh\left(\sqrt{2\pi \bar{f}_j} \right) \right] \left[\cos\left(z\sqrt{\frac{2\pi \bar{f}_j}{\ell^2}} \right) - \cosh\left(z\sqrt{\frac{2\pi \bar{f}_j}{\ell^2}} \right) \right] \\ & - \frac{1}{\sqrt{n_j}} \left[\cos\left(\sqrt{2\pi \bar{f}_j} \right) + \cosh\left(\sqrt{2\pi \bar{f}_j} \right) \right] \left[\sin\left(z\sqrt{\frac{2\pi \bar{f}_j}{\ell^2}} \right) - \sinh\left(z\sqrt{\frac{2\pi \bar{f}_j}{\ell^2}} \right) \right] \end{aligned} \tag{91}$$

$$n_j(z) = \ell \left[\sin\left(\sqrt{2\pi \bar{f}_j} \right) + \sinh\left(\sqrt{2\pi \bar{f}_j} \right) \right]^2 \tag{92}$$

then after integration, we find that the coefficients α_j are given by

$$\begin{aligned} \alpha_j = {} & \left[\frac{F\ell^4}{4\pi^2 \bar{f}_j^4 EI \sqrt{n_j}} \right] \left[\sin\left(\sqrt{2\pi \bar{f}_j} \right) + \sinh\left(\sqrt{2\pi \bar{f}_j} \right) \right] \left[\cos\left(\gamma\sqrt{\frac{2\pi \bar{f}_j}{\ell^2}} \right) - \cosh\left(\gamma\sqrt{\frac{2\pi \bar{f}_j}{\ell^2}} \right) \right] \\ & - \left[\frac{F\ell^4}{4\pi^2 \bar{f}_j^4 EI \sqrt{n_j}} \right] \left[\cos\left(\sqrt{2\pi \bar{f}_j} \right) + \cosh\left(\sqrt{2\pi \bar{f}_j} \right) \right] \left[\sin\left(\gamma\sqrt{\frac{2\pi \bar{f}_j}{\ell^2}} \right) - \sinh\left(\gamma\sqrt{\frac{2\pi \bar{f}_j}{\ell^2}} \right) \right] \end{aligned} \tag{93}$$

Note that although it seems strange, from (93) we can see that the coefficient α_j seems like a scaled version of the eigenmode $\phi_j(z)$ evaluated at $z = \gamma$. To see why this is the case, note that since the solution is static for $t \leq 0$, the solution $u(z,t)$ of (79) satisfies

$$EI \frac{\partial^4 u}{\partial z^4}(z,t) = F\delta(z - \gamma), \qquad t \leq 0 \tag{94}$$

Furthermore, from (83), we find that the solution $u(z,t)$ at some time immediately after 0, that is, $t = 0^+$, is of the form

$$u(z, 0^+) = \sum_{j=1}^{\infty} \alpha_j \phi_j(z) \tag{95}$$

Hence from the continuity of the solution around the time $t = 0$, we find that $u(z, 0^+)$ must also be the solution of (94), that is,

$$EI \sum_{j=1}^{\infty} \alpha_j \frac{\partial^4 \phi_j}{\partial z^4}(z) = F\delta(z - \gamma) \tag{96}$$

where, from (91), we find that

$$\frac{\partial^4 \phi_j}{\partial z^4}(z) = \left[\frac{4\pi^2 \bar{f}_j^2}{\ell^4}\right] \phi_j(z) \tag{97}$$

Thus multiplying (96) by $\phi_j(z)$ and integrating from $z = 0$ to $z = \ell$, we find that

$$EI\alpha_j \left[\frac{4\pi^2 \bar{f}_j^2}{\ell^4}\right] = F\phi_j(\gamma) \tag{98}$$

or simply

$$\alpha_j = \left[\frac{F\ell^4}{4\pi^2 \bar{f}_j^2 EI\sqrt{n_j}}\right] \bar{\phi}_j(\gamma) \tag{99}$$

which is precisely what we found in (93).

Thus the time-varying deflection of a cantilevered-free beam when a point force is released is given by

$$u(z, t) = \left[\frac{F\ell^4}{4\pi^2 EI}\right] \sum_{j=1}^{\infty} \frac{1}{\bar{f}_j^2} \cos\left(2\pi t \bar{f}_j \sqrt{\frac{EI}{\rho A \ell^4}}\right) \phi_j(\gamma)\phi_j(z), \qquad t > 0 \tag{100}$$

7.2 General Solution for all Beam Types

In the previous section, we showed that the time-varying deflection of a cantilevered-free beam when a point force is released could be very concisely be given by (100). It turns out that, for all of the beam types we consider, the response is also of that form. Specifically, for all of the unnormalized eigenmodes in Table 5, we see that

$$\frac{\partial^4 \bar{\phi}_j}{\partial z^4}(z) = \left[\frac{4\pi^2 \bar{f}_j^2}{\ell^4}\right] \bar{\phi}_j(z) \tag{101}$$

Hence, using the same procedure as that for a cantilevered-free beam, we find that, for all of the beam types, the time-varying deflection of a beam when a point force is released at time $t = 0$ is given by

$$u(z, t) = \left[\frac{F\ell^4}{4\pi^2 EI}\right] \sum_{j=1}^{\infty} \frac{1}{\bar{f}_j^2} \cos\left(2\pi t \bar{f}_j \sqrt{\frac{EI}{\rho A \ell^4}}\right) \phi_j(\gamma)\phi_j(z), \qquad t > 0 \tag{102}$$

where $\phi_j(z)$ is the j^{th} beam eigenmode, and $\bar{f}_j$ is the j^{th} specific eigenfrequency. **Note however, that this is only true for the cases which yield a solution in response to a point force (see Table 2), since some beams do not have a solution in response to a point force.** Furthermore, note that $\bar{f}_j$ is the j^{th} positive solution of the characteristic equation displayed in Table 3, and the j^{th} eigenmode $\phi_j(z)$ is given by

$$\phi_j(z) = \frac{1}{\sqrt{n_j}} \bar{\phi}_j(z) \tag{103}$$

where $\bar{\phi}_j(z)$ is the unnormalized eigenmode (Table 5), and n_j is the normalization constant (Table 6).

Finally, note that the coefficients $1/\bar{f}_j^2$ decay very rapidly since $\bar{f}_j$ is proportional to j^2. Specifically, from Table 3, we see that $\bar{f}_j$ is of the form

$$\bar{f}_j = \frac{\pi}{2}\left(j + a\right)^2 \tag{104}$$

for large j, where $a = 1/2$, $a = 1/4$, $a = -1/4$, or $a = 0$ depending on the beam type. Hence the coefficients are approximately proportional to $1/j^4$.

8 Eigenmodes and Characteristic Equation of a Cantilevered-Free Beam with a Point Mass

In this section, we compute the eigenmodes and characteristic equation of a cantilevered-free beam with a point mass m attached to the beam at $z = \zeta$, that is, we compute a basis for the solutions of

$$\left[\rho A + m\delta(z-\zeta)\right]\frac{\partial^2 u}{\partial t^2}(z,t) + EI\frac{\partial^4 u}{\partial z^4}(z,t) = 0 \tag{105}$$

subject to the spatial boundary conditions (52). Using the same approach as before, we will assume that

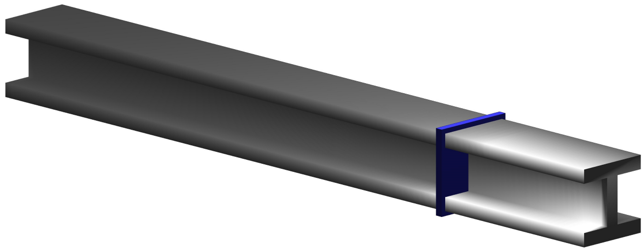


Figure 12: A standard I-Beam with a point mass.

the solution can be written in the form

$$u(z,t) = \sum_{j=1}^{\infty} \psi_j(t)\phi_j(z) \tag{106}$$

where each of the components $u_j(z,t) = \psi_j(t)\phi_j(z)$ is a solution of (105). Hence

$$\left[\rho A + m\delta(z-\zeta)\right]\frac{\partial^2 \psi_j}{\partial t^2}(t)\phi_j(z) + EI\psi_j(t)\frac{\partial^4 \phi_j}{\partial z^4}(z) = 0 \tag{107}$$

and it follows that

$$\frac{EI}{\rho A}\psi_j(t)\left[\frac{\partial^2 \psi_j}{\partial t^2}(t)\right]^{-1} = -\left[1 + \frac{m}{\rho A}\delta(z-\zeta)\right]\phi_j(z)\left[\frac{\partial^4 \phi_j}{\partial z^4}(z)\right]^{-1} \tag{108}$$

Next, since the left-hand side of (110) is only a function of t, and the right-hand side of (110) is only a function of z, it follows that both sides must be equal to some constant value, which we arbitrarily chose to be $\ell^4/(2\pi\bar{f}_j)^2$, that is

$$\frac{EI}{\rho A}\psi_j(t)\left[\frac{\partial^2 \psi_j}{\partial t^2}(t)\right]^{-1} = -\left[1 + \frac{m}{\rho A}\delta(z-\zeta)\right]\phi_j(z)\left[\frac{\partial^4 \phi_j}{\partial z^4}(z)\right]^{-1} = \frac{\ell^4}{(2\pi\bar{f}_j)^2} \tag{109}$$

and hence

$$\frac{\partial^2 \psi_j}{\partial t^2}(t) - \frac{(2\pi\bar{f}_j)^2 EI}{\ell^4 \rho A}\psi_j(t) = 0 \tag{110}$$

$$\frac{\partial^4 \phi_j}{\partial z^4}(z) + \frac{(2\pi\bar{f}_j)^2}{\ell^4}\left[1 + \frac{m}{\rho A}\delta(z-\zeta)\right]\phi_j(z) = 0 \tag{111}$$

Thus the time-varying component $\psi_j(t)$ will be of the form

$$\psi_j(t) = \alpha_j \cos\left(2\pi t\bar{f}_j\sqrt{\frac{EI}{\rho A\ell^4}}\right) + \beta_j \sin\left(2\pi t\bar{f}_j\sqrt{\frac{EI}{\rho A\ell^4}}\right) \tag{112}$$

where $\bar{f}_j$ is the j^{th} *specific eigenfrequency* and, from (112), we can see that the eigenfrequencies f_j are related to the specific eigenfrequencies by

$$f_j = \bar{f}_j \sqrt{\frac{EI}{\rho A \ell^4}} \tag{113}$$

Unfortunately, the eigenmodes $\phi_j(z)$ are a bit more complicated to compute. First, we integrate across the discontinuity at $z = \zeta$, that is, from (111) we find that

$$\frac{\partial^3 \phi_j}{\partial z^3}(\zeta^+) - \frac{\partial^3 \phi_j}{\partial z^3}(\zeta^-) + \frac{(2\pi \bar{f}_j)^2}{\ell^4} \frac{m}{\rho A} \phi_j(\zeta) = 0 \tag{114}$$

where $\zeta^- < \zeta$ and $\zeta^+ > \zeta$ are points infinitesimally before and after ζ. Hence $\phi_j(z)$ is of the form

$$\phi_j(z) = \begin{cases} a_j^- \cos\left(\frac{z\sigma_j}{\ell}\right) + b_j^- \sin\left(\frac{z\sigma_j}{\ell}\right) + c_j^- \cosh\left(\frac{z\sigma_j}{\ell}\right) + d_j^- \sinh\left(\frac{z\sigma_j}{\ell}\right), & 0 \leq z \leq \zeta \\ a_j^+ \cos\left(\frac{z\sigma_j}{\ell}\right) + b_j^+ \sin\left(\frac{z\sigma_j}{\ell}\right) + c_j^+ \cosh\left(\frac{z\sigma_j}{\ell}\right) + d_j^+ \sinh\left(\frac{z\sigma_j}{\ell}\right), & \zeta \leq z \leq \ell \end{cases} \tag{115}$$

$$\sigma_j \triangleq \sqrt{2\pi \bar{f}_j} \tag{116}$$

subject to (114) and the continuity of the lower derivatives, that is,

$$\phi_j(\zeta^+) = \phi_j(\zeta^-), \qquad \frac{\partial \phi_j}{\partial z}(\zeta^+) = \frac{\partial \phi_j}{\partial z}(\zeta^-), \qquad \frac{\partial^2 \phi_j}{\partial z^2}(\zeta^+) = \frac{\partial^2 \phi_j}{\partial z^2}(\zeta^-) \tag{117}$$

Specifically, from the continuity constraints (117), we find that

$$+\left[a_j^+ - a_j^-\right] \cos\left(\frac{\zeta\sigma_j}{\ell}\right) + \left[b_j^+ - b_j^-\right] \sin\left(\frac{\zeta\sigma_j}{\ell}\right) + \left[c_j^+ - c_j^-\right] \cosh\left(\frac{\zeta\sigma_j}{\ell}\right) + \left[d_j^+ - d_j^-\right] \sinh\left(\frac{\zeta\sigma_j}{\ell}\right) = 0 \tag{118}$$

$$-\left[a_j^+ - a_j^-\right] \sin\left(\frac{\zeta\sigma_j}{\ell}\right) + \left[b_j^+ - b_j^-\right] \cos\left(\frac{\zeta\sigma_j}{\ell}\right) + \left[c_j^+ - c_j^-\right] \cosh\left(\frac{\zeta\sigma_j}{\ell}\right) + \left[d_j^+ - d_j^-\right] \sinh\left(\frac{\zeta\sigma_j}{\ell}\right) = 0 \tag{119}$$

$$-\left[a_j^+ - a_j^-\right] \cos\left(\frac{\zeta\sigma_j}{\ell}\right) - \left[b_j^+ - b_j^-\right] \sin\left(\frac{\zeta\sigma_j}{\ell}\right) + \left[c_j^+ - c_j^-\right] \cosh\left(\frac{\zeta\sigma_j}{\ell}\right) + \left[d_j^+ - d_j^-\right] \sinh\left(\frac{\zeta\sigma_j}{\ell}\right) = 0 \tag{120}$$

and from (114) it follows that

$$\begin{aligned} \left[a_j^+ - a_j^-\right] \sin\left(\frac{\zeta\sigma_j}{\ell}\right) &- \left[b_j^+ - b_j^-\right] \cos\left(\frac{\zeta\sigma_j}{\ell}\right) \\ &+ \left[c_j^+ - c_j^-\right] \cosh\left(\frac{\zeta\sigma_j}{\ell}\right) + \left[d_j^+ - d_j^-\right] \sinh\left(\frac{\zeta\sigma_j}{\ell}\right) = -\frac{m\sigma_j}{\rho A} \phi_j(\zeta) \end{aligned} \tag{121}$$

Hence from (118)-(121), we have that

$$-\frac{a_j^+ - a_j^-}{\sin(\zeta\sigma_j)} = +\frac{b_j^+ - b_j^-}{\cos(\zeta\sigma_j)} = +\frac{c_j^+ - c_j^-}{\sinh(\zeta\sigma_j)} = -\frac{d_j^+ - d_j^-}{\cosh(\zeta\sigma_j)} = \frac{m\sigma_j}{2\rho A} \phi_j(\zeta) \tag{122}$$

where $\phi_j(\zeta)$ is a function of all of the coefficients $a_j^-, b_j^-, c_j^-, d_j^-$ and $a_j^+, b_j^+, c_j^+, d_j^+$.

However, this is still not enough to resolve the coefficients $a_j^-, b_j^-, c_j^-, d_j^-$ and $a_j^+, b_j^+, c_j^+, d_j^+$. To do so, we also need the boundary conditions of the beam, which, for a cantilevered-free beam are given by

$$\phi_j(0) = 0, \qquad \frac{\partial \phi_j}{\partial z}(0, t) = 0, \qquad \frac{\partial^2 \phi_j}{\partial^2 z}(\ell) = 0, \qquad \frac{\partial^3 \phi_j}{\partial^3 z}(\ell) = 0 \tag{123}$$

Thus we find that the coefficients of the parameterization must satisfy

$$0 = a_j^- + c_j^- \tag{124}$$

$$0 = b_j^- + d_j^- \tag{125}$$

$$0 = -a_j^+ \cos\left(\sigma_j\right) - b_j^+ \sin\left(\sigma_j\right) + c_j^+ \cosh\left(\sigma_j\right) + d_j^+ \sinh\left(\sigma_j\right) \tag{126}$$

$$0 = +a_j^+ \sin\left(\sigma_j\right) - b_j^+ \cos\left(\sigma_j\right) + c_j^+ \sinh\left(\sigma_j\right) + d_j^+ \cosh\left(\sigma_j\right) \tag{127}$$

Solving this system of equations (after a lot of work), we find that for $z \in [0, \zeta)$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ of the cantilevered-free beam are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\left[\sin\left(\frac{z\sigma_j}{\ell}\right) - \sinh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\cos\left(\frac{\zeta\sigma_j}{\ell}\right) + \cosh\left(\frac{\zeta\sigma_j}{\ell}\right) + \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right)\right. \\ &\left. + \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) - \sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) + \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right] \\ - &\left[\cos\left(\frac{z\sigma_j}{\ell}\right) - \cosh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\zeta\sigma_j}{\ell}\right) + \sinh\left(\frac{\zeta\sigma_j}{\ell}\right) + \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right)\right. \\ &\left. + \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) - \sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) - \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\end{aligned} \tag{128}$$

and for $z \in (\zeta, \ell]$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ of the cantilevered-free beam are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\left[\sin\left(\frac{\bar{z}\sigma_j}{\ell}\right) + \sinh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) - \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) + \cos\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right)\right. \\ &\left. - \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) - \sin\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) - \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right] \\ - &\left[\cos\left(\frac{\bar{z}\sigma_j}{\ell}\right) + \cosh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) - \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) + \cos\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right)\right. \\ &\left. - \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) - \sin\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) + \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\end{aligned} \tag{129}$$

where

$$\sigma_j \triangleq \sqrt{2\pi\bar{f}_j} \tag{130}$$

$$\bar{z} \triangleq \ell - z \tag{131}$$

$$\bar{\zeta} \triangleq \ell - \zeta \tag{132}$$

Furthermore, the characteristic equation is given by

$$\cos\left(\sigma_j\right)\cosh\left(\sigma_j\right) + 1 = -\frac{m\sigma_j}{4\ell\rho A}\bar{\phi}_j(\zeta) \tag{133}$$

The eigenmodes and characteristic equations for all of the beam types are summarized in the next chapter.

9 Eigenmodes and Characteristic Equations of Beams with a Point Mass

In the following sections, we provide the eigenmodes and characteristic equations of beams with a point mass m attached to the beam at $z = \zeta$, that is, the eigenmodes and characteristic equations of

$$\left[\rho A + m\delta(z-\zeta)\right]\frac{\partial^2 u}{\partial t^2}(z,t) + EI\frac{\partial^4 u}{\partial z^4}(z,t) = 0 \tag{134}$$

Specifically, the general solution of (134) is shown to be of the form

$$u(z,t) = \sum_{j=1}^{\infty} \psi_j(t)\phi_j(z) \tag{135}$$

$$\psi_j(t) = \alpha_j \cos\left(2\pi t\bar{f}_j\sqrt{\frac{EI}{\rho A\ell^4}}\right) + \beta_j \sin\left(2\pi t\bar{f}_j\sqrt{\frac{EI}{\rho A\ell^4}}\right) \tag{136}$$

where α_j and β_j are determined by the beam initial conditions, and $\phi_j(t)$ is the j^{th} eigenmode of the beam, which satisfies the spatial boundary constraints, such as (123) for a cantilevered-free beam. Note that every component $u_j(z,t) = \psi_j(t)\phi_j(z)$ of the solution $u(z,t)$ is a solution of (134). In the following, we use the definitions

$$\sigma_j \triangleq \sqrt{2\pi\bar{f}_j} \tag{137}$$

$$\bar{z} \triangleq \ell - z \tag{138}$$

$$\bar{\zeta} \triangleq \ell - \zeta \tag{139}$$

where ℓ is the length of the beam, and $\bar{f}_j$ is the j^{th} specific eigenfrequency, which is the j^{th} nonnegative solution of the characteristic equation. Recall that the eigenfrequencies f_j are given by

$$f_j = \bar{f}_j\sqrt{\frac{EI}{\rho A\ell^4}} = \frac{\sigma_j^2}{2\pi}\sqrt{\frac{EI}{\rho A\ell^4}} \tag{140}$$

As we will see, the specific eigenfrequencies are no longer independent of the system parameters. Specifically, the specific eigenfrequencies are dependent on the ratio of the beam mass $\rho A\ell$ to the point mass m. However, when the point mass m approaches zero, the characteristic equation is that of the beam with no point mass attached (see Table 3).

Note that, in the following, we provide only the unnormalized eigenmodes $\bar{\phi}_j(z)$. The eigenmodes $\phi_j(z)$ are related to the unnormalized eigenmodes by

$$\phi_j(z) = \frac{\bar{\phi}_j(z)}{\sqrt{\int_0^\ell \bar{\phi}_j^2(z)dz}} \tag{141}$$

Hence the eigenmodes satisfy the normalization constraint

$$\int_0^\ell \phi_j^2(z)dz = 1 \tag{142}$$

although the unnormalized eigenmodes most likely do not.

9.1 Cantilevered at $z = 0$ and Cantilevered at $z = \ell$

For $z \in [0, \zeta)$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\left[\sin\left(\frac{z\sigma_j}{\ell}\right) - \sinh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\cos\left(\frac{\zeta\sigma_j}{\ell}\right) + \cosh\left(\frac{\zeta\sigma_j}{\ell}\right) - \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right)\right. \\ &\qquad\left. - \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) + \sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) - \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right] \\ &- \left[\cos\left(\frac{z\sigma_j}{\ell}\right) - \cosh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\zeta\sigma_j}{\ell}\right) + \sinh\left(\frac{\zeta\sigma_j}{\ell}\right) - \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right)\right. \\ &\qquad\left. - \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) + \sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) + \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\end{aligned} \tag{143}$$

and for $z \in (\zeta, \ell]$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\left[\sin\left(\frac{\bar{z}\sigma_j}{\ell}\right) - \sinh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) + \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) - \cos\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right)\right. \\ &\qquad\left. - \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) + \sin\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) - \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right] \\ &- \left[\cos\left(\frac{\bar{z}\sigma_j}{\ell}\right) - \cosh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) + \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) - \cos\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right)\right. \\ &\qquad\left. - \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) + \sin\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) + \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\end{aligned} \tag{144}$$

Furthermore, the characteristic equation is given by

$$2\cos\left(\sigma_j\right)\cosh\left(\sigma_j\right) - 2 = \frac{m\sigma_j}{2\ell\rho A}\bar{\phi}_j(\zeta) \tag{145}$$

9.2 Cantilevered at $z = 0$ and Pinned at $z = \ell$

For $z \in [0, \zeta)$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\left[\sin\left(\frac{z\sigma_j}{\ell}\right) - \sinh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) - \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right] \\ &- \left[\cos\left(\frac{z\sigma_j}{\ell}\right) - \cosh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) - \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right]\end{aligned} \tag{146}$$

and for $z \in (\zeta, \ell]$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\sin\left(\frac{\bar{z}\sigma_j}{\ell}\right)\left[\sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) - \cos\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) + \sin\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right)\right] \\ &+ \sinh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\left[\sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) - \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) + \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\end{aligned} \tag{147}$$

Furthermore, the characteristic equation is given by

$$\cos\left(\sigma_j\right)\sinh\left(\sigma_j\right) - \cosh\left(\sigma_j\right)\sin\left(\sigma_j\right) = \frac{m\sigma_j}{2\ell\rho A}\bar{\phi}_j(\zeta) \tag{148}$$

9.3 Cantilevered at $z = 0$ and Sliding at $z = \ell$

For $z \in [0, \zeta)$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\left[\cos\left(\frac{z\sigma_j}{\ell}\right) - \cosh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) - \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right] \\ &-\left[\sin\left(\frac{z\sigma_j}{\ell}\right) - \sinh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) + \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right]\end{aligned} \tag{149}$$

and for $z \in (\zeta, \ell]$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\cosh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\left[\cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) - \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) + \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right] \\ &- \cos\left(\frac{\bar{z}\sigma_j}{\ell}\right)\left[\cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) - \cos\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) + \sin\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right)\right]\end{aligned} \tag{150}$$

Furthermore, the characteristic equation is given by

$$\cos\left(\sigma_j\right)\sinh\left(\sigma_j\right) + \cosh\left(\sigma_j\right)\sin\left(\sigma_j\right) = \frac{m\sigma_j}{2\ell\rho A}\bar{\phi}_j(\zeta) \tag{151}$$

9.4 Cantilevered at $z = 0$ and Free at $z = \ell$

For $z \in [0, \zeta)$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\left[\sin\left(\frac{z\sigma_j}{\ell}\right) - \sinh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\cos\left(\frac{\zeta\sigma_j}{\ell}\right) + \cosh\left(\frac{\zeta\sigma_j}{\ell}\right) + \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right)\right. \\ &\qquad\left. + \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) - \sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) + \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right] \\ &-\left[\cos\left(\frac{z\sigma_j}{\ell}\right) - \cosh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\zeta\sigma_j}{\ell}\right) + \sinh\left(\frac{\zeta\sigma_j}{\ell}\right) + \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right)\right. \\ &\qquad\left. + \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) - \sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) - \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\end{aligned} \tag{152}$$

and for $z \in (\zeta, \ell]$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &-\left[\sin\left(\frac{\bar{z}\sigma_j}{\ell}\right) + \sinh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) - \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) - \cos\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right)\right. \\ &\qquad\left. + \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) + \sin\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) + \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right] \\ &-\left[\cos\left(\frac{\bar{z}\sigma_j}{\ell}\right) + \cosh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) - \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) + \cos\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right)\right. \\ &\qquad\left. - \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) - \sin\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) + \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\end{aligned} \tag{153}$$

Furthermore, the characteristic equation is given by

$$-2\cos\left(\sigma_j\right)\cosh\left(\sigma_j\right) - 2 = \frac{m\sigma_j}{2\ell\rho A}\bar{\phi}_j(\zeta) \tag{154}$$

9.5 Pinned at $z = 0$ and Cantilevered at $z = \ell$

For $z \in [0, \zeta)$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &-\sin\left(\frac{z\sigma_j}{\ell}\right)\left[\sinh\left(\frac{\zeta\sigma_j}{\ell}\right) - \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) + \sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right)\right] \\ &-\sinh\left(\frac{z\sigma_j}{\ell}\right)\left[\sin\left(\frac{\zeta\sigma_j}{\ell}\right) - \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) + \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\end{aligned} \tag{155}$$

and for $z \in (\zeta, \ell]$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\left[\cos\left(\frac{\bar{z}\sigma_j}{\ell}\right) - \cosh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) - \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right] \\ &-\left[\sin\left(\frac{\bar{z}\sigma_j}{\ell}\right) - \sinh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) - \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\end{aligned} \tag{156}$$

Furthermore, the characteristic equation is given by

$$\cosh\left(\sigma_j\right)\sin\left(\sigma_j\right) - \cos\left(\sigma_j\right)\sinh\left(\sigma_j\right) = \frac{m\sigma_j}{2\ell\rho A}\bar{\phi}_j(\zeta) \tag{157}$$

9.6 Pinned at $z = 0$ and Pinned at $z = \ell$

For $z \in [0, \zeta)$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\bar{\phi}_j(z) = \sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\frac{z\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) - \sinh\left(\frac{z\sigma_j}{\ell}\right)\sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) \tag{158}$$

and for $z \in (\zeta, \ell]$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\bar{\phi}_j(z) = \sin\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\frac{\bar{z}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) - \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) \tag{159}$$

Furthermore, the characteristic equation is given by

$$-\sin\left(\sigma_j\right)\sinh\left(\sigma_j\right) = \frac{m\sigma_j}{2\ell\rho A}\bar{\phi}_j(\zeta) \tag{160}$$

9.7 Pinned at $z = 0$ and Sliding at $z = \ell$

For $z \in [0, \zeta)$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\bar{\phi}_j(z) = \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\frac{z\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) - \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\frac{z\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) \tag{161}$$

and for $z \in (\zeta, \ell]$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\bar{\phi}_j(z) = \cosh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) - \cos\left(\frac{\bar{z}\sigma_j}{\ell}\right)\sin\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) \tag{162}$$

Furthermore, the characteristic equation is given by

$$\cos\left(\sigma_j\right)\cosh\left(\sigma_j\right) = \frac{m\sigma_j}{2\ell\rho A}\bar{\phi}_j(\zeta) \tag{163}$$

9.8 Pinned at $z = 0$ and Free at $z = \ell$

For $z \in [0, \zeta)$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = {} & \sin\left(\frac{z\sigma_j}{\ell}\right)\left[\sinh\left(\frac{\zeta\sigma_j}{\ell}\right) + \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) - \sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right)\right] \\ & + \sinh\left(\frac{z\sigma_j}{\ell}\right)\left[\sin\left(\frac{\zeta\sigma_j}{\ell}\right) + \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) - \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\end{aligned} \tag{164}$$

and for $z \in (\zeta, \ell]$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = {} & \left[\cos\left(\frac{\bar{z}\sigma_j}{\ell}\right) + \cosh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) + \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right] \\ & - \left[\sin\left(\frac{\bar{z}\sigma_j}{\ell}\right) + \sinh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) + \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\end{aligned} \tag{165}$$

Furthermore, the characteristic equation is given by

$$\cosh\left(\sigma_j\right)\sin\left(\sigma_j\right) - \cos\left(\sigma_j\right)\sinh\left(\sigma_j\right) = \frac{m\sigma_j}{2\ell\rho A}\bar{\phi}_j(\zeta) \tag{166}$$

9.9 Sliding at $z = 0$ and Cantilevered at $z = \ell$

For $z \in [0, \zeta)$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned} \bar{\phi}_j(z) = \cosh\left(\frac{z\sigma_j}{\ell}\right)&\left[\cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) - \cos\left(\frac{\zeta\sigma_j}{\ell}\right) + \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right] \\ - \cos\left(\frac{z\sigma_j}{\ell}\right)&\left[\cosh\left(\frac{\zeta\sigma_j}{\ell}\right) - \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) + \sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right)\right] \end{aligned} \tag{167}$$

and for $z \in (\zeta, \ell]$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned} \bar{\phi}_j(z) = &\left[\cos\left(\frac{\bar{z}\sigma_j}{\ell}\right) - \cosh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\cos\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) - \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right] \\ -&\left[\sin\left(\frac{\bar{z}\sigma_j}{\ell}\right) - \sinh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\cos\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) + \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right] \end{aligned} \tag{168}$$

Furthermore, the characteristic equation is given by

$$\cos\left(\sigma_j\right)\sinh\left(\sigma_j\right) + \cosh\left(\sigma_j\right)\sin\left(\sigma_j\right) = \frac{m\sigma_j}{2\ell\rho A}\bar{\phi}_j(\zeta) \tag{169}$$

9.10 Sliding at $z = 0$ and Pinned at $z = \ell$

For $z \in [0, \zeta)$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\bar{\phi}_j(z) = \cosh\left(\frac{z\sigma_j}{\ell}\right)\sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) - \sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\frac{z\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) \tag{170}$$

and for $z \in (\zeta, \ell]$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\bar{\phi}_j(z) = \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) - \cos\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\frac{\bar{z}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) \tag{171}$$

Furthermore, the characteristic equation is given by

$$\cos\left(\sigma_j\right)\cosh\left(\sigma_j\right) = \frac{m\sigma_j}{2\ell\rho A}\bar{\phi}_j(\zeta) \tag{172}$$

9.11 Sliding at $z = 0$ and Sliding at $z = \ell$

For $z \in [0, \zeta)$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\bar{\phi}_j(z) = \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\frac{z\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) + \cosh\left(\frac{z\sigma_j}{\ell}\right)\cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) \tag{173}$$

and for $z \in (\zeta, \ell]$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\bar{\phi}_j(z) = \cos\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\frac{\bar{z}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) + \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) \tag{174}$$

Furthermore, the characteristic equation is given by

$$\sin\left(\sigma_j\right)\sinh\left(\sigma_j\right) = \frac{m\sigma_j}{2\ell\rho A}\bar{\phi}_j(\zeta) \tag{175}$$

9.12 Sliding at $z = 0$ and Free at $z = \ell$

For $z \in [0, \zeta)$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &-\cosh\left(\frac{z\sigma_j}{\ell}\right)\left[\cos\left(\frac{\zeta\sigma_j}{\ell}\right) + \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) + \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right] \\ &-\cos\left(\frac{z\sigma_j}{\ell}\right)\left[\cosh\left(\frac{\zeta\sigma_j}{\ell}\right) + \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) - \sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right)\right]\end{aligned} \tag{176}$$

and for $z \in (\zeta, \ell]$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\left[\sin\left(\frac{\bar{z}\sigma_j}{\ell}\right) + \sinh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\cos\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) - \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right] \\ &-\left[\cos\left(\frac{\bar{z}\sigma_j}{\ell}\right) + \cosh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\cos\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) + \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\end{aligned} \tag{177}$$

Furthermore, the characteristic equation is given by

$$-\cos\left(\sigma_j\right)\sinh\left(\sigma_j\right) - \cosh\left(\sigma_j\right)\sin\left(\sigma_j\right) = \frac{m\sigma_j}{2\ell\rho A}\bar{\phi}_j(\zeta) \tag{178}$$

9.13 Free at $z = 0$ and Cantilevered at $z = \ell$

For $z \in [0, \zeta)$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\left[\sin\left(\frac{z\sigma_j}{\ell}\right) + \sinh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\cosh\left(\frac{\zeta\sigma_j}{\ell}\right) - \cos\left(\frac{\zeta\sigma_j}{\ell}\right) - \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right)\right.\\ &\left. + \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) + \sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) + \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right]\\ &+\left[\cos\left(\frac{z\sigma_j}{\ell}\right) + \cosh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\zeta\sigma_j}{\ell}\right) - \sinh\left(\frac{\zeta\sigma_j}{\ell}\right) + \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right)\right.\\ &\left. - \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) - \sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) + \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\end{aligned} \tag{179}$$

and for $z \in (\zeta, \ell]$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\left[\cos\left(\frac{\bar{z}\sigma_j}{\ell}\right) - \cosh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) + \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) + \cos\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right)\right.\\ &\left. + \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) - \sin\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) - \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\\ &-\left[\sin\left(\frac{\bar{z}\sigma_j}{\ell}\right) - \sinh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) + \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) + \cos\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right)\right.\\ &\left. + \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) - \sin\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) + \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right]\end{aligned} \tag{180}$$

Furthermore, the characteristic equation is given by

$$2\cos\left(\sigma_j\right)\cosh\left(\sigma_j\right) + 2 = \frac{m\sigma_j}{2\ell\rho A}\bar{\phi}_j(\zeta) \tag{181}$$

9.14 Free at $z = 0$ and Pinned at $z = \ell$

For $z \in [0, \zeta)$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\left[\sin\left(\frac{z\sigma_j}{\ell}\right) + \sinh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) + \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\\ &-\left[\cos\left(\frac{z\sigma_j}{\ell}\right) + \cosh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) + \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right]\end{aligned} \tag{182}$$

and for $z \in (\zeta, \ell]$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &-\sin\left(\frac{\bar{z}\sigma_j}{\ell}\right)\left[\sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) + \cos\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) - \sin\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right)\right]\\ &-\sinh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\left[\sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) + \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) - \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\end{aligned} \tag{183}$$

Furthermore, the characteristic equation is given by

$$\cos\left(\sigma_j\right)\sinh\left(\sigma_j\right) - \cosh\left(\sigma_j\right)\sin\left(\sigma_j\right) = \frac{m\sigma_j}{2\ell\rho A}\bar{\phi}_j(\zeta) \tag{184}$$

9.15 Free at $z = 0$ and Sliding at $z = \ell$

For $z \in [0, \zeta)$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\left[\cos\left(\frac{z\sigma_j}{\ell}\right) + \cosh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) + \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right] \\ &-\left[\sin\left(\frac{z\sigma_j}{\ell}\right) + \sinh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) - \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right]\end{aligned} \tag{185}$$

and for $z \in (\zeta, \ell]$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\cosh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\left[\cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) + \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) + \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right] \\ &+\cos\left(\frac{\bar{z}\sigma_j}{\ell}\right)\left[\cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) + \cos\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) - \sin\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right)\right]\end{aligned} \tag{186}$$

Furthermore, the characteristic equation is given by

$$\cos\left(\sigma_j\right)\sinh\left(\sigma_j\right) + \cosh\left(\sigma_j\right)\sin\left(\sigma_j\right) = \frac{m\sigma_j}{2\ell\rho A}\bar{\phi}_j(\zeta) \tag{187}$$

9.16 Free at $z = 0$ and Free at $z = \ell$

For $z \in [0, \zeta)$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\left[\sin\left(\frac{z\sigma_j}{\ell}\right) + \sinh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\cos\left(\frac{\zeta\sigma_j}{\ell}\right) - \cosh\left(\frac{\zeta\sigma_j}{\ell}\right) - \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right)\right. \\ &\left. + \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) + \sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) + \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right] \\ &-\left[\cos\left(\frac{z\sigma_j}{\ell}\right) + \cosh\left(\frac{z\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\zeta\sigma_j}{\ell}\right) - \sinh\left(\frac{\zeta\sigma_j}{\ell}\right) - \cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right)\right. \\ &\left. + \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) + \sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) - \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\end{aligned} \tag{188}$$

and for $z \in (\zeta, \ell]$, the unnormalized eigenmodes $\bar{\phi}_j(z)$ are given by

$$\begin{aligned}\bar{\phi}_j(z) = &\left[\sin\left(\frac{\bar{z}\sigma_j}{\ell}\right) + \sinh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\cos\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) - \cosh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) - \cos\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right)\right. \\ &\left. + \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right) + \sin\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right) + \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right)\right] \\ &-\left[\cos\left(\frac{\bar{z}\sigma_j}{\ell}\right) + \cosh\left(\frac{\bar{z}\sigma_j}{\ell}\right)\right]\left[\sin\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) - \sinh\left(\frac{\bar{\zeta}\sigma_j}{\ell}\right) - \cos\left(\frac{\zeta\sigma_j}{\ell}\right)\sinh\left(\sigma_j\right)\right. \\ &\left. + \cosh\left(\frac{\zeta\sigma_j}{\ell}\right)\sin\left(\sigma_j\right) + \sin\left(\frac{\zeta\sigma_j}{\ell}\right)\cosh\left(\sigma_j\right) - \sinh\left(\frac{\zeta\sigma_j}{\ell}\right)\cos\left(\sigma_j\right)\right]\end{aligned} \tag{189}$$

Furthermore, the characteristic equation is given by

$$2\cos\left(\sigma_j\right)\cosh\left(\sigma_j\right) - 2 = \frac{m\sigma_j}{2\ell\rho A}\bar{\phi}_j(\zeta) \tag{190}$$