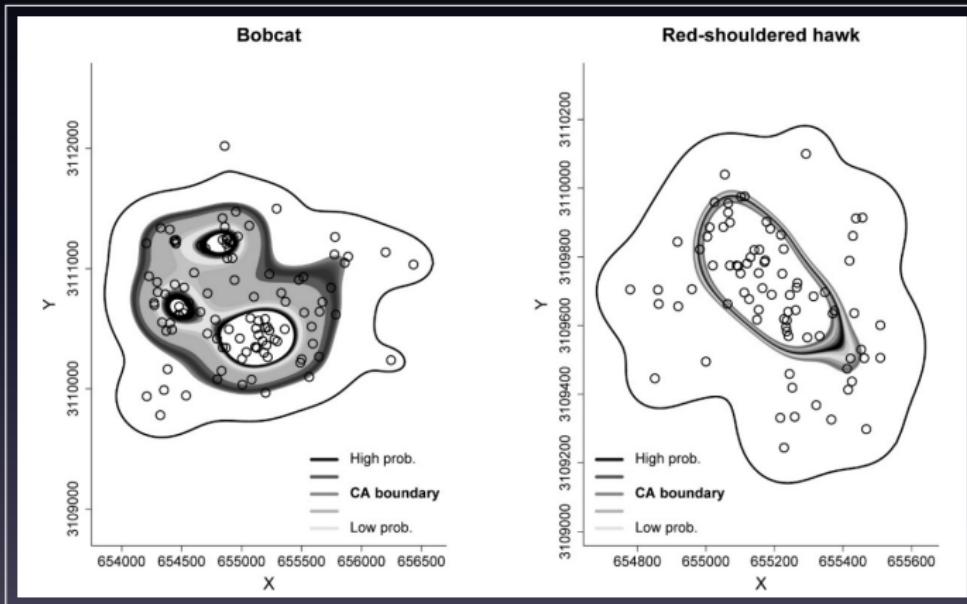


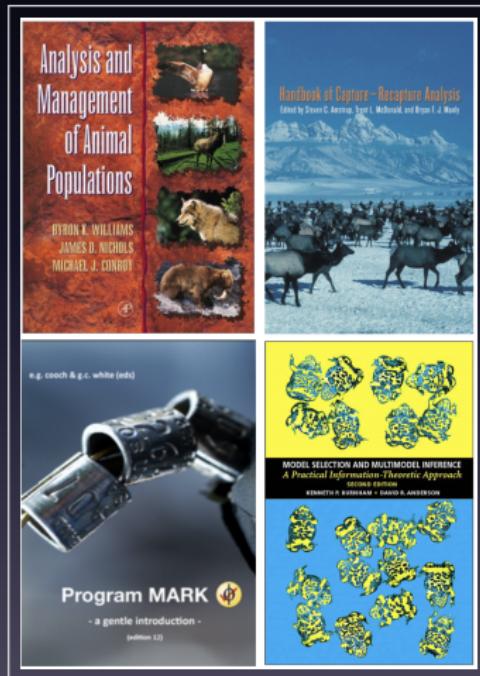
# Geostatistical Capture-Recapture Models

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- Wilson, Hooten, Strobel, Shivik (2010). Accounting for individuals, uncertainty, and multi-scale clustering in core area estimation. *Journal of Wildlife Management*, 74: 1343-1352.

# Wildlife Demographic Models



# CR Model w/ Data Augmentation

## Data Model

$$y_i \sim \begin{cases} \mathbb{1}_{\{y_i=0\}} & , z_i = 0 \\ \text{Binom}(J, p) & , z_i = 1 \end{cases}$$

## Process Model

$$z_i \sim \text{Bern}(\psi)$$

## Derived Quantity

$$N = \sum_{i=1}^M z_i$$

$i$	$\mathbf{Y}$	$y_i$
1	111	3
2	100	1
$\vdots$	$\vdots$	$\vdots$
$n$	101	2
$n+1$	000	0
$\vdots$	$\vdots$	$\vdots$
$M$	000	0

- Royle, Dorazio, Link. (2007). Analysis of multinomial models with unknown index using data augmentation. J Comput Graph Stat 16:67-85.

# Recursive Bayes Perspective

## Posterior

$$\begin{aligned}[p, \psi | \mathbf{y}_{1:n}, \mathbf{y}_{(n+1):M}, n] &\propto \overbrace{[\mathbf{y}_{(n+1):M} | p, \psi, \mathbf{y}_{1:n}, n]} [p, \psi | \mathbf{y}_{1:n}, n] \\ &\propto \left( \prod_{i=1}^n [y_i | p, y_i > 0] \right) [n | p, \psi] [p] [\psi]\end{aligned}$$

## Stage 1

$$\left( \prod_{i=1}^n [y_i | p, y_i > 0] \right) [p] [\psi]$$

## Stage 2

$$r = \frac{[n | p^{(*)}, \psi^{(*)}]}{[n | p^{(k-1)}, \psi^{(k-1)}]}$$

- Hooten, Schwob, Johnson, Ivan. (2023). Multistage hierarchical capture-recapture models. Environmetrics.

# Conditional Model for $n$

## Definition of $n$

$$n = \sum_{i=1}^M \mathbb{1}_{\{y_i > 0\}}$$

## Prob. of Detecting Individual

$$\Pr(\mathbb{1}_{\{y_i > 0\}} = 1 | p, \psi) = \psi(1 - (1 - p)^J)$$

## Resulting Conditional Model for $n$

$$[n | p, \psi] = \text{Binom}(M, \psi(1 - (1 - p)^J))$$

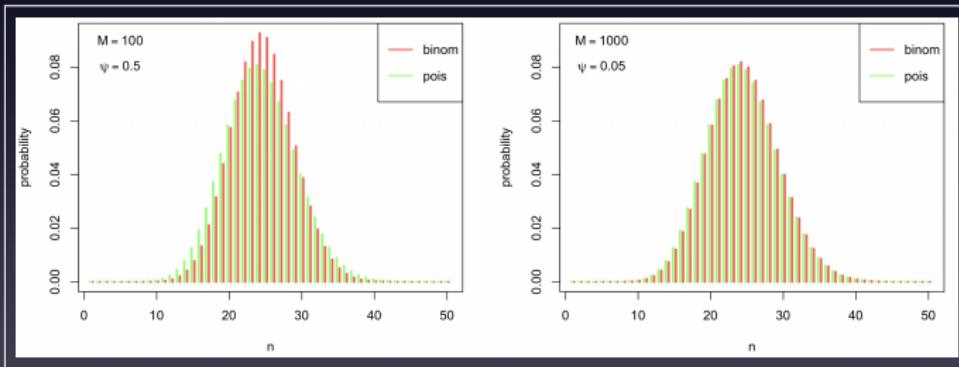
- Hooten, Schwob, Johnson, Ivan. (2023). Multistage hierarchical capture-recapture models. Environmetrics.

# Alternative Conditional Model for $n$

## Poisson Model for $n$

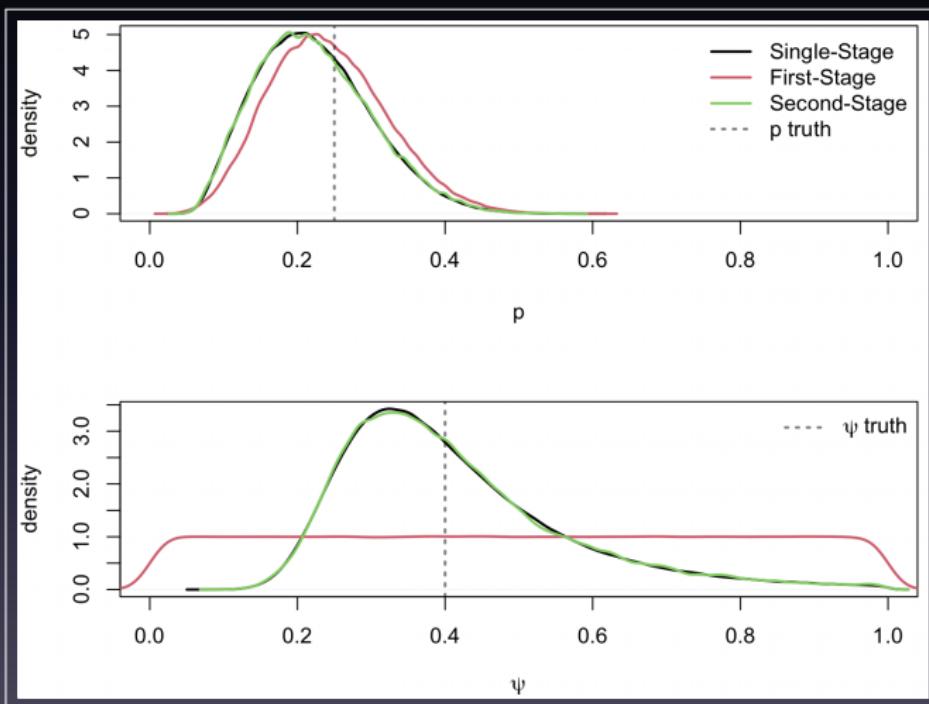
$$[n|p, \psi] = \text{Pois}(M\psi(1 - (1 - p)^J))$$

for  $M \rightarrow \infty$  and  $\psi \rightarrow 0$



- Hooten, Schwob, Johnson, Ivan. (2023). Multistage hierarchical capture-recapture models. Environmetrics.

# Simulation Results

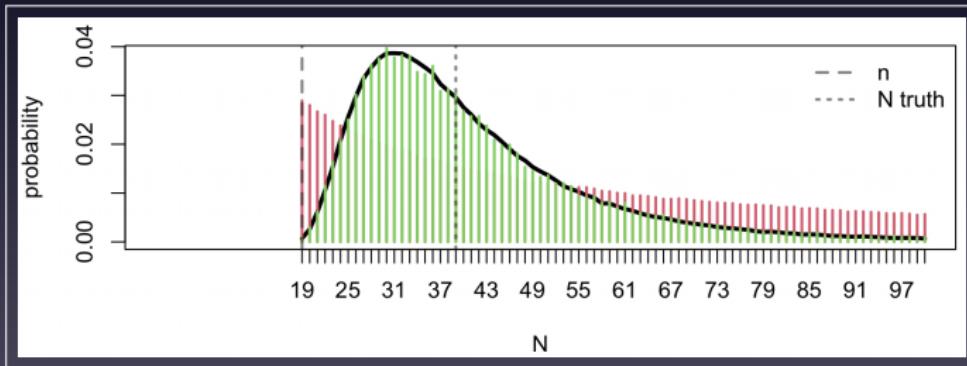


# Inference for $N$

## Stage 3: Sample $N_0$

$$N_0^{(k)} \sim \text{Binom}(M - n, \psi^{(k)}(1 - p^{(k)})^J / (\psi^{(k)}(1 - p^{(k)})^J + 1 - \psi^{(k)}))$$

for  $k = 1, \dots, K$  and let  $N^{(k)} = n + N_0^{(k)}$



# SCR Model w/ Data Augmentation

## Data Model

$$y_{i,l} \sim \begin{cases} \mathbb{1}_{\{y_{i,l}=0\}} & , z_i = 0 \\ \text{Binom}(J, p_{i,l}) & , z_i = 1 \end{cases}$$

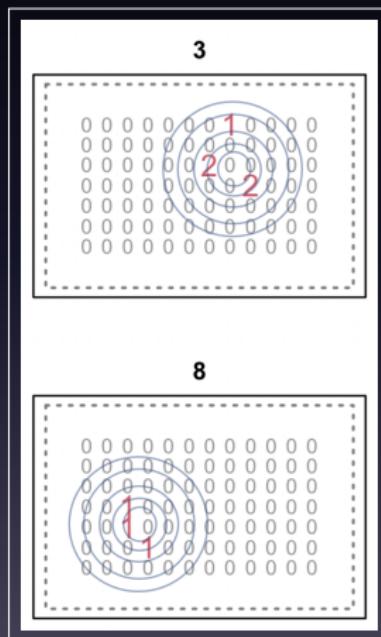
## Process Model

$$z_i \sim \text{Bern}(\psi)$$

## Detection Probability

$$g(p_{i,l}) = \beta_0 + \beta_1 \|\mathbf{c}_i - \mathbf{x}_l\|_2^2$$

$$\mathbf{c}_i \sim \text{Unif}(\mathcal{S})$$



- Royle and Young. (2008). A hierarchical model for spatial capture-recapture data. Ecology, 89: 2281–2289.

# GCR Model w/ Data Augmentation

## Data Model

$$y_{i,l} \sim \begin{cases} \mathbb{1}_{\{y_{i,l}=0\}} & , z_i = 0 \\ \text{Binom}(J, p_{i,l}) & , z_i = 1 \end{cases}$$

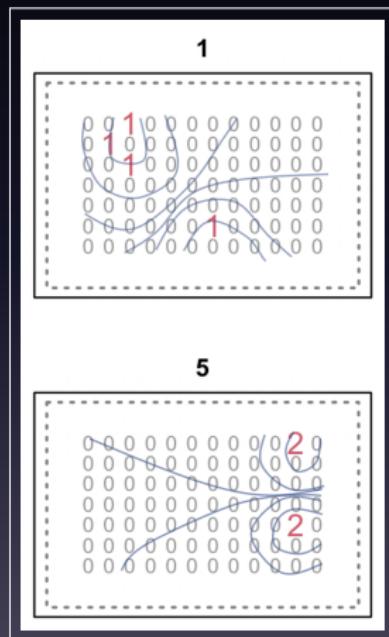
## Process Model

$$z_i \sim \text{Bern}(\psi)$$

## Detection Probability

$$\mathbf{v}_i = g(\mathbf{p}_i) \sim \mathbf{N}(\mu \mathbf{1}, \boldsymbol{\Sigma})$$

$$\boldsymbol{\Sigma} = \sigma^2 \exp(-\mathbf{D}^2/\theta^2)$$



- Hooten, Schwob, Johnson, Ivan. (2023). Geostatistical capture-recapture models. arXiv:2305.04141.

# GCR Distributions

Posterior conditioned on  $n$

$$\begin{aligned} [\mu, \sigma^2, \theta, \psi | \mathbf{Y}_{1:n}, n] &\propto \left( \prod_{i=1}^n [\mathbf{y}_i | \mu, \sigma^2, \theta, \mathbf{y}'_i \mathbf{1} > 0] \right) \\ &\quad \times [n | \mu, \sigma^2, \theta, \psi][\mu][\sigma^2][\theta][\psi] \end{aligned}$$

Integrated model for  $\mathbf{y}_i$

$$[\mathbf{y}_i | \mu, \sigma^2, \theta, \mathbf{y}'_i \mathbf{1} > 0] = \int [\mathbf{y}_i | \mathbf{p}_i, \mathbf{y}'_i \mathbf{1} > 0][\mathbf{v}_i | \mu, \sigma^2, \theta, \mathbf{y}'_i \mathbf{1} > 0] d\mathbf{v}_i$$

Conditional model for  $\mathbf{y}_i$

$$[\mathbf{y}_i | \mathbf{p}_i, \mathbf{y}'_i \mathbf{1} > 0] = \frac{\mathbb{1}_{\{\mathbf{y}'_i \mathbf{1} > 0\}} [\mathbf{y}_i | \mathbf{p}_i]}{1 - \prod_{l=1}^L (1 - p_{i,l})^J}$$

# GCR Distributions (cont)

Conditional Model for  $\mathbf{v}_i$

$$[\mathbf{v}_i | \mu, \sigma^2, \theta, \mathbf{y}'_i \mathbf{1} > 0] = \frac{(1 - \prod_{l=1}^L (1 - p_{i,l})^J) [\mathbf{v}_i | \mu, \sigma^2, \theta]}{\int (1 - \prod_{l=1}^L (1 - p_l)^J) [\mathbf{v} | \mu, \sigma^2, \theta] d\mathbf{v}}$$

Collapsed Integrated Model for  $\mathbf{y}_i$

$$\left[ \mathbf{y}_i | \mu, \sigma^2, \theta, \mathbf{y}'_i \mathbf{1} > 0 \right] = \frac{\int \mathbb{1}_{\{\mathbf{y}'_i \mathbf{1} > 0\}} [\mathbf{y}_i | \mathbf{p}_i] [\mathbf{v}_i | \mu, \sigma^2, \theta] d\mathbf{v}_i}{\int (1 - \prod_{l=1}^L (1 - p_l)^J) [\mathbf{v} | \mu, \sigma^2, \theta] d\mathbf{v}}$$

Integrated Conditional Model for  $n$

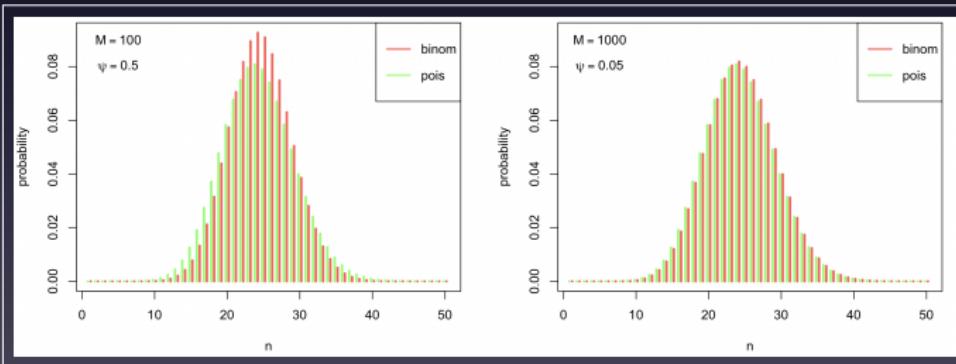
$$[n | \mu, \sigma^2, \theta, \psi] = \int [n | \mathbf{P}, \psi] [\mathbf{V} | \mu, \sigma^2, \theta] d\mathbf{V}$$

# Conditional Model for $n$

## Poisson Model for $n$

$$[n|\mathbf{P}, \psi] = \text{Pois} \left( \sum_{i=1}^M \psi \left( 1 - \prod_{l=1}^L (1 - p_{i,l})^J \right) \right)$$

for  $M \rightarrow \infty$  and  $\psi \rightarrow 0$



- Hooten, Schwob, Johnson, Ivan. (2023). Multistage hierarchical capture-recapture models. Environmetrics.

# Recursive Bayes Perspective

## Posterior

$$\begin{aligned} [\mu, \sigma^2, \theta, \psi | \mathbf{Y}_{1:n}, n] &\propto \left( \prod_{i=1}^n [\mathbf{y}_i | \mu, \sigma^2, \theta, \mathbf{y}'_i \mathbf{1} > 0] \right) \\ &\quad \times [n | \mu, \sigma^2, \theta, \psi][\mu][\sigma^2][\theta][\psi] \end{aligned}$$

## Stage 1: Fit model to only observed individuals

$$\left( \prod_{i=1}^n [\mathbf{y}_i | \mu, \sigma^2, \theta, \mathbf{y}'_i \mathbf{1} > 0] \right) [\mu][\sigma^2][\theta][\psi]$$

## Stage 2: Update using information from $n$

$$r = \frac{[n | \mu^{(*)}, \sigma^{2,(*)}, \theta^{(*)}, \psi^{(*)}]}{[n | \mu^{(k-1)}, \sigma^{2,(k-1)}, \theta^{(k-1)}, \psi^{(k-1)}]}$$

# Inference for $N$

Stage 3: Sample  $N_0$

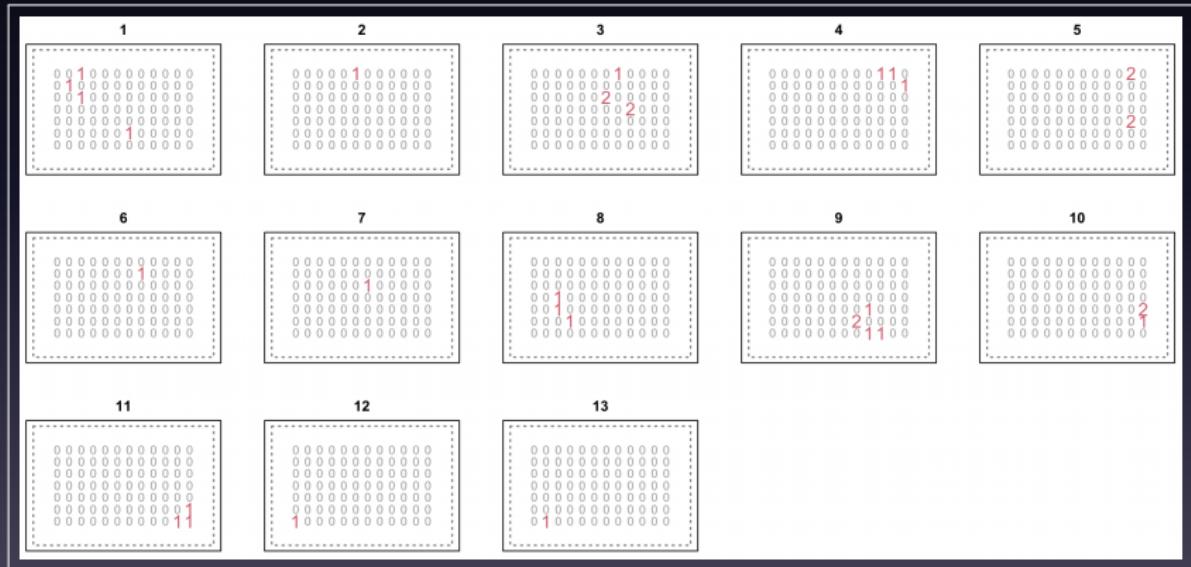
$$N_0^{(k)} \sim \text{Pois}((M - n)\bar{\psi}^{(k)})$$

for  $k = 1, \dots, K$  and let  $N^{(k)} = n + N_0^{(k)}$

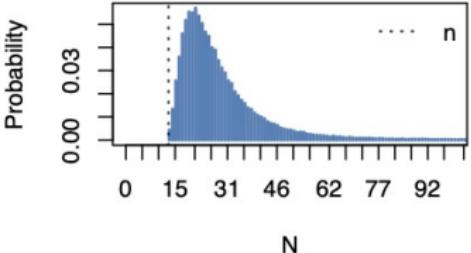
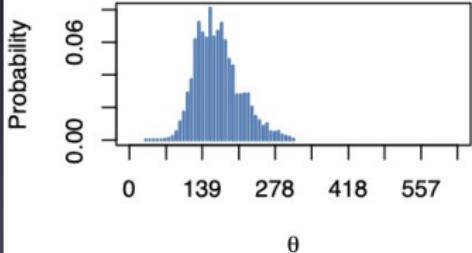
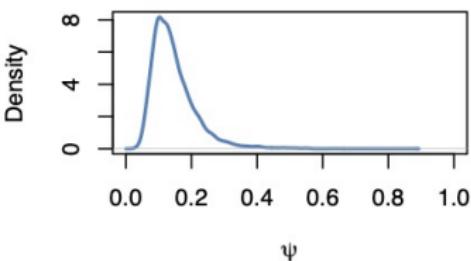
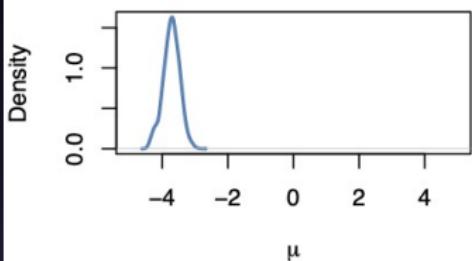
Membership Probability for Undetected Individuals

$$\bar{\psi}^{(k)} = \int \left( \frac{\psi^{(k)} \prod_{l=1}^L (1 - p_l)^J}{\psi^{(k)} \prod_{l=1}^L (1 - p_l)^J + 1 - \psi^{(k)}} \right) [\mathbf{v} | \mu^{(k)}, \sigma^{2(k)}, \theta^{(k)}] d\mathbf{v}$$

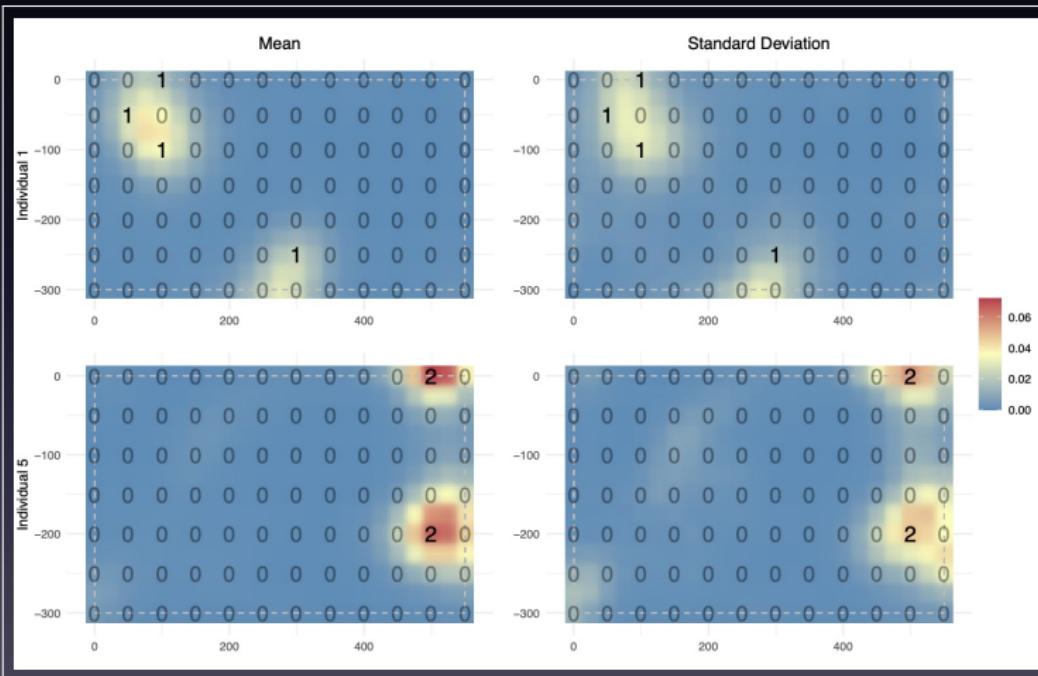
# Snowshoe Hare Data



# Snowshoe Hare Parameters



# Snowshoe Hare Detection Function



# References

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