Master of Science Thesis

Radiative Neutrino Mass Generation and Renormalization Group Running in the Ma-Model

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Abstract

Since 1998, we know from oscillations experiments that neutrinos are not massless, which contradicts the Standard Model and calls for new theoretical models. One way to explain the extremely tiny but non zero mass of the neutrino is to consider radiative models in which this mass is not present at tree-level due to some symmetry, but appears as a quantum correction to the neutrino propagator. In this thesis, we study such a model based on a special case of the two Higgs doublet model, where the additional Higgs doublet does not acquire a vacuum expectation value, and in which an exact and discrete symmetry is imposed. We first review this model and the mechanism generating the mass of the neutrino and derive the expression for the neutrino mass. We show that, in this model, the mass is due to a coupling of the light neutrinos to heavy right-handed neutrinos whose existence is postulated by the model. We also present the whole set of renormalization group equations relevant in the framework of the model. In a second time, we consider the effective theory arising from integrating out the three generations of heavy neutrinos present in the model. We show the expressions for the consistent effective operators and the matching conditions one can use to obtain the running of the parameters throughout the three effective theories and the full theory. Finally, in a top-down approach and in a simplified case, we solve the renormalization group equations and run the parameters down from the GUT scale to the electroweak scale. We then show how, by an adequate tuning of the input parameters of the model, we can obtain low-energy mixing parameters and mass square differences in agreement with the last results from neutrino experiments. The particularly interesting feature of the model is that it gives the mass to the neutrinos as well as introduces new dark matter candidates. Furthermore, it also predicts the existence of three right-handed neutrinos, with mass of the order of TeV, and it is realistic to think that their existence could be validated or invalidated in the next couple of decades at some supercollider.

Key words: Radiative Model, Neutrino Masses, Two-Higgs Doublet Model, Effective Operator, Threshold Effects, Renormalization Group Running.

Preface

This thesis is the result of 7 months of work between September 2011 and March 2012, for the degree of Master of Science in Engineering, in the Theoretical Particle Physics Department of KTH, Sweden. During this period of time, the field of particle physics was quite hectic due to different important results. First, in September 2011, some controversial results were published claiming that the neutrinos could travel faster than light. Secondly, in December 2011, experiments reported a hint for the existence of the Higgs particle in a certain range of mass. Eventually, in March 2012, experimentalists announced that the long-sought third neutrino mixing angle was measured to be non-zero with a good accuracy. This context was rather motivating because showing that, inspite of the extreme complexity of the experiments, these could give some answers or hints regarding the models we work with in theoretical particle physics.

Overview of the thesis

This thesis deals with an extension of the Standard Model that could explain the tiny but non-zero mass of the neutrino and the values of its mixing angles at low energies. It is divided into five chapters and two appendices. Chapter 1 is a short overview of the theoretical basis of the Standard Model, we speak about the quantum field theories, the gauge symmetries, the different particles and the mechanism giving them mass. At the end, we also present the problem regarding the introduction of the neutrino mass and the possible solutions called the seesaw mechanisms. In Chapter 2, we begin with studying radiative neutrino mass models and then go on detailing the regularization and renormalization procedures in the context of the Ma-model. In Chapter 3, we present the principle of the effective theory and we study the effective operators relevant for the Ma-model. Finally, in Chapter 4, we present the results we obtain for the running of the parameters, after using the results from the previous chapters and, in Chapter 5, we summarize and conclude the thesis. Besides, in Appendix A, we present the whole set of RGEs which have been calculated for this thesis, and, in Appendix B, we give some useful results used in the Chapter 2 and the Chapter 3.

Acknowledgments

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Finally, I want to thank my family, my merry fellows the 'Kelliogs', and, more generally, all the persons, dead or alive, who give me hope.

Contents

There are no facts, only interpretations.

-Friedrich Nietzsche

Chapter 1

The Standard Model and Particle Masses

Elementary particle physics was born with the vision to unite the fundamental forces and classify the elementary particles constituting the Universe we live in. The first grand theoretical step in this direction ocurred in 1960 when Sheldon Lee Glashow pointed out [1] some remarkable parallels between electromagnetism and the so-called weak interactions responsible for subatomic particles decay among others. Since then physicists have kept on ordering, predicting, and discovering new particles with a view to completing the big picture and eventually reaching a 'Theory of Everything' [2]. Nowadays, the Standard Model is merely the most accurate description of high-energy physics we have. Almost all its predictions have been experimentally confirmed to a good accuracy, including the existence of the top-quark in 1995 [3] and that of the tauon neutrino in 2000 [4]. In the following we will give a very brief overview of the theoretical machinery on which the Standard Model is built, and afterwards the problem of the introduction of a neutrino mass will be dealt with.

1.1 The Standard Model

1.1.1 Quantum Field Theory

The Standard Model (SM) has gradually become a synonym for a quantum field theory based on the group of symmetry $SU(3)_C \times SU(2)_L \times U(1)_Y$. This sentence carries a lot of concepts, first, Quantum Field Theory (QFT) is a quantum description of the mechanics of fields, which means that we apply the well-known classical equations of Lagrangian Dynamics (namely *Euler-Lagrange* equations) but with the point of view of Quantum Mechanics. The introduction of this class of theories was motivated by the need to create a relativistic version of Quantum

Physics in which the probabilistic description holds. In QFT all the information about the dynamics of fields is contained in a quantity called the action *functional*:

$$S = \int \mathrm{d}^4x \, \mathcal{L} \,,$$

where $\mathcal{L} \equiv \mathcal{L}[\Phi(x)]$ is the Lagrangian density which is a function of all the fields of the theory, and these fields depend on the space-time coordinate x. Once we have this quantity, all the equations of motion can in principle be derived from the *Euler-Lagrange* equations arising from the requirement that S is stationnary:

$$\delta S = 0 \to \partial_\mu \left[\frac{\delta \mathcal{L}}{\partial_\mu \Phi} \right] - \frac{\delta \mathcal{L}}{\partial \Phi} = 0 \,.$$

Although high-energy physics is a topical scientific subject mostly developped since the second half of the 20th century, we can see that the fundamental principle underlying the theory is just a modern rewriting of what Fermat called the *Principle of Least Time* [5] in 1662!

1.1.2 Gauge Symmetries

In light of these facts, the question emerges that in order to characterize a QFT we just need a consistent Lagrangian, but how to constrain its form? First of all, QFTs follow the laws of special relativity and subsequently the Lagrangian of a theory will have to obey certain space-time symmetries *i.e.* it has to be Lorentz invariant. In addition to these constraints it has been observed that particle physics laws obey so-called 'gauge symmetries'. It may seem odd to speak about symmetries as we usually experience symmetry as a geometrical property. However if we think a bit, a symmetry is just an expression of an *invariance* under a given *transformation*. This transformation can by all means be in space (or rather space-time) but it can also occur in another kind of space, the *internal space*. This is what is meant by gauge symmetries, and maybe the most intuitive one is the electromagnetic symmetry: it is merely the expression of the *conservation of the electric charge* and is generated by the group $U(1)$. This group, called the *circle group*, is the ensemble of the 1×1 unitary matrices and is associated with the rotation about the origin in the complex plane. In the unbroken SM, the relevant gauge groups are actually $U(1)_Y$, $SU(2)_L$, and $SU(3)_C$.[1] The first one is linked to the *conservation of hypercharge*, the second one corresponds to *isospin invariance*, and the last one is affiliated with the *colour symmetry*.

[1] The 'unbroken' SM is to distinguish from the SM after symmetry breaking, cf. Sec. 1.2.2. Besides, the 'L' in $SU(2)_L$ actually indicates that this symmetry only affects *left-handed fields* (see Sec. 1.1.4).

Eventually, all the particles included in the Standard Model are characterized by different charges under the groups.[2] These charges define the behaviour of the particles when a symmetry transformation is imposed. When a particle is not affected by a given symmetry, it has no corresponding charge and is referred to as a *singlet* under the corresponding group of transformation. The 'Lagrangian-building' game then merely consists in writing terms that respect the whole set of symmetries we have talked about — and of course whose derivations lead to the correct equations of motion.
To finish with, I would like to stress that the term *'gauge'* is a historical remnant from the time Maxwell came up with his theory uniting electric and magnetic fields in 1865 [6]. From his equations where he expressed these two fields in terms of a scalar and a vector potential, he remarked that while the potentials uniquely determine the fields, the fields do not uniquely determine the potentials. Thus there can be changes of potentials that do not affect the fields. Finally he called such a change a *gauge transformation*, and he characterised the electric and magnetic fields as *gauge invariant*.

1.1.3 The Content

After what has been said in the previous sections, we can now speak about the particle content of the Standard Model. This model contains three kinds of fields sorted according to their spin. The spin-1 fields are *vectors*, the spin-1/2 fields are *spinors*, and the spin-0 fields are *scalars*. We can characterize these fields by the different representations they have in each gauge group. The *physical* fields are:

1. Spin-1 fields, the gauge fields:
   - One $U(1)_Y$ gauge field B_μ with coupling constant g_1. Its gauge representation under $G_{\mathrm{SM}} = SU(3)_C \times SU(2)_L \times U(1)_Y$ is $(\mathbf{1}, \mathbf{1}, 0)$.
   - Three $SU(2)_L$ gauge fields W^i_μ $(i = 1, 2, 3)$ with coupling constant g_2. Their gauge representation is $(\mathbf{1}, \mathbf{3}, 0)$.
   - Eight $SU(3)_C$ gauge fields G^a_μ $(a = 1, \ldots, 8)$ with coupling constant g_3. Their gauge representation is $(\mathbf{8}, \mathbf{1}, 0)$.
2. Spin-1/2 fields, the *fermions* (each of them exists in three generations):
   - An $SU(3)_C$ triplet, $SU(2)_L$ doublet, with $U(1)_Y$ hypercharge $-\frac{1}{3}$, the left-handed quark : $Q_{\mathrm{L}} \sim (\mathbf{3}, \mathbf{2}, -\frac{1}{3})$.
   - An $SU(3)_C$ antitriplet, $SU(2)_L$ singlet, with $U(1)_Y$ hypercharge $+\frac{2}{3}$, the left-handed down-type anti-quark : $\overline{d_{\mathrm{L}}} \equiv (d_{\mathrm{R}}) \sim (\overline{\mathbf{3}}, \mathbf{1}, +\frac{2}{3})$.
   - An $SU(3)_C$ antitriplet, $SU(2)_L$ singlet, with $U(1)_Y$ hypercharge $-\frac{4}{3}$, the left-handed up-type anti-quark : $\overline{u_{\mathrm{L}}} \sim (\overline{\mathbf{3}}, \mathbf{1}, -\frac{4}{3})$.

[2] Actually, all the spin-$\frac{1}{2}$ particles in the SM exist in three different flavours, which are just different copies of the same gauge representation.

- An $SU(3)_C$ singlet, $SU(2)_L$ doublet, with $U(1)_Y$ hypercharge -1, the left-handed lepton : $\ell_{\mathrm{L}} \sim (\mathbf{1}, \mathbf{2}, -1)$.
- An $SU(3)_C$ singlet, $SU(2)_L$ singlet, with $U(1)_Y$ hypercharge 2, the left-handed antilepton : $\overline{e_{\mathrm{L}}} \equiv e_{\mathrm{R}} \sim (\mathbf{1}, \mathbf{2}, +2)$.

3. Spin-0 field:

- An $SU(3)_C$ singlet, $SU(2)_L$ doublet, with hypercharge $+1$, it is the 'famous' Higgs field. Its gauge representation is $(\mathbf{1}, \mathbf{2}, +1)$.

Regarding the fermions, note that even though we have enumerated only the left-handed ones, they all have their CP-conjugate partners (see Sec. 1.1.4) that are right-handed and have opposite quantum numbers. Besides, one can remark that the left-handed antileptons are $SU(3)_C \times SU(2)_L$ singlets and that only the charged ones are present in the SM. Indeed the left-handed anti-neutrino (and the right-handed neutrinos) would be complete singlets under all the gauge groups. In other words it implies that they would not interact with any gauge field and that *a priori* we do not 'need' them to have a consistent model of the particles and their interactions. These fields have indeed not been observed but it is often convenient to assume their existence if we want to explain the masses of the neutrinos (see Sec. 1.2.3).

1.1.4 'L' or 'R', Handedness Matters

One may wonder why we have used the subscripts 'L' and 'R' in the notations for the different fields. This has to do with two notions.
First, all the fermions can be distinguished by their *chirality*, they are said to be *left-handed* or *right-handed*. These states, defined by the action of the projection operators,

$$P_L := \frac{1-\gamma_5}{2} \quad \text{and} \quad P_R := \frac{1+\gamma_5}{2}\,,$$

are seen differently by the *weak interaction* which only affects the left-handed fermions. The tricky thing is that the massive fermions are not eigenstates of such projections, and the *chirality* is hereby not strictly defined for them. Instead they can be decomposed into a left-handed and a right-handed part:

$$\psi = \psi_L + \psi_R\,,$$

where $\psi_L = P_L\,\psi$ and $\psi_R = P_R\,\psi$.
Secondly, there is a close relation between these two states and the notion of antiparticle as the charge conjugate of a left-handed fermion is the corresponding right-handed anti-fermion. However what is an antiparticle? In some sense it is the image of a particle through a mirror. Not the 'everyday life' mirror that only reverses the space direction perpendicular to its plane but a somewhat more

abstract mirror that reverses all the additive quantum numbers characterising the particle. Technically, we define the particle-antiparticle conjugation operator as [7]:

$$\hat{\mathbf{C}}: \quad \psi \to \psi^c = \mathbf{C}\overline{\psi}^T ,$$
$$\text{and} \quad \phi \to \tilde{\phi} = i\tau_2\phi^* . \tag{1.1}$$

where τ_2 is the second *Pauli matrix* and the matrix $\mathbf{C}$ has the properties:

$$\mathbf{C}^\dagger = \mathbf{C}^{-1} = \mathbf{C}^T = -\mathbf{C} ,$$
$$\mathbf{C}\gamma_5 = \gamma_5\mathbf{C} . \tag{1.2}$$

Then we have:

$$(\phi_L)^c = (\phi^c)_R \qquad \text{and} \qquad (\phi_R)^c = (\phi^c)_L . \tag{1.3}$$

Finally the results from Eq. (1.3) show exactly what has been mentioned earlier, say that when we apply the particle-antiparticle conjugation operator to a left-handed fermion field, then we obtain the right-handed anti-field and *vice-versa.*

1.2 The SM Lagrangian and a Massive Problem

1.2.1 Lagrangian Terms

As said earlier the form of the terms in the Lagrangian is constrained by certain symmetries. These terms can be further classified according to the number and types of fields involved. We distinguish the following types of terms:

- The *kinetic* terms, which are quadratic in a single field and involve derivatives (which in quantum mechanics are similar to the momentum), typically: $\overline{\psi}\partial\!\!\!/\psi$.
- The *interaction* terms, which involve at least three fields and can be represented by a vertex in a Feynman diagram. For example, the interaction between the Higgs and two fermions, $\overline{\psi_R}\phi\psi_L$, is called the Yukawa interaction.
- The *mass* terms, which are quadratic in the single fields and do not involve derivatives.[3]

As a matter of fact, in this thesis we are interested mainly in the mass of the neutrino, which is why we will deal specifically with these mass terms. Regarding the dimension of the terms, this can be derived after observing that in *Natural Units*, *i.e.* with $\hbar = c = 1$, the action is dimensionless implying that before integration over space-time the Lagrangian density must have mass dimension 4. More generally in a d-dimensional space-time the Lagrangian density must have (mass) dimension d

[3]Actually there are also 'mixed' mass terms involving two different fields, e.g. B and W_3.

—because $[\mathrm{d}^d x] = -d$. Thus, knowing that $[\partial_\mu] = 1$, the different field dimensions can be derived from the kinetic terms and read:

$$[A_\mu] = 1,\ [\phi] = 1,\ [\psi] = \frac{3}{2}.$$

A Lagrangian containing only terms whose dimensionality is inferior or equal to 4 is called *renormalizable*. Nevertheless some higher dimensionality terms can be added to the Lagrangian in the context of *Effective Theories* (see Sec. 3.1).

1.2.2 A Term for the Mass

All the information is contained in the Lagrangian, but a problem arises when trying to define a term corresponding to the mass of the fermions. This is quite odd that the characteristic which is the most intuitive for us, namely the mass, is somehow the least immediate to introduce in the Lagrangian, whereas it is easy to write down a term for the more abstract interaction between four gluons. The theoretical difficulty regarding the mass was solved in the 60's by the works of Anderson [8], Brout and Englert [9], and Higgs [10]. They hypothesised that some of the gauge symmetries of the Lagrangian were broken by the non-zero vacuum expectation value (VEV) of a scalar field –the Higgs– hereby introducing a mass term in the Lagrangian. This mechanism called the *Higgs mechanism* is based on a certain type of spontaneous symmetry breaking, the electroweak spontaneous symmetry breaking (EWSB):

$$\begin{aligned} SU(3)_C \times SU(2)_L \times U(1)_Y &\longrightarrow SU(3)_C \times U(1)_{em}\,, \\ \phi \quad &\longrightarrow \langle\phi\rangle = v \quad (= 174\mathrm{GeV}). \end{aligned} \tag{1.4}$$

Detailed studies can be found in the literature (see, e.g., [11, 12]), but the main relevant consequence here is that a *Yukawa coupling* such as:

$$Y\overline{\psi}\tilde{\phi}\psi\,,$$

will yield after EWSB:

$$Y v\,\overline{\psi}\psi \;\equiv\; m\,\overline{\psi}\psi\,,$$

which is a valid candidate for a mass term as it is quadratic in the single field and without derivatives.

In light of these facts, the *fermion mass term* looks like:

$$-\mathcal{L}_{\mathrm{Mass}} = m\overline{\psi}\psi\,,$$

which by using the aforementionned notations for the chiral fields, yields:

$$-\mathcal{L}_{\mathrm{Mass}} = m(\overline{\psi_L + \psi_R})(\psi_L + \psi_R)\,.$$

However, as the two chiral states are orthogonal (coming from two orthogonal projectors) and since the Dirac adjoint changes the chirality, only the *mixed terms* remain:

$$-\mathcal{L}_{\text{Mass}} = m(\overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L)\,.$$

Thus, the mass terms couple the *left-handed* and the *right-handed* components of the fermion fields and therefore a massive field must have these two components —and they must be non-zero. This mass term is called the *Dirac mass term*, and it can account for the mass of each fermion provided that its two chiral forms exist. This is the case for all the fermions except the neutrinos, because no right-handed neutrino has ever been observed. The latter must therefore be *massless*.
Besides the *Dirac mass*, a special case can show up. What if a certain fermion is its own antiparticle? This would imply that,

$$\psi^c = \psi\,,$$

where we neglect a possible phase term. Such a field, called *Majorana field*, can also be expressed with the means of chiral fields as:

$$\psi = \psi_L + \psi_R = \psi_L + (\psi^c)_R = \psi_L + (\psi_L)^c\,.$$

We see that this time the field can be expresseed with only one chiral component and we escape the previous condition for a non-zero mass in that a Majorana fermion can now be massive without the simultaneous existence of its two chiral forms. The resulting mass term, called a *Majorana mass term*, reads:

$$-\mathcal{L}_{\text{Majorana}} = \frac{1}{2}m\overline{\psi^c}\psi + \text{h.c.}\,.$$

In this context there are *a priori* two ways of building a fermion mass term in the standard model, but nevertheless some more aspects must be pointed out. The Majorana property is really restrictive because being its own anti-particle means that all the charges that are reversed by the particle-antiparticle operator have to be null. Consequently, a Majorana field cannot carry any electric charge, colour, or even lepton or baryon number.[4] What particle could withstand such a bunch of constraints? The answer is none except a complete gauge singlet, for example a hypothetical right-handed neutrino.

1.2.3 The Neutrino Mass: A Massive Problem

Although the SM has been marvelously successful for the last couple of decades, in 1998 the breakthrough regarding the neutrino flavour oscillations threw a spanner in the works (see [13, 14]). Indeed, the direct implication was that neutrinos could *not* be massless. Then if neutrinos have mass we know they may be either *Dirac* or *Majorana* particles. But the corresponding mass terms are not possible in the SM because [15]:

[4] It can carry isospin because, as stated earlier, $SU(2)$ is broken in the SM.

- Right-handed neutrinos are absent so that a *Dirac mass* is not possible.
- Lepton number is exactly conserved so that a *Majorana mass* is not possible.[5]

Hence, any attempt to generate non-zero neutrino masses has to violate one of the above two assumtions and is by that way demolishing the dream of the Standard Model as a final theory. So, how can we extend the Standard Model so as to accommodate nonzero neutrino masses ? We have mainly three theoretical levers:

- To extend the scalar content (the mass could come from another VEV),
- To extend the fermion content (coupling with heavy particles could generate mass),
- To enlarge the gauge group (*i.e.* add new symmetries).[6]

1.2.3.1 With Right-Handed Neutrinos

The first idea that one can come up with is just to assume that although we have not discovered it yet there could well be a right-handed neutrino. So one can try to cure the SM by introducing this field, actually several generations —say three to conserve some parallel with the other fermions: N_R^i, $i = 1, 2, 3$. The resulting Yukawa coupling is:

$$-\mathcal{L}_{\text{Yuk},\nu} = \overline{\ell_L}\tilde{\phi}Y_\nu N_R + \text{h.c.}\,,$$

which gives after EWSB the following Dirac mass term for the neutrino:

$$-\mathcal{L}_{\text{Dirac},\nu} = \overline{\nu_L}M_D N_R + \text{h.c.}\,,$$

In addition, as we said in Sec. 1.2.2, the right-handed neutrino *can* have a Majorana mass term that reads:

$$-\mathcal{L}_{\text{Majorana},N_R} = \frac{1}{2}\overline{N_R^c}M_R N_R + \text{h.c.}\,,$$

with M_R symmetric. Eventually all these mass terms can be gathered in a general term:

$$-\mathcal{L}_{\text{Mass},\nu} = \overline{\nu_L}M_D N_R + \frac{1}{2}\overline{N_R^c}M_R N_R + \text{h.c.} = \frac{1}{2}\overline{\Psi}\mathcal{M}_\nu\Psi^c\,,$$

where[7]

$$\Psi = \begin{pmatrix}\nu_L \\ N_R^c\end{pmatrix} \quad \text{and} \quad \mathcal{M}_\nu = \begin{pmatrix} 0 & M_D \\ M_D^T & M_R\end{pmatrix} + \text{h.c.} \tag{1.5}$$

Finally, the introduction of right-handed neutrinos leads to a mass term for the left-handed ones. However this is not satisfactory. First, in the case $M_R \ll M_D$,

[5] Actually L is only conserved at the perturbative level due to electroweak sphalerons [16, 17], however the Majorana mass term also breaks $B - L$ (the "Baryon minus Lepton number") which *is* conserved at both the perturbative and non-perturbative level.

[6] This usually also requires to extend the fermion content.

[7] This is actually already the 'seesaw case', cf. Sec 1.3, because we have not introduced a Majorana mass term for the left-handed neutrino.

the Yukawa couplings have to be extremely small: $Y \sim \frac{m_\nu}{v} \sim 10^{-12}$ compared to what they are for quarks $(1-10^{-5})$ or for the charged leptons $(10^{-2}-10^{-5})$. Secondly, if no additional symmetry is imposed the right-handed neutrino singlets will never be observable, which is not an acceptable feature for a theory if we want it to be *falsifiable*.

1.2.3.2 More Generally: With an Effective Operator

As William of Ockham stated in 1495: "Plurality must never be posited without necessity"[18]. In this view, since right-handed neutrinos are not essential in the Standard Model, it is more natural to supply *Majorana* masses to the left-handed neutrinos, thus breaking lepton number by two units.[8] We then just need a term using some combination of fields that breaks the lepton number by two units. Actually these terms or *operators* are not so numerous in the SM, and the one with the lowest dimensionality is the following *unique* dimension-5 operator:

$$\mathcal{L}_\kappa^{d=5} = \frac{1}{4}\kappa_{gf}\overline{\ell_{Lc}^{c}}^{g}\varepsilon_{cd}\phi_d\ell_{Lb}^{f}\varepsilon_{ba}\phi_a + \text{h.c.} \tag{1.6}$$

The problem is that this rather harsh solution also spoils the renormalizability of the theory (see Sec. 1.2.1). However this approach is particularly interesting when considering effective theories, as the term can merely be the remnant of some high-energy interaction with heavy fields. It is exactly one of such possibility that will be studied in the next chapter.[9]

1.3 The Seesaw Models I & II

1.3.1 The Seesaw Model Type I

It is important to talk about this model for it is one of the simplest ways to introduce a neutrino mass. It was introduced in 1980 (see e.g. [19]) to provide an explanation for the smallness of neutrino masses as a direct consequence of the heavyness of right-handed neutrinos. This mass generation through unbalance of right- and left-handed states can be seen as a 'see-saw' phenomenon as we can see on Fig. 1.1. Using Feynman Diagrams it can be sketched as in Fig. 1.2.

This is just a special case of what has been explained in Sec. 1.2.3.1 when we take the Majorana mass M_R to be way larger than the Dirac mass M_D. Although this assumtion may seem artificial it is absolutely possible because the Majorana mass is not linked to EWSB and its value can therefore be far beyond the EWSB

[8]Such a term breaks the lepton number by two units because: L($\overline{\psi^c}$)=L(ψ).

[9]Note that this idea is more general and that the corresponding high-energy theory can potentially be equivalent to the one of the previous case.

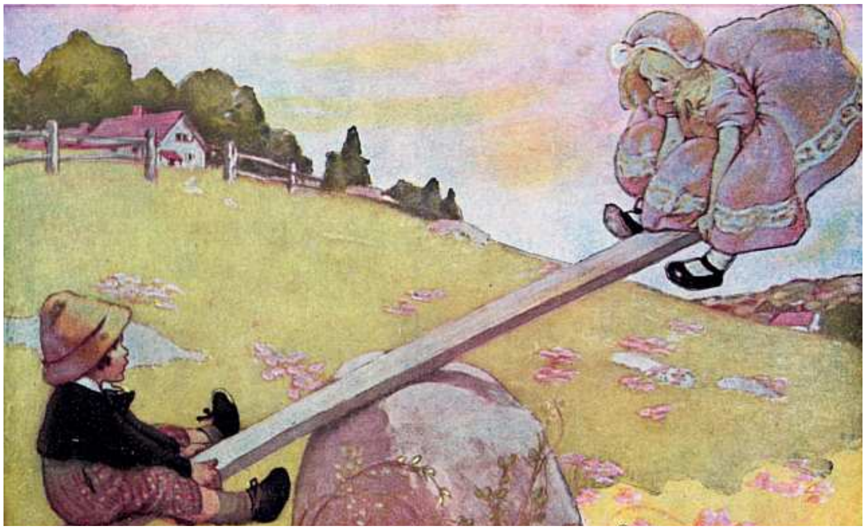


Figure 1.1: *This child game depicts the idea of the Seesaw Model: the little boy represents the elusive but heavy singlets while the little girl plays the role of the light but interacting neutrinos.* © *Jessie Willcox Smith*

scale. Practically, if we take $M_R \gg M_D$, the mass matrix from Eq. (1.5) can be diagonalized as:

$$\mathcal{M}_\nu \longrightarrow U^T \mathcal{M}_\nu U = \begin{pmatrix} -M_D M_R^{-1} M_D^T & 0 \\ 0 & M_R \end{pmatrix}, \tag{1.7}$$

where:

$$U = \begin{pmatrix} \mathbb{1} & \rho \\ -\rho^\dagger & \mathbb{1} \end{pmatrix} \quad \text{and} \quad \rho \equiv M_D M_R^{-1} \ll 1\,. \tag{1.8}$$

Thus, by a simple redefinition of the fields, we obtain two Majorana mass terms for combinations of left- and right-handed neutrinos:

$$\begin{pmatrix} \nu_L' \\ N_R^{c\prime} \end{pmatrix} = U\Psi = \begin{pmatrix} \mathbb{1} & \rho \\ -\rho^\dagger & \mathbb{1} \end{pmatrix} \begin{pmatrix} \nu_L \\ N_R^c \end{pmatrix} = \begin{pmatrix} \nu_L + \rho N_R^c \\ N_R^c - \rho^\dagger \nu_L \end{pmatrix}. \tag{1.9}$$

This means that the left-handed neutrinos 'we know' are actually mixtures containing small 'amounts' of the heavy singlet mass eigenstates, which gives them mass. Conversely the gauge singlets contain small amounts of left-handed neutrino mass eigenstates (but this does not affect their own mass). Finally, the value of this amount is ruled by the ratio between the Dirac and Majorana mass matrices, which is extremely tiny if we assume the neutrino singlets to be far above the EWSB scale (*i.e.* $\sim 10^3$ GeV), hereby explaining the smallness of neutrino masses.

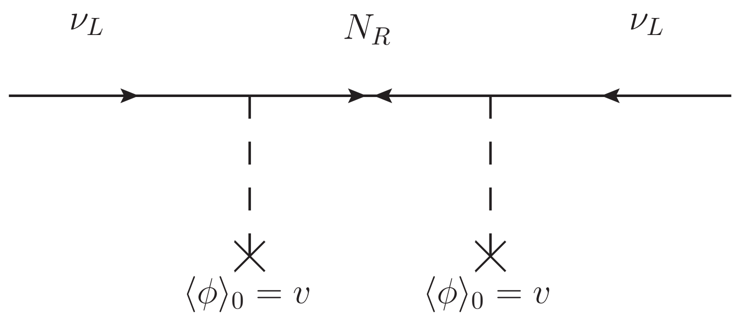


Figure 1.2: *Simplified diagrammatic representation of the Seesaw Type I mechanism (after EWSB).*

1.3.2 The Seesaw Model Type II

What if the neutrino mass originated from another VEV, not the one from the Higgs but one coming from another scalar? The seesaw model type II, introduced in 1981 [20], is a tentative to build an answer to that question. Indeed, instead of extending the fermion content, we can extend the SM with a new heavy scalar which will couple to the neutrino, providing it with a mass (see Fig. 1.3).

This scalar is a triplet: $\Xi \sim (\xi^1, \xi^2, \xi^3)^T$.[10] It can also be expressed in terms of the charge eigenstates, then composed of a doubly charged, a singly charged, and a neutral component:

$$\mathbf{\Xi} = \begin{pmatrix} \xi^{++} \\ \xi^{+} \\ \xi^{0} \end{pmatrix} = \begin{pmatrix} \frac{\xi^1 - i\xi^2}{\sqrt{2}} \\ \xi^3 \\ \frac{\xi^1 + i\xi^2}{\sqrt{2}} \end{pmatrix}.$$

This scalar triplet can couple to fermions, and the yukawa interactions with leptons can be written as:

$$\mathcal{L}_{\Xi,\mathrm{Yuk}} = -\frac{1}{\sqrt{2}}(Y_\Xi)_{fg}\, \overline{\ell^c_L}^{f}\, (\sigma \cdot \mathbf{\Xi})\, \ell^g_L + \text{h.c.}\,, \tag{1.10}$$

where $\sigma = (\sigma^1, \sigma^2, \sigma^3)^T$ with σ^i the Pauli matrices. Regarding the scalar potential, the most general potential is [21]:

$$\begin{aligned} V = {} & m^2\Phi^\dagger\Phi + M^2\Xi^\dagger\Xi + \frac{1}{2}\lambda_1(\Phi^\dagger\Phi)^2 + \frac{1}{2}\lambda_2(\Xi^\dagger\Xi)^2 \\ & \lambda_3(\Phi^\dagger\Phi)(\Xi^\dagger\Xi) + \mu\left(\overline{\xi^0}\phi^0\phi^0 + \sqrt{2}\xi^-\phi^+\phi^0 + \xi^{--}\phi^+\phi^+\right) + \text{h.c.} \end{aligned} \tag{1.11}$$

Besides, similarly to the Higgs mechanism, in order to get a mass term the scalar triplet is given a VEV breaking the symmetries. The requirement for charge

[10] Charge conservation rules out the possibility for a singlet.

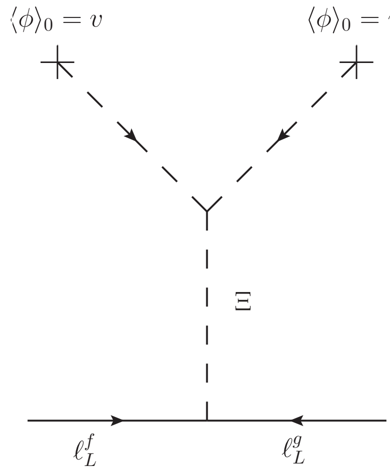


Figure 1.3: *Diagrammatic representation of the Seesaw Type II mechanism.*

conservation fixes uniquely the breaking direction and therefore it is the neutral component which will acquire a VEV.
Moreover, note that the triplet Higgs VEV contributes to the weak boson masses and alters the ρ- parameter from the SM prediction[11], $\rho \simeq 1$, at tree level [22]. As a matter of fact several precision measurements have constrained this deviation to be in the range: $\Delta\rho = \rho - 1 \simeq \dfrac{u}{v} \lesssim 0.01$ (see [23] and for a recent result see [24]). So the scalar triplet VEV has to be extremely tiny compared to the Higg VEV (~ 174 GeV):

$$\langle \xi^0 \rangle = u \ll v = \langle \phi^0 \rangle \,. \tag{1.12}$$

After the symmetry breaking we get a mass term for the neutrinos:

$$\mathcal{L}_{\Xi,\mathrm{Yuk}} = Y_\Xi \, \nu_L^c \langle \xi^0 \rangle \nu_L + \mathrm{h.c.}\,, \tag{1.13}$$

Thus, using the expression for the VEV that can be calculated from the scalar potential, it yields [25]:

$$\mathcal{M}_\nu = -u Y_\Xi \sim +v^2 \mu \frac{Y_\Xi}{M^2}\,. \tag{1.14}$$

In this expression one can notice that the light neutrino mass matrix is inversely proportional to the mass of the scalar triplet, consequently this model can also be seen as a 'Seesaw'.

[11] The so-called $\rho-$parameter is a quantity defined as $\rho = \dfrac{M_W^2}{M_Z^2 \cos^2(\theta_W)}$ and whose value is measured to be close to one in the SM.

Chapter 2

Radiative Neutrino Mass Models

In an attempt to explain the smallness of the neutrino mass compared to the masses of the other fermions in the SM, the radiative models are particularly attractive. Indeed the masses could originate from different mechanisms, and the mass hierarchy could be intrinsic to the nature of the mechanisms. In the following, we will talk about the principle of radiative mass generation, deal with some examples and introduce the particular model we will study later on.

2.1 Overview of Some Famous Radiative Models

In the context of the Glashow-Weinberg-Salam theory of electroweak interactions it is considered that all the fermions (except the neutrinos) acquire mass through Yukawa couplings with the Higgs boson after spontaneous symmetry breaking. However, in a radiative model this term is forbidden by some symmetry and there is no mass at tree-level, *i.e.* the zero order mass vanishes, but the mass is generated at loop-level, *at higher order* in the perturbative expansion.
As the loops are quantum corrections we can say that such masses are *'loop suppressed'* and this feature can account for the impressive hierarchy in mass between neutrinos and other fermions. For example a factor of generally $\frac{1}{16\pi^2}$ arises in front of the one-loop mass term. This is sometimes referred to as a *loop factor* and comes directly from the calculation of the loop integrals as we will see later on.

2.1.1 Zee-Wolfenstein Model

One of the most famous radiative model is the so called Zee-Wolfenstein model [26]. It extends the scalar sector of the SM with two additional scalars: one charged singlet, χ^+ and one doublet $\phi_2 = \binom{\phi_2^+}{\phi_2^0}$ (the SM Higgs doublet remains). Moreover ϕ_2 is assumed not to couple to leptons. In this model it is not possible to have tree-level neutrino masses from a renormalizeable Lagrangian, but it is possible at one loop level. Such one-loop process is shown in Fig. 2.1. Furthermore, the interesting feature of this model is that it suggests a special form for the neutrino mass matrix:

$$\mathcal{M}_\nu = \begin{pmatrix} 0 & f_{\mu e}(m_\mu^2 - m_e^2) & f_{\tau e}(m_\tau^2 - m_e^2) \\ f_{\mu e}(m_\mu^2 - m_e^2) & 0 & f_{\tau \mu}(m_\tau^2 - m_\mu^2) \\ f_{\tau e}(m_\tau^2 - m_e^2) & f_{\tau \mu}(m_\tau^2 - m_\mu^2) & 0 \end{pmatrix}.$$

Unfortunately, although the Zee-Wolfenstein model has been studied extensively, it is now ruled out by experiments [27].

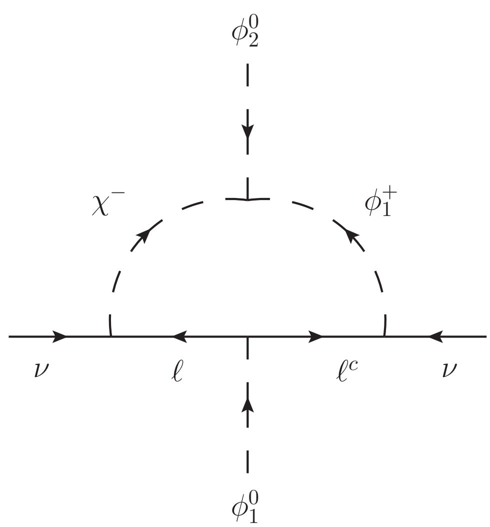


Figure 2.1: *Zee-Wolfenstein one-loop neutrino mass generation.*

2.1.2 Zee-Babu Model

In this model introduced between 1985 and 1987 [15, 28], the Standard Model is extended to include two charged singlet Higgs fields: a singly charged χ^+ and a doubly charged ζ^{++}, while the right-handed neutrinos are not introduced. The outcome is that the resulting neutrino masses are finite and naturally small, since they arise via two-loop diagrams (see Fig. 2.2). The main prediction of the model is the masslessness of one of the neutrinos but it also predicts that some lepton number changing decays such as $\mu \to eee$ and $\tau \to \mu\mu\mu$ occur at tree level via ζ^{++} exchange, which provides a tool for its dicovery (as well as it acts as a contraint).

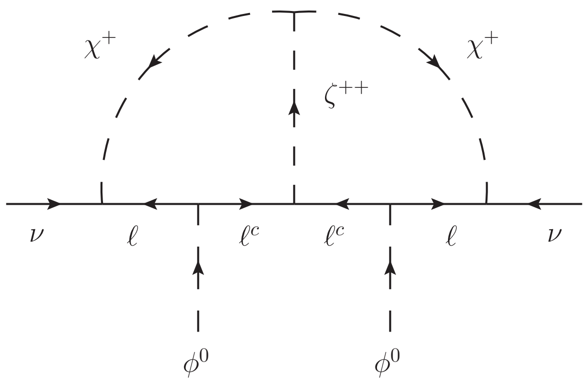


Figure 2.2: *Two-loop neutrino mass generation in the Zee-Babu model.*

2.1.3 Two-W Mechanism

As discussed in Sec. 1.2, the minimal model providing mass for the neutrinos is to add just one neutrino singlet N. In that simple case, only one linear combination of the flavour eigenstates ($\nu_{e,\mu,\tau}$) gets a tree-level Majorana mass term while the two others seem to remain massless. However, these zeros are not protected by any symmetry and nothing forces that it remain as such after radiative corrections. And actually as we can see in Fig. 2.3 there is a two-loop correction generating mass for the other neutrinos. This model introduced in 1988 [29] was formally interesting because although not being totally a radiative model (as one neutrino does have a tree-level mass term) it showed the possibility for the W-boson (discovered only five years earlier) to intervene in the generation of neutrino mass only at higher order. However, this two-loop and doubly GIM suppressed[12] mechanism yields completely unsignificant masses for the relevant neutrinos.

[12]The GIM mechanism or 'Glashow-Ilioupoulos-Maiani' mechanism suppresses the interaction. For further details, see [30].

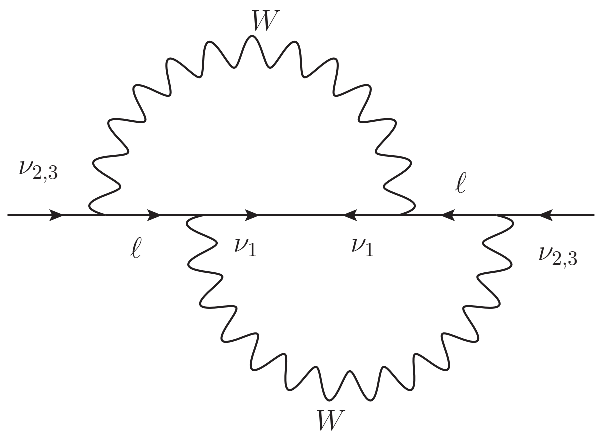


Figure 2.3: *Two-loop radiative generation of neutrino mass through the exchange of two W-bosons.*

2.2 The Ma-Model

2.2.1 Theoretical Aspects of the Ma-Model

The mechanism we will study in details here is a radiative model generating neutrino mass at one loop, namely Ernest Ma's *scotogenic model*.[13] One of the features making this model interesting is that it requires a minimal addition to the Standard Model particle content while some other mechanisms imply a consequent extension with many new particles. This model, introduced in 2006 [31], is based on the well-known two-Higgs doublet (2HDM) model [32, 33] or, more precisely, on a special case of this model sometimes called the *inert doublet model* or *2HDM type I* [34, 35]. It only implies the introduction of two types of fields: A yet to be observed right-handed neutrino N_i (actually three generations) and a hypothetic new scalar doublet sometimes called *inert higgs doublet* [35] or *dark scalar doublet* [36] (the latter will be prefered[14]). Along with these new fields a new type of symmetry is introduced: the Z_2 symmetry. This symmetry is discrete and exact, implying it cannot be spontaneously broken and subsequently forbids rigorously some terms in the Lagrangian of the theory. All the particles described by the standard model are *even* under this symmetry while the new particles are *odd*. The new leptonic and scalar particle content can hereafter be represented as follows under the group

[13] Cf. Sec. 2.2.3 for an explanation of this name.

[14] I will choose not to use the first apellation as it is rather misleading. Indeed neither is η inert as it has gauge and scalar interactions nor it is *Higgs-like* as it does not get a vacuum expectation value, and thus does not intervene in the Higgs mechanism.

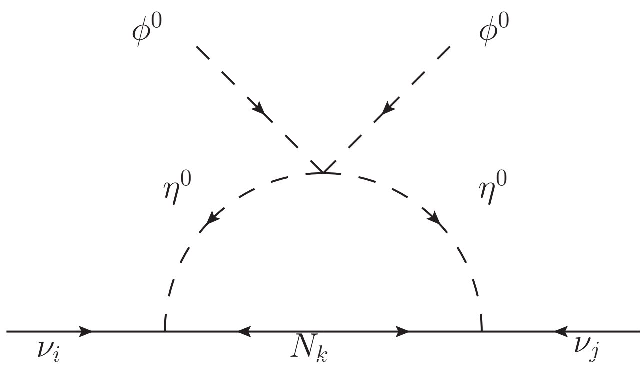


Figure 2.4: *Mechanism generating neutrino mass at one-loop as considered by Ernest Ma [31].*

of symmetries $SU(2)_L \times U(1)_Y \times Z_2$:

$$\begin{aligned} &\begin{pmatrix} \nu_g \\ l_g \end{pmatrix}_L \sim (\mathbf{2}, -1/2, +)\,,\ l_g^c \sim (\mathbf{1}, 1, +)\,,\ \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \sim (\mathbf{2}, 1/2, +)\,, \\ &\quad N_g \sim (\mathbf{1}, 1, -)\,,\ \begin{pmatrix} \eta+ \\ \eta^0 \end{pmatrix} \sim (\mathbf{2}, 1/2, -)\,. \end{aligned} \tag{2.1}$$

The scalar doublets will be written as follows:

$$\eta = \begin{pmatrix} \eta^+ \\ \eta_R^0 + i\eta_I^0 \end{pmatrix}, \Phi = \begin{pmatrix} \phi^+ \\ h + i\varsigma \end{pmatrix}.$$

As the Z_2 symmetry is exact all vertices including new particles must contain an even number of Z_2-*odd* fields. The direct consequences are:

- The Yukawa coupling of the neutrino singlet with the SM Higgs and the left-handed lepton doublet is forbidden, which means there is no Dirac mass term with ν and N: $\overline{N}\phi^\dagger\ell_L$ would break Z_2.

- The similar *Yukawa-like* coupling involving η is allowed, but nevertheless the scalar cannot get a VEV: $(h_\nu)_{gf}\overline{N}^g\widetilde{\eta}^\dagger\ell_L^f$ obeys Z_2 but $(h_\nu)_{gf}\overline{N}^g v_\eta\,\ell_L^f$ would break it.

Summing up, this means that there is no tree-level mass term for ν and that, unlike the Higgs, the dark doublet cannot acquire a VEV. However, with this extended particle content there is a relatively simple way to generate the neutrino mass through a one-loop mechanism. This mechanism is based on the exchange of η

particles and heavy neutrinos. This one-loop diagram is of course just a convenient way to get an idea of the origin of mass, but technically it corresponds to a first order quantum correction of the left-handed neutrinos propagators.

In Fig. 2.4, we can see that the other external fields involved are two neutral Higgs fields. One should note that these two higgs fields will not propagate but will get a VEV after EWSB. Consequently the symmetry breaking is also at the core of the mass in this mechanism even though it is in a different way than for the mass of the other fermions.

As always in QED all the information is contained in the Lagrangian. Let us now illustrate the impact of the introduction of the new fields on the Lagrangian of the theory (before EWSB). A new term regarding the Yukawa interactions of the leptons appears:

$$\mathcal{L}_{\text{Yuk}} = -(Y_\nu)_{gf}\overline{N}^g\widetilde{\eta}^\dagger\ell_L^f\,,$$

as well as a Majorana mass term for the neutrino singlet:[15]

$$\mathcal{L}_{\text{Maj}} = -\frac{1}{2}\overline{N}^g M_{gf}\overline{N}^f + \text{h.c.}$$

As far as the scalar sector is concerned, the new potential is:

$$\begin{aligned} V_{\text{Scalar}} = {} & m_1^2\Phi^\dagger\Phi + m_2^2\eta^\dagger\eta + \frac{1}{2}\lambda_1\left(\Phi^\dagger\Phi\right)^2 + \frac{1}{2}\lambda_2\left(\eta^\dagger\eta\right)^2 \\ & + \lambda_3\left(\Phi^\dagger\Phi\right)\left(\eta^\dagger\eta\right) + \lambda_4\left(\Phi^\dagger\eta\right)\left(\eta^\dagger\Phi\right) + \frac{1}{2}\lambda_5\left[\left(\Phi^\dagger\eta\right)^2 + \text{h.c.}\right]. \end{aligned} \tag{2.2}$$

All the parameters in (2.2) are real by hermicity of the Lagrangian, except for λ_5. However, the exact Z_2 symmetry imposed here forbids the bilinear term $\left(\Phi^\dagger\eta\right)$, so that one can always choose λ_5 real by rotating the relative phase between ϕ and η [37].

We are then left with a set of 6 parameters, namely:[16]

$$\{m_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$$

Furthermore, like in the SM, the $SU(2)_L\times U(1)_Y$ symmetry is spontaneously broken by $\langle\phi^0\rangle = v$ and we are left with one physical Higgs boson h, which resembles the

[15] Unlike the other terms in the Lagrangian the Majorana mass term is not bound to the electroweak scale and the possible values for m could be significantly larger than even the *top-quark* mass: $m_t = 173$ GeV.

[16] m_1^2 is not included in this set because it is constrained by the EWSB condition which in the case of the potential (2.2) reads: $v^2 = \frac{-m_1^2}{\lambda_1}$.

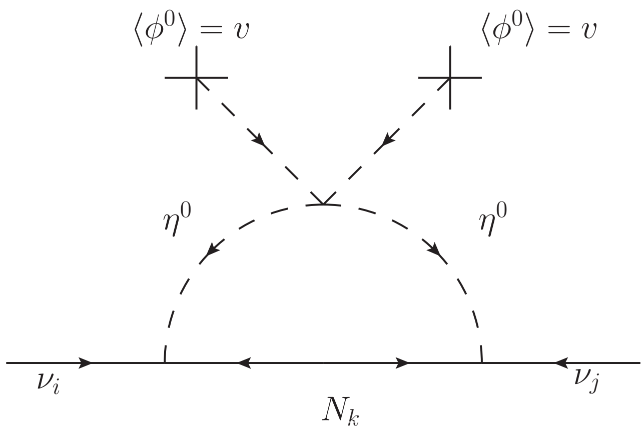


Figure 2.5: *Mechanism after EWSB where the SM-like Higgs has acquired a VEV.*

SM Higgs boson, as well as four dark scalars: the CP *even*-one; η_R^0, the CP *odd*-one; η_I^0; and a pair of charged ones; $\eta^{\pm}$. The masses of these physical scalars are [38]:

$$\begin{aligned} m_h^2 &= -m_1^2 = 2\lambda_1 v^2, \\ m_{\eta^\pm}^2 &= m_2^2 + \lambda_3 v^2, \\ m_{\eta_R^0}^2 &= m_2^2 + (\lambda_3 + \lambda_4 + \lambda_5)\, v^2, \\ m_{\eta_I^0}^2 &= m_2^2 + (\lambda_3 + \lambda_4 - \lambda_5)\, v^2. \end{aligned} \tag{2.3}$$

It is then clearly possible to write all the scalar couplings (except for λ_2) in terms of physical scalar masses and m_2 so that we are free to take the set of 6 independent parameters of the model to be:

$$\left\{ m_2, m_h, m_{\eta_R^0}, m_{\eta_I^0}, m_{\eta^\pm}, \lambda_2 \right\} .$$

2.2.2 Calculation of the Neutrino Mass Matrix

In the following we will derive the radiative mass generated by the aforementioned mechanism. This mass comes from a first-order quantum correction to the neutrino propagator. This correction will shift the pole of the propagator and subsequently the physical mass of the neutrino.[17] If we really want to show that the derivation of the mass is performed in the mass basis after EWSB, the mechanism shown in Fig. 2.5 is more faithfully represented by Fig. 2.6. Not only is this mechanism relatively simple, but the resulting neutrino mass is exactly calculable

[17]For further details, see [39].

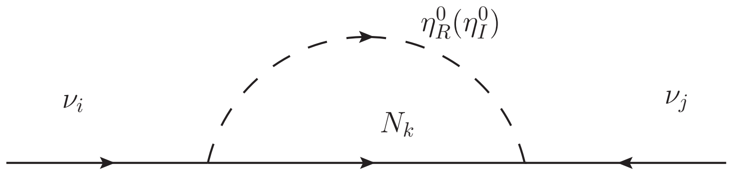


Figure 2.6: *One-loop diagram with exchange of η_R^0 (η_I^0).*

by considering the mechanism after ESWB and splitting it up into two diagrams. This splitting is a direct consequence of representing the *new* scalar doublet as:

$$\eta = \begin{pmatrix} \eta^+ \\ \eta_R^0 + i\eta_I^0 \end{pmatrix},$$

where we separate the real and imaginary part of the neutral component of the scalar. This trick makes the calculation of the mass quite straightforward, as is presented now.

From what has been mentioned above we are left with two one-loop diagrams involving the real and imaginary part of the neutral scalar doublet (see Fig. 2.6). First of all a quick look at the figure indicates that we need only to take care of one diagram, the other being just the same except for η_R^0 that is replaced by η_I^0. Moreover as we are calculating amplitude the factor i in the second diagram will square as a -1 and this will result in subtracting the second diagram from the first. Keeping in mind that what is evaluated here is a radiative correction —and not the invariant matrix element of a scattering event— we apply the Feynman rules to this diagram and we are left with an integral of this form:[18]

$$\begin{aligned} -i\Sigma_{ij}^{\nu} &= -\int \frac{\mathrm{d}^4k}{(2\pi)^4} h_{ik} \frac{i\left(\not{k} + M_k\right)}{k^2 - M_k^2} h_{jk} \frac{i}{(p-k)^2 - m_R^2} \\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^4} h_{ik} \frac{\left(\not{k} + M_k\right)}{k^2 - M_k^2} h_{jk} \frac{1}{(p-k)^2 - m_R^2}. \end{aligned} \tag{2.4}$$

This integral is of a tensorial nature, as the *Feynman slash* notation indicates. Thence it needs to be reduced to a scalar one which in our case is immediate after two observations:

- Firstly, our result must be valid whatever the momentum of the neutrinos, hence we can set the momentum to zero without loss of generality.[19]

[18] The factor $-i\epsilon$ is understood in all squared masses. This trick shifting the poles of the integrals along the imaginary axis is known as the *Feynman prescription.*

[19] A non-zero momentum only shifts $\not{p}$ but not m.

- Secondly, the part of the integral numerator which is linear in $\not{k}$ drops out.[20]

Finally, the integral reduces to a scalar one, which is formally called a scalar two-point loop integral:

$$-i\Sigma^{\nu}_{ij} = -\int \frac{\mathrm{d}^4k}{(2\pi)^4} h_{ik}h_{jk}\frac{M_k}{[k^2 - M_k^2]\,[k^2 - m_R^2]}. \tag{2.5}$$

Clearly this integral is logarithmically divergent since the numerator is proportional to $k^3\mathrm{d}k$ while the denominator is proportional to k^4 for large k. This is the case for most loop-diagrams, but it can nevertheless be calculated in different ways. However, a question may arise, how can we obtain a consistent mass for the neutrinos if it comes from the subtraction of two divergent integrals? Actually, if we look at Fig. 2.4, a rapid power counting indicates that the divergence was not present before EWSB. This would then mean that the infinities are just artificially introduced after EWSB with our splitting of the diagram. Yet the spontaneous symmetry breaking does not physically 'create' divergences and we should land on our feet and obtain a finite value for the mass after calculation. As a matter of fact that is exactly what happens, as we will see the nature of the infinity will be the same in the two integrals and the *'infinities will cancel'* and the mass will be finite. In order to prove elegantly this feature one can notice that this integral can be expressed in terms of a *Passarino-Veltman* function (see Appendix B):

$$I_R = h_{ik}h_{jk}M_k\frac{i}{16\pi^2}B_0\left(p^2 = 0, M_k^2, m_R^2\right). \tag{2.6}$$

Similarly, for the diagram with the imaginary part of the scalar (we deliberately omit the minus sign to show the direct correspondence with the other integral and translate it into a subtraction)

$$I_I = h_{ik}h_{jk}M_k\frac{i}{16\pi^2}B_0\left(p^2 = 0, M_k^2, m_I^2\right). \tag{2.7}$$

The analytical expression for the *Passarino-Veltman* function B_0 can be found in the literature [43], and the remarkably convenient aspect dwells in the fact that the infinite part of this function is *independent* of the variables. Therefore, when subtracting the integrals we get:

$$\begin{aligned} I_R - I_I = {} & h_{ik}h_{jk}M_k\frac{i}{16\pi^2}\left[B_0\left(0, M_k^2, m_R^2\right) - B_0\left(0, M_k^2, m_I^2\right)\right] \\ = {} & i\frac{h_{ik}h_{jk}}{16\pi^2}M_k\left[\left\{\frac{m_R^2}{M_k^2 - m_R^2}\ln\frac{m_R^2}{M_k^2} + \frac{2}{\epsilon}\right\} - \left\{\frac{m_I^2}{M_k^2 - m_I^2}\ln\frac{m_I^2}{M_k^2} + \frac{2}{\epsilon}\right\}\right] \\ = {} & i\frac{h_{ik}h_{jk}}{16\pi^2}M_k\left[\frac{m_R^2}{M_k^2 - m_R^2}\ln\frac{m_R^2}{M_k^2} - \frac{m_I^2}{M_k^2 - m_I^2}\ln\frac{m_I^2}{M_k^2}\right]. \end{aligned} \tag{2.8}$$

[20] In QED one can prove that all terms proportional to $\not{k}$ —where k is the loop momentum— will integrate to zero, see [40–42].

Finally one should understand that physically we have just evaluated a quantum correction to the neutrino propagator, and this must now be linked to a mass, *the radiative mass of the neutrino.* In practice, this translation merely consists in multiplying the result of (2.8) by i and it yields:

$$\mathcal{M}_{ij}^{\nu} = \frac{h_{ik}h_{jk}}{16\pi^2} M_k \left[\frac{m_R^2}{m_R^2 - M_k^2} \ln \frac{m_R^2}{M_k^2} - \frac{m_I^2}{m_I^2 - M_k^2} \ln \frac{m_I^2}{M_k^2} \right]. \tag{2.9}$$

The result in (2.9) is the neutrino mass matrix arising from the radiative model we study —with i and j being the generation indices. The sum is implied over the neutrino generations (sum over k), and one can see that there can well be a mixing between different generations.

2.2.3 Some Remarks on the Model

This mechanism is sometimes referred to in the literature as the *scotogenic* model [44]. This is due to its relation with dark matter —this mystery of physics constituting more than 22% of the mass of the universe (see Fig. 2.7 for an indicative view of the content of the universe). Indeed scotogenic from the greek words $\sigma\kappa o\tau \acute{o}\sigma$ ('skotos') and $\gamma\eta\nu\epsilon\iota\nu$ ('gênein') can be translated by *produced by darkness*, and it goes with the idea getting more and more studied that dark matter may be the origin of a radiative neutrino mass [45, 46]. Hence, any neutrino mass model could be qualified as such provided that they give one or several dark matter candidates involved in the radiative mechanism giving mass to neutrinos. The interest lies in the fact that such mechanisms could simultaneously provide us with a mechanism explaning how the neutrinos acquire mass and shed light on the nature of dark matter. As these problems are two of the main loopholes in the Standard Model (and any consistent theory beyond it should therefore account for that), they may hold some of the answers particle physicists are seeking.

In the scotogenic model, the dark matter candidates (as pointed out in [47]) can be fermionic or bosonic. As always, all depends on the masses – on which we know very little. To generalize it can be said that the dark matter candidate is the lightest neutral particle *odd* under Z_2. This can then be the lightest neutrino mass eigenstate (which will be denoted N_1) or the lightest neutral scalar (directly determined by the sign of the λ_5 coupling constant). For example in the case where η_I^0 is the lightest (which is the case we will consider later on), it is the dark matter candidate [48] and the following chain of decays could be observable [31]:

$$\eta^{\pm} \longrightarrow N_{1,2,3}, \tag{2.10}$$

$$N_2 \longrightarrow \eta^{\pm}\eta^{\mp} N_1, \tag{2.11}$$

$$N_3 \longrightarrow \eta^{\pm}\eta^{\mp} N_{1,2}. \tag{2.12}$$

From these decays the Yukawa couplings may be extracted and compared to the neutrino mass matrix, as soon as this is known completely. In the other case where the scalars are heavier, the lightest Z_2-*odd* particle could be a viable dark matter candidate. This possibility to have sterile neutrinos as a constituent of the dark matter is not a new idea [49,50], and theoretical implications as well as strategies to look for such a type of dark matter have been and are investigated [51]. Eventually let us recall that the sterile neutrino mass in our previous framework would be at the TeV-scale but that there also are studies regarding keV sterile neutrinos as a possible *warm dark matter* constituent [52, 53].

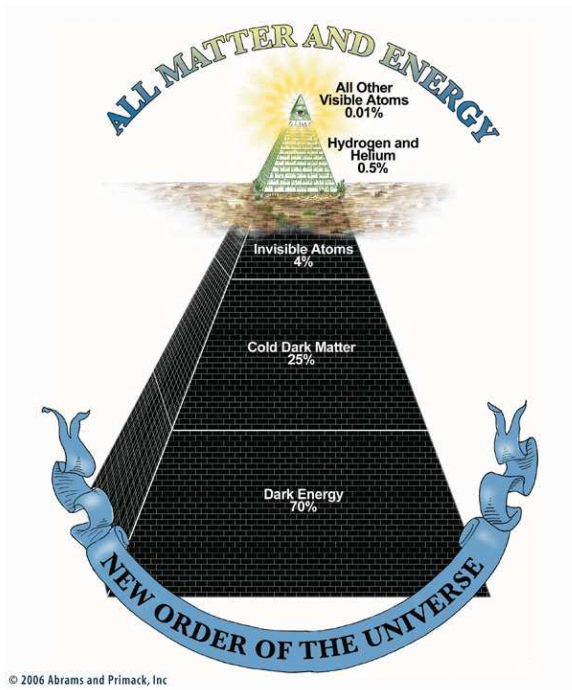


Figure 2.7: *'Cosmic Density Pyramid' (Illustration by Nicolle Rager Fuller. © 2006 Abrams and Primack Inc).*

2.3 Renormalization Group Running

In this section we speak about the problem with the divergences that are found in perturbative calculations in quantum field theory.[21] These divergences are not just a technical problem that we can overlook or forget. They parameterize the dependence on quantum fluctuations at short distance scales (or, equivalently, high

[21] This section is inspired from the lectures on Advanced Quantum Field Theory given in 2006 by Dr. Luty (University of Maryland). These lectures are available online, see [54].

momenta). Historically, it took a long time to understand this. In the 1930's, when these divergences were first discovered in quantum electrodynamics, many physicists believed that almost all the fundamental principles of physics had to be changed to eliminate the divergences. In the end of the 1940's, Bethe, Feynman, Schwinger, Tomonaga, Dyson, and others proposed a program of 'renormalization' that finally gave finite, and therefore, physically sensible results by absorbing the divergences into redefinitions of physical quantities. This leads to calculations that agree with experiments to eight significant digits in QED, the most accurate calculations in all of science![22]

2.3.1 Principles

In all theories in physics, when we calculate a quantity, we only do it to a certain *order* in that we neglect some effects which have weaker contributions. These can be added later as corrections to the main result. In quantum field theories these *corrections* take the form of terms of higher order in a series which can be represented as loops in a Feynman diagram.
The problem is that, when trying to calculate such quantum corrections, divergences may arise. How can the higher-order correction to a physical quantity be divergent? The solution provided by the aforementioned physicists is to absorb the divergences by introducing relevant *counterterms* in the Lagrangian of the theory. We escape the former problem by redefining each physical quantity as the sum of the *bare* quantity and the adequate *counterterm* which are both divergent but whose sum is not.
In practice, the whole procedure is divided in two phases:

1. *Regularization*: We need to 'regulate' the divergences, *i.e.* to make the apparent divergences finite so that we can manipulate them. Actually it is just a parameterization of the sensitivity to short distance physics, modifying the theory at a distance scale of order Λ^{-1} (the cutoff) so that it is well-defined. We say that the theory has been *regularized*. In the theory with the cutoff, the divergences are replaced by sensitivity to Λ^{-1}, which means that the physical quantities diverge in the limit $\Lambda^{-1} \to 0$ (when the couplings is fixed).

2. *Renormalization*: When using the regulated theory to compute physical quantities, we find that these quantities depend only on a combination of the cutoff and the other parameters. In other words, a change in the cutoff can be compensated by a change in the couplings so that all physical quantities are left invariant. This leads to a set of coupled differential equations, the *Renormalization Group Equations* (RGEs), describing the evolution of the parameters of the theory with the energy scale.

[22] For a recent result see [55].

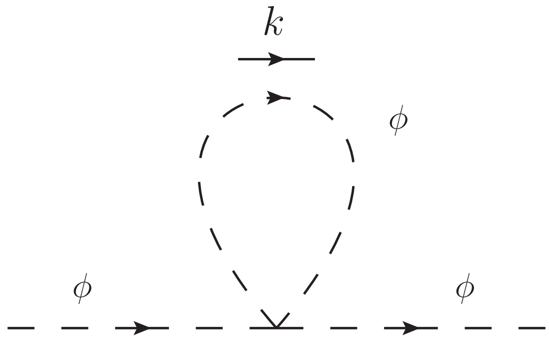


Figure 2.8: *Higgs self-interaction.*

Regularization Procedures Let consider the self-coupling Higgs interaction which is one of the first-order corrections to its mass (Fig. 2.8). The calculation of this diagram yields:

$$\Sigma_\phi \propto \int \frac{\mathrm{d}^4 k}{k^2 - m^2} \overset{k \gg m}{\approx} \int \frac{\mathrm{d}^4 k}{k^2} = 2\pi^2 \int \frac{k^3 \mathrm{d}k}{k^2} \rightarrow \text{Quadratically Divergent} \quad (2.13)$$

We see that this integral diverges for high momenta[23], which is then called an *ultraviolet divergence* (UV divergence) that must be regulated. This can be done using different techniques that are more or less useful according to the context in which they are used. The most intuitive idea is just to to prevent the integral from diverging by imposing a momentum *cutoff*. Some of the other existing techniques are:

- *Lattice regularization*, where we replace continuous spacetime with a lattice. It was introduced as a way of discretizing the path integral to make it well-defined. On a lattice, a quantum field theory becomes a quantum system whose degrees of freedom consist of one field variable ϕ_x at each lattice point x. The problem with the lattice structure is that it does not preserve the spacetime symmetries. (Instead of continous Lorentz and translation symmetries we have the discrete symmetries of the lattice.)

- *Higher-derivative regularization*, where a regulator is obtained by adding to the Lagrangian density a term, e.g. $\delta\mathcal{L} = -\frac{1}{2}\phi\partial_\mu\partial^\mu\phi$, which, when viewed as a part of the kinematic term, modifies the propagator and improves the convergence of the integral.

- *Pauli-Villars regularization*, where we add unphysical scalar fields to the theory. Since, according to the Feynman rules, loops of fermionic fields have

[23]The last equality is actually not trivial but is obtained after changing from Minkowski to Euclidean space and using polar coordinates.

an extra minus sign compared to bosonic fields, we can choose the interactions of the Pauli-Villars fields to make certain loop-diagrams finite.

- *Dimensional regularization*, which is based on the property that quantum field theories are less divergent in lower spacetime dimensions. Motivated by this, one can regulate Feynman diagrams by taking the spacetime dimension d to be a continuous parameter $d = 4 - \epsilon$. This regulator is the most useful one for most practical calculations.

The RGEs calculated in this thesis have been derived using Dimensional Regularization, as it is the most widely used (see [56, 57] for detailed study of this regularization scheme). However this regulator is rather formal and unintuitive because the procedure changes the dimensions of the fields, which are generalized as:

$$[\phi] = [A_\mu] = \frac{d-2}{2}, \quad [\psi] = \frac{d-1}{2}. \tag{2.14}$$

Besides, except the masses, all the parameters have also to be redefined to keep consistent dimensions:

$$\begin{aligned} \lambda_i &\longrightarrow \lambda_{i,B} = \mu^\epsilon \lambda_i \, , \\ g_i &\longrightarrow g_{i,B} = \mu^{\frac{\epsilon}{2}} g_i \, , \\ Y_f &\longrightarrow Y_{f,B} = \mu^{\frac{\epsilon}{2}} Y_f \, , \end{aligned} \tag{2.15}$$

where B stands for 'bare' quantities, *i.e.*, the quantities which are written originally in the Lagrangian but which are divergent and need to be renormalized.

2.3.2 The Need for Running

More than a purely mathematical feature, the knowledge of the running of the parameters is fundamental if we want to perform consistent measurements, gathering information from various experiments throughout the world. Indeed, as it is a fact that parameters depend on the scale at which the experiment is performed: Every experiment with a different energy scale will lead to a different value for the parameters. We can say that 'the answers will depend on how hard we slam the particles into each other'. Without any knowledge about the runnning it is then impossible to make measurements consistent with each other. Besides, when studying new models beyond the SM involving high-energy sectors, we need to have information on how the parameters of this high-energy theory 'look like' at the low-energy scale at which the experiments are performed.
The running is then essential when we want to conceal high-energy predictions and low-energy experiments. Moreover, this somehow unintuitive concept has been confirmed in experiments, as can be seen on Fig. 2.9: the results from different experiments performed at different energies match the analytical solution obtained when solving the RGE for the QCD gauge coupling.

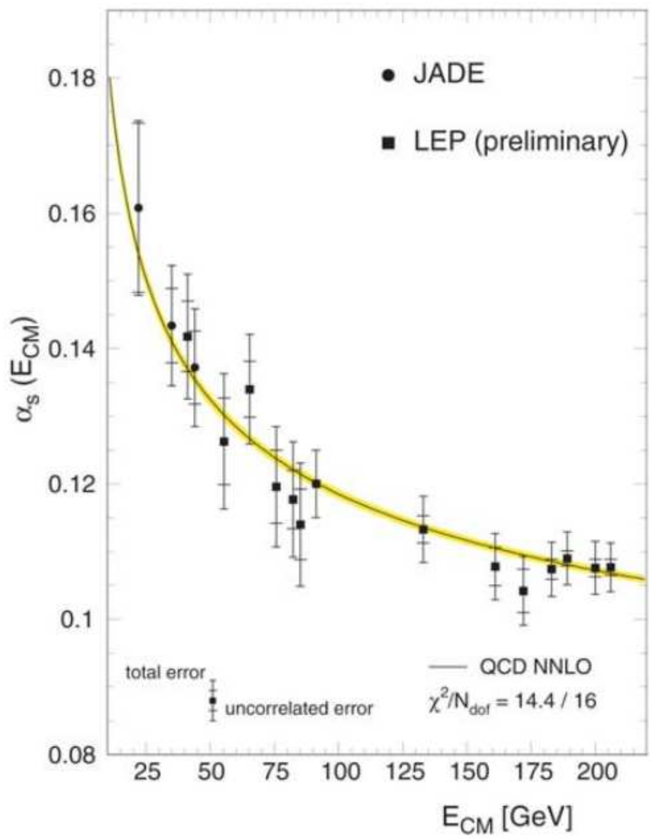
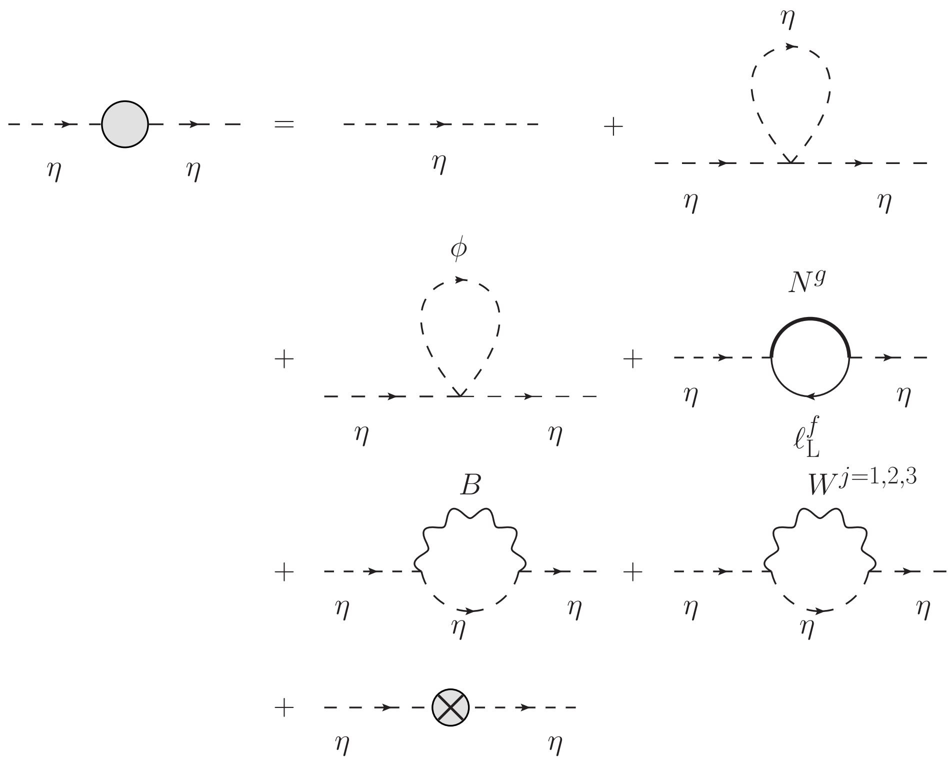


Figure 2.9: *The running of the QCD coupling g_3 from low PETRA to high LEP energies compared with the prediction of asymptotic freedom. Figure taken from Ref. [58].*

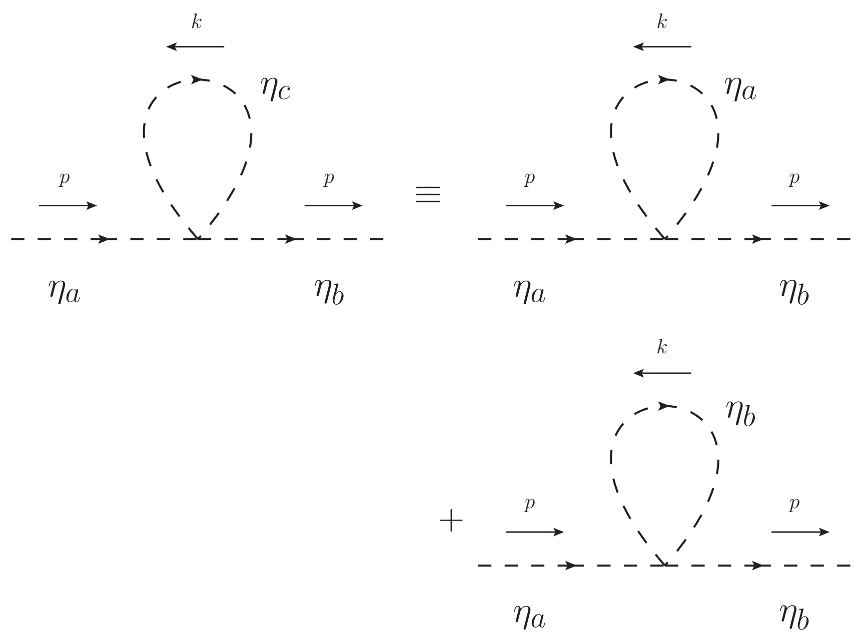
2.3.3 Example of Calculation in the Ma-Model

In the context of the Ma-Model, as in other high-energy models, we must renormalize the theory in order to extrapolate the low-energy behaviour of the newly introduced parameters and thus give some predictions which could be tested at a particle collider. At one-loop, *i.e.* including only the first-order corrrections to all the parameters, this procedure is well known and rather straightforward albeit cumbersome (for a detailed derivation of the one-loop RGEs in the SM see [59]). We will not display here the whole ensemble of calculations but rather an illustrative example. In this example we will derive the RGE for the 'mass', of the dark doublet *i.e.* the parameter denoted as m_2. In the MS scheme, the renormalization constants for the dark scalar wavefunction and mass renormalization are determined by the condition that the counterterms cancel the divergences of the bare dark scalar two-point function. At one-loop level, this is equivalent to calculate the following diagrams:



$$
\begin{aligned}
&= i + i\Sigma_\eta^\eta + i\Sigma_\eta^\phi + i\Sigma_\eta^N + i\Sigma_\eta^B + \sum_{j=1}^{3}\left[i\Sigma_\eta^{W^j}\right] + i(p^2\delta Z_\phi - \delta m^2) \\
&\stackrel{!}{=} \{\text{UV finite terms, } \textit{i.e.}\text{, non-divergent for high loop momentum}\,(k\to\infty)\}\ .
\end{aligned}
\tag{2.16}
$$

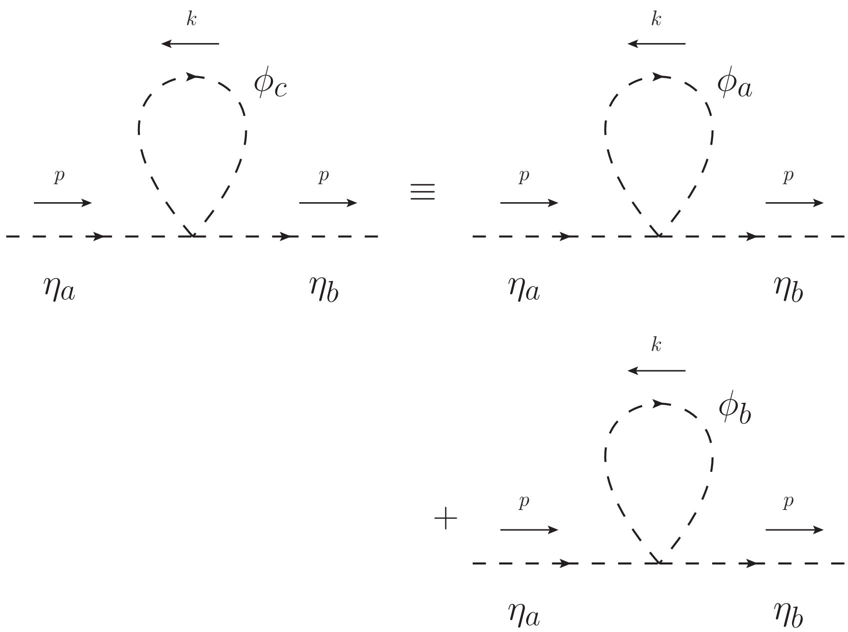
Scalar Self-interactions Let us compute the corrections due to interactions between scalars, *i.e.*, the *self*-interaction of the dark scalar and the *mixed*-interaction between the dark scalar and the Higgs. Note that here all the calculations are performed before EWSB, so that the scalars ϕ and η are doublets associated with the mass parameters m_1 and m_2, respectively. As a direct consequence of these doublet representations, all the calculations will contain $SU(2)$ indices and we have to sum the contributions over the index of the particle involved in the loop. Then, for the dark scalar interaction, the calculation is:[24]



$$
\begin{aligned}
&= i\left(\Sigma^{\eta}_{\eta}\right)^{ba} = \left\{-2i\lambda_2\delta_{ba}\mu^{\epsilon}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{i}{k^2-m_2^2}\right\} \\
&\qquad + \left\{-i\lambda_2\delta_{ba}\mu^{\epsilon}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{i}{k^2-m_2^2}\right\} \\
&= \frac{3i\lambda_2}{16\pi^2}\delta_{ba}\times\frac{2}{\epsilon}\times m_2^2 \qquad \{+\quad \text{UV finite}\}\,.
\end{aligned}
\tag{2.17}
$$

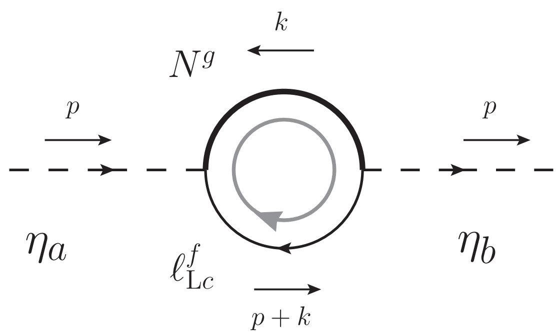
[24] Note that, from now on, all the calculations will utilize the useful results presented in Appendix B.

And for the mixed-interaction:



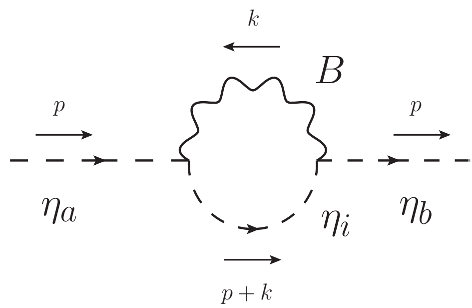
$$
\begin{aligned}
&= i\left(\Sigma_\eta^\phi\right)^{ba} = \left\{-i(\lambda_3+\lambda_4)\delta_{ba}\mu^\epsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{i}{k^2-m_1^2}\right\} \\
&\qquad + \left\{-i\lambda_3\delta_{ba}\mu^\epsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{i}{k^2-m_1^2}\right\} \\
&= \frac{i(2\lambda_3+\lambda_4)}{16\pi^2}\delta_{ba}\times\frac{2}{\epsilon}\times m_1^2 \qquad \{+\quad \text{UV finite}\}\,.
\end{aligned}
\tag{2.18}
$$

Leptonic loop Now we compute the correction due to the *Yukawa-like* coupling of the dark scalar with a neutrino singlet and a lepton doublet. Note that the grey arrow represents the *fermion flow*, as defined in [60].



$$
\begin{aligned}
= i\left(\Sigma_\eta^N\right)^{ba} &= (-1)\left[-i\mu^{\frac{\epsilon}{2}}(h_\nu)_{gf}(\varepsilon^T)_{ac}\right]\left[-i\mu^{\frac{\epsilon}{2}}(h_\nu^\dagger)_{fg}(\varepsilon^T)_{cb}\right] \\
&\quad\times \int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{Tr}\left\{P_\mathrm{L}\frac{-i(\not{p}+\not{k})}{(p+k)^2}P_\mathrm{R}\frac{i(-\not{k}+M_g)}{k^2-M_g^2}\right\} \\
&= -\frac{i}{16\pi^2}(h_\nu^\dagger)_{fg}(h_\nu)_{gf}\delta_{ba}\frac{\mu^\epsilon}{i\pi^2}\int \mathrm{d}^d k\frac{2(p+k)\cdot k}{(k^2-M_g^2)(p+k)^2} \\
&= -\frac{i}{16\pi^2}(h_\nu^\dagger)_{fg}(h_\nu)_{gf}\delta_{ba}\left[2p^2 B_1(p^2,M_g^2,0)+2d\,B_{00}(p^2,M_g^2,0)+2p^2 B_{11}(p^2,M_g^2,0)\right] \\
&= \frac{i}{16\pi^2}(h_\nu^\dagger)_{fg}(h_\nu)_{gf}\delta_{ba}(p^2-2M_g^2)\frac{2}{\epsilon} \qquad \{+\quad \text{UV finite}\}\,. \qquad (2.19)
\end{aligned}
$$

Bosonic loop Let us now take care of the coupling of η with the electroweak bosons B and W^i $(i = 1, 2, 3)$.[25]



$$
\begin{aligned}
&= i\left(\Sigma_\eta^B\right)^{ba} \\
&= \left[-\frac{i}{2}\mu^{\frac{\epsilon}{2}} g_1 \delta_{bi}\right]\left[-\frac{i}{2}\mu^{\frac{\epsilon}{2}} g_1 \delta_{ia}\right] \\
&\quad \times \int \frac{\mathrm{d}^d k}{(2\pi)^d}(2p_\mu + k_\mu) i \frac{-g^{\mu\nu} + (1-\xi_1)\frac{k^\mu k^\nu}{k^2}}{k^2}(2p_\nu + k_\nu)\frac{i}{(p+k)^2 - m_2^2} \\
&= \frac{i}{64\pi^2} g_1^2 \delta_{ba} \frac{\mu^\epsilon}{i\pi^2}\int \mathrm{d}^d k \frac{-(2p+k)^2 + (1-\xi_1)\frac{1}{k^2}\left[k\cdot(2p+k)\right]^2}{k^2\left[(p+k)^2 - m_2^2\right]} \\
&= \frac{i}{16\pi^2} g_1^2 \delta_{ba}(-3p^2 + \xi_1 p^2 - \xi_1 m_2^2)\frac{1}{2\epsilon} \qquad \{+ \quad \text{UV finite}\}\,. \qquad (2.20)
\end{aligned}
$$

The result is similar for the diagrams involving the W^i bosons, because the same loop integral appears in the calculation. In addition to the obvious change in the coupling constant and the gauge fixing parameter ($\xi_1 \to \xi_2$), the main thing which changes is the part carrying the information about the $SU(2)$ structure. Instead of merely having δ_{ba}, the factor is now (when we sum over the W^i):

$$(\tau^i)_{bc}(\tau^i)_{ca} = (\tau^i\tau^i)_{ba} = 3\delta_{ba}.$$

We remark that this yields the same $SU(2)$-diagonal structure (δ_{ba}) as before, but multiplied by three. This seems logical as we now have the contribution from three

[25] The dark scalar η is an $SU(3)$ singlet, so it does not couple to gluons and subsequently these fields do not intervene in the one-loop corrections.

bosons. Anyway, the global result for these diagrams is:

$$i\left(\Sigma_\eta^W\right)^{ba} = \sum_{j=1}^{3}\left[i\left(\Sigma_\eta^{W^j}\right)^{ba}\right] = \frac{3}{32\pi^2}g_2^2\delta_{ba}(-3p^2+\xi_2 p^2-\xi_2 m_2^2)\frac{1}{\epsilon} \quad \{+\text{UV finite}\}\,. \tag{2.21}$$

Now that all the one-loop corrections to the dark scalar two-point function have been derived, the next step in the renormalization process is to solve (2.16) by requiring that the counterterms cancel all the UV divergences which arose in those corrections. Note that we have a factor δ_{ba} in every loop correction so we can drop it, as it just means that the whole expression is diagonal in $SU(2)$. Following these considerations (2.16) yields:

$$\begin{aligned}0 &\overset{!}{=} (\Sigma_\eta^\eta)_{\text{Div}} + (\Sigma_\eta^\phi)_{\text{Div}} + (\Sigma_\eta^N)_{\text{Div}} + (\Sigma_\eta^B)_{\text{Div}} + (\Sigma_\eta^W)_{\text{Div}} + (p^2\delta Z_\phi - \delta m_2^2)\\ &= \frac{3}{16\pi^2}2\lambda_2 m_2^2\frac{1}{\epsilon} + \frac{1}{16\pi^2}(4\lambda_3+\lambda_4)m_1^2\frac{1}{\epsilon} + \frac{1}{8\pi^2}(h_\nu^\dagger)_{fg}(h_\nu)_{gf}(p^2-2M_g^2)\frac{1}{\epsilon}\\ &\quad \frac{1}{32\pi^2}g_1^2(-3p^2+\xi_1 p^2-\xi_1 m_2^2)\frac{1}{\epsilon} + \frac{1}{32\pi^2}g_2^2\delta_{ba}(-3p^2+\xi_2 p^2-\xi_2 m_2^2)\frac{1}{\epsilon}\\ &\quad + p^2\delta Z_\phi - \delta m_2^2\,.\end{aligned} \tag{2.22}$$

From this equation we can extract the *counterterms* absorbing the UV divergences:

$$\delta m_2^2 = \frac{1}{16\pi^2}\left\{\left(6\lambda_2 - \frac{1}{2}\xi_1 g_1^2 - \frac{1}{2}\xi_2 g_2^2\right)m_2^2 \right.$$

$$\left. + (4\lambda_3+2\lambda_4)m_1^2 - (h_\nu^\dagger)_{fg}(h_\nu)_{gf}M_g^2\right\}\frac{1}{\epsilon}\,, \tag{2.23a}$$

$$\delta Z_\eta = \frac{1}{16\pi^2}\left\{2\text{Tr}(h_\nu^\dagger h_\nu) - \frac{1}{2}(3-\xi_1)g_1^2 - \frac{3}{2}(3-\xi_2)g_2^2\right\}\frac{1}{\epsilon}\,. \tag{2.23b}$$

These counterterms are related to the *bare* parameters by the following relations:

$$\eta_{\text{B}} = Z_\eta^{\frac{1}{2}}\eta\,, \tag{2.24a}$$

$$m_{2,\text{B}}^2 = Z_\eta^{-1}(m_2^2+\delta m_2^2)\,, \tag{2.24b}$$

where $\delta Z_\eta \equiv Z_\eta - 1$. Note that more generally we would also have:

$$\phi_{\text{B}} = Z_\phi^{\frac{1}{2}}\phi\,, \tag{2.25a}$$

$$m_{\text{j,B}}^2 = Z_\eta^{-1}(m_j^2+\delta m_j^2), \qquad \text{where } j\in\{1,2\}\,, \tag{2.25b}$$

$$\lambda_{i,B} = \mu^\epsilon Z_\eta^{-2}Z_{\lambda_i}\lambda_i, \qquad \text{where } i\in\{1,2,3,4,5\}\,. \tag{2.25c}$$

Now that these counterterms are known, the next step in the renormalization procedure consists in deriving the so-called renormalization group functions, *i.e.*, the functions describing the evolution of the couplings and masses with the energy scale. The principle used here is the *renormalization group invariance*, *i.e.* the fact that a change in the renormalization scale does not change the theory in itself but only the values of the renormalized parameters. Hence, this means that the bare parameters will remain the same and therefore, by differentiating the Green's function of the perturbation theory, we obtain a differential relation between the remormalized parameters and the renormalized Green's function, which leads to the *Callan-Symanzik* equation [61, 62]:

$$\begin{aligned} &\mu\frac{\mathrm{d}}{\mathrm{d}\mu}G_{\mathrm{B}}^{(n)}(\{p_i\}, g_{\mathrm{B}}, m_{\mathrm{B}}, \xi_{\mathrm{B}}, \epsilon) = 0\,, \\ &\hookrightarrow \left[\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\mu} - \gamma_m m\frac{\partial}{\partial m} - \delta_\xi\frac{\partial}{\partial\xi} + \frac{n}{2}\gamma\right] G^{(n)}(\{p_i\}, g, m, \xi, \epsilon) = 0\,. \end{aligned} \tag{2.26}$$

This equation is rather pretty in that it shows that there exist functions β,γ_m,γ, and δ_ξ of μ, related to the shifts in the coupling and field strength, that compensate for the shift in the renormalization scale. As sketched in Sec. 2.3.2, the strange but direct consequence is that the renormalized parameters depend on the renormalization scale. In this example we focus on the mass of η so we need to derive the renormalization constant for this parameter. This can be done by applying the general formula given in [63]. There it is stated that, if we have a bare (Q_{B}) and a renormalized (Q) parameter, they are generally related by:

$$\begin{aligned} Q_{\mathrm{B}} &= Z_{\phi_1}^{n_1}\ldots Z_{\phi_M}^{n_M}\,[Q+\delta Q]\;\mu^{D_Q\epsilon} Z_{\phi_{M+1}}^{n_{M+1}}\ldots Z_{\phi_N}^{n_N} \\ &= \left(\prod_{i\in I} Z_{\phi_i}^{n_i}\right)[Q+\delta Q]\,\mu^{D_Q\epsilon}\left(\prod_{j\in J} Z_{\phi_j}^{n_j}\right)\,, \end{aligned} \tag{2.27}$$

where:

- $I = \{1, \ldots, M\}$.
- $J = \{M+1, \ldots, N\}$.
- $\mu^{\mathrm{D}_Q\,\epsilon}$ is the renormalization mass scale in which D_Q is introduced to keep the correct mass dimension for the renormalized parameters.[26]
- $Z_{\phi_i} \equiv Z_{\phi_i}(Q(\mu), \omega_a(\mu))$ are wavefunction renormalization constants.
- $\delta Q \equiv \delta Q(Q(\mu), \omega_a(\mu))$ is the counterterm corresponding to the parameter whose β-function is sought.
- $\{\omega_a\}$ are additional parameters, generally μ-dependent and on which Q depends.

[26] For example, $D = \frac{1}{2}$ for a Yukawa coupling and $D = 1$ for a dimension-five effective operator.

Then, the corresponding β-function reads:

$$\begin{aligned}\beta_Q := \mu\frac{\mathrm{d}Q}{\mathrm{d}\mu} = D_Q\left\langle\frac{\mathrm{d}\delta Q_{,1}}{\mathrm{d}Q}|Q\right\rangle + \sum_a D_{\omega_a}\left\langle\frac{\mathrm{d}\delta Q_{,1}}{\mathrm{d}\omega_a}|\omega_a\right\rangle - D_Q\delta Q_{,1}\\ + \sum_{i\in I} n_i\left[D_Q\left\langle\frac{\mathrm{d}\delta Z_{\phi_{i,1}}}{\mathrm{d}Q}|Q\right\rangle + \sum_a D_{\omega_a}\left\langle\frac{\mathrm{d}\delta Z_{\phi_{i,1}}}{\mathrm{d}\omega_a}|\omega_a\right\rangle\right]Q\\ + Q\sum_{j\in J} n_j\left[D_Q\left\langle\frac{\mathrm{d}\delta Z_{\phi_{j,1}}}{\mathrm{d}Q}|Q\right\rangle + \sum_a D_{\omega_a}\left\langle\frac{\mathrm{d}\delta Z_{\phi_{j,1}}}{\mathrm{d}\omega_a}|\omega_a\right\rangle\right],\end{aligned} \tag{2.28}$$

where $Q_{,1}$ is used to specify that we only care for the 'first order' part of the counterterm, *i.e.* the part which is proportional to $\frac{1}{\epsilon}$. This expression is quite cumbersome because it is the most general way to write it, but many terms will vanish because the parameters do not always depend on each other. In the interesting case of the example:

$$m_{2,\mathrm{B}}^2 = Z_\eta^{-1}\left[m_2^2 + \delta m_2^2\right]. \tag{2.29}$$

Therefore the previously introduced parameters are in this case:

- $Q = m_2$,
- $N = 1$,
- $M = 1$,
- $n_1 = -1$,
- $D_Q = 0$,
- $\{\omega_a\} = \{\underbrace{\lambda_1,\,\lambda_3,\,\lambda_4}_{D=1},\ \underbrace{g_1,\,g_2}_{D=\frac{1}{2}},\ \underbrace{h_\nu,\,h_\nu^\dagger}_{D=\frac{1}{2}}\}$.

Hence when applying (2.28), it yields:

$$\beta_{m_2^2} = \frac{1}{16\pi^2}\left\{6\lambda_2 m_2^2 + (4\lambda_3 + 2\lambda_4)m_1^2 + 2\mathrm{Tr}\left(h_\nu^\dagger h_\nu\right)m_2^2 - \frac{3}{2}g_1^2 m_2^2 - \frac{9}{2}g_2^2 m_2^2\right\}. \tag{2.30}$$

Note that, strictly speaking, for a mass parameter, as we can see in (2.26), the relevant renormalization function is not a β-function but rather a 'γ_m-function' called the *anomalous mass dimension*. The definition of this function is slightly different from the β-function and reads:

$$\gamma_m = -\frac{1}{m}\mu\frac{\mathrm{d}m}{\mathrm{d}\mu}\,. \tag{2.31}$$

However, observing that

$$\beta_{m_2^2} = \mu\frac{\mathrm{d}(m^2)}{\mathrm{d}\mu} = \mu\,(2m)\,\frac{\mathrm{d}m}{\mathrm{d}\mu} = -2m^2\gamma_m\,, \tag{2.32}$$

it is clear that the two functions are related by a simple expression and therefore working with one or the other does not really matter.
Finally, we can see that starting from writing all the one-loop corrections to the scalar field propagator we can extract the counterterms and afterwards derive the RGE for a given parameter. Similarly, although not shown here because of the length of the calculation, this procedure has been performed for many other diagrams in order to derive the RGEs relevant for all the other masses and couplings of the model we study. The whole set of relevant RGEs is presented in Appendix A.

Chapter 3

The Ma-Model and Effective Field Theories

Most of the model beyond the SM imply the existence of heavy particles whose production would require a tremendous amount of energy, way beyond the energies at which most experiments are performed. Therefore it seems intuitive to think that their influence is rather insignificant at the *scale* of the experiment. This concept of scale is fundamental in almost all the fields of physics. Indeed when studying orbital motions in the Solar System, one does not worry about the size and geography of planets. Similarly the hydrogen spectrum can be calculated quite precisely whitout knowing that nuclei are composed of quarks and a sea of gluons or even when one weighs oneself there is no need to take in account the rotation of earth nor one's location on it.

In particle physics, this intuitive statement of scale separation translates into a decoupling of the heavy degrees of freedom at low-energy. This concept has actually been theoretically proved in 1975 and it goes under the name of the *Appelquist-Carazzone* theorem [64], and gave birth few years later in 1979 to what we now call effective field theories (EFTs) [65]. The decoupling of heavy states is at the center of *high-energy* physics in the sense that it is the main reason for building high-energy accelerators. Indeed, if quantum field theories were sensitive to all energy scales, it would be much more useful to increase the precision of low-energy experiments instead of building large colliders.[27]

[27] And it would probably be less costly.

3.1 Effective Lagrangian

We saw in the first part that in the context of QFTs we want our calculations to be finite. Indeed a theory is interesting only if it can make some predictions to be tested. However in QFT the quantum corrections may diverge, and it is thanks to a mathematical trick that we can escape this impediment.[28] This procedure called *remormalization* (which has been sytematized by Dyson in 1949 [67]) can only be performed if the theory obeys certain rules. If it does so, then it is called *renormalizable*. Such a renormalizable field theory contains two types of parameters [68]:

- Masses or coupling constants with positive dimensions of mass (due to $m\bar{\Psi}\Psi$ or $\lambda\phi^3$ terms in the Lagrangian).

- Dimensionless coupling constants (due to $\lambda\phi^4$, $g\bar{\Psi}\Psi A_\mu$ or $\lambda\phi\bar{\Psi}\Psi$).

However, in an effective field theory we solely take into account the terms in the Lagrangian related to fields which are *relevant* at the corresponding energy scale. This merely means that we drop the terms involving the momenta of heavy fields. As a consequence the effective Lagrangian of an EFT contains 'higher dimensional terms' (with $d > 4$), which break the renormalizability of the theory.
Mathematically, an effective Lagrangian can be expanded into a finite number of terms of dimension four or less, and a *tower* of terms of dimensions greater than four:

$$\mathcal{L}_{eff} = \mathcal{L}_{d\leq 4} + \sum_i \frac{\mathcal{O}_i}{\Lambda^{dim(\mathcal{O}_i)-4}} \, , \tag{3.1}$$

where Λ is a high-energy scale[29] (typically the mass of the heavy fields) and $dim(\mathcal{O}_i)$ are the dimensions of the operators $\mathcal{O}_i$.

The *higher-dimensional* terms are actually the remnants of the 'heavy' parameters of the full theory (heavy fields), and it is apparent that the heavier these fields are, the more suppressed the corresponding terms in the Lagrangian will be, which can therefore be considered as perturbations of the low-energy theory

[28] This 'mathematical trick' lacks physical motivations and many particle physicists, including Dirac himself, criticized it [66]: *"I must say that I am very dissatisfied with the situation, because this so-called 'good theory' does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it is small - not neglecting it just because it is infinitely great and you do not want it!"*

[29] The scale Λ is often referred to as the *cutoff* of the EFT. This is a somewhat misleading term that is not to be confused with the momentum *cutoff* used in some regularization schemes (noted Λ or μ). Indeed the energy scale to consider varies with the experiment but Λ is intrinsic to the nature of the EFT. Hence one could prefer to call it the *breakdown scale* of the EFT, to recall it is a physical scale that does not depend on the regularization scheme.

— which is the one we know. One can also notice that the higher the dimension of an operator, the smaller its contribution to low-energy observables. Subsequently, despite the sum over higher-dimensional operators being in principle infinite, in practice just a few terms are pertinent if we want to reproduce experiments to finite accuracy. In that sense, *non-renormalizable* field theories are 'as good' as *renormalizable* ones.
Afer what has been said, we understand that effective theories are just a way to 'neglect our ignorance' regarding low-distance physics. To look for answers we can follow two approaches that are as different mathematically as they are conceptually:

- One can imagine one complete renormalizable high-energy theory, run it down to lower energies by gradually decoupling the heavy fields, and eventually obtain low-energy footprints of the high energy regime of this theory. This is the *top-down* approach in which we extract the EFT from a fundamental high-energy theory.

- One can think that Nature may be much more imaginative than we are, and just accept our ignorance by parametrizing it in a useful way as an effective action. This is the *bottom-up* approach in which we start from what we know and extrapolate step by step a more complete description while increasing the energy scale.

Either way it makes use of the *Renormalization Group Equations* (RGEs) which were brought up previously in Sec. 2.3. The second case is the way Fermi came up with a local theory of weak interactions, without knowing anything about the W and Z bosons. Just like his model, the SM can nowadays be considered as an effective theory[30], which we know to be just a part of the theoretical *mosaïque*. One somewhat philosophical question may arise though: Are we going to find the 'Final Theory'[69], from which the SM and General Relativity are subsections, or are we just sealed to dig eternally into an infinity of layers of EFTs vainly trying to open up a bottomless Matryoshka? Who knows, but it might well be that physics is intrinsically arcane.

3.2 The Ma-Model as a Realization of an Effective Operator

3.2.1 The Weinberg Operator

The neutrino mass model we study makes use of the existence of heavy particles which have not been discovered yet. So, in order to appreciate how this model can affect the behaviour of the process we can measure at low-energy, we need to consider what is happening when these heavy particles decouple. This means

[30] Despite being renormalizable, as proved in the works by Veltman and his student 't Hooft which were awarded the Nobel Prize 1999.

that we add to the effective Lagrangian the operators responsible for the mass of the SM neutrino at low-energy. Hence, we can view any high-energy theory as a realization of this low-energy effective operator involving only the fields of the low-energy theory considered. All the information about the short distance physics is hidden in the effective *'Black box'*, and the bottom-up approach implies trying to figure out what is inside the box, just like using a magnifying glass, as shown in Fig. 3.1.

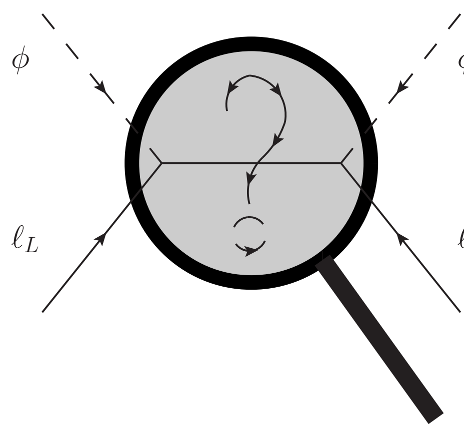


Figure 3.1: *Going from effective to full theory is just like looking through a magnifying glass.*

Following this principle, various models have been built throughout the years which generate mass at tree-level or radiatively (see Secs. 1.3 and 2.1, respectively). Each time, the aim when coming up with a new model is to derive what it can predict for the low energy behaviour of the particles we know and observe at supercolliders such as the LHC. We can then draw conlusions about the consistency of the model, albeit usually we do not rule out the models completely, but rather put bounds on their viability.

In the case of the Ma-Model, we will consider the decoupling of the heavy neutrinos — they will be *integrated out* — and we will then investigate which terms appear in the resulting low-energy effective Lagrangian. One can wonder why we decided to integrate out the heavy neutrinos when we could have chosen the dark scalar doublet or even both. The best answer is that we had to begin with some assumption on the model, and this assumption is that the mass-scale of the neutrino singlets is far beyond that of the dark scalar. This model, as the others, is purely hypothetical until it has made predictions that have been tested by experiments.

When working in the effective theory, one could have several terms of dimension $d > 4$ in the Lagrangian. However, it is reasonable to expect that at energies far below the mass of the heavy neutrinos the dominant effective interactions are those of minimum dimensionality [70]. Hence we will be interested in possible dimension-five operators.
It has been proved in 1979 that the symmetries of the SM allow only one unique dimension-five effective operator which leads to a neutrino mass (and breaks lepton number by two units) [65]. This operator is often reffered to as the *Weinberg Operator*, and it is a combination of four SM fields, two lepton fields and two Higgs fields:

$$\mathcal{L}_\kappa^{d=5} = \frac{1}{4}\kappa_{gf}\,\underbrace{\overline{\ell_{Lc}^{\mathcal{C}}}^{g}\,\varepsilon_{cd}\phi_d\ell_{Lb}^{f}\varepsilon_{ba}\phi_a}_{\mathcal{O}_\kappa:\,\text{effective operator}} + \text{h.c.}\,, \tag{3.2}$$

where the ϵ_{ij} is the two-dimensional antisymmetric tensor used to obtain the correct $SU(2)$ structure.
Moreover, in 1998, Ma showed in a general context [71] that there are only three tree-level realizations of the *Weinberg operator* if we want to use renormalizable interactions only:

- The well-known seesaw type I mechanism with a heavy singlet fermion N_R (or several generations).
- The seesaw type II mechanism with a heavy scalar triplet which naturally acquires a small VEV.
- The seesaw type III involving a fermion triplet in the seesaw.

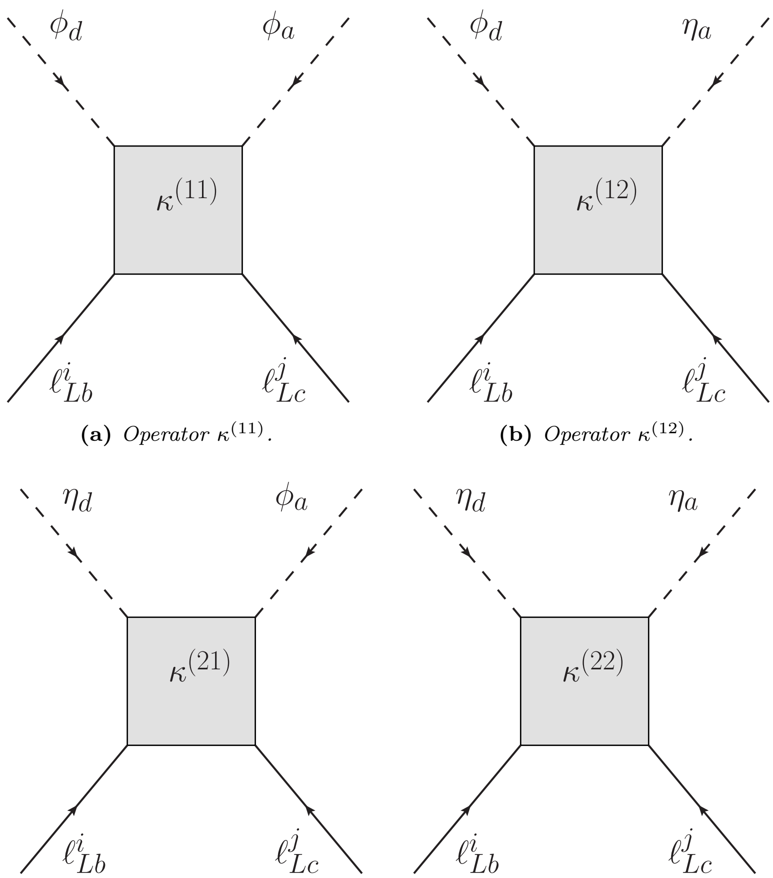
These mechanisms have been studied extensively, see for example [72,73] for type I, [74–76] for type II, and [77] for type III. For a brief overview of type I and II, see Sec. 1.3.

3.2.2 The Effective Operator in our Framework

The effective theory considered here is *not* the SM, but already an extension containing a new symmetry and an additional scalar doublet. Therefore, there are now four ways to combine two scalar fields and two neutrino fields, resulting in four different effective operators of dimension five:

$$\mathcal{L}_\kappa^{d=5} = \sum_{i,j=1}^{2}\mathcal{L}_{\kappa^{(ij)}} = \sum_{i,j=1}^{2}\frac{1}{4}\kappa_{gf}^{(ij)}\,\underbrace{\overline{\ell_{Lc}^{\mathcal{C}}}^{g}\,\epsilon_{cd}\phi_d^{(i)}\ell_{Lb}^{f}\epsilon_{ba}\phi_a^{(j)}}_{\mathcal{O}_\kappa^{(ij)}:\,\text{effective operator}} + \text{h.c.}\,. \tag{3.3}$$

In Eq. (3.3), the notations for the scalar doublets are $\phi^{(1)} = \phi$ (SM) and $\phi^{(2)} = \eta$ (Dark scalar). The effective operator for the neutrino mass is then slightly more complicated in our framework [Eq. (3.3)] than in the SM [Eq. (3.2)]. Nevertheless

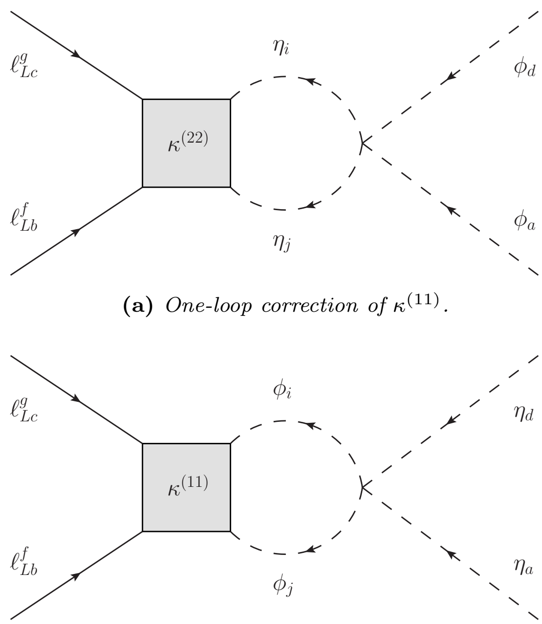


(a) *Operator* $\kappa^{(11)}$. **(b)** *Operator* $\kappa^{(12)}$. **(c)** *Operator* $\kappa^{(21)}$. **(d)** *Operator* $\kappa^{(22)}$.

Figure 3.2: *Diagrammatic representations of the dimension-five operators in a model with two scalar doublets.*

it is clear that the Z_2 symmetry forbids two terms, $\mathcal{O}_\kappa^{(12)}$ and $\mathcal{O}_\kappa^{(21)}$ (the 'mixed' terms shown in Figs. 3.2b and 3.2c) and thus the final remaining operators are $\mathcal{O}_\kappa^{(11)}$ and $\mathcal{O}_\kappa^{(22)}$ (cf. Figs. 3.2a and 3.2d). For convenience these operators will be written as $\kappa^{(11)}$ and $\kappa^{(22)}$, however one has to keep in mind that this is not purely correct as they are just the coefficients of the dimension-five operators, and not the operators themselves.[31]

Actually when we integrate out *by hand* the heavy neutrinos in the Lagrangian, we realize that given the particular behaviour of our particle content under Z_2, only the operator $\kappa^{(22)}$, shown in Fig. 3.2d remains. However, as pointed out in [78], when we renormalize this operator there will be a mixing between the two operators at loop-level (see Fig. 3.3).



(a) *One-loop correction of $\kappa^{(11)}$.*

(b) *One-loop correction of $\kappa^{(22)}$.*

Figure 3.3: *Diagrams contributing to the mixing between the two effective operators $\kappa^{(11)}$ and $\kappa^{(22)}$.*

Subsequently we will have to consider the RGEs of both operators, as these will be coupled. As a matter of fact this mixing was to be expected in the framework of our model where the mass is generated radiatively. Indeed, since the dark scalar does not acquire a VEV, the effective operator $\kappa^{(22)}$ yields a mass equal to zero after EWSB, and it cannot give mass to the neutrino. It is the one-loop mixing

[31] The dimension-five effective operators are often written directly $\kappa^{(ij)}$ or $\mathcal{L}_{\kappa^{(ij)}}$. However, none of them is an operator nor it is of dimension five.

of the latter with the operator involving two Higgs fields which will give mass to the neutrino. This can be expressed as the fact that, our model being a radiative model, the effective operator yielding the neutrino mass arises at loop-level. The relevant RGEs in which the mixing terms appear are presented in Appendix A.

Now we know that our low-energy effective theory will contain two effective operators (whose effects are mixed) accounting for the mass of the neutrinos. We can then renormalize the effective theory and evolve our set of parameters (Yukawa couplings, scalar couplings, scalar masses) up to the scale at which this theory breaks down, typically the mass of the heavy neutrinos. But there is no reason why the three neutrino singlets would be degenerate in mass. Then, instead of integrating out all the generations *at once*, shifting from the effective to the *full theory*, it would seem more general to include a certain hierarchy among the Majorana masses of the singlets, and to integrate out the heavy fields *one by one*. Consequently we will sail through three different effective theories characterized by their own respective scales, the three masses of the heavy neutrinos. Those Majorana masses are unrelated to the EWSB scale and they will be considered to be of the order of TeV. As shown in Fig. 3.3, the three effective theories will be denoted EF1, EF2, and EF3, while the high-energy theory will be denoted by FT for 'full theory'.

3.3 Matching the Theories

Let summarize clearly what will need to be done to get the global running of parameters over the set of theories:

- One starts at a scale above the masses of all the particles, where the effective theory is given simply by the renormalizable theory, *i.e.*, the 'full theory'.
- One then evolves the theory down to lower scales. (As long as no neutrino singlet masses are encountered, this evolution is described by the RGEs of the full theory.)
- When the energy scale considered goes below the mass M_k of one of the neutrino singlets in the theory, we must change the theory to a new theory without that particle. (Non-renormalizable interactions are thus introduced *via* an effective operator.)
- Both the changes in the existing parameters and the coefficients of the new interactions, are computed by 'matching' the physics just below the boundary between the two theories, and this must be done at each mass scale boundary.[32]

[32] In 1991, Georgi expressed a clever remark regarding a way to look at the RGEs from the perspective of the 'matching' [79]:

Therefore, to achieve the first three points, it is just necessary to renormalize each theory, taking into account the changes in the particle content and in the effective operators. But it is then clear that we will end up with parameters whose running is described by piecewise functions. What will happen if there is a mismatch at the junctions between theories, which makes no sense for the running of the physical parameters? How to fix that? This is why we need the fourth point: Indeed, as always in physics, the use of highly simplified models brings some unphysical behaviour. Here it is obvious that the heavy neutrinos do not decouple abruptly but instead their influence weakens slowly until it becomes unsignificant. The exact behaviour is complex but we can fix the potential mismatch of running between theories by imposing so-called *matching conditions*. This procedure has the advantage of being really straightforward, and it consists only in evaluating both theories at the matching scale, *i.e.*, the energy scale at the boundary.
This procedure is used in every study of the running of the paramaters when considering different effective theories (see, e.g., [80]). However it is usually performed at tree-level whereas we must consider it at one-loop level. Indeed, as explained earlier, only the operator $\kappa^{(22)}$ appears in the effective Lagrangian. Yet the mass is generated *via* a one-loop process involving $\kappa^{(11)}$, which turns out to be a quantum correction of $\kappa^{(22)}$. As the running of both operators will be considered it is clear that, for consistency, the matching procedure must be imposed to both of them at tree-level and at one-loop level.

In Fig. 3.3 we can see the different diagrams corresponding to each theory[33], and a rapid power counting indicates that they do not have the same degrees of divergence. Indeed, diagrams in ETs have typically higher degrees of UV divergence, as they contain fewer propagators. For example, for $\kappa^{(11)}$ the diagrams in the ET3 are logarithmically divergent while the diagrams in the FT are finite. This is not an obstacle: we will simply have to regularize each diagram using dimensional regularization. Once this is done, the infinite part will be left aside (as it can be compensated by an adequate counterterm in the Lagrangian), and only the finite scale-dependent part will matter for the matching.

3.3.1 Calculation of the Relevant Diagrams

In the following we will evaluate the relevant diagrams. All the calculations will be performed in a framework before EWSB, where only the scalar doublets and the neutrino singlets have mass. Even if there seems to be a certain number of diagrams, *calculation-wise* only two tree-level and two one-loop diagrams need to be calculated, because only a few couplings will change between the different diagrams. As an example, we will perform in detail the calculation for the diagrams relevant

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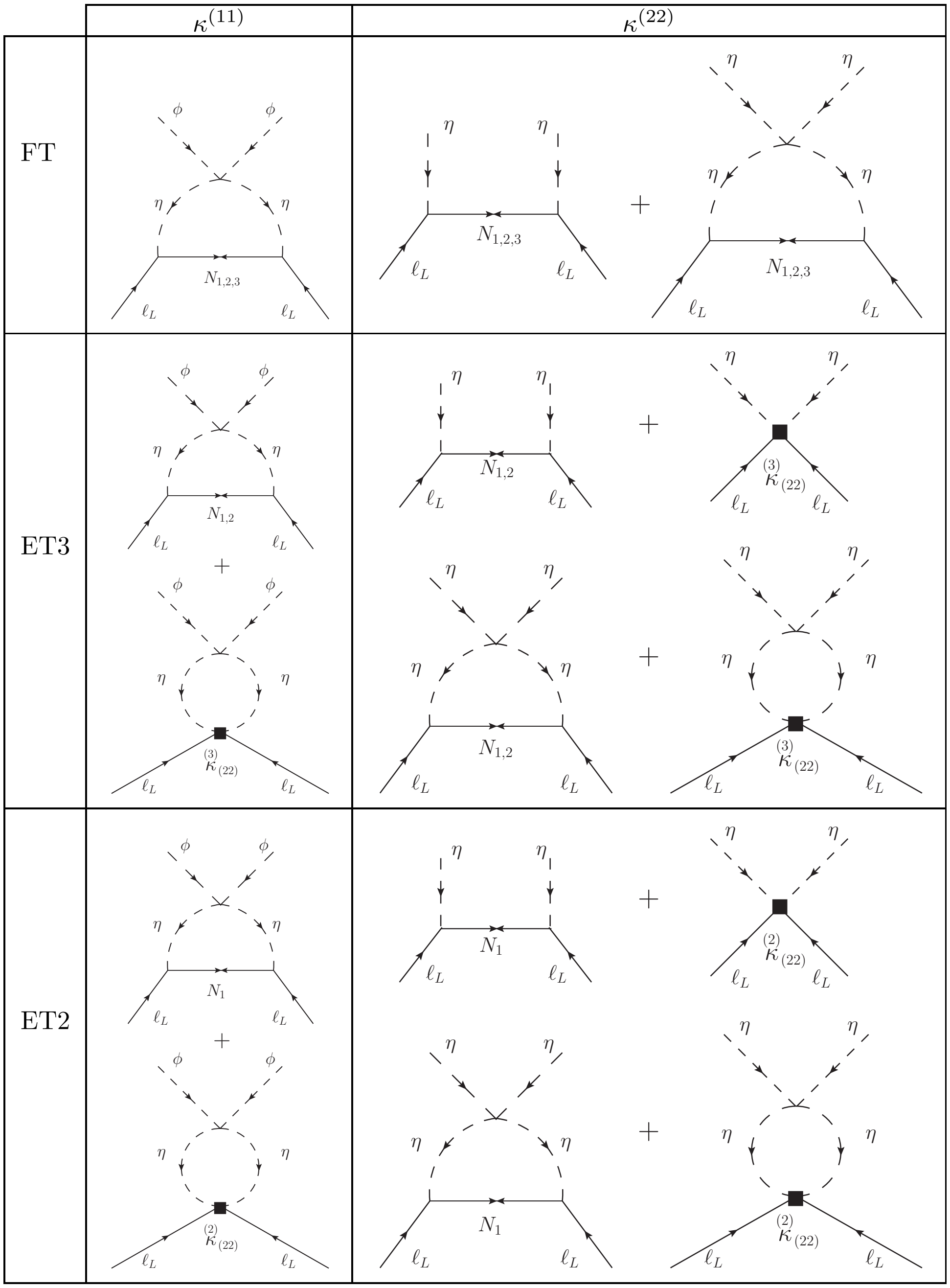
"In a sense, the renormalization group is simply the 'matching' of the theory at the scale μ to the theory at the scale $\mu - d\mu$ without changing the particle content."

[33]For the new notations, see Sec. 3.3.2.


$\kappa^{(11)}$
$\kappa^{(22)}$
FT
ET3
ET2
ϕ
η
$N_{1,2,3}$
$N_{1,2}$
N_1
ℓ_L
$\overset{(3)}{\kappa}_{(22)}$
$\overset{(2)}{\kappa}_{(22)}$
+

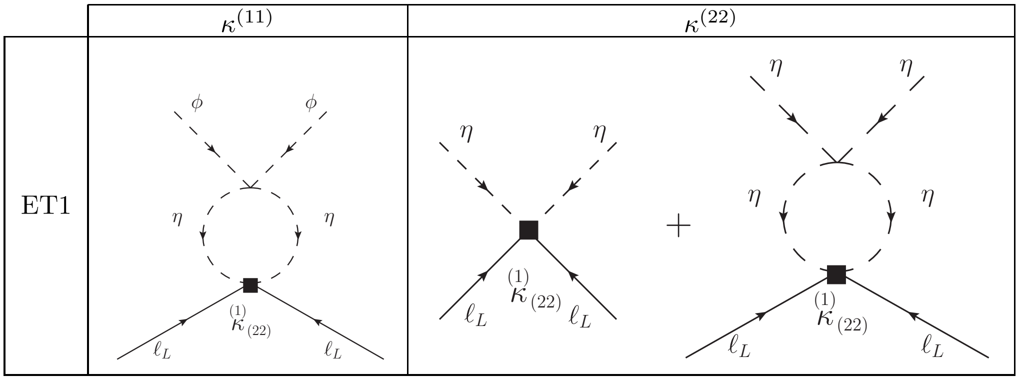


Figure 3.3: *Overview of the diagrams to consider for the matching in the different theories.*

for $\kappa^{(22)}$, *i.e.* , those presented in the second column of Fig. 3.3. Besides, we will consider only two general cases (FT and ET), as the different types of ETs are just particular cases (technically the effect is just on the sum over the neutrino singlet generations).

3.3.1.1 Tree-Level Diagrams

Let us begin with the diagrams relevant for the matching at tree-level (which exist only for the operator $\kappa^{(22)}$). This case is similar to the seesaw type I case, and the result is then well-known (see e.g. [81]). As we can see from Fig. 3.4, the tree-level contribution is actually the sum of two diagrams.

It is clear that:

$$i\left(\Sigma_{\mathrm{FT}}^{\mathrm{Tree}}\right)_{ij}^{abcd} = i(h_\nu^T)_{ik} M_k^{-1} (h_\nu)_{kj} \left(\varepsilon_{bd}\varepsilon_{ca} + \varepsilon_{ba}\varepsilon_{cd}\right) P_L\,. \tag{3.4}$$

In the effective theory, the relevant diagram is just the effective vertex, which is trivially equal to:

$$i\left(\Sigma_{\mathrm{ET}}^{\mathrm{Tree}}\right)_{ij}^{abcd} = i\,\kappa_{ij}\,\tfrac{1}{2}\left(\varepsilon_{bd}\varepsilon_{ca} + \varepsilon_{ba}\varepsilon_{cd}\right) P_L\,. \tag{3.5}$$

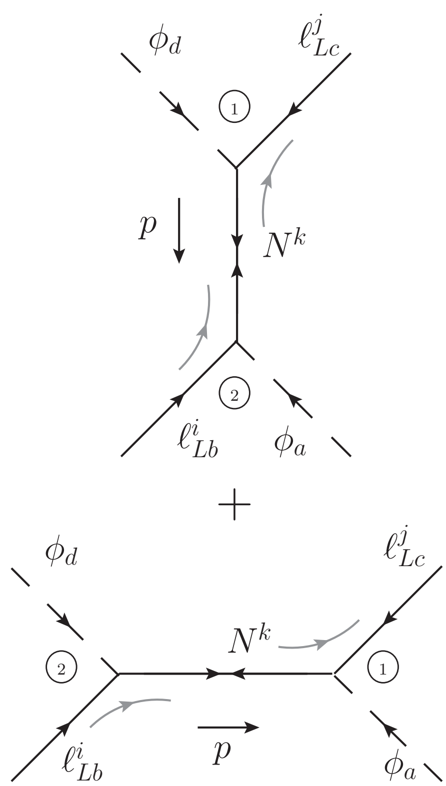


Figure 3.4: *Tree diagrams present in the FT and relevant for the matching.*

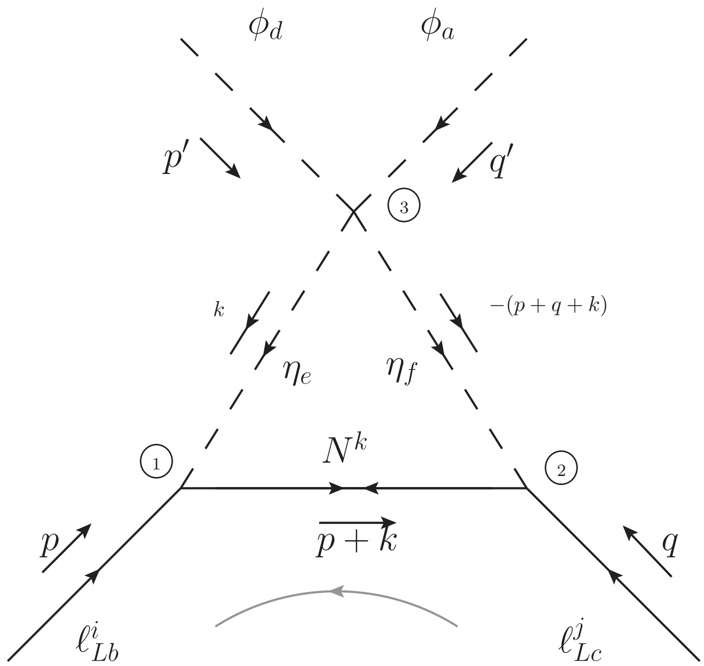


Figure 3.5: *One-loop diagram present in the FT relevant for the matching.*

3.3.1.2 One-Loop Diagrams

Now let us take care of the one-loop diagrams. First, we will deal with the one relevant for the FT. As we can see in Fig. 3.5, it is similar to the one calculated in Sec. 2.2.2. The only difference dwells in the parametrization of the scalar doublet, and in the fact that the calculation was previously performed after EWSB. Hence, our computation will be slightly more general.

The grey arrow in Fig. 3.5 indicates the fermion flow, as defined in [60]. The different vertices are numbered according to the sense of the calculation *i.e.*, in the opposite direction of the fermion flow (which is represented with grey arrows). They are:

① $-i\mu^{\epsilon/2}(h_\nu^T)_{ik}\varepsilon_{be}P_L\,,$

② $-i\mu^{\epsilon/2}(h_\nu)_{kj}\varepsilon^T_{fc}P_L\,,$

③ $-i\mu^{\epsilon}\lambda_5(\delta_{de}\delta_{af}+\delta_{df}\delta_{ae})\,.$

The loop-correction reads:

$$\begin{aligned}
i\mu^\epsilon\,(\Sigma_{\mathrm{FT}})^{abcd}_{ij} &= [-i\mu^{\epsilon/2}(h_\nu^T)_{ik}\varepsilon_{be}][-i\mu^{\epsilon/2}(h_\nu)_{kj}\varepsilon^T_{fc}][-i\mu^{\epsilon/2}\lambda_5(\delta_{de}\delta_{af}+\delta_{df}\delta_{ae})] \\
&\quad\times\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,P_L\,\frac{-i(\not{p}+\not{k}-M_k)}{[(p+k)^2-M_k^2]}\,P_L\,\frac{i}{(p+k+q)^2-m_2^2}\,\frac{i}{k^2-m_2^2} \\
&= i\mu^\epsilon(h_\nu^T)_{ik}(h_\nu)_{kj}\varepsilon_{be}\varepsilon^T_{fc}\lambda_5(\delta_{de}\delta_{af}+\delta_{df}\delta_{ae})M_k\pi^2 \\
&\quad\times\frac{\mu^\epsilon}{i\pi^2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{1}{[k^2-m_2^2][k^2-m_2^2][k^2-M_k^2]}\,P_L\,.
\end{aligned}\tag{3.6}$$

We recognize a scalar loop-integral which can be expressed as a three-point *Passarino-Veltman* function (see Appendix B):

$$\begin{aligned}
i\mu^\epsilon\,(\Sigma_{\mathrm{FT}})^{abcd}_{ij} &= i\frac{1}{16\pi^2}\mu^\epsilon\,(h_\nu^T)_{ik}(h_\nu)_{kj}\,\varepsilon_{be}\varepsilon^T_{fc}\,\lambda_5\,(\delta_{de}\delta_{af}+\delta_{df}\delta_{ae})M_k\,P_L \\
&\quad\times C_0(p_a^2,p_b^2,p_c^2,m_a^2,m_b^2,m_c^2)\,,
\end{aligned}\tag{3.7}$$

where in our case:

$$p_a^2=0,\quad p_b^2=0,\quad p_c^2=0,\quad m_a^2=m_b^2=m_2^2,\quad m_c^2=M_k^2\,.$$

The general analytical expression exists but involves twelve Spence functions (see [43]). However, in our case, where we have set the external momenta to zero, this three-point function may be expressed with the help of two two-point functions:

$$C_0(0,0,0,m_a^2,m_b^2,m_c^2)=\frac{1}{m_a^2-m_b^2}[B_0(0,m_a^2,m_c^2)-B_0(0,m_b^2,m_c^2)]\,. \tag{3.8}$$

A new problem seems to arise as we have here $m_a^2=m_b^2=m_2^2$, and the denominator in (3.8) vanishes.[34] Nevertheless it is clear that this limiting case is just the derivative of $B_0(0,m_2^2,M_k^2)$ with respect to m_2^2:

$$\begin{aligned}
C_0(0,0,0,m_2^2,M_k^2,m_2^2) &= \lim_{m\to m_2}\frac{1}{m^2-m_2^2}[B_0(0,m^2,m_3^2)-B_0(0,m_2^2,m_3^2)] \\
&= \frac{\partial}{\partial(m^2)}\left[B_0(0,m^2,m_3^2)\right]\Big|_{m=m_2} \\
&= \frac{M_k^2}{(m_2^2-M_k^2)^2}\ln\left(\frac{M_k^2}{m_2^2}\right)+\frac{1}{m_2^2-M_k^2}\,.
\end{aligned}\tag{3.9}$$

[34] Of course we could have chosen to change the order of the arguments of C_0 to escape the problem, but the divergence would have then appeared in the definition of one of the B_0 functions. We just chose to deal with the subtlety this way.

Now combining results from (3.7) and (3.9) yields:

$$i\mu^\epsilon \left(\Sigma_{\mathrm{FT}}\right)_{ij}^{abcd} = i\mu^\epsilon \tfrac{\lambda_5}{16\pi^2} (h_\nu^T)_{ik} (h_\nu)_{kj} \left(\varepsilon_{bd}\varepsilon_{ca} + \varepsilon_{ba}\varepsilon_{cd}\right) P_L \times \left[\frac{M_k^3}{(m_2^2 - M_k^2)^2} \ln\left(\frac{M_k^2}{m_2^2}\right) + \frac{M_k}{m_2^2 - M_k^2} \right] . \tag{3.10}$$

This can be written as:

$$i\mu^\epsilon \left(\Sigma_{\mathrm{FT}}\right)_{ij}^{adcd} = i\mu^\epsilon \tfrac{\lambda_5}{16\pi^2} (h_\nu^T)_{ik} M_k^{-1} (h_\nu)_{kj} \left(\varepsilon_{bd}\varepsilon_{ca} + \varepsilon_{ba}\varepsilon_{cd}\right)\, P_L \,\times f(M_k, m_2), \tag{3.11}$$

where we have defined $f(x,y) \equiv \frac{x^4}{(x^2-y^2)^2} \ln\left(\frac{x^2}{y^2}\right) + \frac{x^2}{x^2-y^2}$.

Now let us evaluate the equivalent diagram in the effective theory which is represented in Fig. 3.6. As expected in the earlier remarks, this time the diagram is divergent as we have only two propagators in the loop. Again the vertices are numbered and correspond to:

① $i\mu^\epsilon \kappa_{ij}^{(22)} \frac{1}{2} \left(\varepsilon_{be}\varepsilon_{cf} + \varepsilon_{bf}\varepsilon_{ce}\right) P_L$,

② $-i\mu^\epsilon \lambda_5 (\delta_{de}\delta_{af} + \delta_{df}\delta_{ae})$.

The loop-correction reads:

$$\begin{aligned} i\mu^\epsilon \left(\Sigma_{\mathrm{ET}}\right)_{ij}^{abcd} &= \left[i\mu^\epsilon \kappa_{ij}^{(22)} \frac{1}{2} \left(\varepsilon_{be}\varepsilon_{cf} + \varepsilon_{bf}\varepsilon_{ce}\right)\right] \left[-i\mu^\epsilon \lambda_5 (\delta_{de}\delta_{af} + \delta_{df}\delta_{ae})\right] \\ &\quad \times \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, P_L \frac{i}{k^2 - m_2^2}\, P_L\, \frac{i}{(p+k+q)^2 - m_2^2} \\ &= +\mu^\epsilon \kappa_{ij}^{(22)} \left(\varepsilon_{bd}\varepsilon_{ca} + \varepsilon_{ba}\varepsilon_{cd}\right) \left(\frac{-i\lambda_5}{16\pi^2}\right) B_0\left((p+q)^2, m_2^2, m_2^2\right) . \end{aligned} \tag{3.12}$$

In a similar manner as earlier we have expressed the integral as a (two point) *Passarino-Veltman* function but this time there is a divergent part that we drop and the diagram yields (when we set the external momentum to zero and evaluate the diagram at $\mu = M_k$):

$$i\mu^\epsilon \left(\Sigma_{\mathrm{ET}}\right)_{ij}^{abcd} = i\mu^\epsilon \frac{\lambda_5}{16\pi^2} \kappa_{ij}^{(22)} \left(\varepsilon_{bd}\varepsilon_{ca} + \varepsilon_{ba}\varepsilon_{cd}\right) P_L \ln\left(\frac{m_2^2}{\mu^2}\right) .$$

Finally the result for the diagram can be written as:

$$i\mu^\epsilon \left(\Sigma_{\mathrm{ET}}\right)_{ij}^{abcd} = i\mu^\epsilon \tfrac{\lambda_5}{16\pi^2} \kappa_{ij}^{(22)} \left(\varepsilon_{bd}\varepsilon_{ca} + \varepsilon_{ba}\varepsilon_{cd}\right)\, P_L \,\times g(M_k, m_2) , \tag{3.13}$$

where $g(x,y) \equiv \ln\left(\frac{y^2}{x^2}\right)$.

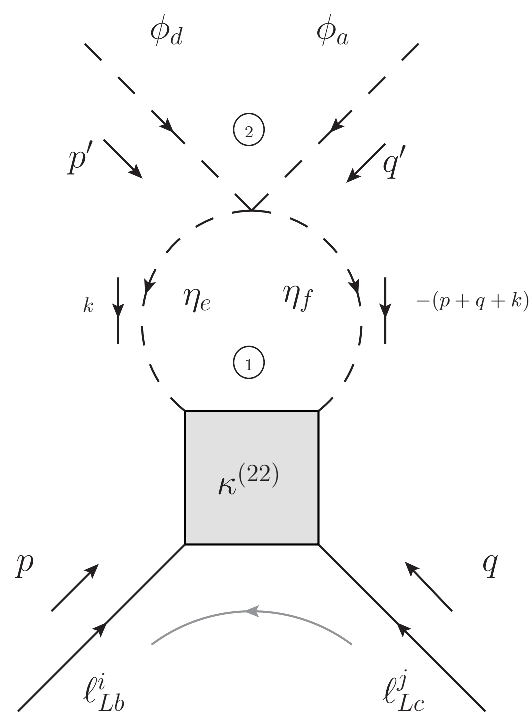


Figure 3.6: *Diagram of the ET relevant for the matching: One-loop correction of the operator* $\kappa^{(11)}$.

3.3.2 Matching Conditions

It is now time to impose the matching conditions. This is a systematic procedure which is done order by order in the loop-expansion. When two theories are compared at a given loop order, the lower order results are included in the matching. Moreover, here one can understand that, despite having a stack of three effective theories, going from one to the other is each time a similar process, namely the decoupling of one heavy neutrino. Therefore, the matching conditions will be analogous and we will write a general expression, valid whichever ET relevant for the energy scale is considered.
Nevertheless, in order to do so, the algebraic formalism must be altered to take into consideration the decoupling of heavy neutrinos. This is why we now introduce some convenient notations already used e.g. in [82]:

- $\kappa^{(ii)} \longrightarrow \overset{(n)}{\kappa}{}^{(ii)}$ is the effective operator in the nth effective theory *i.e.* the operator making up for the low-energy influence of the heavy neutrinos N_k (with $k = n, \ldots, 3$).

- $M_N \longrightarrow \overset{(n)}{M}_N$ is the neutrino mass matrix of the remaining neutrino singlets in ETn. In FT, this matrix is of dimension 3×3 but in ETn, some states have decoupled and their mass is not defined anymore. As a result the Majorana mass matrix in ETn is an $n \times n$ matrix. We denote the largest eigenvalue of $\overset{(n+1)}{M}$ by M_n.

- $h_\nu \longrightarrow \overset{(n)}{h}_\nu$ is the neutrino Yukawa matrix taking in account our integrating out of $(4 - n)$ generations of heavy neutrino singlet. It implies that it is a sort of 'asymmetric' coupling between $(n - 1)$ neutrino singlets and 3 lepton doublets. Since, in the relevant term of the Lagrangian formalism we chose, rows correspond to neutrino singlets and columns correspond to lepton doublets, the coupling technically looks like a $(n - 1) \times 3$ submatrix of the 'full' Yukawa matrix:

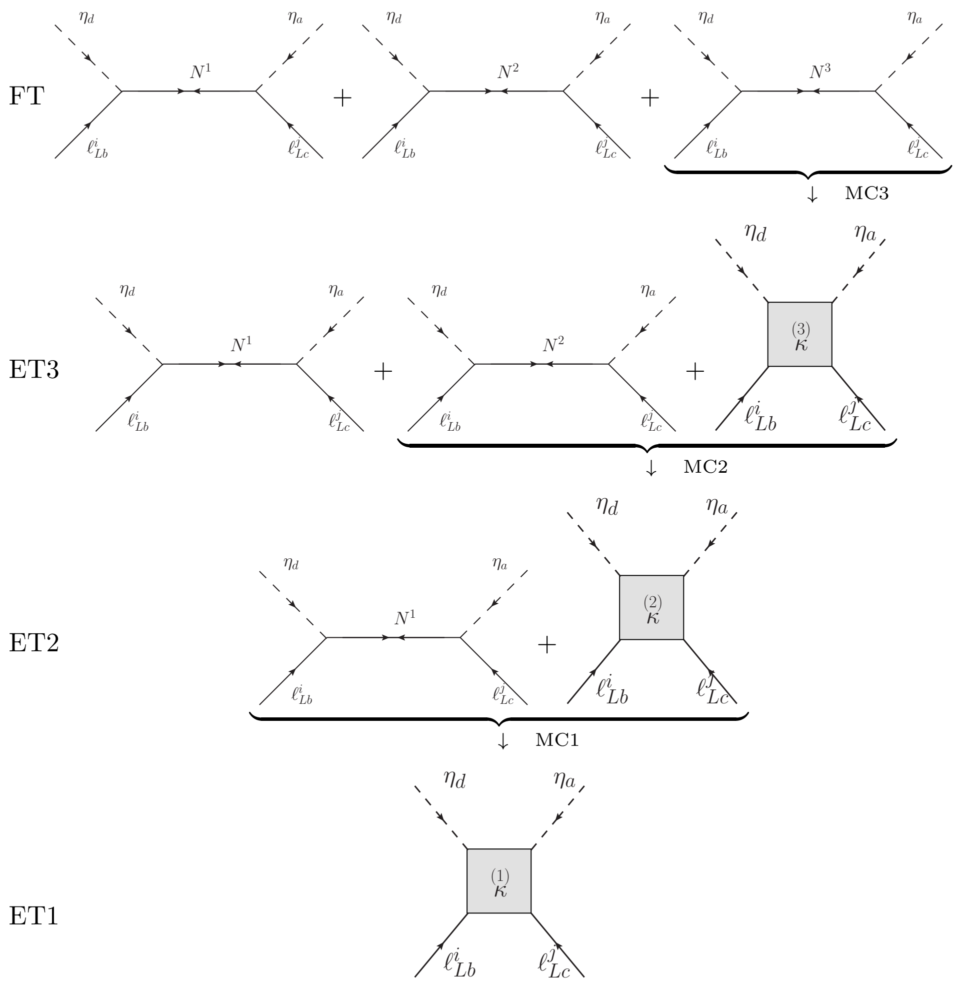
$$h_\nu \equiv \begin{pmatrix} (h_\nu)_{1,1} & (h_\nu)_{1,2} & (h_\nu)_{1,3} \\ \vdots & \ddots & \vdots \\ (h_\nu)_{n-1,1} & (h_\nu)_{n-1,2} & (h_\nu)_{n-1,3} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \rightarrow \overset{(n)}{h}_\nu \\ \\ \\ \\ \end{array}$$

Note that with these notations,

$$\overset{(1)}{\kappa}{}^{(ii)} = \kappa^{(ii)}, \qquad \overset{(n>3)}{\kappa}{}^{(ii)} = 0, \tag{3.14}$$

$$\overset{(4)}{h}_\nu = h_\nu, \qquad \overset{(1)}{h}_\nu = 0. \tag{3.15}$$

The matching procedure to perform can then be schematised as follows:[35]



[35] This scheme has only an indicative value because it is only what would happen if we were dealing solely with a tree-level matching condition. However it is clear that the same kind of scheme could be sketched for one-loop diagrams.

In practice we merely match the values of the diagrams, which change when going from one theory to the other. Considering the scheme above and using Eqs. (3.13), (3.11), and the aforementioned notation, it yields:

1. For $\kappa^{(11)}$:

$$
\begin{aligned}
\mathrm{MC3}\,(\mu = M_3) \longrightarrow \quad \overset{(3)}{\kappa}{}_{ij}^{(11)} &= \frac{f(M_3,m_2)}{g(M_3,m_2)}(\overset{(4)}{h_\nu^T})_{i3} M_3^{-1} (\overset{(4)}{h_\nu})_{3j}\,, \\
\mathrm{MC2}\,(\mu = M_2) \longrightarrow \quad \overset{(2)}{\kappa}{}_{ij}^{(11)} &= \overset{(3)}{\kappa}{}_{ij}^{(11)} + \frac{f(M_2,m_2)}{g(M_2,m_2)}(\overset{(3)}{h_\nu^T})_{i2} M_2^{-1} (\overset{(3)}{h_\nu})_{2j}\,, \\
\mathrm{MC1}\,(\mu = M_1) \longrightarrow \quad \overset{(1)}{\kappa}{}_{ij}^{(11)} &= \overset{(2)}{\kappa}{}_{ij}^{(11)} + \frac{f(M_1,m_2)}{g(M_1,m_2)}(\overset{(2)}{h_\nu^T})_{i1} M_1^{-1} (\overset{(2)}{h_\nu})_{1j}\,.
\end{aligned}
\tag{3.16}
$$

   This set of matching conditions can be written in a general form as follows ($n = 1, 2, 3$):

$$
\mathrm{MC}n\,(\mu = M_n) \longrightarrow \overset{(n)}{\kappa}{}_{ij}^{(11)} = \overset{(n+1)}{\kappa}{}_{ij}^{(11)} + \frac{f(M_n,m_2)}{g(M_n,m_2)}(\overset{(n+1)}{h_\nu^T})_{in} M_n^{-1} (\overset{(n+1)}{h_\nu})_{nj}\,.
\tag{3.17}
$$

2. For $\kappa^{(22)}$:

$$
\begin{aligned}
\mathrm{MC3}\,(\mu = M_3) \longrightarrow \quad \overset{(3)}{\kappa}{}_{ij}^{(22)} &= \left[2 + \frac{f(M_3,m_2)}{g(M_3,m_2)}\right](\overset{(4)}{h_\nu^T})_{i3} M_3^{-1} (\overset{(4)}{h_\nu})_{3j}\,, \\
\mathrm{MC2}\,(\mu = M_2) \longrightarrow \quad \overset{(2)}{\kappa}{}_{ij}^{(22)} &= \overset{(3)}{\kappa}{}_{ij}^{(22)} + \left[2 + \frac{f(M_2,m_2)}{g(M_2,m_2)}\right](\overset{(3)}{h_\nu^T})_{i2} M_2^{-1} (\overset{(3)}{h_\nu})_{2j}\,, \\
\mathrm{MC1}\,(\mu = M_1) \longrightarrow \quad \overset{(1)}{\kappa}{}_{ij}^{(22)} &= \overset{(2)}{\kappa}{}_{ij}^{(22)} + \left[2 + \frac{f(M_1,m_2)}{g(M_1,m_2)}\right](\overset{(2)}{h_\nu^T})_{i1} M_1^{-1} (\overset{(2)}{h_\nu})_{1j}\,,
\end{aligned}
\tag{3.18}
$$

   As before, the set of matching conditions can be written in a general form ($n = 1, 2, 3$):

$$
\mathrm{MC}n\,(\mu = M_n) \longrightarrow \overset{(n)}{\kappa}{}_{ij}^{(22)} = \overset{(n+1)}{\kappa}{}_{ij}^{(22)} + \left[2 + \frac{f(M_n,m_2)}{g(M_n,m_2)}\right](\overset{(n+1)}{h_\nu})_{in} M_n^{-1} (\overset{(n+1)}{h_\nu^T})_{nj}\,.
\tag{3.19}
$$

Although these matching conditions look rather nice, we have somehow cheated to write them in this form, the abuse lying in the fact that these expressions are valid only in the basis where the Majorana mass matrix is diagonal. One may object that we earlier mentioned that we have just chosen to work in this specific basis to solve the RGEs. This is true, but the running does not care for our choice of basis and it will introduce some off-diagonal coefficients in our initially diagonal matrix [83].

Therefore, each time we will integrate out one heavy neutrino at the corresponding threshold, the Majorana mass matrix will have to be diagonalized so that we can use the derived expressions. Besides, this procedure imposes a redefinition of the Yukawa matrix that can be written as follows:[36]

$$\begin{aligned} \overset{(n)}{M} &\equiv U \overset{(n)}{M}_D U^T, \\ \overset{(n)}{h}_\nu &\equiv U^T \overset{(n)}{h}_\nu . \end{aligned} \tag{3.20}$$

Here, U is a unitary matrix since $\overset{(n)}{M}$ is symmetric.
Furthermore, one should notice that the function in Eq. (3.11) that arose when calculating the diagram in Fig. 3.5 involves this mass matrix. Consequently, if we want the relation to hold, we need to apply the function to the whole matrix (non-diagonal during the running and diagonalised at the threshold). This subtlety also appears in the see-saw type I case. However, in that case there is merely the inverse of the mass matrix and not some more complicated function. Eventually, in the present case, for example at the first threshold between ET3 and FT, we have:

$$\overset{(4)}{h}{}_\nu^T\, f(\overset{(3)}{M})\, \overset{(4)}{h}_\nu \overset{Diagonalization}{=} \overset{(4)}{h}{}_\nu^T\, U f(\overset{(3)}{M}_D) U^T\, \overset{(4)}{h}_\nu .$$

3.3.3 Neutrino Mass Matrix

Let us recall that the whole purpose motivating the study hitherto is to compute the evolution of the neutrino mass matrix, in order to be able to run it down to low energies and eventually extract some information — one could say *prediction* — that could be tested directly at, e.g., a particle collider. Consequently, we need to write the general expression of the neutrino mass matrix throughout the stack of EFTs. One can remember that in our framework, when the electroweak symmetry is spontaneously broken and the Higgs doublet acquires a VEV, the operator $\kappa^{(22)}$ leads to a light neutrino mass term appearing as a one-loop correction of $\kappa^{(11)}$ (whose tree-level version does not give any mass for η does not get a VEV). Hence, the light neutrino mass matrix reads:

$$\mathcal{M}_\nu(\mu) = v^2 \left(\frac{\lambda_5}{16\pi^2}\right) \left[g(\mu, m_2) \overset{(n)}{\kappa}{}_{ij}^{(22)} + \overset{(n)}{h}{}_\nu^T f(M_n, m_2) \overset{(n)}{h}_\nu \right] . \tag{3.21}$$

We know that the mixing of neutrino generations implies that there is a mismatch between their mass eigenstates and their flavour eigenstates. This mismatch of basis actually proves that they are not all massless, and in order to quantize this mismatch we parametrize the relevant unitary transformation matrix with the help of the so-called *mixing parameters*. Such a matrix with such a parametrization is nothing

[36] And actually it also imposes a redefinition of the neutrino singlet fields, but this does not affect our results.

else than the well-known PMNS matrix [84]. In order to do so we can observe that at any energy scale μ, the neutrino mass matrix $\mathcal{M}_\nu$ and the charged lepton Yukawa invariant $Y_e^\dagger Y_e$ can be diagonalized by unitary transformations via [85]:

$$U_\nu^T(\mu)\,\mathcal{M}_\nu\,U_\nu(\mu) = \mathrm{Diag}(m_1(\mu), m_2(\mu), m_3(\mu))\,, \tag{3.22}$$

$$U_e^\dagger(\mu)\,Y_e^\dagger Y_e\,U_e(\mu) = \mathrm{Diag}(y_e^2(\mu), y_\mu^2(\mu), y_\tau^2(\mu))\,, \tag{3.23}$$

where U_ν and U_e are unitary matrices and the neutrino mixing matrix will then be given by:

$$U_{\mathrm{PMNS}}(\mu) = U_e^\dagger(\mu) U_\nu(\mu). \tag{3.24}$$

Henceforth, once the running of $\mathcal{M}_\nu$ (and $Y_e^\dagger Y_e$) has been obtained we just have to diagonalize it and extract the parameters from the unitary transformation matrix. A complete parametrization of the lepton mixing matrix U_{PMNS} needs three mixing angles and three CP-violating phases, and is usually chosen to be [86]:

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & -c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{23}s_{13} \end{pmatrix} \begin{pmatrix} e^{i\rho} & 0 & 0 \\ 0 & e^{i\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.25}$$

The mixing parameters extraction can then be done via:

$$\theta_{13} = \arcsin(|U_{13}|)\,, \tag{3.26a}$$

$$\theta_{12} = \begin{cases} \arctan\left(\dfrac{|U_{12}|}{|U_{11}|}\right) & \text{if } U_{11} \neq 0 \\ \frac{\pi}{2} & \text{else} \end{cases}\,, \tag{3.26b}$$

$$\theta_{23} = \begin{cases} \arctan\left(\dfrac{|U_{23}|}{|U_{33}|}\right) & \text{if } U_{33} \neq 0 \\ \frac{\pi}{2} & \text{else} \end{cases}\,, \tag{3.26c}$$

$$\delta = -\arg\left(\frac{\dfrac{U_{11}^* U_{13} U_{31} U_{33}^*}{c_{12}c_{13}^2 c_{23} s_{13}} + c_{12}c_{23}s_{13}}{s_{12}s_{23}}\right), \tag{3.26d}$$

with:

- U_{ij} the coeffficients of the U_{PMNS} matrix.
- δ, ρ, and σ the CP-violating phases.
- c_{ij} and s_{ij} defined as $\cos(\theta_{ij})$ and $\sin(\theta_{ij})$, respectively.

Chapter 4

Numerical Analysis of the RGEs and Conclusion

4.1 Framework of the Study

In this section we will present the results we obtained after performing a numerical analysis of the RGEs in the Ma-model. As can be seen from Appendix A, the RGEs are rather complex and solving the whole set at once would require too much calculation time and processor power than necessary to get a rough idea of the behaviour of the different running in in the Ma-model. That is the reason why we will just try to solve the RGEs in an approximate case, which is not purely correct but makes use of sensible approximations. In this approximate case we do the following:

- The top-quark Yukawa coupling being way larger than the other Yukawa couplings, we will assume that down-quark and charged lepton Yukawa matrices can be neglected compared to the up-quark Yukawa matrix of which only the top-quark component will be considered: (Y_d , $Y_e \ll Y_u \sim y_t$).
- The down-quark and charged lepton Yukawa matrices will be neglected compared to the gauge couplings: (Y_d , $Y_e \ll g_i$ where $i = 1, 2, 3$).
- The scalar couplings λ_i ($i = 1, \ldots, 5$) are taken to be constant.

Moreover, we also make some assumptions:

- As has been said previously, we assume that the dark scalar mass parameter (m_2) is below the masses of the heavy neutrinos ($M_{1,2,3}$), so that we can integrate these while keeping the dark scalar as dynamical particle in the model. We take the mass to be $m_2 = 200$ GeV, which is in agreement with the upper bound derived from the LHC measurements [87]: $m_2 \leq 200$ GeV.

- We assume normal ordering among the light neutrino mass eigenstates.
- We assume that there is bimaximal mixing at the GUT (Grand Unified Theory) scale.

Conceptually speaking, we will perform the study in the view of the *top-down* approach, *i.e.*, we will run the parameters from high-energy to low-energy, using input values at the GUT scale, $M_{GUT} = 10^{16}$ GeV. Finally, the set of RGEs will be solved step by step starting with the simplest non-coupled ones and then going to the coupled ones.

4.2 Gauge Couplings

As the one-loop β-functions for the gauge couplings are decoupled, they are trivially solvable *by hand*. The analytical solutions are:

$$g_1(t) = \sqrt{\frac{g_1^2(0)}{1 - \frac{7}{8\pi^2} g_1^2(0)\, t}}\,, \tag{4.1a}$$

$$g_2(t) = \sqrt{\frac{g_2^2(0)}{1 - \frac{-3}{8\pi^2} g_2^2(0)\, t}}\,, \tag{4.1b}$$

$$g_3(t) = \sqrt{\frac{g_3^2(0)}{1 - \frac{-7}{8\pi^2} g_3^2(0)\, t}}\,. \tag{4.1c}$$

As the gauge couplings are not affected by the heavy neutrinos, there will not be any threshold effects, and the input values can be chosen at any energy scale. Naturally we impose the values of the g_i's at the Z-boson mass scale (those values have been measured experimentally, see [24]). These input values fix the parameters inside (4.1) which are then equal to:

$$\begin{aligned} g_1(0) &= 0.348\,, \\ g_2(0) &= 0.677\,, \\ g_3(0) &= 1.922\,. \end{aligned} \tag{4.2}$$

The previous solutions are displayed and compared with the SM in Fig. 4.1. Note that for the U(1) gauge coupling, we plot the 'GUT renormalized' coupling: $g_1 \to \; g_1^{GUT} = \sqrt{\frac{5}{3}}\, g_1$

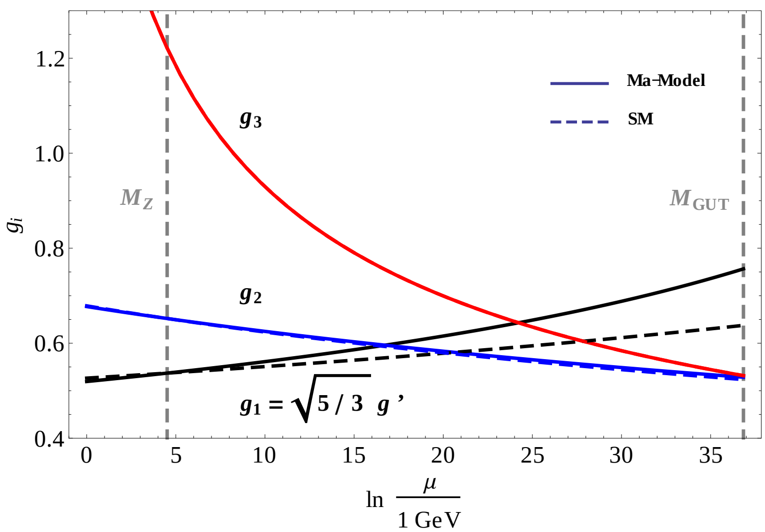


Figure 4.1: *Running of the gauge couplings in the Ma-Model and in the SM.*

4.3 Top-Quark Yukawa Coupling

Then we can solve the simplified RGE for the top-quark Yukawa coupling which is also unaffected when we integrate out the neutrino singlets. The input value is the top-Yukawa coupling (extracted from its mass) at the Z-boson mass scale, and the solution is shown in Fig. 4.2. Although the running of the coupling in itself is not particularly interesting, it is present in the other RGEs so that its running affects almost all the other couplings. We can see that the runnings are similar in the SM and in the Ma-model. This was to be expected, because the running of the top coupling is, as a first approximation, ruled by this of the strong gauge coupling, which is exactly the same in both models. However, the two curves feebly move apart from each other at high energy when the influence of the weak coupling is stronger.

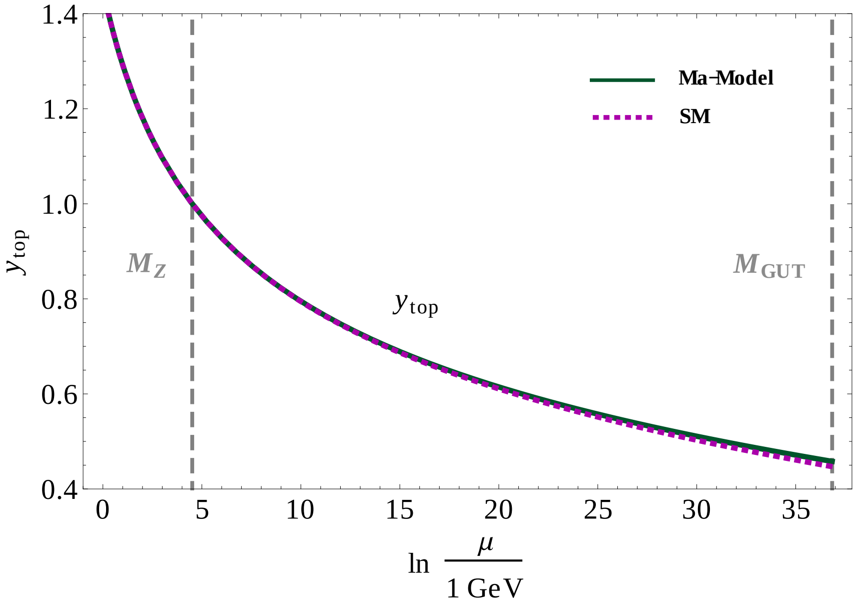


Figure 4.2: *Running of the top Yukawa coupling in the approximate model.*

4.4 Light Neutrino Yukawa Couplings

In the Ma-model, the light neutrinos couple to the dark scalar and the corresponding coupling matrix has been previously defined as h_ν. This is then a matrix of 9 complex coefficients, and therefore solving the relevant RGE implies deriving the running of $9 + 9 = 18$ real parameters. This is hard to compute analytically, but quite straightforward numerically with *Mathematica*. We said earlier that we assume bimaximal mixing at the GUT scale, and consequently the input value for the neutrino Yukawa matrix is merely a diagonal matrix, h_ν^{D}, rotated in the right amount to yield this bimaximal mixing after diagonalizing the light neutrino mass matrix. This leads to define the input neutrino Yukawa matrix as:

$$h_\nu^{\mathrm{in}}(M_{GUT}) = h_\nu^{\mathrm{D}}(M_{GUT})\, U^\dagger_{\mathrm{Bimaximal}} \,. \tag{4.3}$$

As far as the coefficients of h_ν^{D} are concerned, we can choose arbitrary values, as these couplings are new parameters introduced by the model. Nevertheless, in order to be consistent with our previous approximations where we have neglected all but the top-Yukawa coupling, these couplings have to respect:

$$y_{\mathrm{top}} \sim h_{\nu,i} > y_{\mathrm{bottom}} \sim 2.4 \times 10^{-2} \longrightarrow h_{\nu,i} \sim O(0.1)\,.$$

For illustrative purpose, the running of the diagonal coefficients is displayed in Fig. 4.3. Note that, although these couplings can in principle be complex, in our

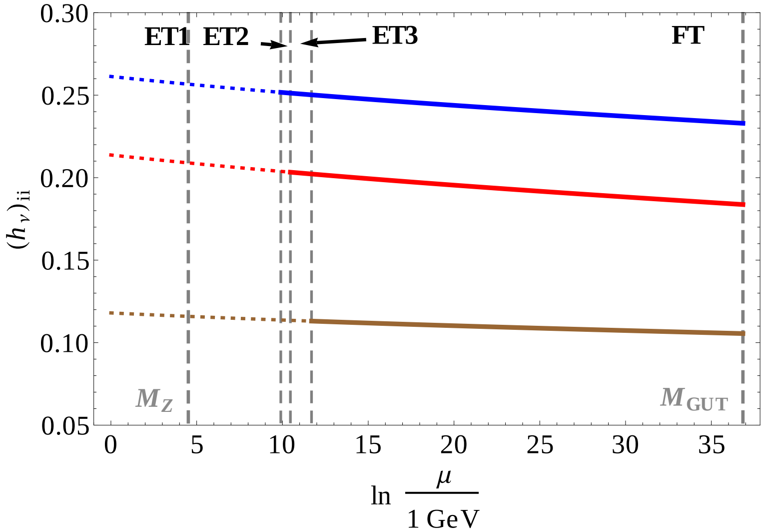


Figure 4.3: *Running of the diagonal coefficients of the light neutrino Yukawa matrix in the approximate model.*

case they are real because we assumed vanishing CP-violating phases in the PMNS matrix (and because we chose real diagonal input values for $h_\nu^{\rm D}$). Besides we can see that, at low-energy, we have represented the runnings with dashed lines. This is to emphasize that, when we successively integrate out the heavy neutrinos, the couplings related to these states cease to de defined, and thus their running does not strictly make sense. This is merely a graphical representation of what was explained in Sec. 3.3.2 regarding how the light neutrino Yukawa matrix is altered in the effective theories.

4.5 Heavy Neutrino Majorana Mass Matrix

The RGE describing the running of the heavy neutrino mass matrix is coupled only to h_ν. It is then possible to derive this running by plugging the previous solution into this RGE. As we can see in Fig. 4.4, the relative running of the mass eigenvalues is weak ($\sim 1\%$). However, absolutely speaking, it is still quite large because this represents about 1 TeV [to compare with the (current) maximal LHC energy $\sim 2 \times 4$ TeV].

In this figure we have also depicted the running in the effective theories with dashed curves, however the dashing is different from the one used in the plot for h_ν because, although the heavy states decouple, they still 'exist' and their mass parameters run

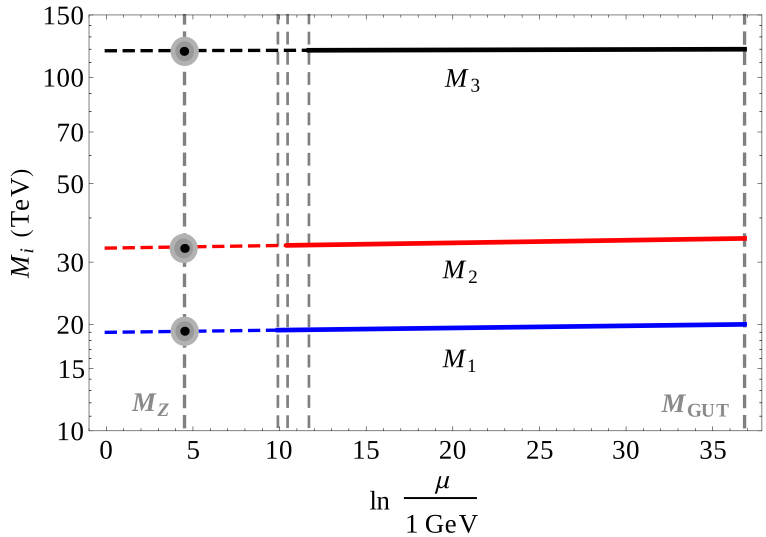


Figure 4.4: *Running of the Majorana masses of the heavy neutrinos.*

even in the effective theories. Subsequently, the masses at low-energy can be read off and they are among the predictions of the Ma-model.

4.6 Light Neutrino Parameters

The most interesting part in a model giving mass to the neutrino is what it predicts for the running of the mixing parameters and the mass square differences. Indeed the three neutrino mixing angles and the two mass square differences are the only neutrino parameters on which we currently have information from experiments. Consequently, in the view of the top-down approach, we would like the running to yield low-energy values for the mass square differences and mixing parameters which are in agreement with the global experimental fits available today. This is neither obvious nor immediate but it can be done by *tuning* the different free parameters of the model.

We then tried to obtain a *specific* scenario which yields consistent results by tuning the parameters of the model. The parameters we tune are:

- The three Majorana masses of the neutrino singlets: M_1, M_2, M_3.
- The three input coefficients of the neutrino Yukawa matrix: $h_{\nu,1}$, $h_{\nu,2}$, $h_{\nu,3}$.
- The scalar coupling λ_5.

This tuning is not trivial because first, we have no information about what starting point to choose to get consistent low-energy values, and second, some parameters

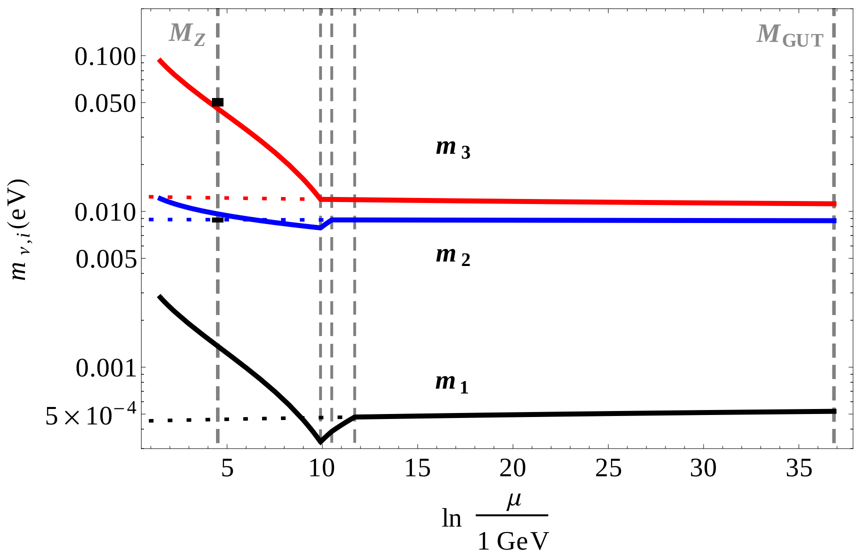


Figure 4.5: *Running of the light neutrino mass eigenstates with the '3σ' regions (black rectangles). The solid curves describe the running in the different effective theories while the dotted curves show the running without integrating out the heavy neutrinos.*

have conflicting effects. This game is therefore quite close to groping in the dark for the best coefficients to put in. The general problem encountered is that, if we want a consistent order of magnitude for the neutrino masses ($\sum m_{\nu,i} < 0.28$ eV [88]), then we need either extremely tiny couplings, which breaks our previous approximations, or tremendous right-handed neutrino masses, which would make the model lose some of its interest (namely to bring the canonical seesaw scale to TeV). The solution was to choose an extremely small value for the fifth scalar coupling: $\lambda_5 \simeq 1 \times 10^{-9}$. This being done, the remaining parameters can be adjusted to reach a somehow satisfying configuration, as displayed in Figs. 4.5 and 4.6.

From these figures, we can see that an adequate tuning of the parameters of the model can yield a configuration in which the low-energy neutrino parameters are consistent with the measurements. In both figures the intervals displayed are the 3σ regions computed from the most recent data currently available. For the mass square differences and the two large mixing angles (θ_{12} and θ_{23}), the data are taken from [89], while the data regarding θ_{13} are taken from the freshly announced results from the Daya Bay experiment [90]:

$$\theta_{12} = 34.0^{\circ}{}^{+2.9^{\circ}}_{-2.7^{\circ}}, \quad \theta_{23} = 46.1^{\circ}{}^{+7.0^{\circ}}_{-7.5^{\circ}}, \quad \theta_{13} = 8.83^{\circ}{}^{+2.1^{\circ}}_{-3.0^{\circ}},$$

$$\Delta m^2_{21}[10^{-5}\mathrm{eV}^2] = 7.59^{+0.60}_{-0.50}, \qquad \Delta m^2_{31}[10^{-3}\mathrm{eV}^2] = 2.50^{+0.26}_{-0.36}. \tag{4.4}$$

Note also that the '3σ' regions for the masses are computed by taking $m_1(M_Z)$ in the low-energy effective theory as a reference.

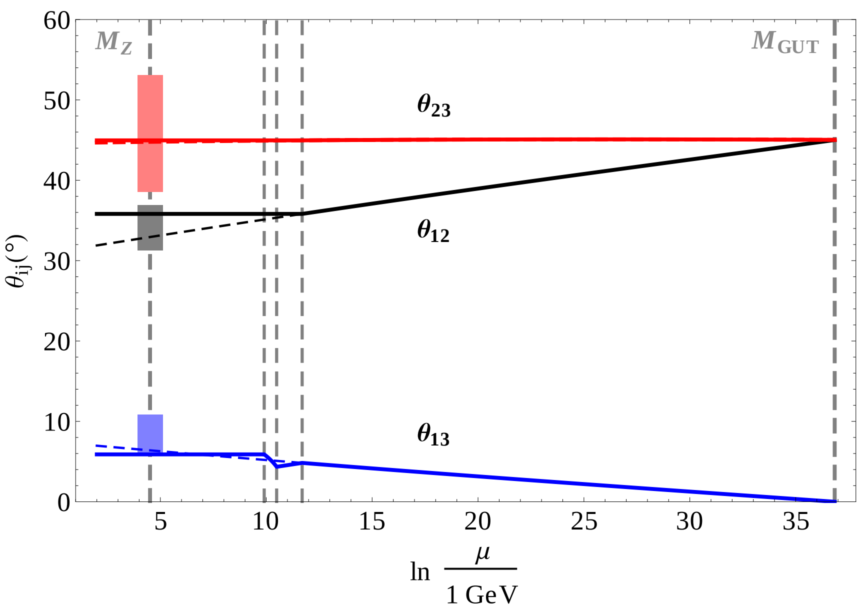


Figure 4.6: *Running of the neutrino mixing angles with the 3σ regions (light rectangles). The solid curves describe the running in the different effective theories while the dashed curves show the hypothetical running without integrating out the heavy neutrinos.*

4.7 Concluding Remarks

Finally, after solving the RGEs relevant for the Ma-model, we have shown that, by an adequate tuning of its different parameters, it is possible to obtain consistent neutrino mixing parameters at low energies. However, the matching is not perfect: the mass square differences are not perfectly respected, and the low-energy θ_{13} value (which is quite hard to fit) hits the extreme bottom of the 3σ interval. Nevertheless it yields a rather satisfactory configuration and a thorough study could certainly improve the matching. The interesting aspect is that, due to the radiative nature of the mass generation in the model, the threshold effects on the light neutrino mass eigenstates are quite important. Indeed, this is a direct consequence of the fact that the running of the effective neutrino mass matrix above and between the thresholds is given by the running of two parts. In our case the coefficients of these two parts are quite different, hereby enhancing the threshold effects. We can remark that the main influence of the thresholds is to change the slope of the running. This leads to a larger hierarchy than at high-energy for the neutrino masses, whereas in the case of the mixing angles these effects are weaker and mainly visible in the running of θ_{12} and θ_{13}. Note also that the running of θ_{23} is almost absent, which is the case in most other models like, e.g., in typical seesaw type I configurations [82]. Summing up we could say that the neutrino mass hierarchy is obtained thanks to the threshold effects, while the 'good' mixing parameters and, in particular, the 'lift of degeneracy' between θ_{12} and θ_{23} is mostly due to the running in the full theory. Another interesting aspect is the behaviour of θ_{13}: Since some of the coefficients in the RGEs do not apparently vanish for a vanishing mixing, a non-zero mixing angle is generated radiatively. And as a matter of fact we can observe that this brings this particular mixing angle to a relatively large value at low-energy, which is rather inspiring after the recent 5.2σ discovery of a non-zero θ_{13} by the Daya Bay collaboration [90].

Finally, let us recall that this is an *approximate* case, and that a more careful study would be required to make precise predictions from the Ma-model. However, this simplified study indicates that it is possible to obtain realistic mixing parameters and masses by considering the Ma-model at high-energy. In our configuration, apart from the dark scalar whose mass has been considered to be 200 GeV, the model predicts three heavy neutrino singlets with masses equal to 19.3 TeV, 34.4 TeV, and 119.1 TeV at M_Z. These masses are still out of reach of the LHC but it seems not unrealistic to think that we could discover such particles in a couple of decades.

Chapter 5

Summary and Conclusions

In this thesis, we have analysed the radiative neutrino mass generation in a specific case of the Two-Higgs Doublet Model called the Ma-Model. In this model, the neutrino mass is generated radiatively, which can explain the smallness of the neutrino mass compared to the mass of other particles. This model is particularly interesting because it provides us with possible Dark Matter candidates, as well as it gives a small mass for the neutrino, but requires only a little extension of the Standard Model. In the Ma-Model, the Standard Model is extended with one additional scalar doublet, three generations of heavy neutrinos, and an additional symmetry, which leads to a set of RGEs different from the Standard Model.
In Chapter 1, we gave an introduction on the theoretical aspects of the Standard Model and emphasized the problem regarding the introduction of the neutrino mass in this framework. In Chapter 2, we introduced the principle of radiative neutrino mass generation and gave an overview of some famous radiative models. Moreover, we presented the Ma-Model, derived the neutrino mass in this context and gave an example of the calculation we performed to derive the Renormalization Group Equations relevant for the model. The whole set of the RGEs we derived is presented in Appendix A, and some useful mathematical results are gathered in Appendix B. In Chapter 3, we spoke about the principles of the Effective Theories and introduced the Effective Operators to consider when we successively integrate out the three heavy neutrinos present in the Ma-model. In Chapter 4, we presented the results of our numerical analysis of the RGEs and, in particular, we showed that it is possible, by an adequate tuning of the parameters of the model, to obtain neutrino mixing parameters in agreement with the experiments. To conclude with, the Renormalization Group running in the Ma-model is different from the one in the Standard Model and this gives interesting results regarding the neutrino mixing parameters. However, our study was performed in an approximate case and we hope that the results of this thesis will be used to perform a more thorough analysis which could determine if this model is viable or not to explain the mass of the neutrinos.

Appendix A

Renormalization Group Equations

A.1 RGEs in the Full Theory

RGEs have been extensively studied and presented in the literature in the context of the SM and the MSSM —extended or not— the reference to these are always numerous yet often messy because of the various notations. We will refer the reader only to the following ones: [91–93], for presentation of the exhaustive set of RGE's. However in the thoretical context of our model these equations are not adequate and a new derivation is required. Below we present the set of RGEs corresponding to the model studied in this thesis. For brevity the abbreviation $\mathcal{D} = 16\pi^2\mu\frac{\mathrm{d}}{\mathrm{d}\mu}$ will be used, and the notations will be the following:

- The Y_x matrices, with $x = u, d, e$, are the Yukawa couplings we are 'familiar' with, corresponding respectively to the *up-quarks*, the *down-quarks*, and the *charged leptons*.

- h_ν is the Yukawa coupling related to the coupling between the dark scalar η and the neutrino singlet. It is similar to what is written Y_ν in the RGEs for the seesaw model. However we stick to our notation to emphasize the fact that it carries some difference with the other Yukawa couplings and to see its influence on the RGEs more easily.

- The g_i's are the gauge couplings: $\underbrace{SU(3)_C}_{g_3} \times \underbrace{SU(2)_L}_{g_2} \times \underbrace{U(1)_Y}_{g_1}$.

- M_N is the Majorana mass matrix of the neutrino singlets.

- m_1 and m_2 are the masses of the Higgs and the *dark scalar* doublets respectively.[37]
- The real parameters λ_i (with $i = 1, 2, 3, 4, 5$) are the scalar couplings.

The RGEs for the Yukawa couplings are:

$$\begin{aligned}\mathcal{D}Y_u = Y_u \Bigg\{ &\frac{3}{2}Y_u^\dagger Y_u - \frac{3}{2}Y_d^\dagger Y_d + \mathrm{Tr}\left(Y_e^\dagger Y_e + 3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d\right) \\ &- \frac{17}{12}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 \Bigg\}\,, \end{aligned} \tag{A.1a}$$

$$\begin{aligned}\mathcal{D}Y_d = Y_d \Bigg\{ &\frac{3}{2}Y_d^\dagger Y_d - \frac{3}{2}Y_u^\dagger Y_u + \mathrm{Tr}\left(Y_e^\dagger Y_e + 3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d\right) \\ &- \frac{5}{12}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 \Bigg\}\,, \end{aligned} \tag{A.1b}$$

$$\begin{aligned}\mathcal{D}Y_e = Y_e \Bigg\{ &\frac{3}{2}Y_e^\dagger Y_e + \frac{1}{2}h_\nu^\dagger h_\nu + \mathrm{Tr}\left(Y_e^\dagger Y_e + 3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d\right) \\ &- \frac{15}{4}g_1^2 - \frac{9}{4}g_2^2 \Bigg\}\,, \end{aligned} \tag{A.1c}$$

$$\mathcal{D}h_\nu = h_\nu \left\{ \frac{3}{2}h_\nu^\dagger h_\nu + \frac{1}{2}Y_e^\dagger Y_e + \mathrm{Tr}\left(h_\nu^\dagger h_\nu\right) - \frac{3}{4}g_1^2 - \frac{9}{4}g_2^2 \right\}. \tag{A.1d}$$

Next we present the RGEs for the gauge couplings. Compared to the SM, these equations remain quite similar because they do *not* depend on the Yukawa and scalar couplings but only on the number of fermion generations and the number of scalar doublets in the theory [94]. One obtains:

$$\mathcal{D}g_1 = 7g_1^3 \tag{A.2a}$$

$$\mathcal{D}g_2 = -3g_2^3 \tag{A.2b}$$

$$\mathcal{D}g_3 = -7g_3^3 \tag{A.2c}$$

One should notice that in all the equations involving the g_1 coupling (sometimes written g'), I have used the SM coupling *i.e.* the one we get from the expression:

$$g_1^2(M_Z) = 4\pi\frac{\alpha_{em}}{\cos^2(\theta_W)} \simeq 0.127\,,$$

where α_{em} is the fine structure constant and θ_W is the *Weinberg angle* at the Z boson mass scale. However, if we take into account the so-called GUT renormalization we have to replace this by $g_1^{GUT} = \sqrt{\frac{5}{3}}g_1$.

[37] These are similar to the paramater μ in the scalar potential of the Standard Model.

Regarding the scalar couplings, the RGEs have been calculated for the Standard Model and its extended version with heavy neutrinos and can easily be found in the litterature [92, 93]. In the context of a theory with two scalar doublets, there are less studies but some references give the RGEs [95, 96]. However in our really specific case implying an additional scalar doublet as well as neutrino singlets and an exact symmetry it is not sufficient and the best way to find consistent results is to consider the derivation of the RGE's in a general quantum field theory as performed in [97–99][38], [100], and [101] (in the latter no assumption regarding the number of scalar doublets is even made). Starting from these general derivations we can obtain the RGEs relevant in our case:

$$\begin{aligned}\mathcal{D}\lambda_1 = {} & 12\lambda_1^2 + 4\lambda_3^2 + 4\lambda_3\lambda_4 + 2\lambda_4^2 + 2\lambda_5^2 + \frac{3}{4}\left(g_1^4 + 3g_2^4 + 2g_1^2 g_2^2\right) \\ & - 3\lambda_1\left[g_1^2 + 3g_2^2 - \frac{4}{3}\mathrm{Tr}\left(Y_e^\dagger Y_e + 3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d\right)\right] \\ & - 4\mathrm{Tr}\left\{Y_e^\dagger Y_e Y_e^\dagger Y_e + 3Y_u^\dagger Y_u Y_u^\dagger Y_u + 3Y_d^\dagger Y_d Y_d^\dagger Y_d\right\}\,,\end{aligned} \tag{A.3a}$$

$$\begin{aligned}\mathcal{D}\lambda_2 = {} & 12\lambda_2^2 + 4\lambda_3^2 + 4\lambda_3\lambda_4 + 2\lambda_4^2 + 2\lambda_5^2 + \frac{3}{4}\left(g_1^4 + 3g_2^4 + 2g_1^2 g_2^2\right) \\ & - 3\lambda_2\left[g_1^2 + 3g_2^2 - \frac{4}{3}\mathrm{Tr}\left(h_\nu^\dagger h_\nu\right)\right] \\ & - 4\mathrm{Tr}\left\{h_\nu^\dagger h_\nu h_\nu^\dagger h_\nu\right\}\,,\end{aligned} \tag{A.3b}$$

$$\begin{aligned}\mathcal{D}\lambda_3 = {} & \left(\lambda_1 + \lambda_2\right)\left(6\lambda_3 + 2\lambda_4\right) + 4\lambda_3^2 + 2\lambda_4^2 + 2\lambda_5^2 + \frac{3}{4}\left(g_1^4 + 3g_2^4 - 2g_1^2 g_2^2\right) \\ & - 3\lambda_3\left[g_1^2 + 3g_2^2 - \frac{2}{3}\mathrm{Tr}\left(h_\nu^\dagger h_\nu + Y_e^\dagger Y_e + 3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d\right)\right] \\ & - 4\mathrm{Tr}\left(h_\nu^\dagger h_\nu Y_e^\dagger Y_e\right)\,,\end{aligned} \tag{A.3c}$$

$$\begin{aligned}\mathcal{D}\lambda_4 = {} & 2\left(\lambda_1 + \lambda_2\right)\lambda_4 + 8\lambda_3\lambda_4 + 4\lambda_4^2 + 8\lambda_5^2 + 3g_1^2 g_2^2 \\ & - 3\lambda_4\left[g_1^2 + 3g_2^2 - \frac{2}{3}\mathrm{Tr}\left(h_\nu^\dagger h_\nu + Y_e^\dagger Y_e + 3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d\right)\right] \\ & + 4\mathrm{Tr}\left(h_\nu^\dagger h_\nu Y_e^\dagger Y_e\right)\,,\end{aligned} \tag{A.3d}$$

$$\begin{aligned}\mathcal{D}\lambda_5 = {} & 2\left(\lambda_1 + \lambda_2\right)\lambda_5 + 8\lambda_3\lambda_5 + 12\lambda_4\lambda_5 \\ & - 3\lambda_5\left[g_1^2 + 3g_2^2 - \frac{2}{3}\mathrm{Tr}\left(h_\nu^\dagger h_\nu + Y_e^\dagger Y_e + 3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d\right)\right]\,.\end{aligned} \tag{A.3e}$$

[38] In these papers the calculation is also performed at the two-loop level.

The RGE required for the running of the heavy neutrino mass matrix is:

$$\mathcal{D}M_N = \left\{ M_N \left(h_\nu h_\nu^\dagger\right)^T + \left(h_\nu h_\nu^\dagger\right) M_N \right\} . \tag{A.4}$$

This result is in agreement with what can be found for example in [102].

Finally, the RGEs for the scalar masses have also been calculated for this thesis and we found:

$$\begin{aligned} \mathcal{D}m_1^2 = {} & 6\lambda_1 m_1^2 + (4\lambda_3 + 2\lambda_4)\, m_2^2 \\ & + m_1^2 \left\{ 2\mathrm{Tr}\left(Y_e^\dagger Y_e + 3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d \right) - \frac{3}{2}\left(g_1^2 + 3g_2^2\right) \right\} , \end{aligned} \tag{A.5a}$$

$$\begin{aligned} \mathcal{D}m_2^2 = {} & 6\lambda_2 m_2^2 + (4\lambda_3 + 2\lambda_4)\, m_1^2 \\ & + m_2^2 \left\{ 2\mathrm{Tr}\left(h_\nu^\dagger h_\nu \right) - \frac{3}{2}\left(g_1^2 + 3g_2^2\right) \right\} . \end{aligned} \tag{A.5b}$$

A.2 RGEs in the Effective Theories

When we integrate out the heavy parameters of the theory (chosen to be the neutrino singlets) we switch from the full theory to the tower of effective theories and the set of RGEs is altered. First here are the RGEs for the dimension-five effective operator which generates mass at low energy. In the context of the 2HDM, such an operator is potentially split into four parts [103]. However the Z_2 symmetry allows only two terms which we renormalize and present the β-functions here:[39]

$$\begin{aligned} \mathcal{D}\kappa^{(11)} = {} & -\frac{3}{2}\left\{ \kappa^{(11)}(Y_e^\dagger Y_e) + (Y_e^\dagger Y_e)^T \kappa^{(11)} \right\} \\ & + 2\mathrm{Tr}\left[Y_e^\dagger Y_e + 3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d \right] \kappa^{(11)} \\ & + 2\lambda_1 \kappa^{(11)} + 2\lambda_5 \kappa^{(22)} - 3g_2^2 \kappa^{(11)} , \end{aligned} \tag{A.6a}$$

$$\begin{aligned} \mathcal{D}\kappa^{(22)} = {} & \frac{1}{2}\left\{ \kappa^{(22)}(Y_e^\dagger Y_e) + (Y_e^\dagger Y_e)^T \kappa^{(22)} \right\} \\ & + 2\lambda_2 \kappa^{(22)} + 2\lambda_5 \kappa^{(11)} - 3g_2^2 \kappa^{(22)} . \end{aligned} \tag{A.6b}$$

[39] The result here is in agreement with the more general case computed in [104]. However, if one wants to compare that calculation with the results presented here it is important to take into account the difference in the definition of the coefficients in the scalar potential. Besides, note that the result given in [103] is wrong as pointed out in [78].

Then we can write the general form of the RGEs troughout the different effective theories. In order to do that we introduce the convenient notations already used in [83]:

- $h_\nu \longrightarrow \overset{(n)}{h}_\nu$ is the neutrino Yukawa matrix taking in account our integrating out of the neutrino singlets.
- $\kappa^{(ii)} \longrightarrow \overset{(n)}{\kappa}{}^{(ii)}$ is the effective operator in the nth effective theories.
- $M_N \longrightarrow \overset{(n)}{M}_N$ is the neutrino mass matrix of the remaining neutrino singlets.

In this section we have presented the whole set of RGEs relevant in the framework of the Ma-model and the effective theories arising from integrating out the three generations of heavy neutrinos. Although the numerical analysis of the RGEs we did in this thesis is partial, one may use these results as such to perform a more thorough and extensive analysis.

Appendix B

The Computation of Loop-Integrals

In this section we will describe the methods used for the calculation of the loop-integrals appearing in the one-loop Feynman diagrams relevant in our framework. When one speaks about this type of integrals, one often refers to them as the *Passarino-Veltman* functions. However in order to be fair it is important to recall that the expressions for all *scalar* one-loop integrals susceptible to appear in one-loop diagrams have been derived originally in January 1979 by 't Hooft and Veltman [43]. They gave the mathematical results to what they called the one-point, two-point, three-point, and four-point functions (corresponding to the number of poles in the integral), and proved that five-point and higher scalar integrals can be reduced to linear combinations of the first four. However, the loop-integrals appearing in one-loop diagrams usually have tensorial expressions involving 'slashed' momenta in the numerator, which prevents us from using straight away the nice formulas given by 't Hooft and Veltman. This difficulty has been adressed few months later in March 1979 by Passarino and Veltman [105]. In their paper one can find a systematic procedure to reduce the tensorial integrals to the previously computed scalar integrals. Consequently it would be more correct to speak about *'t Hooft-Veltman functions* and *Passarino-Veltman tensorial reduction.* Anyway in the following we will present the conventions used for the integrals we required along with some practical results in useful special cases. In the second part we will present a brief summary of the main relations and parameters relevant for the renormalization procedure. Besides, for a modern overview of the systematic procedures one can use to compute loop-integrals, see [106].

B.1 Scalar One-Loop Integrals

In all the following, μ is the renormalization scale and $\Delta = \frac{2}{\epsilon} - \ln(\pi) + \gamma_E$ with $\gamma_E = 0.5772\ldots$ the *Euler-Mascheroni* constant and $\epsilon = 4 - d$ the divergent part.[40]

B.1.0.1 One-Point Function

$$A_0(m^2) = \frac{\mu^\epsilon}{i\pi^2}\int \mathrm{d}^4k \frac{1}{k^2 - m^2} = m^2\left(\Delta - 1 + \ln\left(\frac{m^2}{\mu^2}\right)\right). \tag{B.1}$$

B.1.0.2 Two-Point Function

$$\begin{aligned} B_0(p^2, m_1^2, m_2^2) &= \frac{\mu^\epsilon}{i\pi^2}\int \mathrm{d}^4k \frac{1}{[k^2 - m_1^2][(k+p)^2 - m^2]} \\ &= \Delta - \int_0^1 \mathrm{d}x \ln\left(\frac{x(x-1)p^2 + xm_1^2 + (1-x)m_2^2}{\mu^2}\right). \end{aligned} \tag{B.2}$$

A useful limiting case is when the external momentum is taken to be zero:

$$\begin{aligned} B_0(0, m_1^2, m_2^2) &= \Delta + 1 + \frac{m_1^2}{m_1^2 - m_2^2}\ln\left(\frac{m_1^2}{\mu^2}\right) + \frac{m_2^2}{m_2^2 - m_1^2}\ln\left(\frac{m_2^2}{\mu^2}\right) \\ &= \Delta + 1 + \ln\left(\frac{m_1^2}{\mu^2}\right) + \frac{m_2^2}{m_2^2 - m_1^2}\ln\left(\frac{m_2^2}{m_1^2}\right) \\ &= \frac{m_1^2}{m_1^2 - m_2^2}\left[A_0(m_1^2) - A_0(m_2^2)\right]. \end{aligned} \tag{B.3}$$

Note that B_0 is symmetric in m_1 and m_2.

B.1.0.3 Three-Point Function

$$C_0 = \frac{\mu^\epsilon}{i\pi^2}\int \mathrm{d}^4k \frac{1}{[k^2 - m_1^2][(k+p_1)^2 - m_2^2][(k+p_1+p_2)^2 - m_3^2]}, \tag{B.4}$$

[40] Some differences in the signs can sometimes arise between the results we give and the ones derived elsewhere but it is due to the fact that in part of the literature, the Euler constant is weirdly taken to be negative (which is slightly abusive). In some cases the difference may also originate from the sign of the mass in the propagator.

where $C_0 = C_0(p_1^2, p_2^2, m_1^2, m_2^2, m_3^2)$.
And in the case of vanishing momenta:[41]

$$\begin{aligned} C_0(0,0,m_1^2,m_2^2,m_3^2) &= C_0(m_1^2,m_2^2,m_3^2) \\ &= \frac{1}{m_1^2-m_2^2}\left[B_0(0,m_1^2,m_3^2) - B_0(0,m_2^2,m_3^2)\right] \\ &= \frac{1}{m_1^2-m_2^2}\left(\frac{m_1^2}{m_1^2-m_3^2}\ln\left(\frac{m_1^2}{m_3^2}\right) + \frac{m_2^2}{m_2^2-m_3^2}\ln\left(\frac{m_2^2}{m_3^2}\right)\right). \end{aligned} \tag{B.5}$$

B.1.0.4 Four-Point Function

$$D_0 = \frac{\mu^\epsilon}{i\pi^2}\int \frac{\mathrm{d}^4k}{[k^2-m_1^2][(k+p_1)^2-m_2^2][(k+p_1+p_2)^2-m_3^2][(k+p_1+p_2+p_3)^2-m_4^2]}, \tag{B.6}$$

Where $D_0 = D_0(p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_4^2)$.
And in the case with vanishing momenta:

$$\begin{aligned} D_0(0,0,0,m_1^2,m_2^2,m_3^2,m_4^2) &= D_0(m_1^2,m_2^2,m_3^2,m_4^2) \\ &= \frac{C_0(m_1^2,m_3^2,m_4^2) - C_0(m_2^2,m_3^2,m_4^2)}{m_1^2-m_2^2}. \end{aligned} \tag{B.7}$$

One can remark that we have left the renormalization scale μ^ϵ in the expression for the three- and four-point functions, while we could have dropped it because these functions are clearly ultraviolet finite.

B.2 Tensorial Reduction

The tensorial generalization of the previous functions are written:[42]

$$B_\mu; B_{\mu\nu} = \frac{\mu^\epsilon}{i\pi^2}\int \mathrm{d}^4k \frac{k_\mu; k_\mu k_\nu}{[k^2-m_1^2][(k+p)^2-m^2]}\,, \tag{B.8}$$

$$C_\mu; C_{\mu\nu}; C_{\mu\nu\rho} = \frac{\mu^\epsilon}{i\pi^2}\int \mathrm{d}^4k \frac{k_\mu; k_\mu k_\nu; k_\mu k_\nu k_\rho}{[k^2-m_1^2][(k+p_1)^2-m_2^2][(k+p_1+p_2)^2-m_3^2]}\,. \tag{B.9}$$

Some useful relations for the tensorial reduction can be found in the appendix D of [105]. Moreover, as these computation are usually performed in the context of

[41]For further details regarding the derivation see [107].
[42]The expressions for the tensiorial integrals relative to D_0 are not shown because of a lack of space, but they are obvious.

renormalization, if one uses the MS scheme only the divergent parts of the integrals need to be removed and therefore one just needs to isolate this part. This can be done using the following relations and table B.1:

$$
\begin{aligned}
B_\mu &= p_\mu B_1 \,, \\
B_{\mu\nu} &= \eta_{\mu\nu}\, B_{00} + p_\mu p_\nu\, B_{11} \,, \\
C_\mu &= p_\mu\, C_1 + q_\mu\, C_2 \,, \\
C_{\mu\nu} &= \eta_{\mu\nu}\, C_{00} + p_\mu p_\nu\, C_{11} + (q_\mu q_\nu\, C_{22} + p_\mu q_\nu + q_\mu p_\nu)\, C_{12} \,, \\
C_{\mu\nu\rho} &= (\eta_{\mu\nu} p_\rho + \eta_{\nu\rho} p_\mu + \eta_{\mu\rho} p_\nu)\, C_{001} \\
&\quad + (\eta_{\mu\nu} q_\rho + \eta_{\nu\rho} q_\mu + \eta_{\mu\rho} q_\nu)\, C_{002} \\
&\quad + p_\mu p_\nu p_\rho C_{111} + q_\mu q_\nu q_\rho C_{222} \\
&\quad + (p_\mu p_\nu q_\rho + p_\mu q_\nu p_\rho + q_\mu p_\nu q_\rho)\, C_{112} \\
&\quad + (q_\mu q_\nu p_\rho + q_\mu p_\nu q_\rho + p_\mu q_\nu q_\rho)\, C_{122} \,, \\
D_{\mu\nu\rho\sigma} &= (\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\sigma})\, D_{0000} + \text{finite terms} \,.
\end{aligned}
$$

Table B.1: *Divergent parts of the Passarino-Veltman functions.*

Type	Function	Divergent Part
One-point	$A_0(m^2)$	$\frac{2}{\epsilon}m^2$
Two-point	$B_0(p^2, m_1^2, m_2^2)$	$\frac{2}{\epsilon}$
	$B_1(p^2, m_1^2, m_2^2)$	$-\frac{1}{\epsilon}$
	$B_{00}(p^2, m_1^2, m_2^2)$	$\frac{3m_1^2+3m_2^2-p^2}{6\epsilon}$
	$B_{11}(p^2, m_1^2, m_2^2)$	$\frac{2}{3\epsilon}$
Three-point	C_{00}	$\frac{1}{2\epsilon}$
	C_{001}	$-\frac{1}{6\epsilon}$
	C_{002}	$-\frac{1}{6\epsilon}$
Four-point	D_{0000}	$\frac{1}{12\epsilon}$

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