

Math 425 Assignment 1

Max Horowitz-Gelb

1

2

Lemma A:

Statement: A connected subgraph of n nodes requires at least $n - 1$ edges.

Proof: Consider a graph $G = (V, E)$.

Let G_n be the set of all connected subgraphs of G with n nodes. And let G'_n be a subgraph of G_n such that the number of edges in G'_n is minimized. Then let $E'_n = E(G'_n)$ and $V'_n = V(G'_n)$.

First we shall consider the base case of $n = 1$ nodes. Here it is trivial to see by definition of a graph that $E'_1 = 0$.

Now assume that $E'_b = \alpha$. Then it must be true that $E'_{b+1} \geq \alpha + 1$ since for any subgraph in G_b , by definition of a graph, requires at least one new edge to connect a new node.

Therefore $E'_n \geq n - 1$, hence proving our statement. ■

Lemma B:

Statement: For a spanning tree H , there exists a node v such that $v \in H$ and v has a degree of 1

Proof: Clearly no node can have a degree of 0 or else H would not be connected and therefore not be a tree. So then it is suffice to show that if all nodes in H have a degree greater than 1, then that implies that H has a cycle and therefore is not a tree.

To show this, consider a simple traversal algorithm. In this algorithm one starts at an arbitrary node in H and then traverses an unused edge until it can no longer move because it is stuck at a node with all used edges. If at any point in this algorithm it hits a node already visited then there must be a cycle in the graph since it would imply that there is a path using unique edges from a node back to itself. If this algorithm were run on H it would have to hit a node a second time. This is because for the algorithm to not find a cycle it would have to never hit a node twice and then stop at a final unvisited node because it was stuck. But it can't get stuck at such a node since the degree of all nodes is greater than 1. Therefore there must exist a node in H such that its degree is 1. ■

Lemma C:

Statement: If H is a tree with at least 2 nodes then there exists a node v such that $v \in H$, $H \setminus v$ is a tree and $|E(H)| = |E(H \setminus v)| + 1$

Proof: By Lemma B there must be a node $v \in H$ with a degree of 1. H is a tree and therefore has no cycles so $H \setminus v$ has no cycles either. And since v has a degree of 1, by definition there can be no path in H between any two nodes in $H \setminus v$ that can include the node v . Then since H is connected, by definition $H \setminus v$ is connected. Therefore since $H \setminus v$ is connected and has no cycles it is also a tree. And since v has a degree of 1, then clearly $|E(H)| = |E(H \setminus v)| + 1$. ■

Lemma 2.2

Statement: Let G be a connected graph with n nodes. Then G is a spanning tree if and only if it has exactly $n - 1$ nodes.

Proof: First we shall prove that if G is a spanning tree then it has 1 less edges than nodes.

Let S_n be the set of all spanning trees of all subgraphs of G containing n nodes where a spanning tree is possible. Then let $|E_n|$ be the number of edges of used for each subgraph in S_n assuming they are all the same.

Clearly $|E_1| = 0$ by definition of a graph.

Then if $S_n = n - 1$ then $S_{n+1} = n$. This is because Lemma C implies for all $H \in S_{n+1}$ there exists a node $v \in H$ such that $H \setminus v \in S_n$ and as well $|E_{n+1}| = |E_n| + 1$.

Therefore by induction G is a spanning tree of n nodes then it contains $n - 1$ edges.

Now we shall show that if G is a connected graph with n nodes and $n - 1$ edges

then it is a tree.

If G were connected and not a tree then it must have a cycle. If it had a cycle then there must be an edge $e = (u, v)$ such that there is a path without e from u to v and clearly e can be removed from G without breaking the connectivity of G . But this would imply that there is a connected graph with $n - 2$ edges and n nodes which breaks Lemma A. Therefore G cannot have any cycles and must be a tree. ■

3

Lemma D

Statement: If H is a spanning tree of graph G then there is one unique simple path between all pairs of nodes in H .

Proof: First we shall show that all edge simple paths between nodes are simple paths. If there was an edge simple path between two nodes in H that was not simple, then there is a portion of that path $v_i, e_i, \dots, e_{j-1}, v_j$ such that $v_i = v_j$. This would imply that there is a cycle in H which cannot be true since H is a tree. Therefore all edge simple paths in H are also simple.

For any two nodes in H there is at least one path between them since H is a tree. If that path is not edge simple then it can be transformed into an edge simple path by removing all loops e_i, \dots, e_{j-1}, v_j such that $v_i = v_j, e_i = e_{j-1}$. And as we have shown that edge simple path must be simple. Therefore there is at least one simple path between all pairs of nodes in H .

If there were more than one simple path between two nodes then select two of them and simply remove from both paths $e_k \dots v_n$ such that both removals are equal and remove $v_0 \dots e_i$ such that i is maximized and less than k and v_{i+1} is the same for both paths. When concatenating both paths you would create a cycle which cannot be true since H is a tree.

Therefore for any pair of nodes in H there is only one edge simple path between them and it is simple. ■

Statement: Let $H = (V, T)$ be a spanning tree of $G = (V, E)$. Let $e = vw$ be an edge in $E \setminus T$, and let f be an edge of a simple path in H from v to w . Then (a) the subgraph H' obtained from adding e has a unique cycle containing e , and (b) the subgraph $H'' = (V, T \cup \{e\} \setminus \{f\})$ is a spanning tree of G .

Proof of (a): First we can clearly see that since H is a spanning tree then there is a path from v to w and then adding e creates a cycle by definition. If adding e created more than one unique cycle, that would imply that there exists at least two unique edge simple paths connecting v to w in H . This cannot be true since it would mean that there was a cycle in H which is a tree. Therefore e only introduces one unique cycle. ■

Proof of (b)

First e cannot introduce a cycle because if it did that would imply that there was a simple path from v to w in $H \setminus \{f\}$. But this cannot be true since by Lemma D, removing f breaks the one and only simple path between v and w .

Again by Lemma D only all pairs of nodes in H connected by a simple path containing f will be disconnected in $H \setminus \{f\}$. Since v and w are connected in H using a simple path containing f , then all the disconnected nodes in $H \setminus \{f\}$ must be connected to either v or w . Then all nodes in $H \setminus \{f\}$ are connected to v and or w . Therefore since in $H \cup \{e\} \setminus \{f\}$ v and w are connected, then all pairs of nodes are connected.

Therefore H'' is a spanning tree.