HW 6

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Q1

Let our state space be $[T, C]$. Then we can think of one vehicle following another vehicle as a transition with transition matrix,

$$p_{i,j} = \begin{bmatrix} 1/4 & 3/4 \\ 1/5 & 4/5 \end{bmatrix}$$

We can then solve for the stationary distribution of this transition matrix simply since it is a 2x2 matrix.

$$\pi_T = \frac{1/5}{1/5 + 3/4} = 0.2105, \pi_C = \frac{3/4}{1/5 + 3/4} = 0.7895$$

Then since $p_{i,j} \neq 0$ for all i, j, then clearly p is irreducible and aperiodic. Then since our state space is finite and we have a stationary distribution π, then theorem 1.19 says the probability of being a truck converges to $\pi_T = 0.2105$.

Q2

For this we simply solve for the set of functions

$$\pi_1 - (0.86\pi_1 + 0.05\pi_2 + 0.03\pi_3) = 0$$

$$\pi_2 - (0.08\pi_1 + 0.88\pi_2 + 0.05\pi_3) = 0$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

Then,

$$\pi = [0, 0, 1] \begin{bmatrix} 0.14 & -0.08 & 1 \\ -0.05 & 0.12 & 1 \\ -0.04 & -0.05 & 1 \end{bmatrix}^{-1} = [0.2155477, 0.33215548, 0.45229682]$$

Then since p is finite and $p_{i,j} \neq 1$ for all i, j then p is irreducible and is aperiodic, we may again apply theorem 1.19 to say that in the long run, $\pi_1 = 0.2155477$ fraction of the population will reside in cities, $\pi_2 = 0.33215548$ fraction of people will reside in suburbs and $\pi_3 = 0.45229682$ fraction of people will reside in rural areas.

Q3

a.

Using the same strategy as Q2 we first solve for a stationary distribution to get

$$\pi = [0.36363636, 0.35664336, 0.27972028]$$

Then since p is finite and $p_{i,j} \neq 0$ for all i, j then we again can apply 1.19 to say,

$$\lim_{n\to\infty} p^n(1,2) = 0.35664336$$

b.

Then since p is irreducible and closed, then it is recurrent, and we may apply theorem 1.21 which says,

$$\frac{N_n(2)}{n} \to \frac{1}{E_2 T_2}$$

and as shown in a.,

$$\frac{N_n(2)}{n} \to \pi_2$$

so ,

$$\pi_2 = \frac{1}{E_2 T_2}$$

and

$$E_2 T_2 - 1 = 1/\pi_2 - 1 = 1.8039$$

c.

$$E[\text{walking distance}] = \sum_n p(n) * (0.3 + 0.1n)$$

$$= \sum_n \pi_n * (0.3 + 0.1n) = 0.4916$$

Q4

i

Our state space is $[1, 2, 2']$ where the number indicates the number of light bulbs working and ' indicates we just replaced the light bulbs. Then our transition matrix is

$$\begin{bmatrix} 0.95 & 0 & 0.05 \\ 0.02 & 0.98 & 0 \\ 0.02 & 0.98 & 0 \end{bmatrix}$$

Solving for the stationary distribution we get,

$$\pi = [0.28571429, 0.7, 0.01428571]$$

Then since p is finite and nonzero for all i, j, then it is irreducible and theorem 1.19 says that the long run fraction of time one light bulb is working is $\pi_1 = 0.2857$.

ii

Then we can use theorem 1.21 to say that

$$E_2 T_2 - 1 = 1/\pi_2 - 1 = 0.4$$