HW Week 3

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February 2nd, 2017

Q1

We have shown in class that this probability can be rewritten as for arbitrary $n \geq 1$ as,

$$P(X_{n+2} = 3, X_{n+4} = 4 | X_{n+1} = 9, X_n = 8)$$

Then since X is a THMC this can be written simply as,

$$p(9,3) * p^2(3,4)$$

Q2

a

If $V = \max\{T, U\}$ then V is a stopping time. This is because there are only two cases.

Case 1: $T \geq U$

Since T and U are stopping times, we can check for this case with only $X_1 ... X_U$, which implies that we can check this case with only $X_1 ... X_T$. Then if we are in this case, $V = T$ and since T is a stopping time, V can be determined with only $X_1 ... X_V$.

Case 2: $T < U$

Since T and U are stopping times, we can check this case with only $X_1 .. X_T$ which implies we can tell this case with $X1 ... X_U$. Then in this case, $V = U$ and since U is a stopping time, V can be determined with only $X_1 ... X_V$.

Then by definition V is a stopping time.

b

If $V = \min\{T, U\}$ then again there are the two same cases.

Case 1: $T \geq U$

Again we can tell if we are in this case with only $X_1 ... X_U$. Then if we are in this case, $V = U$ and since U is a stopping time, we can know V with only $X_1 ... X_V$.

Case 2: $T < U$.

Again we can determine if we are in this case with only $X_1 ... X_T$. And if we are in this case, $V = T$, and since T is a stopping time, then we can determine V with only $X_1 ... X_V$.

Therefore V is a stopping time.

Q3

Let n be given. First note that by definition of a THMC

$$P(X_{T+2} = j | X_T = i, T = n) = P(X_{n+2} = j | X_n = i, past) = p^2(i, j)$$

implies,

$$\forall n' P(X_{T+2} = j | X_T = i, T = n') = p^2(i, j)$$

Then note that $P(X_{T+2} = j | X_T = i)$ can be rewritten as,

$$\sum_{n'} P(X_{T+2} = j | X_T = i, T = n') * P(T = n')$$

Which by what we've shown above can be rewritten as,

$$= p^2(i, j) * \sum_{n'} P(T = n')$$

$$= p^2(i, j)$$

Q4

Let $X_0 = E$, and $T = min\{n \geq 1 : X_n = E\}$ Then

$$P(T = \infty) = p(E, N) * \lim_{n \to \infty} \prod_{i=2}^{n} p(N, N)$$

$$= 0.9 * \lim_{n \to \infty} \prod_{i=2}^{n} 0.2$$

$$= 0$$

Therefore $P(T < \infty) = 1$ and E is a recurrent state.

Q5

Let $E : \{X_1..X_n\} \mapsto \mathbb{R}$ and $N : \{X_1...X_n\} \mapsto \mathbb{R}$ be functions such that $E(\{X_1, ...X_n\})$ equals the number of E states in the set $\{X_1...X_n\}$ and $N(\{X_1, ...X_n\})$ equal the number of N states in $\{X_1, ...X_n\}$.

Then $T = n$ if and only if $E(\{X_1, ...X_n\}) \geq 1$, $N(\{X_1, ...X_n\}) \geq 2$ and $E(\{X_1, ...X_{n-1}\}) = 0$ or $N(\{X_1, ...X_{n-1}\}) < 2$. Since this is a function of only $X_1...X_n$ then T is a stopping time.

Now we consider $P(T = +\infty | X_0 = E)$. This is bounded by,

$$P_{n\to\infty}(N(X_1...X_n) < 2) + P_{n\to\infty}(E(X_1...X_n) < 1)$$

For the left part of this bound,

$$P_{n\to\infty}(N(X_1...X_n) < 2) = P_{n\to\infty}(N(X_1...X_n) = 0) + P_{n\to\infty}(N(X_1...X_n) = 1)$$

$$P_{n\to\infty}(N(X_1...X_n) = 0) = \lim_{n\to\infty} p(E, E)^n = \lim_{n\to\infty} 0.1^n = 0$$

$$P_{n\to\infty}(N(X_1...X_n) = 1) = \lim_{n\to\infty} \binom{n}{1} * p(E, N) * p(N, E) * p(E, E)^n$$

$$= \lim_{n\to\infty} n * 0.9 * 0.2 * 0.1^n = 0$$

And for the right part of the bound,

$$P_{n\to\infty}(E(X_1...X_n) < 1) = P_{n\to\infty}(E(X_1...X_n) = 0)$$

$$= \lim_{n\to\infty} p(E, N) * p(N, N)^n = \lim_{n\to\infty} 0.9 * 0.8^n = 0$$

Therefore $P(T = +\infty | X0 = E) = 0$