HW 6

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Q1

Let our state space be $[T, C]$. Then we can think of one vehicle following another vehicle as a transition with transition matrix,

$$p_{i,j} = \begin{bmatrix} 1/4 & 3/4 \\ 1/5 & 4/5 \end{bmatrix}$$

We can then solve for the stationary distribution of this transition matrix simply since it is a 2x2 matrix.

$$\pi_T = \frac{1/5}{1/5 + 3/4} = 0.2105, \pi_C = \frac{3/4}{1/5 + 3/4} = 0.7895$$

Then since $p_{i,j} \neq 0$ for all i, j, then clearly p is irreducible and aperiodic. Then since our state space is finite and we have a stationary distribution π, then theorem 1.19 says the probability of being a truck converges to $\pi_T = 0.2105$.

Q2

For this we simply solve for the set of functions

$$\pi_1 - (0.86\pi_1 + 0.05\pi_2 + 0.03\pi_3) = 0$$

$$\pi_2 - (0.08\pi_1 + 0.88\pi_2 + 0.05\pi_3) = 0$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

Then,

$$\pi = [0, 0, 1] \begin{bmatrix} 0.14 & -0.08 & 1 \\ -0.05 & 0.12 & 1 \\ -0.04 & -0.05 & 1 \end{bmatrix}^{-1} = [0.2155477, 0.33215548, 0.45229682]$$

Then since p is finite and $p_{i,j} \neq 0$ for all i, j then p is irreducible and is aperiodic, we may again apply theorem 1.19 to say that in the long run, $\pi_1 = 0.2155477$ fraction of the population will reside in cities, $\pi_2 = 0.33215548$ fraction of people will reside in suburbs and $\pi_3 = 0.45229682$ fraction of people will reside in rural areas.

Q3

a.

Using the same strategy as Q2 we first solve for a stationary distribution to get

$$\pi = [0.36363636, 0.35664336, 0.27972028]$$

Then since p is finite and $p_{i,j} \neq 0$ for all i, j then we again can apply 1.19 to say,

$$\lim_{n \to \infty} p^n(1, 2) = 0.35664336$$

b.

Then since p is irreducible and closed, then it is recurrent, and we may apply theorem 1.21 which says,

$$\frac{N_n(2)}{n} \to \frac{1}{E_2 T_2}$$

and as shown in a.,

$$\frac{N_n(2)}{n} \to \pi_2$$

so ,

$$\pi_2 = \frac{1}{E_2 T_2}$$

and

$$E_2 T_2 - 1 = 1/\pi_2 - 1 = 1.8039$$

c.

$$E[\text{walking distance}] = \sum_n p(n) * (0.3 + 0.1n)$$

$$= \sum_n \pi_n * (0.3 + 0.1n) = 0.4916$$

Q4

i

Our state space is $[1, 2, 2']$ where the number indicates the number of light bulbs working and ' indicates we just replaced the light bulbs. Then our transition matrix is

$$\begin{bmatrix} 0.95 & 0 & 0.05 \\ 0.02 & 0.98 & 0 \\ 0.02 & 0.98 & 0 \end{bmatrix}$$

Solving for the stationary distribution we get,

$$\pi = [0.2857, 0.7, 0.0143]$$

Then since p is finite and nonzero for all i, j, then it is irreducible and theorem 1.19 says that the long run fraction of time one light bulb is working is $\pi_1 = 0.2857$.

ii

Then we can use theorem 1.21 to say that

$$E_2 T_{2'} - 1 = 1/\pi_{2'} - 1 = 69$$

So the expected number of days between light bulb changes is 69 days.

Q5

First since states $\{1...M\}$ are closed, then for any $x \in \{1...M\}$.

$$\sum_m p(x, m) = \sum_m f((x-m) \mod M) = \sum_{n=0}^{M-1} f(n) = 1$$

Then for any $x \in \{1...M\}$,

$$\sum_m p(m, x) = \sum_m f((m-x) \mod M) = \sum_{n=0}^{M-1} f(n) = 1$$

therefore the THMC is doubly stochastic implying at has a uniform stationary distribution

$$\pi = [1/M...1/M]$$

Clearly our state space is finite. Then since $p(1,2) > 0$ and $p(2,4) > 0)$ then for all x, $p(x, (x+1) \mod M), p(x, (x-1) \mod M), p(x, (x+2) \mod M), p(x, (x-2) \mod M)$ are all positive. So for any integers a, b such that $a + 2b = M$, $p^{a+b}(x, x) \neq 0$. For any $M \geq 4$ the set of $\{a + b\}$ has a gcd of 1. Therefore our THMC is aperiodic and by theorem 1.19

$$lim_{n \to \infty} p^n(1,1) = \pi_1 = 1/M$$

Then using theorem 1.23

$$lim_{n \to \infty} \sum_{m=1}^{n} X_m^2 = \sum_{i=1}^{M} \pi_i * i^2 = 1/M * \sum_{i=1}^{M} i^2$$

Q6

2 already shows us that v is a distribution so we must only show that v is a stationary measure. Note

$$(vp)_j = \sum_i v_i p(i, j)$$

which then using 3.

$$= \sum_i v_j p(j,i) = v_j * \sum_i p(j,i) = v_j$$

Therefore v is a stationary distribution.

Q7

a.

First we solve for a stationary distribution $\pi = [0.5854, 0.2927, 0.0975, 0.0244]$ Clearly $\{X_n\}$ is closed and since $p(1,1) > 0$ and for all x, $x \to 1$ then all states are aperiodic and by theorem 1.19

$$\lim_{n\to\infty} p(x,1) = \pi_1 = 0.5854$$

which is the fraction of the time Dave shaves in the long run.

b.

Yes π has detailed balance condition after verifying, for all i, j

$$\pi_i * p(i,j) = \pi_j * p(j,i)$$

Q8

a.

$$p(i,j) = \begin{cases} \frac{i*(b-i)}{m^2} + \frac{(m-i)(m-b+i)}{m^2} & i = j \\ \frac{(m-i)(b-i)}{m^2} & i = j-1 \\ \frac{i(m-b+i)}{m^2} & i = j+1 \\ 0 & else \end{cases}$$

b.

π is a stationary distribution since it satisfies the detailed balance condition. This can be shown with 3 cases

Case 1 $|i-j| \geq 2$

Then clearly

$$\pi_i * p(i,j) = \pi_j * p(j,i) = 0$$

Case 2 $i = j$

Then clearly

$$\pi_i * p(i,j) = \pi_j * p(j,i)$$

Case 3 $|i - j| = 1$

Without loss of generality assume that $i = j - 1$ since we can simply flip our equation around.

Then,

$$\pi_i * p(i,j) = \binom{b}{i}\binom{2m-b}{m-i} / \binom{2m}{m} \frac{(m-i)(b-i)}{m^2}$$

$$= \frac{b!}{i!(b-i-1)!} \frac{(2m-b)!}{(m-i-1)!(m-b+i)!} \frac{\binom{2m}{m}}{m^2}$$

and,

$$\pi_j * p(j,i) = \binom{b}{j}\binom{2m-b}{m-j} / \binom{2m}{m} \frac{j(m-b+j)}{m^2}$$

$$\frac{b!}{(j-1)!(b-j)!} \frac{(2m-b)!}{(m-j)!(m-b+j-1)!} \frac{\binom{2m}{m}}{m^2}$$

Then since $j = i+1$, plugging in $i+1$ you see that both equations are identical or,

$$\pi_i * p(i,j) = \pi_j * p(j,i)$$

Therefore π satisfies the detailed balance condition and is a stationary distribution.