HW 12

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4/17/17

Q1

(1)

Since $\lambda_i = \sum_{i \neq j} q(i,j)$ then

$$\lambda_i = q(i,i+1) + q(i,i-1) = \lambda + \mathbf{1}_{\{0 \leq i \leq s\}} i\mu + \mathbf{1}_{\{i \geq s\}} s\mu$$

(2)

Then,

$$Q(i,j) = \begin{cases} \lambda & j = i+1 \\ i\mu & 0 \leq i \leq s, \qquad j = i-1 \\ s\mu & i \geq s, \qquad j = i-1 \\ -\lambda_i & i = j \end{cases}$$

Q2

Let λ be the rate of $N(t)$. First note that clearly $q(i,j) = 0$ for $i - j \geq 2$ or $j < i$ then for $j = i+1$

$$q(i,j) = \lim_{t \to 0} \frac{e^{-\lambda t}(\lambda t)^{j-i}}{(j-i)!t} = \lim_{t \to 0} \frac{e^{-\lambda t}\lambda t}{t} = \lambda$$

Then we have

$$Q(i,j) = \begin{cases} \lambda & j = i+1 \\ -\lambda & j = i \\ 0 & else \end{cases}$$

Then by the backward Kolmogorov equation

$$p_t' = Qp_t$$

This is a partial differential equation whose solution is

$$p_t = e^{tQ}$$

Q3

First note that for $i \neq j$

$$q(i,j) = \lim_{t\to 0} \frac{p_t(i,j)u(i,j)}{t}$$

which by what we have shown in two is equal to simply $\lambda u(i,j)$ Then Q is

$$Q(i,j) = \begin{cases} \lambda u(i,j) & i \neq j \\ -\sum_{k\neq j} \lambda u(j,k) & i = j \end{cases}$$

Then

$$\begin{aligned} p'_t(i,j) &= \lim_{h\to 0} \frac{p_{t+h}(i,j) - p_t(i,j)}{h} \\ &= \lim_{h\to 0} \frac{\sum_k \big(p_t(i,k)p_h(k,h)\big) - p_t(i,j)}{h} \\ &= \lim_{h\to 0} \frac{\sum_{k\neq j} \big(p_t(i,k)p_h(k,h)\big) + p_t(i,j)(p_h(j,j)-1)}{h} \\ &= \lim_{h\to 0} \frac{\sum_{k\neq j} \big(p_t(i,k)p_h(k,h)\big)}{h} + \lim_{h\to 0} \frac{p_t(i,j)(p_h(j,j)-1)}{h} \\ &= \sum_{k\neq j} p_t(i,k)q(k,h) - \lim_{h\to 0} \frac{p_t(i,j)(\sum_{k\neq j} p_h(j,k)}{h} \\ &= \sum_{k\neq j} p_t(i,k)q(k,h) + p_t(i,j)(-\sum_{k\neq j} q(j,k)) \\ &= \sum_{k\neq j} p_t(i,k)\lambda u(k,j) + p_t(i,j)\big(-\sum_{k\neq j} \lambda u(j,k)\big) \\ &= p_t(i,\cdot)Q(\cdot,j) \end{aligned}$$

Q4

Let L be the event that an inpatient person leaves, B the event that a bank teller finishes with someone and A the event that a new person enters the bank. First note that even with this new addition to the problem, still $q(i,j) = 0$ for $j - i \geq 2$.

Then

$$q(n,n+1) = \lim_{t\to 0} \frac{P(A)}{t} = \frac{e^{-\lambda t}\lambda t}{t} = \lambda$$

and

$$q(n,n-1) = \lim_{t\to 0} \frac{P(L,\bar{B}) + P(B,\bar{L})}{t}$$

as t goes to 0 the $P(\bar{B}) \to P(\bar{L}) \to 0$ so the above becomes

$$\lim_{t \to 0} \frac{P(L) + P(B)}{t}$$

$$= \lim_{t \to 0} \frac{\left(e^{-\max(n-s,0)\xi t} \max(n-s,0)\xi t\right) + \left(e^{-\min(n,s)\mu t} \min(n,s)\mu t\right)}{t}$$

$$= \max(n-s,0)\xi + \min(n,s)\mu$$

We might think that $q(n, n+2) > 0$, but this is not true since it would require L and B to happen at the exact same time,

$$lim_{t \to 0} \frac{P_t(L, B)}{t} = lim_{t \to 0} \frac{P_t(L)P_t(B)}{t}$$

$$= lim_{t \to 0} \frac{e^{-\xi\mu \min(n,s) \max(n-s,0)t^2} t^2 \xi\mu \min(n,s) \max(n-s,0)}{t} = 0$$

Therefore,

$$Q(i,j) = \begin{cases} \lambda & j = i+1 \\ \max(i-s,0)\xi + \min(i,s)\mu & j = i-1 \\ 0 & |i-j| \geq 2 \\ -\lambda - \max(i-s,0)\xi - \min(i,s)\mu & i = j \end{cases}$$