

# HW1

Max Horowitz-Gelb horowitzgelb@wisc.edu

1/19/17

## Q1

Let  $X_1 \dots X_{40}$  be a set of random variables such that  $X_i = 1$  is the event that box  $i$  is empty after all 80 balls have been placed, and  $X_i = 0$  otherwise. Then the number of boxes empty after all balls are placed, which let's call  $T$ , is

$$T = \sum_{i=1}^{40} X_i$$

Then,

$$E[T] = E\left[\sum_{i=1}^{40} X_i\right] = \sum_{i=1}^{40} E[X_i]$$

The expected value of any  $X_i$  is the same and can be calculated as,

$$E[X_i] = 1 * P(X_i = 1) + 0 * P(X_i = 0) = P(X_i = 1) = \frac{39^{80}}{40} = 0.13193780538$$

Therefore,

$$E[T] = \sum_{i=1}^{40} E[X_i] = 40 * 0.13193780538 = 5.27751221548$$

A similar procedure is used to find the variance of  $T$ . We know that the variance of  $T$  is equal to  $E[T^2] - E[T]^2 = E[T^2] - 27.85213518454$ . Then,

$$E[T^2] = E\left[\sum_{i=1}^{40} \sum_{j=1}^{40} X_i X_j\right] = \sum_{i=1}^{40} \sum_{j=1}^{40} E[X_i X_j]$$

$$E[X_i X_j] = 1 * P(X_i X_j = 1) + 0 * P(X_i X_j = 0) = P(X_i X_j = 1)$$

If  $i = j$  then  $X_i X_j = 1$  if and only if  $X_i = 1$  and then,

$$P(X_i X_j = 1) = P(X_i = 1) = 0.13193780538$$

If  $i \neq j$  then

$$P(X_i X_j = 1) = \frac{38^{80}}{40} = 0.01651537438$$

Therefore,

$$\begin{aligned} E[T^2] &= 40 * P(X_i X_j = 1 | i = j) + 40 * 39 * P(X_i X_j = 1 | i \neq j) \\ &= 40 * 0.13193780538 + 40 * 39 * 0.01651537438 = 31.041496248 \end{aligned}$$

So,

$$Var(T) = E[T^2] - E[T]^2 = 31.041496248 - 27.85213518454 = 3.18936106346$$

## Q2

If  $X$  is a uniform random variable over  $[0, \frac{\pi}{2}]$  then  $X$  has a pdf,

$$f(x) = \frac{2}{\pi} \text{ for } x \in [0, \frac{\pi}{2}]$$

Therefore,

$$E[\sin(X)] = \int_0^{\frac{\pi}{2}} \sin(x) f(x) dx = \frac{2}{\pi}$$

## Q3

At time  $n$  there are  $X_n$  white balls in the left urn,  $5 - X_n$  black balls in the left urn,  $5 - X - n$  white balls in the right urn and  $X_n$  black balls in the right urn. To calculate the transition probability  $P(X_{n+1} = j | X_n = i)$  we must consider 3 cases. If  $i = j$ , then either we must swap a black ball from the left urn with a black ball from the right urn or a white ball from the left urn and a white ball from the right urn. The probability on the black balls is ,

$$\frac{i}{5} * \frac{5-i}{5} = \frac{5i-i^2}{25}$$

and the probability from the white balls is the same,

$$\frac{5-i}{5} * \frac{i}{5} = \frac{5i-i^2}{25}$$

Therefore the total probability for  $j = i$  is,

$$2 * \frac{5i-i^2}{25}$$

If  $j = i - 1$  we must select a white ball from the left urn and a black ball from the right urn. The probability for this is

$$\frac{i}{5} * \frac{i}{5} = \frac{i^2}{25}$$

Finally if  $j = i + 1$  then we must select a black ball from the left urn and a white ball from the right urn. This has a probability,

$$\frac{5-i}{5} * \frac{5-i}{5} = \frac{(5-i)^2}{25}$$

So our complete set of transition probabilities is,

$$P(X_{n+1} = j | X_n = i) = \begin{cases} 2 * \frac{5i-i^2}{25}, i = j \\ \frac{i^2}{25}, j = i - 1 \\ \frac{(5-i)^2}{25}, j = i + 1 \\ 0, \text{ else} \end{cases}$$