Homework 2

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Q1

a.

Are transition matrix $p_{i,j} =$

-	A	B	C
A	0	1/2	1/2
B	3/4	0	1/4
C	3/4	1/4	0

b.

To calculate probabilities of events at time 2 we simply calculate $p(i,j)^2$.

$p(i,j)^2 =$

-	A	B	C
A	3/4	1/8	1/8
B	3/16	7/16	3/8
C	3/16	3/8	7/16

So,

$$P(X_2 = A|X_0 = A) = p(A,A)^2 = 3/4$$
$$P(X_2 = B|X_0 = A) = p(A,B)^2 = 1/8$$
$$P(X_2 = C|X_0 = A) = p(A,C)^2 = 1/8$$

Finally we can also calculate

$$P(X_3 = B|X_0 = A) = p(A,A)^2 * p(A,B) + p(A,B)^2 * p(B,B) + p(A,C)^2 * p(C,B)$$
$$= 3/4 * 1/2 + 1/8 * 0 + 1/8 * 1/4 = 13/32$$

Q2

$$P(X_2 = 3, X_4 = 4|X_7 = 9, X_6 = 8)$$
$$= \frac{P(X_2 = 3, X_4 = 4, X_7 = 9, X_6 = 8)}{P(X_7 = 9, X_6 = 8)}, \text{ using basic defintition of conditional probability}$$
$$= \frac{P(X_7 = 9|X_6 = 8) * P(X_6 = 8|X_4 = 4) * P(X_4 = 4|X_2 = 3) * P(X_2 = 3)}{P(X_7 = 9|X_6 = 8) * P(X_6 = 8)}, \text{ using that} X \text{ is a THMC and chain rule}$$
$$= \frac{p(8,9) * p(4,8)^2 * p(3,4)^2 * p(1,3)^2}{p(8,9) * p(1,8)^6}$$

Q3

Using similar logic to before, we can rewrite $P(X_3 = X_2 + 1 | X_4 = 4)$ as,

$$\frac{P(X_4 = 4, X_3 = X_2 + 1)}{P(X_4 = 4)}$$

$$= \frac{\sum_k P(X_4 | X_3 = k+1) * P(X_2 = k)}{P(X_4 = 4)}$$

$$= \frac{\sum_k p(k+1, 4) * p(k, k+1) * p(1, k)^2}{p(1, 4)^4}$$

Q4

Our probability space is equal to

$$\Omega = \bigcup_{n \geq 0, i} A_{n,i}$$

and,

$$\left\{ \max_{n \geq 1} X_n > m \right\} = \bigcup_{n \geq 1, i > m} A_{n,i}$$

So then,

$$\left\{ \max_{n \geq 1} X_n \leq m \right\} = \left(\bigcup_{n \geq 0, i} A_{n,i} \right) \setminus \left(\bigcup_{n \geq 1, i > m} A_{n,i} \right)$$

Then,

$$\left\{ \max_{n \geq 1} X_n \geq m \right\} = \bigcup_{n \geq 1} A_{n,m}$$

so,

$$\left\{ \max_{n \geq 1} X_n = m \right\} = \left(\bigcup_{n \geq 1} A_{n,m} \right) \setminus \left(\bigcup_{n \geq 1, i > m} A_{n,i} \right)$$

Q5

Let $a = 2, b = 1, c = 0, d = 0$. Then $P(X_8 = a | X_7 \in \{b, c\}, X_6 = d) = 0$. This is because $d = 0$, so $X_6 = 0$, which implies $\forall_{n>6} X_n = 0$. But,

$$P(X_8 = a | X_7 \in \{b, c\}) = P(X_8 = 2 | X_7 = 1) + P(X_8 = 2 | X_7 = 0) = P(X_8 = 2 | X_7 = 1)$$

$P(X_8 = 2 | X_7 = 1) > 0$, so it is not the case that
$P(X_8 = a | X_7 \in \{b, c\}, X_6 = d) = P(X_8 = a | X_7 \in \{b, c\})$.