CS761 Spring 2015 Homework 1

Assigned Jan. 18, due Jan. 20 before class

Instructions:

- Homeworks are to be done individually.
- Typeset your homework in latex using this file as template (e.g. use pdflatex). Show your derivations.
- Hand in the compiled pdf (not the latex file) online. Instructions will be provided. We do not accept hand-written homeworks.
- Homework will no longer be accepted once the lecture starts.
- Fill in your name and email below.

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Each question has 5 points if answered correctly. Wrong answers or blank answers 0 points. However, an explicit "I don't know" answer is worth 1 point.

1. Define inner product on real functions

$$\langle g, h\rangle = \int_0^1 g(x)h(x)\,dx.$$

Orthogonalize the basis $1, x, x^2$.

Answer

I use the Gram Schmidt process for finding orthogonal basis vectors. First let $u_1 = 1$ Then let's define

$$proj_u(v) = \frac{\langle u, v\rangle}{\langle u, u\rangle}u$$

Then let ,

$$u_2 = x - proj_{u1}(x) = x - \frac{\int_0^1 xdx}{\int_0^1 1dx}$$

$$= x - \frac{1}{2}$$

and finally,

$$u_3 = x^2 - proj_{u1}(x^2) - proj_{u2}(x^2)$$

$$= x^2 - \frac{\int_0^1 x^2dx}{\int_0^1 1dx} - \frac{\int_0^1 x^3 - \frac{1}{2}x^2dx}{\int_0^1(x-\frac{1}{2})^2dx}(x - \frac{1}{2})$$

$$= x^2 - x + \frac{1}{6}$$

Giving us our orthogonal basis vectors,

$$u_1 = 1, u_2 = x - \frac{1}{2}, u_3 = x^2 - x + \frac{1}{6}$$

2. Here is a little fact: Let X be a random variable with finite mean μ and finite variance σ^2. Then $\forall t > 0$ we have

$$P(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}.$$

Now let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with mean μ and variance σ^2. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Use the little fact to prove

$$\forall\epsilon > 0, \forall\delta > 0, \exists N > 0, \forall n \geq N, P(|\bar{X}_n - \mu| \leq \epsilon) \geq 1 - \delta.$$

Proof

If $X_i ... X_n$ all have finite variance σ^2 and finite mean μ, then $\bar{X}_n$ has finite variance σ^2/n and finite mean μ. Therefore by our little fact,

$$P(|\bar{X}_n - \mu| \geq t) \leq \frac{\sigma^2}{nt^2}$$

Now let $\epsilon > 0, \delta > 0$ be given. Then set $N = \lceil \frac{\sigma^2}{\delta\epsilon^2} \rceil$

Then for any $n \geq N$

$$P(|\bar{X}_n - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} \leq \frac{\sigma^2}{N\epsilon^2} \leq \delta$$

Then since $P(|\bar{X}_n - \mu| \geq \epsilon) \leq \delta$, we get for it's complement

$$P(|\bar{X}_n - \mu| \leq \epsilon) \geq 1 - \delta$$