Gauge Theories

Based on lectures given by Mrinal Dasgupta, University of Manchester, 2015

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Contents

Chapter 1

Introduction

It is assumed that you have done the pre-requisite Quantum Field Theory course. We will go over a lot of the same material and so if you didn't understand it the first time, now is a time to understand it properly.

There are two main textbooks - textbooks are a personal choice, a book you may like, others may not - it is up to you to look through books and find which ones you like most. Some recommended books are:

- Peskin and Schroeder - this gives a very comprehensive coverage of the course and has many examples which are worth going through
- Aitchison and Hey - this is slightly simpler than Peskin and Schroeder and will be useful for the first sections of this course - anyone looking for a more Heuristic approach will enjoy this book

1.1 Introduction

What is the main motivation for studying Gauge Theories? How will this course build upon what we learnt in QFT? Albeit an ambitious goal - it is to acquire a fundamental understanding of all interactions in nature. The key word here is fundamental. This word implies that it cannot be derived form anything else, it is the basis upon which all else is based. This is only true to-date however, there are many caveats and problems in the theories we will describe.

What interactions do we know about in nature? There are four fundamental forces: strong, electromagnetic, gravity and weak. We are going to study the origin of three, writing a Lagrangian that encompasses them; the last force we don't study is a problem (gravity). There is a common framework that describes three of the forces, but not the last! At the high energies we work with, Gravity does not need to be considered. However the effect of gravity is very strong near the Plank scale (lookup experiment Bicep - Quantum fluctuations).

We are going to see the Standard Model (SM) written in SU(3) x SU(2) x U(1), this is written in terms of the underlying Gauge symmetry groups - this shows just how fundamental symmetries are to this - the whole SM is expressed in terms of underlying Gauge symmetries.

Our goal is to understand the role of symmetries (in order to do that, in the next lecture we will have to study group theory) in generating fundamental forces. You need calculus to manipulate these ideas and so we will need to develop group theoretical tools to understand how to get a calculation handle and once we turn this handle, all of the forces pop out.

We must have some starting point, this is QFT which is derived from taking quantum mechanics and trying to make it consistent with special relativity. The moment you start putting SR into QM, you are inevitably led to the framework of QFT. This is fine for describing freely propagating particles, which is not that useful in itself. You might put interactions in by hand (e.g ϕ^4), but there is an extra intuitive leap to go from QFT to say QCD or QED - this leap is Gauge symmetries. It is a leap as it is not a derivable principle - you cannot derive Gauge symmetries. We impose Gauge symmetries and this leads to interactions. This continues the idea that symmetries are intrinsically linked to the fundamental laws of nature.

For instance conservation of momentum is linked to spatial translation and invariance. It is not fundamental that momentum is conserved, there is no reason why momentum should be conserved - on a fundamental level the conservation of momentum doesn't make physical sense. However, things like translational invariance does make sense. If you do an experiment in one room and do the same experiment (under identical conditions in another room) we expect the same result - this does make sense! So the idea of invariants is sensible whereas conservation laws don't have physical reasoning as to why they should be true.

We have seen gauge invariance before in our studies of EM.

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \rho/\epsilon_0 \\
\nabla \times \mathbf{E} &= -\dot{\mathbf{B}} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dot{\mathbf{E}} \\
\nabla \cdot \mathbf{B} &= 0
\end{aligned}$$

the magnetic field is known to be written as $\mathbf{B} = \nabla \times \mathbf{A}$ where A is the magnetic potential and $\mathbf{E} = -\nabla\phi - \frac{\partial A}{\partial t}$. Now we can add to A, the gradient of a scalar.

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla\Omega$$

and this leaves $\nabla \times \mathbf{A}$ invariant and thus it leaves $\mathbf{B}$ invariant. We can leave the electric field invariant by modifying ϕ at the same time.

$$\phi \rightarrow \phi' = \phi - \frac{\partial \Omega}{\partial t}$$

Doing this at the same time as transforming $\mathbf{A}$ also leaves $\mathbf{E}$ invariant. The punchline is that when we do QFT, we are going to vary the fields by making Gauge transformations on the fields and you are going to look for a place to hide the extra terms you get - very analogous to EM gauge invariance. We will now use covariant notation for the rest of the course.

$$A^{\mu} = \left(\frac{\phi}{c}\mathbf{A}\right)$$
$$\partial^{\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right)$$

and then we can just write the Gauge transforms as

$$A^{\mu} \rightarrow A^{\mu} + \partial^{\mu}\Omega$$

One last thing we can do is to derive a wave equation. Lets start with the curl B equation

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}$$
$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times (\mathbf{A})) = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\left(-\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}\right)}{\partial t}$$
$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla}^2\mathbf{A} = \mu_0\mathbf{J} - \mu_0\epsilon_0\frac{\partial \boldsymbol{\nabla}\phi}{\partial t} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}\boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \boldsymbol{\nabla}^2\mathbf{A} = \mu_0\mathbf{J}$$

the important thing about gauges lies in the first term. This still includes phi, but we can find a gauge so that this term vanishes. This is called the Lorenz gauge $\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}$. The wave equation is $\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \boldsymbol{\nabla}^2\mathbf{A} = \mu_0\mathbf{J}$ and $\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \boldsymbol{\nabla}^2\phi = \frac{\rho}{\epsilon_0}$.

Written in covariant notation these are

$$\partial^2 A^{\nu} = \mu_0 J^{\nu}$$
$$\partial^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2$$
$$A^{\nu} = (\phi/c, \mathbf{A})$$
$$J^{\nu} = (\rho c, \mathbf{J})$$

for $\partial_{\mu}A^{\mu} = 0$ (Lorentz criterion).

Let us define as field strength tensor

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$$

this is invariant under $A_{\nu} \rightarrow A_{\nu} + \partial_{\nu}\Omega$.

$F_{\mu\nu}$ is directly related to the fields

$$F_{0i} = -\frac{1}{c}E_i$$
$$-\frac{1}{2}\epsilon_{ijk}F_{jk} = B_i$$

F is invariant and so the fields are invariant. But this is not true in QCD and corresponding **E** and **B** fields are also not invariant. Why in QCD is the field strength tensor not invariant? In QCD we have colour which means quarks cannot be seen independently, and so the field strength tensor is gauge dependent.

In terms of $F_{\mu\nu}$ the wave equations are simple (trivial)

$$\partial_\mu F^{\mu\nu} = \partial^2 A^\nu - \partial^\nu \left(\partial_\mu A^\mu\right)$$

and in the Lorenz gauge $\partial_\mu A^\mu = 0$. We have already seen what the RHS gives us and we get

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

this equation takes care of both $\nabla \times \mathbf{B}$ and $\nabla \cdot \mathbf{E}$.

1.2 Gauge Invariance of Quantum Electrodynamics

Now we have refreshed our minds of the classical case and gone through some maths, would like to start looking at quantum electrodynamics. We shall talk in terms of the Lagrange density

$$S = \int \mathcal{L} d^4 x$$

The relevant $\mathcal{L}$ that produces Maxwell's equation $\mathcal{L} = -\frac{1}{4} F_{\mu\nu F^{\mu\nu}}$. Make sure (as an exercise) you are able to use the Euler Lagrange equations to reproduce Maxwell's wave equations $\partial^2 A_\nu = \partial_\nu \left(\partial^\mu A_\mu\right)$.

$$\mathcal{L}_{QED} = \mathcal{L}_M + \mathcal{L}_{Dirac}$$

where $\mathcal{L}_{Dirac} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi$ where γ^μ are 4×4 matrices and ψ are four component spinors and $\bar{\psi} = \psi^\dagger \gamma_0$.

$$\gamma_0 = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right)$$

$$\gamma_i = \left(\begin{array}{c|c} 0 & \sigma_i \\ \hline -\sigma_i & 0 \end{array}\right)$$

I are 2×2 identity matrices, 0 are 2×2 null matrices and σ_i are Pauli matrices.

$\mathcal{L}_{Dirac}$ produces the equation of motion for electrons

$$\left(i\gamma \cdot \partial - m\right)\psi = 0 \left(\not{p} - m\right)\psi = 0$$

where in shorthand $\gamma \cdot \partial = \gamma^\mu \partial_\mu$.

We wish to impose a symmetry under transformations of Dirac fields

$$\psi \to \psi' = e^{i\theta}\psi$$

we are going to make this transformation to the fields and say that physics should be invariant under this change of fields. θ can be constant where it doesn't depend on space-time and is called a global transformation or $theta = \theta\left(x^\mu\right)$ and so does depend on space-time, this is called a Gauge (local) transformation. The consequence of invariance under a global transformation is a conserved current (Noether's theorem). The consequence of invariance under a Gauge transformation is what we are going to study.

Let us take $\theta(x) = e\Lambda(x)$ where e is the electron charge and so $\psi \to \psi' = e^{ie\Lambda(x)}\psi$. Let us study the impact of this on $\mathcal{L}$ where from now on $\mathcal{L}$ is taken to mean $\mathcal{L}_{Dirac}$.

$$\begin{aligned}
\mathcal{L} &\to \bar{\psi}' (i\gamma \cdot \partial - m) \psi' \\
\mathcal{L} &\to e^{-ie\Lambda}\bar{\psi} (i\gamma \cdot \partial - m) e^{ie\Lambda}\psi \\
&= \bar{\psi}e^{-ie\Lambda} i \left(\gamma \cdot \partial\psi e^{ie\Lambda} + \gamma \cdot \partial \left(e^{ie\Lambda}\right) \psi\right) - m\bar{\psi}\psi \\
&= \bar{\psi}e^{-ie\Lambda} i \left(\gamma \cdot \partial\psi e^{ie\Lambda} + (\gamma \cdot \partial\Lambda) \left(iee^{ie\Lambda}\right) \psi\right) - m\bar{\psi}\psi \\
&= i\bar{\psi}\gamma \cdot \partial\psi - e\bar{\psi} (\gamma \cdot \partial\Lambda) \psi - m\bar{\psi}\psi
\end{aligned}$$

with $\bar{\psi}' = e^{-ie\Lambda x}\bar{\psi}$. The consequence of a gauge transformation is simple. There is an additional term as taking the partial derivative of the transformation field will produce an extra term with a derivative of Λ. This is exactly the same as we had before, expect for the one problematic piece $e\bar{\psi} (\gamma \cdot \partial\Lambda) \psi$ which is a new term.

We need something that absorbs this term, that absorbs the gradient of a scalar without changing physics. It suggests to us (from what we have done in EM) that we introduce the vector potential A_μ and thus a photon field.

We have introduced the Dirac fields and looked at what happens when transformed under a gauge transformation and observed an additional term. We must introduce a photon field (which can absorb the $\partial_m u\Lambda$ terms) that interacts with electrons (described by the Dirac spinors) to keep physics the same under a gauge transformation. Otherwise when we applied the gauge transform to the Dirac Lagrangian we would change the physic! this is not plausible. The photon field is a required necessity to make the Dirac equation invariant (and thus ensure physics is invariant) under a gauge transformation.

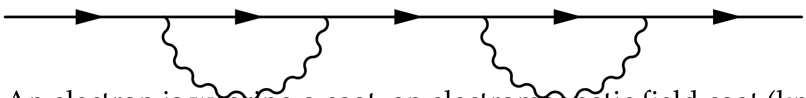
So we write a Lagrangian that looks like

$$\mathcal{L} = \bar{\psi} \left(i\gamma^\mu \left(\partial_\mu - ieA_\mu\right) - m\right) \psi$$

then if we make the transformations $\psi \to e^{ie\Lambda}\psi$ and $A_\mu \to A_\mu + \partial_\mu\Lambda$. $\partial_\mu - ieA_\mu$ is known as the covariant derivative D_μ and $D_\mu\psi$ transforms exactly like ψ. $D_\mu\psi \to D'_\mu\psi' = \left(\partial_\mu - ieA'_\mu\right) e^{ie\Lambda}\psi = e^{ie\Lambda}D_\mu\psi$ with $A'_\mu = A_\mu + \partial_\mu\Lambda$.

$$\bar{\psi}\gamma^\mu \left(iD_\mu - m\right) \psi$$

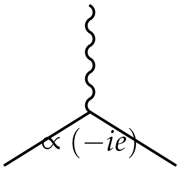
stays invariant. This corresponds to the fact that physical electrons always have photon interactions i.e always have an EM field coat.



An electron is wearing a coat, an electromagnetic field coat (known as its proper field). When an electron for instance interacts with another electron, you smash the EM coat and splinter this field (into a highly virtual state - missing certain components of its field). To become a physical electron again it must regain its proper field through emission of radiation.

$$\mathcal{L}_{QED} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

For derivation of Feynmann rules one point to note is that this Lagrangian contains a term $e\bar{\psi}\gamma^{\mu}A_{\mu}\psi = J^{\mu}A_{\mu}$ with J^{μ} the Dirac current $\bar{\psi}\gamma^{\mu}\psi$. This term tells us that the Dirac field is interacting with the photon field.



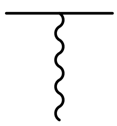
This Lagrangian is not however complete as one piece of the Lagrangian is still missing. One cannot derive a photon propagator from this Lagrangian. Actually we need a gauge-fixing term $-\frac{(\partial\cdot\Lambda)^2}{2\xi}$. ξ is an arbitrary parameter. This is required to derive the photon propagator

$$-i\left(g_{\mu\nu} - (1-\xi)\frac{P_{\mu}P_{\nu}}{P^2}\right)\frac{1}{P^2}$$

where $\xi = 1$ corresponds to the Feynmann gauge. The reason this gauge-fixing is needed is because you have too many degrees of freedom in the Lagrangian. You have a four potential with four components, but the photon has only two polarisation states. In the end current conservation shows that $(1-\xi)\frac{P_{\mu}P_{\nu}}{P^2}$ term drops out and no matter what choice of gauge you make, the physical interpretation (squared amplitude) is the same in the end - the physical behaviour is invariant as we require.

1.3 Feynmann Rules for QED

(You can just look these up in a book - but be consistent as different books use different conventions).

For every	Draw	Write
Internal photon line	μ ν	$= -i\frac{g_{uv}}{p^2+i\epsilon}$
Internal fermion line	α  β α P β	$= -i\frac{g_{uv}}{p^2+i\epsilon}$
Vertex		$-ie(\gamma^{\mu})_{\alpha\beta}$
Outgoing electrons		$= \bar{u}_{\alpha}(s,p)$
Incoming electrons		$= u_{\alpha}(s,p)$
Outgoing positrons		$= v_{\alpha}(s,p)$
Incoming positrons		$= \bar{v}_{\alpha}(s,p)$

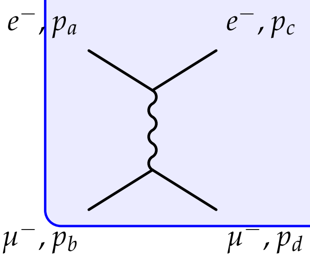
Exercise

Calculate electron-muon scattering in the high energy limit. Use QED Feynmann rules to compute the square matrix element for e^-, μ^- scattering

$$s = (p_a + p_b)^2 = (p_c + p_d)^2 \tag{1.1}$$
$$t = (p_a - p_c)^2 = (p_b - p_d)^2 \tag{1.2}$$
$$u = (p_a - p_d)^2 = (p_b - p_c)^2 \tag{1.3}$$

for $s, u \gg m_e^2, m_\mu^2$. This leads to

$$\frac{1}{4}\sum_{\text{spin}} |M_{fi}|^2 = \frac{2e^4}{t^2}(s^2 + u^2) \tag{1.4}$$



Chapter 2

Group Theory and Lie Groups

Lecture 2

2.1 Introduction to group theory and Lie groups

If you remember what we did last time - we have studied simple gauge transformations which are of the form $\psi \to \psi' = e^{i\theta(x)}\psi$. The interaction of photons and electrons appeared as a consequent for the demand of gauge invariance.

When you talk about other interactions in nature, this gauge transformation is not enough to produce the features of these interactions. For other interactions in the Standard Model, there are some differences (of course otherwise you wouldn't get different interactions). An example is for instance QCD, quark colour charge is not like an electric charge and can be thought of as a vector with three components.

So you can think of a quark as a 3-component vector in colour space and can write the quark wave function as a triplet with the basis vectors red,blue and green

$$\psi_C = \begin{pmatrix} \psi_r \\ \psi_b \\ \psi_g \end{pmatrix} \tag{2.1}$$

this 3-vector will require 3×3 matrices to rotate - transformations in colour space are effected by 3×3 matrices. These matrices look like

$$U = e^{i\theta^a t^a} \tag{2.2}$$

where $a = 1, \ldots 8$ (summed over). We will use summation convention for the rest of the course. These 3×3 t^a matrices form a mathematical entity called the $SU(3)$ group. These are not just arbitrary 3×3 matrices, they have well defined properties and as such form a group. We will first discuss discrete groups before going onto lie groups (which are continuous groups).

Note when you want to look at gauge invariances, you make θ^a a function of spacetime (function of x) and look at the consequences.

2.2 Group Theory Introduction

This is useful is far more than just particle physics and has application in many areas of physics. The main application outside of gauge symmetry is that we know symmetries are linked to powerful physical laws. So we need a 'language' for symmetries.

We would like to transform an object and know if it is invariant under said transformation. We need a single framework to describe rotating an equilateral triangle for instance which also accounts for conducting transformations in field theories. Such a language that lets us deal with a range of transformations within a unified framework is group theory.

2.2.1 What is a group? (Definition)

A group is a set of objects that are combined together using a particular rule (which we will call "composition law" a.k.a group multiplication) such that certain axioms are respected. We have a collection of objects with the property that when we combined them, certain rules must be followed - the combination rule will always be specified.

Lets consider a set of objects

$$G = \{a, b, c, \dots\} \tag{2.3}$$

we have a set of elements G that for each pair, say $a, b \in G$ the composition law assigns an element c. We have that $a \cdot b = c$, this dot is a symbol for group multiplication - it tells us that when we combine a and b we get a third element c. We must check that the axioms are obeyed in this combination. These axioms are

G-1) Closure property
$c \in G, \qquad \forall a, b \in G$

G-2) Associative property
$a \cdot (b \cdot c) = (a \cdot b) \cdot c$

G-3) Existence of an identity element, e
$a \cdot e = e \cdot a = a \qquad \forall a \in G$

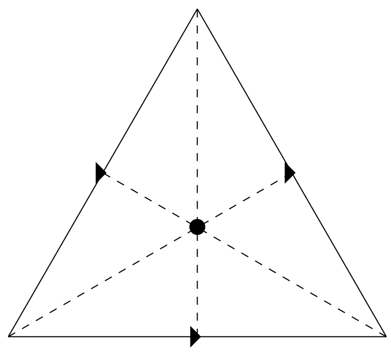
G-4) The existence of an inverse for each element in the group, $a^{-1} \in G, \qquad \forall a \in G$
this holds such that $a^{-1} \cdot a = a^{-1} \cdot a = e$.

Note that we do need $a \cdot b = b \cdot a$, but in cases where this is true (the elements do commute), the groups are known as an Abelian group. We know that all of the differences in QCD (such that quarks are always confined in particles, protons etc.) are down to the non-Abelian nature of QCD.

2.2.2 Examples

Groups can be thought of as either discrete (with a finite number of elements) or continuous (with an infinite number of elements).

C_n are the group of rotations through an angle $\frac{2\pi}{n}$ and we can study the special case of C_3 for example (rotation through angle $\frac{2\pi}{3}$). C_3 is the symmetry group of an equilateral triangle with direction sides. The reason we gave direction to the sides is because there will be extra symmetries that come about if you flip sides - this is know as the D_3 group of which the C_3 group is a subset.



C_3 consists of elements $\{e, c, c^2\}$ where c is a rotation of $\frac{2\pi}{3}$.

Lets check the axioms

$$\begin{aligned} c \cdot c^2 &= e, \qquad c^2 \cdot c^2 = c \text{ (closure)} \\ (c^2)^{-1} &= c \\ c^{-1} &= c^2 \end{aligned}$$

Axioms are all satisfied

We have been talking about these transformations as abstractions, but we can also give it concrete form such as $\frac{2\pi}{3}$ you can either talk about group theory as an abstract transformation or use a concrete mathematical form.

In large calculations it can be helpful to look at things in group representation in an abstract way or select a basis in order to be more specific.

2.2.3 Group representations

In order to practically use groups but need to give the abstract elements and the objects to be rotated a concrete mathematical realisation.

we need a concrete realisation for the object to be rotated as well as the group elements.

We draw inspiration from the rotation of a vector (this is essentially what group theory is all about - if you can do the rotation of a vector you can most of group theory, its really that simple):

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.4}$$

this is the starting point for a group representation. I need to be able to write the objects I want to rotate as a vector space or group elements of a vector, and to do this we must choose a basis. Group theory is about choosing a basis and writing the abstract realisation of a transformation as a matrix in this basis.

This gives us the idea to represent group elements by matrices and too write the objects that are transformed as "vectors" by defining a suitable vector space.

Example: Lets do this for C_3. We can represent C_3 as a set of matrices

$$C_3 \to \left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{D(e)}, \underbrace{\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}}_{D(c)}, \underbrace{\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}}_{D(c^2)} \right\} \tag{2.5}$$

these represent rotations by the angles of the C_3 group. This is a special case of the general idea of group representation.

Definition An n-dimensional representation of a group is a mapping of elements to the group of $n \times n$ non-singular matrices (the fact they are non-singular is important), $g \to D(g)$ such that group multiplication is preserved. This means that the matrix that represents g_1 times g_2 ($D(g_1, g_2)$) is the product of the matrices representing g_1 and g_2 ($D(g_1, g_2) = D(g_1)D(g_2)$).

As an exercise check this! You can always check that the matrix which represents the identity (i.e the identity matrix) obeys $D(c) = D(c) \cdot D(c^2)$.

The matrices must be invertible since the matrix which represent g inverse, $D(g^{-1})$, is the inverse of the matrix which represents g, $[D(g)]^{-1}$, and thus $D(g^{-1}) = [D(g)]^{-1}$.

$$D(g \cdot g^{-1}) = D(e) = I$$
$$D(g) \cdot D(g^{-1}) = I \Rightarrow D(g^{-1}) = D^{-1}(g)$$

But the representations are not unique. This is because the basis choice is not unique. As you can pick from an infinite number of basis to describe the vectors and matrices you have an infinite number of representations (as matrices are just the representation of an object in a particular basis).

2.3 Continuous Groups

Group elements are continuous parameters e.g. the full rotation group representing rotations through any angle θ / represented by a matrix $D(\theta)$ which has an infinite number of elements.

This brings us to the definition of a Lie group. But first lets look at some examples

2.3.1 Examples

SO(N) is the group of rotations in N dimensions through any angle θ. SO(2) is a 2-dimensional rotation and SO(3) is a 3D rotation and so on. We have all met examples of these types of rotations and even met physical quantities related to these rotations; particularly SO(3). The O stands for orthogonal, so the group of rotations must be orthogonal to one another i.e matrices orthogonal to each other. They must be orthogonal to preserve the length of vectors.

Orthogonality

$$\begin{aligned} x \to x' &= Dx \\ (x')^T &= x^T D^T \\ (x')^T x' &= x^T D^T D x = x'x \to D^T D = I \end{aligned}$$

where $D^T = D^{-1}$. The "S" in SO(N) denotes special group, which means that they have unit determinant. In particular reflections are excluded.

One very important concept in groups and lie groups is the concept of a generator - once defined we talk about generators solely from now on. Generators are the building blocks of the group and are a central concept in the context of Lie groups. This is related to infinitesimal transformations - rather than talk about discrete rotations we talk of infinitesimal rotations from which you can build up the full rotations - also these are related to physical quantities. You take an infinitesimal rotation, take the Taylor expansion and first term is your generator. Take SO(2) for instance.

SO(2) matrix is

$$\begin{aligned} R(\phi) &= \begin{pmatrix} cos\phi & -sin\phi \\ sin\phi & cos\phi \end{pmatrix} = \begin{pmatrix} 1-\frac{\phi^2}{2!} & -\phi \\ \phi & 1-\frac{\phi^2}{2!} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \phi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \mathcal{O}(\phi^2) \\ &= I - iX\phi + \mathcal{O}(\phi^2) \end{aligned}$$

where $-iX = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, X is called the generator (where the full rotation can be built up from these infinitesimal rotations) by exponentiating the generator. Here there is just a single generator.

$$R(\phi) = e^{-iX\phi} \tag{2.6}$$

(verify this at home). We should also write that $-iX = \left.\frac{dR}{d\phi}\right|_{\phi=0}$. Here there is just one axis of rotation but in the case of QCD there are 8 axes of rotation and so you will have to sum over 8 generators. Lets talk about how these generators are linked to physical things.

SO(3) generators First identify the matrices that represent rotations about 1, 2 and 3 (x, y, z) axes. These are basically the same as SO(2).

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\phi & -sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \quad R_2 = \begin{pmatrix} cos\phi & 0 & sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}, \quad R_3 = \begin{pmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.7}$$

The generators are

$$-iX_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad -iX_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad -iX_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.8}$$

Exercise: check that they satisfy $[X_i, X_j] = i\epsilon_{ijk}X_k$ and thus rotations about different axes don't commute you can check this physically by rotating a book.

A general rotation about any general (and arbitrary) direction $\hat{\mathbf{n}}$ is $R(\phi) = e^{-i\hat{\mathbf{n}}\cdot\mathbf{x}}\phi$ where $\mathbf{x} = (x_1, x_2, x_3)$.

This commutation relation ($[X_i, X_j] = i\epsilon_{ijk}X_k$) satisfied by generators is called a Lie algebraic and the groups are Lie groups. For every group there is an algebra that characterises that group.

Lets now talk about the physical realisation of SO(3).

2.3.2 Physical realisation of SO(3)

We have the met the algebra in the context of orbital angular momentum. What is orbital angular momentum, this is nothing but a reflection of the way the wave-function transforms under rotations. If you have the Hydrogen GS and you rotate it, the wave-function doesn't change (thus angular momentum is 0), if you have higher states (with the Dumbell shapes) you have definite effects after rotation and definite angular momentum.

The angular momentum operators are just the generators of rotations. Angular momentum exists in classical mechanics and as such you can write the angular momentum as a differential operator in real space (the differential for the z-axis is $-i\hbar\frac{\partial}{\partial\phi}$ and if this is zero, then there is non angular momentum in the z-direction).

Lets consider a wave-function ψ and an infinitesimal rotation through $\delta\phi$. Then

$$\psi \rightarrow \psi' = (I - iX\delta\phi + \mathcal{O}(\delta\phi)^2)\psi \tag{2.9}$$

where $-iX$ is the generator. Thus

$$\delta\psi = -iX\delta\phi\psi + \mathcal{O}(\delta\phi)^2 \tag{2.10}$$

so in the limit that $\delta\phi$ goes to zero

$$\frac{\delta\psi}{\delta\phi} = \frac{\partial\psi}{\partial\phi} = -i\hat{X}\psi \tag{2.11}$$

and so $-i\hat{X} \sim \frac{d}{d\phi}$.

There are 2×2 matrices which satisfy this algebra and we have seen these before <u>SU(2)</u> This is closely related to SO(3). SO(3) relates to orbital angular momentum and this has a classical analogue $D(\phi + 2\pi) = D(\phi)$ (you will not find a classical analogue for spin - spin is purely a 2×2 matrix algebra and that should tell you something). This requirement means that only integer values of the orbital angular momentum quantum number m_l are allowed. Recall the spherical harmonics used for describing Hydrogen, $Y_{lm}(\theta, \phi) \sim e^{im_l\phi}$, the exponential describes the fact that an increase in phase of 2π must bring us back to the same value of the wavefunction if m_l is an integer.

If instead we allow also $\frac{1}{2}$ integer values we get another group which is SU(2). This tells us that spin-space has a periodicity of 4π rather than 2π as in angular momentum-space. Otherwise it shares the same algebra as SO(3). The generators satisfy $[X_i, X_j] = i\epsilon_{ijk}X_k$. Here the X_i generators are just $X_i = \frac{1}{2}\sigma_i$, where σ_i are the Pauli matrices. These matrices are traceless and Hermitian. These matrices act on two-component spinors

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} \tag{2.12}$$

which can be expressed as a superposition of basis vectors, these are taken to be the eigenstates of $\hat{X}_3 = \frac{1}{2}\sigma_3$ with eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$ for $\begin{pmatrix}1\\0\end{pmatrix}$ (spin up) and $\begin{pmatrix}0\\1\end{pmatrix}$ (spin down).

We can write a general SU(2) rotation as

$$U(\theta) = e^{-\frac{1}{2}\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\theta} \tag{2.13}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and $\hat{\mathbf{n}} = (n_x, n_y, n_z)$ with $n_x^2 + n_y^2 + n_z^2 = 1$. Note that U is unitary $U^\dagger = U^{-1}$ (linked to the Hermiticity of σ_i). Also U has a unit determinant linked to the fact that σ is traceless (prove at home). We can also write $U = Icos(\frac{1}{2}\theta) - i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}sin(\frac{1}{2}\theta)$ (again prove at home).

Lastly, we will write a general rotation in SU(2) looks like $\psi \to \psi' = U\psi$ and $\psi_a \to \psi'_a = U_{ab}\psi_b$ (sum over $b = 1,2$) and $U \in SU(2)$.

Another physical realisation is isospin or flavour. The flavour part of the quark wavefunction can be expressed in exactly the same way as the spin part, in general as a two-component spinor

$$q_{flavour} = \begin{pmatrix}u & d\end{pmatrix} = u\begin{pmatrix}1\\0\end{pmatrix} + d\begin{pmatrix}0\\1\end{pmatrix} \tag{2.14}$$

where the first term on the RHS is the isospin up term and the second is the isospin down term and $(1,0)^T$ and $(0,1)^T$ are u and d quarks, eigenstates of I_3. So a proton u,u,d and neutron u,d,d are related by an SU(2) rotation. This explains why they have roughly the same mass, QCD is the main contribution to their mass and it doesn't care about forces.

SU(3) is special unitary group where the fundamental representation consists of 3×3 matrices rotating a 3 component spinor. Therefore the colour part of the wavefunction

$$\psi_C = \begin{pmatrix}\psi_r\\ \psi_b\\ \psi_g\end{pmatrix} \tag{2.15}$$

rotated by 3×3 matrices with hermitian and traceless generators. For SU(N) there are $N^2 - 1$ generators e.g. 3 for SU(2).

Lecture 3

2.4 Lie Groups SU(3)

Gauge group that gives us the strong force.

Structure constants appear in commutator i.e ϵ_{ijk}.

Special unitary group fundamental representation consists of 3×3 matrices that rotate a 3-component spinor.

Chapter 3

SU(3) Group

Lecture 3 contd.

The SU(3) group is a special unitary group where the fundamental representation consists of 3×3 matrices rotating a 3 component spinor (there are 3 dirac spinors).

The physical realisation is quark colour

$$\psi_C = \begin{pmatrix} \psi_r \\ \psi_b \\ \psi_g \end{pmatrix} \tag{3.1}$$

The rotation is unitarity and the generators of this fundamental representation are Hermitian, traceless 3×3 matrices (you can have higher matrices if you put several quarks together. 3 times 3 is the most basic case with one gluon). For SU(N) it can b shown that there are $N^2 - 1$ generators.

This follows from the fact that fundamental representation has $N \times N$ (complex) matrices which in turn have $2N^2$ real numbers (or degrees of freedom).

Unitarity imposes N^2 constraints on these numbers and the fact that the determinant of a unitary matrix is equal to 1, imposes 1 more constraint. Thus there remains

$$2N^2 - (N^2 + 1) = N^2 - 1 \tag{3.2}$$

degrees of freedom.

SU(N) has $N^2 - 1$ degrees of freedom. We can think of these as an $N^2 - 1$ rotation angles $\{\theta_1, \theta_2, \ldots, \theta_{N^2-1}\}$ about $N^2 - 1$ directions, each rotation is associated with/represented by a generator t^a where $a = 1, \ldots, N^2 - 1$.

E.g for SU(2) we had $U(\theta) = e^{-\frac{i}{2}\mathbf{n}\cdot\boldsymbol{\omega}\theta}$. For SU(3) there are 8 angles each corresponding to a rotation axis and hence 8 generations t^a where $a = 1, \ldots, N^2 - 1$. A general rotation is

$$U = e^{i\theta^a t^a} \tag{3.3}$$

where $t^a = \frac{1}{2}\lambda^a$ and λ^a are the Gell-Mann matrices of which there are 8

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots \text{etc.} \tag{3.4}$$

The commutation relations can be expressed as before as

$$[t^a, t^b] = i f^{abc} t^c \tag{3.5}$$

where f^{abc} are the structure constants. These are completely antisymmetric under exchange of indices.

It is conventional to choose the normalisation of the t_a matrices so that

$$Tr(t^a t^b) = Tr\delta^{ab} = \frac{1}{2}\delta^{ab} \tag{3.6}$$

With this choice we have

$$\sum_a (t^a)_{ij}(t^a)_{jk} = \sum_a (t^a)^2 \tag{3.7}$$

$$= C_F \delta_{ik} \tag{3.8}$$

where δ_{ik} is a diagonal matrix and $C_F = \frac{N^2-1}{2N} = \frac{4}{3}$ in SU(3). Later we shall discuss the physical meaning of the Casimir operator. C_F is known as the colour charge.

3.1 General points about Lie Groups

1) **Casimir Operator**
The sum of squares of the generators $\sum_a (t^a)^2$ commutes with all generators t^a and hence (by Schurs's lemma) is proportional to I.

As an example for SU(3)

$$\hat{J} = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 \tag{3.9}$$

with $[J^2, J_z] = 0$.

2) **Adjoint representation**
The structure constants of a group also obey the commutation relations and hence are generators themselves. The representation generated by structure constants is called the adjoints representation i.e we can map the generators

$$T_\alpha \to (D_A(T_\alpha))_{\beta\sigma} = i f_{\alpha\beta\sigma} \tag{3.10}$$

these are 8×8 matrices which rotate an 8-component spinor (SU(3) vectors). The Casimir operator in the adjoint representation is therefore

$$-\sum_\alpha f_{\alpha\beta\sigma} f_{\alpha\sigma\beta} = C_A \delta_{\beta\delta} \tag{3.11}$$

where $C_A = N = 3$ for SU(3). C_A is teh total colour charge for gluons.

Next we shall consider gauge transformations effected by these more general groups and discuss the physical features that emerge.

3.2 Non-Abelian gauge invariance

Let us consider the case of free quarks. Perform a rotation the quark in colour space and demand invariance under this transform: this will be an SU(3) rotation.

In Diracs Lagrangian $\bar{\psi}_i(i\partial\!\!\!/ - m)_{ij}\psi_j$, where i, j are colour indices, the essential piece is $\bar{\psi}\partial_\mu\psi$, so we focus on it - let us rotate it

$$\psi \to \psi' = U\psi \tag{3.12}$$

where U is a 3×3 matrix (in SU(3)), ψ is a 3-component vector and U is also a function of space-time.

$$\psi \to U\psi \Rightarrow \bar{\psi} \to \bar{\psi}U^\dagger \tag{3.13}$$

as $\bar{\psi} = \psi^\dagger\gamma^0$.

$$\begin{aligned}
\bar{\psi}\partial_\mu\psi \to \bar{\psi}U^\dagger\partial_\mu(U\psi) \\
&= \bar{\psi}U^\dagger(\partial_\mu)\psi + \bar{\psi}U^\dagger U\partial_\mu\psi(U^\dagger U = I) \\
&= \bar{\psi}\partial_\mu\psi + \bar{\psi}U^\dagger(\partial_\mu U)\psi
\end{aligned}$$

We see a term involving the gradient of a scalar $\partial_\mu U = \partial_\mu e^{i\theta^a t^a}$, involves $t^a\partial_\mu\theta^a e^{i\theta^a t^a}$ (need 8 analogues of photons). This motives the introduction of a gauge field

$$A_\mu \equiv t^a A^a_\mu \tag{3.14}$$

where $a = 1 \ldots 8$ in SU(3). This is sum over 8 fields A^a_μ or 8 gluon potentials. Then we have the covariant derivative

$$\bar{\psi}(\partial_\mu - iqA_\mu)\psi \tag{3.15}$$

This transforms to $\bar{\psi}'(\partial_\mu - iqA'_\mu)\psi'$, which leads to $\bar{\psi}\partial_\mu\psi + \bar{\psi}U^{-1}(\partial_\mu U)\psi - iq\bar{\psi}U^{-1}A'_\mu U\psi$.

To recover $\bar{\psi}(\partial_\mu - iqA_\mu)\psi$, we set $A'_\mu \to UA_\mu U^{-1} - \frac{i}{q}(\partial_\mu U)U^{-1}$.

For the QED case $U = e^{i\theta}$ and we recover the old result we have introduced an A field so that $D_\mu\psi \to D'_\mu\psi' = UD_\mu\psi$ so that $\bar{\psi}D_\mu\psi$ is invariant.

3.2.1 Field Strength Tensor

Let us go back to QED where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{i}{e}[D_\mu, D_\nu]$.

For a non-abelian theory

$$\frac{i}{q}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - iq[A_\mu, A_\nu] \tag{3.16}$$

and

$$\tilde{F}_{\mu\nu} = \frac{i}{q}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - iq[A_\mu, A_\nu] \tag{3.17}$$

Let us express $\tilde{F}_{\mu\nu}$ in terms of individual gluon field strenght tensors $F^a_{\mu\nu}$.

$$\tilde{F}_{\mu\nu} = t^a F^a_{\mu\nu} \tag{3.18}$$

Let us work out $F^a_{\mu\nu}$

$$\begin{aligned}
\tilde{F}_{\mu\nu} &= t^a\partial_\mu A^a_\nu - t^a\partial_\nu A^a_\mu - iq[t^b A^b_\mu, t^c A^c_\nu] && (3.19)\\
&= t^a\partial_\mu A^a_\nu - t^a\partial_\nu A^a_\mu - iqA^b_\mu A^c_\nu i f^{bca} t^a && (3.20)\\
&= t^a(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + qf^{bca}A^b_\mu A^c_\nu) && (3.21)\\
&= t^a(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + qf^{abc}A^b_\mu A^c_\nu) && (3.22)\\
& && (3.23)
\end{aligned}$$

$$3 \otimes 3 = 8 \oplus 1 \tag{3.24}$$

where there are 8 colours and 1 singlet (bbbar + ggbar + rrbar) and

$$2 \otimes 2 = 3 \oplus 1 \tag{3.25}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A_m u^b A^c_\mu \tag{3.26}$$

$\tilde{F}_{\mu\nu}$ is not gauge invariant (contrast to QED) under gauge transformation

$$\tilde{F}_{\mu\nu} = \frac{i}{q}[D_\mu, D_\nu] \to \frac{i}{q}[D'_\mu, D'_\nu] \tag{3.27}$$

But $D'_\mu \to UD_\mu U^{-1}$ hence $\tilde{F}_{\mu\nu} \to \frac{i}{q}[UD_\mu U^{-1}UD_\nu U^{-1} - UD_\nu U^{-1}UD_\mu U^{-1}] = \frac{i}{q}U[D_\mu, D_\nu]U^{-1}$.

Hence $\tilde{F}_{\mu\nu} \to U\tilde{F}_{\mu\nu}U^{-1}$ and hence is not invariant. We can construct a guage invariant quantity however $F^{\mu\nu,a}F^a_{\mu\nu}$ is invariant since it can be written in terms of $Tr[\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$.

$$\begin{aligned}
Tr[\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} &= Tr[f^a F^{\mu\nu,a} t^b F^b_{\mu\nu}] && (3.28)\\
&= F^{\mu\nu,a}F^b_{\mu\nu} Tr(t^a t^b) && (3.29)\\
&= F^{\mu\nu,a}F^b_{\mu\nu}\frac{1}{2}\delta^{ab} = \frac{1}{2}F^{\mu\nu,a}F^a_{\mu\nu} && (3.30)
\end{aligned}$$

But $Tr[\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}]$ is invariant since it transforms as $Tr[U\tilde{F}_{\mu\nu}U^{-1}U\tilde{F}^{\mu\nu}U^{-1}] = Tr[U\tilde{F}^{\mu\nu}U^{-1}]$.

But the trace is invariant and cyclic permutation

$$Tr(ABC) = Tr(CAB) \tag{3.31}$$

and hence we get

$$Tr[\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}] \tag{3.32}$$

then by showing gauge invariance. So a gauge invariance Lagrange density looks like

$$\bar{\psi}(i\not{D} - m)\psi - \frac{1}{\psi}F^{\mu\nu,a}F^a_{\mu\nu} + \text{ gauge fixing } + \text{ another term} \tag{3.33}$$

It looks like QED superficially but

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + qf^{abc}A^b_\mu A^c_\nu \tag{3.34}$$

where $qf^{abc}A^b_\mu A^c_\nu$ indicates gluons radiating other gluons.

Lecture 4

3.3 Features of the gauge invariant Lagrangian

Last time we discussed SU(3) and we are going to go into a bit more detail today and describe consequences of the gauge invariant Lagrangian. Last time we took free quarks (which are vectors in colour space) and performed an SU(3) gauge transformation.

Demanding invariance under this transformation essentially meant introducing photon-like four potentials $A^{\mu,a}$. There is one of these for each (of the eight) generators t^a. You need 8 generators for the Gauge dependent terms produced.

Physically this implies that there is no such thing as a bare quark. So a quark is an object that continuously emits and reabsorbs gluons. You have the usually picture as with photons, the emission and reabsorption of photons. There is a cloud of photons surrounding a charge - which is constantly emitting and reabsorbing photons, and analogously there is a cloud of quarks, except gluons carry colour charge, whereas photons do not carry electric charge.

Physical quarks always emit and absorb gluons

1

When you scatter quarks (i.e when you collide particles at the LHC), this splinters this cloud surrounding the quarks and gluons and in regenerating this cloud and in becoming a physical object again, this emits gluon radiation - there is a spray of gluons which we see as hadron jets.

3.4 Transformations of quarks and gluons

Quarks transform very simply under what we call the fundamental representation. What are the generators of the fundamental representation? The generators are 3×3 matrices $t^a = \frac{1}{2}\lambda^a$, these matrices follow the $SU(3)$ commutation relations and have the usual properties of tracelessness etc.

$$\psi \to \psi' = e^{i\theta^a t^a}\psi = (I + i\theta^a t^a)\psi + \mathcal{O}(\theta^2) \tag{3.35}$$

where I is the identity matrix and as far as the generator is concerned we only need to expand to 1st order as the generator is the factor in the 1st order term. We want to see what the linear term is in the expansion of gluon equation and whatever matrix occurs in this first order is the generator. Let us work out the generators (these have to be 8×8matrices as they are rotating a 8-component vector). Remembering that the adjoint representation is a mapping from the the t matrices to f_{abc} structure constants and these are the generators of the adjoint representation. Last time we saw that the gluon field (which we derived last time) transforms as

$$A_\mu \to UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \tag{3.36}$$

where $A_\mu = t^a A^a_\mu, U = e^{i\theta^a t^a}$. This is why we must write $UA_\mu U^{-1}$, as these matrices do not commute. We are going to write $U = e^{i\theta^a t^a}$ plug it into the above equation and expand to first order to see what comes out.

$$A_\mu \to (1 + i\theta^a t^a)A_\mu(1 - i\theta^c t^c) - \frac{i}{g}(it^a\partial_\mu\theta^a) + (\theta^2) \tag{3.37}$$

we have expanding the exponential to first order, although we still have some higher order terms, so lets rewrite this to purely first order.

$$A_\mu \to A_\mu + i\theta^a t^a t^b A^b_\mu - i\theta^c A^b_\mu t^b t^c - \frac{i}{g}(it^a\partial_\mu\theta^a) + (\theta^2) \tag{3.38}$$

without loss of generality we can replace c by a in the third term (as they are dummy indices). We then get

$$A_\mu \to A_\mu + i\theta^a(t^a t^b - t^b t^a)A^b_\mu - \frac{i}{g}(it^a\partial^\mu\theta^a) + (\theta^2) \tag{3.39}$$

$$= A_\mu + i\theta^a[t^a, t^b]A^b_\mu - \frac{i}{g}(it^a\partial^\mu\theta^a) + (\theta^2) \tag{3.40}$$

$$= A_\mu - \theta^a f^{abc} A^b_\mu + \frac{1}{g}(it^a\partial^\mu\theta^a) + (\theta^2) \tag{3.41}$$

$$= t^c A^c_\mu - \theta^a f^{abc} t^c A^b_\mu + \frac{1}{g}(t^a\partial^\mu\theta^a) \tag{3.42}$$

where $[t^a, t^b] = if^{abc}t^c$. If we choose the special case of a global SU(3) rotation when I make a transformation in colour space, I change the colour of the gluons - this has nothing to do with the gauge transformation - we can show they are legitimate carriers of colour charge. If we choose the special case of a global SU(3) rotation, $\theta^a =$ constant, $\partial^\mu\theta^a = 0$, but the gluons still transform. So

$$A_\mu = t^c A^c_\mu \to t^c A^c_\mu - \theta^a f^{abc} t^c A^b_\mu \tag{3.43}$$

$$\text{so } A^c_\mu \to A^c_\mu - \theta^a f^{abc} A^b_\mu = A^c_\mu + \theta f^{acb} A^{\mu b} \tag{3.44}$$

this is just the normal rule for the rotation of a vector, this is how the components of a vector components transform. We have also used the fact that $f^{abc} = -f^{acb}$.

$$A^c_\mu \to A^c_\mu + \theta^a f^{acb} A^b_\mu \tag{3.45}$$

this shows that the transformation of gluons in colours space is generated by structure constants which implies that gluons transform in the adjoint representation.

We have learnt that gluons carry colour charge and we would like to explore the consequences of this.

3.5 Feynmann Rules

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu(\partial_\mu - igA_\mu))\psi - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} \tag{3.46}$$

is the Guage invariant Lagrangian. The $\bar{\psi}(i\gamma^\mu(\partial_\mu - igA_\mu))\psi$ section of the Lagrangian looks superficially like that of QED. However to write it out component by component, psi and psibar are in colour space and so would need indices to represent each colour. But besides this matrix structure, we have the same rules that occur between fermions and gluons as occur between fermions and photons. This section produces quark-gluon interactions via an interaction term $g\bar{\psi}\gamma^\mu A_\mu\psi$ which is the interaction of a Dirac current $\bar{\psi}\gamma\psi = j^\mu$ with gluon potential A_μ. The vertex one gets from this is

with a vertex term $-ig\gamma^\alpha t^a$ and a gluon emission with colour A where g is a the strong coupling. $-ig\gamma^\alpha$ this part is identical to the QED vertex term, but there is an additional term t^a which is a colour matrix, describing the colour of the emitted gluon.

However the $F^a_{\mu\nu}F^{\mu\nu,a}$ also contains interactions. Remembering that

$$F^a_{\mu\nu} = \partial_\mu A^{\nu a} - \partial_\nu A^{\mu a} + gf^{abc}A^{\mu b}A^{\nu c} \tag{3.47}$$

Imagine squaring this up, we are going to get a squared second term and cross terms. Therefore, this produces interactions that look like $\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)g^{abc}A^{\mu,b}A^{\nu,c}$. Also there is a term whereby you get $g^2 f_{abc}f_{ade}A^b_\mu A^c_\nu A^{\mu,d}A^{\nu,e}$ you can that there amre no free indices - i.e all indices are summed over and you get a scalar value as you would expect. These correspond to triple gluon and quadruple gluon interactions

$\propto g$

$\propto g^2$

These are the building blocks for higher order corrections. It should be clear that these vertices are a feature of a non-abelian theory. This is because the moment you take the structure constants to zero, these vertices disappear - If the structure constants are zero, this means that the generators commute.

We stated earlier that the sum of squares of generators, called the Casimir operator, commutes with all the generators. Defining $T^2 = t^a t^a$,

$$[T^2, t^b] = [t^a t^a, t^b] = t^a[t^a, t^b] + [t^a, t^b]t^a \tag{3.48}$$

$$= if^{abc}t^a t^c + if^{abc}t^c t^a \tag{3.49}$$

$$= if^{abc}(t^a t^c, t^c t^a) \tag{3.50}$$

this is the contraction of a symmetric tensor with an anti-symmetric tensor, which has the property that it is zero, therefore this vanishes. Therefore the Casimir operators commutes with the geneartors and by Schur's lemma states that any operator that commutes with all the generators is proportional to the identity matrix.

$$T^2 = (t^a)_{ij}(t^a)_{jk} = C_F \delta_{ik} \tag{3.51}$$

now we shall find out what C_F is. The value of C_F, in this case (as we are considerin it in the fundamentla representation), will be the colour charge. Likely for the gluons, the factor will have a different value (as it would be calculated in the adjoint representatino). The value C_F can be obtained by setting $i = k$ and summing. On the RHS we would get a value of N (as we are using the fundamental representation) and the LHS we get the trace i.e

$$Tr[(t^a)^2] = C_F N \tag{3.52}$$

Last time we said that $Tr(t^a t^b) = \frac{1}{2}\delta_{ab}$ and so $Tr(t^a)^2 = \frac{1}{2}\delta^{a,a} = \frac{N^2-1}{2}$ which implies $\frac{N^2-1}{2N} = C_F = \frac{4}{3}$ for SU(3). We can graphically represent this as

You can understand Schurs lemma physically, by breaking down this quark diagram. From the index i to index j, the colour is being effected by the matrix element $(t^a)_i j$.

So when we write $(t^a)_i j(t^a)_j k = C_F \delta_i k$ we are describing colour conservation mathematically. These colour factors (C_F) play a crucial role in the whole theory and these values change when considering quarks and gluons.

To see that C_F represents the strength of the quark emission probability 4 this is the Feynmann diagram you would draw for the emission of a gluon by a quark. Optical theorem tells us that the amplitude of this diagram is proportional to its mirror image attached to this diagram i.e. 5

If you want to calculate something like this, you essentially multiply it by its Hermitian conjugate to form the squared amplitude.

Likewise for gluons 6

Its a bit more complicated to show that C_A turns to just be N which is equal to 3 for SU(3). One of the big issues is how do you distinguish a gluon jet from a quark jet, at the end of hte daay all you get is a spray of pions etc. it is hard to distingh between these kind of jets. One way is to look at the radiation of gluons inside gluon jets and one of the most basic ideas here is that gluons have a different radiation factor simply becasue C_A is different to other factors. Still today people are trying to find ways of distinguishing quark and gluon jets.

3.6 Jacobi Identity

$$[t^a, t^b] = i f^{abc} t^c \tag{3.53}$$

This algebra is also satisfied by the adjoint representation matrices. We get teh following identity

$$f_{dae} f_{ebc} = f_{aec} f_{deb} + f_{abe} f_{dec} \tag{3.54}$$

check this at home as an exercise.

3.7 Fierz Identity

We are gonig to see again and again when we use it for cross-section calculations. Lets start by showing that any $N \times N$ matrix can be represented as a linear combination of the generators and the identity. This means that we can write any M by M matrix as

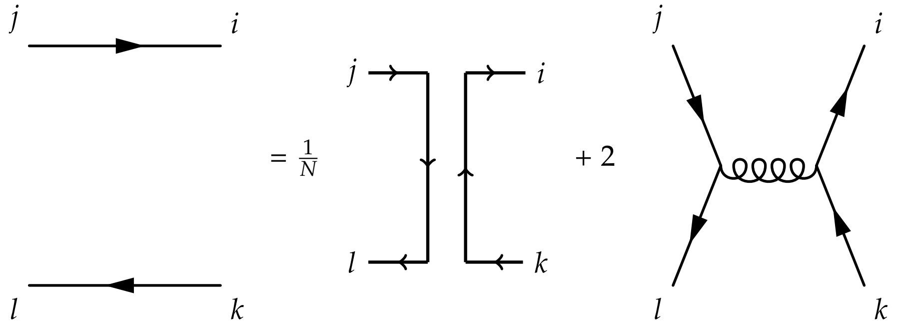
$$M = n_0 I + n^a t^a \tag{3.55}$$

and we need to work out what these coefficents (n_0, n^a) are. Lets multiply both sides by t^b and take the trace

$$Tr(Mt^b) = 0 + n^a Tr(t^a t^b) \tag{3.56}$$

$$= \frac{1}{2}\delta^{ab} n^a = \frac{1}{2} n^b \Rightarrow n^b = 2Tr(Mt^b) \tag{3.57}$$

$$Tr(M) = n_0 N + 0 \Rightarrow n_0 = \frac{1}{N} Tr(M) \tag{3.58}$$



This means that if you give me a matrix M, it can be represented as

$$M = \frac{1}{N} Tr(M) I + 2Tr(Mt^a)t^a \tag{3.59}$$

So the Fierz identity follows frm taking a special case of the matrix M. Let us now choose a special case of M i.e. a matrix with only one non-zero element.

$$M^i_k = \delta^i_{(j)} \delta^{(l)}_k \tag{3.60}$$

where j and l are fixed numbers. Plugging this matrix into the previous equation

$$M^i_k = \delta^i_{(j)} \delta^{(l)}_k \tag{3.61}$$

implies that $\delta^i_j \delta^l_k = \frac{1}{N} \delta^i_k \delta^l_j + 2(t^a)^i_k (t^a)^l_j$. This can be pictorially represented as 7 there is one more way of drawing this 8 (look on phone)

Lecture 5

3.8 Fierz Identity

This is a useful identity when talking about colour algebra, QCD and calculating colour factors, which is important when we talk about actual LHC processes and hteir calculations

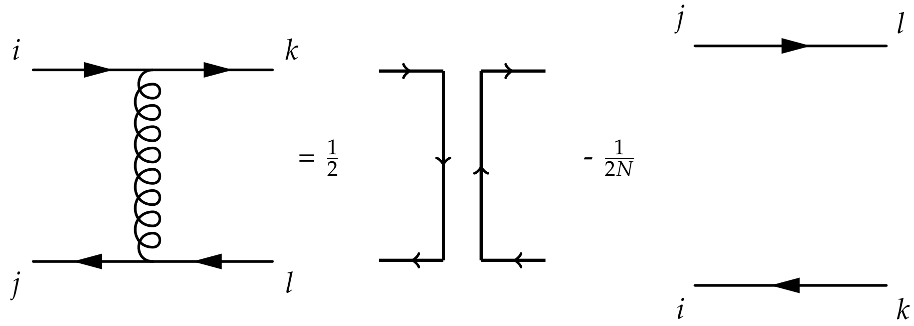
$$\delta^i_j \delta^l_k = \frac{1}{N} \delta^i_k \delta^l_j + 2(t^a)^i_k (t^a)^l_j \tag{3.62}$$

In books you wont see the Kronecker-delta's with mixed upper and lower indices. These upper and lower indices are just to remember (in future use) colour flow directions - we will use a convention whereby in the pictorial representation colour will flow from a lower index to an upper index.

We can either do the algebra without hte pictures or make life easier - it is quite nice to think in terms of pictures and helps your intuition about what is going on. Pictorially

Alternatively

Physially what is going on is, when you consider quark-antiquark scattering via gluon exchange (first image on the left), this gluon can essentially be seen as a pair of colour-anticolour lines.



So the gluon we know carries colour-anticolour charge so the exchange of a gluon is flow of colour-anticolour.

the exchanging of a gluon (first image on left) can be thought of as colour-anticolour (second image on right).

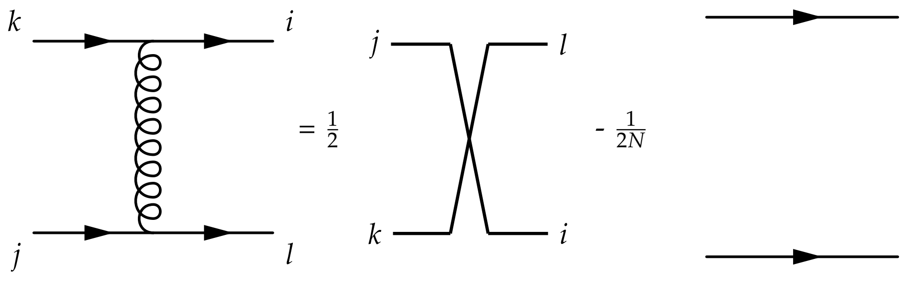
You can think of the gluon as being colour-anticolour except its not quite as colour-anticolour give you 9 components, 3 colour, 3 anticolour, but 8 gluons. So with 9 colours you can build a gluon octet and a singlet and we can see this in the diagram on the far right, colours do not exchange as in the middle part of the diagram.

But with just this contribution we would expect 9 gluon colours, so we must subtract off one possibility where there is no colour, which leaves us with the 8 that we know.

So these diagrams are a representation of gluon colour flow.

Although these diagrams are simple and you can calculate colour flow, when you are doing complicated algebra such as physics at the LHC, there is gluon radiation all over the place and this picture gets very complicated. This is handled often via Monte-Carlo simulations. But these simulations cannot handle colour suppressed terms (such as the last image on the RHS) and so we ignore them - when you square these terms to get amplitudes, you get a $1/N^2$ term which we can ignore, even if we cannot ignore the $1/N$ term (as $1/N^2$ is much smaller).

For qq scattering



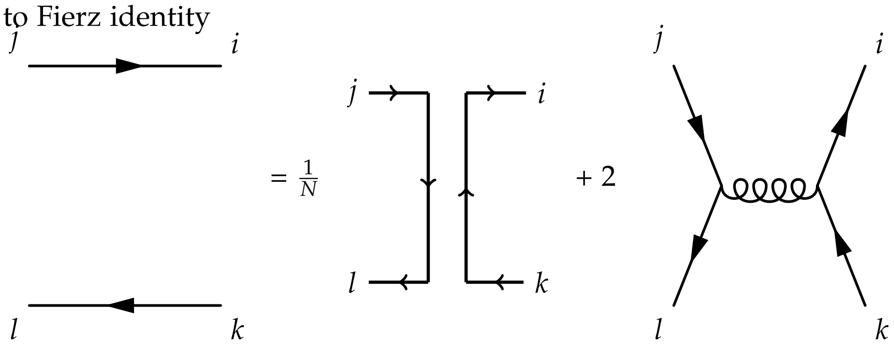
these identities and pictures are useful for colour factor calculations. In other words they are useful for making QCD calculations for process at any collider, in particular the LHC. In the next exercise sheet we will calculate quark-quark scattering, they work the same way as muon-muon scattering as we did in the 1st tutorial sheet.

3.9 Quark Colour Charge

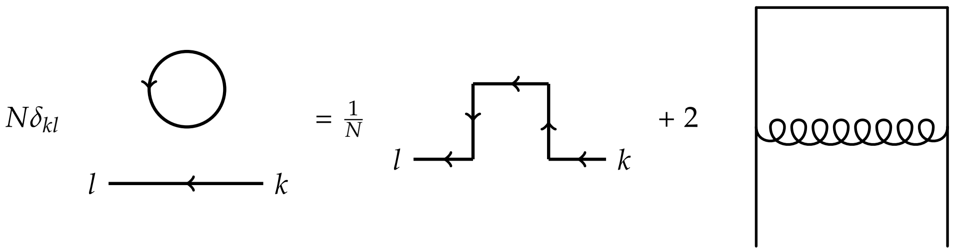
Last time we spoek about what it means to define a quark colour charge -w eknow hte colour charge is actually a vector in colour space.

The quark is a vector in colour space. Last time we said quarks radiate less intensively than gluon's and we define an operator to describe quark/gluon emission. If I sum and square the generators of the fundamental representation, I get something which is proportional to the identity matrix and the constant factor of proportionality was the Casimir factor. Tells you how intensely quarks radiate.

Go back to Fierz identity



Joining up the two ends, i and j, in the image above is the same setting $i = j$. So setting $i = j$ and summing, pictorially we get



where the factor $N\delta_{kl}$ can be broken down as - N comes from δ_{ii}, we already had δ_{kl} and both δ_{ii} and δ_{kl} come from $\delta^i_j\delta^l_k = \frac{1}{N}\delta^i_k\delta^l_j + 2(t^a)\ldots$. The last diagram comes from the proof we saw last lecture - this gives the factor $2C_F\delta_{kl}$ (from the previously seen diagram described by $(t_a)_{ik}(t_a)_{kj} = C_F\delta$).

It is important to remember that these diagrams are not Feynmann diagrams, they are just pictorial representations of the equations we have been working with.

where $C_F = \frac{N^2-1}{2N} = \frac{4}{3}$ for SU(3).

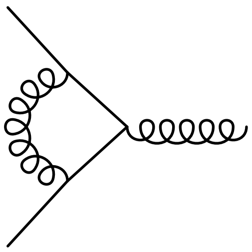
You can do this with the algebra of t^a matrices, but doing it graphically is easier. This graph is always associated with C_F so we have a trivial way of finding C_F. Simple in this case, but can be very difficult in general if you have a more complicated diagram.

When does all of this get used practically? Lets talk about renormalisation in QCD vs. QED and from there we will learn the concept of running coupling.

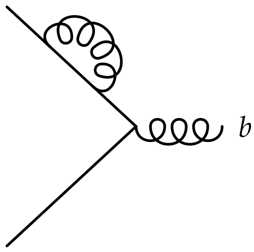
3.10 QCD Vertex Corrections

Lets consider one-loop graphs. These graphs are the vertex and wavefunction renormalisation.

When did QED in the QFT course last semester we wouldh ave come across similar digarms but with photons instead of gluons.



(A) Vertex diagram - vertex correction at one loop

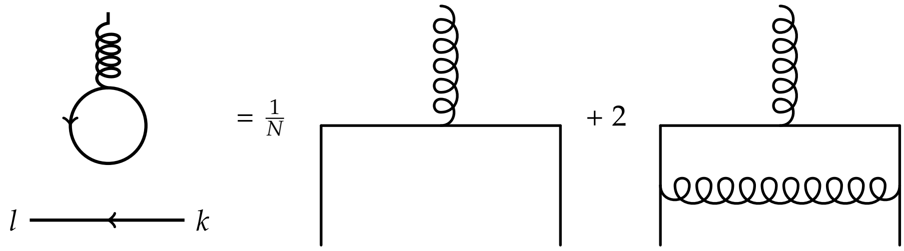


(B) Wavefunction for quark self-energy. Colour factor C_F comes from loop (self-energy contribution) and t^b from the vertex at the gluon-quark intersection - so colour factor is just $C_F t^b$

These one-loop graphs require remonralistaion. In QED we have an electron emitted a photon but in QED we have these kinds of processes going on that modify this basic emission of photons. These graphs give ultra violet divergences due to the facts we have loops.

you hvae to integrate over the momenta in these groups and you pickup a logarithmic divergence. In QED, the logarithmic parts (divergences) of these diagrams cancel each other by virtue of the Ward-Takahashi identity. In QCD it is more tricky than this. First of all we have to worry about the colour factor (the QCD calc. are the same as the QED calc but you have to add colour factors) so unless the two graphs have the same colour factors, they cannot cancel each other out.

To work out the colour factor of the firtst diagram, we do the following. Again we are going to join i and j in the Fierz identity but this time we are going to sandwich a gluon in between.

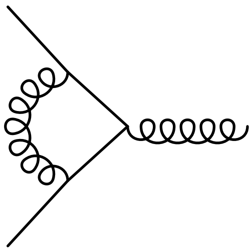
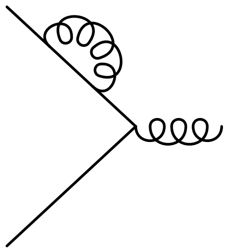


algebraically, these graphs mean that $t^a t^b t^a = -\frac{1}{2N} t^b$. If we were not using pictorial represetnations we would have to compute t^a form the vertex labelled a, t^b from the vertex labelled b on the first diagram on the right(*) and another t^b from the vertex labellled b on the far right (£) with another t^a. Using the diagrams makes colour calculations easier. Infact if we took N to infinity the colour factor of the second graph would become 0.

So these contributions don't cancel each other which at first glance this seems terrible as not only are these graphs divergent, they also appear gauge dependent. We srtart to lose the connectiuon with QED. Note we are talking about these two graphs:

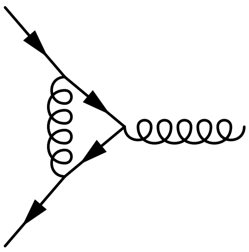
Since both graphs are divergent and their combination gives

$$\frac{N^2-1}{2N} A \ln \Lambda + \frac{1}{2N} A \ln \Lambda \tag{3.63}$$

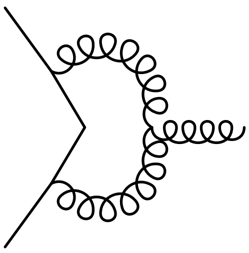
The first terms comes from the wave-function graph and the self-energy graph gives you 1/2N which is the colour factor we just worked out. Λ is a UV cut-off - when you renormalise, you must introduce the UV cut-off and oyu get logarithmic divergences in this cutoff. They are absorbed into bare parameters - this is renormalisation. A is a gauge-dependent constant. There is a left-over contribution and this goes as $\frac{N}{2} A \ln \Lambda$. It is gauge dependent, divergent and in general nasty. You need to worry about how you interpret this in QCD, as this is different to QED (in the way that all the logarthimically divergent contributions dont cancel each ohter) it looks like well have a problem defining QCD charge in the same way as QED charge. But we can it looks as though there is something missing - we will see that gluon-gluon interactions are inevitable.

One thing to notice is that in the large N limit (albeit a slightly artificial limit), the graph

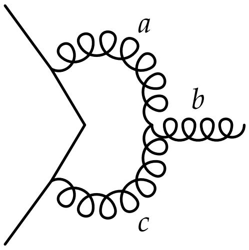


vanishes! Physically this grahp looks like the quark on top transfers its colour to the loop gluon. Its sort of lost its charge to the gluon joining both gluon's and becomes sterile - so it cannt interact with the second gluon. We worry about colour charge conservation, but the colour and colour-charge hasn't disappeared - it has been transferred to the loop gluon. In actual fact colour is conserved and the top quarks colour charge has just been trnafered to the loop gluon.

But in QED this cannot happen as photons don't carry any charge, so by emitting a photon, an electron loses nothing. In the case pictured above, the gluon is clearly a legitimate carrier of colour charge in contrast to the analogous photon. So now one is led to draw



as a graph that must also be considered - this is the missing piece. The colour factor associated with this graph should be $\frac{N}{2}$ to compensate previous mismatch of vertex and wave function.



We have this correction diagram and wewant to show it directly cancels our troublesome terms. From the a gluon, we get a factor of t^a and a factor of t^c from the c gluon. The interaction between the three gluons, gives a structure constant and so we end up with

$$t_a t_c i f_{abc} = \frac{1}{2}([t_a, t_c] + \{t_a, t_c\}) i f_{abc} \tag{3.64}$$

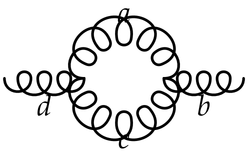
where $\{t_a, t_c\} = t_a t_c + t_c t_a$ and $\{t_a, t_c\} f_{abc} = 0$. Thus

$$t_a t_c i f_{abc} = \frac{1}{2}[t_a, t_c] i f_{abc} = -\frac{1}{2} f_{acd} f_{abc} t_d \tag{3.65}$$

this is using $[t_a, t_c] = i f_{acd} t_d$. This can also be written as

$$= \frac{1}{2} f_{dac} f_{acb} t_d \tag{3.66}$$

which corresponds to the following diagram:



[!h]

which means that

$$\frac{1}{2} f_{dac} f_{acb} t_d = \frac{C_A}{2} \delta_{db} t_d = \frac{N}{2} t_b \tag{3.67}$$

this has just the right colour factor we were looking for to cancel the mismatch. You again get a logarithmic divergence and calculating diagram 10 produces $\frac{C_A}{2}(G - A) \ln \Lambda$ and cancels gauge-dependent $A \ln \Lambda$ term written previously and the residual $G \ln \Lambda$ cancels when combined with the vacuum polarisation diagrams

and so when you consider all of the one-loop diagrams possible, the gauge dependence cancels and the sum over all of these diagrams is finite. In QED there is a different, these diagrams on their own form a gauage invarint subset (these diagrams by themselves are meaninful), and the rest of the diagrams drop out. Here it is all tied up to gether so QCD vaccum polarisitaon diagrams have a gauge dependence which cancels the gauge dependence from the two previous diagrams.

FIGURE 3.3: lowest circle

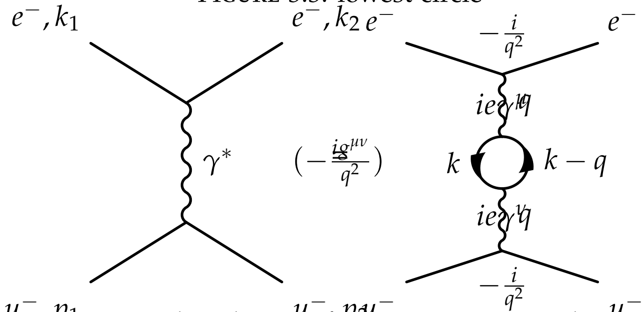


FIGURE 3.4: Correction → but this is infinite as it contains ultraviolet divergence

These calculations produce a left-over result which is written as

$$\delta g_s \propto \frac{g_s^3}{4\pi}(\frac{11}{3}C_A - \frac{2}{3}n_f) \ln \Lambda \tag{3.68}$$

where $\frac{11}{3}C_A - \frac{2}{3}n_f$ is not gauge-dependent. δg_s tells us that we started with a bare copuling which produced infinities and we must add this δg_s to the bare coupling - so the renormalised coupling is $g_s + \delta g_a$, which is finite. You place this coupling into a refinition of the bare coupling. What is important about this number, $\frac{11}{3}C_A - \frac{2}{3}n_f$, this number if not just associated to the lgoarthim divergence, but its also associateds with finite corrections - the finite corerctions to these log divergences. This number is what eventually enters the running coupling constant. At the moment this is just a parameter that has been reshuffled into the bare paremeters, but we havn't yet mentioned that there are finite corretions which are not absrobed int othe renomnralisation but nonetheless effect the QCD coupling constant and this number is what drives these terms. This is what causes the running of the QCD coupling constant. So we can put the infinity into the bare coupling term and get a finite number. It is only because $\frac{11}{3}C_A - \frac{2}{3}n_f$ is the number it is, that QCD works and the world works.

$C_A = N = 3$ and n_f is the number of excited quark flavours form the vacuum. The maximum value this parameter can take is 6 as there are 6 quarks. But typically you work with $n_f = 5$ as we consider energies where top quarks cannot be created. $\frac{11}{3}C_A - \frac{2}{3}n_f$ is known as the QCD β function. The calculation of this number got somebody a Nobel prize. This number established that you can use perturbation theory at high energies but not at low energies. Lets see exactly how this works and discuss the running coupling properly. First we shall discuss QED running coupling.

3.11 Running coupling

Consider e^-, μ^- scattering at tree-level and at one-loop

These are teh two diagramsw e are summing. Of course there are higher orders, but we only do this ti one loop. Everytime we introduce a loop it costs us a factor of charge at two vertices (see diagram), so the left-hand diagrm is supressed by a factor of e^2 compared to the LH diagram (its a correction). We have the problem that this correction diagram is infinite.

The idea of renomrlaisiton is taht the RH diagram is diverging because this loop doesnt make sense. So by reshuffling the infinite part of the second diagram into a dressed charge, rather than a bare charge, you get bvack the first diagram diagram jsut with a proper physical charge, this is telling us we have done something non-physical here. We shuffle the infinity off into a physical electron charge and you are left with a finite corerction which is much smaller than the leading order term. Then you can use perturbation theory. You can negelct higher orders, relative to leading orders, so long as you put the logs in the right place.

The corrections you get to the bare electron charge can be measured in experiment, we have now measured to loops of 7th order.

The amplitude for the second graph is related to the first one by the replacement of the photon propagator $-ig^{\mu\nu}/q^2$ by $(-1)(-i/q^2)\int \frac{d^4}{(2\pi)^4} Tr[ie\gamma^\mu \frac{i}{\not{k}-m} ie\gamma^\nu \frac{i}{\not{k}-\not{q}-m}(-i/q^2)$

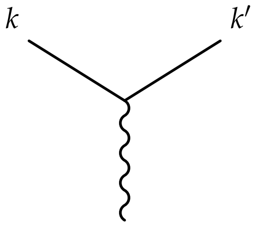
where the factor of (-1) comes for the fact you have a fermion loop, the $(-i/q^2)$ term comes form the photons in the correction graph, and the integral comes form the arbitrary, undetermined momentum flowing in the loop and the trace comes from the fact there is a fermion in the loop and there are two propagators and fermion interactions terms from the loop. Notice also that we pick up an extra factor e^2, which we naively think would suppress this correction by e^2 compared to the leading order diagram, however the loop actually gives us an infinite divergence. The loop-diagram gives a logarithmic divergence. One just absorbs this into the leading order diagram.

$$e^2\bar{u}(k')\gamma_\mu u(k)(-\frac{g^{\mu\nu}}{q^2})\bar{u}(p')\gamma_\nu u(p) \tag{3.69}$$

where we haev currents on either side of a propagator. This gets modified as you take the infinity from the correction diagram and absorb it into the leading order diagram (as the correction has exactly teh same strucutre at the level of ths spinors, the only difference is a logarithmic divergence which you shuffle into the leading order charge) which leads to

$$e^2[1-\underbrace{\frac{\alpha}{3\pi}\ln\frac{\Lambda^2}{m^2}+\chi(q^2)}_{\chi(q^2)\text{ is a finite function of }q^2\text{ and }\chi(q^2\to 0\text{ as }q^2\to 0}]\bar{u}(k')\gamma_\mu u(k)(-\frac{g^{\mu\nu}}{q^2})\bar{u}(p')\gamma_\mu u(p) \tag{3.70}$$

this suggests to me that I just need to redefine the charge at each vertex by the above underlined factor. But now my charge depends on q^2.

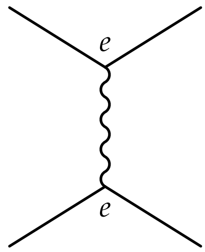


$$\langle k'|\, J_\mu \,|k\rangle = -e\bar{u}(k')\gamma_\mu u(k) \tag{3.71}$$

$$\rightarrow -e[1 - \frac{\alpha}{3\pi}\ln\frac{\Lambda^2}{m^2} + \chi(q^2)]^{\frac{1}{2}}\bar{u}(k')\gamma_\mu u(k) \tag{3.72}$$

This implies that in the scattering one replaces sthe normal dirac current (vertex in digram 15). This tells you its a modification of the charge which enters the current. The square root comes from the fact that we have symmetrically attributed the same factor to each vertex.

i.e



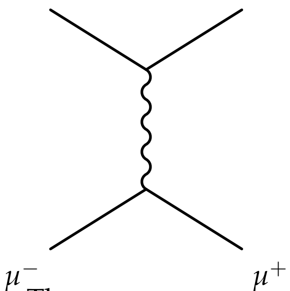
the factor from both vertices (e) i.e $e^2[1 - \frac{\alpha}{3\pi}\ldots]$ but considering one vertex we get $e[\ldots\ldots]^{\frac{1}{2}}$.

We just have a redefinition of the charge

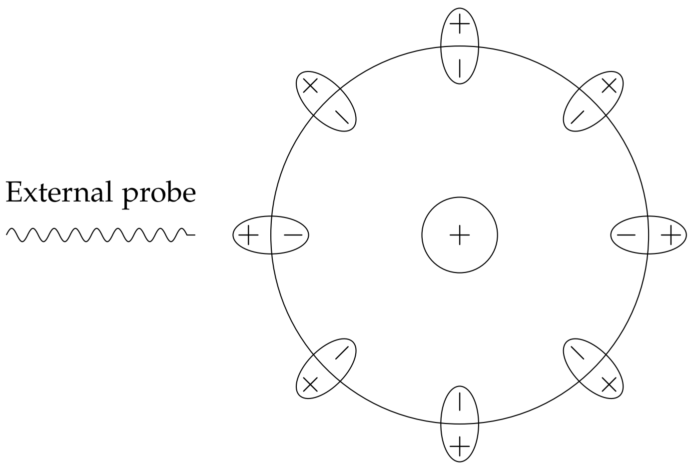
$$e'(q^2) = e[1 - \frac{\alpha}{3\pi}\ln\frac{\Lambda^2}{m^2} + \chi(q^2)]^{\frac{1}{2}} \tag{3.73}$$

where e is the bare charge and e' is the finite charge. But the finite charge doesn't come for free, it picks up a dependence on q^2. We have to accept that from the reshuffling of the ln into the bare charge, we have measurable effects that we can measure in scattering experiments. If you do an scattering experiment to measure the coupling constant you will find it depends on q^2, the momentum which you give to the probing particle. In QED this effect behaves as expected, however in QCD things go ary. The α here can be replaced by the renormalised α' which you will have to worry about again at the next order.

$e'(q^2)$ is related to e'(0) by a calculable function $\chi(q^2)$. So the interaction strength depends on the energy of the probe.



The muons create a cloud of photons and the harder to cut that cloud, causes different effective charge screening effects. For QED as q^2 icnreases, the magnitude of the charge grows. This works exactly like in a dielectric, this is because effecively the vacuum acts as a dielectric - containing electron-positron pairs; this is generated by the emiited cloud of photons.



the harder you cut the cloud (usign the probe particle i.e electron at higher energies determined by q^2) the more charge you see - this is beacue you can get closer to the central charge by cutting past the cloud of charge casued by cloud of photons which act as a screenin effect! This is not the case in QCD however! - we get an anti screening effect.

For QED the coupling α satifies a differential equatino

$$Q^2 \frac{\partial \alpha s}{partial Q^2} = \beta \alpha_s^2 + \text{higher-roder s} \tag{3.74}$$

with $\beta > 0$. This equation says that the QED coupling increases as you go to higher energies. In QCD however, the crucial point is that there is a negative sign in front, which means as you go to higher energies the QCD coupling gets smaller and you get asymptotic freedom. This will describe QCD confinement.

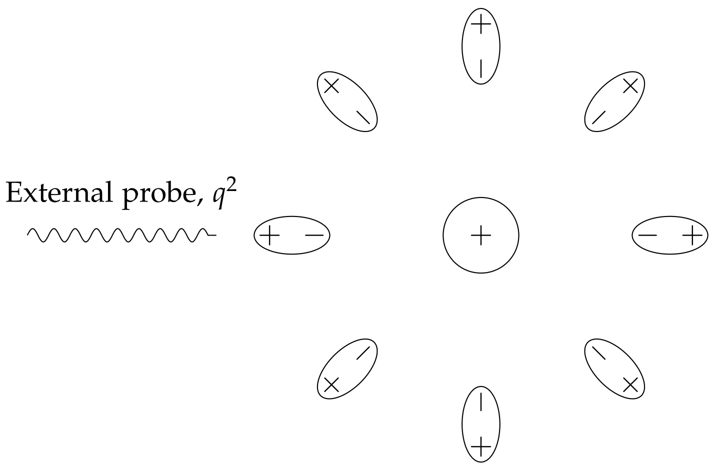
Lecture 6

Last lecture we talked about the running coupling. We showed tehe scatrteing between rtwo charged particels in QED i.e electron-muon scattering via virtual hptoon exchange and vacuum polarisiation effects started at higher orders which screesn the charegs and therefore the actual charges which are seen, the effective interactions strenghts depend on how miuch energy your probe has and how much momentum is transferreed. There is a natural inteaction for this, which comes about by thinking of the vaccum as a dielectric. This is a common feture for all gague theories, the interaction strenght (or coupling) .....

3.12 Running Coupling

In the example sheet we showed there was virtual photon exchange and at higher orders we get vacuum polarisation effects and the actual charges which are seen depend on how hard you crash these things into each other and the momentum that is transferred in the parton scattering. There is a natural explanation for this if you think of the vacuum as being a natural dielectric. This is a common feature in Gauge theories.

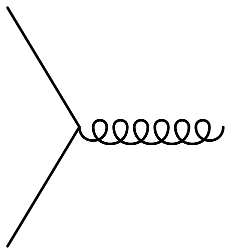
In Gauge theories, the interaction strength of the coupling depends on the interaction scale, or equivalently, the distance form the probed charged. A simple picture of this in QED is take a positive charge and place it in a dielectric. This positive charge will polarised the dielectric and tend to be surrounded by a screening cloud.



We probe this charge with a photon probe of momentum q. In QED a harder probe sees more of the charge as it cuts the cloud more. In QCD however, things go in the other direction - so in QCD this becomes an anti-screening. You could describge the anti-screening as a dielectric effect but it can also be explained as a paramagnetic effect - due to the spin of the gluons. The fact that there are gluon which carry colour charge and spin causes the anti-screening. For QED we can write a differential equation for the running coupling - the running coupling satisfies an equation of the form

$$Q^2 \frac{\partial \alpha}{\partial Q^2} = \beta \alpha^2 + \text{ higher order corrections} \tag{3.75}$$

and the key point is that β is positive. So as you change Q^2 the coupling goes up. The message we get out of this is that until you get to very high scales the QED coupling remains much smaller than 1. So we can use perturbation theory (as it remains valid at all of the energy scales we are currently think about) as a good approximation and to obtain a very high accuracy - $\alpha \ll 1$ ad holds over a very wide range of scales. You can calculate higher and higher order diagrams and get better precision. This isn't the case for QCD however. WE know that partons form hadrons and so perturbtation theory will brek down. For QCD the situation is different - when we first started looking at QCD, we noticed that we never see free quarks, only ever hadrons and assumed we couldn't use perturbation theory with QCD. People set out to prove this wrong and ended up proving QCD is an asymptotically free theory.



Colour charge that is probed at one-loop order depends on

3 [-all of the one-loop re-normalisation diagrams]

at the end of the day after cancellation between diagrams, the gauge invariance is restored. We discussed the cancellation of gauge-dependence of the diagrams in [] last lecture.

In QED thes vaccum poliarsiation diagrasm form a guage invariant subset on their own and you can calculate them by themselves and they contribute to the running coupling. In QCD however, the vacuum polarisations cancel out the gauge dependence of other diagrams and we still see a remaining contribution from such vaccum polarisation diagrams which contribute to running couplings. We discssed teh cancellation of guage dependence of the diagrams in brackets last lecture.

So in the end we are left with finite vacuum polarisation contribution. We have

$$Q^2 \frac{\partial \alpha_s}{\partial Q^2} = -\beta_0 \alpha_s^2 (Q^2)(1) \tag{3.76}$$

β_0 is this famous number we discussed last time and is equal to 6 (add this at home) $\beta_0 = \frac{1}{12\pi}(11C_A - 2nf)$ where nf is at most 6 (due to the six flavours of quark) and usually just considered to equal 5 at high energies and for SU(3) $C_A = 3$. Note that $\beta_0 > 0$ because of this. Tehrefore at higher energies and coupling decreases and the partons become free particles, not interacting with each other.

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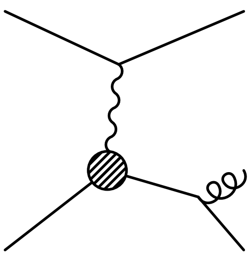
This is a picture of a proton. When people were thinking of this in terms of group theory, they took 3 quarks and qqq can make a colour singlet. Not all combinations of quarks can make a colour singlet i.e. qq, you get a state that transforms as a sexlet and a triplet. You can also make colour singlets from pentaquarks - they are allowed theoretically, but the question is whether they are stable and we can find them experimentally.



When you do an electron proton scattering you exchange a photon (making this a pure EM process) and you basically bang these quarks out - give them a large bash. This sets the scale of the QCD that follows. These quarks can emit gluon's with very low probabilities because the scale with which the quarks are banged out controls all of the physics that follows. You get corrections of scale q^2 but they are very small and much less than one. And to first order this is a purely EM process. This is verified by experiment and so you really see that protons are made of 3 quarks. Then you do this experiment at lower energies and the gluon emission occur with a higher probability and you start to see noticeable discrepancies in experimental data. At high q^2 you have free particels and at low q^2 you get quarks that emit gluon radiation like crazy.

discussion on diagram (pure electromagnetic scattering...)

Let us re-arrange the running coupling equation (1)

$$\int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} = -\beta_0 \int_{\mu^2}^{Q^2} \frac{dx}{x} \tag{3.77}$$

and we get

$$-\frac{1}{\alpha_s(Q^2)} + \frac{1}{\alpha_s(\mu^2)} = -\beta_0 \ln \frac{Q^2}{\mu^2} \tag{3.78}$$

to arrive at

$$\boxed{\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \beta_0 \alpha_s \ln \frac{Q^2}{\mu^2}}(2)} \tag{3.79}$$

if you know the coupling at one scale, you are able to work it out at another scale. If you measure the coupling at one scale and measure it another scale, they will be related by this formula. But remember that this equation is approximate and we have only taken this to the order of one-loop diagrams, there are higher order correction terms. This is one of the fundamental tests of QCD. Usually the reference scale is the Z mass - α_s is taken to be $\alpha_s(M_Z^2)$. Most of the work in this area was done at LEP and at the time, the Z mass was the scale at which α_s was extracted at LEP. The current wold average value of $\alpha_s(M_Z^2) = 0.1184 \pm 0.0007$ which we must to know to feel happy about perturbation theory. It is considerable smaller than one, but although you are happy with this value, smaller scales utlimitealy enter the game. When you produce quark antiquark pari they fly about and radiate gluons. This process becoems softer and softer until partons turn into hardrons and so inevitabley you taken a way from that scale as you radiate gluons and get to lower and lower scales and finally you hit scales where perturbation theory fails. This always happens, if it didn't happen we would be able to see quarks. The important thing is how late this happens, if this happens at a point where your original qqbar turns into to narrow jets then you know this process hasn't messed you about too much. If your final state is completely different to the perturbative configuration you started with, then you know you are in big trouble.

Eventually you will always be hit by this problem, but if you study the correct things (look for the right final states) the series would have already converged by the time this process happens.

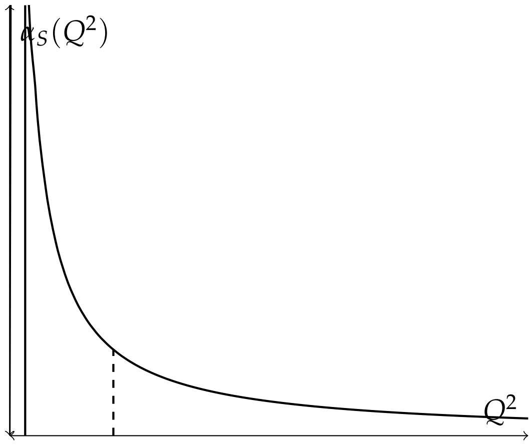
There is another way of writing this running coupling equation, this introduces a new scale Λ. It is not the number itself which is important, but the order of magnitude of the number. Let us take Λ_{QCD} to be the scale at which QCD coupling diverges. Set $\mu^2 = \Lambda_{QCD}$ in (2). We get

$$-\frac{1}{\alpha_s(Q^2)} = -\beta_0 \ln \frac{Q^2}{\Lambda^2_{QCD}} \tag{3.80}$$

leading to

$$\boxed{\alpha_s(Q^2) = \frac{1}{\beta_0 \ln \frac{Q^2}{\Lambda^2_{QCD}}}} \tag{3.81}$$

there is also a £ scale in QED, this is basically the Plank scale. Λ_{QCD} is called the Landau pole (due to the singularity in the ln). If we plot this we see



Note the figure tails off as we get to higher energy scales - $Q^2 \to \infty \Rightarrow \alpha_s \to 0 \Rightarrow$ asymptotic freedom.

as Q^2 increases quarks and gluons become free. Consistent with this formula, there is a pole at the singularity - you might worry about the physics at a singularity and what happens there. This lambda is an artifact - an artifact of using perturbation theory in a region where it shouldn't be used. So Λ is a cut-off after which perturbation theory doesn't work. This is a million dollar question (literally). We don't know what happens in this singular limit and there is a million dollars available to whoever solves it.

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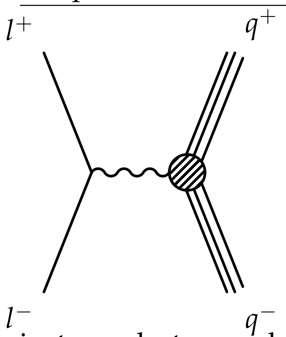
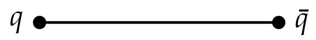
There is some point where we deviate away from this curve. What the actual coupling does at this point is unknown. People are two opinions on this matter. Some people say that we shouldn't talk about it at all as quarks and gluons can't exist at such low energies, however you don't make much progress this way. The other side is lets be a bit braver and try to explore this part of the diagram.

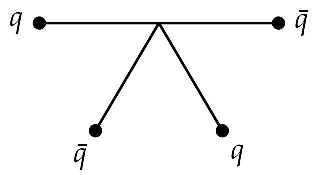
9

Its not as if the transition from partons to hadrons completely wrecks this picture, (finish discussion)



just as a last remark on confinement, it is worth remarking what happens physically when in the region where Q^2 is of order Λ^2_{QCD}. This region marks the onset of confinement effects. We cannot calculate these perturbatively from first principles - it has been done but you have to bend the rules a bit in the process. It has been established by QCD lattice studies etc. that at large distances, QCD has a linearly growing potential. We are not used to linearly growing potentials - in EM and Gravity we have potentials that go as $1/r$.





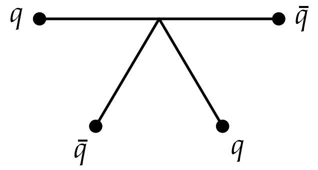
Physically you have a quark and anti-quark bound in a meson (known as the string picture of the meson - imagine they are attached by a string). Lets say you try to pull them apart. When these things are close together they are nice and free, knocking around in the meson, but when you try to pull them apart out of the meson, when you reach the end points which is basically the size of the meson, this is a large distance for QCD (the size that corresponds to Λ_{QCD} and the string snaps.

What happens is there is a very strong colour field created, because of this linearly growing field.

When the string snaps, out of the vacuum you create a qqbar pair and instead of freeing these two things you form another meson.

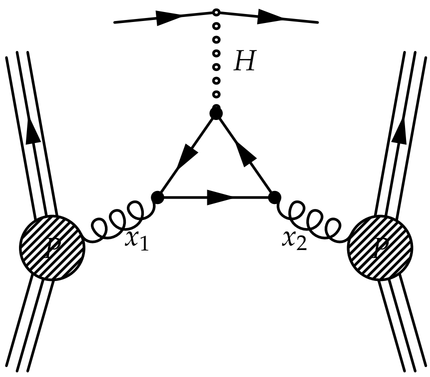
$$V(r) = \underbrace{-\frac{\alpha_s}{r}}_{\text{regular Coulomb-like term}} + \underbrace{kr}_{\text{Confinement effect}} \tag{3.82}$$

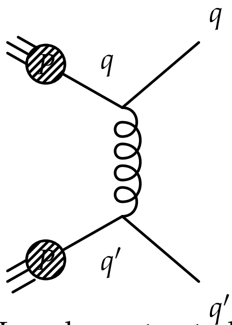
When you pull them apart, the linear term dominates and creates a positive potential. Instead of freeing these quarks, you snap the string and get another meson.



When one tries to separate the meson or baryon constituents, fragmentation occurs into more mesons and baryons.

3.13 Anatomy of a hadron collider calculation





(quark scattering - $qq' \to qq'$)

How do we treat physics in a hadron collider.....

The first step in this is an important one called QCD factorisation. Somebody essentially proved that you can do this up to some corrections which are known as Paol corrections.

3.13.1 QCD Factorisation

You derive that there are these universal functions called parton distribution functions (pdfs). Encode the probability of finding a quark or gluon inside a proton with a specified momentum fraction x, q(x) is a quark distribution, g(x) is a gluon distribution. These functions cannot be calculated from first principles, they must be measured empirically. These are intrinsically non-perturbative functions and thus cannot be calculated. So you do experiments electron-proton scattering at HERA and you extract these functions form the data. These functions are extracted from the data and you know that they are Universal. The fat that such objects exist which are Universal, which let you have this interpretation and probabilities which describe the probability of finding a quark with certain momentum, just to establish this in field theory is an enormous task. It turned out that the factorisation was only studied in specific cases and people have stated to use it beyond the cases where there are no formal proofs, but we use it nonetheless. It turns out that people have found problems with factorisation, such as when you have colours in the final state.

What we really want to emphasise is that this is the first place where you make a leap and everyone accepts it, but this is one area where you have to stop and think about a formal theory.

Although we know ther are problems, if you accept this (and everybody does) we can do the following:

The proton/antiproton four-momenta

$$P = \frac{\sqrt{s}}{2}(1,0,0,1) \tag{3.83}$$

$$\bar{P} = \frac{\sqrt{s}}{2}(1,0,0,-1) \tag{3.84}$$

then the colliding partons have four-momenta that are

$$p_1 = \frac{\sqrt{s}}{2}x_1(1,0,0,1)p_2 \qquad = \frac{\sqrt{s}}{2}x_2(1,0,0,-1)$$

and these are the partonic momenta.

Calculate $\hat{\sigma}$, the parton-parton scattering/annihilation process i.e the partonic cross-section - you convolute the parton distribution functions

$$\sigma = \sum_{a,b} \int f_{a/A}(x_1) f_{b/B}(x_2) \hat{\sigma} d\mathcal{L}_{ips} dx_1 dx_2 \tag{3.85}$$

where $f_{a/A}(x_1)$ is the parton dist. function and it tells us the probability of finding a parton of type a inside hadron A. This is taken from tabulated data. $d\mathcal{L}_{ips}$ is an integration over a Lorentz invariant phase-space measure $d\mathcal{L}_{ips} = \frac{d^3\mathbf{p}_3}{2(2\pi)^3 p_3^0} \frac{d^3\mathbf{p}_4}{2(2\pi)^3 p_4^0} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)$. This is exactly the same as in the case of electron-muon scattering we did earlier, but with the introduction of a parton distribution.

$$\hat{\sigma} = \hat{\sigma}_0 + \frac{\alpha_s}{\pi}\hat{\sigma}_1 + \dots \tag{3.86}$$

partonic cross-section is a series expansion in α_s.

For sensibly chosen observables, partonic calculations should receive non-perturbative power corrections $(\frac{\Lambda^2_{QCD}}{Q^2})^P$, where Q^2 is the hard scale of the process. For $Q^2 \gg \Lambda^2_{QCD}$ perturbation theory should be ok.

That is it for SU(3), now we want to move to talking about electroweak theory SU(2).

Chapter 4

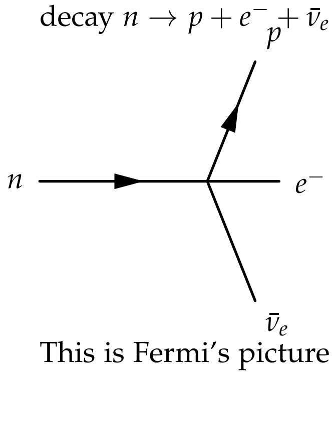
Electroweak

Lecture 6 Contd.

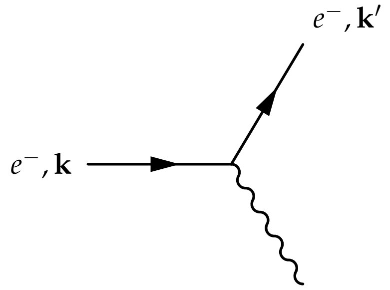
4.1 Electroweak theory and problems with phenomenology of weak interactions

Here, one cold have used a different approach. Taken a book by Peskin and Shroeder and electroweak theory, here's how it works, here is the Higgs Boson. But this is just explaining an established theory. IT is worth discussing what the theory looked like in 1934. There were severe problems with the theory (with Unitarity for instance) and solving these problems produced the current theory of electro weak theory. Its not just a matter of writing down some random Lagrangian and seeing it works and matches nature. What actually happens is you make a theory which works with current data, then at higher energies it is realised that the theory is sick and you can still use the broken theory for a while, ignoring the problems. Then when somebody makes a new collider, you tweak your theory as you get worried that your theory looks silly. You introduce something and do a minimal tweak to agree with new experimental data. This goes on until you must give up and change your Lagrangian.

Essentially, the modern electroweak theory just originates from Fermi's theory of beta decay $n \to p + e^- + \bar{\nu}_e$. Fermi visualised this as a point-like interaction.



This is Fermi's picture which is drawn in analogy with an EM process



The analogy was to treat the electron-neutrino ($e^-\bar{\nu}_e$) pair like an emitted partile - photon emission in the case of QED.

The amplitude of Fermi's theory was assumed to have a weak analogue of the EM current (Fermi didnt understand why these interactions were so weak but you simpl;y use a current calleda weak curernt)

$$\langle p| J^{\mu,h}_{w,k} |n\rangle \tag{4.1}$$

we have that the interactyion whereby a neutron goes to a proton is mediated by a weak interaction current where $J^{\mu,h}_{w,k}$ is a weak hadronic current in analogy with the EM current

$$\langle p| J^{\mu}_{EM} |n\rangle \tag{4.2}$$

The matrix element is

$$M = \langle p| J^{\mu,h}_{w,k} |n\rangle \langle e\bar{\nu}| J^{l}_{\mu,wk} |0\rangle \tag{4.3}$$

which involves this electorn neutrino pair popping out of the vacuum, where μ is a Lorentz index, h denotes a hadron and l denotes a lepton i.e a hadronic current, contracted with a leptonic current - a contraction of J^{μ} with J_{μ} as you get in EM theory i.e contraction of an EM current with another EM curernt. In Fermi's theory these two currents leave to/from a point. This is why Fermi's theory is said to be in a current-current form. In contrast to the below EM diagram with $J_{\mu}g^{\mu\nu}J_{\nu}$ where we have two dirac currents and a propagator. So we can see that there is some structure missing from Fermi's theory - there is no propagator and thus no gauge boson - they didnt know about them at the time.

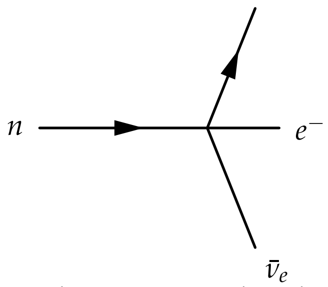
In this process you hvae a dirac current at the top and bottom and a propogator, so the structure is different. There is no such propagator in Fermi's theory as there is no gauge boson to mediate it. This is because Fermi didn't know about the W bosons.

We know that in the correct theory there will be a mediating boson.

This beta decay interaction that we observed was much weaker than the EM interaction and nobody knew why, simply that it must be a different weaker force. You think of gravity and it is much much weaker, we don't know why but it is a different force. It was later found that there was missing structure in this weak theory - the W mass was missing and when you look at the strength of this interaction you get a coupling which is divided by the W mass squared which is what makes the interaction so weak. But they didn't know W existed. Without the W mass, the bare coupling constant of the weak interaction is comparable to the EM interaction.

Lecture 7

Lat time we were talknig about Fermis theroy and we said that Fermis's theory beta was essentially a current-current interaction.



The point was that this isn't mediated by any gauge boson (as Fermi didn't know they existed at the time). THere was a hadronic current and a leptonic current

$$M = \langle P| J_h^\mu |N\rangle \langle e\bar{\nu}| J_{\mu,l} |0\rangle \tag{4.4}$$

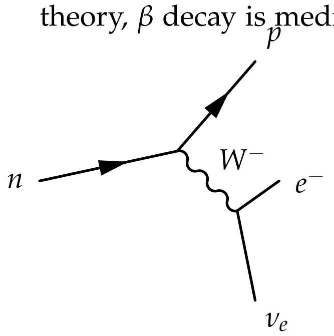
the matrix element of a current contracted with the matrix element of anothe current. We can see the difference with EM as there is no propagator inbetween, whereas in EM you would have current -propagator-current. There was only a small amount of energy relaased in these interaitons and essentially one can, to first approximation, ignore the momentum dependence of this matrix element, which means M is roughy constant. The value of this constant is unsuprisingly called Fermi's constnat, $G_F = 1.14 \times 10^{-5} GeV^{-2}$. This means that before the true trucutre of the theory was realised, this was a theory with a dimensionful coupling - this has issues iwth it. Dimensuonal coupling leads to unitarity violation and bad high-energy behaviour. The corss-sections grow unphysically at high energies, thoyugh Fermi was looking at low energy processes and thus this did not matter - it was a good theory to describe these low-energy weak interactions at the time. If oyu have a weak current makde of hadronic and leptonic terms, you should also expect processes which are hadronic-hadronic and leptopic-leptonic described by this theory. If one writes $J_\mu = J(\mu, l + J_{\mu,h}$ then purely hadronic and leptonic processes should exist:

$$\Lambda^0 \to p + \pi^+, \qquad \mu^- \to e^- \bar{\nu}_e + \nu_\mu \tag{4.5}$$

these exist and have similar rates to β decay (rates propto Fermis constnat). So this theory works quite well for these processes. Time catches you out if oyu ignore issues and cut corners, and whne peeople move experimentally to higher energies, the theory will stop working. The theory had many issues, but the most serious were that it violates unitarity (which means probabilities won't add up to one) and related to this, it has a bad high-energy behaviour. It predicts the corss sections grow unphysically with high energy. There are upper bounds on energies which come from unitarity, which dones't exist in Fermis theory.

Unitarity violation is interesting, even today as it means that new physics exists that we havnt found or theorised. You make a theory and find that when this theory violates unitarity, there is new physics in existence that we haven't described and when you get approach energies which violate unitarity in the current theoretical model, we see new physics popping up in experiment. This was true with the discovery of the Higgs boson, the reason for the LHC's construction. There was unitarity violation in our theory and so we knew there was some new physics beyond this unitarity violating energy scale. Higgs was a no-lose situation as we knew there would be something new at its energy scale as we had untarity violation - we would have fojnd either the Higgs or soemthing else which is new. Now we are not in that situation however, we may not nessecerily see anything new until we hit hte Planck scale! We says it funny that neutrinos dont have mass etc. but what is really NOT funny is that we are not in a positoon where something is clearly wrong, we are not sure what may lay in higher energies and there is no clear motivation ot build higher energy machines. There are alot of discussions about 100 TeV collider in the community, but it wont be approved by governments for a long time as there is little motivation ot spend the money.

What happened to Fermi's theory? It made way for an intermediate vector boosn (IVB) theory. Part of hte motivation for introducing this intermediate vector boson was to cure the issues we see in Fermi's theory - the unphysical growth of cross-sections, as EM didnt have this unphysical growth. A VB solves some of these issues. In this theory, β decay is mediated by a charged vector boson



where W is a hypothetical particle for the time. Using the propagator for a massive vector boson, we can write (now we want a form that looks like a curret, massive vector boson propagator and another current)

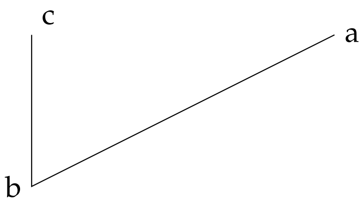
$$M \sim J_\mu \left(\frac{-g^{\mu\nu} + q^\mu q^\nu / M_W^2}{q^2 - M_W^2} \right) J_\nu \tag{4.6}$$

this is what replaces the current matrix element. The second term of the propagator is due to longituduinal polarisations which are there for massive particles but not massless - photons only have transverse polarisation - the lognitudunal polarisation exists for massive vector boson.

This is the form that was assume and the idea is that you use the ME and compare the cross-sections to data to get some idea of what M_W may be. For tpyicaly β decay energies, you get back Fermi's theroy. If the energy is small and so momentum transfer is small, you can basically neglect the q-dependent terms and so the M_W^2 in the demoninator can dominate at energies of roughly a few GeV i.e $q^2 \ll M_W^2$. At this energy scale one just gets back Fermis theroy with $G_F \sim \frac{g^2}{M_W^2}$ where g is now the coupling of the theory - note this is a dimensionless coupling. Once you realise that there is an M_W in the game, you realsie that what you were calling a coupling is actually a different, dimensionless coupling divided by the M_W mass.It was realised why weak interactions are weak, because were mising the srtucture involving the $1/M_W^2$ term. Though if you take g to be of the same order as the electon charge, you get the correct mass of the W boson. THis is the first hint that there is some electroweak unification at some energy scale. It is the largeness of M_W that makes the interactions weak, not the size of the coupling which many had been fooled into beleiving was the perpetrator. In fact with $g \sim e$, one obtains a value of $M_W \sim 90GeV$. So you get something whivh is basically correct.

This theory was successful for a while and this cured soem of the problems at the high-energy scale. $q_\mu q_\nu / q^2$ goes to one at large energies which is fine, the overall propagator dies off with q^2. IF you forget about the $q^\mu q^\nu / M_W^2$, we see taht there is a $1/q^2$ suprression as you get to higher energies. This is like the Photon propgator, it gives the right fall-off with energy. So we get the correct behaviour, given that you ignore the second term and there are a set of interactions where you can ignore them (internal W lines), but for the best-part of the theory you cannot ignore them (where W are external lines) and so we still ahve a problem!

Whilst this new theory (IVB) appeared to cure the problems with Fermi's theory, there were ongoing experiments at the time which observed parity violation. This was an issue as there was no parity violation built into this theory. Parity violation was experimetnally observed by Wu in the 1960's. The main observation is that hte nagular distriubtion of electrons in the β deacy of $^{6}0Co$ depedns on the spin of the polarisation of the decaying nucelus.



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This lead to parity violation as we observed that things were dependent on the polarisation of S. If we denote the polriasiton vector by $\mathbf{S}$, then the electron angular distributon dpeends on $\mathbf{S} \cdot \mathbf{P}_e$, in additiona to the usual terms that one gets when you calculate matrix elements, involing dot products of 4-momenta.

It was clear form this experimental observaiton, that the martix elements should include terms like $\mathbf{S} \cdot \mathbf{P}$ rather than just $p_1 \cdot p_2$ etc.. Under parity $\mathbf{r} \to -\mathbf{r}$ i.e spatial components are inverted, however $\mathbf{S}$ is a so-called Pseudo-vector like angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ which is invariant under a parity trnasfomration; these are also known as axial vectors. The dot product of a pseudo vector, with a regular vector gives you a psuedo scalar, which does change sign under a parity trnaasfomraton (in contrast to a normal scalar produced from two normal vectors which is invariant). $\mathbf{S} \cdot \mathbf{p}$ is a pseudo scalar and thus changes sign under a parity trnasfomration.

The β deacy angular distribution contained both scalar and psuedo-scalar terms and this is what leads to parity violation; part of the equation flips sign. This implies that hte currents in Fermi's theory are not complete, there is some furhter unknown structure. They are clearly not pure vector currents (like in EM) as if you have purely vector currents, you get scalar terms when you calculate the matrix element and no parity violation exists.

At the time people like Gellmann and Feynmann were thikning up ways of fixing these currents. The currents in Fermi's theory appear to be a mixture of vector and axial (pseudo) vectors. This is known as V,A (vector, axial vector) theory. If you read books such as one of Feynmanns biographies, there was a lot of excitement at the time and Gellmann was the first realise that it was infact a V,-A form.

The dirac EM current is a pure vector current which emans the current under parity should behave like a vector and flip sign.

4.1.1 Vector and axial vector currents

In Dirac theory, γ^0 plays the role of the parity operator when it comes to Dirac spin, we will soon show how this happens. Under parity, a spinor transforms as such $u(\mathbf{k}, s) \rightarrow_P \eta u(-\mathbf{k}, s)$. We allow a phase factor (η) to multiply the second term which can only be ± 1 ($\eta = \pm 1$) - called intrinsic parity - so that if you act with the parity operator twice, you get back to your original state.

The matrix

$$\gamma_0 = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right) \tag{4.7}$$

and we would like to considet his acting on $u(\mathbf{k}, s)$. The most common representaiton we have is

$$u(\mathbf{k}, s) = \sqrt{\omega + m}\begin{pmatrix} \phi_s \\ \frac{\sigma \cdot \mathbf{k}}{\omega + m} \end{pmatrix} \tag{4.8}$$

where

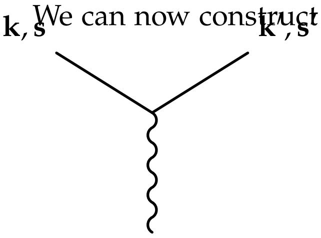
$$\phi_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for spin } \uparrow \tag{4.9}$$

$$\phi_s = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ for spin } \downarrow \tag{4.10}$$

and so

$$\gamma^0 u(k, s) = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \phi_s \\ \frac{\sigma \cdot \mathbf{k}}{\omega + m} \end{pmatrix} = \begin{pmatrix} \phi_s \\ -\frac{\sigma \cdot \mathbf{k}}{\omega + m} \end{pmatrix} = u(-\mathbf{k}, s) \tag{4.11}$$

We can now construct vector and axial-vector currents:



which is represented mathematically as

$$V^\mu = \bar{u}(\mathbf{k}', s)\gamma^\mu u(\mathbf{k}, s) \tag{4.12}$$

and under parity trnasofmration we get

$$V^\mu = \bar{u}(-\mathbf{k}', s)\gamma^\mu u(-\mathbf{k}, s)(1) \tag{4.13}$$

and we want to show that hte vector part of this,$\mu = 1, 2, 3$, comes out with a minus sign relative to the original current. The first step is to write

$$u(-\mathbf{k}, s) = \gamma^0 u(\mathbf{k}, s) \tag{4.14}$$

and taking the † of both sides (the hermitian conjugate) we get

$$u^\dagger(\mathbf{k}, s)\gamma^0 = u^\dagger(-\mathbf{k}, s) \tag{4.15}$$

and this is because ${\gamma^0}^\dagger = \gamma^0$ and so we continue

$$\bar{u}(\mathbf{k}, s) = u^\dagger(-\mathbf{k}, s) \tag{4.16}$$

then multiply by γ^0 from the RHS to get

$$\bar{u}(\mathbf{k}, s)\gamma^0 = u^\dagger(-\mathbf{k}, s)\gamma^0 = \bar{u}(-\mathbf{k}, s) \tag{4.17}$$

(1) can be written as

$$\bar{u}(\mathbf{k}', s')\gamma^0\gamma^\mu\gamma^0 u(\mathbf{k}, s) \tag{4.18}$$

This is the behaviour of V^μ under parity. But $V^\mu(V^0, \mathbf{V})$ and so $V^0 \to^P \bar{u}(\mathbf{k}', s)\gamma^0\gamma^0\gamma^0 u(\mathbf{k}, s)$ where $(\gamma^0)^2 = I$. Thus $V^0 \to^P \bar{u}(\mathbf{k}', s)\gamma^0 u(\mathbf{k}, s)$ so it transforms to itself and is thus invariant.

For the vector components ($\mu \neq 0$) we use the anticommutator relation $\{\gamma^\mu, \gamma^n u\} = 2g^{\mu\nu} I$, so $\gamma^\mu\gamma^0 = -\gamma^0\gamma^\mu$ for $\mu \neq 0$

$$\bar{u}(\mathbf{k}', s')\gamma^0\gamma^\mu\gamma^0 u(\mathbf{k}, s) - \bar{u}(\mathbf{k}', s')\gamma^\mu u(\mathbf{k}, s) \tag{4.19}$$

and so $V^\mu \to^p (V_0, -\mathbf{V})$ under parity. So there is a change of sign as you would expect with a vector. Hence it is a vector curernt. Thus, a pure vector current will not explain parity violation. We need something else - just a normal EM current with a γ^μ is not what we need in electroweak theory - we need something else. This motivates the construction of an axial current, lets construct one.

4.1.2 Axial current

This involves defining $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and this has the proeprty that $(\gamma^5)^2 = I$ and $\{\gamma^5, \gamma^\mu\} = 0$.

Lets consider

$$A^\mu = \bar{u}\gamma^\mu\gamma_5 u \tag{4.20}$$

whereas before we called the vector current V^μ, we will now denote the axial vector as A^μ. This will give an axial-vector which does not change sign under a parity transformation. Under parity this becomes

$$\bar{u}(-\mathbf{k}', s')\gamma^\mu\gamma_5 u(-\mathbf{k}, s) = \bar{u}(\mathbf{k}', s')\gamma^0\gamma^\mu\gamma_5\gamma^0 u(\mathbf{k}, s) \tag{4.21}$$

Under parity

$$A^0 \to^P \bar{u}(\mathbf{k}', s')\gamma_5\gamma^0 u(\mathbf{k}, s) = -\bar{u}(\mathbf{k}', s')\gamma^0\gamma_5 u(\mathbf{k}, s) = -A^0 \tag{4.22}$$

and so under parity transformation the time like component of this current flips sign and thus has a pseudo-scalar like behaviour, but the space like component will not flip sign.

For $\mu \neq 0$, we will anticommute γ^0 through twice to get $\bar{u}(\mathbf{k}', s')\gamma^\mu \gamma_5 u = A^\mu$ and tehrefore we now have a current that under parity acts as $A^\mu \rightarrow^P (-A^0, +\mathbf{A})$. We now have an axial vector and in Fermi's theory we can put in a linear combination of vector and axial vector current and to be consistent with the data what we found was this is simply a V,-A. So we take the vector current and subtract the axial-vector. So parity violation is pretty easy to accomodate once we include the above construction. Parity violation is explained by a V,-A current $\bar{\psi}\gamma^\mu(1-\gamma_5)\psi$.

Its simple enough to build parity violation into the theory, but there is a more serious issue with the theory which will pave the way towards the Higgs mechanism.

4.2 Search for a guage theory

The issue is that IVB theory resolved some of the probems with unitarity violation in Fermi's theory. What do we mean by it solved some of the problems? It means the problems fell into two kinds, those which were resolved and those which were found later and weren't resolved. There are '2 situations':

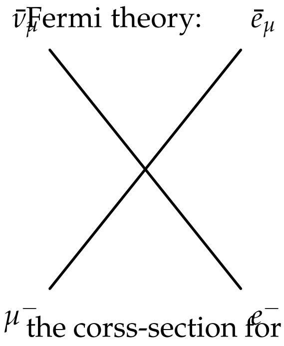
a) violation of unitarity is cured by an IVB
   this is what made people happy with this for a while and tehre were processes where the problem went away

b) processes were found later where IVB itself is sick and has its own problems

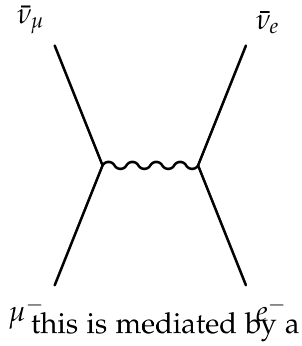
4.2.1 Example

$$\bar{\nu}_\mu + \mu^- \rightarrow \bar{\nu}_e + e^- \tag{4.23}$$

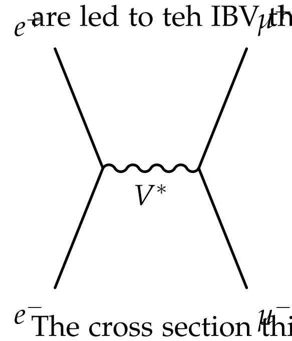
Fermi theory:



the corss-section for this process goes as $G_F^2 E^2$ where E is the center-of-mass energy. The problem was that the corss-section grows uncontrollably with E^2 and has no upper limit. THe IVB theory looks as such



this is mediated by a W^- boson. There is an analogous process to this in QED and this has a perfectly fine behaviour. So just by contrasting Fermis theory to QED we are led to teh IBV theory.



The cross section this time however has a different behaviour, $\sigma \sim \frac{4\pi\alpha^2}{3E^2}$. The cross-section falls of as E^2. This is a dimensionful coupling (with dimension GeV^{-2}) and falls-off with $1/E^2$ and is physically acceptable - i.e does not violate unitarity. There is a $1/q^2$ in the propagator which leaves the $1/E^2$ in the final answer. This is what Fermi's theory is missing and this would cure the high-energy unitarity violation. The problem is in QED we have a massless gauge boosn and in IVB with a massive vector boson - this makes a huge difference.

In IVB theory the amplitude is

$$i\frac{g^2}{2}\bar{u}(e)\gamma_\mu(1-\gamma_5)u(\nu_e)(\frac{-g^{\mu\nu}+k^\mu k^\nu/M_W^2}{k^2-M_W^2})\bar{v}(\nu_\mu)\gamma_\nu\frac{(1-\gamma_5)}{2}v(\mu) \tag{4.24}$$

this is the same as for QED processes, but we now have the introduced $(1-\gamma_5)$ and the progpagator now has the mass of the W boson in it and it has that behaviour with a pole at M_W. At high-energies the $1/E^2$ factor coems from the propagator (and its factor $1/k^2$). We have this $1/k^2$ limit and if the $k^\mu k^\nu$ terms were absent we would get a decent high-energy behaviour exactly like QED as the propagator would be the same ($k^2 - M_W^2$ at high energy is just k^2). The issue what happens to the $k^\mu k^\nu$ at high energies as they have the potnetial to cancel the k^2 on the bottom thus removing the $1/E^2$ dependence in our final result and violating unitarity. This is where what process you're considering makes a difference. If the W's occur as internal lines in Feymann graphs tehse $k^\mu k^\nu$ terms are harmless - they can be traded off for electron and muon masses and so they produce nothign significant. However if you have external W's emitted into the final state we cannot do this and so we cannot eliminate these terms and stop uncontrolled growth of cross-section.

g is now a dimensionless coupling and we can compare this with the QED amplitude for $e^+e^- \to \mu^+\mu^-$

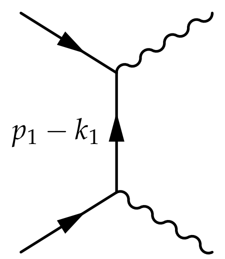
$$ie^2\bar{V}(e)\gamma_\mu u(e)(-\frac{g^{\mu\nu}}{k^2})\bar{u}(\mu)\gamma_\nu V(\mu) \tag{4.25}$$

Comparing this to the previous amplitude in IVB theory there is an additional, problematic factor of $k^\mu k^\nu/M_W^2$. Interestingly, lets also check what happens when we take the M_W mass to zero. In this limit we get a divergence, this suggests that something real is going on. You cant take the mass to 0 and recover QED. Alarm bells should start ringing as we have a problematic term which hasa divergence when the mass is taken to 0 and does not reproduce QED when you tkae the mass to 0 as is true with photons.

You dont have to worry about this is some processes, but there are certain processes with external lines where this factor is an issue.

Using momentum conservation, you can replace these k^μ factors and write thme in terms of the electron and neutrino momenta ($k^\mu = P_e^\mu + P_\nu^\mu$ - final state momenta) through momentum conservation. Once you introduce these momenta and replace the k^μ with these momenta, the dirac equation comes into play $(\not{p} - m)\psi = 0$. We can see from this, that $(\not{p} - m)u = 0$ and we can replace the $k^\mu k^\nu$ terms by those proportional to the electron mass m_e (ignoring neutrino mass terms as they are so small). Therefore at high energies, this is just a negligible contribution (as $m_e \simeq 511 MeV/c^2$ and is small). So infact, the high energy behaviour is $1/E^2$ like QED. Lets try to draw a diagram where this is not the case

c) Lets consider processes where W's are present as external lines e.g $\nu_\mu \bar{\nu}_\mu \to W^+ W^-$ via μ^- exchange.



Here the amplitude is, for given polarisation states λ_1 and λ_2 of the W's,

$$M_{\lambda_1,\lambda_2} = g^2 \epsilon_\mu^{-*}(k_2, \lambda_2)\epsilon_\nu^{+*}(k_1, \lambda_1)\bar{v}(p_2)\gamma^\mu(1-\gamma_5)\frac{(\not{p}_1 - \not{k}_1)}{(p_1 - k_1)^2}\gamma^\nu(1-\gamma_5)u(p_1) \tag{4.26}$$

For $\sum_{\lambda_1,\lambda_2} |M|^2$ i.e for the cross-section we get... - now when we take the conjugate of the M (as needed for the absolute of the matrix element) the epsilon stars become epsilons, so for each massive gaauge boson we get $\epsilon^*\epsilon$, there is a sum over the polarisation sates $\sum_\lambda \epsilon_\mu(k,\lambda)\epsilon_\nu^*(k,\lambda)$

The result for a massive boson (spin 1) is

$$\sum_\lambda \epsilon_\mu(k,\lambda)\epsilon^*(k,\lambda) = (-g_{\mu\nu} + \frac{k_\mu k_\nu}{M_W^2}) \tag{4.27}$$

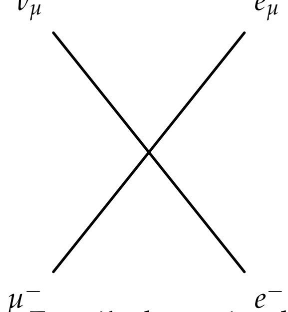
but now the $k^\mu k^\nu$ produces a bad high-energy behaviour proportional to E^2. So now there is no escape from the $k^\mu k^\nu$ terms and we cannot use the dirac spinor to trade thme off for massives which are negligible at high energy. $k_\mu k_\nu$ come from the longitudinal components of the polarisation vectors i.e they come from $\epsilon_\mu \sim \frac{k_\mu}{M_W}$. There is no longitundla polarisation for a massless particle. For an electron, such terms would be gauge dpeendent and there would be a guage fixing term here and thus would be unphysical terms. At the moment as they are, these polarisations look like they are physcial paraenters, but after we introduce a guage dependent term we will see a solution.

This is because the longitidunal modes have not produced properly into the theory, they have been introduced in a very ad-hoc way.

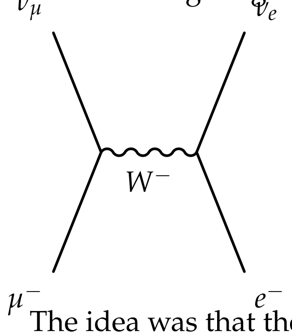
Lecture 8

Recap of last lecture:

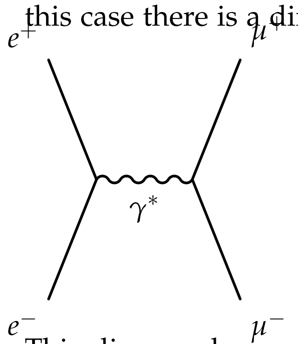
Last lecture we were getting towards the Higgs mechanism and the need for massive particles. To see the reasons why people started about these things, you need to look back to Fermi's theory.



Fermi's theory is a low-energy effective theory with a bad behaviour at high-energy - this leads to violated unitarity $\sigma \sim G_F^2 E^2$. This was extended to the IVB theory with the following diagrams

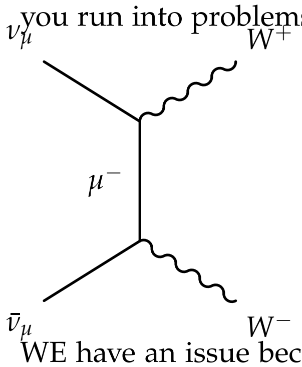


The idea was that the propagator ($\frac{-g^{\mu\nu}+k^\mu k^\nu/M_W^2}{k^2-M_W^2}$) for this massive particle would fix the high-energy problems with its characteristic $1/q^2$ fall-off. But the terms $k^\mu k^\nu$, which come from the longitudinal polarisation states (ϵ_L^μ), can spoil the potential to fix the high-energy corrections. Whne these particles are present as internal liens, you are safe and the $k^\mu k^\nu$ terms play no role and can be traded for fermion mass terms (using the Dirac equation). For amplitudes where W's are internal the $k^\mu k^\nu$ are hamrless. In this case there is a direct anaology with a QED process



This diagram has a good high-energy behaviour and the cross section falls of as E^2 ($\omega \sim \frac{4\pi\alpha^2}{3E^2}$) so there is no violation of unitarity. The $1/E^2$ comes from the $1/q^2$ photon propagator. So we haev a succesful analogy with QED when the W bosons are internal liens.

But when you have these W particles as external lines (i.e emitted in the final state), you run into problems



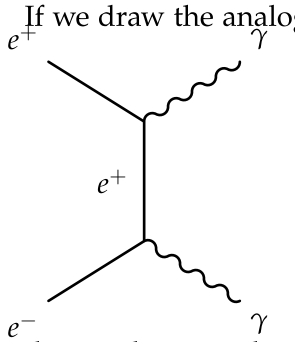
WE have an issue because of the $k^\mu k^\nu$ terms and this time we dont have a propagator as W is not an internal line, but we still have the sum of hte polarisations over these W bosons and so we meet the longitudinal polarisations again.

The W^+ poliarisaiton sum produces

$$\sum_\lambda \epsilon_\mu(k,\lambda)\epsilon^*_\nu(k,\lambda) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{M_W^2} \tag{4.28}$$

the last term gives you the problem as they cannot be traded off for mass temrs as you can for a propagator. So now we have bad high-energy behaviour which is proportional to E^2.

If we draw the analogous QED process



due to the two photons, the polarisaitons are purely transverse. The sum over polarisiations of the massless γ's does not have this problme as there is no longitudinal poliarisaiton. So we are moving towards realising that massive vector particles and their longitdunal polarisitaons are causing all our problems.

One can also compare propagators for QED and IVB theory, we get

$$\frac{-g^{\mu\nu} + k^\mu k^\nu / M_W^2}{k^2 - M_W^2}, \qquad \frac{1}{k^2}(-g_{\mu\nu} + (1-\epsilon)\frac{k_\mu k_\nu}{k^2}) \tag{4.29}$$

Its intersting that you cant just take the 0 mass limut of the W propagator to get the QED proapgator. This tells us where some of our problems are coming from!

The $M_W \to 0$ limit does not gibe us back our massless theory (QED) and therefore is not sensible (as we know QED works). Its interesting that the same variable which does not sensibly go to a massless limit, is also causing our high-energhy problems with its longitudinal polarisations. We also expect there to be a gauge fixing parameter in IVB propagator and its just not there. We are looking for a gauge invariant theroy so that our troublesome terms drop out in a guge invariance theory just as in QED. In

QED, the 1-ϵ term just drops out due to gauge invariance where epsiolin is a guage praremter. 1-ϵ $k_\mu k_\nu$ are just guage artifcats and so cannot effect physical observables - physical observables are not dependent on gauge. The gauge invariance is conducted to make sure that your theory is not invariant, as in realiyt physics is not invaraint.

We have introduced the mass of W and stukc it in by hand. We have wrecked gauge invariance in the IVB theroy and perhaps we should move torwads making it gauge invariant and perhaps we will introduce a gauge parameter which will cause the unitarity violating to drop out. This suggests that gauge invariance has something to do with the problem.

Though quickly, instead of making the IVB gauge invariant, we could go in the other direction and require that the photon has mass

$$m^2 A_\mu A^\mu \tag{4.30}$$

But if this term is added to the Lagrangian, this destroys gauge invariance. Clearly adding in a mass term by hand is not the way forward.

So we look to construct a gauge invariant theory with massive particles which is delicate as massive particles can only interact over a short range. We are on the way to introducing the Higgs mechanism.

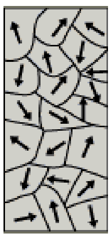
4.3 Higgs Mechanism

The spontaneous symmetry breaking mechanism on which the Gihhs mechanism relies, has already been observed in other fiedls. But to use it within particle physics to solve the probelem of gauge invariance with amssive particles was the extra step required.

The Higgs mechanism is that by which the gauge bosons acquire mass. This mechanism acts through spontaneous symmetry breaking which was already a known phenomenon in nature e.g

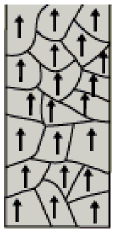
1) Block of ferromagnetic material
   The hamiltonian has rotational symmetry. The block of ferrogmagnetic material has a symmetry, but the GS doesn't share that symmetry. The ground state above the Curie temperature, T_c, is illustrated below there is a random alignment of spins

   FIGURE 4.1: In bulk material the domains usually cancel, leaving the material unmagnetised

   (magnetic moments) with no preferred direction. This is consistent with the Hamiltonian and the net magnetisation is zero on average, $M = 0$. But below T_c, there is a phase transition and the spins align to a preferred direction spins align and

FIGURE 4.2: The domains are now aligned



so $M \neq 0$. Rotational symmetry is lost and not shared by the ground state and this happens spontaneously once cooled below T_c. Rotational symmetry is lost i.e spontaneous symmetry breaking. When you reheat the magnet above T_c, the spins form domains, whereby a group of spins align, but the different groups have different orientations, so there is an artifact of the original symmetry. The Higgs is analogous to this. The maths that accompanies these theories is very similar to the Higgs mechanism and these theories were written decades prior to the Higgs mechanism.

2) Another example is superconductors
What happens in a super conductor is that the magnetic fields for example decay exponentially inside the super conductor.

$$\mathbf{B} = \mathbf{B_0} e^{-\mu x} \tag{4.31}$$

This is consisten with a short ranged force or an effectively massive photon, i.e a short ranged photon field. The photon appears to be massive because of the short-ranged field. So we have a massless particle in a semiconductor which acts like/'appears' to be massive.

In a semicodnuctor e^+e^- pairs (so-called cooper pairs) play the role of a scalar field and after spontaneous symmetry breaking photon acquires an effective mass. This is extremely similar to the Higgs mechanism.

You take a vacuum and you take the propagation of these fields through vacuum. So the Higgs mechanism is a scalar field which acts like the electron-positron pairs to give vector bosons mass.

When Higgs came up with the Higgs mechanism, people were thinking of other spontaneous symmetry breaking mechanisms as a fundamental scalar field (such as the Higgs field (Cooper pairs)) had not been discovered. So although the theory was already in place, particle physics theorists need to work out how to use it in the context of our theory to ensure that the vector bososn (W and Z) acquire mass from this fundamental field.

Before discussing the full Higgs mechanism, we consider

1. Spontaneous symmetry breaking of a discrete symmetry
2. Spontaneous symmetry breaking of a continuous global symemtry
3. Spontaneous symmetry breaking of a local (gauge) symmetry

4.3.1 Discrete symmetry

Lets write down the Lagrangian of the discrete symmetry (ϕ^4 theory). Again until the Higgs was discovered ϕ^4 theory was not one that people considered. The first example of this is actually seen in nature in elementary particle physics and nature actually makes use of ϕ^4.

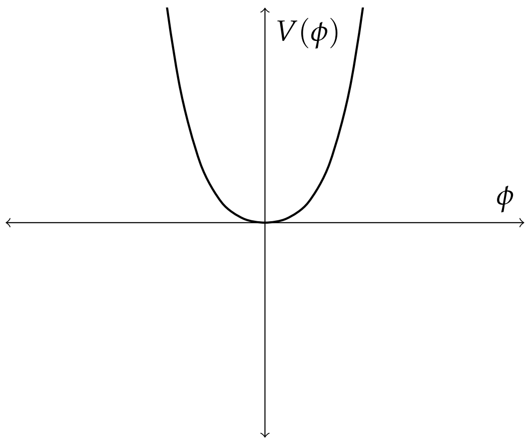
Start with

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\mu^2\phi^2 - \frac{1}{4}\lambda\phi^4 \tag{4.32}$$

the fact of $\frac{1}{2}$ is chosen for convenience for the time being, remember that any additional term that crops up could be assimilated into the ϕ term. The corresponds to

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4 \tag{4.33}$$

We can see that a Z_2 symmetry exists i.e under $\phi \to -\phi$. The potential looks like



This is quadratic near the origin and quartic elsewhere. This has a minimum at $\phi = 0$ - this is the ground (vacuum) state, which is one where the field vanishes i.e $\langle 0| \phi |0\rangle$.

Now lets do the analogous thing that happens when you cool the ferromangetic below the curie temperature - we get a phase transition. Here the phase transition is one that changes the sign of the μ^2 term.

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4 \tag{4.34}$$

This is different, it still has Z_2 symmetry but we wish to work out what the extrema positions of these potentials. So

$$V'(\phi) = -\mu^2\phi + \lambda\phi^3 = 0 \tag{4.35}$$

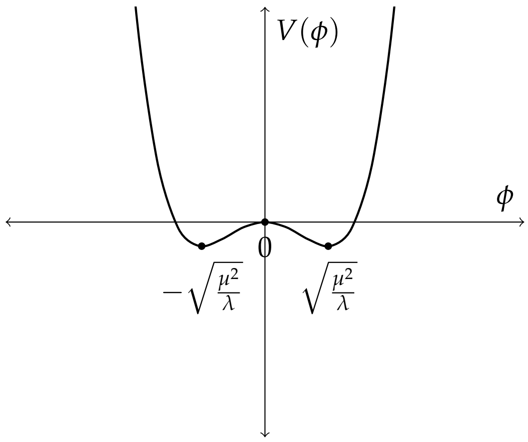
which gives

$$\phi = 0 \text{ or } \pm\sqrt{\frac{\mu^2}{\lambda}} \tag{4.36}$$

Taking the second derivative

$$V''(\phi) = -\mu^2 + 3\lambda\phi^2 = 0 \tag{4.37}$$

you find that $V'' < 0$ for $\phi = 0$ i.e this extrema is a maximum whereas at $\pm\sqrt{\frac{\mu^2}{\lambda}}$, $V''(\phi) = 2\mu^2 > 0$, which indicates these extrema are minima. This is illustated below



Now the ground states are at $\phi = \pm\sqrt{\frac{\mu^2}{\lambda}}$ and so $\langle 0|\,\phi\,|0\rangle = \pm V$ where $V = \pm\sqrt{\frac{\mu^2}{\lambda}}$. One problem with this Lagrangian is that we get a complex mass, this is the not the physical mass of a phi particle. Essentially the phi field is not the physical field here. the physical particle spectrum is the one that appears when you perturb the true vacuum, so the excitations of the true vacuum are the modes which correspond to physical particles. We introduce a modification to get the true vacuum. Instead of expanding about $\phi = 0$ we expand about one of the minima and the right physics will be found around the true vacuum value. (finish discussion)

Lets us expand about a chosen vacuum, at $\phi = V$, and define a shifted field $\xi = \phi - V$ where $\langle 0|\,\xi\,|0\rangle = 0$. We want to write our Lagrangian in terms of the ξ field and we get

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \xi)(\partial^\mu \xi) + \frac{1}{2}\mu^2(\xi + V)^2 - \frac{\lambda}{4}(\xi + V)^4 \tag{4.38}$$

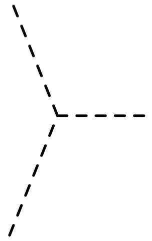
$$= \frac{1}{2}(\partial_\mu \xi)(\partial^\mu \xi) + \frac{1}{2}\mu^2(\xi^2 + V^2 + 2\xi V)^2 - \frac{\lambda}{4}(\xi^4 + V^4 + 6\xi^2 V^2 + 4\xi V^3 + 4\xi^3 V)^4 \tag{4.39}$$

just pure constant terms in the Lagrangian are playing no physical role (i.e V^4, V^2 etc.) so we discard them and we write the $-\xi V^3\lambda$ as $-\xi V\frac{\mu^2}{\lambda}\lambda$ (since $V^2 = \mu^2/\lambda$) $= -\xi V\mu^2$ and we get cancellation. The result is

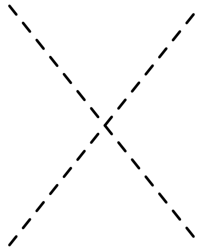
$$\frac{1}{2}(\partial_\mu \xi)(\partial^\mu \xi) + \xi^2(\frac{1}{2}\mu^2 - \frac{3}{2}\lambda V^2) - \xi^3 V\lambda - \frac{\lambda}{4}\xi^4 \tag{4.40}$$

where we have collected the terms into powers of ξ, as we know the ξ^2 term will correspond to the mass. Using $V^2 = \mu^2/\lambda$, the coefficient of ξ^2 is just $-\mu^2$. Note the ξ^2 terms produces a $-\mu^2\xi^2$ term and so now it has the correct sign to be a mass term. The ξ^2 term correspond to the $-\frac{1}{2}m_\xi^2\xi^2$ usual mass term. This straight away gives us m_ξ, where $m_\xi = \sqrt{2}\mu$ - a physical mass. We have learnt that if you expand about the true vacuum (true grondstate) in a theory with a spontaneously broken symmetry, you get the correct theory. (Note: expanding about the true ground state is obviously important as QFT models particles as excitations of fields from their groundstaes (vacuum states)).

It is important to observe that there are two types of interaction terms (vertices) that weren't there originally



which is the (λV) cubic coupling and



which is the $\frac{\lambda}{4}$quartic coupling. If this were teh actual Higgs mechansim in the SM, we would be able to measure these couplings in experiment.

Although we wont do the actual Higgs mechanism until later, there are some eky points in this example that will become evident when we derive the Higgs mechanism. Our Lagrangian has lost its Z_2 symmetry and only signs of the symmetry breaking mechanism which are left after we have done all of this is a relationship between these 2 couplings. What we are doing in the Higgs case as well, is when we discover the Higgs we don't just say right we have the Nobel prize its all over. We needs to establish the symmetry breaking pattern which is the relationships between couplings etc. which then is a sign of how the symmetry has been broken. In this toy example we see there is a relationship between cubic and quartic couplings and the mass of the physical particle (m_ξ in this case) but we will see that the are not independent of each other. The original Z_2 symmetry is lost, but there are signs of the underlying symmetry breaking mechanism still prevalent in the relationship between couplings and masses.

This means for isntance that hte ratio of the square of the cubic coupling $(\lambda^2 V^2)$ to the quartic coupling $(\frac{\lambda}{4})$

$$\frac{g_3^2}{g_4} = \frac{\lambda^2 V^2}{\lambda} = \lambda V^2 = \mu^2 \tag{4.41}$$

i.e the relationship between the interaction couplings is μ^2 - this could be one of the tests in experiment to make sure we have understood the symmetry breaking mechanism. We have also reabsorbed the $\frac{1}{4}$ factor into a redefinition of the coupling - the constant factors are not important.

Ultimately we want gauge invariant symmetries, but as intermediate step along the way (which gives you further physical insight) we shall consider the case of continuous global symmetries.

4.3.2 Continuous global symmetry

Lets takteh the field ϕ and geneaalise the example we saw earlier to something more exotic, i.e a vector in N dimensions and consider rotations of this undere $SO(N)$ for example. This is what model builder do..... (finish discussion)

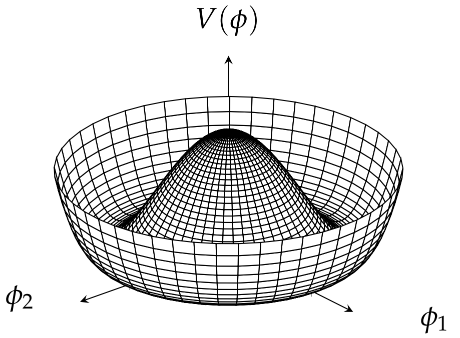
Lets say the Lagrangian $\mathcal{L}$ has a symmetry under a Lie group transformation. The simplst example is to let ϕ be a complex field. Then the Lagrangian looks like

$$\mathcal{L} = (\partial^\mu \phi)^* (\partial_\mu \phi) + \mu^2 |\phi|^2 - \lambda |\phi|^4 \tag{4.42}$$

we can tthink of this as one complex field or two real fields representing the real and complex parts of the field. The potnetial is a function of $|\phi|$ now

$$V = V(|\phi|) \tag{4.43}$$

you can do the same exercise we did previously and we will get minima at $|\phi| = \sqrt{\frac{\mu^2}{2\lambda}}$. How many vacuum states does this correspond to? Previously we had two, but not it corresponds to an infinite number as you could introduce a phase factor which disappears when you take the modulus i.e $\phi = \sqrt{\frac{\mu^2}{2\lambda}} e^{i\theta}$. There is a circle (in the ϕ^1, ϕ^2 plane) of degenerate minima values i.e infinite possible vacuua with a radius of $|\phi|$.



$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \tag{4.44}$$

the Lagrangian there is a symmetry under $\phi \to e^{i\theta}\phi$ and thats not repsected by teh vacuum because you go from one vacuum to another in this transfomration. Just like the previous case from phi to -phi we go from one vacuum to the other.

Lets choose our vaccum state with $\theta = 0$, so its all along the ϕ_1 axis say. In this choice we have

$$\phi_1 = \frac{V}{\sqrt{2}}, \qquad V = \sqrt{\frac{\mu^2}{\lambda}}, \qquad \phi_2 = 0 \tag{4.45}$$

so the entoire choice of modulus is down to ϕ_1.

so we have our vacuum choice, and we must expand around this vacuum. But we can expand in two different directions. You can span in teh radial direction giving phi_1 a small cahnge or the tangential direction and in both cases we will get particles. We can see that moving in the radial direction we are trying to push up against the potential, which will push us back. However when we expand in the tangential direction, there is no resistance from the potential and so all the tangential directions are equivalent.

This suggest that in the radial direction we will get the physics we have seen before and we get a mass, the mass is teh interia that is felt by rolling up a potential and it acts as a massive particle due to the effects of the potential and inertia. However in the tangential direction we get massless particles as tehre is no interia, so we get a massless particle in the tangential direction.

Expand about the vacuum, where $\phi = \frac{1}{\sqrt{2}}(V + \xi + i\chi)$. In terms of the ϕ_1 and ϕ_2 fields, $\mathcal{L}$ becomes

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi_1 - i\partial^\mu\phi_2)(\partial_\mu\phi_1 + i\partial_\mu\phi_2) + \mu^2|\phi|^2 - \lambda|\phi|^4 \tag{4.46}$$

$$= \frac{1}{2}(\partial_\mu\phi_1)(\partial^\mu\phi_1) + \frac{1}{2}(\partial_\mu\phi_2)(\partial^\mu\phi_2) + \mu^2|\phi|^2 - \lambda|\phi|^4 \tag{4.47}$$

Using $\phi = \frac{1}{\sqrt{2}}(V + \xi + i\chi)$ we get

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) + \frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi) + \frac{1}{2}]((V+\xi)^2 + \chi^2) - \frac{\lambda}{4}((V+\xi)^2 + \chi^2)^2 \tag{4.48}$$

$$= \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) + \frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi) + \frac{\mu^2}{2}(V^2 + \xi^2 + \chi^2 + 2V\chi) - \frac{\lambda}{4}(V^2 + \xi^2 + \chi^2 + 2V\xi)^2 \tag{4.49}$$

$$= \overbrace{\frac{1}{2}\underline{(\partial_\mu\xi)}(\partial^\mu\xi) + \frac{1}{2}\underline{(\partial_\mu\chi)}(\partial^\mu\chi)}^{\text{kinetic terms}} + \frac{\mu^2}{2}(V^2 + \xi^2 + \chi^2 + 2V\chi) - \frac{\lambda}{4}(V^2 + \xi^2 + \chi^2 + 2V\xi)^2 \tag{4.50}$$

$$\mathcal{L} = \text{kinetic terms} + \frac{\lambda V^2}{2}(\xi^2 + \chi^2 + 2V\xi) - \frac{\lambda}{4}[(\xi^2 + \chi^2)^2 + 4V^2\xi^2 + 4V\xi^3 + 4V\xi\chi^2 \tag{4.51}$$

$$+ 4V^3\xi + 2V^2(\xi^2 + \chi^2)] \tag{4.52}$$

where in the last line we have substituted for μ^2 and we have discarded constants. When you expand all of this out you get the following

$$\mathcal{L} = \text{kinetic terms} + \frac{\lambda V^2}{2}(\xi^2 + \chi^2 + 2V\xi) - \frac{\lambda}{4}(\xi^2 + \chi^2)^2 - \lambda V^2\xi^2 - \lambda V\xi^3 - \lambda V\xi\chi^2 \tag{4.53}$$

$$- \lambda V^3\xi - \frac{\lambda V^2}{2}(\xi^2 + \chi^2) \tag{4.54}$$

and now we cancel terms to arrive at

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) + \frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi) + \frac{\lambda}{4}(\xi^2 + \chi^2)^2 - \lambda V^2\xi^2 - \lambda V\xi^2 - \lambda V\xi\chi^2 \tag{4.55}$$

where the mass term of the ξ field is $-\lambda V^2\xi^2$, there are a bunch of interaction temrs, <u>but</u> there is no mass term for the χ field. Ouyr theory predicts that there are two scalar particles (fields), where one is massfull and the other is massless and there are interactions between these fields. We started with a single field and sponteanlusly broke symmetry and we have ended up with a physical particle spectrum with two particles and there is no mass for one of them. The fact that there is no mass temr for one of the fields is no accident. This is a special case of what is known as Goldstones theorem and χ is an example of what is known as the Goldstone boson. In theories where there is a global symmetry, these massless Goldstone bosons arise. The reason we don't hear about Goldstone bosons at the LHC is because in a gauge theory these Goldstone bosons don't appear as final states, but they are intermediate interaction bosons which describe the longitudinal polarisations of the W boson in a theory with gauge symmetry and spontaneous symmetry breaking, the Goldstone particles act as artifacts in a gauge theory.

4.3.3 Goldstones theorem

States that if a symmetry G, some continuous global symmetry, is spontaneously broken so that the residual symmetry of the vacuum (the vacuum still has a symmetry, but is less than that of the Lagrangian, so if L has a big symmetry group G, the vacuum is a subset symmetry group) is some subgroup H, then the number of generators which are broke is equal to the number of massless scalars in the theory (and these massless scalars are called Goldstone bosons). So what we did was a special case of this, a simple case where we had a symmetry which was $\phi \to \phi e^{i\theta}$ (otherwise known as an SO(2) symmetry) and the symmetry has been broken so the vacuum is no longer symmetric - but as SO(2) has 1 generator, this no longer works and so we get one Goldstone boson.

Lecture 9

Last lecture we covered sponteaneous breaking of discrete and continuous global symmetries. Soemthing different happene in the case of continuous glboal symmetry that doesn't happen in the case of discrete symmetries. We took a complex scalar field and worked out what happened.

What is spontaneous symmetry breaking? The symmetry of the ground state does not share that of the rest of the system. The symmetry of the Hamiltonian is lost. The ground state for instance may choose a preferred direction and the symmetry is lost.

The difference between discrete and continuous symmetry was the emergence of a massless particle (Goldstone boson). We discussed Goldstones theorem and in the example we look at last lecture U(1) symmetry, we got a massless particle pop out. We saw that every broken generator gives a massless particle and in the example we did last time, there was one generator associated with a rotation in the ϕ_1 ϕ_2 plane which was broken.

Today we shall move on and discuss what happens when you have a gauge symmetry and here again the physics is different. Lets go back to the previous example for a second. We took the example of a complex scalar field

$$V(\phi) = -\mu^2 |\phi|^2 + \lambda |\phi|^4 \tag{4.56}$$

with a minima at $|\phi| = \sqrt{\frac{\mu^2}{2\lambda}} = \frac{V}{\sqrt{2}}$. This ios modulus phi, so we actually get an infinite set as we can multiply by an arbitrary phase factor $\phi = \frac{V}{\sqrt{2}} e^{i\theta}$. We get a whole set of minima and the relationship between the different vacuum states reflects the original symmetry. As you go around phi 1 phi 2 plane, as you go in the tangential direction you transit the vacuum states and that is just the original symmetry - the potential is symmetric. To derive the particle spectrum of the theory, we make a choice of vacuum $\phi = \frac{V}{\sqrt{2}}$ (we choose the vacuum to be real, say along the ϕ_1 axis) and then expand about this. We get

$$\phi = \frac{V + \xi}{\sqrt{2}} e^{i\chi/V} \tag{4.57}$$

ξ is in there as we can change the modulus, so we can have a field ξ which his pushing you in the radial direction and we have a phase factor which moves us around in the tangential direction, which is the most general thing we can do. Te factor of V in the exponential is chosen for convenience to cancel other factors. But in general the point is the two things we can do are to change the modulus and phase. Then in order to get the particle spectrum we have to consider $\xi, \chi \ll 1$ and make a first order expansion, so one has

$$\phi \sim \frac{1}{\sqrt{2}}(V + \xi + i\chi) \tag{4.58}$$

last lecture we used this form. We plugged it into the Lagrangian and derived what it looked like. We found the ξ has a physical mass and the χ field was massless.

So today we want to move beyond this into a situation where we have gauge symmetry. This is the case that is going to be interestign to us with teh SM. WE want to derive the gauge boson masses this is why we started palying this game in the first place. we had issues with massive force carrying particles violating unitarity. Therefore the whole reason for dealing wit hte Higgs mechanism is not to play with a toy model but to solve a problem and that invollves working with a Lagrangian that has a gauge symmerty and seeignt what hjappens as a consequence of that in relation to the gauge bosons. So far we have been dwelling on the phi (Higgs) field, the scalar field - i.e using sclar field theory (which will soon introduce the Higgs) and we have been looking at what happens when you intrdocue scalars but when you have a gauge field arond, we must also what happens to this gauge field (i.e photon field) in the presecne of broken symmertrty.

4.4 Guage symmetry

The first step will be to write down a gauge invariant Lagrangian. The original Lagrangian we wrote down before the gauge symmetries was

$$\mathcal{L} = (\partial_\mu \phi) j (\partial^\mu \phi) - V(|\phi|) \tag{4.59}$$

We would eventually like to consider SU(2), but for simplicity we consider the case of Abelian U(1) gauge transform i.e $\phi \rightarrow e^{i\theta}\phi$ and we want symmetry under this where the fact that its a gauge transform specifically means that $\theta = \theta(x)$, the transformation is a function of space-time. Now this Lagrangian as it stands is not symmetric under a gauge transformation, so what do we have to do to this Lagrangian to get it gauge invariant?

What we do is introduce another field which absorbs the gradient of a scalar (the extra terms which come about due to the gauge transformation) and we introduce the covariant derivative D_μ to counteract the effects of this transformation. We now have the Lagrangian

$$\mathcal{L} = (D_\mu \phi) j (D^\mu \phi) - V(|\phi|) \tag{4.60}$$

Then one repeats the same exercise we did before. But now we have to plug in for the fields. Again we have the same starting point. Start with $\phi = \frac{V}{\sqrt{2}}$ as our vacuum and expanding around this we get

$$\phi = \frac{1}{\sqrt{2}}(V+H)e^{i\theta/V} \simeq \frac{1}{\sqrt{2}}(V+H+i\theta) \tag{4.61}$$

where H and θ are just like ξ and χ from the previous example (as we are now introducing the Higgs mechanism). This is the same as we had before. We write the potential in terms of the H and θ fields as

$$V = \frac{\lambda}{4}(H^2+\theta^2)^2 + \lambda V^2 H^2 + \lambda V H^3 + \lambda V H \theta^2 = \frac{\lambda}{4}(H^4+\theta^4+2H^2\theta^2) + \lambda V^2 H^2 \tag{4.62}$$

$$+ \lambda V H^3 + \lambda V H \theta^2 \tag{4.63}$$

and using $\lambda V^{=} \mu^2$, which is basically the Higgs mass, we have

$$\frac{\lambda}{4}(H^2+\theta^4+2H^2\theta^2) + \mu^2 H^2 + \sqrt{\lambda}\mu(H^3 + H\theta^2) \tag{4.64}$$

this is a massive Higgs field H with mass $M_H = \sqrt{2}\mu$. One of the consequences is that the Higgs has mass. We are still not doing the full Standard model, but this is the basic idea. We have to consider kinetic terms as these will be different. The kinetic terms $((D_\mu\phi)^*(D^\mu\phi))$ now become a bit more complicated, but this is where the power of the Higgs mechanism really comes in. One of the consequences of the kinetic terms is that the Gauge bosons, while reserving Gauge invariance, still acquire mass - which is otherwise not the case, we would have to put the masses in by hand and this breaks unitarity.

$$(D_\mu\phi)^*(D^\mu\phi) = [(\partial_\mu + igA_\mu)\frac{1}{\sqrt{2}}(V+H+i\theta)]^*[(\partial^\mu + igA^\mu)\frac{1}{\sqrt{2}}(V+H+i\theta)] \tag{4.65}$$

$$= \frac{1}{2}(\partial_\mu - igA_\mu)(V+H-i\theta)(\partial^\mu + igA^\mu)(V+H+i\theta) \tag{4.66}$$

$$= \frac{1}{2}(\partial_\mu H - i\partial_\mu\theta - igA_\mu V - igA_\mu H - g\theta A_\mu)(\partial^\mu H + i\partial^\mu\theta \tag{4.67}$$

$$+ igA^\mu V + igA^\mu H - g\theta A^\mu) \tag{4.68}$$

This is basically a complex vector contracted with its complex conjugate. So you're going to get the square of the real part plus the square of the imaginary (whereby square we mean the 4-vector squared) so you can do this virtually by inspection and we get

$$= \frac{1}{2}[(\partial_\mu H - g\theta A_\mu)^2 + (\partial_\mu\theta + gA_\mu V + gA_\mu H)^2] \tag{4.69}$$

we can see that all of the physics is here, but to see the separate terms we must expand it out to see what is going on

$$= \frac{1}{2}[(\partial_\mu H)(\partial^\mu H) + g^2\theta^2 A_\mu A^\mu - 2g\theta(\partial^\mu H)A_\mu + (\partial_\mu \theta)(\partial^\mu \theta) + g^2 V^2 A_\mu A^\mu \tag{4.70}$$

$$+ g^2 H^2 A_\mu A^\mu + 2gVA_\mu(\partial^\mu \theta) + 2gHA_\mu \partial^\mu \theta + 2g^2 VHA_\mu A^\mu] \tag{4.71}$$

the $g^2V^2A_\mu A^\mu$ term is the one where we see a photon 'acquiring' mass. The photons don't acquire mass in reality, as we don't have a U(1) symmetric Higgs sector we have SU(2), so that will give rise to 3 gauge boson masses corresponding to generators of SU(2), but in this model the photon acquires mass - infact in a superconductor this is exactly the physics. The rest of the terms are now interaction terms between the Higgs and the gauge bosons - the LHC are currently trying to measure those kinds of interactions. Rewriting this equation into a more convenient form we get (writing the kinetic terms, mass terms together etc.)

$$= \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{1}{2}(\partial_\mu \theta)(\partial^\mu \theta) + \frac{1}{2}g^2 v^2 A_\mu A^\mu + \frac{1}{2}g^2(H^2 + \theta^2)A_\mu A^\mu \tag{4.72}$$

$$- gA_\mu(\theta\partial^\mu H - H\partial^\mu \theta) + gVA_\mu \partial^\mu \theta + g^2 VA_\mu A^\mu H \tag{4.73}$$

we can see that the θ field has no mass, as we would expect from Goldstones theorem. We can see that the Gauge boson has a mass but there are also terms which mix the Gauge field and the θ field and this is something we will discuss in a bit more detail later on.

The way to go from the action to the Feynmann rules, you can essentially just right down the rules straight from the Lagrangian. The way to identify propagators in particles is to look at terms which are bilinear in the field, like the term which involes an A field and theta field. When you have ugly terms which mix two fields, this means physical somethign weird is going on because tehse states cannot mix during propagation which means you cant define the propagator ofd a gauge field with this term still there. It suggests that these kinds of terms are unphysical and will turn out later to be gauge artifacts.

The A_μ field has acquired a mass, $m_A = gV$, but apart form that how do you interpret the terms $gVA_\mu\partial^\mu\theta + g^2VA_\mu A^\mu H$ - the second is a Higgs γ, γ inteaction with coupling strength g^2? These are interaction terms involving the couplings of Higgs and gauge fields like $g^2VA_\mu A^\mu H$, and one of the features of Higgs physics is that we have the coupling $g^2V = g \cdot gV = g \cdot m_A$. This corresponds to a well-known feature of Higgs physics - that the interaction strength is proportional to particle masses.

However, there is still a Goldstone boson, θ, flying around - we have looked for and found the Higgs particle but why are we not looking for a θ particle at the LHC? There are mixing terms involing the θ and gauge fields, $gVA_\mu\partial^\mu\theta = m_A A_\mu \partial^\mu \theta$, which we have trouble interpreting, which suggests taht this theta field is not physical. If you count the number of degrres of freedom before and after symmetry beraking, before we had a mssless photon with two polarisations states (2 DOF), a complex scaalr field with 2 DOF (with a total of 4 DOF) and after symmetry breaking we have a amssive gauge particle, one higgs field and a theta field (5 DOF), so we have gained a DOF. Actually that is not really the case as the theta field is not a genuinely physical field. Why? Because it is still a Goldstone boson and the terms that mix the Goldstone boson and gauge fields $gVA_\mu\partial^\mu\theta = m_A A_\mu\partial^\mu\theta$ is actually equal to $-m_A\theta\partial_\mu A^\mu$ after an integration by parts.

Recall:

Terms that are bilinear in fields contribute to the propagators and such mixing terms make the interpretation of a propagator for A_μ difficult.

There is a mixing of states and these terms must be eliminated to enable us to get the physical spectrum. Also if one counts the degrees of freedom, then before symmetry breaking we have a complex scalar field (with 2 degrees of freedom) and a massless photon (2 polarisation states and thus 2 DOF). There is thus a total of 4 DOF before symmetry breaking. However after symmetry breaking occurs we have a H field, θ field and a massive gauge boson (3 polarisation states and thus 3 DOF) giving a total of 5 DoF.

What might happen to this theta field. In the previous case, the χ field was just there and we said it was a goldstone boson with 0 mass. But this goldstone boson has mixing terms with the gauge field which would predict interactions and measurable consequences, what is the difference here? The θ field isn't a real particle (not a physical field), but what is the reasoning behind this.

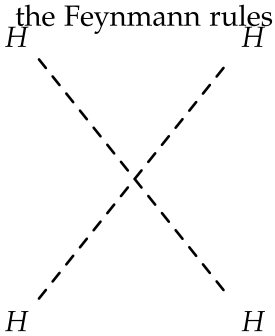
$$\phi = \frac{V+H}{\sqrt{2}}e^{i\theta(x)/V} \tag{4.74}$$

Trying to thikn about what is different? The obvious line of though is to think about what have we done differently this time, that we didnt do before. Last time we discussed continuous global symmetry so what it is that is different about this case? In this case there is a particular symmetry we are dealing with which we didn't have last time, we didn't write a Lagrangian that had this particular symmetry, what is the symmetry we are talking about? How is this exercise different from the continuous case before Easter? The difference between the Lagrangian we used in lecture 8 and this lecture is gauge invariance. We introduce the theta field as we have a compelx field nd a vacuum, take it along real axis and then expand about this vacuum. The theta field was introduced as we coudl change the modulus or chagne hte phase of the vacuum, we can exapnd by having a small extension in the radial direction or tangential direction and the theta field was a little perturbation in the tangential direction (change in the phase). This, alogn with the fact that there is a gauge invariant lagrangian, whjat does gauge invariance let us do? we can make an arbitrary chagne of phase, because now we have a potential where we can put these arbitrary changes of phase int he covariant derivative and I am guaranteed that such changes of phase will not effect my physics so I can always make a gauge transformation $\phi \to \phi' = e^{-i\theta(x)/V}\phi$ and $A_\mu \to A'_\mu = A_\mu + \frac{1}{gV}(\partial_\mu\theta)$. There is no physics in the phase, it can be shuffled off into a redefinition of the potential which has no consequences because you can always add a gradient of a scalar to your potential and everything stays the same. That means the theta fields corresponds to a choice of gauge - i.e we can gauge it away. The θ field is a gauge artifact and in a suitable gauge it can be removed entirely. This is called the unitary gauge. This also means that Goldstone bosons are a gauge artifact (in this case). If somebody doing model building actually wants a massless scalar around, they would use a different gauge that doesn't rid the model of Goldstone bosons. However as the theta field is troublesome, we can just gauge it away. Thus, simply dropping the θ field gives

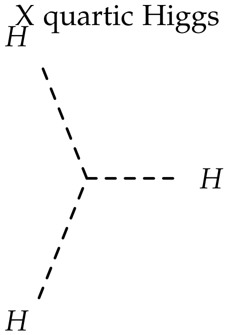
$$\mathcal{L} = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{1}{2}m_A^2 A_\mu A^\mu + \frac{1}{2}g^2 H^2 A_\mu A^\mu + g m_A A^\mu A_\mu H - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{4.75}$$

$$- \frac{\lambda H^4}{4} - \frac{1}{2}m_H^2 H^2 - \sqrt{\frac{\lambda}{2}} H^3 m_H \tag{4.76}$$

where $m_H^2 = 2\mu^2$. This Lagrangian as it is has the advantage of giving you the physical particle spectrum, we only work with physical particles and no theta. This is wonderful, at least up to a point. This gauge is economical you get rid of non-physical particles and deal with the real interactions of physical particles and we can read off the Feynmann rules from this Lagrangian, so life seems good. We get

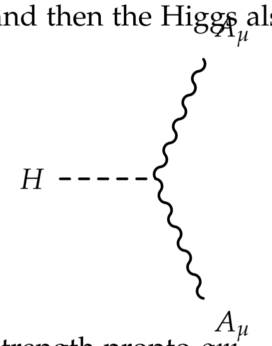


X quartic Higgs self-couplings with coupling stregth proportional to lambda

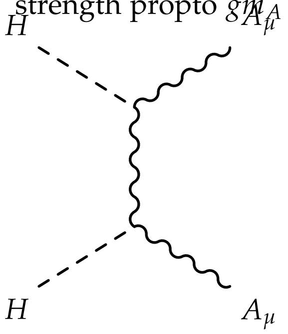


cubic Higgs coupling, with streght propto $\sqrt{\frac{\lambda}{2}} m_H$

and then the Higgs also couples to the gauge bosons



strength propto $g m_A$



strength propto g^2

This is the unitary gauge picture, it looks quite nice, you can work with these physical particles and calculate cross sections based on these interactions only and there is no theta field. This is good to a point but what one things look like one day, don't look like the other day. You get a bit ambitious and want to calculate things beyond the tree level and then we run into problems with the unitary gauge. One of the issues with the unitary gauge is an issue we have met before when we wrote down the propagator for a massive gauge particle (by hand), in the 0 limit we don't recover massless propagators. The unitary has this exact problem, but luckily we can escape from this problem by choosing another gauge.

The unitary gauge theory also suffers from the same problem we had before we introduced a gauge, but luckily we can manipulate the gauge in this case to resolve the problem. From the Lagrangian one can read off the propagator. The recipe to do this (derive the propagator from the Lagrangian) is to transform to momentum space and focus on bilinear and quadratic terms in A_μ. You essentially get the inverse of the propagator is the coefficient of those bilinear terms.

The first step is to take your Lagrangian and write the fields in terms of their Fourier transforms which means you can take an x-space expression for the photon field and just write it in terms of the Fourier transform in the usual way

$$A_\mu(x) = \int e^{ipx}\tilde{A}_\mu(p)d^4p \tag{4.77}$$

then you integrate over x and everything is transformed into momentum space and one gets

$$A_\mu(-p)(-g^{\mu\nu}p^2 + p^\mu p^\nu + g^{\mu\nu}m_A^2)A_\nu(p) \tag{4.78}$$

this is what the bilinear terms become in momentum space. The propagator is the inverse of the coefficient of $A_\mu(-p)A_\nu(p)$ which gives

$$-i(g_{\mu\nu} - \frac{p_\mu p_\nu]}{m_A^2})\frac{1}{p^2 - m_A^2} \tag{4.79}$$

where we have left out some normalisation. This propagator has the issue we identified earlier. What happens in the limit of very large momentum/energy? As $p \to \infty$ it goes as p^0 i.e goes to 1. You would like the propagator to fall off as $1/p^2$ to have good energy behaviour, but this propagator does not do this and so produces a bad high-energy behaviour. So this Unitary gauge is not a good choice for precise calculations beyond tree level (whereby tree-level we mean there are no loop calculations). However to first order (tree level) this propagator is fine and the Feynmann rules we wrote down will give you right answer at leading order. If we would like to include loops we need to manipulate our gauge and give the Goldstone bosons an artificial mass.

For rough calculations (to first order) this gauge is fine, but in order to do precise calculations (to higher order) we must abandon our gauge for another choice of gauge. The problem with this is that the Goldstone bosons will reappear, we will have to live with it and it becomes an intermediate step in the calculation just like a gauge-fixing parameter. You then need to pretend that these Goldstone bosons are real particles and use them in the calculations with the Feynmann rules and in the end you get the same answer whatever gauge you employ. Its just in the unitary gauge you cant do loop calculations. We ultimately use our gauge freedom to get rid of Goldstone bosons, but for accurate calculations we still need them around.

We therefore have a motivation to look for a different gauge. Goldstone Bosons give us the problematic term with mixing between the A and theta field. This means we cannot define a propagator for teh A field. So we woudl still like to do that even though we may not want to completely remove the Goldstone bosons as we don't want to work in the unitary gauge, we want to work in another gauge but still want to eliminate the mixing terms (to define a propagator). Can we think of other gauges that do this job. Another way is simply to remove the whole theta, but when you do that it is only good for tree level. This motivates other gauges where we get rid of the mixing terms, but still have good high-energy behaviour.

4.4.1 $R\xi$ gauges

This is a more conventional gauge choice than unitary gauge

$$\mathcal{L} \to \mathcal{L} + \mathcal{L}_k \tag{4.80}$$

$$\mathcal{L}_k = -\frac{1}{2(1-\xi)}(\partial_\mu A^\mu - (1-\xi)m_A\theta)^2 \tag{4.81}$$

$$= -\frac{1}{2(1-\xi)}\partial_\mu A^\mu \partial_\nu A^\nu - \frac{1-\xi}{2}m_A^2\theta^2 + m_A\theta\partial_\mu A^\mu \tag{4.82}$$

we have seen a similar gauge fixing in the case of QED, where to define the propagator we need a gauge fixing term. The $m_A\theta\partial_\mu A^\mu$ term cancels the mixing term, so we can define a propagator for the A field, theta field and there is no mixing between them. The theta field also acquires a mass, there is a mass term that comes out for the theta field with the same mass as that for A. The Goldstone boson in these kinds of gauges also acquires a mass ($= m_A$ in this case). So now we can go through this exercise again, take this Lagrangian and work out the terms which are bilinear in the field to get a propagator. We can repeat the derivation of the propagator by collecting bilinear terms to get

$$-\frac{1}{2}\tilde{A}^\mu(-p)(-g_{\mu\nu}(p^2 - m_A^2) + p_\mu p_\nu - \frac{p_\mu p_\nu}{1-\xi})\tilde{A}^\nu(p) \tag{4.83}$$

which gives the propagator

$$-\frac{i}{(p^2 - m_A^2)}(g_{\mu\nu} - \frac{\xi p_\mu p_\nu}{p^2 - (1-\xi)m_A^2}) \tag{4.84}$$

This is what comes out in defining the propagator for the A field in this gauge. It has some differences to the last case. Now what happens in the limit that p goes to infinity? This propagator goes to 0. In the limit that the mass goes to 0, we recover the propagator for the gauge-fixed photon, we have recovered QED.

Now it appears that the theory is consistent. You can do calculations in this $R\xi$ gauge - and we should get the same answers in whatever gauge we use - it has a good high-energy behaviour and the only issue is you have to live with is the existence of these Goldstone gauge bosons and their interactions. We have to treat the Goldstone bosons as real particles, even though we know they are a gauge artifact that only exists in this gauge. The choice of gauge goes hand in hand with choosing whether the Goldstone bosons exist or not.

We now have in $R\xi$ gauges, Goldstone bosons and they interact with gauge bosons, with the Higgs and themselves. These interactions all need to be included in a consistent calculation in this gauge. These gauges are useful for higher-order calculations where they preserve the re-normalisability of the theory which is otherwise lost.

Chapter 5

Chapter 5

5.1 The Standard Model (Glashow, Weinberg, Salam)

We need to take the matter content of the theory (quarks and leptons) and gauge bosons (force carrying particles), we want to write a Dirac Lagrangian for the fermionic part, a Maxwell-like Lagrangian for the gauge bosons (γ's, W's, Z's) and want to account for gauge boson masses in a gauge invariant way, which means we don't just write them as a Lagrangian but also have to consider the Higgs sector and we want to account for Fermion masses. As we will see in a minute, you cant just stick fermion masses in the Lagrangian. You have to start with a massless Dirac Lagrangian i.e massless fermions and because a fermion mass term in the Lagrangian also violates invariance, we cannot write down Fermion masses directly in the L and these again will have to come from the Higgs sector.

One of the key experimental observations which brings us to write the correct Lagrangian is parity violation. We have already discussed this in some detail when we were discussing beta decay. This means that only left handed states participate in charged current weak interactions. This means we will have to give several interactions to the left and right-handed sates - we will treat the LH and RH states on different footing. This implies that we are going to have to treat interactions for the left-handed and right-handed states differently. Just as a reminder, left-handed and right-handed states refer to states that in the massless limit are eigenstates of an operator called the Helicity, $\hat{S} = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E}$ with eigenvalues of -1 and $+1$ respectively. The helictiy is just the component of the spin in the direction of motion. The RH state is the one with spin along the direction of motion. Only the LH states will go with the charge-current interaction. We shall give the LH SU(2) interactions with W bosons etc. and the RH states will be treated differently.

The first thing we do is split the Dirac Lagrangian into a part with LH components and a part with RH components. Then we have an issue with fermion masses as fermion mass terms mix the LH and RH in a way we will discuss now. If you we want to give separate interactions to LH and RH then fermion mass terms screw up gauge invariance.

Firstly, we can project out the left and right -handed states by using the appropriate projection operators

$$\hat{P}_L = \frac{(I - \gamma_5)}{2} \tag{5.1}$$

$$\hat{P}_R = \frac{(I + \gamma_5)}{2} \tag{5.2}$$

these are the operators that project out the left and right-handed fields. What does that mean? They can be used to define

$$\psi_L = \frac{(I - \gamma_5)}{2}\psi \tag{5.3}$$

$$\psi_R = \frac{(I + \gamma_5)}{2}\psi \tag{5.4}$$

so a general ψ can be written as a sum of left and right -handed components.

Generally $\psi = \psi_L + \psi_R$, i.e ψ can be written as a linear superposition of left and right -handed fermion fields and for a massless spinor, ψ_L and ψ_R should be eigenstates of the Helicity operator ($\hat{S} = \frac{\boldsymbol{\sigma}\cdot\mathbf{P}}{E}$) with eigenvalues -1 and $+1$ - CHECK THIS AT HOME. Hint: To show this use the standard representation for the Dirac spinor,

$$u(p) = \frac{1}{\sqrt{E}}\begin{pmatrix} E\chi \\ \boldsymbol{\sigma}\cdot\mathbf{P}\chi \end{pmatrix} \tag{5.5}$$

where χ is a two component spinor and use $\gamma_5 = ...copyinfromearlier(dottedmatrix$ and $\hat{S} = \frac{1}{E}\left(\begin{array}{c|c} \boldsymbol{\sigma}\cdot\mathbf{P} & 0 \\ \hline 0 & \boldsymbol{\sigma}\cdot\mathbf{P} \end{array}\right)$.

For the kinetic terms of the Dirac Lagrangian

$$\bar{\psi}\gamma^\mu\partial_\mu\psi = \bar{\psi}_L\gamma^\mu\partial_\mu\psi_L + \bar{\psi}_R\gamma^\mu\partial_\mu\psi_R \tag{5.6}$$

splits into separate left and right-handed terms. So this is one of the terms in the Dirac Lagrangian, the kinetic term, ultimately this will become the covariant derivative and we will put separate gauge fields in the first part and separate gauge fields in the second part. So the first transforms under SU(2), whilst the second term doesn't. So there are separate interactions in the LH apart and RH part and so the LH part has purely SU(2) and RH part purely U(1). There are also mass terms of the form m$\bar{\psi}\psi$ and if we write these we could get our masses for free (as we stuck the masses in by hand), but this is where problems start to emerge. Although the kinetic terms separate into purely LH and RH, when splitting into LH and RH components a mass term $m\bar{\psi}\psi = m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$, mixes the left and right fields and if I require the LH states to transform under SU(2), but not the RH states, then this term is not gauge invariant as the right handed spinor cannot compensate the gauge transformation of the LH spinor. Normally with our mass terms, the $\bar{\psi}\psi$ terms are gauge invariant as one cancels out the extra terms found in the other, but not here as we wish one half to transform under SU(2) and not the other.

Also, in $m\bar{\psi}\psi = m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$ what happens to terms like $\bar{\psi}_L\psi_L$? They cancel out. Why? You can demonstrate, using the definitions ψ_L and ψ_R.

You would expect extra terms in the above mass term, but we will now show why they become 0 (do the exercise at home)

Note

Terms like $\bar{\psi}_L\psi_L$ and $\bar{\psi}_R\psi_R$ vanish

$$\psi_L^\dagger = [(\frac{(I-\gamma_5)}{2}\psi)]^\dagger = \psi^\dagger\frac{(I-\gamma_5)}{2} \qquad as(\gamma_5^\dagger = \gamma_5) \tag{5.7}$$

$$\bar{\psi}_L = \psi_L^\dagger\gamma^0 = \psi^\dagger\frac{(I-\gamma_5)}{2}\gamma^0 = \psi^\dagger\gamma^0\frac{(I+\gamma_5)}{2} = \bar{\psi}\frac{(I+\gamma_5)}{2} \qquad as(\{\gamma^5,\gamma^\mu\} = 0) \tag{5.8}$$

$$\bar{\psi}_L\psi_L = \bar{\psi}\frac{(I+\gamma_5)}{2}\frac{(\gamma-\gamma_5)}{2}\psi = 0 \qquad as\gamma_5^2 = I \tag{5.9}$$

So we have an issue with gauge invariance n the mass terms, and we are going to have to invoke another method to generate fermion masses. this answer comes again from the coupling of the Higgs.

So we arrange the LH states into SU(2) doublets. We start with only a single generation of quarks and leptons (later on we can add additional generations after we get the general idea)

$$\text{Quark flavour doublet } \begin{pmatrix} u_L \\ d_L \end{pmatrix} = q_L \tag{5.10}$$

$$\text{Lepton doublet } \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = l_L \tag{5.11}$$

$$\tag{5.12}$$

what sort of components are these? It is SU(2), so we know the generators are Pauli matrices, we say they are doublets, what sort are they? What are the up and down quarks? these are isospin doublets. Remaining states use u_R, d_R, e_R - you can theoretically write down ν_R happily, but as of yet there is no experimental evidence to justify this and we shall treat neutrinos as the SM was originally formulated, with no mass. Under SU(2) rotations (in isospin space)

$$U = e^{-i\omega^a\tau^a}, where \tau_a = \frac{1}{2}\sigma_a \tag{5.13}$$

$$l_L \to l_L' = e^{-i\omega^a\tau^a}l_L \tag{5.14}$$

this is just a transfomration of the lepton doublet by 2x2 matrices. But RH states do not transform in this way, they aer gauge singlets and so dont transform under SU(2) rotations, they transform as

$$e_R \to e_R' = e_R \tag{5.15}$$

and similarly for quarks.

Now these are three matrices, so there are three generators, when we work out the gauge invariance - when you want a gauge invariant theory under SU(2), how many gauge fields are we going to have to invoke. We will invoke 3, a field for each of these rotations. We will go through the process of introducing a field ot cancel the gradient of a scalar (of which there are 3 terms as there are 3 generators) and so we need 3 guage fields and thus 3 gauge bosons. How many gauge bosons do we know of in EW theory? We know of 4. So there is an additional gauge field and this corresponds to another U(1) symmetry. Tehse particels do not just carry isospins, they carry charge.

In addition to the 4 known gauge bosons fro electroweak theory, there is another U(1) symmetry around and thus another associated gauge field.

An this U(1) symmetry is associated not with charge but what we call hypercharge. Then the hypercharge combines with isospin to produce what we call charge. So initially the U(1) of the SM and SU(2) x U(1) is a hyper charge so we have a hypercharge symmetry. This corresponds to a hypercharge quantum number. A rotation in hypercharge space is a U(1) rotation.

Under this U(1) transform the fields transform as $\psi \to \psi' = e^{-i\omega Y}\psi$ where Y is the hypercharge. It is just a change in phase, as you would expect with a U(1) transform. Of course this phase depends on space-time, we will go throug hthe recipe again and produce gradients of this scalar and we need another compensating field, which is the hyper charge fiedl and we will se how this combination of hypercahrge and isospin adn associated gauge fields will give us the photon.

You can trade off the hypercahrge, by talking in terms of the charge and that is what we are more familiar with. To reproduce the standard model interactions hypercharge is chosen so as to give the right charge to the particles

$$q = I_3 + Y \tag{5.16}$$

where I_3 is the z-component (the eigenvalue of I_3) of isospin. This must add up so the hyper is what we usually know as charge minus the isospin.

The hypercharges are as follows

$$\text{lepton doublet } Y_{l_L} = -\frac{1}{2} \tag{5.17}$$

$$Y_{e_R} = -1 \text{ as there is no isospin} \tag{5.18}$$

$$\text{quark doublet } Y_{q_L} = \frac{1}{6} \tag{5.19}$$

$$Y_{u_R} = \frac{2}{3} \tag{5.20}$$

$$Y_{d_R} = -\frac{1}{3} \tag{5.21}$$

Taking the l_L doublet as an example. The top component of the doublet has isospin $+\frac{1}{2}$ and so $Y + I_3 = 0$, corresponding to the fact that we know the neutrino doesn't have charge.

So now what does the SM Lagrangian look like? Now we have accommodated our Dirac fields in LH doublets, the RH ones are left separately. We can give the LH ones SU(2) gauge transformations, everything transforms under hyercharge according ot a U(1) transformation. So next lecture we will write the full SM Lagrangian, but we are not there just yet. The standard model Lagrangian should then look like

$$\mathcal{L} = \mathcal{L}_{dirac} + \mathcal{L}_{gauge-bosons} + \mathcal{L}_{Higgs} + \text{ fermion mass terms} \tag{5.22}$$

$\mathcal{L}_{dirac}$ includes only the kinetic terms, no mass terms, $\mathcal{L}_{gauge-bosons}$ would have their part which corresponds to the propagation of gauge bosons - for a photon this piece would look like $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ and we have to write a similar term for gluons, the SU(2) field and hypercharge. There are a whole bunch of terms corresponding to the gauge bosons of that kind. Then $\mathcal{L}_{Higgs}$ is responsible for giving mass to the gauge bosons and also the fermions. Then we have to add terms which should generate the fermion mass terms which involve coupling to the Higgs - we will do this term later. We srtarted to meet the Higgs part last time when we did the U(1) Higgs model, but now we must do this in SU(2). So the mass generating terms are the only things we havn't seen yet.

5.2 Higgs Sector

We have done the Higgs sector in the U(1) case and we did this so that we don't feel too puzzled by the SU(2) case. The idea here i that the Higgs is a complex SU(2) doublet

$$\phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \tag{5.23}$$

instead of being a complex scalar field, it is a complex SU(2) doublet - it is a two-vector in SU(2) space. $\phi_1 + i\phi_2$ is the up I_3 component and $\phi_3 + i\phi_4$ is the down I_3 component. This matrix transforms under 2×2 SU(2) matrices and the hyper charge we will see is $Y = \frac{1}{2}$ and we will see later that this corresponds to a neutral scalar particle. The Higgs potential, with spontaneous symmetry breaking now looks like this (V has to be a funtion of an SU(2) scalar, which $\phi^\dagger\phi$ - which is in contrast to the $|\phi|$ we used last lecture)

$$V(\phi^\dagger\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \tag{5.24}$$

where $\phi^\dagger\phi = abs\phi^2 = \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2$. The minima are in the usual place at $\phi^\dagger\phi = \frac{\mu^2}{2\lambda} = \frac{V^2}{2}$. Note before we had $|\phi|^2$ in place of $\phi^\dagger\phi$ in the U(1) case. One can choose the vacuum so that $\phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ V \end{pmatrix}$. So we choose it be aligned along this component as we normally do when we make a choice of vacuum. We choose this modulus but we have to chose it pointing along a particular direction. This is a special choice of vacuum and when we expand about this vacuum, we must introduce a fields which can either change by phase of modulus just as we did last lecture. Last time we had the modulus coresponding to V/sqrt(2) and there was a phase factor which we could times teh modulus by, but here you can give a phase to this, but this is just our choice of vacuum.

The original Lagrangian had a symmetry under the generators of $SU(2) \times U(1)$. The original Lagrangian has a symmetry under SU(2) x U(1), this is obvious that this potential is an SU(2) scalar, $\phi^\dagger\phi$ is not a vector its a dot product in SU(2) space, its an SU(2) scalar, it is independent of rotations in SU(2) case, it is an U(1) scalar as well because the phase cancels with $\phi^\dagger\phi$. We want to see, what is the symmetry of the vacuum because what is the definition of spontaneous symmetry breaking? You want to see that the GS does not share the symmetry of the potential. So what is the symmetry of this state? we will find out

Let us examine the symmetry of ϕ_0 under $SU(2) \times U(1)$. What is $SU(2) \times U(1)$? A general $SU(2) \times U(1)$ rotation can be expressed in terms of the generators of SU(2) whih are $t_a = \frac{1}{2}\sigma_a$ (Pauli matrices) and the generator of U(1) is YI (the hypercharge of the particles times by the identity matrix). A linear combination of generators in any particle group is also a generator in the same group, thus linear generators of the above are also generators. We want to make a particular combination, leaving t_1 and t_2 alone. We shall define the generators $t_3 \pm YI$ so we start koving towards the generator of charge - the genetor which creates the charge quantum number.

So consider $t_1, t_2, t_3 + YI$ and $t_3 - YI$. I take linear combinations of t_3 and YI. $t_3 + YI$ is particularly interesting because it will correspond to the generator which is to do with the charge and which will couple to photons.

$$t_3 + YI = \frac{1}{2}(\sigma_3 + I) \text{ (since } Y = \frac{1}{2} \text{ for Higgs) } = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.25}$$

and we want to check the action of this on the chosen vacuum

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ V \end{pmatrix} = 0 \tag{5.26}$$

Therefore $t_3 + YI$ annihilates the vacuum. This means that the exponential of this will leave the vacuum invariant, thus the vacuum has a symmetry under this operation. Thus $e^{i\omega(t_3+YI)}$ leaves the vacuum invariant. This means that the vacuum has a symmetry, its not like we have lost all the symmetry. Before we had symmetry under SU(2) x U(1), but now here you have only one generator which is a U(1) symmetry. So the vacuum has a residual symmetry U(1). This means that, and you will hear this statement made very often, $SU_I(2) \times U_Y(1)$ gets spontaneously broken down to $U_{EM}(1)$, the reason they say electromagnetism is because this corresponds now to charge. Because the third component of isospin plus hyperchage gives us charge. So the eigenstates of operator $e^{i\omega(t_3+YI)}$ gives us charge. This means there is a symmetry under $U_{EM}(1)$, so how many generators are broken? There are 3 broken generators. Thus there are three Goldstone bosons, but what happens to Goldstone bosons in gauge theory? Goldstone bosons are degrees of freedom which ultimately get eaten by the gauge bosons and give them longitudinal polarisations. So 3 gauge bosons acquire longitudinal polarisations and what does that mean about the masses? They are non-zero, so we have 3 massive gauge bosons. We have one more gauge field which is massless which comes from the $e^{i\omega(t_3+YI)}$ generator. This is the generator of charge and we know photons couple ot charge. The don't couple to Higgs boson as the Higgs field is invariant under rotations in this space, it doesn't carry charge, so there is no coupling between the Higgs and photons so the photons remain massless. But we do have 3 generators broken, 3 longitudinal polarisations emerge and they go into W,Z bosons.

1

3 massive gauge bosons and a massless photon. A general choice of vacuum would look like

$$\phi = \frac{1}{\sqrt{2}} e^{i(\omega^a t^a - \omega^3 YI)} \begin{pmatrix} 0 \\ V + H \end{pmatrix} \tag{5.27}$$

and in the unitary gauge $\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ V+H \end{pmatrix}$ Now we are going to stick in this unitary gauge and get he gauge boson masses out, like we did for the U(1) case before. We know that gauge boson masses will come from this part of the Lagrangian

$$(D^\mu\phi)^\dagger(D_\mu\phi) \tag{5.28}$$

we know that in these covariant derivatives, we will have our gauge fields and these will couple to the Higgs to give you the masses of the gauge bosons

$$D_\mu = \partial_\mu I + igt^a W^a_\mu + ig'YB_\mu \tag{5.29}$$

where B_μ is the hypercharge field. Each gauge field is associated with a gauge coupling where $igt^aW^a_\mu$ is associated with SU(2) and $ig'YB_\mu$ is associated with U(1)

$$D_\mu\phi = \frac{1}{\sqrt{2}}(\partial_\mu I + igt^aW^a_\mu + ig'YB_\mu I)\begin{pmatrix} 0 \\ V+H \end{pmatrix} \tag{5.30}$$

where

$$t^aW^a_\mu = \frac{1}{2}(\sigma^1W^1_\mu + \sigma^2W^2_\mu + \sigma^3W^3_\mu) \tag{5.31}$$

$$= \frac{1}{2}[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}W^1_\mu + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}W^2_\mu + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}W^3_\mu] \tag{5.32}$$

$$= \frac{1}{2}(\begin{pmatrix} W^3_\mu & W^1_\mu - iW^2_\mu \\ W^1_\mu + iW^2_\mu & -W^3_\mu \end{pmatrix}) = \frac{1}{2}\begin{pmatrix} W^3_\mu & \sqrt{2}W^+_\mu \\ \sqrt{2}W^-_\mu & -W^3_\mu \end{pmatrix} \tag{5.33}$$

and so

$$D_\mu\phi = \frac{1}{\sqrt{2}}[\begin{pmatrix} 0 \\ \partial_\mu H \end{pmatrix} + \frac{ig}{2}\begin{pmatrix} \sqrt{2}W^+_\mu(V+H) \\ -W^3_\mu(V+H) \end{pmatrix} + \frac{ig'}{2}B_\mu\begin{pmatrix} 0 \\ V+H \end{pmatrix}] \tag{5.34}$$

where we have used $Y_H = \frac{1}{2}$ and we have defined $W^+_\mu = \frac{W^1_\mu - iW^2_\mu}{\sqrt{2}}$ and $W^-_\mu = (W^+_\mu)^*$.

(I want B_μ spinor to only have a down component i.e 0 & V+H (on bottom) (finish discussion)

$$D_\mu = \sqrt{1}\sqrt{2}\begin{pmatrix} \frac{ig}{\sqrt{2}}W^+_\mu(V+H) \\ \partial_\mu H + \frac{ig'}{2}B_\mu(V+H) - \frac{ig}{2}W^3_\mu(V+H) \end{pmatrix} \tag{5.35}$$

and so

$$(D^\mu\phi)^\dagger(D_\mu\phi) = \frac{1}{2}\left(-\frac{ig}{\sqrt{2}}W^{\mu,-}(V+H), \quad \partial^\mu H - \frac{ig'}{2}B^\mu(V+H) + \frac{ig}{2}W^{\mu,3}(V+H)\right)\left(\begin{matrix} \frac{ig}{\sqrt{2}}W^+_\mu(\\ \partial_\mu H + \frac{ig'}{2}B_\mu(V+ \end{matrix}\right. \tag{5.36}$$

and multiplying this out we get (finish DISCUSSION)

$$= \frac{1}{2}(\frac{g^2}{2}W^{\mu,-}W^+_\mu V^2 + (\partial_\mu H)(\partial_\mu H) + \frac{V^2}{4}(g'B^\mu - gW^{\mu,3})^2) + \text{ interaction terms} \tag{5.37}$$

$$= \frac{g^2V^2}{4}W^{\mu,-}W^+_\mu + \frac{1}{2}(\partial_\mu H)^2 + \frac{V^2}{8}\underbrace{(g'B^\mu - gW^{\mu,3})^2}_{Z_\mu Z^\mu} \tag{5.38}$$

(finish DISCUSSION.......)

5.3 Lecture 11

From lloking at $(D^\mu\phi)^\dagger(D_\mu\phi)$ we got the mass spectrum which looked like

$$\frac{g^2v^2}{4}W^{\mu-}w^+_\mu + \frac{1}{2}(\partial^\mu H)^2 + \frac{V^2}{8}(g'B^\mu - gW^{\mu,3})^2 \tag{5.39}$$

where the B fields are the field associated with hypercharge and when we.... we shall define the linear combination of fields .... into a single field and we know, so what is physically relevant is the (finish DISCUSSION)

wE can either think of, $g'B^\mu - gW^{\mu,3}$ as a linear combination as a new field or visualise it as a rotation. One can introduce "rotated" fields

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \tag{5.40}$$

we can see here that when we multiply these matrices out we are going to get

$$= \begin{pmatrix} W^3_\mu\cos\theta_W - B_\mu\sin\theta_W \\ W^3_\mu\sin\theta_w + B_\mu\cos\theta_W \end{pmatrix} \tag{5.41}$$

and we are talking about, that is the Z field and photon are

$$Z_\mu = W^3_\mu\cos\theta_W - B_\mu\sin\theta_W \tag{5.42}$$

$$A_\mu = W^3_\mu\sin\theta_W + B_\mu\cos\theta_W \tag{5.43}$$

with $\tan\theta_W = \frac{g'}{g}$. This means that the Z gets a mass, whereas the photon does not get a mass. The $\frac{V^2}{8}(g'B^\mu - gW^{\mu,3})^2$ becomes $\frac{g^2V^2}{8\cos^2\theta_W}(\sin\theta_W B^\mu - \cos\theta_W W^{\mu,3})^2$; this is essential just $(Z_\mu)^2$. All we have done is define a combination of fields, Z_μ and perform a rotation.

$$\frac{g^2V^2}{8\cos^2\theta_W}(\sin\theta_W B^\mu - \cos\theta_W W^{\mu,3})^2 = \frac{g^2V^2}{8\cos^2\theta_W}Z_\mu Z^\mu \Rightarrow M_Z^2 = \frac{g^2V^2}{4\cos^2\theta_W} \tag{5.44}$$

and if you go back and look at the W mass terms, which looked like $\frac{g^2V^2}{4}W^{\mu,-}W^{\mu,+}$. You can write htis separately in terms of hte W_1 etc. and so it can be written as $\frac{g^2V^2}{8}(|W^-|^2 + |W^+|^2)$. W_1 and W_2 are the complex conjugate parts. From this we can see the mass of the W boson and this can be related to the mass of the Z boson.

Hence, $M_W^2 = M_Z^2\cos^2\theta_W$. This is interesting as we know that $tantheta_W$ is the ratio of the couplings and if you experiemtnally measure somehow the coupling ratios, then they should come out to be the ratio of the W and Z masses. Thus we can test the consitency of the standard model. However, $M_W^2 = M_Z^2 \cos^2\theta_W$ is only true at tree level. In general $M_W^2 = \rho M_Z^2 \cos^2\theta_W$ where $\rho = 1$ at tree level and receive corrections that depend logarithmically on m_H and m_t etc. They receive higher order corrections when we look beyond tree level.

At one point experimental measurements of the rho parameter were used to place limits on the Higgs mass as rho has a logarithmic relationship to the Higgs mass (which appears in higher order corrections).

We now know the mass of Higgs boson and and top quark and the deviations of experiential rho measurements from calculated values will be used to further test consistency of the standard model.

We may have subtler signs of BSM, such as through loop corrections. These are called indirect searches of BSM and use precision measurements to look at slight deviation from theory.

5.4 Fermion masses

Cannot introduce fermion mass directly into $\mathcal{L}_S M$, reminder of why (finish discussion)...... What oyu can do however is write the following sort of guage ivnariant, inteaction term which involves itneractions of the Higgs boson with fermions. THis is called the Yukawa term.

$$\mathcal{L}_{yukawa} = -Y_e \bar{l}_{Li}\phi_i e_R + \text{ h.c} \tag{5.45}$$

so there is a dot produce in SU(2) space between the LH dobulet and the Higgs doublet. e_R is the righthadned electron field. We are making our way to the t-shirt which had a couple of mistakes on it as well. This term is an SU(2) scalar, it doesnt transfomr under SU(2) and this kind of temr has zero net hypercharge. That means that this is totally gauge invariant under SU(2) X U(1) and we are happy to introduce this. Infact some would be mystified if this didnt exist as .... (finish discussion)

If we actually work this out, what does it become.

$$-\frac{Y_e}{\sqrt{2}}\begin{pmatrix}\bar{v} & \bar{e}_L\end{pmatrix}\begin{pmatrix}0 \\ V+H\end{pmatrix} e_R + \text{ h.c.} = -\frac{Y_e}{\sqrt{2}}\bar{e}_L(V+H)e_R + \text{ h.c.} \tag{5.46}$$

and in full, this can be written as

$$= -\frac{Y_e}{\sqrt{2}}(\bar{e}_L V e_R + \bar{e}_R V e_L) + \text{ interaction terms} \tag{5.47}$$

where the interaeciton temrs are proportional to higgs and thus proprotional to yukawa and thus proportional to the fermion mass terms themsevles. Continuing we get

$$= -\frac{Y_e}{\sqrt{2}} V\bar{e}e \tag{5.48}$$

the electron has acquired a mass which is $m_e = \sqrt{Y_e}\sqrt{2}V$ or in other words we can express teh Yukawa coupling in temrs of $Y_e = \sqrt{2}\frac{m_e}{V}$. We have worked out a way to get fermion mass. (finish DISCUSSION) about quarks.....

For d quarks one can write an identical kind of term which works

$$-Y_d \bar{q}_{Li} \phi_i d_R + \text{ h.c.} \tag{5.49}$$

and we get $m_d = \frac{Y_d}{\sqrt{2}} V$ where V is the Higgs vacuum expectation value. To give the u quarks a mass one has to work a bit harder and somehow flip th Higgs frm having only a low component to only an upper component. This is done in SU(2) space via the following. We write a term $-Y_u \epsilon_{ij} \bar{q}_{Li} \phi_j^* u_R + \text{ h.c.}$. (finish Discussion).... the epsilon rotates the qbar...... Again this is an SU(2) $\times$ U(1) scalar.

We want a matrix representation of the epslion object and so

$$\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.50}$$

whic hsi known as the metric tensor of SU(2). (finish Discussion) metric tensor...... Therefore

$$\epsilon_{ij} \phi_j^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{V+H}\sqrt{2} \end{pmatrix} = \begin{pmatrix} \sqrt{V+H}\sqrt{2} \\ 0 \end{pmatrix} \tag{5.51}$$

this has flipped the loer component of the Higgs fieldsi nto an upper component and if we now put this into the dot product we mentioend before, this comes out as

$$= -Y_u \epsilon_{ij} \bar{q}_{Li} \phi_j^* u_R + \text{ h.c.} \tag{5.52}$$

$$= -\frac{Y_\mu}{\sqrt{2}} (\bar{u}_L V u_R + \bar{u}_R V u_L) \tag{5.53}$$

$$= -\frac{Y_\mu}{\sqrt{2}} V(\bar{u}u) \tag{5.54}$$

$$\tag{5.55}$$

and so $m_u = \frac{Y_\mu}{\sqrt{2}} V$.

The Yukawa terms in $\mathcal{L}$ for a single generation are

$$\mathcal{L}_{yukawa} = -Y_e \bar{l}_{Li} \phi_i e_R - Y_d \bar{q}_{Li} \phi_i d_R - Y_u \epsilon_{ij} \bar{q}_{Li} \phi_j^* u_R \tag{5.56}$$

5.5 Kinetic terms for fermions

So far we havenŵritten down the kinetic terms for the Dirac part of the Lagrangian

$$\mathcal{L}_{fermion-kinetic} = i\bar{l}_L^T \gamma^\mu D_\mu l_L + i\bar{e}_R \gamma^\mu D_\mu e_R + i\bar{q}_L^T \gamma^\mu D_\mu q_L + i\bar{d}_R \gamma^\mu d_R + i\bar{u}_R \gamma^\mu D_\mu u_R \tag{5.57}$$

what we want to put in these covariant derivatives is give the LH terms the SU(2) covariant derivatives and give RH only U(1).

$$D_\mu \text{ for } l_L = \partial_\mu I + igt^a W_\mu^a + ig' Y(l_L) B_\mu I \tag{5.58}$$

$$\text{for } e_R = \partial_\mu + ig' Y(e_R) B_\mu \tag{5.59}$$

$$\text{for } d_R = \partial_\mu + ig_s t_s^a A_\mu^a + ig' Y(d_R) B_\mu \tag{5.60}$$

where g_s is the strong interaction coupling.

This produces the following interaction terms between leptons and gauge bosons

$$-\frac{g}{2}\begin{pmatrix}\bar{\nu}_L\\ \bar{e}_L\end{pmatrix}^T\gamma^\mu[\begin{pmatrix}W^3_\mu & \sqrt{2}W^+_\mu\\ \sqrt{2}W^-_\mu & W^3_\mu\end{pmatrix} - B_\mu I\tan\theta_W]\begin{pmatrix}\nu_L\\ e_L\end{pmatrix} \tag{5.61}$$

sometime we want to (finish discussion)..... . this has the same structure as the electroweak current, though this time the current is more complicated.

$$= -\frac{g}{2}(\bar{\nu}_L\bar{e}_L)\gamma^\mu[\begin{pmatrix}W^3_\mu - B_\mu\tan\theta_W & \sqrt{2}W^+\\ \sqrt{2}W^-_\mu & -W_m u^3 - B_\mu\tan\theta_W\end{pmatrix}]\begin{pmatrix}\nu_L\\ e_L\end{pmatrix} \tag{5.62}$$

$$= -\frac{g}{2}[\bar{\nu}_L\gamma^\mu(W^3_\mu - B_\mu\tan\theta_W)\nu_L + \bar{\nu}_L\gamma^\mu\sqrt{2}W^+_\mu e_L + \bar{e}_L\gamma^\mu\sqrt{2}W^-_\mu\nu_L - \bar{e}_L\gamma^\mu(W^3_\mu + B_\mu\tan\theta_W)e_L] \tag{5.63}$$

5.5.1 Interaction terms

A) $\bar{\nu}_L\gamma^\mu(W^3_\mu - B_\mu\tan\theta_W)\nu_L$: usign the definition of the Z field, $Z_\mu = W^3_\mu\cos\theta_W - B_\mu\sin\theta_W = Z_\mu$ (finish discussion)..... This produces the vertex: 1

$$= -\frac{g}{2}\cos\theta_W\bar{\nu}_L\gamma^\mu Z_\mu\nu_L \tag{5.64}$$

B) $-\frac{g}{2}\bar{\nu}_L\gamma^\mu\sqrt{2}W^+_\mu e_L$: 2 this is a charge raising process.

C) $-\frac{g}{2}\bar{e}_L\gamma^\mu\sqrt{2}W^-_\mu\nu_L$ 3 charge lowering process

D) $\frac{g}{2}\bar{e}_L\gamma^\mu(W^3_\mu + B_\mu\tan\theta_W)e_L$ We know that if we use our defintions (from the last couple of lectures)

$$W^3_\mu = Z_\mu\cos\theta_W + A_\mu\sin\theta_W \tag{5.65}$$

$$B_\mu = -Z_\mu\sin\theta_W + A_\mu\cos\theta_W \tag{5.66}$$

so if we look at interactions with photons (and this the photon-field, A_μ) we get

$$g\sin\theta_W\bar{e}_L\gamma^\mu A_\mu e_L \tag{5.67}$$

which leads to $g\sin\theta_W = e$, this is a sing of a unified theroy.... (finish discussion) so you recover all of hte known inteactions. If we had doen thins the other way rpund and done the kinetic terms first instead of the amsses, then it would be acase of getting ther ighti nteractions and finding fields which igve me the correct interactions... (finish discussion)

For right-handed fields

$$i\bar{e}_R\gamma^\mu(ig'Y(e_R)B_\mu)e_R \tag{5.68}$$

where $Y(e_R) = -1$, so one gets

$$g'\bar{e}_R\gamma^\mu B_\mu e_R = g'\bar{e}_R\gamma^\mu A_\mu\cos\theta_W e_R \tag{5.69}$$

but $g' = g\tan\theta_W$ and so we get

$$= g\sin\theta_W\bar{e}_R\gamma^\mu A_\mu e_R \tag{5.70}$$

Combining e_L and e_R terms

$$g\sin\theta_W A^\mu(\bar{e}_L\gamma_\mu e_L + \bar{e}_R\gamma_\mu e_R) = g\sin\theta_W A^\mu(\bar{e}\gamma_\mu e) \tag{5.71}$$

and this is what we know - A^μ is the photon field, $\bar{e}\gamma_\mu e$ is the Dirac current and $g\sin\theta_W$ is the electron charge so we recover everything we want to recover.

5.6 Summary

$$\mathcal{L}_{fermion-kinetic} = \bar{e}_L(i\partial\!\!\!/)e_L + \bar{e}_R(i\partial\!\!\!/)e_R + \bar{q}_L(i\partial\!\!\!/)q_L + \bar{u}_R(i\partial\!\!\!/)u_R + \bar{d}_R(i\partial\!\!\!/)d_R + g(W^+_\mu J^\mu_{W^+} + W^-_\mu J^\mu_{W^-} + Z_\mu J^\mu_Z + \tag{5.72}$$

we have separated the Lagrangian into different kinds of currents .... (finish discussion) and where

$$J^\mu_{W^+} = \frac{1}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_L + \bar{u}_L\gamma^\mu d_L) \tag{5.73}$$

$$J^\mu_{W^-} = \frac{1}{\sqrt{2}}(\bar{e}_L\gamma^\mu \nu_L + \bar{d}_L\gamma^\mu u_L) \tag{5.74}$$

$$J^\mu_{EM} = [\bar{e}\gamma^\mu(-1)e + \bar{u}\gamma^\mu(\frac{2}{3})u + \bar{d}\gamma^\mu(-\frac{1}{3})d] \tag{5.75}$$

however Z couples to everything, so we have am ore complcated structure for the Z current

$$J^\mu_Z = \frac{1}{\cos\theta_W}[\bar{\nu}_L\gamma^\mu\frac{1}{2}\nu_L + \bar{e}_L\gamma^\mu(-\frac{1}{2} + \sin^2\theta_W)e_L\bar{e}_R\gamma^\mu\sin^2\theta_W e_R + \bar{u}_L\gamma^\mu(\frac{1}{2} - \frac{2}{3}\sin^2\theta_W)u_L + \bar{u}_R\gamma^\mu(-\frac{2}{3})\sin \tag{5.76}$$

this essentially covers the whole picture for one geenration, QCD, Electroweak, Higgs, leptoons and quarks.

5.7 How to extend this theory to all geneartions

(finish discussion)

5.8 Open Questions (p)

1) Dark matter $\rightarrow$ SUSY extensions?
2) Neutrinos (could be linked DM via streile neutrios etc.), their mass?, their nature?
3) Gravity is incompatible with SM - we describe all itneracitons in a single way, but cant yet describe gravity - Strings/Brain theory or Extra dimensions
4) GUT (Planck scales)
5) Confinement - we dont know about what gluons and quarks do at very small distances - it is possible there is a string solution to this problem