HW1

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Q1

Let $X_1 ... X_{40}$ be a set of random variables such that $X_i = 1$ is the event that box i is empty after all 80 balls have been placed, and $X_i = 0$ otherwise. Then the number of boxes empty after all balls are placed, which let's call T, is

$$T = \sum_{i=1}^{40} X_i$$

Then,

$$E[T] = E[\sum_{i=1}^{40} X_i] = \sum_{i=1}^{40} E[X_i]$$

The expected value of any X_i is the same and can be calculated as,

$$E[X_i] = 1 * P(X_i = 1) + 0 * P(X_i = 0) = P(X_i = 1) = \frac{39}{40}^{80} = 0.13193780538$$

Therefore,

$$E[T] = \sum_{i=1}^{40} E[X_i] = 40 * 0.13193780538 = 5.27751221548$$

A similar procedure is used to find the vaiance of T. We know that the variance of T is equal to $E[T^2] - E[T]^2 = E[T^2] - 27.85213518454$. Then,

$$E[T^2] = E[\sum_{i=1}^{40} \sum_{j=1}^{40} X_i X_j] = \sum_{i=1}^{40} \sum_{j=1}^{40} E[X_i X_j]$$

$$E[X_i X_j] = 1 * P(X_i X_j = 1) + 0 * P(X_i X_j = 0) = P(X_i X_j = 1)$$

If $i = j$ then $X_i X_j = 1$ if and only if $X_i = 1$ and then,

$$P(X_i X_j = 1) = P(X_i = 1) = 0.13193780538$$

If $i \neq j$ then

$$P(X_iX_j = 1) = \frac{38}{40}^{80} = 0.01651537438$$

Therefore,

$$E[T^2] = 40 * P(X_iX_j = 1|i = j) + 40 * 39 * P(X_iX_j = 1|i \neq j)$$

$$= 40 * 0.13193780538 + 40 * 39 * 0.01651537438 = 31.041496248$$

So,

$$Var(T) = E[T^2] - E[T]^2 = 31.041496248 - 27.85213518454 = 3.18936106346$$

Q2

If X is a uniform random variable over $[0, \frac{\pi}{2}]$ then X has a pdf,

$$f(x) = \frac{2}{\pi} \text{ for } x \in [0, \frac{\pi}{2}]$$

Therefore,

$$E[sin(X)] = \int_0^{\frac{\pi}{2}} sin(x)f(x)dx = \frac{2}{\pi}$$

Q3

At time n there are X_n white balls in the left urn, $5 - X_n$ black balls in the left urn, $5 - X_n$ white balls in the right urn and X_n black balls in the right urn. To calculate the transition probability $P(X_{n+1} = j|X_n = i)$ we must consider 3 cases. If $i = j$, then either we must swap a black ball from the left urn with a black ball from the right urn or a white ball from the left urn and a white ball from the right urn. The probability on the black balls is ,

$$\frac{5-i}{5} * \frac{i}{5} = \frac{5i - i^2}{25}$$

and the probability from the white balls is the same,

$$\frac{i}{5} * \frac{5-i}{5} = \frac{5i - i^2}{25}$$

Therefore the total probability for $j = i$ is,

$$2 * \frac{5i - i^2}{25}$$

If $j = i - 1$ we must select a white ball from the left urn and a black ball from the right urn. The probability for this is

$$\frac{i}{5} * \frac{i}{5} = \frac{i^2}{25}$$

Finally if $j = i + 1$ then we must select a black ball from the left urn and a white ball from the right urn. This has a probability,

$$\frac{5-i}{5} * \frac{5-i}{5} = \frac{(5-i)^2}{25}$$

So our complete set of transition probabilities is,

$$P(X_{n+1} = j | X_n = i) = \begin{cases} 2 * \frac{5i-i^2}{25}, i = j \\ \frac{i^2}{25}, j = i - 1 \\ \frac{(5-i)^2}{25}, j = i + 1 \\ 0, \text{ else} \end{cases}$$