HW 13

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Q1

First note that for $i \neq j$

$$q(i,j) = \lim_{t \to 0} \frac{p_t(i,j)u(i,j)}{t}$$

Then by definition of p' and the Chapman-Kolmogorov equation,

$$p'_t(i,j) = \lim_{h \to 0} \frac{p_{t+h}(i,j) - p_t(i,j)}{h}$$

$$= \lim_{h \to 0} \frac{\sum_k \big(p_t(i,k)p_h(k,j)\big) - p_t(i,j)}{h}$$

$$= \lim_{h \to 0} \frac{\sum_{k \neq j} \big(p_t(i,k)p_h(k,j)\big) + p_t(i,j)(p_h(j,j) - 1)}{h}$$

$$= \lim_{h \to 0} \frac{\sum_{k \neq j} \big(p_t(i,k)p_h(k,j)\big)}{h} + \lim_{h \to 0} \frac{p_t(i,j)(p_h(j,j) - 1)}{h}$$

$$= \sum_{k \neq j} p_t(i,k)q(k,j) - \lim_{h \to 0} \frac{p_t(i,j)(\sum_{k \neq j} p_h(j,k)}{h} \qquad \text{By definition of } q$$

$$= \sum_{k \neq j} p_t(i,k)q(k,j) + p_t(i,j)(-\sum_{k \neq j} q(j,k))$$

$$= \sum_{k \neq j} p_t(i,k)q(k,j) + p_t(i,j)(-\lambda_j)$$

$$= p_t(i,\cdot)Q(\cdot,j)$$

Q2

Note

$$
\begin{aligned}
\frac{\delta}{\delta t} p_t = \frac{\delta}{\delta t} e^{Qt} \\
= Q \sum_{n=1}^{\infty} \frac{nQ^{n-1} t^{n-1}}{n!} \\
= Q \sum_{n=1}^{\infty} \frac{Q^{n-1} t^{n-1}}{(n-1)!} \\
= Q p_t \qquad \text{by the givien solution of KBE}
\end{aligned}
$$

Q3

$$
\begin{aligned}
(\tilde{\pi}\tilde{p})_i = \frac{1}{\sum_{k \in S} \pi_k \lambda_k} \sum_{k \neq i} \pi_k \lambda_k \tilde{p}(k, i) \\
= \frac{1}{\sum_{k \in S} \pi_k \lambda_k} \sum_{k \neq i} \pi_k \lambda_k \frac{q(k, i)}{\lambda_k} \\
= \frac{1}{\sum_{k \in S} \pi_k \lambda_k} \sum_{k \neq i} \pi_k q(k, i)
\end{aligned}
$$

We know that $\pi Q = 0$ so This is equal to

$$
\begin{aligned}
\frac{-\pi_i Q_{i,i}}{\sum_{k \in S} \pi_k \lambda_k} \\
= \frac{\pi_i \lambda_i}{\sum_{k \in S} \pi_k \lambda_k}
\end{aligned}
$$

so $\tilde{\pi}$ is a stationary measure, and clearly by the normalizing term it is also a distribution.

Q4

i

Note

$$\lim_{t \to \infty} P_t(B, C) = \pi_C$$

Then we solve for
$$\pi \begin{bmatrix} Q_{A,A} & Q_{A,B} & 1 \\ Q_{B,A} & Q_{B,B} & 1 \\ Q_{C,A} & Q_{C,B} & 1 \end{bmatrix} = [0,0,1]$$
and we get $\pi = [1/2, 1/4, 1/4]$ so $\pi_C = 1/4$.

ii

The limiting fraction for the time in each city is simply the stationary distribution. So the limiting fraction for Atlanta is 1/2, the limiting fraction for Boston is 1/4 and the limiting fraction for Chicago is 1/4.

iii

Let $N(t)$ be the number of cities visited t years, X_i be the amount of time spent in the i^{th} city and $Z_{B_i} = 1$ if city i is Boston.

The average number of visits to Boston per year is equal to

$$\lim_{t\to\infty} \frac{\sum_{i=1}^{N(t)} Z_{B,i}}{t}$$

$$= \lim_{t\to\infty} \frac{N(t)}{t} \frac{\sum_{i=1}^{N(t)} Z_{B_i}}{N(t)}$$

$$\lim_{t\to\infty} \frac{N(t)}{t} \lim_{t\to\infty} \frac{\sum_{i=1}^{N(t)} Z_{B_i}}{N(t)}$$

We know from previously by solving the stationary distribution that,

$$\lim_{t\to\infty} \frac{\sum_{i=1}^{N(t)} Z_{B,i} X_i}{t} = \pi_B = 1/4$$

Then note

$$E[Z_{B,i}X_i] = E[Z_{B,i}]E[X_i|Z_{B,i}=1] = E[Z_{B,i}] * 1/4$$

Then

$$\lim_{t\to\infty} \frac{\sum_{i=1}^{N(t)} Z_{B,i} X_i}{t}$$

$$= \lim_{t\to\infty} \frac{N(t)}{t} \frac{\sum_{i=1}^{N(t)} Z_{B_i} X_i}{N(t)}$$

$$\lim_{t\to\infty} \frac{N(t)}{t} \lim_{t\to\infty} \frac{\sum_{i=1}^{N(t)} Z_{B_i} X_i}{N(t)}$$

Which by the Strong Law of Large Numbers is equal to

$$\lim_{t\to\infty} \frac{N(t)}{t} E[Z_{B,i} X_i]$$

$$= \lim_{t\to\infty} \frac{N(t)}{t} E[Z_{B,i}]0.25$$

We know this value is equal to $\pi_B = 0.25$ which implies

$$\lim_{t\to\infty} \frac{N(t)}{t} E[Z_{B,i}] = 1$$

which is the expected number of visits per year in Boston.

iv

The expected number of times she flies from Boston to Atlanta is

$$E[\sum_{i=1}^{N(1)} Z_{B,i}] * P(B, A) = 3/4$$

Q5

$Q =$

	0	1	2	3
0	-1	0	1	0
1	2	-3	0	1
2	0	2	-2	0
3	0	0	2	-2

Solving for $\pi Q = 0$ we get

$$\pi = [0.4, 0.2, 0.3, 0.1]$$

Q6

a

There are 5 states in this model,

0 is when no machines are broken.

1 is when machine 1 is broken.

2 is when machine 2 is broken.

12 is when both machines are broken but 1 broke first.

21 is when both machines are broken but 2 broke first.

0 can transition to 1 or 2

1 can transition to 0, or 12

12 can transition to 2

21 can transition to 1

b

This gives a Q matrix of

	0	1	2	12	21
0	-4	1	3	0	0
1	2	-5	0	3	0
2	4	0	-5	0	1
12	0	0	2	-2	0
21	0	4	0	0	-4

Solving for $\pi Q = 0$ we get

$$\pi = [0.34108527, 0.12403101, 0.27906977, 0.18604651, 0.06976744]$$